ERDŐS–SZEKERES THEOREM WITH FORBIDDEN ORDER TYPES. II

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Abstract. According to the Erdős-Szekeres theorem, every set of n points in the plane contains roughly $\log n$ in convex position. We investigate how this bound changes if our point set does not contain a subset that belongs to a fixed order type.

1. Introduction

Throughout this paper we will always assume that every point set is in general position in the plane, that is, no three points of the configuration are collinear. Two such configurations are said to be of the same *order type*, if there is a one-to-one correspondence between them which preserves the orientation of each triple. Thus, order types are equivalence classes of configurations. For example, given an integer $n \geq 3$, the vertex set of any convex n-gon belongs to the same order type we denote by $\mathcal{C}_n$. It is clear that the order type of a configuration stays invariant

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under orientation preserving non-singular affine transformations and also under a wide class of projective transformations. The cardinality $|\mathcal{T}|$ of an order type $\mathcal{T}$ is the common cardinality of all configurations contained in it. We will say that the order type $\mathcal{T}$ contains the order type $\mathcal{S}$ if some (hence any) configuration in $\mathcal{T}$ contains a subset which belongs to $\mathcal{S}$. We denote this relation by $\mathcal{S} \hookrightarrow \mathcal{T}$ and will use this notation on the level of configurations as well. All configurations belonging to the same order type possess the same separation properties. Thus, several notions of discrete geometry carry over to order types in a natural way. For example, if the convex hull of some element of $\mathcal{T}$ is a convex n-gon, then it is true for every element of $\mathcal{T}$, in which case it makes sense to say that $\text{conv}\mathcal{T} = \mathcal{C}_n$. An order type is convex, if it equals its convex hull.

Ramsey theoretic aspects of order types have been studied by Nešetřil and Valtr in [17]. Order types play an important role in canonical versions of the Erdős–Szekeres theorem [8]. A connection was first established via the so called 'same type lemma' by Bárány and Valtr [4], see also [3] for a survey.

According to the Erdős–Szekeres theorem, there is an integer N_0 such that every order type $\mathcal{T}$ with $|\mathcal{T}| \geq N_0$ contains $\mathcal{C}_n$. Denoting the smallest such number by $F(n)$, it is known [9, 18] that

$$2^{n-2} + 1 \leq F(n) \leq \binom{2n-5}{n-2} + 1,$$

the lower bound conjectured to be tight. This is a truly Ramsey-type result whose relation to Ramsey's theorem is widely explored e.g. in [16]. Motivated by a conjecture of Erdős and Hajnal [7] in graph Ramsey theory, Gil Kalai [13] suggested the following problem. For a fixed non-convex order type $\mathcal{T}$, define $F_{\mathcal{T}}(n)$ as the smallest integer N_0 such that any order type of size at least N_0 that does not contain $\mathcal{T}$ necessarily contains $\mathcal{C}_n$. Note that $F_{\mathcal{T}}$ is an increasing function. Is it always true that $F_{\mathcal{T}}(n)$ is bounded above by a polynomial function of n? Somewhat surprisingly, the analogue with graph Ramsey theory breaks here. In [14] we have shown the existence of an order type $\mathcal{T}$ with $F_{\mathcal{T}}(n) > 2^{n-2}$, in contrast with the original Erdős–Hajnal problem where a sub-exponential upper bound is known [7]. For more on this problem, see [1, 5, 10, 11].

Our proof however was based on a general result of Nešetřil and Valtr [17] from which it is not easy to extract a concrete order type $\mathcal{T}$ with the above property. One novelty in the present paper is the exhibition of explicit order types $\mathcal{T}$ for which $F_{\mathcal{T}}(n)$ is exponentially large (Theorem 1). Such an order type of size 6 can be obtained, for example, by putting an extra point at the centre of a regular pentagon. Large order types containing neither this, nor $\mathcal{C}_n$ can be constructed by a doubling process we call 'twin construction', similar to the one found in [15]. We discuss these constructions in Section 2.

On the other hand, several families of order types satisfy the analogue of the Erdős–Hajnal conjecture, see [14]. In Section 3 we exhibit new families of order types $\mathcal{T}$ for which the function $F_{\mathcal{T}}$ has polynomial growth (Theorems 4 and 5).

