ANALYSIS AND DESIGN OF PERFECT RECONSTRUCTION FILTER BANKS SATISFYING SYMMETRY CONSTRAINTS

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Abstract: Perfect reconstruction filter banks are analysis-synthesis systems with M channels and subsampling by N that have an input-output transfer function equal to a delay. In this paper, FIR filter structures are given that will guarantee perfect reconstruction and meet additional constraints like linear phase and/or frequency symmetries. Recursive structures are derived that will generate filter banks of any order satisfying the desired properties. The computational complexity of these structures is studied as well.

I Introduction

An analysis-synthesis system with M channels subsampled by N is depicted in figure 1. The case of most interest appears when $N = M$, that is, the system is critically subsampled, and we will be only considering this case in what follows. Furthermore, we will restrict both the analysis and the synthesis filters to be FIR. While such systems have been used in speech coding for quite sometime with so-called quadrature mirror filters (QMF) [3], solutions allowing perfect reconstruction are more recent [8,9,10,14,15,12].

In the simple, yet important case of two channels ($N = 2$), there are basically 2 solutions of interest: one which leads to a pair of minimum/maximum phase filters [8,9], and one which yields linear phase filters [14]. The first case yields a matrix of polyphase components [3,15] which is paraunitary [12], while it is easy to show that the second case is not, and that actually there is no paraunitary and linear phase solution for the case $N = 2$.

When the polyphase matrices are paraunitary, then they can be factored into cascades of unitary matrices and diagonal matrices of delays [1,12]. Therefore, nice cascaded lattice structures can be obtained for perfect reconstruction filter banks based on these paraunitary polyphase matrices [12,13]

In what follows, we first derive a cascade structure that generates perfect reconstruction, linear phase filter banks for $N = 2$. Then, this structure is generalized to $N > 2$, where it turns out that linear phase paraunitary solutions exist. As a further design constraint, we then require that the i-th and $(N - i)$-th filters are related by modulation. Again, this is possible for $N > 2$, even with an additional linear phase constraint. Finally, the computational complexity of the various structures is discussed.

II Analysis Framework

Let us consider a z-transform analysis of the system in figure 1. A filter with z-transform $H_i(z)$ followed by a subsampling by N is best described by its decomposition into polyphase components $H_{i,k}(z^N)$ [3,15,12].

$$H_i(z) = \sum_{k=0}^{N-1} H_{i,k}(z^N) z^{-k} \tag{1a}$$

$$H_{i,k}(z^N) = \sum_{n=0}^{\infty} h_{i,k+nN} z^{-nN} \tag{1b}$$

where $h_{i,n}$ are the elements of the impulse response of the i-th filter. For example, an unit impulse at time $-k$ will generate an output in the subsampled domain equal to the k-th polyphase component, that is $H_{i,k}(z)$. We can now define the following polyphase component matrix for the analysis filter bank:

$$\mathbf{H}_p(z) = \begin{pmatrix} H_{0,0}(z) & \dots & H_{0,N-1}(z) \\ H_{1,0}(z) & \dots & H_{1,N-1}(z) \\ \vdots & \cdot & \vdots \\ H_{M-1,0}(z) & \dots & H_{M-1,N-1}(z) \end{pmatrix} \tag{2a}$$

and, with an inversion of the order of the polyphase components, the polyphase matrix for the synthesis filter bank:

$$\mathbf{G}_p(z) = \begin{pmatrix} G_{0,N-1}(z) & \dots & G_{0,0}(z) \\ G_{1,N-1}(z) & \dots & G_{1,0}(z) \\ \vdots & \cdot & \vdots \\ G_{M-1,N-1}(z) & \dots & G_{M-1,0}(z) \end{pmatrix} \tag{2b}$$

Note that we always number row and columns starting from 0. It can be verified that a sufficient condition so that the analysis/synthesis system of fig. 1 is a perfect reconstruction system is that [12,15]:

$$[\mathbf{G}_p(z)]^T \cdot \mathbf{H}_p(z) = z^{-l} \cdot \mathbf{I} \ . \tag{3}$$

