

Emergence of Concentration Effects in the Variational Analysis of the N -Clock Model

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Abstract

We investigate the relationship between the N -clock model (also known as planar Potts model or \mathbb{Z}_N -model) and the XY model (at zero temperature) through a Γ -convergence analysis of a suitable rescaling of the energy as both the number of particles and N diverge. We prove the existence of rates of divergence of N for which the continuum limits of the two models differ. With the aid of Cartesian currents we show that the asymptotics of the N -clock model in this regime features an energy that may concentrate on geometric objects of various dimensions. This energy prevails over the usual vortex-vortex interaction energy. © 2021 The Authors. *Communications on Pure and Applied Mathematics* published by Wiley Periodicals LLC.

1 Introduction

Classical ferromagnetic spin systems on lattices represent fundamental models to understand phase transition phenomena. On the one hand, the study of their properties has motivated the introduction of new mathematical tools which have provided useful insights for a number of problems arising in different fields. On the other hand, many techniques borrowed from probability theory, mathematical analysis, topology, and geometry have contributed to a better understanding of the properties of these systems.

In this paper we make use of fine concepts in geometric measure theory and in the theory of Cartesian currents to understand the relationship between the XY model and the N -clock model (also known as the planar Potts model or \mathbb{Z}_N -model) within a variational framework. The N -clock model is a two-dimensional nearest-neighbors ferromagnetic spin model on the square lattice in which the spin field is constrained to take values in a set of N equispaced points of \mathbb{S}^1 . For N

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large enough, it is usually considered as an approximation of the XY (planar rotator) model, for which the spin field can attain all the values of \mathbb{S}^1 . The asymptotic behavior of the N -clock model for large N has been considered by Fröhlich and Spencer in the seminal paper [34]. There the authors have proved that both the N -clock model (for N large enough) and the XY model present Berezinskii-Kosterlitz-Thouless transitions, i.e., phase transitions mediated by the formation and interaction of topological singularities. The microscopic picture leading to the emergence of such topological phase transitions (first introduced in [16, 45, 46]) is a result of a nontrivial interplay between entropic and energetic effects that takes place at different length scales.

This paper contributes to precisely relating the N -clock model and the XY model at zero temperature. Specifically, we show that the enhancement of symmetry, from the discrete one of the N -clock model to the continuous one of the XY model, comes along with concentration of energy on geometric objects of various dimension. This is achieved by studying a suitably rescaled version of the energy of the N -clock model as N diverges, through a coarse graining procedure that is made rigorous by Γ -convergence; see [20, 32]. A crucial step of this analysis is the choice of the topologies that best identify the relevant variables of the coarse-grained model and lead to the effective description of the microscopic/mesoscopic geometry of the spin field. In contrast to the XY model, the sole study of the distributional Jacobian of the spin field turns out to provide not enough information on the concentration effects of the energy; we shall see how these effects can be detected by Cartesian currents, for the first time introduced in the context of lattice spin models.

In what follows we present the model and our main result. We consider a bounded, open set with Lipschitz boundary $\Omega \subset \mathbb{R}^2$. Given a small parameter $\varepsilon > 0$, we consider $\Omega_\varepsilon := \Omega \cap \varepsilon\mathbb{Z}^2$. The classical XY model is defined on spin fields $u: \Omega_\varepsilon \rightarrow \mathbb{S}^1$ by

$$(1.1) \quad - \sum_{\langle i, j \rangle} \varepsilon^2 u(\varepsilon i) \cdot u(\varepsilon j),$$

where the sum is taken over ordered pairs of nearest neighbors $\langle i, j \rangle$, i.e., $(i, j) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ such that $|i - j| = 1$ and $\varepsilon i, \varepsilon j \in \Omega_\varepsilon$. The variational analysis of the XY model is part of a larger program devoted to the study of systems of spins with continuous symmetry [4, 5, 8, 13–15, 25, 27, 30, 31, 50].

Here we consider an additional parameter $N_\varepsilon \in \mathbb{N}$ or, equivalently, $\theta_\varepsilon := \frac{2\pi}{N_\varepsilon}$. The admissible spin fields we consider here are only those taking values in the discrete set $\mathcal{S}_\varepsilon := \{\exp(ik\theta_\varepsilon): k = 0, \dots, N_\varepsilon - 1\} \subset \mathbb{S}^1$; i.e., we consider the energy

$$F_\varepsilon(u) := - \sum_{\langle i, j \rangle} \varepsilon^2 u(\varepsilon i) \cdot u(\varepsilon j) \quad \text{if } u: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon,$$

extended to $+\infty$ otherwise. For $N_\varepsilon = N \in \mathbb{N}$, with N independent of ε , the spin system described by the energy F_ε is usually referred to as N -clock model;

cf. [34]. The particular case where $N = 2$ is the so-called Ising system, recently analyzed in [3, 19, 23, 26, 42]. See also [2, 6, 9, 21] for the long-range case.

As (1.1) is minimized on constant spin fields, one refers the energy to its minimum

$$XY_\varepsilon(u) = - \sum_{(i,j)} \varepsilon^2 (u(\varepsilon i) \cdot u(\varepsilon j) - 1) = \frac{1}{2} \sum_{(i,j)} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2.$$

Analogously, we set

$$(1.2) \quad E_\varepsilon(u) := F_\varepsilon(u) - \min F_\varepsilon = XY_\varepsilon(u) \quad \text{if } u: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon,$$

extended to $+\infty$ otherwise, and we find the scalings $\kappa_\varepsilon \rightarrow 0$ for which $\frac{1}{\kappa_\varepsilon} E_\varepsilon$ has a nontrivial variational limit. These are affected by N_ε , as it emerges in the two limiting scenarios $N_\varepsilon = 2$ and $\mathcal{S}_\varepsilon = \mathbb{S}^1$ (formally corresponding to $N_\varepsilon = +\infty$). If $N_\varepsilon = 2$, $\frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon)$ approximates an anisotropic interfacial energy between the phases $(1, 0)$ and $(-1, 0)$; see [3]. In contrast, for the XY system (i.e., $\mathcal{S}_\varepsilon = \mathbb{S}^1$), it has been shown in [4, example 1] that no interfacial-type energy emerges at any scaling $\kappa_\varepsilon \gg \varepsilon^2$. Indeed, if u_ε interpolates (linearly in the angle) from $u^- = \exp(i\varphi^-)$ to $u^+ = \exp(i\varphi^+)$ on a length-scale of size η_ε , the energy amounts to

$$(1.3) \quad \frac{1}{\kappa_\varepsilon} XY_\varepsilon(u_\varepsilon) \sim \left(1 - \cos\left(\frac{\varepsilon}{\eta_\varepsilon}(\varphi^+ - \varphi^-)\right)\right) \frac{\eta_\varepsilon}{\kappa_\varepsilon} \sim \frac{\varepsilon^2}{\eta_\varepsilon \kappa_\varepsilon},$$

which goes to 0 if $\eta_\varepsilon \gg \frac{\varepsilon^2}{\kappa_\varepsilon}$. This construction may not be feasible when $\mathcal{S}_\varepsilon \neq \mathbb{S}^1$ if the minimal angle θ_ε satisfies $\theta_\varepsilon \gtrsim \frac{\kappa_\varepsilon}{\varepsilon}$. In the constrained case, choosing the largest possible length-scale $\eta_\varepsilon = |\varphi^+ - \varphi^-| \frac{\varepsilon}{\theta_\varepsilon}$, one gets (denoting by $d_{\mathbb{S}^1}$ the geodesic distance on \mathbb{S}^1)

$$(1.4) \quad \begin{aligned} \frac{1}{\kappa_\varepsilon} E_\varepsilon(u_\varepsilon) &\sim (1 - \cos(\theta_\varepsilon)) \frac{\varepsilon}{\theta_\varepsilon \kappa_\varepsilon} |\varphi^+ - \varphi^-| \sim \frac{\varepsilon \theta_\varepsilon}{\kappa_\varepsilon} |\varphi^+ - \varphi^-| \\ &\sim \frac{\varepsilon \theta_\varepsilon}{\kappa_\varepsilon} d_{\mathbb{S}^1}(u^+, u^-), \end{aligned}$$

which suggests that $\kappa_\varepsilon = \varepsilon \theta_\varepsilon$ leads to an energy proportional to a BV total variation (in the sense of [11, formula (2.11)]). In fact, in Proposition 4.1, we prove that sublevel sets of $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon$ are precompact in $BV(\Omega; \mathbb{S}^1)$ equipped with the L^1 topology.

Given $u_\varepsilon: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ with $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$ we have

$$(1.5) \quad \frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon) = \frac{\varepsilon \theta_\varepsilon}{\varepsilon^2 |\log \varepsilon|} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \sim \frac{\theta_\varepsilon}{\varepsilon |\log \varepsilon|}.$$

As is known from the theory of the XY model [4, 5] (see also [1, 10, 18, 43, 44, 47–49] for the Ginzburg-Landau theory), boundedness of $\frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon)$ implies flat compactness (i.e., with respect to the norm induced by duality with compactly supported Lipschitz functions) of the discrete vorticity measure μ_{u_ε} , which counts the winding number of u_ε at each point of $\varepsilon \mathbb{Z}^2$, cf. (2.4). If $\varepsilon |\log \varepsilon| \ll \theta_\varepsilon$, (1.5)

gives no bound on $\frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon)$ and suggests that μ_{u_ε} does not play a role in the asymptotics of $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon$. In fact, in [28] we prove that $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon$ Γ -converges to an anisotropic total variation in $BV(\Omega; \mathbb{S}^1)$, while in [29] we identify the Γ -limit for N fixed and the asymptotic behavior of the limit functionals when $N \rightarrow +\infty$ in any dimension. Here we are interested in regimes for which the limit cannot be exhaustively described in BV .

We start by assuming $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$, which by (1.5) implies $\mu_{u_\varepsilon} \xrightarrow{f} 0$. This constraint will induce a Γ -limit (possibly strictly) larger than the anisotropic total variation in $BV(\Omega; \mathbb{S}^1)$. To prove this fact, our idea is to associate to u_ε with $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$ the current G_{u_ε} given by the extended graph in $\Omega \times \mathbb{S}^1$ of its piecewise constant interpolation; see Section 3.5. Since $\partial G_{u_\varepsilon} = -\mu_{u_\varepsilon} \times \llbracket \mathbb{S}^1 \rrbracket^1$ and $\mu_{u_\varepsilon} \xrightarrow{f} 0$, the limit T of the currents G_{u_ε} satisfies $\partial T = 0$ and, more precisely, is a Cartesian current in $\text{cart}(\Omega \times \mathbb{S}^1)$. For this reason, the limit of $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon$ in this regime shares strong similarities with the L^1 -relaxation of the $W^{1,1}$ -norm of maps in $C^1(\Omega; \mathbb{S}^1)$; cf. [35, 38]. The Γ -limit, cf. Propositions 4.8 and 4.16, features a term reminiscent of the BV -type concentration of $|Du_\varepsilon|$ possibly induced by the topological constraint $\mu_{u_\varepsilon} \xrightarrow{f} 0$. This term is, in general, not expressible as an integral functional on the limit of u_ε .

Our main theorem concerns the regime in which u_ε displays simultaneously vortex-type and BV -type concentration effects. The discretization (in the domain and the codomain) $v_\varepsilon: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ of a vortex $\frac{x-x_0}{|x-x_0|}$ satisfies

$$\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(v_\varepsilon) \sim 2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \rightarrow +\infty \quad \text{if } \theta_\varepsilon \ll \varepsilon |\log \varepsilon|;$$

cf. (4.73).

To obtain a finer description of the limit, we renormalize E_ε by removing the diverging energy of M vortices and by studying the excess energy $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon}$. A bound on the latter energy yields (cf. Proposition 4.10)

$$\mu_{u_\varepsilon} \xrightarrow{f} \mu = \sum_{h=1}^N d_h \delta_{x_h}, \quad d_h \in \mathbb{Z},$$

and $|\mu|(\Omega) \leq M$. If $|\mu|(\Omega) = M$, the diverging energy $2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon}$ has been saturated by μ and a finite energy $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon$ is still accessible to the system. This might lead to BV -type concentration effects, detected by the current T , the limit of the extended graphs G_{u_ε} . Since $\partial G_{u_\varepsilon} = -\mu_{u_\varepsilon} \times \llbracket \mathbb{S}^1 \rrbracket$ and $\mu_{u_\varepsilon} \xrightarrow{f} \mu$, T satisfies the nontrivial constraint $\partial T = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$. This condition couples the vortex-type

¹By $\llbracket \mathbb{S}^1 \rrbracket$ we mean the current given by the integration over \mathbb{S}^1 oriented counterclockwise.

and BV -type concentration effects displayed by the spin field, resulting in a term $\mathcal{J}(\mu, u; \Omega)$ in the Γ -limit.² This leads to our main result.³

THEOREM 1.1. *Assume that $\varepsilon \ll \theta_\varepsilon \ll \varepsilon |\log \varepsilon|$. Then the following results hold true:*

(i) *Compactness.* Let $M \in \mathbb{N}$ and let $u_\varepsilon: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ be such that

$$\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \leq C.$$

Then there exists a measure $\mu = \sum_{h=1}^N d_h \delta_{x_h}$, $x_h \in \Omega$, $d_h \in \mathbb{Z}$ such that (up to a subsequence) $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ and $|\mu|(\Omega) \leq M$. If, in addition, $|\mu|(\Omega) = M$, then there exists a function $u \in BV(\Omega; \mathbb{S}^1)$ such that (up to a subsequence) $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$.

(ii) Γ -liminf inequality. Let $u_\varepsilon: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ and let

$$\mu = \sum_{h=1}^N d_h \delta_{x_h}, \quad x_h \in \Omega, \quad d_h \in \mathbb{Z} \quad \text{with } |\mu|(\Omega) = M.$$

Assume that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$. Let $u \in BV(\Omega; \mathbb{S}^1)$ be such that $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$. Then

$$\int_{\Omega} |\nabla u|_{2,1} dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega) \leq \liminf_{\varepsilon \rightarrow 0} \left(\frac{E_\varepsilon(u_\varepsilon)}{\varepsilon \theta_\varepsilon} - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right).$$

(iii) Γ -limsup inequality. Let $\mu = \sum_{h=1}^N d_h \delta_{x_h}$, $x_h \in \Omega$, $d_h \in \mathbb{Z}$ with $|\mu|(\Omega) = M$, and let $u \in BV(\Omega; \mathbb{S}^1)$. Then there exists a sequence $u_\varepsilon: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$, $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$, and

$$\limsup_{\varepsilon \rightarrow 0} \left(\frac{E_\varepsilon(u_\varepsilon)}{\varepsilon \theta_\varepsilon} - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) \leq \int_{\Omega} |\nabla u|_{2,1} dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega).$$

The case $\theta_\varepsilon \sim \varepsilon |\log \varepsilon|$ is studied in [28]. If $\theta_\varepsilon \ll \varepsilon$, in [28] we prove that $\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon|$ approximates the renormalized and core energies obtained in the first-order analysis of the XY model carried out in [8]. Instead, Theorem 1.1 points out that the N -clock and XY models exhibit different asymptotic behaviors if $\varepsilon \ll \theta_\varepsilon \ll \varepsilon |\log \varepsilon|$. This is due to the arising of a surprising interaction between

²It is given by $\mathcal{J}(\mu, u; \Omega) := \inf\{\int_{J_T} \ell_T(x) |v_T(x)|_1 d\mathcal{H}^1(x) : T \in \text{Adm}(\mu, u; \Omega)\}$, where $\text{Adm}(\mu, u; \Omega)$, defined in (4.18), is a suitable class of currents T satisfying, in particular, the constraint $\partial T = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$. Here J_T is the 1-codimensional jump-concentration set of T oriented by the normal v_T . At each point $x \in J_T$, the current T has a vertical part, given by a (not necessarily geodesic) arc in \mathbb{S}^1 of length $\ell_T(x)$ which connects the traces of u on the two sides of J_T . The set-function $\mathcal{J}(\mu, u; \cdot)$ is not subadditive.

³In Theorem 1.1 the matrix norm $|\cdot|_{2,1}$ reflects the anisotropy of the lattice, see Section 2.

vortex-type and BV -type concentration effects. Coexistence of singularities of two different dimensions has been already observed in other models, e.g. [15, 39]. The difference is that here they naturally appear as a result of both the dependence on ε of the codomain and of topological obstructions.

We highlight here some of the main technical difficulties in the very delicate construction of the recovery sequence in the proof of Theorem 1.1. Given $u \in BV(\Omega; \mathbb{S}^1)$, we define its recovery sequence following a gradual approximation procedure, which involves a series of steps of increasing complexity. At each of these steps, the map u is modified without essentially changing the energy.

The first main issue is to regularize the map u . Maps in $BV(\Omega; \mathbb{S}^1)$ cannot always be approximated in energy by \mathbb{S}^1 -valued smooth functions (in general they cannot be lifted without increasing the BV -norm [24, 41]). Nonetheless, the result in [17] (see also [7]) guarantees the density of \mathbb{S}^1 -valued maps that are smooth outside finitely many point-singularities. These are related to the vorticity measure μ using the approximation theorem for Cartesian currents; cf. Lemma 4.17. The next main issue is to construct a recovery sequence u_ε for such a regularization of u . Close to each singularity, u_ε is defined by discretizing (in domain and codomain) a proper translation of $\frac{x}{|x|}$. The energy carried by this discrete spin field close to a singularity diverges as $2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon}$. Far from the singularities, the problem reduces to the construction of a recovery sequence for a smooth \mathbb{S}^1 -valued map. This can be further simplified to the case of a piecewise constant \mathbb{S}^1 -valued map by introducing a mesoscopic scale into the problem; see Lemma 4.13. For such maps, the construction is a refinement of the one described above to obtain (1.4).

The most delicate step is to merge the different parts of the recovery sequence close to and far from the singularities. This is achieved in the proof of Proposition 4.22 (step 2) by a careful interpolation on dyadic layers of mesoscopic squares, whose size is chosen to be smaller for layers closer to the singularity. At each layer generation, $\frac{x}{|x|}$ is sampled at a different mesoscopic length-scale. The latter is optimized in order to provide the correct control on the energy in progressing from each layer to the next one.

2 Notation and Preliminaries

We denote the imaginary unit by ι . We shall identify \mathbb{R}^2 with \mathbb{C} . Given $a = (a_1, a_2) \in \mathbb{R}^2$, its 1-norm is $|a|_1 = |a_1| + |a_2|$. We define the $|\cdot|_{2,1}$ -norm of a matrix $A = (a_{ij}) \in \mathbb{R}^{2 \times 2}$ by $|A|_{2,1} := (a_{11}^2 + a_{21}^2)^{1/2} + (a_{12}^2 + a_{22}^2)^{1/2}$.

If $u, v \in \mathbb{S}^1$, their geodesic distance on \mathbb{S}^1 is denoted by $d_{\mathbb{S}^1}(u, v)$. It is given by the angle in $[0, \pi]$ between the vectors u and v , i.e., $d_{\mathbb{S}^1}(u, v) = \arccos(u \cdot v)$. Observe that

$$(2.1) \quad \frac{1}{2}|u - v| = \sin\left(\frac{1}{2}d_{\mathbb{S}^1}(u, v)\right) \quad \text{and} \quad |u - v| \leq d_{\mathbb{S}^1}(u, v) \leq \frac{\pi}{2}|u - v|.$$

Given two sequences α_ε and β_ε , we write $\alpha_\varepsilon \ll \beta_\varepsilon$ if $\lim_{\varepsilon \rightarrow 0} \frac{\alpha_\varepsilon}{\beta_\varepsilon} = 0$. We will use the notation $\deg(u)(x_0)$ to denote the topological degree of a continuous map

$u \in C(B_\rho(x_0) \setminus \{x_0\}; \mathbb{S}^1)$, i.e., the topological degree of its restriction $u|_{\partial B_r(x_0)}$, independent of $r < \rho$. We let $I_\lambda(x)$ be the half-open square given by $I_\lambda(x) = x + [0, \lambda)^2$.

By $BV(\Omega; \mathbb{S}^1)$ we denote the space of \mathbb{S}^1 -valued BV -functions. We refer the reader to [12] for a detailed introduction to the theory of BV -functions.

2.1 Results for the classical XY model

We recall here some results when the spin field $u_\varepsilon: \Omega_\varepsilon \rightarrow \mathbb{S}^1$ is not constrained to take values in a discrete set. Following [5], in order to define the discrete vorticity of u_ε , we introduce the projection $Q: \mathbb{R} \rightarrow 2\pi\mathbb{Z}$ defined by

$$(2.2) \quad Q(t) := \operatorname{argmin}\{|t - s| : s \in 2\pi\mathbb{Z}\},$$

with the convention that, if the argmin is not unique, then we choose the one with minimal modulus. Then for every $t \in \mathbb{R}$ we define $\Psi(t) := t - Q(t) \in [-\pi, \pi]$.

Let $u: \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$, and let $\varphi: \varepsilon\mathbb{Z}^2 \rightarrow [0, 2\pi)$ be the phase of u defined by the relation $u = \exp(i\varphi)$. The discrete vorticity of u is defined for every $\varepsilon i \in \varepsilon\mathbb{Z}^2$ by

$$(2.3) \quad d_u(\varepsilon i) := \frac{1}{2\pi} \left[\Psi(\varphi(\varepsilon i + \varepsilon e_1) - \varphi(\varepsilon i)) + \Psi(\varphi(\varepsilon i + \varepsilon e_1 + \varepsilon e_2) - \varphi(\varepsilon i + \varepsilon e_1)) \right. \\ \left. + \Psi(\varphi(\varepsilon i + \varepsilon e_2) - \varphi(\varepsilon i + \varepsilon e_1 + \varepsilon e_2)) + \Psi(\varphi(\varepsilon i) - \varphi(\varepsilon i + \varepsilon e_2)) \right].$$

As already noted in [5], it holds that $d_u \in \{-1, 0, 1\}$; i.e., only singular vortices can be present in the discrete setting. The discrete vorticity measure associated to u is given by

$$(2.4) \quad \mu_u := \sum_{\varepsilon i \in \varepsilon\mathbb{Z}^2} d_u(\varepsilon i) \delta_{\varepsilon i + (\varepsilon, \varepsilon)}.$$

We recall the following compactness and lower bound for the XY model.

PROPOSITION 2.1. *Let $u_\varepsilon: \Omega_\varepsilon \rightarrow \mathbb{S}^1$ and assume that $\frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon) \leq C$ for some $C > 0$. Then there exists a measure $\mu \in \mathcal{M}_b(\Omega)$ of the form $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$ and $x_h \in \Omega$, and a subsequence (not relabeled) such that $\mu_{u_\varepsilon} \llcorner \Omega \xrightarrow{f} \mu$. Moreover,*

$$2\pi |\mu|(\Omega) \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2 |\log \varepsilon|} XY_\varepsilon(u_\varepsilon).$$

Remark 2.2. Observe that in the regime $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$ the bound $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$ and Proposition 2.1 imply that $\mu_{u_\varepsilon} \llcorner \Omega \xrightarrow{f} 0$.

3 Currents

For the theory of currents we refer to [33, 36, 37]. We recall here some basic facts.

3.1 Definitions and basic facts

Given an open set $O \subset \mathbb{R}^d$, we denote by $\mathcal{D}^k(O)$ the space of k -forms $\omega: O \mapsto \Lambda^k \mathbb{R}^d$ with $C_c^\infty(O)$ -coefficients. A k -current $T \in \mathcal{D}_k(O)$ is an element of the dual of $\mathcal{D}^k(O)$ and we write $T(\omega)$ for the duality. The *boundary* of a k -current T is the $(k-1)$ -current $\partial T \in \mathcal{D}_{k-1}(O)$ defined by $\partial T(\omega) := T(d\omega)$ for every $\omega \in \mathcal{D}^{k-1}(O)$ (or $\partial T := 0$ if $k = 0$). The *support* of a current T is the smallest relatively closed set K in O such that $T(\omega) = 0$ if ω is supported outside K . Given a smooth map $f: O \rightarrow O' \subset \mathbb{R}^{N'}$ such that f is proper,⁴ $f^\# \omega \in \mathcal{D}^k(O)$ denotes the pull-back of a k -form $\omega \in \mathcal{D}^k(O')$ through f . The *push-forward* of a k -current $T \in \mathcal{D}_k(O)$ is the k -current $f_\# T \in \mathcal{D}_k(O')$ defined by $f_\# T(\omega) := T(f^\# \omega)$. Given a k -form $\omega \in \mathcal{D}^k(O)$, we can write it via its components $\omega = \sum_{|\alpha|=k} \omega_\alpha dx^\alpha$ with $\omega_\alpha \in C_c^\infty(O)$, where the expression $|\alpha| = k$ denotes all multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$ with $1 \leq \alpha_i \leq d$, and $dx^\alpha = dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k}$. The norm of $\omega(x)$ is denoted by $|\omega(x)|$, and it is the Euclidean norm of the vector with components $(\omega_\alpha(x))_{|\alpha|=k}$. The *total variation* of $T \in \mathcal{D}_k(O)$ is defined by

$$|T|(O) := \sup\{T(\omega) : \omega \in \mathcal{D}^k(O), |\omega(x)| \leq 1\}.$$

If $T \in \mathcal{D}_k(O)$ with $|T|(O) < \infty$, then we can define the measure $|T| \in \mathcal{M}_b(O)$ by

$$|T|(\psi) := \sup\{T(\omega) : \omega \in \mathcal{D}^k(O), |\omega(x)| \leq \psi(x)\}, \quad \psi \in C_0(O), \psi \geq 0.$$

Due to Riesz's representation theorem (see [36, 2.2.3, theorem 1]) there exists a $|T|$ -measurable function $\vec{T}: O \mapsto \Lambda_k \mathbb{R}^d$ with $|\vec{T}(x)| = 1$ for $|T|$ -a.e. $x \in O$ such that

$$(3.1) \quad T(\omega) = \int_O \langle \omega(x), \vec{T}(x) \rangle d|T|(x)$$

for every $\omega \in \mathcal{D}^k(O)$. If T has finite total variation, then it can be extended to a linear functional acting on all forms with bounded, Borel-measurable coefficients via the dominated convergence theorem. In particular, in this case the push-forward $f_\# T$ can be defined also for $f \in C^1(O, O')$ with bounded derivatives; cf. the discussion in [36, p. 132].

A set $\mathcal{M} \subset O$ is a countably \mathcal{H}^k -rectifiable set if it can be covered, up to an \mathcal{H}^k -negligible subset, by countably many k -manifolds of class C^1 . As such, it admits at \mathcal{H}^k -a.e. $x \in \mathcal{M}$ a tangent space $\text{Tan}(\mathcal{M}, x)$ in a measure-theoretic sense. A current $T \in \mathcal{D}_k(O)$ is an *integer multiplicity (i.m.) rectifiable current* if it is representable as

$$(3.2) \quad T(\omega) = \int_{\mathcal{M}} \langle \omega(x), \xi(x) \rangle \theta(x) d\mathcal{H}^k(x) \quad \text{for } \omega \in \mathcal{D}^k(O),$$

⁴This means that $f^{-1}(K)$ is compact in O for all compact sets $K \subset O'$.

where $\mathcal{M} \subset O$ is an \mathcal{H}^k -measurable and countably \mathcal{H}^k -rectifiable set, $\theta: \mathcal{M} \rightarrow \mathbb{Z}$ is locally $\mathcal{H}^k \llcorner \mathcal{M}$ -summable, and $\xi: \mathcal{M} \rightarrow \Lambda_k \mathbb{R}^d$ is a \mathcal{H}^k -measurable map such that $\xi(x)$ spans $\text{Tan}(\mathcal{M}, x)$ and $|\xi(x)| = 1$ for \mathcal{H}^k -a.e. $x \in \mathcal{M}$. We use the short-hand notation $T = \tau(\mathcal{M}, \theta, \xi)$. One can always remove from \mathcal{M} the set $\theta^{-1}(\{0\})$, so that we may always assume that $\theta \neq 0$. Then the triple $(\mathcal{M}, \theta, \xi)$ is uniquely determined up to \mathcal{H}^k -negligible modifications. Moreover, one can show, according to the Riesz's representation in (3.1), that $\vec{T} = \xi$ and the total variation⁵ is given by $|T| = |\theta| \mathcal{H}^k \llcorner \mathcal{M}$.

If T_j are i.m. rectifiable currents and $T_j \rightarrow T$ in $\mathcal{D}_k(O)$ with $\sup_j (|T_j|(V) + |\partial T_j|(V)) < +\infty$ for every $V \Subset O$, then by the closure theorem [36, 2.2.4, theorem 1] T is an i.m. rectifiable current, too. By $\llbracket \mathcal{M} \rrbracket$ we denote the current defined by integration over \mathcal{M} .

3.2 Currents in product spaces

We recall some notation for currents defined on the product space $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Let us denote by (x, y) the points in this space. The standard basis for \mathbb{R}^{d_1} is $\{e_1, \dots, e_{d_1}\}$, while $\{\bar{e}_1, \dots, \bar{e}_{d_2}\}$ is the standard basis for \mathbb{R}^{d_2} . Given $O_1 \subset \mathbb{R}^{d_1}$, $O_2 \subset \mathbb{R}^{d_2}$ open sets, $T_1 \in \mathcal{D}_{k_1}(O_1)$, $T_2 \in \mathcal{D}_{k_2}(O_2)$, and a $(k_1 + k_2)$ -form $\omega \in \mathcal{D}^{k_1+k_2}(O_1 \times O_2)$ of the type

$$\omega(x, y) = \sum_{\substack{|\alpha|=k_1 \\ |\beta|=k_2}} \omega_{\alpha\beta}(x, y) dx^\alpha \wedge dy^\beta,$$

the product current $T_1 \times T_2 \in \mathcal{D}_{k_1+k_2}(O_1 \times O_2)$ is defined by

$$T_1 \times T_2(\omega) := T_1 \left(\sum_{|\alpha|=k_1} T_2 \left(\sum_{|\beta|=k_2} \omega_{\alpha\beta}(x, y) dy^\beta \right) dx^\alpha \right),$$

while $T_1 \times T_2(\phi dx^\alpha \wedge dy^\beta) = 0$ if $|\alpha| + |\beta| = k_1 + k_2$ but $|\alpha| \neq k_1$, $|\beta| \neq k_2$.

3.3 Graphs

Let $O \subset \mathbb{R}^d$ be an open set and $u: \Omega \rightarrow \mathbb{R}^2$ a Lipschitz map. Then we can consider the d -current associated to the graph of u given by $G_u := (\text{id} \times u)_\# \llbracket O \rrbracket \in \mathcal{D}_2(O \times \mathbb{R}^2)$, where $\text{id} \times u: O \rightarrow O \times \mathbb{R}^2$ is the map $(\text{id} \times u)(x) = (x, u(x))$. Note that

$$G_u(\omega) = \int_O \langle \omega(x, u(x)), M(\nabla u(x)) \rangle dx$$

for all $\omega \in \mathcal{D}^d(O \times \mathbb{R}^2)$, with the d -vector

$$(3.3) \quad M(\nabla u) = (e_1 + \partial_{x^1} u^1 \bar{e}_1 + \partial_{x^1} u^2 \bar{e}_2) \wedge \dots \wedge (e_d + \partial_{x^d} u^1 \bar{e}_1 + \partial_{x^d} u^2 \bar{e}_2).$$

⁵For i.m. rectifiable currents, the total variation coincides with the so-called mass. Hence, we will not distinguish between these two concepts.