Our final result (Theorem 8) in Section 4 concerns a complete characterization of order types whose convex hull is a triangle according to the behavior of the function $F_{\mathcal{T}}(n)$. They each fall in one of the following three categories:

(i) $F_{\mathcal{T}}(n)$ is bounded by a linear function in n;
(ii) $F_{\mathcal{T}}(n)$ is at least quadratic in n but bounded by a polynomial in n;
(iii) $F_{\mathcal{T}}(n)$ is exponentially large in n.

Part of this result originates in [14]. Besides that and the methods involved therein the most crucial element is a new construction that we obtain via modification of Horton's well-known example [12]. Somewhat surprisingly these 'neo-Horton' sets can be also obtained by the twin construction.

2. Explicite Constructions

There exist three different non-convex order types $\mathcal{T}$ of size less than 6. It was shown in [14], that for any of them $F_{\mathcal{T}}(n)$ is bounded from above by a polynomial function in n. In contrast, we have the following result.

Theorem 1. *The 6-element order types $\mathcal{T} = \mathcal{A}$ and $\mathcal{T} = \mathcal{P}$ depicted below satisfy $F_{\mathcal{T}}(n) > 2^{n/2-1}$.*

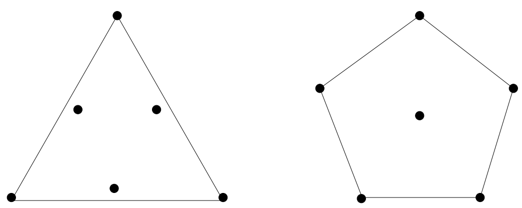

Figure 1. *Order types $\mathcal{A}$ and $\mathcal{P}$.*

A common feature of these order types is that they have the following *separation property*: any two of their points can be separated by a line determined by two other points of theirs.

To prove the theorem we first introduce the twin construction. Define sets T_k, $k \geq 0$ recursively. T_0 consists of one point. Suppose we already have defined T_{k-1}. Take a line ℓ which is not parallel to any line determined by the points of T_{k-1}. Replace each point $p \in T_{k-1}$ by two points, p', p'', both very close to p, such that the line $p'p''$ is parallel to ℓ. The points p' and p'' are called the *twins* of each other and p is the *parent* of them.

For a more formal and somewhat more restricted definition, choose k different unit vectors $v_1, \ldots, v_k$ such that no two of them add up to zero, and a small positive number ε. Given T_i, define T_{i+1} as $T_i \cup (T_i + \varepsilon^i v_{i+1})$. For all sufficiently small values of ε the 2^k-element set T_k thus constructed will belong to the same order type, which by a slight abuse of notation we denote by $\mathcal{T}_k = \mathcal{T}(v_1, \ldots, v_k)$. Note that different choice of unit vectors may yield the same order type. On the other hand, reordering of a given sequence of unit vectors usually results in a different order type. Observe that $\mathcal{T}_k$ does not have the separation property, since two twins cannot be separated by any line determined by other points.

Lemma 2. *For $k \geq 1$, no order type $\mathcal{T}_k$ contains $\mathcal{C}_{2k+1}$.*

Proof. We need to prove that T_k does not contain more than $2k$ points in convex position, which clearly holds for $k = 1$. Suppose it holds for $k - 1$ and let $p_1, p_2, \ldots, p_m \in T_k$ be a sequence of m points in convex position, in clockwise order. If p_i and p_j are twins of each other, then they are consecutive points. Therefore there can be at most two pairs of twins among $p_1, p_2, \ldots, p_m$. Replacing each point by its parent in T_{k-1} we find at least $m - 2$ points of T_{k-1} in convex position. By the induction hypothesis, $m-2 \leq 2k-2$. It follows that $m \leq 2k$. □

Since both $\mathcal{A}$ and $\mathcal{P}$ have the separation property, Theorem 1 is an immediate consequence of the following claim.

Lemma 3. *Suppose that the order type $\mathcal{S}$ has the separation property. Then $F_{\mathcal{S}}(2n + 1) > 2^n$.*

Proof. Since $|\mathcal{T}_k| = 2^k$, in view of the previous lemma it will be sufficient to show that T_k does not contain a subset whose order type is $\mathcal{S}$. We prove it by induction on k. It is obviously true for $k = 1$. Suppose that the statement holds for $k - 1$.

