Recursive Algorithms for Identification in Closed Loop: A Unified Approach and Evaluation*

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A unified framework for recursive plant model identification in closed loop is provided. A comparative evaluation of various algorithms on the basis of theoretical analysis and experimental results is presented.

Key Words—Identification algorithms; closed-loop identification; validation; robustness; stability; convergence.

Abstract—A unified presentation of recursive algorithms for plant model identification in closed loop is given. From the basic formulation of the problem of finding the plant model that gives the best predictor for the closed-loop system, two families of algorithms using either a reparameterized predictor for the closed loop or a plant predictor operating on filtered data are presented, and their asymptotic properties are examined. Validation tests for the models identified in closed loop are proposed. A comparative evaluation of the various algorithms in simulation and on real-time experiments concludes the paper. © 1997 Elsevier Science Ltd.

ABBREVIATIONS

CLOE	closed-loop output-error method;
F-CLOE	filtered closed-loop output-error method;
AF-CLOE	adaptive filtered closed-loop output-error method;
X-CLOE	extended closed-loop output-error method;
G-CLOE	generalized closed-loop output-error method;
FOL	filtered open-loop identification method.

1. INTRODUCTION

In practice there are many situations in which identification in open loop is difficult or simply not feasible. This includes for example the case of plants having an integrator behaviour or being

unstable in open loop, as well as the case of plants subject to significant drift in open-loop operation. Identification in closed loop is also used when a controller already exists on the system and a validation of the current tuning or a redesign of the controller is considered.

The practical importance of plant model identification in closed loop has been recognized for many years, and a number of methods have been developed and analyzed (Gustavsson *et al.*, 1977; Ljung, 1987; Söderström and Stoica, 1989). In recent years a revival of the interest in plant model identification occurred in the context of the iterative combination of identification in closed loop and control redesign (Gevers, 1993; van den Hof and Schrama, 1998). Figure 1 illustrates the basis of this iterative procedure (for the case of an RST digital controller). The upper part represents the true closed-loop system and the lower part represents the design system. The objective is to minimize the error between the two systems by using new data acquired in closed-loop operation. Since the problem is not directly tractable, the idea is first to improve the plant model estimation and then to redesign the controller based on the new model. This sequence of operations is carried out one or several times. However, a key point is that the new plant model estimation should be done in order to reduce the error between the two systems. In fact, in this context the objective of the plant model identification in closed loop is to get a better predictor for the closed loop via a better estimation of the plant model. While this idea in an embryonic form was presented in Ljung (1987, p. 393, Problem 14T.2) and theoretical tools for its development were available, it has not been explored in detail until recently, particularly in the area of recursive identification, algorithms.

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Fig. 1. Closed-loop identification scheme.

The objective of this paper is to give a unified presentation and a comparative evaluation of a number of recursive identification algorithms for plant model identification in closed loop that were developed starting from the point of view mentioned above.

The problem of plant model identification in closed loop as discussed above can be viewed in two different ways, leading, however to similar types of algorithms.

- *MRAS point of view.* The true closed-loop system corresponds to a reference model, and a parallel adjustable system having a feedback configuration is built up. This adjustable feedback system contains a fixed controller and an adjustable model of the plant. The problem is to design a parameter adaptation algorithm assuring the global asymptotic stability of the closed-loop prediction error (or driving the closed-loop prediction error to zero asymptotically in the absence of noise). This is a dual problem with respect to the classical model reference adaptive control problem.
- *Identification point of view.* Construct a reparameterized adjustable predictor for the closed loop system in terms of a known fixed controller and of an adjustable plant model.

In the context of recursive algorithms the problem of plant model identification in closed loop can be formulated as follows. Under the assumptions that

- the controller is constant,
- the input-output part of the plant to be identified belongs to the model set, and
- an external excitation is applied to the closed-loop system;

identify a plant model such that

- global asymptotic stability is assured for any initial parameter estimates and initial error between the output of the true system and that of the closed loop predictor (in the absence of noise);
- an asymptotically optimal predictor of the true closed-loop system is obtained for specific noise disturbances;
- under appropriate richness conditions, asymptotically unbiased estimation of the plant model parameters is obtained (in the presence of noise).

Furthermore, for the case of plant-model order mismatch, bounded prediction errors and bounded parameters error should be assured.

The external excitation can either be applied on the reference (as in Fig. 1) or added to the output of the controller. This does not change the structure of the algorithm; it just affects the external quantity added to the input of the plant (or of the estimated plant model).

From this basic scheme (Fig. 1), two families of recursive algorithms for plant model identification in closed loop can be derived. The first approach starts from the rewriting of the closed-loop system output equation in terms of the regressor of the predictor (i.e. in terms of $\hat{u}(t)$ and $\hat{y}(t)$) and the previous values of the closed-loop prediction error (see Fig. 1). This approach generates the family of 'closed-loop output-error recursive identification algorithms' (Landau and Karimi, 1996, 1997), where the parameters of the estimated model in Fig. 1 are driven by a parameter adaptation algorithm using the closed-loop prediction error ϵ_{CL} as adaptation error.

The second approach starts from the rewriting of the predictor output equation in terms of the measurement vector (i.e. in terms of $u(t)$ and $y(t)$) and the previous values of the closed-loop prediction error. This approach leads to an expression for the closed-loop prediction error as a filtered least-square-type prediction error for plant model identification from input-output measurements (Åström, 1993). This allows one to use many of the open-loop recursive identification algorithms with an appropriate filtering of the data. This family will be called 'filtered open-loop recursive identification algorithms'. Note that this is the type of method that has been used in indirect adaptive control for plant model parameter estimation.

With respect to the various approaches for identification in closed loop mentioned in Söderström and Stoica (1989), the algorithms considered can be viewed as a variation of the 'direct approach' using a special kind of instrumental variable generated on line either in the predictor itself or by appropriate filtering of the data. They can also be viewed as a modification of the 'indirect method' in which the plant model and the closed loop are simultaneously identified.

Once an identification procedure has been carried out using one or several algorithms, the question arises as to what is the best model. The answer may be given by the comparative validation of the identified models in closed-loop operation. The best plant model will be the one that allows the best prediction of the closed-loop behaviour to be obtained. As will be shown, coherent statistical validation tests can be defined and further enhanced by a direct comparison of the computed closed-loop poles and the achieved ones (which are obtained by identifying the closed-loop system).