Later on we use the orientation of the graph of a smooth function $u: O \subset \mathbb{R}^2 \rightarrow \mathbb{S}^1$ (cf. [36, 2.2.4]). For such maps we have $|G_u| = \mathcal{H}^2 \llcorner \mathcal{M}$, where $\mathcal{M} = (\text{id} \times u)(\Omega)$, and

$$(3.4) \quad \begin{aligned} & \sqrt{1 + |\nabla u(x)|^2} \tilde{G}_u(x, y) \\ &= e_1 \wedge e_2 + \partial_{x_2} u^1(x) e_1 \wedge \bar{e}_1 + \partial_{x_2} u^2(x) e_1 \wedge \bar{e}_2 \\ & \quad - \partial_{x_1} u^1(x) e_2 \wedge \bar{e}_1 - \partial_{x_1} u^2(x) e_2 \wedge \bar{e}_2 \quad \text{for all } (x, y) \in \mathcal{M}. \end{aligned}$$

3.4 Cartesian currents

Let $O \subset \mathbb{R}^d$ be a bounded, open set. We recall that the class of *Cartesian currents* in $O \times \mathbb{R}^2$ is defined by

$$\text{cart}(O \times \mathbb{R}^2) := \left\{ T \in \mathcal{D}_d(O \times \mathbb{R}^2) : T \text{ is i.m. rectifiable, } \partial T|_{O \times \mathbb{R}^2} = 0, \right. \\ \left. \pi_{\#}^O T = \llbracket O \rrbracket, T|_{dx} \geq 0, |T| < +\infty, \|T\|_1 < +\infty \right\},$$

where $\pi^O: O \times \mathbb{R}^2 \rightarrow O$ denotes the projection on the first component, $T|_{dx} \geq 0$ means that $T(\phi(x, y)dx) \geq 0$ for every $\phi \in C_c^\infty(O \times \mathbb{R}^2)$ with $\phi \geq 0$, and

$$\|T\|_1 = \sup\{T(\phi(x, y)|y|dx) : \phi \in C_c^\infty(O \times \mathbb{R}^2), |\phi| \leq 1\}.$$

Note that, if for some function u

$$(3.5) \quad T(\phi(x, y)dx) = \int_O \phi(x, u(x))dx \quad \text{then} \quad \|T\|_1 = \int_O |u|dx.$$

The class of *Cartesian currents* in $O \times \mathbb{S}^1$ is

$$\text{cart}(O \times \mathbb{S}^1) := \{T \in \text{cart}(O \times \mathbb{R}^2) : \text{supp}(T) \subset \bar{O} \times \mathbb{S}^1\}$$

(cf. [37, 6.2.2] for this definition). We recall the following approximation theorem which explains that Cartesian currents in $O \times \mathbb{S}^1$ are precisely those currents that arise as limits of graphs of \mathbb{S}^1 -valued smooth maps. The proof can be found in [35, theorem 7].⁶

THEOREM 3.1 (Approximation theorem). *Let $T \in \text{cart}(O \times \mathbb{S}^1)$. Then there exists a sequence of smooth maps $u_h \in C^\infty(O; \mathbb{S}^1)$ such that*

$$G_{u_h} \rightharpoonup T \text{ in } \mathcal{D}_d(O \times \mathbb{R}^2) \quad \text{and} \quad |G_{u_h}|(O \times \mathbb{R}^2) \rightarrow |T|(O \times \mathbb{R}^2).$$

We state an extension result for Cartesian currents, which we could not find in the literature. For a proof we refer the interested reader to [28].

LEMMA 3.2 (Extension of Cartesian currents). *Let $O \subset \mathbb{R}^d$ be a bounded, open set with Lipschitz boundary, and let $T \in \text{cart}(O \times \mathbb{S}^1)$. Then there exist an open set $\tilde{O} \ni O$ and a current $T \in \text{cart}(\tilde{O} \times \mathbb{S}^1)$ such that*

$$\tilde{T}|_{O \times \mathbb{R}^2} = T \quad \text{and} \quad \tilde{T}|(\partial O \times \mathbb{R}^2) = 0.$$

⁶Notice that some results in [35] require O to have smooth boundary. This is not the case for this theorem, which is based on a local construction regularizing a local lifting of T .

We will also use the structure theorem for Cartesian currents in $O \times \mathbb{S}^1$ [35, sec. 3, theorems 1, 5, 6].⁷ To simplify notation, from now on we focus on dimension 2. Recall that $\Omega \subset \mathbb{R}^2$ is a bounded, open set with Lipschitz boundary. To state the theorem, we recall the following decomposition for a current $T \in \text{cart}(\Omega \times \mathbb{S}^1)$. Letting \mathcal{M} be the countably \mathcal{H}^2 -rectifiable set where T is concentrated, we denote by $\mathcal{M}^{(a)}$ the set of points $(x, y) \in \mathcal{M}$ at which the tangent plane $\text{Tan}(\mathcal{M}, (x, y))$ does not contain vertical vectors (namely, the Jacobian of the projection π^Ω restricted to $\text{Tan}(\mathcal{M}, (x, y))$ has maximal rank), by $\mathcal{M}^{(jc)} := (\mathcal{M} \setminus \mathcal{M}^{(a)}) \cap (J_T \times \mathbb{S}^1)$, where $J_T := \{x \in \Omega : \frac{d\pi^\Omega|_{T_1}}{d\mathcal{H}^1}(x) > 0\}$, and by $\mathcal{M}^{(c)} := \mathcal{M} \setminus (\mathcal{M}^{(a)} \cup \mathcal{M}^{(jc)})$. Then we can split the current via $T = T^{(a)} + T^{(c)} + T^{(jc)}$, where $T^{(a)} := T \llcorner \mathcal{M}^{(a)}$, $T^{(c)} := T \llcorner \mathcal{M}^{(c)}$, $T^{(jc)} := T \llcorner \mathcal{M}^{(jc)}$ are mutually singular measures, and we denote by \llcorner the restriction of the Radon measure T . Hereafter we use the notation $\hat{x}^1 = x^2$ and $\hat{x}^2 = x^1$.

THEOREM 3.3 (Structure theorem for $\text{cart}(\Omega \times \mathbb{S}^1)$). *Let $T \in \text{cart}(\Omega \times \mathbb{S}^1)$. Then there exists a unique map $u_T \in BV(\Omega; \mathbb{S}^1)$ and a (not unique) i.m. rectifiable 1-current $L_T = \tau(\mathcal{L}, k, \bar{L}_T) \in \mathcal{D}_1(\Omega)$ such that $T^{(jc)} = T^{(j)} + L_T \times \llbracket \mathbb{S}^1 \rrbracket$ and*

$$(3.6) \quad T(\phi(x, y) dx) = T^{(a)}(\phi(x, y) dx) = \int_{\Omega} \phi(x, u_T(x)) dx,$$

$$(3.7) \quad T^{(a)}(\phi(x, y) d\hat{x}^l \wedge dy^m) = (-1)^{2-l} \int_{\Omega} \phi(x, u_T(x)) de_{x^l}^{(a)} u_T^m(x) dx,$$

$$(3.8) \quad T^{(c)}(\phi(x, y) d\hat{x}^l \wedge dy^m) = (-1)^{2-l} \int_{\Omega} \phi(x, \tilde{u}_T(x)) d\partial_{x^l}^{(c)} u_T^m(x),$$

$$(3.9) \quad \begin{aligned} & T^{(j)}(\phi(x, y) d\hat{x}^l \wedge dy^m) \\ &= (-1)^{2-l} \int_{J_{u_T}} \left\{ \int_{\gamma_x} \phi(x, y) dy^m \right\} v_{u_T}^l(x) d\mathcal{H}^1(x) \end{aligned}$$

for every $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$, γ_x being the (oriented) geodesic arc in \mathbb{S}^1 that connects $u_T^-(x)$ to $u_T^+(x)$ and u_T being the precise representative of u_T .⁸

⁷ As for the approximation theorem, no boundary regularity is required for this result.

⁸ In [35, theorem 6] the structure of $T^{(j)}$ is formulated slightly differently with the counter-clockwise arc $\gamma_{\varphi^-, \varphi^+}$ between $(\cos(\varphi^-), \sin(\varphi^-))$, $(\cos(\varphi^+), \sin(\varphi^+))$, $\varphi^- < \varphi^+$, and replacing J_{u_T} by J_φ , where $\varphi \in BV(\Omega)$ is such that $T = \chi_\# G_\varphi$, where $\chi(x, \vartheta) = (x, \cos(\vartheta), \sin(\vartheta))$ and $G_\varphi \in \text{cart}(\Omega \times \mathbb{R})$ is the boundary of the subgraph of φ . To explain (3.9), we recall the local construction in [35]: for every $x \in J_\varphi$ one chooses $p^+(x) \in \mathbb{R}$ and $k'(x) \in \mathbb{N}$ such that $\varphi^+(x) = p^+(x) + 2\pi k'(x)$ and $0 \leq p^+(x) - \varphi^-(x) < 2\pi$. Then, locally, the 1-current L'_T in [35, theorem 6] is given by $L'_T = \tau(\mathcal{L}', k'(x), \bar{L}'_T)$, where $\mathcal{L}' \subset J_\varphi$ is the set of points with $k'(x) \geq 1$ and $\bar{L}'_T = v_\varphi^2 e_1 - v_\varphi^1 e_2$. To obtain the representation via geodesics, we let

$$(q^+(x), k(x)) = \begin{cases} (p^+(x), k'(x)) & \text{if } p^+(x) - \varphi^-(x) < \pi, \\ (p^+(x) - 2\pi, k'(x) + 1) & \text{if } p^+(x) - \varphi^-(x) > \pi, \end{cases}$$

It is convenient to recast the jump-concentration part of $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ in the following way. Let $L_T = \tau(\mathcal{L}, k, \vec{L}_T)$ as in Theorem 3.3. We introduce for \mathcal{H}^1 -a.e. $x \in J_T$ the normal $v_T(x)$ to the 1-rectifiable set $J_T = J_{u_T} \cup \mathcal{L}$ as

$$(3.10) \quad v_T(x) = \begin{cases} v_{u_T}(x) & \text{if } x \in J_{u_T}, \\ (-\vec{L}_T^2(x), \vec{L}_T^1(x)) & \text{if } x \in \mathcal{L} \setminus J_{u_T}, \end{cases}$$

where we choose $v_{u_T}(x) = (-\vec{L}_T^2(x), \vec{L}_T^1(x))$ if $x \in \mathcal{L} \cap J_{u_T}$. For \mathcal{H}^1 -a.e. $x \in J_T$ we consider the curve γ_x^T given by the following arcs:

- if $x \in J_{u_T} \setminus \mathcal{L}$, the (oriented) geodesic arc γ_x connecting $u_T^-(x)$ to $u_T^+(x)$ (in the sense of footnote 8 for antipodal points);
- if $x \in \mathcal{L} \setminus J_{u_T}$, the whole \mathbb{S}^1 turning $k(x)$ times;
- if $x \in J_{u_T} \cap \mathcal{L}$, the sum (in the sense of currents)⁹ of the oriented geodesic arc γ_x and of \mathbb{S}^1 with multiplicity $k(x)$.

Then

$$(3.11) \quad \begin{aligned} & T^{(j_c)}(\phi(x, y) d\hat{x}^l \wedge dy^m) \\ &= (-1)^{2-l} \int_{J_T} \left\{ \int_{\gamma_x^T} \phi(x, y) dy^m \right\} v_T^l(x) d\mathcal{H}^1(x). \end{aligned}$$

The integration over γ_x^T with respect to the form dy^m in the formula above is intended with the correct multiplicity of the curve γ_x^T defined for \mathcal{H}^1 -a.e. $x \in J_T$ by the integer

$$(3.12) \quad m(x, y) := \begin{cases} \pm 1, & \text{if } x \in J_{u_T} \setminus \mathcal{L}, y \in \text{supp}(\gamma_x), \\ k(x), & \text{if } x \in \mathcal{L} \setminus J_{u_T}, y \in \mathbb{S}^1, \\ k(x) \pm 1, & \text{if } x \in \mathcal{L} \cap J_{u_T}, y \in \text{supp}(\gamma_x), \\ k(x), & \text{if } x \in \mathcal{L} \cap J_{u_T}, y \in \text{supp}(\gamma_x^T) \setminus \text{supp}(\gamma_x), \end{cases}$$

The case $p^+(x) - \varphi^-(x) = \pi$ needs special care. In this case we let $\tilde{\varphi}^\pm(x) := \varphi^\pm(x) \bmod 2\pi \in [0, 2\pi)$, and we set

$$(q^+(x), k(x)) = \begin{cases} (p^+(x), k'(x)) & \text{if } \tilde{\varphi}^+(x) - \tilde{\varphi}^-(x) = \pi, \\ (p^+(x) - 2\pi, k'(x) + 1) & \text{if } \tilde{\varphi}^+(x) - \tilde{\varphi}^-(x) = -\pi. \end{cases}$$

Replacing $(p^+(x), k'(x))$ by $(q^+(x), k(x))$, one proves (3.9) as in [35, pp. 107–108]. The curves $\gamma_{\varphi^-, \varphi^+}$ are then replaced by the more intrinsic geodesic arcs γ_x . Exchanging $u_T^-(x)$ and $u_T^+(x)$ will change the orientation of the arc and of the normal $v_{u_T}(x)$, making (3.9) invariant.

⁹In this case, a more elementary definition of γ_x^T is the following: let $\gamma_x: [0, 1] \rightarrow \mathbb{S}^1$ be the geodesic arc, and let $\varphi_x: [0, 1] \rightarrow \mathbb{R}$ be a continuous function (unique up to translations of an integer multiple of 2π) such that $\gamma_x(t) = \exp(i\varphi_x(t))$. Then $\gamma_x^T(t) = \exp(i((1-t)\varphi_x(0) + t(\varphi_x(1) + 2\pi k(x)))$.

where $\pm = +/-$ if the geodesic arc γ_x is oriented counterclockwise/clockwise, respectively. More precisely,

$$(3.13) \quad \int_{\gamma_x^T} \phi(y) dy^m = (-1)^m \int_{\text{supp}(\gamma_x^T)} \phi(y) \hat{y}^m \mathbf{m}(x, y) d\mathcal{H}^1(y).$$

Remark 3.4. Note that we constructed $\mathbf{m}(x, y)$ based on the orientation (3.10) of ν_T . As discussed in footnote 8, changing the orientation of ν_{u_T} changes the orientation of the geodesic γ_x , while a change of the orientation of \vec{L}_T switches the sign of $k(x)$. Hence changing the orientation of $\nu_T(x)$ changes $\mathbf{m}(x, y)$ into $-\mathbf{m}(x, y)$. If we choose locally $\nu_T = \nu_\varphi$ as in footnote 8, our construction above yields $\mathbf{m}(x, y) \geq 0$.

Finally, we recall the following result, proven in [35, sec. 4].

PROPOSITION 3.5. *For $u \in BV(\Omega; \mathbb{S}^1)$ there exists $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ such that $u_T = u$ a.e.*

3.5 Currents associated to discrete spin fields

We introduce the piecewise constant interpolations of spin fields. For every set S , we put

$$\mathcal{PC}_\varepsilon(S) := \{u: \mathbb{R}^2 \rightarrow S : u(x) = u(\varepsilon i) \text{ if } x \in \varepsilon i + [0, \varepsilon)^2 \text{ for some } i \in \varepsilon \mathbb{Z}^2\}.$$

Given $u: \Omega_\varepsilon \rightarrow \mathbb{S}^1$, we can always identify it with its piecewise constant interpolation belonging to $\mathcal{PC}_\varepsilon(\mathbb{S}^1)$, arbitrarily extended to \mathbb{R}^2 .

To $u \in \mathcal{PC}_\varepsilon(\mathbb{S}^1)$ we associate the current $G_u \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ defined by

$$(3.14) \quad G_u(\phi(x, y) dx^1 \wedge dx^2) := \int_{\Omega} \phi(x, u(x)) dx,$$

$$(3.15) \quad \begin{aligned} & G_u(\phi(x, y) d\hat{x}^l \wedge dy^m) \\ & := (-1)^{2-l} \int_{J_u} \left\{ \int_{\gamma_x} \phi(x, y) dy^m \right\} \nu_u^l(x) d\mathcal{H}^1(x), \end{aligned}$$

$$(3.16) \quad G_u(\phi(x, y) dy^1 \wedge dy^2) := 0,$$

for every $\phi \in C_c^\infty(\Omega \times \mathbb{R}^2)$, where J_u is the jump set of u , $\nu_u(x)$ is the normal to J_u at x , and $\gamma_x \subset \mathbb{S}^1$ is the (oriented) geodesic arc that connects the two traces $u^-(x)$ and $u^+(x)$. If $u^+(x)$ and $u^-(x)$ are opposite vectors, the choice of the geodesic arc $\gamma_x \subset \mathbb{S}^1$ is done consistently with the choice made in (2.2) for the values $\Psi(\pi)$ and $\Psi(-\pi)$ as follows: let $\varphi^\pm(x) \in [0, 2\pi)$ be the phase of $u^\pm(x)$; if $\Psi(\varphi^+(x) - \varphi^-(x)) = \pi$, then γ_x is the arc that connects $u^-(x)$ to $u^+(x)$ counterclockwise; if $\Psi(\varphi^+(x) - \varphi^-(x)) = -\pi$, then γ_x is the arc that connects $u^-(x)$ to $u^+(x)$ clockwise. Note that the choice of the arc γ_x is independent of the orientation of the normal $\nu_u(x)$.

We define for \mathcal{H}^1 -a.e. $x \in J_u$ the integer number $\mathbf{m}(x) = \pm 1$, where $\pm = +/-$ if the geodesic arc γ_x is oriented counterclockwise/clockwise, respectively.

Then

$$(3.17) \quad \int_{\gamma_x} \phi(y) dy^m = (-1)^m m(x) \int_{\text{supp}(\gamma_x)} \phi(y) \hat{y}^m d\mathcal{H}^1(y).$$

The proof of the following proposition is standard.

PROPOSITION 3.6. *Let $u \in \mathcal{PC}_\varepsilon(\mathbb{S}^1)$ and let $G_u \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the current defined in (3.14)–(3.16). Then G_u is an i.m. rectifiable current and, according to the representation formula (3.1), $G_u = \vec{G}_u |G_u|$, where $|G_u| = \mathcal{H}^2 \llcorner \mathcal{M}$,*

$$\mathcal{M} = \mathcal{M}^{(a)} \cup \mathcal{M}^{(j)} = \{(x, u(x)) : x \in \Omega \setminus J_u\} \cup \{(x, y) : x \in J_u, y \in \gamma_x\},$$

and, for \mathcal{H}^2 -a.e. $(x, y) \in \mathcal{M}^{(a)}$,

$$(3.18) \quad \vec{G}_u(x, y) = e_1 \wedge e_2$$

while for \mathcal{H}^2 -a.e. $(x, y) \in \mathcal{M}^{(j)}$ we have

$$(3.19) \quad \begin{aligned} \vec{G}_u(x, y) = \text{sign}(m(x)) & \left[-v_u^2(x) y^2 e_1 \wedge \bar{e}_1 + v_u^2(x) y^1 e_1 \wedge \bar{e}_2 \right. \\ & \left. + v_u^1(x) y^2 e_2 \wedge \bar{e}_1 - v_u^1(x) y^1 e_2 \wedge \bar{e}_2 \right]. \end{aligned}$$

We now relate the boundary of the current G_u with the discrete vorticity μ_u . The interested reader can find a detailed proof in [28].

PROPOSITION 3.7. *Let $u \in \mathcal{PC}_\varepsilon(\mathbb{S}^1)$ and let $G_u \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the current defined in (3.14)–(3.16). Then $\partial G_u|_{\Omega \times \mathbb{R}^2} = -\mu_u \times \llbracket \mathbb{S}^1 \rrbracket$, where μ_u is the discrete vorticity measure defined in (2.4) for $u|_{\varepsilon\mathbb{Z}^2} : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$.*

The proof of the following fact follows essentially from the definitions.

LEMMA 3.8. *Assume $\mu_\varepsilon \xrightarrow{f} \mu$ in Ω . Then $\mu_\varepsilon \times \llbracket \mathbb{S}^1 \rrbracket \rightarrow \mu \times \llbracket \mathbb{S}^1 \rrbracket$ in $\mathcal{D}_1(\Omega \times \mathbb{R}^2)$.*

4 Proofs

Now we give the proof of our main Theorem 1.1. In what follows, for $A \subset \mathbb{R}^2$ we shall use the localized energy

$$E_\varepsilon(u; A) := \frac{1}{2} \sum_{\substack{\langle i, j \rangle \\ \varepsilon i, \varepsilon j \in A}} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2.$$

4.1 Compactness and lower bound in absence of vortices

In this subsection we consider a generic sequence $u_\varepsilon : \varepsilon\mathbb{Z}^2 \rightarrow \mathbb{S}^1$ such that $\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon)$ is bounded. First we prove that such sequences are compact in $L^1(\Omega)$ with limits in $BV(\Omega; \mathbb{S}^1)$.

PROPOSITION 4.1 (Compactness in BV). *Assume that $\theta_\varepsilon \ll 1$ and $\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$. Then there exists a subsequence (not relabeled) and a function $u \in BV(\Omega; \mathbb{S}^1)$ such that $u_\varepsilon \rightarrow u$ in $L^1(\Omega)$ and $u_\varepsilon \xrightarrow{*} u$ in $BV_{\text{loc}}(\Omega; \mathbb{R}^2)$.*

PROOF. Fix $A \Subset \Omega$. Since $d_{\mathbb{S}^1}(u(\varepsilon i), u(\varepsilon j)) \geq \theta_\varepsilon$ when $u(\varepsilon i) \neq u(\varepsilon j)$, (2.1) implies that

$$\begin{aligned}
 (4.1) \quad C &\geq \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) = \frac{1}{2\varepsilon\theta_\varepsilon} \sum_{\langle i,j \rangle} \varepsilon^2 |u(\varepsilon i) - u(\varepsilon j)|^2 \\
 &= \frac{1}{2\varepsilon\theta_\varepsilon} \sum_{\langle i,j \rangle} \varepsilon^2 2 \sin\left(\frac{1}{2} d_{\mathbb{S}^1}(u(\varepsilon i), u(\varepsilon j))\right) |u(\varepsilon i) - u(\varepsilon j)| \\
 &\geq \frac{\sin(\frac{\theta_\varepsilon}{2})}{\theta_\varepsilon} \sum_{\langle i,j \rangle} \varepsilon |u(\varepsilon i) - u(\varepsilon j)| \geq \frac{2 \sin(\frac{\theta_\varepsilon}{2})}{\theta_\varepsilon} |Du_\varepsilon|(A).
 \end{aligned}$$

Hence u_ε is bounded in $BV(A; \mathbb{S}^1)$ and we conclude that (up to a subsequence) $u_\varepsilon \rightarrow u$ in $L^1(A)$ and $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(A; \mathbb{R}^2)$ for some $u \in BV(A; \mathbb{S}^1)$ with $|Du|(A) \leq C$. Since $A \Subset \Omega$ was arbitrary and the constant C does not depend on A , the claim follows from a diagonal argument and the equi-integrability of u_ε . \square

In the next lemma we prove a lower bound for the energy still at the discrete level.

LEMMA 4.2. *Assume that $\theta_\varepsilon \ll 1$ and let $\sigma \in (0, 1)$. Then for ε small enough we have*

$$(4.2) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)|^2 \geq (1 - \sigma)\theta_\varepsilon d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)).$$

In particular,

$$(4.3) \quad \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \geq (1 - \sigma) \frac{1}{2} \sum_{\langle i,j \rangle} \varepsilon d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)).$$

PROOF. By (2.1) we have that $|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = 2 \sin(\frac{1}{2} d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)))$. Since u_ε takes values in \mathcal{S}_ε there exists $k \in \mathbb{N}$ (depending on i, j , and ε) such that $d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)) = k\theta_\varepsilon$. We can assume $k \neq 0$. Moreover, note that $k\theta_\varepsilon \leq \pi$.

Due to Taylor's formula there exists a $\zeta \in [0, k\frac{\theta_\varepsilon}{2}]$ such that

$$\sin(k\frac{\theta_\varepsilon}{2}) = k\frac{\theta_\varepsilon}{2} - \frac{1}{6} \cos(\zeta) (k\frac{\theta_\varepsilon}{2})^3 \geq k\frac{\theta_\varepsilon}{2} - \frac{1}{6} (k\frac{\theta_\varepsilon}{2})^3$$

Dividing by $\sqrt{k\frac{\theta_\varepsilon}{2}}$ we get that

$$(4.4) \quad \frac{\sin(k\frac{\theta_\varepsilon}{2})}{\sqrt{k\frac{\theta_\varepsilon}{2}}} \geq \sqrt{k} \left[1 - \frac{1}{6} (k\frac{\theta_\varepsilon}{2})^2 \right]$$

If $k \geq 9$, using the fact that $k\theta_\varepsilon \leq \pi \leq 4$ we obtain that

$$(4.5) \quad \frac{\sin(k\frac{\theta_\varepsilon}{2})}{\sqrt{k\frac{\theta_\varepsilon}{2}}} \geq \sqrt{k} \frac{1}{3} \geq 1.$$

Otherwise, if $k \leq 8$, (4.4) directly implies

$$(4.6) \quad \frac{\sin\left(k \frac{\theta_\varepsilon}{2}\right)}{\sqrt{k \frac{\theta_\varepsilon}{2}}} \geq 1 - 3\theta_\varepsilon^2 \geq 1 - \sigma,$$

for ε small enough. Squaring both sides in (4.5) and (4.6) (notice that $k\theta_\varepsilon \in [0, \pi]$ implies $\sin\left(k \frac{\theta_\varepsilon}{2}\right) \geq 0$) we have that $4 \sin^2\left(k \frac{\theta_\varepsilon}{2}\right) \geq (1 - \sigma)k\theta_\varepsilon^2$. We conclude the proof of (4.2) by replacing $k\theta_\varepsilon = d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j))$ in the last inequality and by (2.1). The estimate (4.3) follows from (4.2) by summation. \square

We now recast the energy as a parametric integral of the currents G_{u_ε} . To do so, we define the convex and positively 1-homogeneous function $\Phi: \Lambda_2(\mathbb{R}^2 \times \mathbb{R}^2) \mapsto \mathbb{R}$ by

$$(4.7) \quad \Phi(\xi) := \sqrt{(\xi^{21})^2 + (\xi^{22})^2} + \sqrt{(\xi^{11})^2 + (\xi^{12})^2}$$

for every $\xi = \xi^{00}e_1 \wedge e_2 + \xi^{21}e_1 \wedge \bar{e}_1 + \xi^{22}e_1 \wedge \bar{e}_2 + \xi^{11}e_2 \wedge \bar{e}_1 + \xi^{12}e_2 \wedge \bar{e}_2 + \xi^{00}\bar{e}_1 \wedge \bar{e}_2$.

LEMMA 4.3. *For every open set $A \Subset \Omega$ and ε small enough we have*

$$\frac{1}{2} \sum_{(i,j)} \varepsilon d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)) \geq \int_{A \times \mathbb{R}^2} \Phi(\vec{G}_{u_\varepsilon}) d|G_{u_\varepsilon}|.$$

PROOF. By the explicit formulas (3.18)–(3.19) for the orientation of G_{u_ε} , we infer that

$$\begin{aligned} \Phi(\vec{G}_{u_\varepsilon})(x, y) &= \mathbb{1}_{J_{u_\varepsilon}}(x) \left[|v_{u_\varepsilon}^2(x)| \sqrt{(y^2)^2 + (y^1)^2} + |v_{u_\varepsilon}^1(x)| \sqrt{(y^2)^2 + (y^1)^2} \right] \\ &= \mathbb{1}_{J_{u_\varepsilon}}(x) |v_{u_\varepsilon}(x)|_1. \end{aligned}$$

Moreover, we recall that $|G_{u_\varepsilon}| = \mathcal{H}^2 \llcorner \mathcal{M}_\varepsilon$, where

$$\begin{aligned} \mathcal{M}_\varepsilon &= \mathcal{M}_\varepsilon^{(a)} \cup \mathcal{M}_\varepsilon^{(j)} \\ &= \{(x, u_\varepsilon(x)) : x \in \Omega \setminus J_{u_\varepsilon}\} \cup \{(x, y) : x \in J_{u_\varepsilon}, y \in \text{supp}(\gamma_x^\varepsilon)\}, \end{aligned}$$

γ_x^ε being the geodesic arc that connects $u_\varepsilon^-(x)$ to $u_\varepsilon^+(x)$. Therefore

$$\begin{aligned} \int_{A \times \mathbb{R}^2} \Phi(\vec{G}_{u_\varepsilon}) d|G_{u_\varepsilon}| &= \int_{A \times \mathbb{R}^2} |v_{u_\varepsilon}|_1 d\mathcal{H}^2 \llcorner \mathcal{M}_\varepsilon^{(j)} \\ &= \int_{J_{u_\varepsilon} \cap A} \left\{ \int_{\text{supp}(\gamma_x^\varepsilon)} d\mathcal{H}^1(y) \right\} |v_{u_\varepsilon}(x)|_1 d\mathcal{H}^1(x) \\ &= \int_{J_{u_\varepsilon} \cap A} d_{\mathbb{S}^1}(u_\varepsilon^-, u_\varepsilon^+) |v_{u_\varepsilon}|_1 d\mathcal{H}^1 \\ &\leq \frac{1}{2} \sum_{(i,j)} \varepsilon d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)). \quad \square \end{aligned}$$

Next we show that energy bounds also yield compactness for the associated currents G_{u_ε} .