Assume that $\{p_1, p_2, \ldots p_m\} \subseteq T_k$ belongs to $\mathcal{S}$. Consider any two points p_i and p_j. Since they are separated by some line $p_u p_v$, they cannot be twins, so their parents $\bar{p}_i$ and $\bar{p}_j$ are different. The set of parents $\bar{p}_1, \bar{p}_2, \ldots \bar{p}_m$ thus form an m-element subset in T_{k-1} whose order type is again $\mathcal{S}$, which contradicts the induction hypothesis. This concludes the proof. □

In Section 4 we will make use of the following special case of the twin construction. For any sequence of vectors $v_1, \ldots, v_k$ whose tangents form a decreasing sequence of positive numbers, the order type $\mathcal{T}(v_1, \ldots, v_k)$ will be the same. We denote this order type by $\mathcal{RNH}_k$ and its mirror image by $\mathcal{LNH}_k$. The order type $\mathcal{RNH}_k$ resemble very much the order type $\mathcal{H}_k$ of Horton's famous construction H_k (see [12]). The difference lies therein, that $\mathcal{RNH}_k$ is obtained as the order type of $T_k = T(v_1, \ldots, v_k)$ using very small values of ε, whereas $\mathcal{H}_k$ can be obtained using vectors v_i of slope $\alpha_i = \pi/4^i$, and using $(2^i \cos \alpha_i)^{-1} \approx (1/2)^i$ instead of ε^i for very small ε. This order type is indeed different from $\mathcal{RNH}_k$, which can be easily seen from the following explicit construction.

For any nonnegative integer m let $m = \sum_{i=0}^{\infty} a_i 2^i$ ($a_i \in \{0, 1\}$) be its unique binary representation. Given that, define $\overline{m} = \sum_{i=0}^{\infty} a_i 2^{2^{2i}}$. Identifying the plane with $\mathbb{R}^2$, put $p_m = (m, \overline{m})$. The set $RNH_k = \{p_m \mid 0 \leq m < 2^k\}$ gives a concrete configuration whose order type is $\mathcal{RNH}_k$. Reflecting it in the second axis we obtain a configuration $LNH_k \in \mathcal{LNH}_k$. One can readily check that these 'neoHorton' sets have the following remarkable properties:

- (i) each set RNH_k is centrally symmetric;
- (ii) for any $k < n$, RNH_n is the disjoint union of 2^{n-k} translated copies $RNH_k(1), \ldots, RNH_k(2^{n-k})$ of $RNH_k = RNH_k(1)$ such that for every $i < j$, the whole set $RNH_k(j)$ lies above and to the right of $RNH_k(i)$;
- (iii) for $i < j$ and $x, y \in RNH_k(i)$, the whole set $RNH_k(j)$ lies above the line xy, thus every point of $RNH_k(j)$ sees the points of $RNH_k(i)$ so that if $m_1 < m_2$, then p_{m_1} precedes p_{m_2} in counterclockwise order;
- (iv) for $j < 2i-1 < 2i$, every point of $RNH_k(2i)$ sees the points of $RNH_k(j)$ later than the points of $RNH_k(2i-1)$.

These properties will be used for the proof of Lemma 9 in Section 4 without any partial reference.

3. Order Types with the Erdős–Hajnal Property

We say that the order type $\mathcal{T}$ has the Erdős–Hajnal property, if $F_{\mathcal{T}}$ is bounded from above by a polynomial function. Here we exhibit three families of order types with this property. The notations we apply here slightly deviate from those used in [14].

First, for any $k \geq 1$, consider a configuration $E = \{a, b, c, p_1, \ldots, p_k\}$ such that the points $p_1, \ldots, p_k$ lie inside the triangle abc and the points $b, p_1, \ldots, p_k, c$ are in convex position. E belongs to a unique order type we denote by $\mathcal{E}_k$. Thus, $F_{\mathcal{E}_1}(n) = n$. In general $F_{\mathcal{E}_k}$ is bounded form above by a linear funcion, see [14].

Next, for any $k \geq 3$, consider a configuration $F = \{a, b, c, p_1, \ldots, p_k\}$ such that the points $p_1, \ldots, p_k$ lie inside the triangle abc, the points $p_2, \ldots, p_{k-1}$ lie inside the convex quadrilateral bp_1p_kc, the points $p_1, \ldots, p_k$ are in convex position, and no line defined by two of them intersects the segment bc. The order type of F we denote by $\mathcal{F}_k$.