The paper is organized as follows. In Section 2 the basic equations associated with the identification system will be given. Section 3 will briefly present the family of the closed-loop output-error recursive identification algorithms and their asymptotic properties. Section 4 will present a robustness analysis of the various algorithms in the presence of plant model order mismatch. Section 5 will briefly present the family of filtered open-loop recursive identification algorithms and their asymptotic properties. The bias distributions of the estimates for different algorithms are presented in Section 6. Section 7 will discuss the validation of a plant model identified in closed loop. Comparative evaluation of the various algorithms on simulated examples and on a real system (a very flexible transmission system) will be presented in

Section 8, and hints for practical use of the algorithms will be given in Section 9.

2. THE BASIC EQUATIONS

The objective is to estimate the parameters of the plant model defined by the transfer operator

$$G(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})}, \quad (1)$$

where

$$\begin{aligned} B(q^{-1}) &= b_1 q^{-1} + \dots + b_{n_B} q^{-n_B} \\ &= q^{-1} B^*(q^{-1}), \end{aligned} \quad (2)$$

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \dots + a_{n_A} q^{-n_A} \\ &= 1 + q^{-1} A^*(q^{-1}). \end{aligned} \quad (3)$$

The plant is operated in closed loop with an RST digital controller (without loss of generality) (Landau, 1990). The output of the plant operating in closed loop is given by (see Fig. 1)

$$\begin{aligned} y(t+1) &= -A^*y(t) + B^*u(t-d) \\ &\quad + Aw(t+1) = \theta^T \varphi(t) + Aw(t+1), \end{aligned} \quad (4)$$

where $u(t)$ is the plant input, $y(t)$ is the plant output, $w(t)$ is the output disturbance noise and

$$\theta^T = [a_1 \dots a_{n_A} \ b_1 \dots b_{n_B}], \quad (5)$$

$$\begin{aligned} \varphi^T(t) &= [-y(t) \dots -y(t-n_A+1) \\ &\quad u(t-d) \dots u(t-n_B+1-d)], \end{aligned} \quad (6)$$

$$u(t) = -\frac{R}{S}y(t) + r_u(t), \quad (7)$$

where $r_u(t)$ is the external excitation added to the output of the controller ($r_u(t) = (T/S)r(t)$ if the external excitation is applied on the reference).

For a fixed value of the estimated parameters, the predictor of the closed loop (the design system) is described by

$$\hat{y}(t+1) = -\hat{A}^*\hat{y}(t) + \hat{B}^*\hat{u}(t-d) = \hat{\theta}^T \phi(t), \quad (8)$$

where

$$\hat{\theta}^T = [\hat{a}_1 \dots \hat{a}_{n_A} \ \hat{b}_1 \dots \hat{b}_{n_B}], \quad (9)$$

$$\begin{aligned} \phi^T(t) &= [-\hat{y}(t) \dots -\hat{y}(t-n_A+1) \\ &\quad \hat{u}(t-d) \dots \hat{u}(t-n_B+1-d)], \end{aligned} \quad (10)$$

$$\hat{u}(t) = -\frac{R}{S}\hat{y}(t) + r_u(t). \quad (11)$$

The closed-loop prediction (output) error is defined as

$$\varepsilon_{CL}(t+1) = y(t+1) - \hat{y}(t+1). \quad (12)$$

Note that, as clearly results from Fig. 1, for

constant values of the estimated parameters the predictor regressor vector $\phi(t)$ depends only upon the external excitation. Therefore, under the assumption that the external excitation ($r(t)$ or $r_u(t)$) and the stochastic disturbance $w(t)$ are independent, $\phi(t)$ and $w(t)$ are not correlated. The scheme has the structure of an output error prediction. If, furthermore, a model for the noise $w(t)$ is assumed then a predictor can be built such that, under exact knowledge of the input–output plant model and of the noise model, the closed-loop prediction errors tends asymptotically towards white noise.

For all the methods, the parameter adaptation algorithm (PAA) has the general form

$$\hat{\Theta}(t+1) = \hat{\Theta}(t) + F(t)\Phi(t)v(t+1), \quad (13)$$

$$F^{-1}(t+1) = \lambda_1(t)F^{-1}(t) + \lambda_2(t)\Phi(t)\Phi^T(t), \quad (14)$$

$$0 < \lambda_1(t) \leq 1, \quad 0 \leq \lambda_2(t) < 2, \quad F(0) > 0,$$

$$F^{-1}(t) > \alpha F^{-1}(0), \quad 0 < \alpha < \infty,$$

$$F(t+1) = \frac{1}{\lambda_1(t)} \times \left[F(t) - \frac{F(t)\Phi(t)\Phi^T(t)F(t)}{\lambda_1(t)/\lambda_2(t) + \Phi^T(t)F(t)\Phi(t)} \right] \quad (15)$$

$$v(t+1) = \frac{v^o(t+1)}{1 + \Phi^T(t)F(t)\Phi(t)}, \quad (16)$$

where $v^o(t+1) = f_1(\hat{\Theta}(t), \hat{\Theta}(t-1), \dots, y(t+1), v(t), v(t-1), \dots)$ is the a priori adaptation error, $v(t+1) = f_2(\hat{\Theta}(t+1), \hat{\Theta}(t), \dots, y(t+1), v(t), v(t-1), \dots)$ is the a posteriori adaptation error and $\Phi(t)$ is the observation vector.

For each recursive identification algorithm Θ , Φ , $v^o(t+1)$ will have a specific expression. Note that the sequences $\lambda_1(t)$ and $\lambda_2(t)$ allow one to have different laws for the time evolution of the adaptation gain $F(t)$ (Landau, 1990). For convergence analysis in the stochastic environment, it is assumed that a PAA with decreasing adaptation gain is used (i.e. $\lambda_1(t) \equiv 1$, $\lambda_2(t) > 0$ or $\lambda_2(t) = \lambda_2 > 0$).

3. CLOSED-LOOP OUTPUT-ERROR ALGORITHMS

The key observation is that the output of the closed loop system given by (4) can be expressed as

$$\begin{aligned} y(t+1) &= \theta^T \varphi(t) + Aw(t+1) \\ &= \theta^T \phi(t) - \left(A^* + \frac{q^{-d}B^*R}{S} \right) \varepsilon_{CL}(t) \\ &\quad + Aw(t+1), \end{aligned} \quad (17)$$

and that

$$1 + q^{-1} \left(A^* + \frac{q^{-d}B^*R}{S} \right) = \frac{AS + q^{-d}BR}{S} = \frac{P}{S}, \quad (18)$$

where $P = AS + q^{-d}BR$ defines the poles of the true closed-loop system.