PROPOSITION 4.4 (Compactness in $\text{cart}(\Omega \times \mathbb{S}^1)$). *Assume that $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$ as well as $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$. Let $G_{u_\varepsilon} \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the currents associated to u_ε defined as in (3.14)–(3.16). Then there exists a subsequence (not relabeled) and a current $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ such that $G_{u_\varepsilon} \rightharpoonup T$ in $\mathcal{D}_2(\Omega \times \mathbb{R}^2)$. Moreover, $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ and $u_T = u$ a.e. in Ω , where u_T is the BV function associated to T given by Theorem 3.3 and $u \in BV(\Omega; \mathbb{S}^1)$ is the function given by Proposition 4.1.*

PROOF. Let us fix an open set $A \Subset \Omega$. From the elementary inequality $\Phi(\xi) \geq \sqrt{(\xi^{21})^2 + (\xi^{22})^2 + (\xi^{11})^2 + (\xi^{12})^2}$, we deduce the estimate

$$\begin{aligned}
 & |G_{u_\varepsilon}|(A \times \mathbb{R}^2) \\
 &= |G_{u_\varepsilon}|(\mathcal{M}^{(a)} \cap A \times \mathbb{R}^2) + |G_{u_\varepsilon}|(\mathcal{M}^{(j)} \cap A \times \mathbb{R}^2) \\
 &\leq |A| + \int_{A \times \mathbb{R}^2} \Phi(\vec{G}_{u_\varepsilon}) d|G_{u_\varepsilon}| \leq |\Omega| + \frac{1}{2} \sum_{(i,j)} \varepsilon d_{\mathbb{S}^1}(u_\varepsilon(\varepsilon i), u_\varepsilon(\varepsilon j)) \\
 (4.8) \quad &\leq |\Omega| + \frac{2}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C,
 \end{aligned}$$

where in the last inequality we employed (4.3) with $\sigma = 1/2$. By the compactness theorem for currents [36, 2.2.3, prop. 2, theorem 1-(i)] we deduce that there exists a subsequence (not relabeled) and a current $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ with $|T| < \infty$ such that $G_{u_\varepsilon} \rightharpoonup T$ in $\mathcal{D}_2(\Omega \times \mathbb{R}^2)$. Due to Proposition 3.7 we have $\partial G_{u_\varepsilon}|_{\Omega \times \mathbb{R}^2} = -\mu_{u_\varepsilon} \llbracket \mathbb{S}^1 \rrbracket$. By Remark 2.2 and Lemma 3.8, and since $\partial G_{u_\varepsilon} \rightharpoonup \partial T$ in $\mathcal{D}_1(\Omega \times \mathbb{R}^2)$, we conclude that $\partial T|_{\Omega \times \mathbb{R}^2} = 0$. The other properties to show that $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ follow from [28, prop. 4.1]. Finally, it is easy to see that $u = u_T$ a.e. in Ω . \square

PROPOSITION 4.5 (Lower bound for the parametric integral). *Assume that $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$ and that $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C$. Let $G_{u_\varepsilon} \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be the currents associated to u_ε defined as in (3.14)–(3.16) and assume that $G_{u_\varepsilon} \rightharpoonup T$, where $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ is a current given by Proposition 4.4, represented as $T = \vec{T}|T|$. Then for every open set $A \Subset \Omega$*

$$(4.9) \quad \int_{A \times \mathbb{R}^2} \Phi(\vec{T}) d|T| \leq \liminf_{\varepsilon \rightarrow 0} \int_{A \times \mathbb{R}^2} \Phi(\vec{G}_{u_\varepsilon}) d|G_{u_\varepsilon}|.$$

PROOF. The statement is a consequence of the lower semicontinuity of parametric integrals with respect to mass bounded weak convergence of currents, [37, 1.3.1, theorem 1]. \square

We can write explicitly the parametric integral in the left-hand side of (4.9) in terms of the limit u of the sequence u_ε . By (3.11) the jump-concentration part of

T is given by

$$T^{(jc)}(\phi(x, y) d\hat{x}^l \wedge dy^m) = (-1)^{2-l} \int_{J_T} \left\{ \int_{\gamma_x^T} \phi(x, y) dy^m \right\} v_T^l(x) d\mathcal{H}^1(x).$$

For \mathcal{H}^1 -a.e. $x \in J_T$ we define the number

$$(4.10) \quad \ell_T(x) := \text{length}(\gamma_x^T) = \int_{\text{supp}(\gamma_x^T)} |\mathfrak{m}(x, y)| d\mathcal{H}^1(y),$$

where $\mathfrak{m}(x, y)$ is the integer defined in (3.12). Notice that by $\text{length}(\gamma_x^T)$ we mean the length of the curve γ_x^T counted with its multiplicity and not the \mathcal{H}^1 Hausdorff measure of its support. Observe that, in particular, $\ell_T(x) = d_{\mathbb{S}^1}(u^-(x), u^+(x))$ if $x \in J_u \setminus \mathcal{L}$, whilst $\ell_T(x) = 2\pi|k(x)|$ if $x \in \mathcal{L} \setminus J_u$. The full form of the parametric integral is contained in the lemma below. For a detailed proof, see [28].

LEMMA 4.6. *Let $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ and $u \in BV(\Omega; \mathbb{S}^1)$ be as in Proposition 4.4, and let Φ be the parametric integrand defined in (4.7). Then*

$$\int_{\Omega \times \mathbb{R}^2} \Phi(\vec{T}) d|T| = \int_{\Omega} |\nabla u|_{2,1} dx + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_T} \ell_T(x) |v_T(x)|_1 d\mathcal{H}^1(x).$$

Remark 4.7. In the presence of vortices, we will work with Cartesian currents on punctured open sets. Given a measure $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ and an open set A , we adopt the notation

$$A_\mu := A \setminus \text{supp}(\mu) = A \setminus \{x_1, \dots, x_N\}$$

and $A_\mu^\rho := A \setminus \bigcup_{h=1}^N B_\rho(x_h)$. We observe that a current $T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1)$ can be extended to a current $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$. Indeed, since $T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1)$, it can be represented as

$$T(\omega) = \int_{\Omega_\mu \times \mathbb{R}^2} \langle \omega, \xi \rangle \theta d\mathcal{H}^2 \llcorner \mathcal{M} \quad \text{for } \omega \in \mathcal{D}^2(\Omega_\mu \times \mathbb{R}^2),$$

according to the notation in (3.2), where $\mathcal{M} \subset \Omega_\mu \times \mathbb{S}^1$ \mathcal{H}^2 -a.e. The integral above can be extended to a linear functional on forms $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$, namely,

$$T(\omega) = \int_{\Omega \times \mathbb{R}^2} \langle \omega, \xi \rangle \theta d\mathcal{H}^2 \llcorner \mathcal{M} \quad \text{for } \omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2).$$

To prove the continuity of this extension, fix $\omega \in \mathcal{D}^2(\Omega \times \mathbb{R}^2)$ with $\sup_x |\omega(x)| \leq 1$. Then

$$(4.11) \quad \begin{aligned} |T(\omega)| &\leq |T((1-\zeta)\omega)| + \left| \int_{\Omega \times \mathbb{R}^2} \zeta \langle \omega, \xi \rangle \theta d\mathcal{H}^2 \llcorner \mathcal{M} \right| \\ &\leq |T|(\Omega_\mu \times \mathbb{R}^2) + \sum_{h=1}^N \int_{B_\rho(x_h) \times \mathbb{R}^2} |\theta| d\mathcal{H}^2 \llcorner \mathcal{M} \end{aligned}$$

where $\zeta \in C_c^\infty(\Omega)$ is such that $0 \leq \zeta \leq 1$, $\text{supp}(\zeta) \subset \bigcup_{h=1}^N B_\rho(x_h)$, and $\zeta \equiv 1$ on $B_{\rho/2}(x_h)$ for every $h = 1, \dots, N$. Letting $\rho \rightarrow 0$ in the inequality above, we

get $|T(\omega)| \leq |T|(\Omega_\mu \times \mathbb{R}^2)$ since $\mathcal{H}^2(\mathcal{M} \cap (\{x_h\} \times \mathbb{R}^2)) \leq \mathcal{H}^2(\{x_h\} \times \mathbb{S}^1) = 0$ for $h = 1, \dots, N$ and θ is $\mathcal{H}^2 \llcorner \mathcal{M}$ -summable. This shows that $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$.

Moreover, from the arbitrariness of ω in (4.11) we deduce that $|T|(\Omega \times \mathbb{R}^2) = |T|(\Omega_\mu \times \mathbb{R}^2)$ and

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^2} \Phi(\vec{T}) d|T| &= \int_{\Omega_\mu \times \mathbb{R}^2} \Phi(\vec{T}) d|T| \\ &= \int_{\Omega} |\nabla u|_{2,1} dx + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_T} \ell_T(x) |v_T(x)|_1 d\mathcal{H}^1(x). \end{aligned}$$

To state the final lower bound result when $M = 0$, for every $u \in BV(\Omega; \mathbb{S}^1)$ we introduce

$$(4.12) \quad \mathcal{J}(u; \Omega) := \inf \left\{ \int_{J_T} \ell_T(x) |v_T(x)|_1 d\mathcal{H}^1(x) : T \in \text{cart}(\Omega \times \mathbb{S}^1), u_T = u \text{ a.e. in } \Omega \right\}.$$

PROPOSITION 4.8 ($M = 0$, lower bound). *Assume*

$$\theta_\varepsilon \ll \varepsilon |\log \varepsilon| \quad \text{and} \quad \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq C.$$

If $u_\varepsilon \rightarrow u$ in $L^1(\Omega)$, where $u \in BV(\Omega; \mathbb{S}^1)$ is as in Proposition 4.1, then

$$(4.13) \quad \int_{\Omega} |\nabla u|_{2,1} dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(u; \Omega) \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon).$$

PROOF. Let $\sigma \in (0, 1)$ and $A \Subset \Omega$ be open. By Lemma 4.2 and Lemma 4.3, we deduce that

$$(1 - \sigma) \int_{A \times \mathbb{R}^2} \Phi(\vec{G}_{u_\varepsilon}) d|G_{u_\varepsilon}| \leq \frac{1}{\varepsilon \theta_\varepsilon} E(u_\varepsilon).$$

Passing to the limit as $\varepsilon \rightarrow 0$, Proposition 4.5 implies

$$(1 - \sigma) \int_{A \times \mathbb{R}^2} \Phi(\vec{T}) d|T| \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E(u_\varepsilon).$$

Letting $\sigma \rightarrow 1$ and $A \rightarrow \Omega$, by Lemma 4.6 we conclude that (4.13) holds true. \square

Remark 4.9. The lower bound (4.13) dominates the anisotropic total variation; namely, for $u \in BV(\Omega; \mathbb{S}^1)$ it holds that

$$\mathcal{J}(u; \Omega) \geq \int_{J_u} d_{\mathbb{S}^1}(u^-, u^+) |v_u|_1 d\mathcal{H}^1.$$

This can be seen using the definition of $\ell_T(x)$ for a given $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ with $u_T = u$: for \mathcal{H}^1 -a.e. $x \in J_u \cap \mathcal{L}$, we have $d_{\mathbb{S}^1}(u^-(x), u^+(x)) \leq \text{length}(\gamma_x^T) = \ell_T(x)$, since γ_x^T is a curve connecting $u^-(x)$ and $u^+(x)$ in \mathbb{S}^1 .

4.2 Compactness and lower bound in presence of vortices

Next we extend the results of the previous subsection to the case of M vortices. Again we consider a general sequence $u_\varepsilon: \varepsilon\mathbb{Z}^2 \rightarrow \mathcal{S}_\varepsilon$ and the associated current G_{u_ε} .

PROPOSITION 4.10 (*M vortices, compactness*). *Assume that $\varepsilon \ll \theta_\varepsilon \ll \varepsilon|\log \varepsilon|$ and that there exist $M \in \mathbb{N}$ and $C > 0$ such that*

$$(4.14) \quad \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \leq C.$$

Then there exists $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ with $d_h \in \mathbb{Z}$ such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ (up to a subsequence) and $|\mu|(\Omega) \leq M$. If, in addition, $|\mu|(\Omega) = M$, then there exist $u \in BV(\Omega; \mathbb{S}^1)$ and $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ such that

- (i) $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$ and $u_\varepsilon \overset{*}{\rightharpoonup} u$ weakly* in $BV_{\text{loc}}(\Omega_\mu; \mathbb{R}^2)$;
- (ii) $T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1)$ and $u_T = u$ a.e. in Ω ;
- (iii) $G_{u_\varepsilon} \rightharpoonup T$ in $\mathcal{D}_2(\Omega_\mu \times \mathbb{R}^2)$ (up to a subsequence);
- (iv) $\partial T|_{\Omega \times \mathbb{R}^2} = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$.

PROOF. From (4.14) it follows that $\frac{1}{\varepsilon^2 |\log \varepsilon|} E_\varepsilon(u_\varepsilon) \leq 2\pi M + C \frac{\theta_\varepsilon}{\varepsilon |\log \varepsilon|}$, so that by Proposition 2.1 we get that (up to a subsequence)

$$\mu_{u_\varepsilon} \xrightarrow{f} \mu = \sum_{h=1}^N d_h \delta_{x_h} \quad \text{and} \quad |\mu|(\Omega) \leq M.$$

From now on we assume that $|\mu|(\Omega) = M$, that is $\sum_{h=1}^N |d_h| = M$.

Let $\rho > 0$ small enough such that the balls $B_\rho(x_h)$ are pairwise disjoint and $\overline{B_\rho(x_h)} \subset \Omega$. Recall the localized lower bound for the XY -model [8, theorem 3.1], which states that

$$(4.15) \quad \liminf_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) - 2\pi |d_h| \log \frac{\rho}{\varepsilon} \right] \geq \tilde{C} \quad \text{for some } \tilde{C} \in \mathbb{R}.$$

From this inequality and the fact that $\varepsilon \ll \theta_\varepsilon$ we deduce that

$$(4.16) \quad \begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) - 2\pi |d_h| |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right] \\ &= \liminf_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) - 2\pi |d_h| |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} - 2\pi |d_h| \log \rho \frac{\varepsilon}{\theta_\varepsilon} \right] \\ &= \liminf_{\varepsilon \rightarrow 0} \frac{\varepsilon}{\theta_\varepsilon} \left[\frac{1}{\varepsilon^2} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) - 2\pi |d_h| \log \frac{\rho}{\varepsilon} \right] \geq 0. \end{aligned}$$

Summing over $h = 1, \dots, N$, the superadditivity of the \liminf yields

$$(4.17) \quad \liminf_{\varepsilon \rightarrow 0} \left[\sum_{h=1}^N \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right] \geq 0.$$

Therefore the bound (4.14) implies

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon; \Omega_\mu^\rho)}{\varepsilon \theta_\varepsilon} \\ & \leq C - \liminf_{\varepsilon \rightarrow 0} \left[\sum_{h=1}^N \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon; B_\rho(x_h)) - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right] \leq C, \end{aligned}$$

so that, for ε small enough, $\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon; \Omega_\mu^\rho) \leq 2C$, where C is independent of ρ . By Proposition 4.1 and Proposition 4.4, with a diagonal argument we obtain that there exist $u \in BV(\Omega; \mathbb{S}^1)$ and $T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1)$ such that $u_\varepsilon \xrightarrow{*} u$ weakly* in $BV_{\text{loc}}(\Omega_\mu; \mathbb{S}^1)$, $G_{u_\varepsilon} \rightharpoonup T$ in $\mathcal{D}_2(\Omega_\mu \times \mathbb{R}^2)$ up to a subsequence, and $u_T = u$ a.e. in Ω . Since u_ε is equi-integrable, the local weak* BV -convergence implies strong $L^1(\Omega)$ -convergence. Thus (i)–(iii) hold true.

By Remark 4.7, the current T can be extended to a current $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$. Thus, it only remains to prove (iv). The argument is local, and we can work close to a single atom x_h of μ . Without loss of generality, assume that $x_h = 0$ and $\Omega = B := B_1(0)$. First of all, let us note that $\text{supp}(\partial T) \subset \{0\} \times \mathbb{S}^1$. Indeed, on the one hand if $\omega \in \mathcal{D}^1(B \times \mathbb{R}^2)$ is such that $\text{supp}(\omega) \subset (B \times \mathbb{R}^2) \setminus (\{0\} \times \mathbb{R}^2)$, then $\partial T(\omega) = 0$, since $T \in \text{cart}((B \setminus \{0\}) \times \mathbb{S}^1)$; on the other hand, if $\omega \in \mathcal{D}^1(B \times \mathbb{R}^2)$ is such that $\text{supp}(\omega) \subset (B \times \mathbb{R}^2) \setminus (B \times \mathbb{S}^1)$, then $\text{supp}(d\omega) \subset (B \times \mathbb{R}^2) \setminus (B \times \mathbb{S}^1)$ and thus $\partial T(\omega) = T(d\omega) = 0$, since $\text{supp}(T) \subset B \times \mathbb{S}^1$. In conclusion $\text{supp}(\partial T) \subset (\{0\} \times \mathbb{R}^2) \cap (B \times \mathbb{S}^1) = \{0\} \times \mathbb{S}^1$. Being ∂T a boundaryless 1-current with support in a 1-dimensional manifold, the constancy theorem [36, 5.3.1, theorem 2] gives that $\partial T|_{B \times \mathbb{R}^2} = -c \delta_0 \times \llbracket \mathbb{S}^1 \rrbracket$ for some $c \in \mathbb{R}$.

Now fix a function $\zeta \in C_c^\infty(B)$ with $\zeta \equiv 1$ in the ball $B_{1/2}(0)$ and define the 1-form $\omega = \zeta \omega_{\mathbb{S}^1}$, $\omega_{\mathbb{S}^1}$ being the 0-homogeneous extension of the volume form of \mathbb{S}^1 to $\mathbb{R}^2 \setminus \{0\}$. Since $d\omega \in \mathcal{D}^2((B \setminus \{0\}) \times \mathbb{R}^2)$, the convergence in (ii), Proposition 3.7, and the flat convergence $\mu_{u_\varepsilon} \xrightarrow{f} \mu$ yield the claimed equality $c = \mu(\{0\})$. Indeed,

$$\begin{aligned} -c2\pi &= \partial T(\omega) = T(d\omega) = \lim_{\varepsilon \rightarrow 0} G_{u_\varepsilon}(d\omega) = \lim_{\varepsilon \rightarrow 0} \partial G_{u_\varepsilon}(\omega) \\ &= \lim_{\varepsilon \rightarrow 0} -\langle \mu_{u_\varepsilon}, \zeta \rangle 2\pi = -\langle \mu, \zeta \rangle 2\pi. \quad \square \end{aligned}$$

We now prove the lower bound for M vortices. Let us define the set of admissible currents:

$$(4.18) \quad \begin{aligned} & \text{Adm}(\mu, u; \Omega) \\ & := \left\{ T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2) : \begin{aligned} & T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1), \partial T|_{\Omega \times \mathbb{R}^2} = -\mu \times \llbracket \mathbb{S}^1 \rrbracket, \\ & u_T = u \text{ a.e.} \end{aligned} \right\} \end{aligned}$$

and, similarly to (4.12), the energy

$$(4.19) \quad \mathcal{J}(\mu, u; \Omega) := \inf \left\{ \int_{J_T} \ell_T(x) |v_T(x)|_1 \, d\mathcal{H}^1(x) : T \in \text{Adm}(\mu, u; \Omega) \right\}$$

for every $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ and $u \in BV(\Omega; \mathbb{S}^1)$ with $\ell_T(x)$ defined in (4.10)¹⁰.

PROPOSITION 4.11 (*M vortices, lower bound*). *Assume that $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$ and (4.14) holds. Assume further that $\mu_{u_\varepsilon} \xrightarrow{f} \mu = \sum_{h=1}^N d_h \delta_{x_h}$ with $|\mu|(\Omega) = M$, $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$ with $u \in BV(\Omega; \mathbb{S}^1)$ as in Proposition 4.10. Then*

$$(4.20) \quad \begin{aligned} & \int_{\Omega} |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega) \\ & \leq \liminf_{\varepsilon \rightarrow 0} \left[\frac{E_\varepsilon(u_\varepsilon)}{\varepsilon \theta_\varepsilon} - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right]. \end{aligned}$$

PROOF. Let us fix $A \Subset \Omega_\mu$ and $\sigma \in (0, 1)$. Then there exists $\rho > 0$ such that $A \Subset \Omega_\mu^\rho$. Thanks to (4.3) and Lemma 4.3, for ε small enough, we infer that

$$(1 - \sigma) \int_{A \times \mathbb{R}^2} \Phi(\vec{G}_{u_\varepsilon}) d|G_{u_\varepsilon}| \leq \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon; \Omega_\mu^\rho).$$

Passing to a subsequence, we have that $G_{u_\varepsilon} \rightharpoonup T$ in $\mathcal{D}_2(\Omega_\mu \times \mathbb{R}^2)$ for some $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ given by Proposition 4.10. As in Proposition 4.5 and from (4.17) we infer that

$$\begin{aligned} & (1 - \sigma) \int_{A \times \mathbb{R}^2} \Phi(\vec{T}) d|T| \\ & \leq \liminf_{\varepsilon \rightarrow 0} (1 - \sigma) \int_{A \times \mathbb{R}^2} \Phi(\vec{G}_{u_\varepsilon}) d|G_{u_\varepsilon}| \\ & \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon; \Omega_\mu^\rho) \leq \liminf_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right]. \end{aligned}$$

Letting $A \rightarrow \Omega_\mu$ and $\sigma \rightarrow 0$ we conclude that

$$\int_{\Omega_\mu} \Phi(\vec{T}) d|T| \leq \liminf_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right].$$

By Proposition 4.10 (ii) and (iv), we have $T \in \text{Adm}(\mu, u; \Omega)$, so that (4.20) is a direct consequence of Lemma 4.6 and Remark 4.7. \square

¹⁰ $\text{Adm}(\mu, u; \Omega)$ is nonempty. Indeed, by Proposition 3.5 there exists $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ such that $u_T = u$. Let $\gamma_1, \dots, \gamma_N$ be pairwise disjoint unit speed Lipschitz curves such that γ_h connects x_h to $\partial\Omega$. Define L_h to be the 1-current $\tau(\text{supp}(\gamma_h), -d_h, \dot{\gamma}_h)$, so that $\partial L_h = d_h \delta_{x_h}$. Then $T + \sum_{h=1}^N L_h \times \llbracket \mathbb{S}^1 \rrbracket \in \text{Adm}(\mu, u; \Omega)$.

4.3 Upper bound in the absence of vortices

To reduce notation, for $u \in BV(\Omega; \mathbb{S}^1)$ we set

$$\mathcal{E}(u) := \int_{\Omega} |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(u; \Omega)$$

with $\mathcal{J}(u; \Omega)$ given by (4.12). The proof of the Γ -limsup inequality is done in several steps, which gradually simplify the map $u \in BV(\Omega; \mathbb{S}^1)$ that we want to approximate.

In the next proposition we approximate the map u with a sequence of smooth maps.

PROPOSITION 4.12. *Let $u \in BV(\Omega; \mathbb{S}^1)$. Then there exist an open set $\tilde{\Omega} \ni \Omega$ and a sequence $u_h \in C^\infty(\tilde{\Omega}; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ such that $u_h \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^2)$ and*

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h|_{2,1} \, dx \leq \int_{\Omega} |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(u; \Omega).$$

PROOF. Let $\eta > 0$ and let $T \in \text{cart}(\Omega \times \mathbb{S}^1)$ with $u_T = u$ a.e. in Ω be such that

$$(4.21) \quad \int_{J_T} \ell_T(x) |v_T(x)|_1 \, d\mathcal{H}^1(x) \leq \mathcal{J}(u; \Omega) + \eta.$$

Note that by Lemma 4.6

$$\int_{\Omega} |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \int_{J_T} \ell_T(x) |v_T(x)|_1 \, d\mathcal{H}^1(x) = \int_{\Omega \times \mathbb{R}^2} \Phi(\vec{T}) d|T|,$$

where Φ is the parametric integrand defined in (4.7). By Lemma 3.2 we can extend the current T to $\tilde{\Omega} \times \mathbb{S}^1$ for some $\tilde{\Omega} \ni \Omega$ such that $T \in \text{cart}(\tilde{\Omega} \times \mathbb{S}^1)$ and $|T|(\partial\Omega \times \mathbb{R}^2) = 0$.

Thanks to the approximation theorem 3.1 we find a sequence $u_h \in C^\infty(\tilde{\Omega}; \mathbb{S}^1)$ such that $G_{u_h} \rightarrow T$ in $\mathcal{D}_2(\tilde{\Omega} \times \mathbb{R}^2)$ and $|G_{u_h}|(\tilde{\Omega} \times \mathbb{R}^2) \rightarrow |T|(\tilde{\Omega} \times \mathbb{R}^2)$. In particular, since $|T|$ does not charge $\partial\Omega \times \mathbb{R}^2$, we have $|G_{u_h}|(\Omega \times \mathbb{R}^2) \rightarrow |T|(\Omega \times \mathbb{R}^2)$ and the convergence $u_h \rightarrow u_T = u$ in $L^1(\Omega; \mathbb{R}^2)$. Therefore, by Reshetnyak's continuity theorem [12, theorem 2.39] we have

$$\int_{\Omega \times \mathbb{R}^2} \Phi(\vec{G}_{u_h}) d|G_{u_h}| \rightarrow \int_{\Omega \times \mathbb{R}^2} \Phi(\vec{T}) d|T|.$$

By (3.4) and the area formula, we can write

$$\int_{\Omega \times \mathbb{R}^2} \Phi(\vec{G}_{u_h}) d|G_{u_h}| = \int_{\Omega} |\nabla u_h|_{2,1}(x) \, dx.$$

This implies that

$$\limsup_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h|_{2,1}(x) \, dx \leq \int_{\Omega} |\nabla u|_{2,1} \, dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(u; \Omega) + \eta.$$

Since $\eta > 0$ was arbitrary, we conclude the proof. \square

The next lemma states that we can discretize on a lattice $\lambda_n \mathbb{Z}^2$ any smooth map with values in \mathbb{S}^1 in such a way that the anisotropic BV norm does not increase. The discretized maps u_n satisfy in addition an “almost continuity property”, cf. (4.23), which states that for λ_n small enough the constant values of u_n in two neighboring cubes are close.

LEMMA 4.13 (Discretization of smooth \mathbb{S}^1 -valued maps). *Assume that $\lambda_n := 2^{-n}$, $n \in \mathbb{N}$, and let O, \tilde{O} be bounded, open sets such that $O \Subset \tilde{O}$. Assume that $u \in C^\infty(\tilde{O}; \mathbb{S}^1) \cap W^{1,1}(\tilde{O}; \mathbb{S}^1)$. Then there exist a sequence of piecewise constant maps $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$ such that $u_n \rightarrow u$ strongly in $L^1(O; \mathbb{R}^2)$ as $n \rightarrow +\infty$ and*

$$(4.22) \quad \limsup_{n \rightarrow +\infty} \int_{J_{u_n} \cap O^{\lambda_n}} d_{\mathbb{S}^1}(u_n^+, u_n^-) |v_{u_n}|_1 d\mathcal{H}^1 \leq \int_O |\nabla u|_{2,1} dx,$$

where O^{λ_n} is the union of half-open squares given by

$$O^{\lambda_n} := \bigcup \{I_{\lambda_n}(\lambda_n z) : z \in \mathbb{Z}^2 \text{ such that } I_{\lambda_n}(\lambda_n z) \cap O \neq \emptyset\}.$$

Moreover, for every $\delta > 0$ there exists $\bar{n} = \bar{n}(u, \delta, \tilde{O})$ such that for every $n \geq \bar{n}$ and for every $z_1, z_2 \in \mathbb{Z}^2$ with $\bar{I}_{\lambda_n}(\lambda_n z_1) \cap \bar{I}_{\lambda_n}(\lambda_n z_2) \neq \emptyset$ and $I_{\lambda_n}(\lambda_n z_i) \cap O \neq \emptyset$ we have

$$(4.23) \quad d_{\mathbb{S}^1}(u_n(\lambda_n(z_1)), u_n(\lambda_n(z_2))) \leq \delta.$$

PROOF. Let O' be an open set such that $O \Subset O' \Subset \tilde{O}$ and let \bar{n} be so large that for every $n \geq \bar{n}$ we have $O^{\lambda_n} \subset O'$. For every $z \in \mathbb{Z}^2$ such that $I_{\lambda_n}(\lambda_n z) \cap O \neq \emptyset$ we define

$$u_n(\lambda_n z) := u\left(\lambda_n\left(z + \frac{e_1}{2} + \frac{e_2}{2}\right)\right),$$

$\lambda_n(z + \frac{e_1}{2} + \frac{e_2}{2})$ being the center of the square $I_{\lambda_n}(\lambda_n z)$. The definition is well-posed, since $\lambda_n(z + \frac{e_1}{2} + \frac{e_2}{2}) \in \tilde{O}$. Then we extend u_n to $\lambda_n \mathbb{Z}^2$ by choosing an arbitrary value in \mathbb{S}^1 . This defines a piecewise constant map $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$.

Since u is continuous on O' , it follows that $u_n \rightarrow u$ pointwise on O and thus also strongly in $L^1(O; \mathbb{R}^2)$ by dominated convergence. Next we show (4.22). For $i \in \{1, 2\}$ define

$$\mathcal{Z}_i(\lambda_n) := \{z \in \mathbb{Z}^2 : I_{\lambda_n}(\lambda_n z) \cap O \neq \emptyset \text{ and } I_{\lambda_n}(\lambda_n(z + e_i)) \cap O \neq \emptyset\}.$$

Let $z \in \mathcal{Z}_i(\lambda_n)$. Since u is C^∞ in the interior of the rectangle $I_{\lambda_n}(\lambda_n z) \cup I_{\lambda_n}(\lambda_n(z + e_i))$, it admits a C^∞ lifting φ such that $u = \exp(i\varphi)$ in the interior of $I_{\lambda_n}(\lambda_n z) \cup I_{\lambda_n}(\lambda_n(z + e_i))$. Then, by the fundamental theorem of calculus

and the definition of u_n ,

$$\begin{aligned}
 & d_{\mathbb{S}^1}(u_n(\lambda_n(z + e_i)), u_n(\lambda_n z)) \\
 & \leq \left| \varphi\left(\lambda_n\left(z + e_i + \frac{e_1}{2} + \frac{e_2}{2}\right)\right) - \varphi\left(\lambda_n\left(z + \frac{e_1}{2} + \frac{e_2}{2}\right)\right) \right| \\
 (4.24) \quad & \leq \lambda_n \int_0^1 \left| \partial_i \varphi\left(\lambda_n\left(z + te_i + \frac{e_1}{2} + \frac{e_2}{2}\right)\right) \right| dt \\
 & = \lambda_n \int_0^1 \left| \partial_i u\left(\lambda_n\left(z + te_i + \frac{e_1}{2} + \frac{e_2}{2}\right)\right) \right| dt.
 \end{aligned}$$

We notice, in addition, that for every $t \in [0, 1]$ and $z \in \mathcal{Z}_i(\lambda_n)$,

$$\begin{aligned}
 & \left| \int_{I_{\lambda_n}(\lambda_n z)} |\partial_i u(x)| dx - \lambda_n^2 \left| \partial_i u\left(\lambda_n\left(z + te_i + \frac{e_1}{2} + \frac{e_2}{2}\right)\right) \right| \right| \\
 & \leq \int_{I_{\lambda_n}(\lambda_n z)} \left| \partial_i u(x) - \partial_i u\left(\lambda_n\left(z + te_i + \frac{e_1}{2} + \frac{e_2}{2}\right)\right) \right| dx \\
 & \leq 2\lambda_n^3 \|\nabla^2 u\|_{L^\infty(O')} =: C(u)\lambda_n^3.
 \end{aligned}$$

From (4.24) and the previous estimate it follows that

$$\begin{aligned}
 & \int_{J_{u_n} \cap O^{\lambda_n}} d_{\mathbb{S}^1}(u_n^+, u_n^-) |v_{u_n}|_1 d\mathcal{H}^1 \\
 & \leq \sum_{i=1}^2 \sum_{z \in \mathcal{Z}_i(\lambda_n)} \lambda_n d_{\mathbb{S}^1}(u_n(\lambda_n(z + e_i)), u_n(\lambda_n z)) \\
 & \leq \sum_{i=1}^2 \int_0^1 \sum_{z \in \mathcal{Z}_i(\lambda_n)} \lambda_n^2 \left| \partial_i u\left(\lambda_n\left(z + te_i + \frac{e_1}{2} + \frac{e_2}{2}\right)\right) \right| dt \\
 & \leq \sum_{i=1}^2 \int_0^1 \sum_{z \in \mathcal{Z}_i(\lambda_n)} \left\{ \int_{I_{\lambda_n}(\lambda_n z)} |\partial_i u(x)| dx + C(u)\lambda_n^3 \right\} dt \\
 & \leq \int_{O'} |\nabla u|_{2,1} dx + C(u)|O'|\lambda_n.
 \end{aligned}$$

We conclude the proof of (4.22) by letting $n \rightarrow +\infty$ and then $O' \searrow O$.