Finally, let $k \geq 4$, $l, m \geq 0$ be arbitrary integers. Two configurations X and Y are said to be *mutually avoiding* if any line determined by two points of X has all points of Y on the same side, and vice versa. Consider a configuration $G = \{p_1, \ldots, p_k, q_1, \ldots, q_l, r_1, \ldots, r_m\}$ with the following properties. The points $p_1, \ldots, p_k$ are in convex position, the points $q_1, \ldots, q_l, r_1, \ldots, r_m$ lie inside the convex polygon $p_1p_2 \ldots p_k$, the points $p_1, q_1, \ldots q_l, p_2$ are in convex position such that $Q = \{p_1, q_1, \ldots q_l, p_2\}$ and $G \setminus Q$ are mutually avoiding, and similarly, $p_3, r_1, \ldots r_m, p_4$ are in convex position such that $R = \{p_3, r_1, \ldots r_m, p_4\}$ and $G \setminus R$ are mutually avoiding. Depending on the orientation of the convex polygon $p_1p_2 \ldots p_k$, G belongs one of (at most) two different order types, which we denote by $\mathcal{G}_{k;l,m}$.

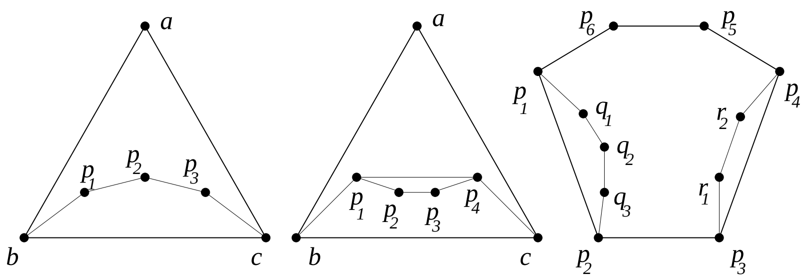

Figure 2. *Order types $\mathcal{E}_3$, $\mathcal{F}_4$, and $\mathcal{G}_{6;3,2}$.*

We will prove that all these order types have the Erdős–Hajnal property. With a small modification of an idea in [14] we have the following result.

Theorem 4. *Every order type $\mathcal{F}_k$ with $k \geq 3$ has the Erdős–Hajnal property.*

Theorem 4 in [14] asserts that the order types $\mathcal{G}_{k;l,0}$ also have this property. Here we claim the following more general result.

Theorem 5. *Every order type $\mathcal{G}_{k;l,m}$, where $k \geq 4$, $l, m \geq 0$ and not both l and m are zero, has the Erdős–Hajnal property.*

Both proofs utilize a result of Erdős and Szekeres concerning caps and cups. The points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$ with $x_1 < \ldots < x_n$ form an *n-cap* if

$$\frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_2}{x_3 - x_2} > \ldots > \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

Similarly, they form an *n-cup* if

$$\frac{y_2 - y_1}{x_2 - x_1} < \frac{y_3 - y_2}{x_3 - x_2} < \ldots < \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

Lemma 6. [8] *Let f(a,b) denote the smallest integer such that any set of $f(a,b)$ points in general position in the plane, no two on a vertical line, contains either an a-cap or a b-cup. Then*

$$f(a,b) = \binom{a+b-4}{a-2} + 1.$$

Proof of Theorem 4. We prove that for $\mathcal{T} = \mathcal{F}_k$, the function $F_{\mathcal{T}}$ is bounded from above by a polynomial of degree $3k-5$. Let $\mathcal{X}$ be any order type of cardinality $|\mathcal{X}| > n\binom{n+k-4}{k-2}^3$. Assuming that $\mathcal{C}_n \not\hookrightarrow \mathcal{X}$, we prove that $\mathcal{F}_k \hookrightarrow \mathcal{X}$. Let $X \in \mathcal{X}$, then conv$X$ has less than n vertices. Triangulating it we find $a, b, c \in X$ such that more than $\binom{n+k-4}{k-2}^3$ points of X lie inside triangle abc. Denote by P the set of these points. Define a partial ordering $\prec_{ab}$ on P as follows: For $p, q \in P$, let $p \prec_{ab} q$ if and only if the ray pq intersects side bc and the ray qp intersects side ac of the triangle. One can readily check that the relation $\prec_{ab}$ is indeed transitive. Partial orders $\prec_{ac}$ and $\prec_{bc}$ can be introduced in a similar manner. Note that any two points of P are related by exactly one of these three relations. Thus, a repeated application of Dilworth's theorem [6] gives that there is a subset P' of P of size $|P'| > \binom{n+k-4}{k-2}$, which is linearly ordered with respect to one of the three partial orders, say $\prec_{bc}$.