Subtracting (8) from (17), one obtains in the deterministic case ($w(t+1) \equiv 0$):

$$\begin{aligned} \varepsilon_{CL}(t+1) &= (\theta - \hat{\theta})^T \phi(t) \\ &\quad - \left(A^* + \frac{q^{-d}B^*R}{S} \right) \varepsilon_{CL}(t). \end{aligned} \quad (19)$$

Taking in account the relationship (18), (19) can be rewritten as

$$\varepsilon_{CL}(t+1) = \frac{S}{P} (\theta - \hat{\theta})^T \phi(t). \quad (20)$$

3.1. The CLOE algorithm

Now replacing the fixed predictor of the closed loop (8) by an adjustable predictor, i.e., a priori,

$$\hat{y}^o(t+1) = \hat{y}([t+1 | \hat{\theta}(t)]) = \hat{\theta}^T(t)\phi(t), \quad (21)$$

a posteriori,

$$\hat{y}(t+1) = \hat{y}[t+1 | \hat{\theta}(t+1)] = \hat{\theta}^T(t+1)\phi(t), \quad (22)$$

and defining the a priori prediction error as

$$\varepsilon_{CL}^o(t+1) = y(t+1) - \hat{y}^o(t+1) \quad (23)$$

and the a posteriori prediction error as

$$\varepsilon_{CL}(t+1) = y(t+1) - \hat{y}(t+1), \quad (24)$$

the equation for the a posteriori prediction error becomes

$$\varepsilon_{CL}(t+1) = \frac{S}{P} [\theta - \hat{\theta}(t+1)]^T \phi(t). \quad (25)$$

Theorem A.1 (Appendix A) suggests a PAA with

$$\hat{\Theta}(t) = \hat{\theta}(t),$$

$$\Phi(t) = \phi(t),$$

$$v^o(t+1) = \varepsilon_{CL}^o(t+1).$$

This is termed the CLOE *closed-loop output error* algorithm (Landau and Karimi, 1997). Using the results of Theorem A.1, it can be shown that in a deterministic environment the sufficient condition for $\lim_{t \rightarrow \infty} \varepsilon_{CL}(t+1) = 0$ together with the boundedness of $\varepsilon_{CL}(t+1)$ for any initial condition is that

$$H'(z^{-1}) = \frac{S(z^{-1})}{P(z^{-1})} - \frac{\lambda_2}{2} \quad (26)$$

is strictly positive real (where $\max_t \lambda_2(t) \leq \lambda_2 < 2$). Since $\varepsilon_{CL}(t)$ is bounded, this allows one to

conclude that $\hat{y}(t)$ and $\hat{u}(t)$ are bounded (under the hypothesis that S is stable) and therefore also $\lim_{t \rightarrow \infty} \varepsilon_{CL}(t+1) = 0$.

In a stochastic environment using the ODE approach (Ljung, 1977; Dugard and Landau, 1980), the equation of the prediction error for a fixed value of estimated parameter $\hat{\theta}$ takes the form

$$\varepsilon_{CL}(t+1, \hat{\theta}) = \frac{S}{P}(\theta - \hat{\theta})^T \phi(t, \hat{\theta}) + \frac{AS}{P}w(t+1). \quad (27)$$

Under the hypothesis that $w(t+1)$ is independent with respect to the external excitation $r_u(t)$, $\phi(t, \hat{\theta})$ and $w(t+1)$ are independent; therefore Theorem B.1 (Appendix B) can be used (for $\lambda_1(t) \equiv 1$, $\lambda_2(t) \equiv \lambda_2$), and the same positive-real condition of (26) assures asymptotic unbiased estimates under the richness condition (Landau and Karimi, 1997).

3.2. The F-CLOE and AF-CLOE algorithms

However, (20) can also be rewritten as

$$\begin{aligned} \varepsilon_{CL}(t+1) &= \frac{S}{P} \frac{\hat{P}}{S} (\theta - \hat{\theta})^T \frac{S}{\hat{P}} \phi(t) \\ &= \frac{\hat{P}}{P} (\theta - \hat{\theta})^T \phi(t), \end{aligned} \quad (28)$$

where

$$\phi_t(t) = \frac{S}{\hat{P}} \phi(t), \quad (29)$$

$$\hat{P} = \hat{A}S + q^{-d}\hat{B}R. \quad (30)$$

In (30) $\hat{P}$ is an estimation of the true closed-loop poles based on an initial estimation of the plant model (for example using an open-loop experiment). This formulation leads to the *filtered closed-loop output-error algorithm* (F-CLOE) (Landau and Karimi, 1997) which uses the same adjustable predictor as CLOE (see (21) and (22) and the PAA with

$$\begin{aligned} \hat{\Theta}(t) &= \hat{\theta}(t), \\ \Phi(t) &= \phi_t(t), \\ v^o(t+1) &= \varepsilon_{CL}^o(t+1). \end{aligned}$$

It can be shown that neglecting the non-commutativity of time-varying operators (an exact algorithm can, however, be derived), under the sufficient condition that

$$H'(z^{-1}) = \frac{\hat{P}(z^{-1})}{P(z^{-1})} - \frac{\lambda_2}{2} \quad (31)$$

is strictly positive-real, both asymptotic stability in a deterministic environment and asymptotic

unbiasedness in a stochastic environment is assured (Landau and Karimi, 1997).

If in the F-CLOE algorithm one replaces in the data filter $\hat{P}(q^{-1})$ by $\hat{P}(q^{-1}, t)$ (the current estimate of closed loop polynomial), the *adaptive filtered closed-loop output-error* AF-CLOE identification algorithm is obtained. But for this method only local convergence results can be derived and a stability test should be added.