In order to prove (4.23), observe that the condition $\bar{I}_{\lambda_n}(\lambda_n z_1) \cap \bar{I}_{\lambda_n}(\lambda_n z_2) \neq \emptyset$ implies that $|\lambda_n(z_1 + \frac{1}{2}e_1 + e_2) - \lambda_n(z_2 + \frac{1}{2}e_1 + \frac{1}{2}e_2)| \leq \sqrt{2}\lambda_n$, so the claim follows from the Lipschitz continuity of u on the larger set \tilde{O} . \square

Now we can construct a recovery sequence $u_\varepsilon: \varepsilon\mathbb{Z}^2 \rightarrow \mathcal{S}_\varepsilon$. Due to the previous simplifications, it suffices to approximate the energy of piecewise constant maps on the lattice $\lambda_n\mathbb{Z}^2$ that come from Lemma 4.13. To define the recovery sequence, we shall construct a minimal transition (in \mathbb{S}^1) between two constant values of \mathbb{S}^1 .

First we introduce some notation about geodesics in \mathbb{S}^1 and state an elementary stability property, whose proof we omit.

DEFINITION 4.14. For $u^1, u^2 \in \mathbb{S}^1$ denote by $\text{Geo}[u^1, u^2]: [0, d_{\mathbb{S}^1}(u^1, u^2)] \rightarrow \mathbb{S}^1$ the (in case of nonuniqueness counterclockwise rotating) unit speed geodesic between u^1 and u^2 that we extend by $\text{Geo}[u^1, u^2](t) = u^1$ for $t < 0$ and $\text{Geo}[u^1, u^2](t) = u^2$ for $t > d_{\mathbb{S}^1}(u^1, u^2)$. As such the geodesics are 1-Lipschitz continuous functions on \mathbb{R} . We further set $\text{mid}(u^1, u^2) = \text{Geo}(\frac{1}{2}d_{\mathbb{S}^1}(u^1, u^2))$ as the midpoint on that geodesic.

LEMMA 4.15. *There exists a constant $c > 0$ such that whenever $u^1, u^2, b \in \mathbb{S}^1$ are such that $u^1, u^2 \in B_c(b)$, then for all $t \in \mathbb{R}$*

$$|\text{Geo}[u^1, b](t) - \text{Geo}[u^2, b](t)| \leq d_{\mathbb{S}^1}(u^1, u^2).$$

We introduce a map that will be used to project vectors of \mathbb{S}^1 on \mathcal{S}_ε . Given $u \in \mathbb{S}^1$ we let $\varphi_u \in [0, 2\pi)$ be the unique angle such that $u = \exp(i\varphi_u)$. We define $\mathfrak{P}_\varepsilon: \mathbb{S}^1 \rightarrow \mathcal{S}_\varepsilon$ by

$$(4.25) \quad \mathfrak{P}_\varepsilon(u) = \exp\left(i\theta_\varepsilon \left[\frac{\varphi_u}{\theta_\varepsilon} \right]\right).$$

Combined with Propositions 4.1 and 4.8 the next result completes the proof of Theorem 1.1 when $M = 0$.

PROPOSITION 4.16 ($M = 0$, upper bound). *Assume $\varepsilon \ll \theta_\varepsilon \ll 1$. Let $u \in BV(\Omega; \mathbb{S}^1)$. Then there exist $u_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ such that $u_\varepsilon \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^2)$ and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) \leq \int_{\Omega} |\nabla u|_{2,1} dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(u; \Omega).$$

PROOF. By Proposition 4.12, Lemma 4.13, and the L^1 -lower semicontinuity of the Γ -limsup, it is enough to prove that for $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$

$$(4.26) \quad \Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_n) \leq \int_{J_{u_n} \cap \Omega^{\lambda_n}} d_{\mathbb{S}^1}(u_n^+, u_n^-) |v_{u_n}|_1 d\mathcal{H}^1.$$

Since u_n is fixed in the following discussion, to simplify the notation we denote u_n by u and λ_n by λ , always assuming that $\lambda \ll 1$.

We will define a recovery sequence locally on each half-open cube $I_\lambda(\lambda z)$ for $z \in \mathbb{Z}^2$. First, we define a boundary condition on $\partial I_\lambda(\lambda z)$. For a side $S = \{\lambda z' + te_i : t \in [0, \lambda]\}$ with $z' \in \mathbb{Z}^2$ and $i \in \{1, 2\}$, and three values $v =$

$(v^1, v^2, v^3) \in (\mathbb{S}^1)^3$, we set $b_S^\varepsilon[v]: S \rightarrow \mathbb{S}^1$ for $x = \lambda z' + t e_i$ as

$$(4.27) \quad b_S^\varepsilon[v](x) = \begin{cases} v^1 & \text{if } t \in c_0 \frac{\varepsilon}{\theta_\varepsilon} [0, 1), \\ \text{Geo}[v^1, v^2] \left(\frac{d_{\mathbb{S}^1}(v^1, v^2) \theta_\varepsilon}{c_0 \varepsilon} (t - c_0 \frac{\varepsilon}{\theta_\varepsilon}) \right) & \text{if } t \in c_0 \frac{\varepsilon}{\theta_\varepsilon} [1, 2), \\ v^2 & \text{if } t \in [2c_0 \frac{\varepsilon}{\theta_\varepsilon}, \lambda - 2c_0 \frac{\varepsilon}{\theta_\varepsilon}), \\ \text{Geo}[v^2, v^3] \left(\frac{d_{\mathbb{S}^1}(v^2, v^3) \theta_\varepsilon}{c_0 \varepsilon} (t - \lambda + 2c_0 \frac{\varepsilon}{\theta_\varepsilon}) \right) & \text{if } t \in \lambda - c_0 \frac{\varepsilon}{\theta_\varepsilon} (1, 2], \\ v^3 & \text{if } t \in \lambda - c_0 \frac{\varepsilon}{\theta_\varepsilon} [0, 1]. \end{cases}$$

The particular choice of the constant c_0 is not important. For this proof it suffices that $c_0 > 2\pi$. This condition will be clear only after (4.35). (However, to apply this construction also in the proof of Proposition 4.22, we need to choose a larger constant, namely $c_0 = 393$.) Since $\frac{\varepsilon}{\theta_\varepsilon} \rightarrow 0$ by assumption, the function $b_S[v]$ can be interpreted as follows: in a small neighborhood of the two endpoints of S we set the two values v^1 and v^3 , while in a contiguous small neighborhood we use the geodesic for a transition to the value v^2 , which is taken on most of the side.

Next, given $u \in \mathcal{PC}_\lambda(\mathbb{S}^1)$ and a side S as above, we specify the values

$$(4.28) \quad v_S(u) = (u(\lambda z'), \text{mid}(u_S^-, u_S^+), u(\lambda(z' + e_i))),$$

where u_S^- and u_S^+ denote the (constant) traces of u along the side S and the midpoint is given by Definition 4.14. The boundary values $b_{z,\varepsilon}: \partial I_\lambda(\lambda z) \rightarrow \mathbb{S}^1$ are then defined by

$$b_{z,\varepsilon}(x) = b_S^\varepsilon[v_S(u)](x) \quad \text{if } x = \lambda z' + t e_i \in S \text{ for some } z' \in \mathbb{Z}^2 \text{ and } t \in [0, \lambda].$$

Note that this function is also well-defined in the corners with $b_{z,\varepsilon}(\lambda z_0) = u(\lambda z_0)$ for all $z_0 \in \mathbb{Z}^2$. Moreover, since we have chosen unit speed geodesics and $c_0 > 2\pi$, on each side S the function $b_S^\varepsilon[v_S(u)]$ satisfies a Lipschitz-estimate of the form

$$(4.29) \quad |b_S^\varepsilon[v_S(u)](x) - b_S^\varepsilon[v_S(u)](y)| \leq \frac{1}{2} \frac{\theta_\varepsilon}{\varepsilon} |x - y|, \quad x, y \in S.$$

Repeating the construction on every half-open cube we obtain a continuous function on the skeleton $\bigcup_{z \in \mathbb{Z}^2} \partial I_\lambda(\lambda z)$.

We are now in a position to define the recovery sequence of u . We will interpolate between the constant $u(\lambda z)$ and the boundary value $b_{z,\varepsilon}$ in $I_\lambda(\lambda z)$. This will be done on a mesoscale towards the boundary $\partial I_\lambda(\lambda z)$. Let $P: I_\lambda(\lambda z) \rightarrow \partial I_\lambda(\lambda z)$ be a function satisfying $|P(x) - x| = \text{dist}(x, \partial I_\lambda(\lambda z))$ for all $x \in I_\lambda(\lambda z)$ (such a function can be defined globally by periodicity). To reduce notation, let $u_z = u(\lambda z)$. Set $\bar{u}_\varepsilon: \varepsilon \mathbb{Z}^2 \cap I_\lambda(\lambda z) \rightarrow \mathbb{S}^1$ as

$$(4.30) \quad \bar{u}_\varepsilon(\varepsilon i) = \text{Geo}[b_{z,\varepsilon}(P(\varepsilon i)), u_z](\theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon i, \partial I_\lambda(\lambda z))),$$

with the extended geodesics given by Definition 4.14. Note that in general $\bar{u}_\varepsilon(\varepsilon i) \notin \mathcal{S}_\varepsilon$. Hence we define $u_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ by $u_\varepsilon := \mathfrak{P}_\varepsilon(\bar{u}_\varepsilon)$ with \mathfrak{P}_ε given by (4.25).

We claim that u_ε converges to u in $L^1(\Omega; \mathbb{R}^2)$. Indeed, for all $\varepsilon i \in I_\lambda(\lambda z)$ we have by Definition 4.14

$$(4.31) \quad u_\varepsilon(\varepsilon i) = \mathfrak{P}_\varepsilon(u_z) \quad \text{if } \text{dist}(\varepsilon i, \partial I_\lambda(\lambda z)) \geq \pi \frac{\varepsilon}{\theta_\varepsilon},$$

so that the assumptions $\frac{\varepsilon}{\theta_\varepsilon} \rightarrow 0$ and $\theta_\varepsilon \rightarrow 0$ yield that $u_\varepsilon \rightarrow u_z$ in measure on $I_\lambda(\lambda z)$. Here we used that $|\mathfrak{P}_\varepsilon - I| \leq \theta_\varepsilon$. Vitali's theorem then implies $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$. Next we bound the differences $u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)$ for all $i, j \in \mathbb{Z}^2$ with $|i - j| = 1$.

Step 1. Interactions within one cube.

We start with $\varepsilon i, \varepsilon j \in \varepsilon \mathbb{Z}^2 \cap I_\lambda(\lambda z)$ for the same z and $|i - j| = 1$. Let us write $I = I_\lambda(\lambda z)$ for short. One has to distinguish several cases:

Case 1. If $\text{dist}(\varepsilon i, \partial I) \geq \pi \varepsilon \theta_\varepsilon^{-1}$ and $\text{dist}(\varepsilon j, \partial I) \geq \pi \varepsilon \theta_\varepsilon^{-1}$, then (4.31) yields

$$|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = 0.$$

Since for neighboring lattice points it holds that

$$(4.32) \quad \frac{\theta_\varepsilon}{\varepsilon} |\text{dist}(\varepsilon i, \partial I) - \text{dist}(\varepsilon j, \partial I)| \leq \theta_\varepsilon,$$

for the remaining cases we can assume that

$$(4.33) \quad \max\{\text{dist}(\varepsilon i, \partial I), \text{dist}(\varepsilon j, \partial I)\} < (\pi + 1)\varepsilon \theta_\varepsilon^{-1}.$$

Case 2. We next analyze when $P(\varepsilon i)$ and $P(\varepsilon j)$ lie on different 1-dimensional boundary segments $S_i \neq S_j$ of I . We claim that $P(\varepsilon i)$ and $P(\varepsilon j)$ are then close to a node of the lattice $\lambda \mathbb{Z}^2$. Indeed, denote by Π_{S_i} and Π_{S_j} the projections onto the subspaces spanned by the segments S_i and S_j , respectively. Since by (4.33)

$$|P(\varepsilon i) - P(\varepsilon j)| \leq \varepsilon |i - j| + \text{dist}(\varepsilon i, \partial I) + \text{dist}(\varepsilon j, \partial I) \leq (2\pi + 2)\varepsilon \theta_\varepsilon^{-1} + \varepsilon,$$

for ε small enough the sides S_i and S_j cannot be parallel. Hence the point $\lambda z_{i,j} := \Pi_{S_i}(\Pi_{S_j}(\varepsilon i))$ belongs to $S_i \cap S_j \subset \lambda \mathbb{Z}^2$ and therefore the 1-Lipschitz continuity of Π_{S_i} and Π_{S_j} combined with (4.33) implies

$$(4.34) \quad \begin{aligned} \text{dist}(P(\varepsilon i), \lambda \mathbb{Z}^2) &= \text{dist}(\Pi_{S_i}(\varepsilon i), \lambda \mathbb{Z}^2) \leq |\varepsilon i - \Pi_{S_j}(\varepsilon i)| \\ &\leq |\varepsilon i - \varepsilon j| + |\varepsilon j - \Pi_{S_j}(\varepsilon j)| + |\Pi_{S_j}(\varepsilon j) - \Pi_{S_j}(\varepsilon i)| \\ &\leq 2\varepsilon + (\pi + 1)\varepsilon \theta_\varepsilon^{-1}. \end{aligned}$$

Exchanging the roles of i and j we derive by the same argument the bound

$$(4.35) \quad \text{dist}(P(\varepsilon j), \lambda \mathbb{Z}^2) \leq 2\varepsilon + (\pi + 1)\varepsilon \theta_\varepsilon^{-1}.$$

For ε small enough both terms can be bounded by $2\pi \varepsilon \theta_\varepsilon^{-1}$. In particular, the distance to $\lambda \mathbb{Z}^2$ of both $P(\varepsilon i)$ and $P(\varepsilon j)$ is realized by the point $\lambda z_{i,j}$, which is an endpoint of both the sides S_i and S_j . Hence from the definition of the boundary condition $b_{z,\varepsilon}$ in (4.27) and (4.28) and the fact that $c_0 > 2\pi$, we deduce that

$$b_{z,\varepsilon}(P(\varepsilon i)) = b_{z,\varepsilon}(P(\varepsilon j)) = u_{z_{i,j}}.$$

Equation (4.32), the 1-Lipschitz continuity of $\text{Geo}[u_{z_{i,j}}, u_z]$, and the construction of \bar{u}_ε yield $|\bar{u}_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon$. Due to the definition of the function \mathfrak{F}_ε this inequality implies

$$(4.36) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon.$$

Moreover, note that by (4.33), (4.34), and (4.35), for ε small enough,

$$(4.37) \quad \begin{aligned} \text{dist}(\varepsilon j, \lambda\mathbb{Z}^2) &\leq |P(\varepsilon j) - \varepsilon j| + \text{dist}(P(\varepsilon j), \lambda\mathbb{Z}^2) \\ &\leq (2\pi + 2)\frac{\varepsilon}{\theta_\varepsilon} + 2\varepsilon < 2c_0\frac{\varepsilon}{\theta_\varepsilon}, \\ \text{dist}(\varepsilon i, \lambda\mathbb{Z}^2) &\leq |P(\varepsilon i) - \varepsilon i| + \text{dist}(P(\varepsilon i), \lambda\mathbb{Z}^2) \\ &\leq (2\pi + 2)\frac{\varepsilon}{\theta_\varepsilon} + 2\varepsilon < 2c_0\frac{\varepsilon}{\theta_\varepsilon}. \end{aligned}$$

These inequalities will be used in Step 3 to count how many interactions fall under Case 2.

Case 3. Now consider points i and j such that $P(\varepsilon i) = \Pi_{S_i}(\varepsilon i)$ and $P(\varepsilon j) = \Pi_{S_i}(\varepsilon j)$ and assume additionally that $\text{dist}(P(\varepsilon j), \lambda\mathbb{Z}^2) \geq 3c_0\varepsilon\theta_\varepsilon^{-1}$. Since Π_{S_i} is 1-Lipschitz, this implies that $\text{dist}(P(\varepsilon i), \lambda\mathbb{Z}^2) \geq 2c_0\varepsilon\theta_\varepsilon^{-1}$ for ε small enough. Hence by the definition of the boundary condition (cf. (4.27) and (4.28))

$$b_{z,\varepsilon}(P(\varepsilon i)) = b_{z,\varepsilon}(P(\varepsilon j)) = \text{mid}(u_{S_i}^-, u_{S_i}^+).$$

Using again the 1-Lipschitz-continuity of the geodesic $\text{Geo}[\text{mid}(u_{S_i}^-, u_{S_i}^+), u_z]$, similarly to (4.36) we obtain that

$$(4.38) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon.$$

However, we need to analyze more accurately which points yield a nonzero difference. On the one hand, the projection property of P and the definition of \bar{u}_ε yield the implication

$$(4.39) \quad \begin{aligned} \mathbf{if} \quad \text{dist}(\varepsilon j, S_i) = \text{dist}(\varepsilon j, \partial I) \geq d_{S_i}(\text{mid}(u_{S_i}^-, u_{S_i}^+), u_z)\varepsilon\theta_\varepsilon^{-1} \\ \mathbf{then} \quad \bar{u}_\varepsilon(\varepsilon j) = u_z. \end{aligned}$$

The same conclusion holds true for εi . Hence for Case 3 the estimate (4.38) needs to be taken into account only for (i, j) such that one of them violates the condition in (4.39), while for other couples $\varepsilon i, \varepsilon j$ the difference vanishes as in Case 1.

On the other hand, using that $P(\varepsilon i) = \Pi_{S_i}(\varepsilon i)$ and $P(\varepsilon j) = \Pi_{S_i}(\varepsilon j)$, one can show the following implication (where \parallel means parallel):

$$(4.40) \quad \begin{aligned} (\varepsilon i - \varepsilon j) \parallel S_i = 0 &\implies \text{dist}(\varepsilon i, \partial I) = \text{dist}(\varepsilon j, \partial I) \\ &\implies |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = 0. \end{aligned}$$

Case 4. It remains to treat the case of points i and j such that $P(\varepsilon i) = \Pi_{S_i}(\varepsilon i)$ and $P(\varepsilon j) = \Pi_{S_i}(\varepsilon j)$, but $\text{dist}(P(\varepsilon j), \lambda\mathbb{Z}^2) < 3c_0\varepsilon\theta_\varepsilon^{-1}$. Here we use the Lipschitz-continuity of $b_{z,\varepsilon}$ on S_i and the stability estimate of Lemma 4.15. For the latter, we need that $b_{z,\varepsilon}(P(\varepsilon i))$ and $b_{z,\varepsilon}(P(\varepsilon j))$ are sufficiently

close to u_z . Since on S_i the boundary condition $b_{z,\varepsilon}$ is defined by geodesic interpolation between the elements of the vector $v_{S_i}(u) \in (\mathbb{S}^1)^3$ defined in (4.28) and $u_z \in \{u_{S_i}^-, u_{S_i}^+\}$, we know that

$$\begin{aligned}
& |b_{z,\varepsilon}(P(\varepsilon i)) - u_z| \\
& \leq d_{\mathbb{S}^1}(b_{z,\varepsilon}(P(\varepsilon i)), u_z) \\
& \leq \max_{r=1,3} d_{\mathbb{S}^1}((v_{S_i}(u), e_r), \text{mid}(u_{S_i}^-, u_{S_i}^+)) + d_{\mathbb{S}^1}(\text{mid}(u_{S_i}^-, u_{S_i}^+), u_z) \\
& = \max_{r=1,3} d_{\mathbb{S}^1}((v_{S_i}(u), e_r), \text{mid}(u_{S_i}^-, u_{S_i}^+)) + \frac{1}{2} d_{\mathbb{S}^1}(u_{S_i}^-, u_{S_i}^+) \\
& \leq \max_{r=1,3} d_{\mathbb{S}^1}((v_{S_i}(u), e_r), u_z) + d_{\mathbb{S}^1}(u_{S_i}^-, u_{S_i}^+).
\end{aligned}$$

Recall that the first and third component of $v_{S_i}[u]$ are given by the evaluation of u at the endpoints of S_i . Hence by the almost continuity estimate (4.23) we deduce for $\lambda \ll 1$ that $|b_{z,\varepsilon}(P(\varepsilon i)) - u_z| < c$, where c is the constant given by Lemma 4.15. Repeating the argument one proves the analogue estimate for $P(\varepsilon j)$. To reduce notation, we set $d_{\varepsilon,i} = \theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon i, \partial I)$ and $d_{\varepsilon,j} = \theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon j, \partial I)$. Then by (4.29) and the applicable Lemma 4.15 we have

$$\begin{aligned}
|\bar{u}_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)| & \leq |\text{Geo}[b_{z,\varepsilon}(P(\varepsilon i)), u_z](d_{\varepsilon,i}) - \text{Geo}[b_{z,\varepsilon}(P(\varepsilon i)), u_z](d_{\varepsilon,j})| \\
& \quad + |\text{Geo}[b_{z,\varepsilon}(P(\varepsilon i)), u_z](d_{\varepsilon,j}) - \text{Geo}[b_{z,\varepsilon}(P(\varepsilon j)), u_z](d_{\varepsilon,j})| \\
& \leq |d_{\varepsilon,i} - d_{\varepsilon,j}| + d_{\mathbb{S}^1}(b_{z,\varepsilon}(P(\varepsilon i)), b_{z,\varepsilon}(P(\varepsilon j))) \\
& \leq \theta_\varepsilon + \frac{\pi}{4} \theta_\varepsilon \varepsilon^{-1} |\Pi_{S_i}(\varepsilon i) - \Pi_{S_i}(\varepsilon j)| \leq 2\theta_\varepsilon.
\end{aligned}$$

Hence in Case 4 we deduce the slightly weaker bound

$$(4.41) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq 2\theta_\varepsilon.$$

Finally, the location condition on j and (4.33) imply that

$$(4.42) \quad \text{dist}(\varepsilon j, \lambda \mathbb{Z}^2) \leq |P(\varepsilon j) - \varepsilon j| + \text{dist}(P(\varepsilon j), \lambda \mathbb{Z}^2) < 4c_0 \varepsilon \theta_\varepsilon^{-1}.$$

Step 2. Interactions between different cubes.

Now we consider points $\varepsilon i \in I_\lambda(\lambda z_i)$ and $\varepsilon j \in I_\lambda(\lambda z_j)$ with $z_i \neq z_j$ and $|i - j| = 1$. By the definition of \bar{u}_ε via geodesics and by the 1-Lipschitz continuity of the latter we have

$$\begin{aligned}
|\bar{u}_\varepsilon(\varepsilon i) - b_{z_i,\varepsilon}(P(\varepsilon i))| & = |\bar{u}_\varepsilon(\varepsilon i) - \text{Geo}[b_{z_i,\varepsilon}(P(\varepsilon i)), \bar{u}_{z_i}](0)| \\
& \leq \frac{\theta_\varepsilon}{\varepsilon} \text{dist}(\varepsilon i, \partial I_\lambda(\lambda z_i)), \\
|\bar{u}_\varepsilon(\varepsilon j) - b_{z_j,\varepsilon}(P(\varepsilon j))| & = |\bar{u}_\varepsilon(\varepsilon j) - \text{Geo}[b_{z_j,\varepsilon}(P(\varepsilon j)), \bar{u}_{z_j}](0)| \\
& \leq \frac{\theta_\varepsilon}{\varepsilon} \text{dist}(\varepsilon j, \partial I_\lambda(\lambda z_j)).
\end{aligned}$$

Note that there exists a boundary segment S_{ij} of $\partial I_\lambda(\lambda z_j)$ such that the line segment $[\varepsilon i, \varepsilon j]$ intersects S_{ij} orthogonally and moreover $S_{ij} \subset \partial I_\lambda(\lambda z_i)$. In particular,

$$\text{dist}(\varepsilon i, \partial I_\lambda(\lambda z_i)) + \text{dist}(\varepsilon j, \partial I_\lambda(\lambda z_j)) \leq \varepsilon.$$

Summing the previous two estimates then yields

$$(4.43) \quad |\bar{u}_\varepsilon(\varepsilon i) - b_{z_i, \varepsilon}(P(\varepsilon i))| + |\bar{u}_\varepsilon(\varepsilon j) - b_{z_j, \varepsilon}(P(\varepsilon j))| \leq \theta_\varepsilon.$$

We claim that either $P(\varepsilon i) \in S_{ij}$ and $P(\varepsilon j) \in S_{ij}$ or that both $P(\varepsilon i)$ and $P(\varepsilon j)$ are close to $\lambda \mathbb{Z}^2$. Indeed, first assume that $P(\varepsilon j) \notin S_{ij}$. Then there exists another facet S_j of $I_\lambda(\lambda z_j)$ such that $P(\varepsilon j) \in S_j$. Since $\text{dist}(\varepsilon j, S_j) \leq \varepsilon$ and $\text{dist}(\varepsilon j, S_{ij}) \leq \varepsilon$, the sides S_j and S_{ij} cannot be parallel. Denoting by Π_S the projection onto the subspace spanned by a segment S , we deduce that $\Pi_{S_j}(\Pi_{S_{ij}}(\varepsilon j)) \in S_j \cap S_{ij} \subset \lambda \mathbb{Z}^2$. Hence

$$\text{dist}(P(\varepsilon j), \lambda \mathbb{Z}^2) = \text{dist}(\Pi_{S_j}(\varepsilon j), \lambda \mathbb{Z}^2) \leq |\varepsilon j - \Pi_{S_{ij}}(\varepsilon j)| \leq \varepsilon.$$

For $P(\varepsilon i)$ we check two possibilities. First consider $P(\varepsilon i) \in S_{ij}$. Then we may assume that $P(\varepsilon j) \notin S_{ij}$ as above. From the Lipschitz-continuity of $\Pi_{S_{ij}}$ we infer

$$\begin{aligned} \text{dist}(P(\varepsilon i), \lambda \mathbb{Z}^2) &= \text{dist}(\Pi_{S_{ij}}(\varepsilon i), \lambda \mathbb{Z}^2) \leq |\Pi_{S_{ij}}(\varepsilon i) - \Pi_{S_{ij}}(\Pi_{S_j}(\varepsilon j))| \\ &\leq |\varepsilon i - \varepsilon j| + |\varepsilon j - \Pi_{S_j}(\varepsilon j)| \leq 2\varepsilon. \end{aligned}$$

On the contrary, if $P(\varepsilon i) \notin S_{ij}$, then there exists a facet $S_i \neq S_{ij}$ of $I_\lambda(\lambda z_i)$ such that $P(\varepsilon i) \in S_i$. Since S_i and S_{ij} are both sides of the cube $I_\lambda(\lambda z_i)$ that cannot be parallel, we deduce that $\Pi_{S_i}(\Pi_{S_{ij}}(\varepsilon i)) \in \text{span}(S_i) \cap \text{span}(S_{ij}) \subset \lambda \mathbb{Z}^2$ and thus the defining property of S_{ij} implies that

$$\text{dist}(P(\varepsilon i), \lambda \mathbb{Z}^2) = \text{dist}(\Pi_{S_i}(\varepsilon i), \lambda \mathbb{Z}^2) \leq |\varepsilon i - \Pi_{S_{ij}}(\varepsilon i)| \leq \varepsilon.$$

It remains to establish an estimate for $\text{dist}(P(\varepsilon j), \lambda \mathbb{Z}^2)$ when $P(\varepsilon i) \notin S_{ij}$ and $P(\varepsilon j) \in S_{ij}$. In this case we have

$$\begin{aligned} \text{dist}(P(\varepsilon j), \lambda \mathbb{Z}^2) &= \text{dist}(\Pi_{S_{ij}}(\varepsilon j), \lambda \mathbb{Z}^2) \leq |\Pi_{S_{ij}}(\varepsilon j) - \Pi_{S_{ij}}(\Pi_{S_i}(\varepsilon i))| \\ &\leq |\varepsilon j - \varepsilon i| + |\varepsilon i - \Pi_{S_i}(\varepsilon i)| \leq 2\varepsilon. \end{aligned}$$

To sum up, we have proved the following two alternatives:

- (i) $P(\varepsilon i), P(\varepsilon j) \in S_{ij}$;
- (ii) $\max\{\text{dist}(P(\varepsilon i), \lambda \mathbb{Z}^2), \text{dist}(P(\varepsilon j), \lambda \mathbb{Z}^2)\} \leq 2\varepsilon$.

Again we treat the two cases separately.

Case 5. Note that the condition in (ii) above implies that the unique points $\lambda \bar{z}_i, \lambda \bar{z}_j \in \lambda \mathbb{Z}^2$ realizing the minimal distance satisfy

$$\begin{aligned} &|\lambda \bar{z}_i - \lambda \bar{z}_j| \\ &\leq |\lambda \bar{z}_i - P(\varepsilon i)| + |P(\varepsilon i) - \varepsilon i| + \varepsilon|i - j| + |\varepsilon j - P(\varepsilon j)| + |P(\varepsilon j) - \lambda \bar{z}_j| \\ &\leq 7\varepsilon, \end{aligned}$$

so that necessarily $\bar{z}_i = \bar{z}_j$ for ε small enough. In particular, the construction of the boundary condition forces $b_{z_i, \varepsilon}(P(\varepsilon i)) = b_{z_j, \varepsilon}(P(\varepsilon j)) = u_{\bar{z}_i}$. From (4.43) we infer

$$|\bar{u}_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)| \leq |\bar{u}_\varepsilon(\varepsilon i) - u_{\bar{z}_i}| + |\bar{u}_\varepsilon(\varepsilon j) - u_{\bar{z}_j}| \leq \theta_\varepsilon,$$

which by the definition of \mathfrak{F}_ε allows to conclude that

$$(4.44) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon.$$

Furthermore, we know that

$$(4.45) \quad \text{dist}(\varepsilon j, \lambda \mathbb{Z}^2) \leq \text{dist}(P(\varepsilon j), \lambda \mathbb{Z}^2) + \text{dist}(\varepsilon j, \partial I_\lambda(\lambda z)) \leq 3\varepsilon.$$

Case 6. We now analyze the case $P(\varepsilon i), P(\varepsilon j) \in S_{ij}$. By the symmetric definition, $b_{z_i, \varepsilon}$ and $b_{z_j, \varepsilon}$ coincide on S_{ij} . Since by assumption the segment $[\varepsilon i, \varepsilon j]$ is orthogonal to S_{ij} and $S_{ij} \subset \partial I_\lambda(\lambda z_i) \cap \partial I_\lambda(\lambda z_j)$, we have

$$P(\varepsilon i) = \Pi_{S_{ij}}(\varepsilon i) = \Pi_{S_{ij}}(\varepsilon j) = P(\varepsilon j).$$

Hence estimate (4.43) yields

$$|\bar{u}_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)| = |\bar{u}_\varepsilon(\varepsilon i) - b_{z_i, \varepsilon}(P(\varepsilon i))| + |b_{z_j, \varepsilon}(P(\varepsilon j)) - \bar{u}_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon,$$

which again can be turned into an estimate for u_ε that reads

$$(4.46) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon.$$

Moreover, we can give an estimate for the location of εj by

$$(4.47) \quad \text{dist}(\varepsilon j, S_{ij}) = \text{dist}(\varepsilon j, \partial I_\lambda(\lambda z_j)) \leq \varepsilon.$$

Step 3. Energy estimates.