Consider a Cartesian system whose horizontal axis meets the rays ab and ac at equal angles and the points of P' lie in the upper half plane. The first coordinates of the elements of P' follow each other according to the linear order $\prec_{bc}$. Since X, and hence P' too, does not contain an n-cap, according to Lemma 6 it must contain a k-cup $p_1, \ldots, p_k$. It is clear that $\{a, b, c, p_1, \ldots, p_k\} \in \mathcal{F}_k$. Thus, $\mathcal{F}_k \hookrightarrow \mathcal{X}$ as claimed. $\square$

Given a family of sets $Y_1, Y_2, \ldots, Y_m$, a *transversal* of this family is an m-element set $\{y_1, y_2, \ldots, y_m\}$ such that $y_i \in Y_i$ for $i = 1, 2, \ldots, m$. One key to the proof of Theorem 5 is the following 'same type lemma' due to Bárány and Valtr.

Lemma 7. [4] *For every integer t there is a positive c_t with the following property. Assume that $X_1, X_2, \ldots, X_t$ are planar point sets such that $X_1 \cup X_2 \cup \cdots \cup X_t$ in general position. Then there are subsets $Y_i \subset X_i$ with $|Y_i| \geq c_t|X_i|$, such that all transversals of $Y_1, Y_2, \ldots, Y_t$ belong to the same order type.*

Proof of Theorem 5. Let $\mathcal{X}$ be an order type of cardinality $|\mathcal{X}| > c_0 n^{\alpha}$, which does not contain $\mathcal{C}_n$. We prove that if $c_0 = c_0(k, l, m)$ and $\alpha = \alpha(k, l, m)$ are sufficiently large, then $\mathcal{G}_{k;l,m} \hookrightarrow \mathcal{X}$. Let $X \in \mathcal{X}$, and assume that no two points of X lie on a vertical line. Choose a large enough integer $t = t(k, l, m)$ whose value will be specified later. According to a result of Aronov et al. [2], every configuration of N points contains two mutually avoiding subsets of size at least $\sqrt{N}/10$. By a repeated application, we can obtain pairwise mutually avoiding subsets $X_1, X_2, \ldots X_t$, such that $|X_i| > c_1 n^{\beta}$ holds for every $1 \leq i \leq t$ with $\beta > \alpha/2t$. Using Lemma 7, we can find subsets $X_i' \subset X_i$, $|X_i'| > c_2 n^{\beta}$ such that any transversal of $X_1', X_2', \ldots X_t'$ is of the same order type. In view of the Erdős–Szekeres theorem (Lemma 6), there is a sequence $i_1, i_2, \ldots i_s$ such that $s \geq \log_4 t$, and any transversal of $X_{i_1}', X_{i_2}', \ldots X_{i_s}'$ is in convex position. For simplicity, we denote X_{i_j}' by Y_j.

Consider now any ordered pair (Y_i, Y_j), $1 \leq i, j \leq s$. Define a binary relation on the points of Y_i. For $p, q \in Y_i$, let $p \prec q$ if and only if p has smaller x-coordinate than q, and all points of Y_j lie *above* the line pq. It is not hard to see that $\prec$ is a partial ordering. According to Dilworth's theorem, there is either a chain or an antichain of size $\sqrt{|Y_i|} > c_3 n^{\beta/2}$. Suppose that $C \subset Y_i$ is such a chain (resp. antichain). Then all points of Y_j are *above* (resp. *below*) every line determined by C. Delete all points of Y_i which are not in that chain (resp. antichain).

Proceed analogously for each ordered pair (Y_i, Y_j). Denote the resulting sets by $Z_i \subset Y_i$, $i = 1, \dots s$. Now we have the family $Z_1, Z_2, \dots Z_s$, such that any transversal of $Z_1, Z_2, \dots Z_s$ is in convex position, in this counterclockwise order, for any pair (Z_i, Z_j), Z_j is either above, or below *every* line determined by Z_i, and $|Z_i| > c_4 n^\gamma$ holds for every $1 \le i \le s$ with $\gamma = \beta/2^{t-1}$.

Define now a four-coloured complete graph on the vertex set $\{1, \dots, s\}$ as follows. For any $i < j$, we know that Z_j is either above, or below every line determined by Z_i, and Z_i is either above, or below every line determined by Z_j. So we have four possibilities for the pair (Z_i, Z_j), that determines the color of the edge ij. Call the correponding colours aa, ab, ba, and bb, respectively. By Ramsey's theorem, there is a complete monochromatic subgraph of size $r \ge \log_{256} s$. Suppose without loss of generality that its vertices are $1, \dots, r$.