3.3. The X-CLOE algorithm

For the case $w(t+1) = (C/A)e(t+1)$, where $e(t+1)$ is a zero-mean Gaussian white noise and $C(q^{-1}) = 1 + q^{-1}C^*(q^{-1})$ is an asymptotically stable polynomial, an extended output-error prediction model can be defined:

$$\begin{aligned} \hat{y}(t+1) &= -\hat{A}^* \hat{y}(t) + \hat{B}^* \hat{u}(t-d) + \hat{H}^* \frac{\varepsilon_{CL}(t)}{S} \\ &= \hat{\theta}^T \phi(t) + \hat{H}^* \frac{\varepsilon_{CL}(t)}{S} = \hat{\theta}_e^T \phi_e(t). \end{aligned} \quad (32)$$

Equation (17) for the plant output becomes in this case

$$\begin{aligned} y(t+1) &= \theta^T \phi(t) + H^* \frac{\varepsilon_{CL}(t)}{S} \\ &\quad - C^* \varepsilon_{CL}(t) + Ce(t+1), \end{aligned} \quad (33)$$

where

$$\begin{aligned} H^* &= h_1 + h_2 q^{-1} + \dots + h_{n_H} q^{-n_H+1} \\ &= C^*S - A^*S - q^{-d}B^*R, \end{aligned} \quad (34)$$

$$H = 1 + q^{-1}H^* = 1 + CS - P, \quad (35)$$

and, subtracting (32) from (33), one obtains the following expression for the closed-loop prediction error (for details see Landau and Karimi, 1996):

$$\varepsilon_{CL}(t+1) = \frac{1}{C}(\theta_e - \hat{\theta}_e)^T \phi_e(t) + e(t+1), \quad (36)$$

where

$$\theta_e^T = [\theta^T \quad h_1 \quad \dots \quad h_{n_H}], \quad (37)$$

$$\hat{\theta}_e^T = [\hat{\theta}^T \quad \hat{h}_1 \quad \dots \quad \hat{h}_{n_H}], \quad (38)$$

$$\phi_e^T(t) = [\phi^T(t) \quad \varepsilon_{CL}(t) \quad \dots \quad \varepsilon_{CL}(t-n_H+1)], \quad (39)$$

$$\varepsilon_{CL}(t) = \frac{1}{S} \varepsilon_{CL}(t). \quad (40)$$

Equation (36) clearly shows that for $\hat{\theta}_e = \theta_e$ the closed-loop prediction error tends asymptotically towards $e(t+1)$.

Replacing the fixed predictor (32) by an adjustable one, a recursive identification algo-

rithm (X-CLOE) can be obtained by using a PAA with

$$\begin{aligned}\hat{\Theta}(t) &= \hat{\theta}_e(t), \\ \Phi(t) &= \phi_e(t), \\ v^\circ(t+1) &= \varepsilon_{\text{CL}}^\circ(t+1) \\ &= y(t+1) - \hat{\theta}_e^T(t)\phi_e(t).\end{aligned}$$

The analysis in the deterministic case using Theorem A.1 shows that global asymptotic stability is assured without any positive real condition (since the a posteriori closed-loop prediction error equation in this case is $\varepsilon_{\text{CL}} = [\theta_e - \hat{\theta}_e(t+1)]^T \phi_e(t)$). In the stochastic environment the equation of the a posteriori closed-loop prediction error takes the form

$$\varepsilon_{\text{CL}}(t+1) = \frac{1}{C}(\theta_e - \hat{\theta}_e)^T \phi_e(t) + e(t+1). \quad (41)$$

Asymptotically unbiased estimates in a stochastic environment can be obtained under the sufficient condition (Landau and Karimi, 1996)

$$H'(z^{-1}) = \frac{1}{C(z^{-1})} - \frac{\lambda_2}{2} \quad (42)$$

is strictly positive-real (where $\max \lambda_2(t) \leq \lambda_2 < 2$) by using either the ODE approach (Theorem B.1) or the martingale convergence theorem (Theorem C.1 in Appendix C), together with Lemma C.1.

3.4. The G-CLOE algorithm

For the case of narrow-band disturbances it is interesting to consider a disturbance model of the form

$$w(t+1) = \frac{C}{DA} e(t+1),$$

where $e(t+1)$ is a zero-mean white noise, $C(q^{-1}) = 1 + q^{-1}C^*(q^{-1})$ and

$$\begin{aligned}D(q^{-1}) &= 1 + q^{-1}D^*(q^{-1}) \\ &= 1 + d_1 q^{-1} + \dots + d_{n_D} q^{-n_D+1}.\end{aligned}$$

$C(q^{-1})$ and $D(q^{-1})$ are asymptotically stable polynomials.

The same procedure as for X-CLOE can be used for deriving a recursive parameter estimation algorithm. The output of the plant operating in closed loop can be expressed as

$$y(t+1) = -A^*y(t) + B^*u(t-d) + \frac{C}{D}e(t+1) \quad (43)$$

$$\begin{aligned}y(t+1) &= -A^*\hat{y}(t) + B^*\hat{u}(t-d) - D^*\alpha(t) \\ &\quad + H^*\frac{\varepsilon_{\text{CL}}(t)}{S} + Ce(t+1) - C^*\varepsilon_{\text{CL}}(t) \\ &\quad + q^{-1}D^*[-(A^* - \hat{A}^*)\hat{y}(t) \\ &\quad + (B^* - \hat{B}^*)\hat{u}(t-d)],\end{aligned} \quad (44)$$

where

$$\alpha(t) = \hat{A}^*\hat{y}(t) - \hat{B}^*\hat{u}(t-d), \quad (45)$$

$$H^* = C^*S - A^*S - q^{-d}B^*R - D^*AS - D^*q^{-d}BR, \quad (46)$$

$$H = 1 + q^{-1}H^* = 1 + CS - DP, \quad (47)$$

and therefore the asymptotically optimal predictor will take the form

$$\begin{aligned}\hat{y}(t+1) &= -\hat{A}^*\hat{y}(t) + \hat{B}^*\hat{u}(t-d) \\ &\quad - \hat{D}^*\alpha(t) + \hat{H}^*\frac{\varepsilon_{\text{CL}}(t)}{S}.\end{aligned} \quad (48)$$

Subtracting (48) from (44), one obtains the following expression for the closed-loop prediction error:

$$\begin{aligned}\varepsilon_{\text{CL}}(t+1) &= \frac{D}{C}(\theta - \hat{\theta})^T \phi(t) + \frac{1}{C}(H^* - \hat{H}^*)\frac{\varepsilon_{\text{CL}}(t)}{S} \\ &\quad - \frac{1}{C}(D^* - \hat{D}^*)\alpha(t) + e(t+1).\end{aligned} \quad (49)$$

Equation (49) clearly shows that for $\hat{\theta} = \theta$, $\hat{H}^* = H^*$ and $\hat{D}^* = D^*$ the closed-loop prediction error is equal to $e(t+1)$.