Let us first sum up our analysis hitherto. The interactions of couples $(\varepsilon i, \varepsilon j)$ with $|i - j| = 1$ and at least one point in an half-open cube $I_\lambda(\lambda z)$ can be grouped as follows:

- (1) In Case 1 it holds that $|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = 0$.
- (2) In Cases 2, 4, and 5 we have for ε small enough $\text{dist}(\varepsilon j, \lambda \mathbb{Z}^2) \leq 4c_0\varepsilon\theta_\varepsilon^{-1}$ (see (4.37), (4.42), and (4.45)) and by (4.36), (4.41), and (4.44) the continuity estimate

$$|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq 2\theta_\varepsilon.$$

- (3) In Cases 3 and 6, according to (4.38)–(4.40), respectively (4.46)–(4.47), we know that there exists a side S of $I_\lambda(\lambda z)$ such that, setting

$$\kappa_S(z) := \frac{1}{2} \text{d}_{\mathbb{S}^1}(u_S^-, u_S^+) = \text{d}_{\mathbb{S}^1}(\text{mid}(u_S^-, u_S^+), u_z),$$

it holds that

$$|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq \begin{cases} \theta_\varepsilon & \text{if } \text{dist}(\varepsilon j, S) < \kappa_S(z) \frac{\varepsilon}{\theta_\varepsilon} + \varepsilon \text{ and } (i - j) \perp S, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can estimate the discrete energy. Due to (1)–(3) above it is bounded by $\frac{E_\varepsilon(u_\varepsilon)}{\varepsilon\theta_\varepsilon}$

$$\begin{aligned} &\leq \sum_{I_\lambda(\lambda z) \cap \Omega \neq \emptyset} \sum_{S \subset \partial I_\lambda(\lambda z)} \varepsilon\theta_\varepsilon \# \left(\varepsilon\mathbb{Z}^2 \cap I_\lambda(\lambda z) \cap \left\{ \text{dist}(\cdot, S) < \frac{\varepsilon}{\theta_\varepsilon} \kappa_S(z) + 2\varepsilon \right\} \right) \\ &\quad + C\varepsilon\theta_\varepsilon \# (\Omega_\varepsilon \cap \{ \text{dist}(\cdot, \lambda\mathbb{Z}^2) \leq C\varepsilon\theta_\varepsilon^{-1} \}), \end{aligned}$$

where we used that each point in \mathbb{Z}^2 has four neighbors, but for (3) we have to count only half of the interactions. We claim that the first right-hand side term vanishes when $\varepsilon \rightarrow 0$. To this end, fix a large cube Q such that $\Omega \Subset Q$. For ε small enough we have

$$\varepsilon^2 \# (\Omega_\varepsilon \cap \{ \text{dist}(\cdot, \lambda\mathbb{Z}^2) \leq C\varepsilon\theta_\varepsilon^{-1} \}) \leq \sum_{x \in \lambda\mathbb{Z}^2 \cap Q} |B_{2C\varepsilon\theta_\varepsilon^{-1}}(x)| \leq C|Q|\lambda^{-2}\varepsilon^2\theta_\varepsilon^{-2}.$$

Inserting this estimate into the first term, we obtain

$$(4.48) \quad \varepsilon\theta_\varepsilon \# (\Omega_\varepsilon \cap \{ \text{dist}(\cdot, \lambda\mathbb{Z}^2) \leq C\varepsilon\theta_\varepsilon^{-1} \}) \leq C|Q|\lambda^{-2} \frac{\varepsilon}{\theta_\varepsilon},$$

which vanishes when $\varepsilon \rightarrow 0$ due to the assumption $\varepsilon \ll \theta_\varepsilon$.

Now we treat the second term. Since each segment S has length λ , for any fixed $\kappa > 0$ it holds that

$$\varepsilon\theta_\varepsilon \# \left(\varepsilon\mathbb{Z}^2 \cap I_\lambda(\lambda z) \cap \left\{ \text{dist}(\cdot, S) < \frac{\varepsilon}{\theta_\varepsilon} \kappa + 2\varepsilon \right\} \right) \leq \frac{\theta_\varepsilon}{\varepsilon} \left(\lambda + \frac{2\varepsilon}{\theta_\varepsilon} \kappa + 6\varepsilon \right) \left(\frac{\varepsilon}{\theta_\varepsilon} \kappa + 4\varepsilon \right)$$

and, for only finitely many cubes $I_\lambda(\lambda z)$ intersecting Ω , we can insert this estimate with $\kappa = \kappa_S(z)$, pass to the limit in ε , and obtain by (4.48) that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon) &\leq \sum_{I_\lambda(\lambda z) \cap \Omega \neq \emptyset} \sum_{S \subset \partial I_\lambda(\lambda z)} \frac{\lambda}{2} d_{\mathbb{S}^1}(u_S^-, u_S^+) \\ (4.49) \quad &= \int_{J_u \cap \Omega^\lambda} d_{\mathbb{S}^1}(u^-, u^+) |v_u|_1 d\mathcal{H}^1. \end{aligned}$$

This estimate agrees with (4.26) and hence concludes the proof. \square

4.4 Upper bound in the presence of vortices

Also in the case of vortices the construction of the recovery sequence is done by gradually simplifying the map $u \in BV(\Omega; \mathbb{S}^1)$, following the main idea of Section 4.3. However, due to the presence of the vortex measure $\mu = \sum_{h=1}^N d_h \delta_{x_h}$, in general the map u cannot be approximated by smooth maps with values in \mathbb{S}^1 . This requires additional steps in the simplification of u . For notational convenience, set

$$\mathcal{E}(\mu, u) = \int_{\Omega} |\nabla u|_{2,1} dx + |D^{(c)}u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega)$$

with $\mathcal{J}(\mu, u; \Omega)$ given by (4.19).

We start with the approximation result for currents T with boundary $\partial T = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$, which appear in the definition of the limit functional.

LEMMA 4.17 (Approximations creating finitely many singularities). *Let the measure $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ and $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$ be such that $T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1)$ and $\partial T|_{\Omega \times \mathbb{R}^2} = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$. Then there exist an open set $\tilde{\Omega} \ni \Omega$ and a sequence $u_k \in C^\infty(\tilde{\Omega}_\mu; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ such that*

$$(4.50) \quad u_k \rightarrow u_T \quad \text{in } L^1(\Omega; \mathbb{R}^2),$$

$$(4.51) \quad G_{u_k} \rightharpoonup T \quad \text{in } \mathcal{D}_2(\Omega_\mu \times \mathbb{R}^2),$$

$$(4.52) \quad |G_{u_k}|(\Omega \times \mathbb{R}^2) \rightarrow |T|(\Omega \times \mathbb{R}^2),$$

$$(4.53) \quad \deg(u_k)(x_h) = d_h \quad \text{for } h = 1, \dots, N.$$

PROOF. Let Ω' and Ω'' be open sets with Lipschitz boundary such that $\Omega' \Subset \Omega'' \Subset \Omega$ and $\{x_1, \dots, x_N\} \subset \Omega'$, and let us define the open set $O := \Omega \setminus \bar{\Omega}'$. Since $\partial T|_{O \times \mathbb{R}^2} = 0$, we have $T \in \text{cart}(O \times \mathbb{S}^1)$. By Lemma 3.2 there exist an open set $\tilde{O} \ni O$ and a current $\tilde{T} \in \text{cart}(\tilde{O} \times \mathbb{S}^1)$ such that $\tilde{T}|_{O \times \mathbb{R}^2} = T|_{O \times \mathbb{R}^2}$ and $|\tilde{T}|(\partial O \times \mathbb{R}^2) = 0$. In particular,

$$(4.54) \quad \tilde{T}|_{(\Omega'' \setminus \bar{\Omega}') \times \mathbb{R}^2} = T|_{(\Omega'' \setminus \bar{\Omega}') \times \mathbb{R}^2}.$$

This allows us to glue together the currents T and \tilde{T} . To do so, we define the set $\tilde{\Omega} := \Omega \cup \tilde{O}$ and the current $S \in \mathcal{D}_2(\tilde{\Omega} \times \mathbb{R}^2)$ as follows. Fix a cutoff function $\zeta \in C_c^\infty(\Omega'')$ such that $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on a neighborhood of $\bar{\Omega}'$. For every $\omega \in \mathcal{D}^2(\tilde{\Omega} \times \mathbb{R}^2)$ we put $S(\omega) := T(\zeta \omega) + \tilde{T}((1 - \zeta)\omega)$. Then by (4.54) it follows that $S|_{\Omega \times \mathbb{R}^2} = T$, $S|_{(\tilde{\Omega} \setminus \bar{\Omega}') \times \mathbb{R}^2} = \tilde{T}|_{(\tilde{\Omega} \setminus \bar{\Omega}') \times \mathbb{R}^2}$, and $|S|(\partial \Omega \times \mathbb{R}^2) = 0$. In particular, using the product rule for the exterior derivative, for any 1-form $\omega \in \mathcal{D}^1(\tilde{\Omega} \times \mathbb{S}^1)$ we find that

$$\begin{aligned} \partial S(\omega) &= T(\zeta d\omega) + \tilde{T}((1 - \zeta) d\omega) \\ &= \partial T(\zeta \omega) - T(d\zeta \wedge \omega) + \partial \tilde{T}((1 - \zeta)\omega) + \tilde{T}(d\zeta \wedge \omega) = -\mu \times \llbracket \mathbb{S}^1 \rrbracket(\omega), \end{aligned}$$

where we used that $\zeta \equiv 1$ on $\text{supp}(\mu)$ and $d\zeta \wedge \omega \in \mathcal{D}^2(\Omega'' \setminus \bar{\Omega}' \times \mathbb{R}^2)$. Hence $S \in \text{cart}(\tilde{\Omega}_\mu \times \mathbb{S}^1)$ and $\partial S|_{\tilde{\Omega} \times \mathbb{R}^2} = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$.

Since $S \in \text{cart}(\tilde{\Omega}_\mu \times \mathbb{S}^1)$, by the approximation theorem for Cartesian currents (Theorem 3.1) there exists a sequence $u_k \in C^\infty(\tilde{\Omega}_\mu; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ such that $G_{u_k} \rightharpoonup S$ in $\mathcal{D}_2(\tilde{\Omega}_\mu \times \mathbb{R}^2)$ and $|G_{u_k}|(\tilde{\Omega}_\mu \times \mathbb{R}^2) \rightarrow |S|(\tilde{\Omega}_\mu \times \mathbb{R}^2)$. In particular, we get (4.51) and thus (4.50). Moreover,

$$(4.55) \quad |G_{u_k}|(\tilde{\Omega} \times \mathbb{R}^2) \rightarrow |S|(\tilde{\Omega} \times \mathbb{R}^2),$$

since G_{u_k} and S do not charge the sets $\{x_h\} \times \mathbb{R}^2$ (being i.m. rectifiable 2-currents concentrated on a subset of $\mathbb{R}^2 \times \mathbb{S}^1$, see also Remark 4.7).

Thanks to the convergence in (4.51) we can prove (4.53). Indeed, let $h = 1, \dots, N$. For $\rho > 0$ small enough we have (e.g., by [35, sec. 6, prop. 1])

$$\partial G_{u_k}|_{B_\rho(x_h) \times \mathbb{R}^2} = -\deg(u_k)(x_h) \delta_{x_h} \times \llbracket \mathbb{S}^1 \rrbracket.$$

Fix a cutoff function $\zeta \in C_c^\infty(B_\rho(x_h))$ such that $\zeta \equiv 1$ on $B_{\rho/2}(x_h)$ and define the 1-form $\omega = \zeta \omega_{\mathbb{S}^1}$, $\omega_{\mathbb{S}^1}$ being the 0-homogeneous extension of the volume form of \mathbb{S}^1 to $\mathbb{R}^2 \setminus \{0\}$. Observing that $d\omega \in \mathcal{D}^2((B_\rho(x_h) \setminus \{x_h\}) \times \mathbb{R}^2)$, the convergence in (4.51) implies that

$$-\deg(u_k)(x_h) = \partial G_{u_k}(\omega) = G_{u_k}(d\omega) \rightarrow T(d\omega) = \partial T(\omega) = -d_h.$$

Then $\deg(u_k)(x_h)$ is a sequence of integer numbers that converges to the integer number d_h . Thus for k large enough $\deg(u_k)(x_h) = d_h$. To conclude, we observe that (4.55) and $|S|(\partial\Omega \times \mathbb{R}^2) = 0$ imply (4.52). \square

The next result shows how to reduce the analysis to singularities with degree ± 1 .

LEMMA 4.18 (Splitting of the degree). *Let $V := \{x_1, \dots, x_N\} \subset \Omega$ and let $u \in C^\infty(\Omega \setminus V; \mathbb{S}^1) \cap W^{1,1}(\Omega; \mathbb{S}^1)$ be such that $\deg(u)(x_h) \neq 0$ for $h = 1, \dots, N$. Then for $0 < \tau \ll 1$ there exist a set $V_\tau = \{x_1^\tau, \dots, x_{N_\tau}^\tau\} \subset \Omega$ and a function $u^\tau \in C^\infty(\Omega \setminus V_\tau; \mathbb{S}^1) \cap W^{1,1}(\Omega; \mathbb{S}^1)$ such that $u^\tau \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^2)$, $|\deg(u^\tau)(x_h^\tau)| = 1$ for $h = 1, \dots, N_\tau$ and*

$$(4.56) \quad \lim_{\tau \rightarrow 0} \int_{\Omega} |\nabla u^\tau|_{2,1} dx = \int_{\Omega} |\nabla u|_{2,1} dx,$$

$$(4.57) \quad N_\tau = \sum_{h=1}^{N_\tau} |\deg(u^\tau)(x_h^\tau)| = \sum_{h=1}^N |\deg(u)(x_h)|.$$

Moreover, defining the measures $\mu^\tau = \sum_{h=1}^{N_\tau} \deg(u^\tau)(x_h^\tau) \delta_{x_h^\tau}$, we have that

$$\mu^\tau \xrightarrow{f} \sum_{h=1}^N \deg(u)(x_h) \delta_{x_h} \quad \text{as } \tau \rightarrow 0.$$

Finally, if $u \in C^\infty(\tilde{\Omega} \setminus V; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ for some $\tilde{\Omega} \ni \Omega$, then one can additionally choose $u^\tau \in C^\infty(\tilde{\Omega} \setminus V; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$.

Remark 4.19. In this section we shall apply Lemma 4.18 to functions given by Lemma 4.17; cf. (4.53). In [28] we consider $u \in C^\infty(\Omega \setminus V; \mathbb{S}^1) \cap W^{1,1}(\Omega; \mathbb{S}^1)$ without assuming that $\deg(u)(x_h) \neq 0$ for every $h = 1, \dots, N$. In that case, the statement of the lemma holds true, but (4.57) needs to be adapted to

$$N_\tau = \sum_{h=1}^N |\deg(u)(x_h)| + 2\#\{x_h : \deg(u)(x_h) = 0\}.$$

The argument in the proof remains unchanged.

PROOF OF LEMMA 4.18. Via an iterative construction we create $|\deg(u)(x_h)|$ different singularities out of one singularity whenever $|\deg(u)(x_h)| > 1$ in such a way that the new function is close in energy. To reduce notation, we assume that $x_1 = 0$ and $\deg(u)(x_1) > 1$ (the case of a negative degree less than -1 can be treated similarly). We equip \mathbb{R}^2 with the complex product, which we denote by \odot . Given $0 < \tau \ll 1$ we set $u^\tau \in C^\infty(\Omega \setminus (V \cup \{\tau e_1\}); \mathbb{S}^1)$ as

$$u^\tau(x) = u(x) \odot \left(\frac{x}{|x|} \right)^{-1} \odot \frac{(x - \tau e_1)}{|x - \tau e_1|}.$$

Defined as above, it follows that $u^\tau \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$ by dominated convergence. Next, we estimate its anisotropic gradient norm. By the product rule, for $i = 1, 2$ we obtain

$$\begin{aligned} \partial_i u^\tau(x) &= \partial_i u(x) \odot \left(\frac{x}{|x|} \right)^{-1} \odot \frac{(x - \tau e_1)}{|x - \tau e_1|} \\ &\quad + u(x) \odot \left\{ \partial_i \left(\frac{x}{|x|} \right)^{-1} \odot \frac{(x - \tau e_1)}{|x - \tau e_1|} + \left(\frac{x}{|x|} \right)^{-1} \odot \partial_i \frac{(x - \tau e_1)}{|x - \tau e_1|} \right\}. \end{aligned}$$

A straightforward computation shows that, for a.e. $x \in \Omega$, we have

$$\begin{aligned} \lim_{\tau \rightarrow 0} \partial_i u^\tau(x) &= \partial_i u(x) + u \odot \left\{ \partial_i \left(\frac{x}{|x|} \right)^{-1} \odot \frac{x}{|x|} + \left(\frac{x}{|x|} \right)^{-1} \odot \partial_i \frac{x}{|x|} \right\} \\ &= \partial_i u(x) + u \odot \partial_i \left\{ \left(\frac{x}{|x|} \right)^{-1} \odot \frac{x}{|x|} \right\} = \partial_i u(x). \end{aligned}$$

In order to use dominated convergence, we observe that $\left(\frac{x}{|x|} \right)^{-1} = \frac{1}{|x|}(x_1, -x_2)$, so that

$$\begin{aligned} |\partial_i u^\tau(x)| &\leq |\partial_i u(x)| + \left| \partial_i \left(\frac{x}{|x|} \right)^{-1} \right| + \left| \partial_i \frac{(x - \tau e_1)}{|x - \tau e_1|} \right| \\ &\leq |\partial_i u(x)| + \frac{2}{|x|} + \frac{2}{|x - \tau|}. \end{aligned}$$

The right-hand side is equi-integrable on $\Omega \subset \mathbb{R}^2$, so that we conclude

$$\lim_{\tau \rightarrow 0} \int_{\Omega} |\nabla u^\tau|_{2,1} \, dx = \int_{\Omega} |\nabla u|_{2,1} \, dx.$$

Finally, we need to compute the degree of u^τ . Let us introduce the complex-valued functions

$$\begin{aligned} \tilde{u}(x) &= u_1(x) + \iota u_2(x), \quad f(x) = \frac{1}{|x|}(x_1 - \iota x_2), \\ g(x) &= \frac{1}{|x - \tau e_1|}((x_1 - \tau) + \iota x_2). \end{aligned}$$

In \mathbb{R}^2 the degree around a point \bar{x} can be expressed via the winding number, which means

$$\begin{aligned} (2\pi\iota) \deg(u^\tau)(\bar{x}) &= \int_{\partial B_r(\bar{x})} \frac{d(\tilde{u}fg)}{\tilde{u}fg} = \int_{\partial B_r(\bar{x})} \frac{(d\tilde{u})fg + \tilde{u}(df)g + \tilde{u}f(dg)}{\tilde{u}fg} \\ &= 2\pi\iota(\deg(u)(\bar{x}) - \delta_0(\bar{x}) + \delta_{\tau e_1}(\bar{x})). \end{aligned}$$

We deduce that the degree of u^τ is of the form (recalling that $x_1 = 0$)

$$\deg(u^\tau)(\bar{x}) = \begin{cases} \deg(u)(x_1) - 1 & \text{if } \bar{x} = x_1, \\ 1 & \text{if } \bar{x} = x_1 + \tau e_1, \\ \deg(u)(\bar{x}) & \text{otherwise,} \end{cases}$$

where for the second equality we used that $\deg(u)(x_1 + \tau e_1) = 0$ due to the local smoothness of u around $x_1 + \tau e_1$ (see [22, cor. 8]). Repeating this construction (with $\tau/2$, $\tau/4$, and so on) we find a finite set $V_\tau = \{x_1^\tau, \dots, x_{N_\tau}^\tau\}$ and a sequence $u^\tau \in C^\infty(\Omega \setminus V_\tau; \mathbb{S}^1) \cap W^{1,1}(\Omega; \mathbb{S}^1)$ such that $u^\tau \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$, $\deg(u^\tau)(x_h^\tau) \in \{\pm 1\}$, and (4.56)–(4.57) hold true. The claim on the flat convergence follows by the construction. \square

In the next lemma we move the singularities onto a lattice $\lambda_n \mathbb{Z}^2$ that makes them compatible with a piecewise constant approximation $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$.

LEMMA 4.20 (Moving singularities on a lattice). *Let $V := \{x_1, \dots, x_N\} \subset \Omega$, let $\tilde{\Omega} \ni \Omega$, and let $u \in C^\infty(\tilde{\Omega} \setminus V; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$. Then for every $\lambda > 0$ there exist a set $V_\lambda = \{x_1^\lambda, \dots, x_N^\lambda\} \subset \lambda \mathbb{Z}^2 \cap \Omega$ and a map $u^\lambda \in C^\infty(\tilde{\Omega} \setminus V_\lambda; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ such that $u^\lambda \rightarrow u$ strongly in $W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ as $\lambda \rightarrow 0$. Moreover, $\deg(u^\lambda)(x_h^\lambda) = \deg(u)(x_h)$ for $h = 1, \dots, N$ for λ small enough and, defining the measures*

$$\mu^\lambda = \sum_{h=1}^N \deg(u^\lambda)(x_h^\lambda) \delta_{x_h^\lambda},$$

it holds that $\mu^\lambda \xrightarrow{f} \sum_{h=1}^N \deg(u)(x_h) \delta_{x_h}$.

PROOF. We set $u^\lambda := u \circ \psi_\lambda$, where $\psi_\lambda: \tilde{\Omega} \rightarrow \tilde{\Omega}$ is a suitable diffeomorphism with $\psi_\lambda(x_h^\lambda) = x_h$ (see, e.g., [40, p. 210] for a construction). The details are omitted as they are standard. \square

We modify the target sequence one last time close to the singularities.

LEMMA 4.21 (Modification near a singularity). *Let $\rho \leq 1$ and let*

$$u \in C^\infty(B_\rho \setminus \{0\}; \mathbb{S}^1) \cap W^{1,1}(B_\rho; \mathbb{S}^1) \quad \text{with } |\deg(u)(0)| = 1.$$

Then for every $\sigma > 0$ there exist $\tilde{u} \in C^\infty(B_\rho \setminus \{0\}; \mathbb{S}^1) \cap W^{1,1}(B_\rho; \mathbb{S}^1)$ and a radius $\eta_0 \in (0, \rho^2)$ such that

$$(i) \quad \tilde{u}(x) = \frac{(x_1, \pm x_2)}{|x|} \text{ for every } x \in \bar{B}_{\eta_0} \setminus \{0\};$$

- (ii) $\tilde{u}(x) = u(x)$ for every $x \in B_\rho \setminus \bar{B}_{\sqrt{\eta_0}}$;
- (iii) it holds that

$$\int_{B_\rho} |\nabla \tilde{u}|_{2,1} \, dx \leq \int_{B_\rho} |\nabla u|_{2,1} \, dx + \sigma.$$

PROOF. We give a proof in the case $\deg(u)(0) = 1$, the case $\deg(u)(0) = -1$ being completely analogous. We also assume, without loss of generality, that $\rho = 1$, and we denote B_ρ simply by B . Let us consider the set $\Sigma := \{(x_1, 0) : 0 \leq x_1 \leq 1\}$. To modify the map u we will actually modify its lifting φ . We start by discussing some useful properties of φ .

Since $B \setminus \Sigma$ is simply connected, the map $u: B \setminus \Sigma \rightarrow \mathbb{S}^1$ admits a lifting $\varphi: B \setminus \Sigma \rightarrow \mathbb{R}$, i.e., a function satisfying $u = \exp(i\varphi)$. The function φ is unique up to integer multiples of 2π and has the same regularity of u , namely,

$$\varphi \in C^\infty(B \setminus \Sigma; \mathbb{R}) \cap W^{1,1}(B \setminus \Sigma; \mathbb{R}).$$

The fact that $u \in C^\infty(B \setminus \{0\}; \mathbb{S}^1)$ can be translated in terms of the regularity of φ as follows: for every $x \in \Sigma \cap (B \setminus \{0\})$ and $r > 0$ such that $B_r(x) \Subset B \setminus \{0\}$, we have

$$(4.58) \quad \mathbb{1}_{\bar{B}_r^+(x)}\varphi + \mathbb{1}_{B_r^-(x)}(\varphi - 2\pi) \in C^\infty(B_r(x)),$$

where $B_r^\pm(x) = \{x \in B_r(x) : \pm x_2 > 0\}$.

To show this, note that $u \in C^\infty(B_r(x); \mathbb{S}^1)$; thus it admits a lifting $\varphi_x \in C^\infty(B_r(x))$. Up to adding an integer multiple of 2π to φ_x , by the uniqueness of the lifting up to integer multiples of 2π we have $\varphi = \varphi_x$ in $B_r^+(x)$, and there exists a $k_x \in \mathbb{Z}$ such that $\varphi = \varphi_x + 2\pi k_x$ in $B_r^-(x)$, thus

$$\mathbb{1}_{\bar{B}_r^+(x)}\varphi + \mathbb{1}_{B_r^-(x)}(\varphi - 2\pi k_x) = \varphi_x \in C^\infty(B_r(x)).$$

To prove that $k_x = 1$, we observe that the proven regularity implies, in particular, that the restrictions $\varphi|_{B_r^\pm(x)}$ admit traces φ^\pm on Σ in the classical sense. To compute the jump $\varphi^+ - \varphi^- = -2\pi k_x$ at a point $x \in \Sigma \cap (B \setminus \{0\})$, we parametrize the circle $\partial B_{|x|}$ counterclockwise with the closed path $\gamma(t) = |x| \exp(it2\pi t)$, $t \in [0, 1]$. Observing that $\nabla \varphi = u_1 \nabla u_2 - u_2 \nabla u_1$ in $B \setminus \Sigma$, we infer that

$$\begin{aligned} \varphi^+(x) - \varphi^-(x) &= \varphi(\gamma(0^+)) - \varphi(\gamma(1^-)) = - \int_0^1 \nabla \varphi(\gamma(t)) \cdot \dot{\gamma}(t) \, dt \\ &= - \int_\gamma (u_1 \nabla u_2 - u_2 \nabla u_1) \cdot \tau \, d\mathcal{H}^1 = -2\pi \deg(u)(0) = -2\pi. \end{aligned}$$

This proves $k_x = 1$, and in turn (4.58).

Finally, it holds that $\varphi \in L^2(B)$ due to the Sobolev embedding theorem.

We are now in a position to define a modification $\tilde{\varphi}$ of φ . Let us fix $\sigma > 0$. By a classical capacity argument, we find $\eta_0 > 0$ small enough and a cutoff function

$\zeta \in C_c^\infty(B_{\sqrt{\eta_0}})$, $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on \bar{B}_{η_0} satisfying

$$\|\nabla \zeta\|_{L^2(B)}^2 \leq \frac{C}{\log \frac{\sqrt{\eta_0}}{\eta_0}} \leq \frac{C}{|\log \eta_0|} < \sigma^2.$$

We use the cutoff ζ to interpolate between φ and the principal argument \arg . The function \arg is defined in polar coordinates (ρ, ϑ) by $\arg(\rho, \vartheta) = \vartheta$ and satisfies $\frac{x}{|x|} = \exp(i \arg(x))$ in $B \setminus \{0\}$. In particular, also $\arg \in C^\infty(B \setminus \Sigma; \mathbb{R}) \cap W^{1,1}(B \setminus \Sigma; \mathbb{R})$, and it satisfies the regularity property as in (4.58). Let us define

$$\tilde{\varphi} := \zeta \arg + (1 - \zeta)\varphi, \quad \tilde{u} := \exp(i\tilde{\varphi}).$$

Since $\tilde{\varphi} \in C^\infty(B \setminus \Sigma; \mathbb{R})$ and for every $x \in \Sigma \cap (B \setminus \{0\})$ and $r > 0$ such that $B_r(x) \Subset B \setminus \{0\}$, we have $\mathbb{1}_{\bar{B}_r^+(x)}\tilde{\varphi} + \mathbb{1}_{B_r^-(x)}(\tilde{\varphi} - 2\pi) \in C^\infty(B_r(x); \mathbb{R})$, we deduce that $\tilde{u} \in C^\infty(B \setminus \{0\}; \mathbb{S}^1)$. By definition \tilde{u} satisfies (i) and (ii). To prove (iii), let us compute

$$\begin{aligned} \int_B |\nabla \tilde{u}|_{2,1} dx &= \int_{B \setminus \Sigma} |\nabla \tilde{\varphi}|_1 dx \leq \int_{B \setminus \Sigma} |\nabla \tilde{\varphi} - \nabla \varphi|_1 dx + \int_{B \setminus \Sigma} |\nabla \varphi|_1 dx \\ &= \int_{B \setminus \Sigma} |\nabla \tilde{\varphi} - \nabla \varphi|_1 dx + \int_B |\nabla u|_{2,1} dx; \end{aligned}$$

thus it only remains to estimate the first integral in the right-hand side. We have

$$\begin{aligned} &\int_{B \setminus \Sigma} |\nabla \tilde{\varphi} - \nabla \varphi|_1 dx \\ &\leq \sqrt{2} \int_{B \setminus \Sigma} |\nabla \zeta| |\arg - \varphi| dx + \sqrt{2} \int_{B \setminus \Sigma} \zeta |\nabla \arg - \nabla \varphi| dx \\ &\leq \sqrt{2} \|\nabla \zeta\|_{L^2(B)} (\|\arg\|_{L^2(B)} + \|\varphi\|_{L^2(B)}) \\ &\quad + \sqrt{2} \int_{B_{\sqrt{\eta_0}} \setminus \Sigma} (|\nabla \arg| + |\nabla \varphi|) dx \\ &\leq (\sqrt{2} \|\arg\|_{L^2(B)} + \sqrt{2} \|\varphi\|_{L^2(B)} + 1)\sigma, \end{aligned}$$

if $\eta_0 > 0$ is also chosen small enough such that

$$\sqrt{2} \int_{B_{\sqrt{\eta_0}} \setminus \Sigma} (|\nabla \arg| + |\nabla \varphi|) dx < \sigma. \quad \square$$

Now we are in a position to construct the discrete recovery sequence.

PROPOSITION 4.22. *Assume that $\varepsilon \ll \theta_\varepsilon \ll \varepsilon |\log \varepsilon|$. Let $\lambda_n := 2^{-n}$, $n \in \mathbb{N}$, and let $V := \{x_1, \dots, x_N\} \subset \lambda_n \mathbb{Z}^2 \cap \Omega$. Assume that $u \in C^\infty(\tilde{\Omega} \setminus V; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ with $\tilde{\Omega} \ni \Omega$ has the following structure: $|\deg(u)(x_h)| = 1$ for all $1 \leq h \leq N$ and*

$$u(x) = \begin{pmatrix} 1 & 0 \\ 0 & \deg(u)(x_h) \end{pmatrix} \frac{x - x_h}{|x - x_h|} \quad \text{in } B_{\eta_0}(x_h)$$

for some $\eta_0 > 0$. Set moreover $\mu = \sum_{h=1}^N \deg(u)(x_h) \delta_{x_h}$. Then there exists a sequence $u_\varepsilon: \Omega_\varepsilon \rightarrow \mathcal{S}_\varepsilon$ such that $\mu_{u_\varepsilon} \xrightarrow{f} \mu$, $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$, and

$$\limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon) - 2\pi |\mu|(\Omega) |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) \leq \int_{\Omega} |\nabla u|_{2,1} dx.$$

Remark 4.23. We emphasize that the construction presented below also works under the sole assumption $\varepsilon \ll \theta_\varepsilon$. The scaling $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$ will be used only after (4.123) on, where we identify the flat limit of the discrete vorticity measure. This observation will be useful to study the regimes $\theta_\varepsilon \sim \varepsilon |\log \varepsilon|$ and $\varepsilon |\log \varepsilon| \ll \theta_\varepsilon$ in [28].