A necessary and sufficient condition for perfect reconstruction filter banks where the analysis and synthesis filters are equal (within time reversal) is that $\mathbf{H}_p(z)$ satisfies [12,16]:

$$[\mathbf{H}_p(z^{-1})]^T \cdot \mathbf{H}_p(z) = \mathbf{I} \tag{4}$$

Obviously in this case $\mathbf{G}_p(z)$ can be chosen as:

$$\mathbf{G}_p(z) = z^{-m} \cdot \mathbf{H}_p(z^{-1}) \tag{5}$$

where m is chosen so that $\mathbf{G}_p(z)$ leads to causal synthesis filters, and therefore, (3) is satisfied with $l = m$. Conversely if $\mathbf{G}_p(z)$ satisfies (5), i.e. perfect reconstruction is achieved with identical analysis and synthesis filter, then $\mathbf{H}_p(z)$ satisfies (4). In the case of critical sampling ($M = N$) a matrix $\mathbf{H}_p(z)$ that satisfies (4) is called a paraunitary matrix [12] and the product in (4) is commutative since $\mathbf{H}_p(z)$ is square. In the following, only critically sampled systems ($M = N$) will be considered.

III The Two Channel Case, $N = 2$

Let us first recall the paraunitary solution: in that case, the polyphase matrix $\mathbf{H}_p(z)$ of the analysis filters can be written as [12,13]:

$$\mathbf{H}_p(z) = \begin{pmatrix} 1 & \alpha_0 \\ -\alpha_0 & 1 \end{pmatrix} \cdot \prod_{k=1}^{K-1} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha_k \\ -\alpha_k & 1 \end{pmatrix} \quad (6)$$

This leads to minimum/maximum phase filters of length $L = 2K$. Note that (6) is a denormalized version, since the rotation matrices don't have unit length vectors. This can be taken care of at the reconstruction, with the polyphase matrix of the synthesis filters chosen as [12,13]:

$$\mathbf{G}_p(z) = \frac{1}{1+\alpha_0^2} \cdot \begin{pmatrix} 1 & \alpha_0 \\ -\alpha_0 & 1 \end{pmatrix} \cdot \prod_{k=1}^{K-1} \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_k \\ -\alpha_k & 1 \end{pmatrix} \cdot \frac{1}{1+\alpha_k^2} \quad (7)$$

The product $[\mathbf{G}_p(z)]^T \cdot \mathbf{H}_p(z)$ (see (3)) is equal to:

$$[\mathbf{G}_p(z)]^T \cdot \mathbf{H}_p(z) = z^{-(K-1)} \cdot \mathbf{I} \quad (8)$$

and perfect reconstruction is achieved with a delay of $L-1$ samples. Let us now consider linear phase filters, that is, for $N = 2$, two filters $H_0(z)$ and $H_1(z)$ obtained from $\mathbf{H}_p(z)$:

$$\begin{pmatrix} H_0(z) \\ H_1(z) \end{pmatrix} = \begin{pmatrix} H_{0,0}(z^2) & H_{1,0}(z^2) \\ H_{1,0}(z^2) & H_{1,1}(z^2) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} \quad (9)$$

and that are symmetric or antisymmetric (one of each). Then, it is easy to verify that $\mathbf{H}_p(z)$ has to satisfy (assuming even length filters):

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot [z^{-k} \cdot \mathbf{H}_p(z^{-1})] \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{H}_p(z) \quad (10)$$

where k is the highest degree in $\mathbf{H}_p(z)$ (note that a sign change in (10) is possible). We will now develop a recursive procedure to obtain linear phase filters of any even length which guarantee perfect reconstruction:
i) assume $\mathbf{H}_p(z)$ satisfies (10)
ii) then $\mathbf{H}'_p(z)$ given by:

$$\mathbf{H}'_p(z) = \mathbf{H}_p(z) \cdot \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} \quad (11)$$

satisfies (10) as well.

The proof is straightforward by replacing (11) into (10) and verifying that it holds indeed. Note that for $L = 2$, the two only possible filters are given by $H_0(z) = 1 + z^{-1}$ and $H_1(z) = 1 - z^{-1}$ (or scaled versions thereof) and therefore, a possible way to obtain length $L = 2K$ linear phase perfect reconstruction filters is by writing $\mathbf{H}_p(z)$ as:

$$\mathbf{H}_p(z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \prod_{k=1}^{K-1} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha_k \\ \alpha_k & 1 \end{pmatrix} \quad (12)$$

and $\mathbf{G}_p(z)$ as:

$$\mathbf{G}_p(z) = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \prod_{k=1}^{K-1} \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha_k \\ -\alpha_k & 1 \end{pmatrix} \cdot \frac{1}{1-\alpha_k^2} \quad (13)$$