In the deterministic case, using an adjustable predictor instead of (48), the a posteriori equation of the prediction error takes the form

$$\varepsilon_{\text{CL}}(t+1) = [\theta_{e0} - \hat{\theta}_e(t+1)]^T \phi_e(t), \quad (50)$$

where

$$\theta_{e0}^T = [\theta^T \quad h_1 \quad \dots \quad h_{n_H} \quad d_1 \quad \dots \quad d_{n_D}], \quad (51)$$

$$\begin{aligned}\phi_e^T(t) &= [\phi^T(t) \quad \varepsilon_{\text{CLf}}(t) \quad \dots \quad \varepsilon_{\text{CLf}}(t - n_H + 1) \\ &\quad - \alpha(t) \quad \dots \quad -\alpha(t - n_D + 1)],\end{aligned} \quad (52)$$

$$\begin{aligned}\hat{\theta}_e^T(t) &= [\hat{\theta}^T(t) \quad \hat{h}_1(t) \quad \dots \quad \hat{h}_{n_H}(t) \\ &\quad \hat{d}_1(t) \quad \dots \quad \hat{d}_{n_D}(t)],\end{aligned} \quad (53)$$

$$\begin{aligned}H_0^* &= h_1 + h_2 q^{-1} + \dots + h_{n_H} q^{-n_H+1} \\ &= -A^*S - B^*R,\end{aligned} \quad (54)$$

$$D^* = 0, \quad (55)$$

$$\varepsilon_{\text{CLf}}(t) = \frac{1}{S}\varepsilon_{\text{CL}}(t), \quad (56)$$

$$\alpha(t) = \hat{A}(t)\hat{y}(t) - \hat{B}^*(t)\hat{u}(t-d-1). \quad (57)$$

Taking into account the form of (50), Theorem A.1 suggests the use of the following PAA:

$$\hat{\Theta}(t) = \hat{\theta}_e(t),$$

$$\Phi(t) = \phi_e(t),$$

$$v^\circ(t+1) = \varepsilon_{\text{CL}}^\circ(t+1) = y(t+1) - \hat{\theta}_e^T(t)\phi_e(t).$$

Therefore, using Theorem A.1, the boundedness of $\varepsilon_{CL}(t+1)$ and $\lim_{t \rightarrow \infty} \varepsilon_{CL}(t+1) = 0$ is assured. While one can conclude from this that $\varepsilon_{CL}(t)$, $\hat{y}(t)$ and $\hat{u}(t)$ are bounded, it can be concluded that $\alpha(t)$ is bounded using (57).

Because of the form of (49), the ODE approach cannot be directly applied in order to obtain a clear convergence condition.

These algorithms are summarized in Table 1.

4. ROBUSTNESS ANALYSIS

The stability of the parameter adaptation algorithms is guaranteed under the matching assumption that the model order is equal to the order of the plant (Appendix A). Since this assumption may be violated in applications, it is

important to determine the robustness of the algorithms with respect to the unmodeled dynamics. Such a robustness analysis for output error algorithm used for plant model identification in open loop has been discussed in the literature (see e.g. Ioannou and Johnson, 1981; Ioannou and Kokotovic, 1982; Anderson and Johnson, 1982; Naik and Kumar, 1992). It was shown in Anderson and Johnson (1982) that for a constant adaptation gain, given a persistent excitation, the maximum a posteriori adaptation error is bounded by the maximum of the modeling error. In Naik and Kumar (1992) the robustness of the algorithm in the case of scalar adaptation gain in terms of preserving mean-square boundedness of all signals was demon-

Table 1. Summary of closed-loop output-error algorithms

		CLOE	F-CLOE	X-CLOE	G-CLOE
Plant + noise model		$y = \frac{z^{-d}B}{A}u + w$		$y = \frac{z^{-d}B}{A}u + \frac{C}{A}e$	$y = \frac{z^{-d}B}{A}u + \frac{C}{AD}e$
Adjustable parameter vector		$\hat{\Theta}^T(t) = [\hat{a}^T(t) \ \hat{b}^T(t)]$ $\hat{a}^T(t) = [\hat{a}_1 \ \dots \ \hat{a}_{n_A}]$ $\hat{b}^T(t) = [\hat{b}_1 \ \dots \ \hat{b}_{n_B}]$		$\hat{\Theta}^T(t) = [\hat{a}^T(t) \ \hat{b}^T(t) \ \hat{h}^T(t)]$ $\hat{a}^T(t) = [\hat{a}_1 \ \dots \ \hat{a}_{n_A}]$ $\hat{b}^T(t) = [\hat{b}_1 \ \dots \ \hat{b}_{n_B}]$ $\hat{h}^T(t) = [\hat{h}_1 \ \dots \ \hat{h}_{n_H}]$	$\hat{\Theta}^T(t) = [\hat{a}^T(t) \ \hat{b}^T(t) \ \hat{h}^T(t) \ \hat{d}^T(t)]$ $\hat{a}^T(t) = [\hat{a}_1 \ \dots \ \hat{a}_{n_A}]$ $\hat{b}^T(t) = [\hat{b}_1 \ \dots \ \hat{b}_{n_B}]$ $\hat{h}^T(t) = [\hat{h}_1 \ \dots \ \hat{h}_{n_H}]$ $\hat{d}^T(t) = [\hat{d}_1 \ \dots \ \hat{d}_{n_D}]$
Predictor -regressor vector		$\psi^T(t) =$ $[-\hat{y}(t) \ \dots \ -\hat{y}(t-n_A+1)$ $\hat{u}(t-d) \ \dots \ \hat{u}(t-d-n_B+1)]$ $\hat{u}(t) = -\frac{R}{S}\hat{y}(t) + r_u$		$\psi^T(t) =$ $[-\hat{y}(t) \ \dots \ -\hat{y}(t-n_A+1)$ $\hat{u}(t-d) \ \dots \ \hat{u}(t-d-n_B+1)$ $\varepsilon_{CL}(t) \ \dots \ \varepsilon_{CL}(t-n_H+1)]$ $\hat{u}(t) = -\frac{R}{S}\hat{y}(t) + r_u$ $\varepsilon_{CL} = \frac{1}{S}\varepsilon_{CL}(t)$	$\psi^T(t) =$ $[-\hat{y}(t) \ \dots \ -\hat{y}(t-n_A+1)$ $\hat{u}(t-d) \ \dots \ \hat{u}(t-d-n_B+1)$ $\varepsilon_{CL}(t) \ \dots \ \varepsilon_{CL}(t-n_H+1)$ $-\alpha(t) \ \dots \ -\alpha(t-n_D+1)]$ $\hat{u}(t) = -\frac{R}{S}\hat{y}(t) + r_u$ $\varepsilon_{CL} = \frac{1}{S}\varepsilon_{CL}(t)$ $\alpha(t) = \hat{A}(t)\hat{y}(t) - \hat{B}(t)\hat{u}(t-d)$
Predictor	a priori	$\hat{y}^o(t+1) = \hat{\Theta}^T(t)\psi(t)$			
Output	a posteriori	$\hat{y}(t+1) = \hat{\Theta}^T(t+1)\psi(t)$			
Prediction	a priori	$\varepsilon_{CL}^o(t+1) = y(t+1) - \hat{y}^o(t+1)$			
Error	a posteriori	$\varepsilon_{CL}(t+1) = y(t+1) - \hat{y}(t+1)$			
Adaptation error		$v^o(t+1) = \varepsilon_{CL}^o(t+1)$			
Observation vector		$\Phi(t) = \psi(t)$	$\Phi(t) = \frac{S}{P}\psi(t)$ $\hat{P} = \hat{A}S + z^{-d}\hat{B}R$	$\Phi(t) = \psi(t)$	$\Phi(t) = \psi(t)$
Stability condition		$\frac{S}{P} - \frac{\lambda_2}{2}$	$\frac{\hat{P}}{P} - \frac{\lambda_2}{2}$	None	None
Deterministic env.		SPR	SPR		
Convergence condition		$\frac{S}{P} - \frac{\lambda_2}{2}$	$\frac{\hat{P}}{P} - \frac{\lambda_2}{2}$	$\frac{1}{C} - \frac{\lambda_2}{2}$	Unknown
Stochastic env.		SPR	SPR		