PROOF OF PROPOSITION 4.22. We divide the proof into several steps. First we define a good approximation very close to the singularity. Then we define an interpolation between this construction and the piecewise constant approximations provided by Lemma 4.13 far from the singularities. In the third step we estimate the energy of this interpolation. In a final step we bound the energy and identify the flat limit of the discrete vorticity measures.

Step 1. Local discrete approximation of (degree ± 1)-singularities.

We define a local recovery sequence close to the first singularity x_1 . In the whole proof we will assume that $\deg(u)(x_1) = 1$ so that

$$(4.59) \quad u(x) = \frac{x - x_1}{|x - x_1|} \quad \text{in } B_{\eta_0}(x_1).$$

(If, instead, $\deg(u)(x_1) = -1$, we have $u(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{x - x_1}{|x - x_1|}$ and the construction below is adapted accordingly.) For simplification, we specify $u(x_1) := e_1$. Next we partition $\mathbb{R}^2 \setminus \{0\}$ according to the value of the angle in polar coordinates. More precisely, for $k = 0, 1, \dots, N_\varepsilon - 1$ we set

$$(4.60) \quad S_{k,\varepsilon} := \{x = r \exp(i\varphi) \in \mathbb{R}^2 : r > 0, \varphi \in [k\theta_\varepsilon, (k+1)\theta_\varepsilon)\}.$$

Based on this partition, we approximate the functions u with sequences $v_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ defined on $\varepsilon\mathbb{Z}^2 \setminus \{x_1\}$ by

$$(4.61) \quad v_\varepsilon(\varepsilon i) = \exp(i k \theta_\varepsilon) \quad \text{if } \varepsilon i - x_1 \in S_{k,\varepsilon}$$

while $v_\varepsilon(\varepsilon i) = e_1$ if $\varepsilon i = x_1$. Note that $v_\varepsilon = \mathfrak{P}_\varepsilon(u)$ by definition of \mathfrak{P}_ε ; see (4.25). Then, writing $\varepsilon i - x_1 = |\varepsilon i - x_1| \exp(i(k_i^\varepsilon \theta_\varepsilon + \phi_i^\varepsilon))$ with $\phi_i^\varepsilon \in [0, \theta_\varepsilon)$, it holds that

$$(4.62) \quad |u(\varepsilon i) - v_\varepsilon(\varepsilon i)| \leq |\exp(i(k_i^\varepsilon \theta_\varepsilon + \phi_i^\varepsilon)) - \exp(i k_i^\varepsilon \theta_\varepsilon)| \leq \theta_\varepsilon$$

for $\varepsilon i \in B_{\eta_0}(x_1)$. We define the radius $r_\varepsilon = 4\varepsilon\theta_\varepsilon^{-1}$ (its role will become clear below). We start estimating the energy of v_ε in $B_{2r_\varepsilon}(x_1)$. By a change of variables we may assume that $x_1 = 0$ in order to reduce the notation. Observe that for any two vectors $a, b \in \mathbb{R}^2$ we have

$$(4.63) \quad |a|^2 - |b|^2 \leq |a - b|(|a| + |b|) \leq 2|b||a - b| + |a - b|^2.$$

Hence, using also (4.62), we obtain

$$\begin{aligned}
 & \frac{E_\varepsilon(v_\varepsilon; B_{2r_\varepsilon})}{\varepsilon\theta_\varepsilon} \\
 & \leq \frac{E_\varepsilon(u; B_{2r_\varepsilon})}{\varepsilon\theta_\varepsilon} + \frac{1}{2\varepsilon\theta_\varepsilon} \sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in B_{2r_\varepsilon}}} \varepsilon^2 \left(|v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon j)|^2 - |u(\varepsilon i) - u(\varepsilon j)|^2 \right) \\
 (4.64) \quad & \leq \frac{E_\varepsilon(u; B_{2r_\varepsilon})}{\varepsilon\theta_\varepsilon} + \frac{C}{\varepsilon\theta_\varepsilon} \sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in B_{2r_\varepsilon}}} \varepsilon^2 (\theta_\varepsilon |u(\varepsilon i) - u(\varepsilon j)| + \theta_\varepsilon^2).
 \end{aligned}$$

We prove that the last sum vanishes, while the first right-hand side term scales as $2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon}$ in the sense that the difference vanishes. To this end, we estimate finite differences of u away from the singularity. Since for $t \in [0, 1]$ and $i, j \in \mathbb{Z}^2$ with $|i - j| = 1$ we have

$$|(1-t)\varepsilon i + t\varepsilon j| \geq |\varepsilon i| - \varepsilon,$$

the regularity of u in $B_{\eta_0} \setminus \{0\}$ implies that for any $\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \setminus B_{2\varepsilon}$ with $|i - j| = 1$

$$|u(\varepsilon i) - u(\varepsilon j)| \leq \int_0^1 |\nabla u(t\varepsilon i + (1-t)\varepsilon j)(\varepsilon i - \varepsilon j)| dt.$$

Since $i - j \in \{\pm e_1, \pm e_2\}$, a direct computation yields the two cases

$$(4.65) \quad |u(\varepsilon i) - u(\varepsilon j)| \leq \begin{cases} \int_0^1 \frac{|i \cdot e_2|}{|ti + (1-t)j|^2} dt & \text{if } (i - j) \parallel e_1, \\ \int_0^1 \frac{|i \cdot e_1|}{|ti + (1-t)j|^2} dt & \text{if } (i - j) \parallel e_2. \end{cases}$$

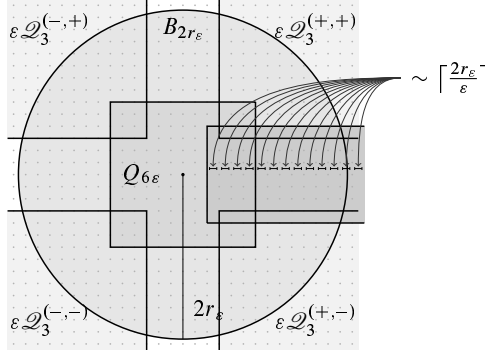


FIGURE 4.1. The trimmed quadrants $\varepsilon \mathcal{Q}_3^s$ used to bound the energy in (4.67).

To further simplify the energy, given a couple of signs $s = (s_1, s_2) \in \{\pm\}^2$ and $n \in \mathbb{N}$, we define the trimmed quadrants \mathcal{Q}_n^s as

$$(4.66) \quad \mathcal{Q}_n^s := \{x \in \mathbb{R}^2 : s_1 x \cdot e_1 \geq n, s_2 x \cdot e_2 \geq n\}.$$

Applying Jensen's inequality in (4.65) and specifying $n = 3$, we can bound the energy via

$$(4.67) \quad \begin{aligned} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u; B_{2r_\varepsilon}) &\leq \sum_s \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u, \varepsilon\mathcal{Q}_3^s \cap B_{2r_\varepsilon}) + \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u; Q_{6\varepsilon}) \\ &\quad + \frac{C}{\varepsilon\theta_\varepsilon} \sum_{k=2}^{\lceil 2r_\varepsilon/\varepsilon \rceil} \frac{\varepsilon^2 k^2}{(k-1)^4}; \end{aligned}$$

see Figure 4.1. The last sum is converging with respect to k , so that the second and third terms can be estimated by

$$(4.68) \quad \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u; Q_{6\varepsilon}) + \frac{C}{\varepsilon\theta_\varepsilon} \sum_{k=2}^{\lceil 2r_\varepsilon/\varepsilon \rceil} \varepsilon^2 \frac{k^2}{(k-1)^4} \leq C \frac{\varepsilon}{\theta_\varepsilon},$$

where for the first term we used the estimate $|u(\varepsilon i) - u(\varepsilon j)|^2 \leq 4$. On the trimmed quadrants \mathcal{Q}_3^s we can use again Jensen's inequality in (4.65) and a monotonicity argument to deduce

$$(4.69) \quad \begin{aligned} E_\varepsilon(u; \varepsilon\mathcal{Q}_3^s \cap B_{2r_\varepsilon}) &= \sum_{\substack{\varepsilon i \in \varepsilon\mathbb{Z}^2 \cap B_{2r_\varepsilon} \\ i \in \mathcal{Q}_3^s}} \sum_{r=1,2} \varepsilon^2 |u(\varepsilon(i + s_r e_1)) - u(\varepsilon i)|^2 \\ &\leq \sum_{\substack{\varepsilon i \in \varepsilon\mathbb{Z}^2 \cap B_{2r_\varepsilon} \\ i \in \mathcal{Q}_3^s}} \varepsilon^2 \frac{\varepsilon^2}{|\varepsilon i|^2} \leq \int_{\varepsilon\mathcal{Q}_3^s \cap B_{2r_\varepsilon}} \frac{\varepsilon^2}{|x|^2} dx. \end{aligned}$$

Note that we shifted the trimming in the last inequality to pass from discrete to continuum. We sum (4.69) over all possible s and, after multiplying with $\frac{1}{\varepsilon\theta_\varepsilon}$, we infer that

$$(4.70) \quad \sum_s \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u; \varepsilon\mathcal{Q}_3^s \cap B_{2r_\varepsilon}) \leq \frac{1}{\varepsilon\theta_\varepsilon} \int_{B_{2r_\varepsilon} \setminus B_\varepsilon} \frac{\varepsilon^2}{|x|^2} dx \leq 2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon},$$

since $r_\varepsilon = 4\varepsilon\theta_\varepsilon^{-1} < 1$. The combination of (4.67), (4.68), and (4.70) yields

$$(4.71) \quad \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u; B_{2r_\varepsilon}) - 2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \leq C \frac{\varepsilon}{\theta_\varepsilon}.$$

To bound the remaining sum in (4.64), note that on the one hand $r_\varepsilon = 4\varepsilon\theta_\varepsilon^{-1}$ implies that

$$(4.72) \quad \frac{1}{\varepsilon\theta_\varepsilon} \sum_{\substack{(i,j) \\ \varepsilon i, \varepsilon j \in B_{2r_\varepsilon}}} \varepsilon^2 \theta_\varepsilon^2 \leq C \frac{\theta_\varepsilon}{\varepsilon} (2r_\varepsilon + 2\varepsilon)^2 \leq C \frac{\varepsilon}{\theta_\varepsilon}.$$

On the other hand, using the trimmed quadrants \mathcal{Q}_3^s to split the sum as in (4.67) yields

$$\begin{aligned} & \frac{1}{\varepsilon\theta_\varepsilon} \sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in B_{2r_\varepsilon}}} \varepsilon^2 \theta_\varepsilon |u(\varepsilon i) - u(\varepsilon j)| \\ & \leq \frac{\sqrt{2}}{\varepsilon\theta_\varepsilon} \int_{B_{2r_\varepsilon} \setminus B_\varepsilon} \theta_\varepsilon \frac{\varepsilon}{|x|} dx + C\varepsilon + \frac{C}{\varepsilon\theta_\varepsilon} \sum_{k=2}^{\lceil 2r_\varepsilon/\varepsilon \rceil} \frac{\varepsilon^2 \theta_\varepsilon k}{(k-1)^2}. \end{aligned}$$

Note that due to the nonquadratic structure we have the additional constant $\sqrt{2}$ in front of the integral. Moreover, the last sum diverges logarithmically, but we have an additional factor θ_ε that compensates this growth. We conclude that

$$\frac{1}{\varepsilon\theta_\varepsilon} \sum_{\substack{\langle i,j \rangle \\ \varepsilon i, \varepsilon j \in B_{2r_\varepsilon}}} \varepsilon^2 \theta_\varepsilon |u(\varepsilon i) - u(\varepsilon j)| \leq C(r_\varepsilon^2 + \varepsilon + \varepsilon |\log \theta_\varepsilon|).$$

Since $\varepsilon \ll \theta_\varepsilon \ll 1$, the right-hand side vanishes when $\varepsilon \rightarrow 0$. Thus this estimate, (4.64), (4.71), and (4.72) imply that

$$(4.73) \quad \limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(v_\varepsilon; B_{2r_\varepsilon}) - 2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) \leq 0.$$

Next we control the energy in $B_\eta(x_1) \setminus B_{r_\varepsilon}(x_1)$ for $0 < \eta < \eta_0$, where η_0 is given by the assumptions. To this end, we need to examine the precise behavior of the sequence v_ε for $i, j \in \mathbb{Z}^2$ satisfying $|i - j| = 1$ and $|\varepsilon j - x_1|, |\varepsilon i - x_1| \geq r_\varepsilon$. The basic idea is that for many such pairs the energy contribution vanishes. Indeed, write such points as

$$(4.74) \quad \varepsilon i - x_1 = r_i^\varepsilon \exp(\iota(k_i^\varepsilon \theta_\varepsilon + \phi_i^\varepsilon)), \quad \varepsilon j - x_1 = r_j^\varepsilon \exp(\iota(k_j^\varepsilon \theta_\varepsilon + \phi_j^\varepsilon)),$$

with $k_i^\varepsilon, k_j^\varepsilon \in \{0, \dots, N_\varepsilon - 1\}$ and $\phi_i^\varepsilon, \phi_j^\varepsilon \in [0, \theta_\varepsilon)$. By (2.1) we obtain

$$\begin{aligned} \varepsilon &= |r_i^\varepsilon \exp(\iota(k_i^\varepsilon \theta_\varepsilon + \phi_i^\varepsilon)) - r_j^\varepsilon \exp(\iota(k_j^\varepsilon \theta_\varepsilon + \phi_j^\varepsilon))| \\ &\geq r_\varepsilon |\exp(\iota(k_i^\varepsilon \theta_\varepsilon + \phi_i^\varepsilon)) - \exp(\iota(k_j^\varepsilon \theta_\varepsilon + \phi_j^\varepsilon))| - |r_i^\varepsilon - r_j^\varepsilon| \\ &\geq \min_{n \in \{0, \pm 1\}} \frac{2r_\varepsilon}{\pi} |(k_i^\varepsilon - k_j^\varepsilon) \theta_\varepsilon + \phi_i^\varepsilon - \phi_j^\varepsilon + 2\pi n| - \varepsilon \\ &\geq \min_{n \in \{0, \pm 1\}} \frac{2r_\varepsilon \theta_\varepsilon}{\pi} (|k_i^\varepsilon - k_j^\varepsilon + N_\varepsilon n| - 1) - \varepsilon. \end{aligned}$$

Inserting $r_\varepsilon = 4\varepsilon\theta_\varepsilon^{-1}$, the above estimate can be rearranged into

$$\frac{\pi}{4} \geq \min_{n \in \{0, \pm 1\}} (|k_i^\varepsilon - k_j^\varepsilon + N_\varepsilon n| - 1).$$

Since $k_i^\varepsilon - k_j^\varepsilon + N_\varepsilon n$ is an integer, we get the following two possibilities:

- (i) $k_i^\varepsilon - k_j^\varepsilon - N_\varepsilon n = 0$, which is possible only for $n = 0$ since $0 \leq k_i^\varepsilon, k_j^\varepsilon \leq N_\varepsilon - 1$. This yields $v_\varepsilon(\varepsilon i) = v_\varepsilon(\varepsilon j)$ since εi and εj belong to the same sector;
- (ii) $k_i^\varepsilon - k_j^\varepsilon = \pm 1 \pmod{N_\varepsilon}$, which implies $|v_\varepsilon(\varepsilon i) - v_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon$. Moreover, since $k_i^\varepsilon \neq k_j^\varepsilon$ we infer that $\text{dist}(\varepsilon i - x_1, \partial S_{k_i^\varepsilon, \varepsilon}) \leq \varepsilon$.

With this information, we can estimate the energy by bounding the number of all points in $\varepsilon\mathbb{Z}^2 \cap B_\eta(x_1)$ that are ε -close to one of the lines in $\bigcup_{k=0}^{N_\varepsilon-1} \partial S_{k, \varepsilon} + x_1$. Since $N_\varepsilon \leq C\theta_\varepsilon^{-1}$, this leads to

$$\begin{aligned} & E_\varepsilon(v_\varepsilon; B_\eta(x_1) \setminus B_{r_\varepsilon}(x_1)) \\ & \leq C\theta_\varepsilon^2 \sum_{k=0}^{N_\varepsilon-1} \varepsilon^2 \#(\varepsilon\mathbb{Z}^2 \cap B_\eta(x_1) \cap \{\text{dist}(\cdot, \partial S_{k, \varepsilon} + x_1) \leq \varepsilon\}) \\ & \leq C\theta_\varepsilon(\eta + 2\varepsilon)\varepsilon. \end{aligned}$$

Dividing the inequality by $\varepsilon\theta_\varepsilon$ we obtain for ε small enough that

$$\begin{aligned} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(v_\varepsilon; B_\eta(x_1)) & \leq \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(v_\varepsilon; B_{2r_\varepsilon}(x_1)) + \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(v_\varepsilon; B_\eta(x_1) \setminus B_{r_\varepsilon}(x_1)) \\ & \leq \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(v_\varepsilon; B_{2r_\varepsilon}(x_1)) + C\eta, \end{aligned}$$

where we used that $r_\varepsilon \gg \varepsilon$ to split the energy via changing the inner radius from r_ε to $2r_\varepsilon$. Subtracting the term $2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon}$ and using (4.73), we proved that for some $C < +\infty$

$$(4.75) \quad \limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(v_\varepsilon; B_\eta(x_1)) - 2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) \leq C\eta.$$

Step 2. An interpolation between singular and piecewise constant approximations.

We do the construction in the case where the singularity lies in the origin. The case of singularities contained in $\lambda\mathbb{Z}^2$ will be treated with a translation argument. Consider a cube $Q(\lambda) = [-2^{m(\lambda)}\lambda, 2^{m(\lambda)}\lambda]^2$, where $\lambda = \lambda_k$ with $k \geq n$ will be small, but fixed in this step, and $1 \ll m(\lambda) \in \mathbb{N}$ is chosen maximal such that $Q(\lambda) \subset B_{\eta/2}$ with fixed $0 < \eta < \eta_0$. Note that the corners of $Q(\lambda)$ belong to $\lambda\mathbb{Z}^2$. Define then a sequence of dyadically shrinking cubes by $Q_k = [(-2^{m(\lambda)} + (2 - 2^{-k}))\lambda, (2^{m(\lambda)} - (2 - 2^{-k}))\lambda]^2$ for $k \geq 0$. Here the factor $2 - 2^{-k}$ is chosen as the value of the geometric sum $\sum_{l=0}^k 2^{-l}$. For notational reasons we also set $Q_{-1} := Q(\lambda)$ and $Q_{-2} := [-(2^{m(\lambda)} + 1)\lambda, (2^{m(\lambda)} + 1)\lambda]$. Then for $k \geq 0$ the layer $L_k = Q_{k-1} \setminus Q_k$ can be decomposed into finitely many closed cubes with disjoint interior and side lengths $2^{-k}\lambda$. Indeed, those cubes are given by the closures of the half-open cubes belonging to the family

$$\mathcal{Q}_k := \{q_k^z = (2^{-k}\lambda z + [0, 2^{-k}\lambda]^2) : z \in \mathbb{Z}^2, q_k^z \subset L_k\}.$$

With a slight abuse of notation we also set

$$\mathcal{Q}_{-1} := \{q_{-1}^z = (\lambda z + [0, \lambda]^2) : z \in \mathbb{Z}^2, q_{-1}^z \subset L_{-1}\}.$$

For $k \geq -1$, a generic element of \mathcal{Q}_k is of the form

$$q_k^z = 2^{-\max\{k,0\}} \lambda z + [0, 2^{-\max\{k,0\}} \lambda]^2 \subset L_k$$

(see Figure 4.2). We introduced the square Q_{-2} and the family of cubes \mathcal{Q}_{-1} since they will be useful later to glue in the layer $L_{-1} = Q_{-2} \setminus Q_{-1}$ the construction of the recovery sequence u_ε inside $Q(\lambda)$ and outside $Q(\lambda)$. The construction of u_ε outside $Q(\lambda)$ will be based, as in Proposition 4.16, on a piecewise constant approximation of u on the $\lambda\mathbb{Z}^2$ lattice, and its boundary value on $\partial Q(\lambda)$ will agree with that of the construction from the inside. For this reason the cubes in \mathcal{Q}_{-1} have volume λ^2 , like those of \mathcal{Q}_0 , instead of the notationally more consistent volume $(2\lambda)^2$.

We further choose $k_\varepsilon \in \mathbb{N}$ as the unique number such that

$$(4.76) \quad 2^{-k_\varepsilon} \leq \theta_\varepsilon < 2^{-k_\varepsilon + 1}.$$

Note that, in particular, we have that

$$(4.77) \quad Q_{k_\varepsilon} \supset B_{(2^{m(\lambda)-2})\lambda}.$$

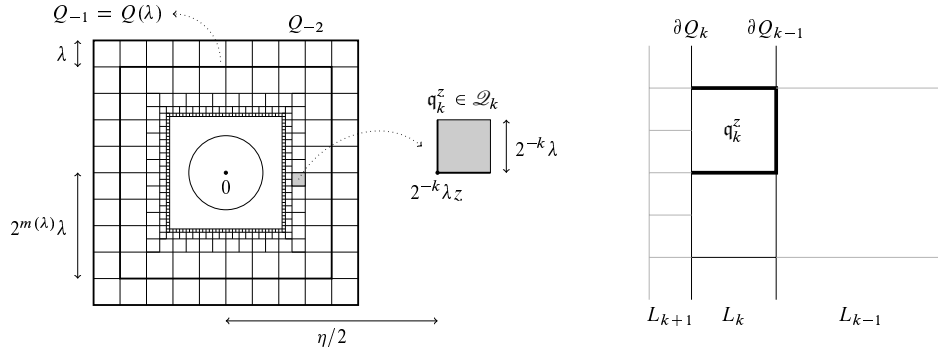


FIGURE 4.2. On the left: Dyadic decomposition of $Q(\lambda)$ and example of a square belonging to the family \mathcal{Q}_k (in the picture, $k = 1$). The ball contained in all squares Q_k is given by (4.77). On the right: Sides of a cube q_k^z contained in the layer L_k where we define the boundary conditions.

To each (nonempty, half-open) cube

$$q_k^z = 2^{-\max\{k,0\}} \lambda z + [0, 2^{-\max\{k,0\}} \lambda]^2 \cap L_k \in \mathcal{Q}_k,$$

we associate the value

$$u_{k,\varepsilon}^z = u\left(2^{-\max\{k,0\}} \lambda \left(z + \frac{1}{2}e_1 + \frac{1}{2}e_2\right)\right),$$

where $2^{-\max\{k,0\}}\lambda(z + \frac{1}{2}e_1 + \frac{1}{2}e_2)$ is the midpoint of the cube q_k^z . We use these values $u_{k,\varepsilon}^z$ to define an interpolation similar to the one in the proof of Proposition 4.16, but on a family of shrinking cubes. In order to obtain quantitative energy estimates, we need a bound on the differences of the values $u_{k,\varepsilon}^z$ between cubes that touch at their boundaries. A key ingredient will be the estimate

$$(4.78) \quad \left| \frac{x}{|x|} - \frac{y}{|y|} \right| \leq \frac{|x|y| - x|x| + x|x| - y|y|}{|x||y|} \leq 2 \frac{|x-y|}{|y|},$$

which is valid for $x, y \in \mathbb{R}^2 \setminus \{0\}$. Due to (4.77) it holds that $0 \notin q_k^z$ for $-1 \leq k \leq k_\varepsilon$. Hence, for two touching cubes $q_{k_1}^{z_1}$ and $q_{k_2}^{z_2}$ with $-1 \leq k_1, k_2 \leq k_\varepsilon$ (i.e., $\bar{q}_{k_1}^{z_1} \cap \bar{q}_{k_2}^{z_2} \neq \emptyset$), the estimate (4.78) implies the bound

$$(4.79) \quad |u_{k_1,\varepsilon}^{z_1} - u_{k_2,\varepsilon}^{z_2}| \leq 2 \frac{\sqrt{2}\lambda(2^{-k_1} + 2^{-k_2})}{\min_{l=1,2} |2^{-\max\{k_l,0\}}\lambda(z_l + \frac{1}{2}e_1 + \frac{1}{2}e_2)|},$$

where we used that the distance between midpoints is bounded by the sum of the diameters of the cubes. Assuming that $m(\lambda) \geq 2$, the inclusion (4.77) implies that the denominator can be estimated from below via

$$\left| 2^{-\max\{k_l,0\}}\lambda \left(z_l + \frac{1}{2}e_1 + \frac{1}{2}e_2 \right) \right| \geq (2^{m(\lambda)} - 2)\lambda \geq 2^{m(\lambda)-1}\lambda.$$

In combination with (4.79) and the bound $2^{-k_\varepsilon} \leq \theta_\varepsilon$ (cf. (4.76)), we obtain

$$(4.80) \quad |u_{k_1,\varepsilon}^{z_1} - u_{k_2,\varepsilon}^{z_2}| \leq 2^{3-m(\lambda)}(2^{-k_1} + 2^{-k_2}) \leq 16 \max_{l=1,2} 2^{k_\varepsilon - k_l - m(\lambda)}\theta_\varepsilon.$$

Next, we define the piecewise constant function $\bar{w}_\varepsilon: Q_{-2} \setminus Q_{k_\varepsilon} \rightarrow \mathcal{S}_\varepsilon$ via

$$(4.81) \quad \bar{w}_\varepsilon(x) = u_{k,\varepsilon}^z \\ \text{if } x \in (2^{-\max\{k,0\}}\lambda z + [0, 2^{-\max\{k,0\}}\lambda]^2) \cap L_k, \quad -1 \leq k \leq k_\varepsilon.$$

Note that this function is pointwise well-defined except on parts of ∂Q_{-2} since we consider half-open cubes. In order to define an interpolation between cubes that approximates the piecewise constant function \bar{w}_ε , we introduce again boundary conditions. In each cube $q_k^z \in \mathcal{Q}_k$, we define the boundary conditions only on those sides that are not contained in ∂Q_k (recall that ∂Q_k is the inner part of the boundary of the layer L_k). On the side contained in ∂Q_k (if there is any) we define the boundary condition via the cubes in L_{k+1} (cf. Figure 4.2). To fix ideas, in what follows one can use an iterative definition starting with $k = k_\varepsilon$, for which we neglect the inner boundary.

For a generic side $S = \{2^{-k}\lambda z' + te_i : t \in [0, 2^{-k}\lambda]\}$ with $z' \in \mathbb{Z}^2$, $k \geq 0$, and $i \in \{1, 2\}$, and three values $w = (w^1, w^2, w^3) \in (\mathbb{S}^1)^3$, we set $b_{S,k}^\varepsilon[w]: S \rightarrow \mathbb{S}^1$ as

$$b_{S,k}^\varepsilon[w](x) = b_{2^k S}^\varepsilon[w](2^k x),$$

where $b_{S'}^\varepsilon[w]$ is defined in (4.27) for every side $S' = \{\lambda z' + te_i : t \in [0, \lambda]\}$. In this proof we work with the constant $c_0 := 393$ in (4.27); this choice will be clear only after formula (4.97). Given a side S as above satisfying additionally $S \subset L_k$ with $k \geq 0$ and $S \not\subset \partial Q_k$ (recall that the layer L_k is closed), we specify the three values $w = w_S^\varepsilon$ on S by

$$(4.82) \quad w_S^\varepsilon = (\bar{w}_\varepsilon(2^{-k}\lambda z'), \text{mid}((\bar{w}_\varepsilon)_S^-, (\bar{w}_\varepsilon)_S^+), \bar{w}_\varepsilon(2^{-k}\lambda(z' + e_i))),$$

where $(\bar{w}_\varepsilon)_S^-$ and $(\bar{w}_\varepsilon)_S^+$ denote the (constant) traces along the side S of the function \bar{w}_ε defined in (4.81) (note that on sides in ∂Q_k the trace from outside L_k may be nonconstant because the cubes shrink). It is only here where we have to use the values in the layer L_{-1} .

Fix a cube $q_k^z \in \mathcal{Q}_k$ ($k \geq 0$) and define the boundary values $b_{k,\varepsilon}[z]: \partial q_k^z \setminus \partial Q_k \rightarrow \mathbb{S}^1$ by

$$b_{k,\varepsilon}[z](x) = b_{S,k}^\varepsilon[w_S^\varepsilon](x) \\ \text{if } x = 2^{-k}\lambda z' + te_i \in S \text{ for some } z' \in \mathbb{Z}^2, t \in [0, 2^{-k}\lambda].$$

Having in mind the definition (4.27), on each side S the function $b_{k,\varepsilon}[z]$ satisfies the Lipschitz-estimate

$$(4.83) \quad \begin{aligned} |b_{k,\varepsilon}[z](x) - b_{k,\varepsilon}[z](y)| &= |b_{2^k S}^\varepsilon[w_S^\varepsilon](2^k x) - b_{2^k S}^\varepsilon[w_S^\varepsilon](2^k y)| \\ &\leq \max_{i=1,3} d_{\mathbb{S}^1}((w_S^\varepsilon)^i, (w_S^\varepsilon)^2) \frac{2^k \theta_\varepsilon}{c_0 \varepsilon} |x - y| \\ &\leq \max_{i=1,3} |(w_S^\varepsilon)^i - (w_S^\varepsilon)^2| \frac{\pi 2^{k-1} \theta_\varepsilon}{c_0} \frac{1}{\varepsilon} |x - y|, \end{aligned}$$

where we used (2.1) in the last inequality. We continue with estimating the right-hand side of (4.83). On the one hand, equation (4.80) implies

$$(4.84) \quad \begin{aligned} |(w_S^\varepsilon)^2 - (\bar{w}_\varepsilon)_S^\pm| &\leq d_{\mathbb{S}^1}(\text{mid}((\bar{w}_\varepsilon)_S^+, (\bar{w}_\varepsilon)_S^-), (\bar{w}_\varepsilon)_S^\pm) = \frac{1}{2} d_{\mathbb{S}^1}((\bar{w}_\varepsilon)_S^+, (\bar{w}_\varepsilon)_S^-) \\ &\leq \frac{\pi}{4} |(\bar{w}_\varepsilon)_S^+ - (\bar{w}_\varepsilon)_S^-| \leq C_1 2^{k_\varepsilon - k - m(\lambda)} \theta_\varepsilon, \end{aligned}$$

with $C_1 = 32$, where in the last inequality we also used that due to the definition (4.81) and the fact that $S \subset \partial q_k^z$ we have $(\bar{w}_\varepsilon)_S^\pm = u_{k_\pm, \varepsilon}^{z_\pm}$ for some $z_\pm \in \mathbb{Z}^2$ and $-1 \leq k_\pm \leq k_\varepsilon$ with $|k - k_\pm| \leq 1$. On the other hand, observe that in the definition of w_S^ε in (4.82) the points $2^{-k}\lambda z'$ and $2^{-k}\lambda(z' + e_i)$ belong to S . Thus the cubes $q_{k_1}^{z_1}$ and $q_{k_3}^{z_3}$ used in the definition (4.81) for $(w_S^\varepsilon)^1 = \bar{w}_\varepsilon(2^{-k}\lambda z')$ and $(w_S^\varepsilon)^3 = \bar{w}_\varepsilon(2^{-k}\lambda(z' + e_i))$, respectively, must touch both the cubes $q_{k_\pm}^{z_\pm}$ used in the definition (4.81) for $(\bar{w}_\varepsilon)_S^\pm$. Hence, again due to (4.80),

$$(4.85) \quad |(w_S^\varepsilon)^i - (\bar{w}_\varepsilon)_S^\pm| \leq C_1 2^{k_\varepsilon - k - m(\lambda)} \theta_\varepsilon \quad \text{for } i = 1, 3.$$

Combining the last two estimates with (4.83) and the bound $2^{k_\varepsilon} \theta_\varepsilon \leq 2$ yields

$$(4.86) \quad |b_{k,\varepsilon}[z](x) - b_{k,\varepsilon}[z](y)| \leq \frac{\theta_\varepsilon}{\varepsilon} |x - y|,$$

where we used that $2C_1 2^{k_\varepsilon - k - m(\lambda)} \theta_\varepsilon \frac{\pi 2^{k-1}}{c_0} \leq 1$. Next observe that the locally defined boundary values yield a function

$$(4.87) \quad b_\varepsilon: \bigcup_{0 \leq k \leq k_\varepsilon} \bigcup_{q_k^z \in \mathcal{Q}_k} \partial q_k^z \setminus \partial Q_{k_\varepsilon} \rightarrow \mathbb{S}^1,$$

$$x \mapsto b_\varepsilon(x) := b_{k,\varepsilon}[z](x) \quad \text{if } x \in \partial q_k^z \setminus \partial Q_k.$$

We briefly explain the idea how to construct the recovery sequence. In Q_{k_ε} we put the value of the function v_ε used in Step 1 and defined in (4.61), namely, we approximate $\frac{x}{|x|}$ close to its singularity. In the first layer L_{k_ε} we keep this construction and then we start an interpolation scheme with respect to the cubes q_k^z , where we put the value $u_{k,\varepsilon}^z$ in most of the cube. The boundary conditions $b_{k,\varepsilon}[z]$ help to control interactions between different cubes. In the estimates we can allow for multiplicative constants since the total contribution will be proportional to $2^{m(\lambda)} \lambda \sim \eta$. However, a precise dependence on the energy with respect to the layer number k is crucial since we have to sum over all layers.