Therefore, the product (3) is satisfied and equal to (8). Note that while this is not a paraunitary solution, the synthesis filters are simply related to the analysis filters by a modulation with $(-1)^n$:

$$G_0(z) = H_1(-z) \qquad G_1(z) = -H_0(-z) \quad (14)$$

The first few filters obtained from (12) are given in table I. Note that if one wants odd length filters (one of length $2N+1$ and the other of length $2N-1$) a zero can be exchanged between $H_0(z)$ and $H_1(-z)$ since this will not alter the perfect reconstruction property. For example, starting with $H_0(z) = 1 + 3z^{-1} + 3z^{-2} + z^{-3}$ and $H_1(z) = 1 + 3z^{-1} - 3z^{-2} - z^{-3}$, one can obtain $H'_0(z) = 1 + 2z^{-1} + 6z^{-2} + 2z^{-3} + z^{-4}$ and $H'_1(z) = 1 - 2z^{-1} + z^{-2}$ by exchanging a zero at $z = -1$ in $H_0(z)$ into a zero in $H'_1(-z)$. This technique has been used in [6] to derive good and efficient filters for sub-band coding of images.

Having derived a possible generic structure for linear phase filters, one may want to know its generality. Because $\mathbf{H}_p(z)$ is not paraunitary, there is no factorization theorem that can be used. Also, very particular perfect reconstruction pairs can be generated (for example, via the complementary filter method [15]) that would lead to singular factors in (12). However, such cases turn out to be degenerated and thus of little practical interest.

In conclusion of this section on the two channel case, note that while previous techniques existed to find linear phase perfect reconstruction filters (like the complementary filter method or the factorization method [15]) the above recursive form structurally guarantees perfect reconstruction (similarly to the paraunitary case [13]). Also, the factors α_i can be chosen so as to minimize the resulting hardware complexity of an implementation.

IV Linear Phase Solutions, $N > 2$

Similarly to (10), a polyphase matrix that leads to linear phase filters has to satisfy (we assume N even, and that the first N/2 filters are symmetric while the last $N/2$ are antisymmetric):

$$\begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \cdot [z^{-k} \cdot \mathbf{H}_p(z^{-1})] \cdot \mathbf{J} = \mathbf{H}_p(z) \quad (15)$$

where $\mathbf{I}$ is the identity matrix of size $N/2$ and $\mathbf{J}$ is the antidiagonal matrix of size N. In order to recursively generate polyphase matrices that satisfy (15), we write a new polyphase matrix $\mathbf{H}'_p(z)$ as:

$$\mathbf{H}'_p(z) = \mathbf{H}_p(z) \cdot \mathbf{D}(z) \cdot \mathbf{R} \quad (16)$$

where $\mathbf{H}_p(z)$ satisfies (15), $\mathbf{D}(z)$ is a diagonal matrix of delays and $\mathbf{R}$ is a unitary matrix. In that case, the following two conditions are necessary and sufficient for $\mathbf{H}'_p(z)$ to satisfy (15) as well:

$$z^{-1} \cdot \mathbf{J} \cdot \mathbf{D}(z^{-1}) \cdot \mathbf{J} = \mathbf{D}(z) \tag{17}$$

$$\mathbf{J} \cdot \mathbf{R} \cdot \mathbf{J} = \mathbf{R} \tag{18}$$

As a starting matrix, one can take any unitary transform that has $N/2$ symmetric and $N/2$ antisymmetric vectors, like for example the Walsh-Hadamard ($N = 2^m$) or the discrete cosine transform. There is obviously a large set of possible diagonal matrices of delays that satisfy (17). For example, in the case $N = 4$, we list the 4 possible matrices $\mathbf{D}(z)$:

$$\{\mathbf{D}(z)\} = \left\{ \begin{pmatrix} 1 & & & \\ & z^{-1} & & \\ & & z^{-1} & \\ & & & z^{-1} \end{pmatrix}, \begin{pmatrix} z^{-1} & & & \\ & z^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 1 & & & \\ & z^{-1} & & \\ & & 1 & \\ & & & z^{-1} \end{pmatrix}, \begin{pmatrix} z^{-1} & & & \\ & 1 & & \\ & & z^{-1} & \\ & & & 1 \end{pmatrix} \right\} \tag{19}$$