strated, and it was shown that the algorithm is robustly stable whenever the denominator of the nominal model satisfies a strictly positive real condition. The results of such studies depend on the characterization of the model-plant mismatch. These results can be utilized for the family of the closed-loop output-error identification algorithms, with some modification in the hypotheses and proofs. In this paper we assume that the unmodeled portion of the plant is norm-bounded, and we show that for the family of closed-loop identification algorithms with a matrix adaptation gain all of the closed-loop signals remain norm-bounded. The use of a matrix adaptation gain, which is the typical algorithm used in applications instead of a scalar one (which was considered in Naik and Kumar, 1992), leads to a positive real condition identical to that for the non-mismatch-order case and does not require an 'orthonormal' projection of the estimated parameter vector $\hat{\theta}(t)$ for ensuring the norm-boundedness of the signals.

Consider the plant output to be described by

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t-d) + y_u(t) + w(t), \quad (58)$$

where $w(t)$ is a bounded disturbance and $y_u(t)$ is the unmodeled portion of the plant output. It can be assumed that $y_u(t)$ is equal to $\Delta u(t)$ (additive uncertainty) where Δ is an unknown stable linear time-invariant system that captures the unmodeled dynamics of the system. It is easy to show that $(AS/P)y_u(t)$ (which acts on the closed-loop output) is norm-bounded if the input of the closed-loop system is norm-bounded and the closed-loop system is stable. It should be mentioned that while the unmodeled portion of the plant should be stable, the reduced-order model (described by B/A) can be unstable. In the following theorem, for notational ease, unmodeled dynamics and bounded disturbances are presented by a single norm-bounded variable $\eta(t)$.

Theorem 4.1. Consider the parameter adaptation algorithm of (13) and (14) with $\lambda_1(t) = 1$ and assume that the plant output can be described by

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t-d) + \eta(t), \quad (59)$$

where $\eta(t)$ represents the effect of unmodeled dynamics and disturbances. Under the assumptions that

- the degree of $B(q^{-1})$ and $A(q^{-1})$ are upper-bounded by n_B and n_A respectively and d is known,
- the parameters of a stable RST controller that

stabilizes the closed loop system are known, and

- the input to the closed-loop system $r(t)$ and the disturbances $\eta(t)$ are norm-bounded, i.e.

$$\lim_{N \rightarrow \infty} \sum_{t=0}^N r^2(t) \leq \varepsilon^2, \\ \lim_{N \rightarrow \infty} \sum_{t=0}^N \eta^2(t) \leq \alpha^2, \quad \varepsilon^2, \alpha^2 < \infty,$$

the a posteriori prediction error $v(t+1)$, the predicted output $\hat{y}(t+1)$ and the regressor vector $\phi(t)$ are norm-bounded if

$$H'(z^{-1}) = H(z^{-1}) - \frac{\lambda}{2}, \quad \sup_t \lambda_2(t) \leq \lambda \leq 2, \quad (60)$$

is strictly positive-real, where the transfer function $H(z^{-1})$ is equal to S/P , $\hat{P}/P$ and 1 for the CLOE, F-CLOE and X-CLOE algorithms respectively.

Proof. See Appendix D.

Corollary 4.1. If the closed-loop input signal, disturbances and unmodeled dynamics are mean-square-bounded, i.e.

$$\lim_{N \rightarrow \infty} \sum_{t=0}^N r^2(t) \leq \varepsilon^2 N + k_r, \\ \lim_{N \rightarrow \infty} \sum_{t=0}^N \eta^2(t) \leq \alpha^2 N + k_\eta, \quad (61)$$

where k_r and k_η are the positive constants, then all of the closed-loop signals remain mean-square-bounded.

Proof. The proof is similar to that of Theorem 4.1, with the difference that the bound for the norm of $v(t+1)$ in (D.16) will depend on $\alpha^2 N + k_\eta$ instead of α^2 , which consequently leads to the mean-square-boundedness of $v(t+1)$ and of the other signals. $\square$

Remark. The proof of Theorem 4.1 shows that the positive-real condition concerns the reduced-order model, not the true system with unmodeled dynamics. Therefore S/P_i (where P_i is the true closed-loop polynomial) may be non SPR, while the S/P computed with the reduced-model should be SPR (see Example 4 in Section 8).