Now we should distinguish four cases. Since reflection in the x-axis interchanges the "above" and "below" relations, is will be enough to consider two cases.

Case 1: All edges are coloured with colour aa.

Case 2: All edges are coloured with colour ab.

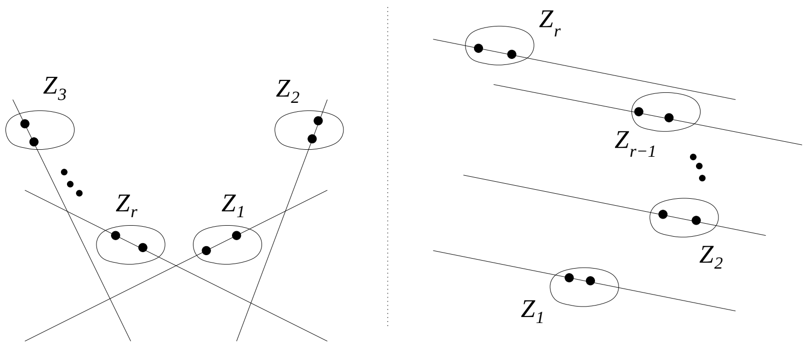

Figure 3. *Cases 1 and 2.*

Now we assume that t is big enough so that $r \ge k - 2$. We choose the value of c_0 and α so that

$$c_4 n^\gamma > \max\left\{\binom{n+l-2}{l}, \binom{n+m-2}{m}\right\}.$$

X does not contain n points in convex position, therefore Z_1 does not contain an n-cup. It follows from Lemma 6 that in either case Z_1 contains an $(m+2)$-cap $C_1 = \{p_3, r_1, \dots, r_m, p_4\}$. For $i = 2, \dots k-3$, choose a point $p_{i+3} \in Z_i$.

In Case 1, we use the fact that Z_{k-2} does not contain an n-cup, therefore it must contain an $(l+2)$-cap $C_{k-2} = \{p_1, q_1, \dots, q_l, p_2\}$. In Case 2, we use the fact that Z_{k-2} does not contain an n-cap, therefore it must contain an $(l+2)$-cup $C_{k-2} = \{p_2, q_l, \dots, q_1, p_1\}$.

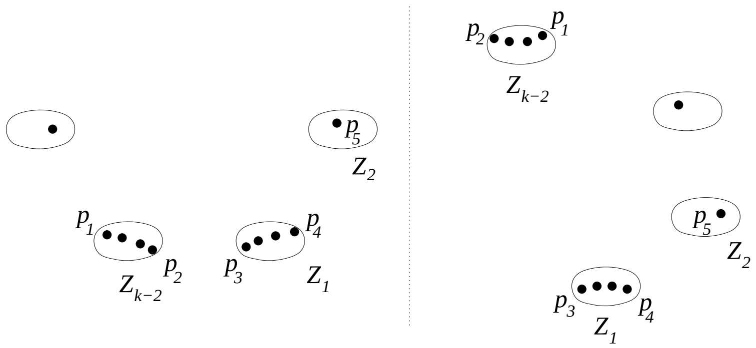

Figure 4. *Finding* $\mathcal{G}_{k;l,m}$.

In either case, the set $C_{k-2} \cup C_1 \cup \{p_5, \dots, p_k\}$ is a configuration whose order type is $\mathcal{G}_{k;l,m}$. It is proved that $\mathcal{G}_{k;l,m} \hookrightarrow \mathcal{X}$. □

4. Order Types with Triangular Convex Hull

The following result describes the growth of the function $F_{\mathcal{T}}(n)$ for order types $\mathcal{T}$ whose convex hull has three vertices.

Theorem 8. *Let* $\mathcal{T} \neq \mathcal{C}_3$ *be an order type whose convex hull is* $\mathcal{C}_3$.

(i) *If* $\mathcal{T} = \mathcal{E}_k$ *for some integer* $k \geq 1$, *then* $F_{\mathcal{T}}(n)$ *is bounded by a linear function in* n.

(ii) *If* $\mathcal{T} = \mathcal{F}_k$ *for some integer* $k \geq 3$, *then* $F_{\mathcal{T}}(n)$ *is at least quadratic in* n *but bounded by a polynomial in* n.