The condition (18) on the matrix $\mathbf{R}$ is even more relaxed. For example, all symmetric Toeplitz matrices satisfy (18). A closer look shows that matrices satisfying (18) are of the form:

$$\mathbf{R} = \begin{pmatrix} \mathbf{M}_0 & \mathbf{M}_1 \\ \mathbf{J}\mathbf{M}_1\mathbf{J} & \mathbf{J}\mathbf{M}_0\mathbf{J} \end{pmatrix} \tag{20}$$

where $\mathbf{M}_0$ and $\mathbf{M}_1$ are size $N/2$ by $N/2$ matrices, and $\mathbf{J}$ is the antidiagonal matrix of size $N/2$. One can verify that (18) is satisfied since:

$$\begin{pmatrix} \mathbf{0} & \mathbf{J} \\ \mathbf{J} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{M}_0 & \mathbf{M}_1 \\ \mathbf{J}\mathbf{M}_1\mathbf{J} & \mathbf{J}\mathbf{M}_0\mathbf{J} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{J} \\ \mathbf{J} & \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{M}_0 & \mathbf{M}_1 \\ \mathbf{J}\mathbf{M}_1\mathbf{J} & \mathbf{J}\mathbf{M}_0\mathbf{J} \end{pmatrix} \tag{21}$$

where we used the fact that $\mathbf{J}^2 = \mathbf{I}$. If $\mathbf{R}$ is required to be unitary, that is (assuming real coefficients):

$$\mathbf{R}^T \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I} \tag{22}$$

then it can be verified that the matrices $\mathbf{M}_0$ and $\mathbf{M}_1$ have to satisfy:

$$\mathbf{M}_0 \cdot \mathbf{M}_0^T + \mathbf{M}_1 \cdot \mathbf{M}_1^T = \mathbf{I} \tag{23a}$$

$$\mathbf{M}_1 \cdot \mathbf{J} \cdot \mathbf{M}_0^T + \mathbf{M}_0 \cdot \mathbf{J} \cdot \mathbf{M}_1^T = 0 \tag{23b}$$

which corresponds to $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$. Since the product (22) is commutative, $\mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$ leads to another but equivalent set of conditions. Note that (23a) is the usual orthonormality of the first $N/2$ rows of $\mathbf{R}$, while (23b) captures the inherent "symmetry" of $\mathbf{R}$ that is required in order to meet (18). A simple example, for $N = 4$, would be:

$$\mathbf{R} = \frac{1}{\sqrt{2(1+\alpha^2)}} \cdot \begin{pmatrix} -1 & 1 & \alpha & \alpha \\ 1 & 1 & -\alpha & \alpha \\ \alpha & -\alpha & 1 & 1 \\ \alpha & \alpha & 1 & -1 \end{pmatrix} \tag{24}$$

which satisfies (23) as can be checked. Instead of a post-multiplication in (16), one may want a pre-multiplication. It turns out that that no delay matrix $\mathbf{D}(z)$ (except trivial ones) will satisfy (15), but that any matrix $\mathbf{R}$ which is block-diagonal with blocks of size $N/2$ by $N/2$ will. Therefore, any solution $\mathbf{H}_p(z)$ can be modified by a pre-multiplication with such a block-diagonal matrix. This is similar to techniques used in so-called "lapped orthogonal transforms" (LOT) [7], which can be seen as filter banks where the filters are restricted to $L = 2N$ [16].

In summary, this section showed a possible way, for more than two channels ($N > 2$), to generate paraunitary filter matrices of any degree meeting the additional constraint of linear phase of the resulting filters.

V Filters Satisfying Frequency Symmetry Constraints

In a bank of N filters, it is often desired that the higher frequency filters are obtained by modulation of the lower frequency ones, and this both for computational reasons and because of the desired frequency responses. The extreme case is when a single prototype low-pass filter is modulated over the whole frequency range, and the simplest case appears when pairs of high and low frequency filters are related by a modulation with $(-1)^n$. We will consider only the latter case here, the former having been addressed in [14,11]. Let us assume the following relation between filters (N is even):

$$H_{N-i-1}(z) = H_i(-z) \quad i = 0 \cdots N/2 - 1 \tag{25}$$

For $N = 2$, this is the classical QMF case. Now, the condition in (25) forces symmetries in the polyphase matrix $\mathbf{H}_p(z)$, since even numbered polyphase components of $H_i(z)$ and $H_{N-i-1}(z)$ are equal, while odd numbered ones have opposite sign. Define a matrix $\mathbf{T}$ of size N by N as:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & \dots & 0 & 0 & -1 \\ 0 & 1 & 0 & \dots & \dots & 0 & 1 & 0 \\ . & . & . & \dots & \dots & . & . & . \\ 0 & \dots & 0 & 1 & -1 & 0 & \dots & 0 \end{pmatrix} \tag{26}$$