5. FILTERED OPEN-LOOP RECURSIVE IDENTIFICATION ALGORITHMS

The key observation leading to the use of open-loop recursive identification algorithms for

identification in closed loop is that the output of the predictor given by (8) can be expressed as

$$\begin{aligned}\hat{y}(t+1) &= \hat{\theta}^T \phi(t) \\ &= \hat{\theta}^T \varphi(t) + \left(\hat{A}^* + \frac{q^{-d} \hat{B}^* R}{S} \right) \varepsilon_{CL}(t),\end{aligned}\quad (62)$$

and that

$$1 + q^{-1} \left(\hat{A}^* + \frac{q^{-d} \hat{B}^* R}{S} \right) = \frac{\hat{A}S + \hat{B}R}{S} = \frac{\hat{P}}{S}, \quad (63)$$

where $\hat{P}$ (see also (30)) is an estimation of the true closed-loop poles based on the available plant model parameter estimates. Subtracting (62) from (4), one gets

$$\begin{aligned}\varepsilon_{CL}(t+1) &= \frac{S}{\hat{P}} (\theta - \hat{\theta})^T \varphi(t) + \frac{AS}{\hat{P}} w(t+1) \\ &= \frac{S}{\hat{P}} \varepsilon_{LS}(t+1),\end{aligned}\quad (64)$$

where

$$\begin{aligned}\varepsilon_{LS}(t+1) &= (\theta - \hat{\theta})^T \varphi(t) + Aw(t+1) \\ &= y(t+1) - \bar{y}(t+1)\end{aligned}\quad (65)$$

represents the prediction error when using a standard least-squares predictor for the plant defined as

$$\begin{aligned}\bar{y}(t+1) &= -\hat{A}^* y(t) + \hat{B}^* u(t-d) \\ &= \hat{\theta}^T \varphi(t).\end{aligned}\quad (66)$$

Therefore for fixed parameter estimates *the closed-loop prediction error is equal to the filtered least-squares prediction error for the plant output* (ignoring the feedback). This observation has been made by Åström (1993). However, (64) can be rewritten as

$$\varepsilon_{CL}(t+1) = (\theta - \hat{\theta})^T \varphi_f(t) + \frac{AS}{\hat{P}} w(t+1), \quad (67)$$

where

$$\begin{aligned}\varphi_f(t) &= \frac{S}{\hat{P}} \varphi(t) \\ &= [-y_f(t) \quad \dots \quad -y_f(t-n_A+1) \\ &\quad u_f(t-d) \quad \dots \quad u_f(t-d-n_B+1)],\end{aligned}\quad (68)$$

which immediately leads to the use of the standard recursive least-squares algorithm but on filtered inputs and outputs. Note that the filter to be used has a very appealing characteristic, namely it will filter out high-frequency signals beyond the estimated closed-loop bandwidth, and if the controller contains an integrator (i.e. $S = (1 - q^{-1})S'(q^{-1})$), the d.c. components will be also filtered out. However, asymptotic unbiased estimates can be obtained only for particular noise model structures and closed-loop

poles (i.e. for the cases when $(AS/\hat{P})w(t+1) = e(t+1)$). To account for the effect of noise upon the parameter estimates, one can use other open-loop recursive identification algorithms belonging to the category of algorithms based on the asymptotic whitening of the prediction error like

- extended recursive least squares (ELS);
- output error with extended prediction model (OEEPM);
- recursive maximum-likelihood (RML).

However, from a theoretical point of view (using, for example, the ODE method; Ljung, 1987), even when $w(t+1) = (C/A)e(t+1)$, the filtering of the data modifies the conditions for the asymptotic unbiasedness of the estimates. The equation of the filtered output for $w(t+1) = (C/A)e(t+1)$ is

$$\begin{aligned}y_f(t+1) &= -A^* y_f(t) + B^* u_f(t) + \frac{CS}{\hat{P}} e(t+1), \\ &\approx -A^* y_f(t) + B^* u_f(t) + C'e(t+1),\end{aligned}\quad (69)$$

where $C'(q^{-1})$ is the result of the polynomial division of CS by $\hat{P}$ (typically of higher order than C). The corresponding extended least-squares predictor will have the form

$$\begin{aligned}\bar{y}(t+1) &= -\hat{A}^* y_f(t) + \hat{B}^* u_f(t-d) \\ &\quad + \hat{C}^* \varepsilon_{LSf}(t),\end{aligned}\quad (70)$$

but with an order for $\hat{C}^*$ higher than in the open-loop case. The resulting prediction-error equation will take the form

$$\varepsilon_{LSf}(t+1) \approx \frac{1}{C'} (\theta_e - \hat{\theta}_e)^T \varphi_{fe}(t) + e(t+1), \quad (71)$$

where

$$\varphi_{fe}^T(t) = [\varphi_f^T(t) \quad \varepsilon_{LSf}(t) \quad \varepsilon_{LSf}(t-1) \quad \dots], \quad (72)$$

and the sufficient condition for asymptotic unbiased estimates becomes

$$H'(z^{-1}) = \frac{1}{C'} - \frac{\lambda_2}{2} \quad (73)$$

is strictly positive real (where C' is an approximation of $CS/\hat{P}$).

In practice, all of the recursive identification algorithms based on the asymptotic whitening of the prediction error can be used on the filtered input-output data through $S/\hat{P}$. Therefore an initial estimate of the plant model is necessary in order to implement the data filter, which is not the case for 'closed-loop output-error' identification methods.

Output-error recursive algorithms for open-loop identification can also be used for plant

model identification in closed loop by using an appropriate data filter (Voda and Landau, 1995). The open-loop output-error predictor is

$$\begin{aligned}\bar{y}(t+1) &= -\hat{A}^* \bar{y}(t) + \hat{B}^* u(t-d) \\ &= \hat{\theta}^T \varphi_{OE}(t),\end{aligned}\quad (74)$$

where

$$\varphi_{OE}^T(t) = [\bar{y}(t) \quad \bar{y}(t-1) \quad \dots \quad u(t-d) \quad \dots \quad u(t-d-n_B+1)]. \quad (75)$$

Neglecting the effect of noise, the output-error prediction obtained by subtracting (76) from (4) is given by

$$\begin{aligned}\varepsilon_{OE}(t+1) &= \frac{1}{A} (\theta - \hat{\theta})^T \varphi_{OE}(t) \\ &= \frac{1}{\hat{A}} (\theta - \hat{\theta})^T \varphi(t) \\ &= \frac{1}{\hat{A}} \varepsilon_{LS}(t+1).\end{aligned}\quad (76)$$