(iii) *If* $\mathcal{T} \neq \mathcal{E}_k, \mathcal{F}_k$, *then* $F_{\mathcal{T}}(n)$ *is exponentially large in* n.

Proof. Part (i) and the lower bound in (ii) is contained in Theorems 3 and 6, respectively, of the earlier paper [14]. The upper bound in (ii) can be obtained with a slight modification of Theorem 4 therein, which we described in the previous section. The key to the third satatement is the following lemma, whose proof we postpone until the end of this section.

Lemma 9. *Let n be a positive integer and let $\mathcal{T}$ be an order type of a configuration of 3 points contained in the convex hull of other 3 points. If $\mathcal{T}$ is contained in both $\mathcal{LNH}_n$ and $\mathcal{RNH}_n$, then $\mathcal{T} = \mathcal{E}_3$, or $\mathcal{T} = \mathcal{F}_3$.*

Let T be a configuration of at least 4 points such that its convex hull is a triangle abc, and the relations $T \hookrightarrow LNH_n$, $T \hookrightarrow RNH_n$ hold for some positive integer n. If $|T| \leq 5$, then $T \in \mathcal{E}_1 = \mathcal{F}_1$, or $T \in \mathcal{E}_2 = \mathcal{F}_2$. If $6 \leq |T| = k+3$, then Lemma 9 implies that $S \in \mathcal{E}_3 \cup \mathcal{F}_3$ holds for every 6-element configuration $S \subseteq T$. It follows that line pq intersects the same two sides, say ac and bc, of triangle abc for each pair of two different points $p, q \in T \setminus \{a,b,c\}$. That is, the elements of $T \setminus \{a,b,c\}$ can be ordered as $p_1, \ldots, p_k$ so that the rays p_ip_j and p_jp_i intersect sides bc and ac, respectively, for every $1 \leq i < j \leq k$. Assume that $T_i = \{a,b,c,p_i,p_{i+1},p_{i+2}\} \in \mathcal{F}_3$ and $T_{i+1} = \{a,b,c,p_{i+1},p_{i+2},p_{i+3}\} \in \mathcal{E}_3$ for some $1 \leq i \leq k-3$. Then points $p_{i+1}, p_{i+2}, p_{i+3}$ lie inside triangle p_ibc so that line $p_{i+1}p_{i+2}$ intersects sides p_ib and bc, whereas line $p_{i+2}p_{i+3}$ intersects sides p_ic and bc of the triangle (Fig. 1), a contradiction. By symmetry, it is not possible that $T_i \in \mathcal{E}_3$ and $T_{i+1} \in \mathcal{F}_3$. Therefore T_i must belong to the same order type, either $\mathcal{E}_3$ or $\mathcal{F}_3$, for every $1 \leq i \leq k-3$. Accordingly, $T \in \mathcal{E}_k$ or $T \in \mathcal{F}_k$.

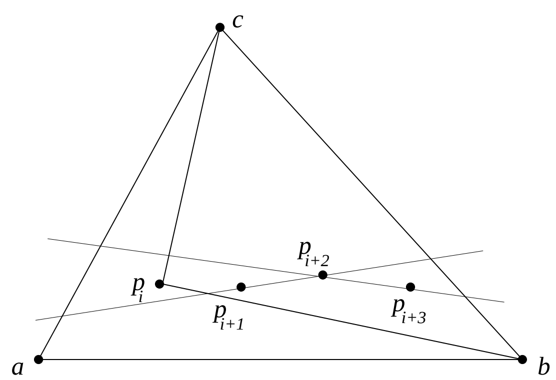

Figure 5. $S = \{b, c, p_i, p_{i+1}, p_{i+2}, p_{i+3}\} \notin \mathcal{E}_3 \cup \mathcal{F}_3$.

Thus we proved that if $\mathcal{T} \neq \mathcal{E}_k, \mathcal{F}_k$, then either $\mathcal{LNH}_n$ or $\mathcal{RNH}_n$ does not contain $\mathcal{T}$ for every integer $n \geq 1$. From Lemma 2 it follows that $f_{\mathcal{T}}(2n+1) > 2^n$ and $f_{\mathcal{T}}$ has exponential growth as claimed. □

It remains to prove Lemma 9. If $\mathcal{T} \neq \mathcal{E}_3, \mathcal{F}_3$, then $\mathcal{T}$ is either one of the four order types depicted on Fig. 6, or one of the mirror images $\mathcal{C}^\top$, $\mathcal{D}^\top$.