Now, because of the symmetry in (25), we can verify that:

$$[\mathbf{T} \cdot \mathbf{H}_p(z)]_{i,j} = 2H_{i,j}(z), \quad i + j \text{ even}$$

$$= 0, \quad i + j \text{ odd} \tag{27}$$

where i, j is the row/column index with the numbering starting from 0. As in previous sections, we would like to be able to generate recursively polyphase matrices satisfying (27). More precisely, assume we have a matrix $\mathbf{H}_p(z)$ that satisfies (27), and that we obtain a matrix $\mathbf{H}'_p(z)$ by pre- or post-multiplying $\mathbf{H}_p(z)$ with a delay or "rotation" matrix, then how should we choose these so that $\mathbf{H}'_p(z)$ satisfies (27) as well. Note that the delay and "rotation" matrix should be independent of $\mathbf{H}_p(z)$. Let us consider the following 4 cases.

a) Pre-multiplication by a delay matrix $\mathbf{D}(z)$: It is easy to verify that $\mathbf{D}(z)$ (which is diagonal by assumption) has to meet:

$$[\mathbf{D}(z)]_{i,i} = [\mathbf{D}(z)]_{N-i-1,N-i-1} \tag{28}$$

b) Pre-multiplication by a rotation matrix $\mathbf{R}$: We call this a "rotation" matrix because it will be in general chosen as a unitary matrix (but not necessarily). Then the rows of $\mathbf{R}$ have to be related by:

$$row_{N-i-1} = row_i \cdot \mathbf{J} \tag{29}$$

so that $\mathbf{H}'_p(z)$ meets (27).
c) Post-multiplication by a delay matrix $\mathbf{D}(z)$: No condition is necessary, since as long as $\mathbf{D}(z)$ is diagonal, $\mathbf{H}'_p(z)$ will meet (27).
d) Post-multiplication by a rotation matrix $\mathbf{R}$: In that case, it is necessary and sufficient that

$$\mathbf{R}_{i,j} = 0, \quad i+j \ odd \tag{30}$$

Then, $\mathbf{H}'_p(z)$ will meet (27) as well.

Note that a)-d) give conditions so that (27) is met recursively, and this independently of the previous terms in the cascade. Now, if the matrices $\mathbf{D}(z)$ and $\mathbf{R}$ meet additional constraints, like the ones required for linear phase or paraunitarity, then the resulting polyphase matrix will yield linear phase filters or be paraunitary (on top of leading to the frequency symmetry given by (25)). As a simple example, look at the following post-multiplication matrix:

$$\mathbf{R}_\alpha = \frac{1}{\sqrt{1+\alpha^2}} \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & -\alpha \\ -\alpha & 0 & 1 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix} \tag{31}$$

This matrix satisfies (18) (linear phase constraint), (22) (it is a unitary matrix) and (30) (frequency symmetry). When used together with delay matrices as in (19) as well as a starting matrix that satisfies (27) (like the Walsh-Hadamard or the discrete cosine transform matrix of size 4 by 4), this rotation matrix will lead to a perfect reconstruction, linear phase filter bank with frequency symmetry ($N = 4$, arbitrary filter length and identical analysis and synthesis filters). Of course, (31) is very constrained, and does not lead to very interesting filters. The point was to show that solutions to such heavily constrained filter banks exist and can be constructed recursively.

VI Computational Complexity

Because of their special structure the filter banks introduced in this paper can have very low computational complexity. We will concentrate mainly on the two channel case. Let us first review the paraunitary case (see (6)), where the computational blocks are 2 by 2 rotation matrices [13]. Since a rotation matrix can always be written as [2] (with $a = \cos(\alpha)$ and $b = \sin(\alpha)$):

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & & \\ & a+b & \\ & & b-a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{32}$$

it takes 3 multiplications per rotation. A bank with 2 filters of length $L = 2K$ has K such blocks and 2 input values produce 2 new output values (one in each channel), thus the computational complexity of a paraunitary 2 channels filter bank is $(3K/2)$ multiplications per input sample. This was already noted in [5] without using factorization. Now, if one uses denormalized blocks (for example, dividing (32) by a) then each block takes only two multiplications, and one multiplication is required at the end in each channel in order to renormalize the output. The computational complexity per input sample is therefore equal to $(K+1)$. This result was noted in [4].