Now let us start with the details. For the moment fix $0 \leq k < k_\varepsilon$. Given a cube $q_k^z \in \mathcal{Q}_k$, let $P_{k,z}: q_k^z \rightarrow \partial q_k^z$ be any function such that $|P_{k,z}(x) - x| = \text{dist}(x, \partial q_k^z)$ for all $x \in q_k^z$. Set $\bar{u}_\varepsilon: \varepsilon \mathbb{Z}^2 \cap q_k^z \rightarrow \mathbb{S}^1$ as

$$\bar{u}_\varepsilon(\varepsilon i) = \text{Geo}[b_\varepsilon(P_{k,z}(\varepsilon i)), u_{k,\varepsilon}^z](\theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon i, \partial q_k^z)),$$

with the extended geodesics given by Definition 4.14 and b_ε given by (4.87). Since in general $\bar{u}_\varepsilon(\varepsilon i) \notin \mathcal{S}_\varepsilon$, we project it. The function u_ε in the square $Q(\lambda)$ is then given by

$$(4.88) \quad u_\varepsilon(\varepsilon i) := \begin{cases} v_\varepsilon(\varepsilon i) & \text{if } \varepsilon i \in Q_{k_\varepsilon-1}, \\ \mathfrak{P}_\varepsilon(\bar{u}_\varepsilon(\varepsilon i)) & \text{if } \varepsilon i \in q_k^z \text{ for some } q_k^z \in \mathcal{Q}_k \text{ with } 0 \leq k < k_\varepsilon, \end{cases}$$

with the operator \mathfrak{P}_ε defined in (4.25). In this step we are interested in the energy restricted to $Q(\lambda)$, and for this reason we defined u_ε only in $Q(\lambda)$. The sequence u_ε will be defined later in Step 4 outside $Q(\lambda)$, that means, far from the singularity, as in Proposition 4.16.

First let us identify the $L^1(Q(\lambda))$ -limit of u_ε . To this end, observe that for all $\varepsilon i \in q_k^z$ with $0 \leq k < k_\varepsilon$ we have by Definition 4.14

$$(4.89) \quad u_\varepsilon(\varepsilon i) = \mathfrak{P}_\varepsilon(u_{k,\varepsilon}^z) \quad \text{if } \theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon i, \partial q_k^z) \geq d_{\mathbb{S}^1}(b_\varepsilon(P_{k,z}(\varepsilon i)), u_{k,\varepsilon}^z).$$

We need to quantify the dependence on k in the right-hand side. Let $S \subset \partial q_k^z$ be a side such that $P_{k,z}(\varepsilon i) \in S$. Since the boundary datum b_ε restricted to S interpolates via geodesic arcs between the three elements of the vector w_S^ε defined

in (4.82) and by construction $u_{k,\varepsilon}^z \in \{(\bar{w}_\varepsilon)_S^-, (\bar{w}_\varepsilon)_S^+\}$, it follows from (4.84) and (4.85) (with $k+1$ in place of k if $S \subset \partial Q_k$, which improves the estimate) that

$$\begin{aligned}
 & d_{\mathbb{S}^1}(b_\varepsilon(P_{k,z}(\varepsilon i)), u_{k,\varepsilon}^z) \\
 & \leq d_{\mathbb{S}^1}(b_\varepsilon(P_{k,z}(\varepsilon i)), \text{mid}((\bar{w}_\varepsilon)_S^+, (\bar{w}_\varepsilon)_S^-)) + C_1 2^{k_\varepsilon - k - m(\lambda)} \theta_\varepsilon \\
 & \leq \max_{i=1,3} d_{\mathbb{S}^1}((w_\varepsilon)_S^i, \text{mid}((\bar{w}_\varepsilon)_S^+, (w_\varepsilon)_S^-)) + C_1 2^{k_\varepsilon - k - m(\lambda)} \theta_\varepsilon \\
 (4.90) \quad & \leq C_2 2^{k_\varepsilon - k - m(\lambda)} \theta_\varepsilon,
 \end{aligned}$$

for $C_2 = (\frac{\pi}{2} + 1)C_1 \leq 96$. In particular, the condition (4.89) implies that

$$(4.91) \quad u_\varepsilon(\varepsilon i) = \mathfrak{P}_\varepsilon(u_{k,\varepsilon}^z) \quad \text{if } \text{dist}(\varepsilon i, \partial q_k^z) \geq C_2 2^{k_\varepsilon - k} \varepsilon.$$

Since $2^{k_\varepsilon} \theta_\varepsilon \leq 2$ (cf. (4.76)), the term $2^{k_\varepsilon} \varepsilon$ vanishes when $\varepsilon \rightarrow 0$. As the measure of each q_k^z is $2^{-k} \lambda$, we deduce from (4.62) that a.e. in $Q(\lambda)$ (and thus in $L^1(Q(\lambda))$) it holds that

$$(4.92) \quad u_\varepsilon \rightarrow u_0^\lambda = \begin{cases} \frac{x}{|x|} & \text{on } Q_\infty := [(-2^{m(\lambda)} + 2)\lambda, (2^{m(\lambda)} - 2)\lambda]^2, \\ \frac{2^{-k-1}\lambda(2z+e_1+e_2)}{|2^{-k-1}\lambda(2z+e_1+e_2)|} & \text{if } x \in q_k^z \cap L_k \text{ for some } k \in \mathbb{N} \cup \{0\}. \end{cases}$$

Notice that $Q_\infty = \bigcap_{k=0}^\infty Q_k$ and that $u_0^\lambda = \frac{x}{|x|}$ except in the layer $Q(\lambda) \setminus Q_\infty$, whose thickness is 2λ , thus infinitesimal when $\lambda \rightarrow 0$.

Below we bound the differences $u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)$ for all $\varepsilon i, \varepsilon j \in \varepsilon \mathbb{Z}^2 \cap Q(\lambda)$ with $|i - j| = 1$.

Substep 2.1 (Interactions within a single cube)

Consider first $\varepsilon i, \varepsilon j \in \varepsilon \mathbb{Z}^2 \cap q_k^z$ with $0 \leq k < k_\varepsilon$ and $|i - j| = 1$. We treat several cases:

Case 1. If $\text{dist}(\varepsilon i, \partial q_k^z) \geq C_2 2^{k_\varepsilon - k} \varepsilon$ and $\text{dist}(\varepsilon j, \partial q_k^z) \geq C_2 2^{k_\varepsilon - k} \varepsilon$, then by (4.91)

$$|u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| = |\mathfrak{P}_\varepsilon(u_{k,\varepsilon}^z) - \mathfrak{P}_\varepsilon(u_{k,\varepsilon}^z)| = 0.$$

By the Lipschitz continuity of $\text{dist}(\cdot, \partial q_k^z)$, we can from now on assume that

$$(4.93) \quad \max\{\text{dist}(\varepsilon i, \partial q_k^z), \text{dist}(\varepsilon j, \partial q_k^z)\} < (C_2 + 1) 2^{k_\varepsilon - k} \varepsilon.$$

Case 2. We first analyze when $P_{k,z}(\varepsilon i)$ and $P_{k,z}(\varepsilon j)$ lie on different 1-dimensional boundary segments $S_i \neq S_j$ of q_k^z . We claim that $P_{k,z}(\varepsilon i)$ and $P_{k,z}(\varepsilon j)$ are then close to a node of the lattice $2^{-k} \lambda \mathbb{Z}^2$. Indeed, denote by Π_{S_i} and Π_{S_j} the projections onto the subspaces spanned by the segments S_i and S_j , respectively. Assumption (4.93) and the defining property of $P_{k,z}$ imply that

$$|P_{k,z}(\varepsilon i) - P_{k,z}(\varepsilon j)| \leq \varepsilon |i - j| + \text{dist}(\varepsilon i, \partial q_k^z) + \text{dist}(\varepsilon j, \partial q_k^z) \leq 2(C_2 + 1) 2^{k_\varepsilon - k} \varepsilon + \varepsilon.$$

Hence for ε small enough the sides S_i and S_j cannot be parallel because $2^{k_\varepsilon} \varepsilon \ll \lambda$. Therefore the point $\Pi_{S_i}(\Pi_{S_j}(\varepsilon i))$ belongs to $S_i \cap S_j \subset 2^{-k} \lambda \mathbb{Z}^2$. We claim that

$$(4.94) \quad b_\varepsilon(P_{k,z}(\varepsilon i)) = b_\varepsilon(P_{k,z}(\varepsilon j)) = \bar{w}_\varepsilon(\Pi_{S_i}(\Pi_{S_j}(\varepsilon i))).$$

Indeed, denote by $0 \leq k_i^*, k_j^* \leq k_\varepsilon$ the layer numbers and by $S_i^* \subset S_i$ and $S_j^* \subset S_j$ the sides satisfying

$$\begin{aligned} b_\varepsilon(P_{k,z}(\varepsilon i)) &= b_{2^{k_i^*} S_i^*}^\varepsilon [w_{S_i^*}^\varepsilon](2^{k_i^*} P_{k,z}(\varepsilon i)), \\ b_\varepsilon(P_{k,z}(\varepsilon j)) &= b_{2^{k_j^*} S_j^*}^\varepsilon [w_{S_j^*}^\varepsilon](2^{k_j^*} P_{k,z}(\varepsilon j)). \end{aligned}$$

(The sides S_i^* and S_j^* are needed due to the fact that S_i or S_j may be contained in ∂Q_k , where b_ε is defined using the cubes that decompose the layer L_{k+1} ; if, for instance, S_i is not contained in ∂Q_k , then $k_i^* = k$ and $S_i^* = S_i$.) Since by the dyadic construction S_i^* either agrees with S_i or is exactly one-half of the side S_i , it follows that $\Pi_{S_i}(\Pi_{S_j}(\varepsilon i))$ is an endpoint of S_i^* .

By the same reasoning it is also an endpoint of S_j^* . Since $\Pi_{S_i}(\Pi_{S_j}(\varepsilon i)) = \Pi_{S_j}(\Pi_{S_i}(\varepsilon j))$, it then suffices to show that $2^{k_i^*} P_{k,z}(\varepsilon i)$ and $2^{k_j^*} P_{k,z}(\varepsilon j)$ are sufficiently close to $2^{k_i^*} \Pi_{S_i}(\Pi_{S_j}(\varepsilon i))$ and $2^{k_j^*} \Pi_{S_j}(\Pi_{S_i}(\varepsilon j))$, respectively, since by construction the boundary datum is constant in a neighborhood of the endpoints of a side. The 1-Lipschitz continuity of Π_{S_i} and Π_{S_j} combined with (4.93) yields

$$\begin{aligned} &|P_{k,z}(\varepsilon i) - \Pi_{S_i}(\Pi_{S_j}(\varepsilon i))| \\ &\leq |\varepsilon i - \Pi_{S_j}(\varepsilon i)| \\ &\leq |\varepsilon i - \varepsilon j| + |\varepsilon j - \Pi_{S_j}(\varepsilon j)| + |\Pi_{S_j}(\varepsilon i) - \Pi_{S_j}(\varepsilon j)| \\ (4.95) \quad &\leq 2\varepsilon + (C_2 + 1)2^{k_\varepsilon - k} \varepsilon. \end{aligned}$$

Similarly, we can derive the estimate

$$(4.96) \quad |P_{k,z}(\varepsilon j) - \Pi_{S_j}(\Pi_{S_i}(\varepsilon j))| \leq 2\varepsilon + (C_2 + 1)2^{k_\varepsilon - k} \varepsilon.$$

By (4.76), both terms can be bounded by $2^{-k+1}(C_2 + 2)\varepsilon\theta_\varepsilon^{-1}$ for ε that is small enough. Since $k \leq k_i^*, k_j^* \leq k + 1$, multiplying (4.95) by $2^{k_i^*}$ and (4.96) by $2^{k_j^*}$ yields

$$(4.97) \quad \max\{|2^{k_i^*} P_{k,z}(\varepsilon i) - 2^{k_i^*} \Pi_{S_i}(\Pi_{S_j}(\varepsilon i))|, |2^{k_j^*} P_{k,z}(\varepsilon j) - 2^{k_j^*} \Pi_{S_j}(\Pi_{S_i}(\varepsilon j))|\} \leq C_3 \varepsilon \theta_\varepsilon^{-1},$$

where $C_3 = 4(C_2 + 2) = 392 < c_0$, c_0 being the constant in the definition (4.27) (thus explaining the choice $c_0 = 393$). The estimate (4.97) thus implies (4.94).

Having in mind that $\frac{\theta_\varepsilon}{\varepsilon} |\text{dist}(\varepsilon i, \partial I) - \text{dist}(\varepsilon j, \partial I)| \leq \theta_\varepsilon$, the 1-Lipschitz continuity of $\text{Geo}[\bar{w}_\varepsilon(\Pi_{S_i}(\Pi_{S_j}(\varepsilon i))), u_{k,\varepsilon}^z]$ and the formula for \bar{u}_ε yield that $|\bar{u}_\varepsilon(\varepsilon i) -$

$|\bar{u}_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon$. From the definition of the function \mathfrak{P}_ε and the previous estimate we infer that

$$(4.98) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon.$$

Case 3. It remains to treat the case of points i and j such that $P_{k,z}(\varepsilon i) = \Pi_{S_i}(\varepsilon i)$ and $P_{k,z}(\varepsilon j) = \Pi_{S_j}(\varepsilon j)$. Here we use the Lipschitz-continuity of b_ε on S_i . Note that b_ε might be defined separately on two smaller sides contained in S_i , but nevertheless the Lipschitz property (4.86) holds on the whole S_i due to convexity. Moreover, we want to apply the stability estimate of Lemma 4.15. To this end, observe that by (4.90) and (4.76) we have

$$|b_\varepsilon(P_{k,z}(\varepsilon i)) - u_{k,\varepsilon}^z| \leq C_2 2^{k_\varepsilon - k - m(\lambda)} \theta_\varepsilon \leq 2C_2 2^{-m(\lambda)},$$

and the right-hand side can be made arbitrarily small (specifically, $2C_2 2^{-m(\lambda)} < c$, where c is the constant given in Lemma 4.15) since $m(\lambda) \gg 1$ for small λ . The same estimate holds with i replaced by j . To reduce notation, we set $d_{\varepsilon,i} = \theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon i, \partial q_k^z)$ and $d_{\varepsilon,j} = \theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon j, \partial q_k^z)$. Then by the triangle inequality, (4.86), and Lemma 4.15 we have

$$\begin{aligned} & |\bar{u}_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)| \\ & \leq |\text{Geo}[b_\varepsilon(P_{k,z}(\varepsilon i)), u_{k,\varepsilon}^z](d_{\varepsilon,i}) - \text{Geo}[b_\varepsilon(P_{k,z}(\varepsilon i)), u_{k,\varepsilon}^z](d_{\varepsilon,j})| \\ & \quad + |\text{Geo}[b_\varepsilon(P_{k,z}(\varepsilon i)), u_{k,\varepsilon}^z](d_{\varepsilon,j}) - \text{Geo}[b_\varepsilon(P_{k,z}(\varepsilon j)), u_{k,\varepsilon}^z](d_{\varepsilon,j})| \\ & \leq |d_{\varepsilon,i} - d_{\varepsilon,j}| + \text{d}_{\mathbb{S}^1}(b_\varepsilon(P_{k,z}(\varepsilon i)), b_\varepsilon(P_{k,z}(\varepsilon j))) \\ & \leq \theta_\varepsilon + \frac{\pi}{2} \theta_\varepsilon \varepsilon^{-1} |\Pi_{S_i}(\varepsilon i) - \Pi_{S_j}(\varepsilon j)| \leq \left(1 + \frac{\pi}{2}\right) \theta_\varepsilon. \end{aligned}$$

Hence we deduce the weaker but still sufficient bound

$$(4.99) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq 3\theta_\varepsilon.$$

Substep 2.2 (Interactions between different cubes)

Now we consider lattice points $\varepsilon i \in \mathfrak{q}_{k_i}^{z_i}$ and $\varepsilon j \in \mathfrak{q}_{k_j}^{z_j}$ with $\mathfrak{q}_{k_i}^{z_i} \neq \mathfrak{q}_{k_j}^{z_j}$ and $|i - j| = 1$. In this substep we assume that $0 \leq k_i, k_j \leq k_\varepsilon - 1$, that means, we consider only the layers where we interpolate. Assume without loss of generality that $k_i \leq k_j$. We also have to consider the numbers k_i^* and k_j^* characterized by the property

$$P_{k_i,z_i}(\varepsilon i) \in L_{k_i^*} \setminus \partial Q_{k_i^*}, \quad P_{k_j,z_j}(\varepsilon j) \in L_{k_j^*} \setminus \partial Q_{k_j^*},$$

that means, those values which determine the rescaling of the boundary conditions. Note that from the definition of $P_{k,z}$ it follows that

$$(4.100) \quad k_i \leq k_i^* \leq k_i + 1, \quad k_j \leq k_j^* \leq k_j + 1.$$

Since all cubes q_k^z are half-open and oriented along the coordinate axes, there exists a side S_{ij} of $\partial q_{k_j}^{z_j}$ such that the segment $[\varepsilon i, \varepsilon j]$ intersects S_{ij} orthogonally and additionally

$$(4.101) \quad S_{ij} \subset \partial q_{k_i}^{z_i} \cap \partial q_{k_j}^{z_j},$$

where we used that $k_j \geq k_i$ to ensure the inclusion. In particular,

$$(4.102) \quad \text{dist}(\varepsilon i, \partial q_{k_i}^{z_i}) + \text{dist}(\varepsilon j, \partial q_{k_j}^{z_j}) \leq \varepsilon,$$

which implies that

$$(4.103) \quad \begin{aligned} & |P_{k_i, z_i}(\varepsilon i) - P_{k_j, z_j}(\varepsilon j)| \\ & \leq |P_{k_i, z_i}(\varepsilon i) - \varepsilon i| + |\varepsilon i - \varepsilon j| + |\varepsilon j - P_{k_j, z_j}(\varepsilon j)| \leq 2\varepsilon. \end{aligned}$$

Moreover, in analogy to the estimate (4.43), we deduce the bound

$$(4.104) \quad |\bar{u}_\varepsilon(\varepsilon i) - b_\varepsilon(P_{k_i, z_i}(\varepsilon i))| + |\bar{u}_\varepsilon(\varepsilon j) - b_\varepsilon(P_{k_j, z_j}(\varepsilon j))| \leq \theta_\varepsilon.$$

Note that the above estimate does not give information on $|\bar{u}_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)|$ since, a priori, $b_\varepsilon(P_{k_i, z_i}(\varepsilon i))$ might differ from $b_\varepsilon(P_{k_j, z_j}(\varepsilon j))$. We will show that this is not the case. To this end, we shall prove two alternatives:

$$(4.105) \quad \begin{aligned} & \text{(i) } P_{k_i, z_i}(\varepsilon i) \in S_{ij} \text{ and } P_{k_j, z_j}(\varepsilon j) \in S_{ij}, \\ & \text{(ii) } \text{dist}(2^{k_i^*} P_{k_i, z_i}(\varepsilon i), \lambda \mathbb{Z}^2) \leq 2^{k_i+3} \varepsilon \\ & \quad \text{and } \text{dist}(2^{k_j^*} P_{k_j, z_j}(\varepsilon j), \lambda \mathbb{Z}^2) \leq 2^{k_j+2} \varepsilon. \end{aligned}$$

Indeed, first assume that $P_{k_j, z_j}(\varepsilon j) \notin S_{ij}$. Then there exists another facet S_j of $q_{k_j}^{z_j}$, $S_j \neq S_{ij}$, such that $P_{k_j, z_j}(\varepsilon j) \in S_j$. Since $\text{dist}(\varepsilon j, S_{ij}) \leq \varepsilon$ and $\text{dist}(\varepsilon j, S_j) \leq \varepsilon$, the sides S_{ij} and S_j cannot be parallel since the distance between parallel sides of $\partial q_{k_j}^{z_j}$ is given by $2^{-k_j} \lambda \geq \frac{1}{2} \theta_\varepsilon \lambda \gg \varepsilon$. Hence $\Pi_{S_j}(\Pi_{S_{ij}}(\varepsilon j)) \in S_j \cap S_{ij} \subset 2^{-k_j} \lambda \mathbb{Z}^2$, so that

$$(4.106) \quad \begin{aligned} \text{dist}(P_{k_j, z_j}(\varepsilon j), 2^{-k_j} \lambda \mathbb{Z}^2) &= \text{dist}(\Pi_{S_j}(\varepsilon j), 2^{-k_j} \lambda \mathbb{Z}^2) \\ &\leq |\varepsilon j - \Pi_{S_{ij}}(\varepsilon j)| \leq \varepsilon. \end{aligned}$$

In particular, applying (4.100) we deduce from the above estimate that

$$(4.107) \quad \begin{aligned} & \text{dist}(2^{k_j^*} P_{k_j, z_j}(\varepsilon j), \lambda \mathbb{Z}^2) \\ &= 2^{k_j^*} \text{dist}(P_{k_j, z_j}(\varepsilon j), 2^{-k_j^*} \lambda \mathbb{Z}^2) \\ &\leq 2^{k_j+1} \text{dist}(P_{k_j, z_j}(\varepsilon j), 2^{-k_j} \lambda \mathbb{Z}^2) \leq 2^{k_j+1} \varepsilon, \end{aligned}$$

where we used that $2^{-k_j} \lambda \mathbb{Z}^2 \subset 2^{-k_j^*} \lambda \mathbb{Z}^2$. For the point $P_{k_i, z_i}(\varepsilon i)$ consider first the case $P_{k_i, z_i}(\varepsilon i) \in S_{ij}$. Due to what we aim to prove, we then assume that

$P_{k_j, z_j}(\varepsilon j) \in S_j \setminus S_{ij}$ as above, so that (4.107) holds. Hence (4.103) and (4.106) imply

$$\text{dist}(P_{k_i, z_i}(\varepsilon i), 2^{-k_j} \lambda \mathbb{Z}^2) \leq \text{dist}(P_{k_j, z_j}(\varepsilon j), 2^{-k_j} \lambda \mathbb{Z}^2) + 2\varepsilon \leq 3\varepsilon < 2^2 \varepsilon.$$

In order to conclude the claimed estimate, observe that the condition $P_{k_i, z_i}(\varepsilon i) \in S_{ij} \subset \partial \mathfrak{q}_{k_j}^{z_j} \subset L_{k_j}$ forces $k_i^* \geq k_j$ so that

$$\begin{aligned} & \text{dist}(2^{k_i^*} P_{k_i, z_i}(\varepsilon i), \lambda \mathbb{Z}^2) \\ &= 2^{k_i^*} \text{dist}(P_{k_i, z_i}(\varepsilon i), 2^{-k_i^*} \lambda \mathbb{Z}^2) \\ (4.108) \quad & \leq 2^{k_i+1} \text{dist}(P_{k_i, z_i}(\varepsilon i), 2^{-k_j} \lambda \mathbb{Z}^2) \leq 2^{k_i+3} \varepsilon, \end{aligned}$$

where we used that $2^{-k_j} \lambda \mathbb{Z}^2 \subset 2^{-k_i^*} \lambda \mathbb{Z}^2$. On the contrary, if $P_{k_i, z_i}(\varepsilon i) \notin S_{ij}$, denote by S_i a facet of $\mathfrak{q}_{k_i}^{z_i}$ such $P_{k_i, z_i}(\varepsilon i) \in S_i$, $S_i \neq S_{ij}$. Then S_i and S_{ij} do not lie on the same straight line. To show this, we argue by contradiction. Assume that $S_i \subset \text{span}(S_{ij})$. Since the segment $[\varepsilon i, \varepsilon j]$ is orthogonal to S_{ij} , this would imply the false statement

$$\Pi_{S_i}(\varepsilon i) = \Pi_{S_{ij}}(\varepsilon i) = \Pi_{S_{ij}}(\varepsilon j) \in S_{ij},$$

where the last inclusion holds since $\varepsilon j \in \mathfrak{q}_{k_j}^{z_j}$ and S_{ij} is a side of the cube $\mathfrak{q}_{k_j}^{z_j}$. Since neither S_i nor S_{ij} can be parallel for ε small enough, we conclude that $\Pi_{S_i}(\Pi_{S_{ij}}(\varepsilon i)) \in \text{span}(S_i) \cap \text{span}(S_{ij})$. Since, by (4.101), $S_{ij} \subset \partial \mathfrak{q}_{k_i}^{z_i}$, we know that $\Pi_{S_i}(\Pi_{S_{ij}}(\varepsilon i)) \in 2^{-k_i} \lambda \mathbb{Z}^2$. Thus the defining property of S_{ij} yields

$$\begin{aligned} & \text{dist}(P_{k_i, z_i}(\varepsilon i), 2^{-k_i} \lambda \mathbb{Z}^2) \\ &= \text{dist}(\Pi_{S_i}(\varepsilon i), 2^{-k_i} \lambda \mathbb{Z}^2) \leq |\Pi_{S_{ij}}(\varepsilon i) - \varepsilon i| \leq |\varepsilon i - \varepsilon j| = \varepsilon. \end{aligned}$$

Again in combination with (4.100) this inequality implies the estimate

$$\begin{aligned} & \text{dist}(2^{k_i^*} P_{k_i, z_i}(\varepsilon i), \lambda \mathbb{Z}^2) \\ (4.109) \quad & \leq 2^{k_i+1} \text{dist}(P_{k_i, z_i}(\varepsilon i), 2^{-k_i} \lambda \mathbb{Z}^2) \leq 2^{k_i+1} \varepsilon. \end{aligned}$$

What remains is to establish an estimate for $\text{dist}(2^{k_j^*} P_{k_j, z_j}(\varepsilon j), \lambda \mathbb{Z}^2)$ when $P_{k_i, z_i}(\varepsilon i) \notin S_{ij}$ and $P_{k_j, z_j}(\varepsilon j) \in S_{ij}$. In this case we have

$$\begin{aligned} \text{dist}(P_{k_j, z_j}(\varepsilon j), 2^{-k_j} \lambda \mathbb{Z}^2) &= \text{dist}(\Pi_{S_{ij}}(\varepsilon j), 2^{-k_i} \lambda \mathbb{Z}^2) \\ &\leq |\Pi_{S_{ij}}(\varepsilon j) - \Pi_{S_{ij}}(\Pi_{S_i}(\varepsilon i))| \\ &\leq |\varepsilon j - \varepsilon i| + |\varepsilon i - \Pi_{S_i}(\varepsilon i)| \leq 2\varepsilon, \end{aligned}$$

where we used the inclusion $2^{-k_i} \lambda \mathbb{Z}^2 \subset 2^{-k_j} \lambda \mathbb{Z}^2$ (recall the assumption $k_i \leq k_j$ at the beginning of Substep 2.2). From the above inequality we deduce the estimate

$$\begin{aligned} & \text{dist}(2^{k_j^*} P_{k_j, z_j}(\varepsilon j), \lambda \mathbb{Z}^2) \\ (4.110) \quad & \leq 2^{k_j+1} \text{dist}(P_{k_j, z_j}(\varepsilon j), 2^{-k_j} \lambda \mathbb{Z}^2) \leq 2^{k_j+2} \varepsilon. \end{aligned}$$

Combining the estimates (4.107), (4.108), (4.109), and (4.110) we have proved the claimed alternatives (i) or (ii) in (4.105). We analyze them separately below.

Case 5. Assume

$$\text{dist}(2^{k_i^*} P_{k_i, z_i}(\varepsilon i), \lambda \mathbb{Z}^2) \leq 2^{k_i+3\varepsilon} \quad \text{and} \quad \text{dist}(2^{k_j^*} P_{k_j, z_j}(\varepsilon j), \lambda \mathbb{Z}^2) \leq 2^{k_j+2\varepsilon}$$

(that means, alternative (ii)) and denote by $\lambda \bar{z}_i, \lambda \bar{z}_j \in \lambda \mathbb{Z}^2$ points realizing the minimal distance. We start by observing that $2^{-k_i^*} \lambda \bar{z}_i = 2^{-k_j^*} \lambda \bar{z}_j$. Indeed, on the one hand we use (4.103) to estimate

$$\begin{aligned} |2^{-k_i^*} \lambda \bar{z}_i - 2^{-k_j^*} \lambda \bar{z}_j| &\leq |2^{-k_i^*} \lambda \bar{z}_i - P_{k_i, z_i}(\varepsilon i)| + |P_{k_j, z_j}(\varepsilon j) - 2^{-k_j^*} \lambda \bar{z}_j| + 2\varepsilon \\ &\leq 14\varepsilon. \end{aligned}$$

On the other hand, since both $2^{-k_i^*} \lambda \bar{z}_i$ and $2^{-k_j^*} \lambda \bar{z}_j$ belong to

$$2^{-\max\{k_i^*, k_j^*\}} \lambda \mathbb{Z}^2 \subset 2^{-k_j-1} \lambda \mathbb{Z}^2 \quad \text{and} \quad 2^{-k_j-1} \lambda \geq \frac{1}{2} \theta_\varepsilon \lambda \gg \varepsilon$$

by (4.76), we infer that $2^{-k_i^*} \lambda \bar{z}_i = 2^{-k_j^*} \lambda \bar{z}_j$. We set $p_{ij} := 2^{-k_i^*} \lambda \bar{z}_i = 2^{-k_j^*} \lambda \bar{z}_j$.