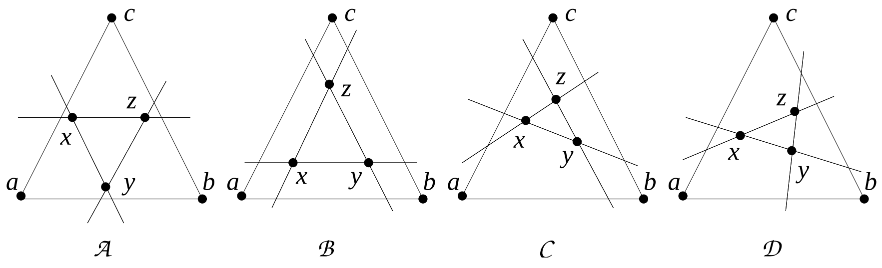

Figure 6. *Order types of six points.*

We must prove that neither of these six order types is contained in both $\mathcal{LNH}_n$ and $\mathcal{RNH}_n$. Since $\mathcal{A}$ has the separation property, it is not conteined in any twin construction. Therefore neither $\mathcal{LNH}_n$ nor $\mathcal{RNH}_n$ does contain $\mathcal{A}$. Next we prove that $\mathcal{B}$ is not contained in $\mathcal{RNH}_n$. Assume that on the contrary, a configuration $\{a, b, c, x, y, z\} \in \mathcal{B}$ is contained in RNH_n. Condider the smallest k such that $\{x, y, z\}$ is contained in $RNH_k(i)$ for some $1 \leq i \leq 2^{n-k}$. Both $RNH_{k-1}(2i-1)$ and $RNH_{k-1}(2i)$ must contain at least one of x, y, z. Because of the threefold symmetry of $\mathcal{B}$ and the central symmetry of RNH_n, without any loss of generality we may suppose that $x, y \in RNH_{k-1}(2i-1)$ and $z \in RNH_{k-1}(2i)$. Note that z is inside triangle xyc. Now $c \in RNH_{k-1}(j)$ for some $1 \leq j \leq 2^{n-k+1}$. Here $j \geq 2i$, for otherwise any vertical line that separates $RNH_{k-1}(2i-1)$ and $RNH_{k-1}(2i)$ would separate $\{x, y, c\}$ from z. It is equally impossible that $j = 2i$, since in that case both x and y would lie below the line cz, so cz would not separate x and y. Finally, were $j > 2i$, both x and y would lie left to the line cz, again a contradiction.

To see that $\mathcal{C}$ is not contained in $\mathcal{LNH}_n$, we assume that a configuration $\{a, b, c, x, y, z\} \in \mathcal{C}$ is contained in LNH_n and that k is the smallest integer such that $\{x, y, z\} \subset LNH_k(i)$ for some $1 \leq i \leq 2^{n-k}$. Having lost the threefold symmetry, we must distinguish three subcases.

Case 1: $z \in LNH_{k-1}(2i-1)$ and $x, y \in LNH_{k-1}(2i)$. For z is inside triangle xyc, mirroring the previous argument we find that c cannot be in any subset $LNH_{k-1}(j)$.

Case 2: $y \in LNH_{k-1}(2i-1)$ and $x, z \in LNH_{k-1}(2i)$. Now we use the fact that y is inside triangle zxb. We arrive at a contradiction as before: there is no place for the point b.

Case 3: $x \in LNH_{k-1}(2i-1)$ and $y, z \in LNH_{k-1}(2i)$. Because of the orientation of triangle xyz, the points x, y, z follow each other from left to right in this order. Since the orientation of both triangles yzb and yzc is clockwise, both b and c must lie under any horizontal line ℓ that separates $LNH_{k-1}(2i-1)$ and $LNH_{k-1}(2i)$. Point x sees y, z and c in this order, therefore c must lie in $LNH_{k-1}(2i)$. For triangle abc to contain z, point a must lie above ℓ. But then line ax cannot separate z and c, a contradiction.

Thus we have proved that $\mathcal{C}$ is indeed not contained in $\mathcal{LNH}_n$. By symmetry, $\mathcal{C}^\top$ is not contained in $\mathcal{RNH}_n$. A similar argument demonstrating that $\mathcal{RNH}_n$ does not contain $\mathcal{D}$ completes the proof. We omit the technical details.

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