Let us consider the linear phase case next (see (12)) . Each computational block can be written as:

$$\frac{1}{\sqrt{1-a^2}} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1+a}{2\sqrt{1-a^2}} & \\ & \frac{1-a}{2\sqrt{1-a^2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad a \neq \{1,-1\} \tag{33}$$

The factor $1/\sqrt{1-a^2}$ was used to "normalize "the matrix (in a loose sense since the matrix is not orthogonal). Since a block written as (33) requires 2 multiplications and that there are $K-1$ blocks, the multiplicative complexity of a length $L = 2K$ filter bank is $(K-1)$ multiplications per input sample. Now, if (33) is "denormalized", that is:

$$\frac{2}{1+a} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ & \frac{1-a}{1+a} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad a \neq \{1,-1\} \tag{34}$$

then each block requires only 1 multiplication plus one at the end to renormalize the result in each channel, that is a total of $(K+1)/2$ multiplication per input sample. Note that this is half as much as in the paraunitary case. The number of additions in (34) can actually be reduced by 1 [4]. A hardware structure implementing a linear phase perfect reconstruction filter bank is shown in figure 2.

Finally, and for completeness, we review the "classical" QMF case. In that case, $H_1(z) = H_0(-z)$, and $H_0(z)$ is a linear phase (symmetric) filter. It can be verified that the corresponding polyphase matrix can be written as:

$$H_p(z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \prod_{k=1}^{K-1} \begin{pmatrix} (1+\alpha_k z^{-1}) & \\ & (\alpha_k + z^{-1}) \end{pmatrix} \tag{35}$$

Note that perfect reconstruction can only be approximated since the determinant of $H_p(z)$ is not a delay (nor a minimum phase filter). Now (35) takes 2 multiplications per block, that is assuming a normalization at the output of each channel, a complexity of K multiplication per input sample. This result is well known [15] even without going through a factorization such as (35), but the form of the diagonal matrix in (35) is such that no "denormalization" will reduce the complexity further as it did in the other cases.

Table II summarizes the various computational complexities derived in this section. Note that the restrictions put on the building blocks have actually reduced the computational complexity, a result that holds for $N > 2$ as well.

VII Conclusion

This paper has shown sufficient conditions to generate recursively perfect reconstruction filter banks meeting additional constraints, like linear phase and/or frequency symmetries.

The method used was to cascade independent blocks, where the constraints were met at each intermediate stage as well. While the condition that a perfect reconstruction filter bank can be generated recursively with independent blocks is sometimes too restrictive (for ex., some solutions are not reachable in the case $N = 2$ with linear phase), cases of interest have been generated with the proposed cascade forms. Additionally, these cascade forms are very convenient for synthesis and implementation purposes.

Using these structures, it was shown that even very restricted designs of filter banks can exist, like paraunitary solutions with both linear phase and frequency symmetry for $N = 4$. Future work is necessary however in order to design filters meeting desired frequency responses in addition to the above, structurally enforced properties of the global filter bank.

Finally, the computational complexity of these constrained filter banks was addressed, and shown to be fairly low. For example, a linear phase two channel perfect reconstruction filter bank with filters of length L requires only about $L/4$ multiplications per input sample. This is about half as much as that required by the classical QMF or the minimum/maximum phase filter solutions.

In conclusion, new cascade solutions have been demonstrated for constrained, perfect reconstruction filter bank systems with arbitrary long FIR filters.

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L	Scaling factor	Analysis & synthesis Filters
2	2	$H_0(z) = 1 + z^{-1}$ $H_1(z) = 1 - z^{-1}$
4	$2(1-\alpha_1^2)$	$H_0(z) = 1 + \alpha z^{-1} + \alpha z^{-2} + z^{-3}$ $H_1(z) = 1 + \alpha z^{-1} - \alpha z^{-2} - z^{-3}$
6	$2(1-\alpha_1^2)(1-\alpha_2^2)$	$H_0(z) = 1 + \alpha_2 z^{-1} + (\alpha_1 + \alpha_1\alpha_2)z^{-2} + (\alpha_1 + \alpha_1\alpha_2)z^{-3} + \alpha_2 z^{-4} + z^{-5}$ $H_1(z) = 1 + \alpha_2 z^{-1} + (-\alpha_1 + \alpha_1\alpha_2)z^{-2} + (\alpha_1 - \alpha_1\alpha_2)z^{-3} - \alpha_2 z^{-4} - z^{-5}$

Table I: Linear phase perfect reconstruction filters obtained from the cascade form (12). The scaling factor is the term that has to be divided out between analysis and synthesis. The synthesis filters are given by $G_0(z) = H_1(-z)$ and $G_1(z) = -H_0(-z)$.