Let now S_i and S_j be the sides of the cubes in $\mathcal{Q}_{k_i^*}$ and $\mathcal{Q}_{k_j^*}$, respectively, such that $P_{k_i, z_i}(\varepsilon i) \in S_i$, $P_{k_j, z_j}(\varepsilon j) \in S_j$, and

$$(4.111) \quad \begin{aligned} b_\varepsilon(P_{k_i, z_i}(\varepsilon i)) &= b_{2^{k_i^*} S_i}^\varepsilon [w_{S_i}^\varepsilon](2^{k_i^*} P_{k_i, z_i}(\varepsilon i)), \\ b_\varepsilon(P_{k_j, z_j}(\varepsilon j)) &= b_{2^{k_j^*} S_j}^\varepsilon [w_{S_j}^\varepsilon](2^{k_j^*} P_{k_j, z_j}(\varepsilon j)). \end{aligned}$$

We claim that $p_{ij} \in S_i \cap S_j$. Indeed, since by assumption

$$|p_{ij} - P_{k_i, z_i}(\varepsilon i)| \leq 8\varepsilon, \quad |p_{ij} - P_{k_j, z_j}(\varepsilon j)| \leq 4\varepsilon,$$

and for a side $S \subset \partial \mathfrak{q}_k^z$ with $0 \leq k < k_\varepsilon$ and $z \in \mathbb{Z}^2$ it holds that

$$\text{dist}(S, 2^{-k} \lambda \mathbb{Z}^2 \setminus S) \geq 2^{-k} \lambda \gg \varepsilon,$$

the claim follows by a triangle inequality argument. Moreover, recalling that $k_i, k_j \leq k_\varepsilon - 1$, property (4.76) yields $2^{k_i+3\varepsilon} \leq 4\varepsilon \theta_\varepsilon^{-1} < c_0 \varepsilon \theta_\varepsilon^{-1}$ and $2^{k_j+2\varepsilon} \leq 2\varepsilon \theta_\varepsilon^{-1} < c_0 \varepsilon \theta_\varepsilon^{-1}$, c_0 being the constant used in the definition (4.27). Thus we conclude from (4.111) and the definition of the boundary condition that

$$b_\varepsilon(P_{k_i, z_i}(\varepsilon i)) = b_\varepsilon(p_{ij}) = b_\varepsilon(P_{k_j, z_j}(\varepsilon j)),$$

where we used that p_{ij} must be an endpoint of S_i and S_j . Combined with (4.104) we infer

$$\begin{aligned} |\bar{u}_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)| &\leq |\bar{u}_\varepsilon(\varepsilon i) - b_\varepsilon(P_{k_i, z_i}(\varepsilon i))| + |\bar{u}_\varepsilon(\varepsilon j) - b_\varepsilon(P_{k_j, z_j}(\varepsilon j))| \\ &\leq \theta_\varepsilon, \end{aligned}$$

which by the definition of \mathfrak{P}_ε allows us to conclude that

$$(4.112) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon.$$

Case 6. Finally, we analyze the case $P_{k_i, z_i}(\varepsilon i) \in S_{ij}$ and $P_{k_j, z_j}(\varepsilon j) \in S_{ij}$ (that means, alternative (i)). Since by assumption the line segment $[\varepsilon i, \varepsilon j]$ intersects S_{ij} orthogonally and S_{ij} is a side of both cubes $q_{k_i}^{z_i}$ and $q_{k_j}^{z_j}$, we know that $P_{k_i, z_i}(\varepsilon i) = P_{k_j, z_j}(\varepsilon j)$. In combination with estimate (4.104) we therefore obtain

$$|\bar{u}_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)| = |\bar{u}_\varepsilon(\varepsilon i) - b_\varepsilon(P_{k_i, z_i}(\varepsilon i))| + |b_\varepsilon(P_{k_j, z_j}(\varepsilon j)) - \bar{u}_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon,$$

which yields the estimate

$$(4.113) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq \theta_\varepsilon.$$

Substep 2.3 (Interactions between $Q_{k_\varepsilon-1}$ and the layers)

In this step we consider the case where $\varepsilon i \in \varepsilon \mathbb{Z}^2 \cap Q_{k_\varepsilon-1}$ but $\varepsilon j \in \varepsilon \mathbb{Z}^2 \setminus Q_{k_\varepsilon-1}$. Since $|\varepsilon i - \varepsilon j| = \varepsilon$, it follows that $\varepsilon i \in L_{k_\varepsilon}$ and $\varepsilon j \in L_{k_\varepsilon-1}$, that means, the last and the last but one layers. Indeed, the thickness of the last layer is $2^{-k_\varepsilon} \lambda \geq \frac{1}{2} \theta_\varepsilon \lambda \gg \varepsilon$, so that the claim follows by a triangle inequality argument. Let $z_j \in \mathbb{Z}^2$ be such that $\varepsilon j \in q_{k_\varepsilon-1}^{z_j}$. From the definition of u_ε in (4.88) and (4.62) we infer

$$\begin{aligned} |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| &= |v_\varepsilon(\varepsilon i) - \mathfrak{P}_\varepsilon(\bar{u}_\varepsilon(\varepsilon j))| \\ &\leq |v_\varepsilon(\varepsilon i) - \bar{u}_\varepsilon(\varepsilon j)| + \theta_\varepsilon \\ &\leq |v_\varepsilon(\varepsilon i) - b_\varepsilon(P_{z_j, k_\varepsilon-1}(\varepsilon j))| + |b_\varepsilon(P_{z_j, k_\varepsilon-1}(\varepsilon j)) - \bar{u}_\varepsilon(\varepsilon j)| + \theta_\varepsilon \\ &\leq |v_\varepsilon(\varepsilon i) - b_\varepsilon(P_{z_j, k_\varepsilon-1}(\varepsilon j))| + 2\theta_\varepsilon, \end{aligned}$$

where the last inequality can be proven like the estimates (4.43) and (4.104). Using the general estimate (4.90) with $k = k_\varepsilon - 1$ we can further bound the last term to conclude that

$$(4.114) \quad \begin{aligned} |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| &\leq |v_\varepsilon(\varepsilon i) - u_{k_\varepsilon-1, \varepsilon}^{z_j}| + C\theta_\varepsilon \\ &= |v_\varepsilon(\varepsilon i) - u(2^{-k_\varepsilon+1} \lambda(z_j + \frac{1}{2}e_1 + \frac{1}{2}e_2))| + C\theta_\varepsilon. \end{aligned}$$

Recall that $2^{-k_\varepsilon+1} \lambda(z_j + \frac{1}{2}e_1 + \frac{1}{2}e_2)$ is the midpoint of the cube $q_{k_\varepsilon-1}^{z_j}$, so that by (4.76)

$$\begin{aligned} |\varepsilon i - 2^{-k_\varepsilon+1} \lambda(z_j + \frac{1}{2}e_1 + \frac{1}{2}e_2)| &\leq |\varepsilon i - \varepsilon j| + |\varepsilon j - 2^{-k_\varepsilon+1} \lambda(z_j + \frac{1}{2}e_1 + \frac{1}{2}e_2)| \\ &\leq \varepsilon + 2^{-k_\varepsilon+1} \lambda \leq \varepsilon + 2\theta_\varepsilon \lambda. \end{aligned}$$

We insert this bound with the estimates (4.62) and (4.78) in (4.114) to obtain

$$\begin{aligned} |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| &\leq |u(\varepsilon i) - u(2^{-k_\varepsilon+1}\lambda(z_j + \frac{1}{2}e_1 + \frac{1}{2}e_2))| + C\theta_\varepsilon \\ &\leq 2 \frac{|\varepsilon i - 2^{-k_\varepsilon+1}\lambda(z_j + \frac{1}{2}e_1 + \frac{1}{2}e_2)|}{|2^{-k_\varepsilon+1}\lambda(z_j + \frac{1}{2}e_1 + \frac{1}{2}e_2)|} + C\theta_\varepsilon \\ &\leq C \frac{\varepsilon + 2\theta_\varepsilon\lambda}{(2^{m(\lambda)} - 2)\lambda} + C\theta_\varepsilon, \end{aligned}$$

where for the last inequality we used the set inclusion (4.77). Since $\varepsilon \ll \theta_\varepsilon$, for $\lambda > 0$ fixed we can assume that $\varepsilon \leq \theta_\varepsilon\lambda$, so that the last estimate turns into the bound

$$(4.115) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq C\theta_\varepsilon.$$

Step 3. (Energy estimates in $Q(\lambda)$)

Let us first summarize what we have proven so far. By our choice of $m(\lambda)$ at the beginning of Step 2 we have $Q(\lambda) \subset B_{\eta/2}$ and (4.77). Hence we can use the bound (4.75) of Step 1 to control the energy due to interactions with both points in $Q_{k_\varepsilon-1}$, where $u_\varepsilon = v_\varepsilon$; cf. (4.88). For the interactions with at least one point in $Q(\lambda) \setminus Q_{k_\varepsilon-1}$, we showed in Substeps 2.1–2.3 (cf. (4.98), (4.99), (4.112), (4.113), and (4.115)) that the bound

$$(4.116) \quad |u_\varepsilon(\varepsilon i) - u_\varepsilon(\varepsilon j)| \leq C\theta_\varepsilon$$

holds with a uniform constant $C < +\infty$. In order to obtain precise estimates on the energy due to interactions with at least one point in $Q(\lambda) \setminus Q_{k_\varepsilon-1}$, we have to count the number of lattice points $\varepsilon i, \varepsilon j$ satisfying $u_\varepsilon(\varepsilon i) \neq u_\varepsilon(\varepsilon j)$. For such points (4.116) will suffice.

Fix such $\varepsilon i, \varepsilon j$. Then there exists a cube $q_k^z \in \mathcal{Q}_k$ with $0 \leq k < k_\varepsilon$ and $z \in \mathbb{Z}^2$ with

$$(4.117) \quad \text{dist}(\varepsilon i, \partial q_k^z) \leq C 2^{k_\varepsilon - k} \varepsilon.$$

Indeed, if $\varepsilon i, \varepsilon j \in Q(\lambda) \setminus Q_{k_\varepsilon-1}$ and they belong to the same cube of \mathcal{Q}_k (Substep 2.1), then this is a consequence of (4.93). If $\varepsilon i, \varepsilon j \in Q(\lambda) \setminus Q_{k_\varepsilon-1}$, but they belong to two different cubes (Substep 2.2), then this follows from (4.102). Finally, if, for instance, $\varepsilon i \in Q_{k_\varepsilon-1}$ and $\varepsilon j \in Q(\lambda) \setminus Q_{k_\varepsilon-1}$ (Substep 2.3), then this is a consequence of the fact that $\varepsilon i \in L_{k_\varepsilon}$ and $\varepsilon j \in L_{k_\varepsilon-1}$ (see also (4.102)). Therefore it suffices to count lattice points that satisfy (4.117).

From a covering argument with cubes of volume ε^2 and (4.76) we infer that

$$\begin{aligned} \varepsilon^2 \#(\varepsilon \mathbb{Z}^2 \cap \{\text{dist}(\cdot, \partial q_k^z) \leq C 2^{k_\varepsilon - k} \varepsilon\}) \\ \leq 8(2^{-k}\lambda + 2(C 2^{k_\varepsilon - k} \varepsilon + \varepsilon))(C 2^{k_\varepsilon - k} \varepsilon + \varepsilon) \\ \leq C(2^{-k}\lambda + \varepsilon(2^{-k}\theta_\varepsilon^{-1} + 1))(2^{-k}\theta_\varepsilon^{-1} + 1)\varepsilon \\ \leq C 2^{-2k}\lambda \varepsilon \theta_\varepsilon^{-1}, \end{aligned}$$

where in the last inequality we also used that $\varepsilon\theta_\varepsilon^{-1} \ll \lambda$ for ε small enough. Next recall that the number of cubes q_k^z in the layer L_k can be roughly bounded by

$$\#\mathcal{Q}_k \leq C 2^{m(\lambda)} 2^k.$$

Combining the previous two estimates with (4.116) we can estimate the energy of u_ε via

$$\begin{aligned} \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon; Q(\lambda)) &\leq \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(v_\varepsilon; B_\eta) + \frac{C\theta_\varepsilon^2}{\varepsilon\theta_\varepsilon} 2^{m(\lambda)} \lambda \sum_{k=0}^{k_\varepsilon} 2^{-k} \varepsilon\theta_\varepsilon^{-1} \\ &\leq \frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(v_\varepsilon; B_\eta) + C 2^{m(\lambda)} \lambda. \end{aligned}$$

Due to the choice of $m(\lambda)$ it holds that $2^{m(\lambda)} \lambda \leq \eta$. Subtracting the diverging term $2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon}$ and inserting the upper bound (4.75) we conclude that

$$(4.118) \quad \limsup_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon\theta_\varepsilon} E_\varepsilon(u_\varepsilon; Q(\lambda)) - 2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) \leq C\eta.$$

We emphasize that $Q(\lambda)$ implicitly depends on η through the quantity $2^{m(\lambda)} \lambda$.

Step 4. (From local to global constructions)

We are now in a position to define u_ε globally. In this step we stress again the dependence on n of λ_n . We start by repeating the construction presented in Step 2 around each singularity x_h of u , by defining u_ε as in (4.88) (combined with a reflection if $\deg(u)(x_h) = -1$) in the squares $Q(\lambda_n, x_h) = Q(\lambda_n) + x_h$.

To define u_ε outside the squares $Q(\lambda_n, x_h)$, we first observe that the square $x_h + [-2^{m(\lambda_n)+1}\lambda_n, 2^{m(\lambda_n)+1}\lambda_n]$ is not contained in $B_{\eta/2}(x_h)$, since $m(\lambda_n)$ has been chosen as the maximal integer such that

$$Q(\lambda_n, x_h) = x_h + [-2^{m(\lambda_n)}\lambda_n, 2^{m(\lambda_n)}\lambda_n] \subset B_{\eta/2}(x_h).$$

This yields $\eta/4 \leq 2^{m(\lambda_n)+1}\lambda_n$ and thus, by (4.77),

$$(4.119) \quad Q_k^{\lambda_n}(x_h) = x_h + Q_k \supset B_{(2^{m(\lambda_n)-2})\lambda}(x_h) \supset B_{\eta/16}(x_h).$$

Note that here we stress the dependence of $Q_k^{\lambda_n}(x_h)$ on λ_n , in contrast to the notation adopted for Q_k in Step 2. We recall that $Q_{-1}^{\lambda_n}(x_h) = Q(\lambda_n, x_h)$.

Applying Lemma 4.13 with

$$O = \Omega \setminus \bigcup_{h=1}^N \bar{B}_{\eta/16}(x_h) \quad \text{and} \quad \tilde{O} = \tilde{\Omega} \setminus \bigcup_{h=1}^N \bar{B}_{\eta/32}(x_h)$$

to $u \in C^\infty(\tilde{\Omega} \setminus \bigcup_{h=1}^N \bar{B}_{\eta/32}(x_h); \mathbb{S}^1)$, we get a sequence of piecewise constant functions $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$ such that $u_n \rightarrow u$ strongly in

$$L^1\left(\Omega \setminus \bigcup_{h=1}^N \bar{B}_{\eta/16}(x_h); \mathbb{R}^2\right)$$

and, by (4.119),

$$(4.120) \quad \begin{aligned} & \limsup_{n \rightarrow +\infty} \int_{J_{u_n} \cap (\Omega \setminus \bigcup_{h=1}^N Q_0^{\lambda_n}(x_h))^{\lambda_n}} d_{\mathbb{S}^1}(u_n^-, u_n^+) |v_{u_n}|_1 d\mathcal{H}^1 \\ & \leq \int_{\Omega} |\nabla u|_{2,1} dx. \end{aligned}$$

Notice that the squares $Q_0^{\lambda_n}(x_h)$ have vertices on the lattice $\lambda_n \mathbb{Z}^2$.

Let us fix n large enough. For $\varepsilon i \in \varepsilon \mathbb{Z}^2 \setminus \bigcup_{h=1}^N Q_0^{\lambda_n}(x_h)$ we define $u'_\varepsilon(\varepsilon i)$ as the recovery sequence given in the proof of Proposition 4.16 for the piecewise constant function $u_n \in \mathcal{PC}_{\lambda_n}(\mathbb{S}^1)$ with the constant $c_0 = 393$ in (4.27). Then we define $u_\varepsilon(\varepsilon i) := u'_\varepsilon(\varepsilon i)$ for

$$\varepsilon i \in \varepsilon \mathbb{Z}^2 \setminus \bigcup_{h=1}^N Q(\lambda_n, x_h) = \varepsilon \mathbb{Z}^2 \setminus \bigcup_{h=1}^N Q_{-1}^{\lambda_n}(x_h).$$

This completes the definition of u_ε in $\varepsilon \mathbb{Z}^2$.

We claim that

$$(4.121) \quad \begin{aligned} & \varepsilon i \in \varepsilon \mathbb{Z}^2 \cap Q_{-1}^{\lambda_n}(x_h) \quad \text{and} \\ & \text{dist}(\varepsilon i, \partial Q_{-1}^{\lambda_n}(x_h)) \leq \varepsilon \implies u_\varepsilon(\varepsilon i) = u'_\varepsilon(\varepsilon i), \end{aligned}$$

which means that the two constructions given by Step 2 and Proposition 4.16 are identical. Indeed, first note that the assumptions on εi above imply that $\varepsilon i \in L_0$. Hence we find $q_0^{z_0} \in \mathcal{Q}_0$ such that $\varepsilon i \in q_0^{z_0}$. We now consider the two cases $P_{0,z_0}(\varepsilon i) \in \partial q_0^{z_0} \setminus \partial Q_{-1}^{\lambda_n}(x_h)$ and $P_{0,z_0}(\varepsilon i) \in \partial Q_{-1}^{\lambda_n}(x_h)$. If $P_{0,z_0}(\varepsilon i) \in \partial q_0^{z_0} \setminus \partial Q_{-1}^{\lambda_n}(x_h)$, let $S_i \subset \partial q_0^{z_0}$ be the side such that $P_{0,z_0}(\varepsilon i) \in S_i$. By the assumption in (4.121), S_i is not contained in $\partial Q_0^{\lambda_n}(x_h)$ and thus it intersects a side S_0 of $q_0^{z_0}$ such that $S_0 \subset \partial Q_{-1}^{\lambda_n}(x_h)$. In particular, $\Pi_{S_i}(\Pi_{S_0}(\varepsilon i)) \in \lambda_n \mathbb{Z}^2$ is an endpoint of S_i and, by (4.121),

$$\begin{aligned} |P_{0,z_0}(\varepsilon i) - \Pi_{S_i}(\Pi_{S_0}(\varepsilon i))| &= |\Pi_{S_i}(\varepsilon i) - \Pi_{S_i}(\Pi_{S_0}(\varepsilon i))| \\ &\leq |\varepsilon i - \Pi_{S_0}(\varepsilon i)| \leq \varepsilon \ll c_0 \frac{\varepsilon}{\theta_\varepsilon}, \end{aligned}$$

where we used that S_0 is the side such that $\text{dist}(\varepsilon i, S_0) = \text{dist}(\varepsilon i, \partial Q_{-1}^{\lambda_n}(x_h)) \leq \varepsilon$. Since $P_{0,z_0}(\varepsilon i)$ is close enough to the corner $p_{i,0} := \Pi_{S_i}(\Pi_{S_0}(\varepsilon i)) \in \lambda_n \mathbb{Z}^2$,

the boundary condition used for the definition of u_ε at $P_{0,z_0}(\varepsilon i)$ agrees with its value at the corner; cf. (4.27). Thus

$$\begin{aligned} u_\varepsilon(\varepsilon i) &= \text{Geo}[b_\varepsilon(p_{i,0}), u_{0,\varepsilon}^{z_0}](\theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon i, \partial q_0^{z_0})) \\ &= \text{Geo}\left[u\left(p_{i,0} + \lambda_n \left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right)\right), u\left(\lambda_n \left(z_0 + \frac{1}{2}e_1 + \frac{1}{2}e_2\right)\right)\right] \\ &\quad \cdot (\theta_\varepsilon \varepsilon^{-1} \text{dist}(\varepsilon i, \partial q_0^{z_0})). \end{aligned}$$

The same holds true for u'_ε . This concludes the proof of (4.121) when $P_{0,z_0}(\varepsilon i) \in \partial q_0^{z_0} \setminus \partial Q_{-1}^{\lambda_n}(x_h)$. If, instead, $P_{0,z_0}(\varepsilon i) \in \partial Q_{-1}^{\lambda_n}(x_h)$, let S_i be the side of $q_0^{z_0}$ such that $P_{0,z_0}(\varepsilon i) \in S_i$. Then the two 3-tuples of values $v_{S_i}(u)$ and $w_{S_i}^\varepsilon$ defined in (4.28) and (4.82), respectively, coincide (note that, by definition, both cubes in \mathcal{Q}_0 and \mathcal{Q}_{-1} have size λ). Then (4.121) follows in this case too.

Taking into account (4.92) on each $Q(\lambda_n, x_h)$, the function $u_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{S}_\varepsilon)$ converges in $L^1(\Omega; \mathbb{R}^2)$ to the function $\mathring{u}_n \in L^1(\Omega; S^1)$ defined by

$$\mathring{u}_n(x) := \begin{cases} u_n(x) & \text{if } x \in \Omega \setminus \bigcup_{h=1}^N Q(\lambda_n, x_h), \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \deg(u)(x_h) u_0^{\lambda_n}(x - x_h) & \text{if } x \in Q(\lambda_n, x_h) \text{ for some } 1 \leq h \leq N. \end{cases}$$

We remark that the precise structure for fixed λ_n is not important. Just note that due to the fact that $Q(\lambda_n, x_h) \subset B_{\eta/2}(x_h)$ and (4.92), the layer in each $Q(\lambda_n, x_h)$ where \mathring{u}_n differs from $\left(\frac{x-x_h}{|x-x_h|}\right)^{\pm 1}$, and thus from u , is of thickness $2\lambda_n$. Consequently,

$$(4.122) \quad \mathring{u}_n \rightarrow u \quad \text{in } L^1(\Omega; \mathbb{R}^2) \text{ as } n \rightarrow +\infty.$$

It remains to estimate the energy of u_ε in terms of λ_n and η . In particular, we need to estimate the interactions between the square $Q(\lambda_n, x_h)$ and the exterior. Thanks to (4.121) we can split the energy as

$$\begin{aligned} & \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon; \Omega) - 2\pi N |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \\ & \leq \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u'_\varepsilon; \Omega \setminus \bigcup_{h=1}^N Q_0^{\lambda_n}(x_h)) \\ & \quad + \sum_{h=1}^N \left(\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon; Q(\lambda_n, x_h)) - 2\pi |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right). \end{aligned}$$

By (4.118) and (4.49) we can pass to the limit in ε and conclude that

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \left(\frac{E_\varepsilon(u_\varepsilon; \Omega)}{\varepsilon \theta_\varepsilon} - 2\pi N |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) \\ & \leq \int_{J_{u_n} \cap (\Omega \setminus \bigcup_{h=1}^N Q_0^{\lambda_n}(x_h))^{\lambda_n}} \mathbf{d}_{\mathbb{S}^1}(u_n^-, u_n^+) |v_{u_n}|_1 \, d\mathcal{H}^1 + C\eta. \end{aligned}$$

Before we can conclude, we have to identify the flat limit of the vorticity measures μ_{u_ε} associated with the sequence u_ε . Since the right-hand side in (4.123) is finite, Proposition 4.10 implies that (up to a subsequence) $\mu_{u_\varepsilon} \xrightarrow{f} \bar{\mu}$ for some $\bar{\mu} = \sum_{k=1}^N d_k \delta_{y_k}$ with $d_k \in \mathbb{Z}$ and $|\bar{\mu}|(\Omega) \leq N$ (we allow $d_k = 0$ in order to sum from 1 to N). We claim that $\bar{\mu} = \mu$ with μ defined in the statement of the proposition.

Here comes the argument. Fix $x_0 \in \Omega \setminus \{x_1, \dots, x_N\}$. Since the singular part $2\pi N |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon}$ of the estimate (4.123) is concentrated in the set $\bigcup_{h=1}^N B_{2r_\varepsilon}(x_h)$ (cf. (4.73)) and $r_\varepsilon \rightarrow 0$, we deduce that for $0 < \rho \ll \lambda$ small enough we have $\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon(u_\varepsilon; B_\rho(x_0)) < +\infty$. Since we assume here that $\theta_\varepsilon \ll \varepsilon |\log \varepsilon|$, Remark 2.2 yields that $\mu_{u_\varepsilon} \llcorner B_\rho(x_0) \xrightarrow{f} 0$. Testing this convergence with a Lipschitz-function $\varphi \in C_c^{0,1}(B_\rho(x_0))$ such that $\varphi(x_0) = 1$ we obtain that $x_0 \notin \{y_1, \dots, y_N\}$ (or $x_0 = y_k$ for some k with $d_k = 0$). Since $x_0 \in \Omega \setminus \{x_1, \dots, x_N\}$ was arbitrary, we can write $\bar{\mu} = \sum_{h=1}^N d_h \delta_{x_h}$.

It remains to prove that $d_h = \deg(u)(x_h)$ for all $1 \leq h \leq N$. Note that for $\rho \ll \lambda$ it holds that $u_\varepsilon = v_\varepsilon$ on each $B_\rho(x_h)$, where v_ε is defined in (4.61). Due to (4.75) we have for ε small enough

$$(4.123) \quad \frac{1}{\varepsilon^2} E_\varepsilon(v_\varepsilon; B_\rho(x_h)) \leq C \eta \frac{\theta_\varepsilon}{\varepsilon} + 2\pi |\log \varepsilon| \leq C |\log \varepsilon|.$$

Hence we can apply [5, Proposition 5.2], which states that in dimension 2 the flat convergence of $\mu_{v_\varepsilon} \llcorner B_\rho(x_h)$ is equivalent to the flat convergence of the (normalized) Jacobians of the piecewise affine interpolation of v_ε on $B_\rho(x_h)$. Denote this piecewise affine interpolation and the one associated to the function u on $B_\rho(x_h)$ by \hat{v}_ε and $\hat{u}(\varepsilon)$, respectively. Inserting the estimate (4.62) in the definition of the piecewise affine interpolation one can show that

$$(4.124) \quad |\hat{v}_\varepsilon(x) - \hat{u}(\varepsilon)(x)| \leq C \theta_\varepsilon \quad \text{for all } x \in B_\rho(x_h).$$

Taking into account one more time the estimate (4.123), we conclude that

$$(4.125) \quad \begin{aligned} & \|\hat{v}_\varepsilon - \hat{u}(\varepsilon)\|_{L^2(B_{\rho/2}(x_h))} (\|\nabla \hat{v}_\varepsilon\|_{L^2(B_{\rho/2}(x_h))} + \|\nabla \hat{u}(\varepsilon)\|_{L^2(B_{\rho/2}(x_h))}) \\ & \leq C \theta_\varepsilon \left(\frac{1}{\varepsilon^2} E_\varepsilon(v_\varepsilon; B_\rho(x_h)) + \frac{1}{\varepsilon^2} E_\varepsilon(u; B_\rho(x_h)) \right)^{\frac{1}{2}} \leq C \theta_\varepsilon |\log \varepsilon|^{\frac{1}{2}}, \end{aligned}$$

where the bound $\frac{1}{\varepsilon^2} E_\varepsilon(u; B_\rho(x_h)) \leq C |\log \varepsilon|$ can be proven with similar arguments used to show (4.71). The above right-hand side vanishes when $\varepsilon \rightarrow 0$. Thus [5, lemma 3.1] implies that the Jacobians fulfill $J \hat{v}_\varepsilon - J \hat{u}(\varepsilon) \xrightarrow{f} 0$. Recalling that $u = \left(\frac{x-x_h}{|x-x_h|}\right)^{\pm 1}$ on $B_\rho(x_h)$, it follows from Step 1 of the proof of [4, theorem 5.1(ii)] that $\frac{1}{\pi} J \hat{u}(\varepsilon) \xrightarrow{f} \deg(u)(x_h) \delta_{x_h}$. Fixing again $\varphi \in C_c^{0,1}(B_\rho(x_h))$ such that $\varphi(x_h) = 1$, the above arguments imply

$$d_h = \langle \mu, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \langle \mu_{u_\varepsilon}, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \langle \frac{1}{\pi} J \hat{v}_\varepsilon, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \langle \frac{1}{\pi} J \hat{u}(\varepsilon), \varphi \rangle = \deg(u)(x_h)$$

as claimed.

Since the limit measure equals μ for all λ , we deduce from the $L^1(\Omega)$ -lower semicontinuity of the Γ -lim sup, (4.122), and (4.120) that

$$\Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon - 2\pi N |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) (u, \mu) \leq C\eta + \int_{\Omega} |\nabla u|_{2,1} \, dx.$$

The claim then follows by the arbitrariness of $0 < \eta < \eta_0$ (recall that $|\mu|(\Omega) = N$). \square

Together with Propositions 4.10 and 4.11 the next result finishes the proof of Theorem 1.1.

PROPOSITION 4.24 (M vortices, upper bound). *Assume that $\varepsilon \ll \theta_\varepsilon \ll \varepsilon |\log \varepsilon|$. Let $\mu = \sum_{h=1}^N d_h \delta_{x_h}$ with $|\mu|(\Omega) = M \in \mathbb{N}$ and let $u \in BV(\Omega; \mathbb{S}^1)$. Then*

$$\Gamma\text{-lim sup}_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon \theta_\varepsilon} E_\varepsilon - 2\pi M |\log \varepsilon| \frac{\varepsilon}{\theta_\varepsilon} \right) (u, \mu) \leq \int_{\Omega} |\nabla u|_{2,1} \, dx + |\mathbf{D}^{(c)} u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega).$$

PROOF. Fix $\sigma > 0$. By definition (4.19) of \mathcal{J} there exists a $T \in \mathcal{D}_2(\Omega \times \mathbb{R}^2)$, with $T \in \text{cart}(\Omega_\mu \times \mathbb{S}^1)$, $\partial T|_{\Omega \times \mathbb{R}^2} = -\mu \times \llbracket \mathbb{S}^1 \rrbracket$, and $u_T = u$, such that

$$(4.126) \quad \int_{\Omega \times \mathbb{R}^2} \Phi(\vec{T}) d|T| \leq \int_{\Omega} |\nabla u|_{2,1} \, dx + |\mathbf{D}^{(c)} u|_{2,1}(\Omega) + \mathcal{J}(\mu, u; \Omega) + \sigma.$$

In the previous inequality we applied Lemma 4.6 with Ω_μ in place of Ω (cf. Remark 4.7).

Due to Lemma 4.17 we find an open set $\tilde{\Omega} \ni \Omega$ and a sequence of maps $u_k \in C^\infty(\tilde{\Omega}_\mu; \mathbb{S}^1) \cap W^{1,1}(\tilde{\Omega}; \mathbb{S}^1)$ such that $u_k \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$, $|G_{u_k}|(\Omega \times \mathbb{R}^2) \rightarrow |T|(\Omega \times \mathbb{R}^2)$, and $\deg(u_k)(x_h) = d_h$ for $h = 1, \dots, N$. Reshetnyak's continuity theorem implies that

$$(4.127) \quad \int_{\Omega} |\nabla u_k|_{2,1} \, dx = \int_{\Omega \times \mathbb{R}^2} \Phi(\vec{G}_{u_k}) d|G_{u_k}| \leq \int_{\Omega \times \mathbb{R}^2} \Phi(\vec{T}) d|T| + \sigma$$

for k large enough. In the first equality we applied Lemma 4.6 and Remark 4.7 to u_k in Ω_μ . Applying Lemmata 4.18, 4.20, and 4.21 we reduce to the assumptions in Proposition 4.22. By the lower semicontinuity of the Γ -lim sup with respect to the strong L^1 -convergence of u and the flat convergence of μ , we conclude the proof. \square

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