Type of Filter bank	Normalized block	Denormalized block
min/max phase (paraunitary)	3K/2	K+1
Linear Phase	K-1	(K+1)/2
Classical QMF	K	-

Table II: Number of multiplications for two channel filter banks (for each new input and filters of length $L = 2K$).

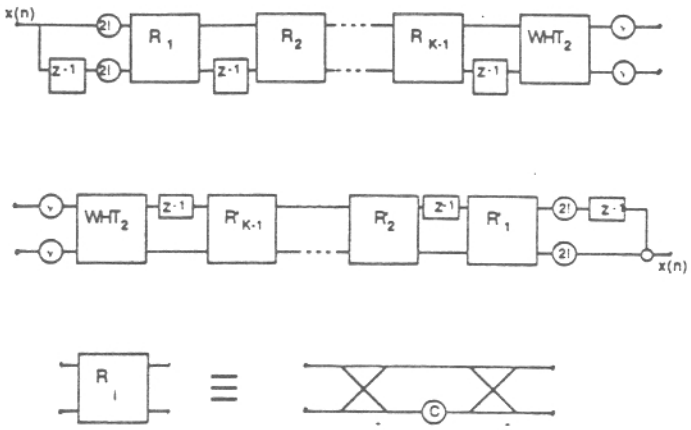

Fig. 2: a) Linear phase, perfect reconstruction two channel analysis filter bank
b) Equivalent synthesis filter bank
c) One multiplier elementary block implementation

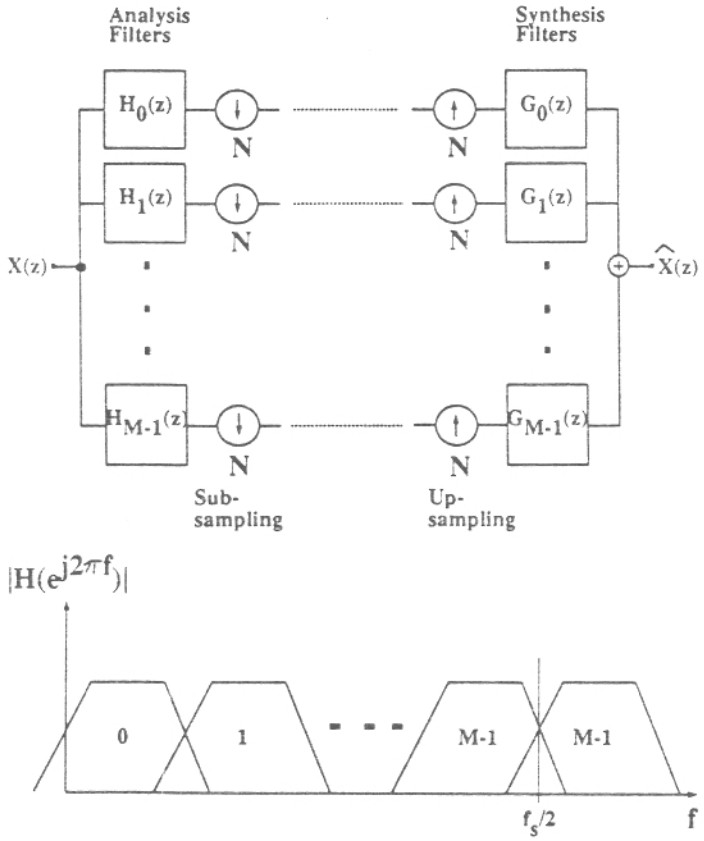

Fig. 1: Analysis/synthesis system with M channels and subsampling by N, as well as typical frequency responses of the filters.

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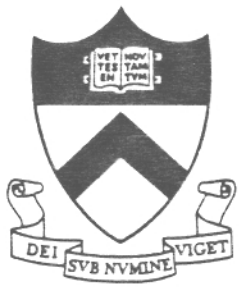

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