Therefore

$$\begin{aligned}\varepsilon_{CL}(t+1) &= \frac{S\hat{A}}{\hat{P}} \varepsilon_{OE}(t+1) \\ &= \frac{S\hat{A}}{\hat{P}} \frac{1}{A} (\theta - \hat{\theta})^T \varphi_{OE}(t) \\ &= \frac{1}{A} (\theta - \hat{\theta})^T \varphi_{OEF}(t) = \varepsilon_{OEF},\end{aligned}\quad (77)$$

where

$$\varphi_{OEF}^T(t) = \frac{S\hat{A}}{\hat{P}} \varphi_{OE}(t), \quad (78)$$

$$\varepsilon_{OEF}(t+1) = y_f(t+1) - \hat{\theta}^T \varphi_{OEF}(t). \quad (79)$$

In other words, the open-loop output-error recursive algorithm can be used for plant model identification in closed loop provided that the input-output data are filtered through the estimated output sensitivity function ($S\hat{A}/\hat{P}$). However, since in closed-loop operation u and w are correlated, the parameter estimates will be asymptotically unbiased only for particular noise model structures and closed-loop poles ($\hat{A}S/\hat{P}w = e$ for $w = (C/A)e$, $(CS/\hat{P})$, $\hat{A}/A = 1$).

6. THE BIAS DISTRIBUTION

The asymptotic expression for the bias distribution of transfer-function estimates plays an important role in the analysis of different identification methods, particularly when the true system does not belong to the model set. In this section the bias distributions of the estimates for different algorithms are presented using the framework considered in Ljung (1987). For more details see Karimi and Landau (1997).

A discrete-time SISO system with an additive disturbance $p(t)$ is considered:

$$y(t) = G(q^{-1})u(t) + p(t), \quad (80)$$

$$p(t) = W(q^{-1})e(t), \quad (81)$$

where $y(t)$ is the output signal, $u(t)$ is the input signal and $e(t)$ is a zero-mean unit-variance white-noise signal. $G(q^{-1})$ and $W(q^{-1})$ are the transfer functions of the plant and noise model respectively.

6.1. Filtered open-loop identification

In these algorithms an open-loop-type identification (direct method) is performed on the filtered input/output data, which leads to the following bias distribution for the estimates (Ljung, 1993):

$$\begin{aligned}\hat{\theta}^* &= \arg \min_{\hat{\theta}} \int_{-\pi}^{\pi} |W^{-1}(e^{j\omega}, \hat{\theta})L(e^{j\omega})|^2 \left\{ |G(e^{j\omega}) \right. \\ &\quad - G(e^{j\omega}, \hat{\theta})|^2 |S_{yp}(e^{j\omega})|^2 \phi_r(\omega) \\ &\quad + \left| [G(e^{j\omega}) - G(e^{j\omega}, \hat{\theta})] S_{up}(e^{j\omega}) W(e^{j\omega}) \right. \\ &\quad \left. \left. + \left[W(e^{j\omega}) - \frac{W(e^{j\omega}, \hat{\theta})}{L(e^{j\omega})} \right] \right|^2 \phi_e(\omega) \right\} d\omega,\end{aligned}\quad (82)$$

where

$G(e^{j\omega}, \hat{\theta})$ is the estimate of the plant model, $W(e^{j\omega}, \hat{\theta})$ is the estimate of the noise model, $L(e^{j\omega})$ is the input/output data filter, $S_{yp}(e^{j\omega}) = \left(1 + G \frac{R}{S}\right)^{-1}$ is the output sensitivity function, $S_{up}(e^{j\omega}) = \frac{-R}{S} \left(1 + G \frac{R}{S}\right)^{-1}$ is the input sensitivity function, $\phi_r(\omega)$ is the spectrum of the reference signal (added to the controller output), $\phi_e(\omega)$ is the spectrum of the noise signal.

To simplify the notation, $e^{j\omega}$ will be omitted in the sequel. The following comments can be made about (82).

- The estimation of the noise model is biased. In fact, $W(\hat{\theta})$ tends asymptotically to LW instead of W . This incorrect estimation leads to a biased estimation of G , even when G is in the model set.
- In the ideal case when $W(\hat{\theta}) = LW$ the parameter estimates are given by

$$\begin{aligned}\hat{\theta}^* &= \arg \min_{\hat{\theta}} \int_{-\pi}^{\pi} |W^{-1}|^2 \\ &\quad \times \{ |G - G(\hat{\theta})|^2 [|S_{yp}|^2 \phi_r(\omega) \\ &\quad + |S_{up} W|^2 \phi_e(\omega)] \} d\omega,\end{aligned}\quad (83)$$

which means that the data filtering by L will be compensated by the estimated noise model $W(\hat{\theta})$, and the bias distribution is almost identical to the nonfiltered case.

- In the particular case when an ARX model is considered (i.e. $W(\hat{\theta}) = 1/\hat{A}$) and $L = S/\hat{P}$ we have

$$\begin{aligned} \hat{\theta}^* = \arg \min_{\hat{\theta}} \int_{-\pi}^{\pi} |\hat{S}_{yp}^*|^2 \{ |G - G(\hat{\theta})|^2 |S_{yp}|^2 \phi_{r_u}(\omega) \\ + |[G - G(\hat{\theta})]S_{up}W \\ + [W - (\hat{S}_{yp}^*)^{-1}]^2 \phi_e(\omega) \} d\omega, \end{aligned} \quad (84)$$

where $\hat{S}_{yp}^* = \hat{A}S/\hat{P}$ is an estimation of the output sensitivity function S_{yp} . It can be observed that $G - G(\hat{\theta})$ is weighted by both the real output sensitivity function and the estimation of the output sensitivity function, which should improve the accuracy of the identified model in critical frequency regions where the Nyquist plot is closest to $(-1, j0)$. Unfortunately this positive effect is countered by the noise.

For the case of output-error structure (i.e. $p(t) = w(t)$) using an estimation of the output sensitivity function $\hat{S}_{yp}^*$ as a data filter, the bias distribution is given by (Voda and Landau, 1995)

$$\begin{aligned} \hat{\theta}^* = \arg \min_{\hat{\theta}} \int_{-\pi}^{\pi} |\hat{S}_{yp}^*|^2 \{ |G - G(\hat{\theta})|^2 |S_{yp}|^2 \phi_{r_u}(\omega) \\ + |[G - G(\hat{\theta})]S_{up} + 1|^2 \phi_w(\omega) \} d\omega, \end{aligned} \quad (85)$$

where $\phi_w(\omega)$ is the spectrum of the disturbances. Of course the estimation of the plant model using the above criterion, as in the previous case, is biased.

6.2. Closed-loop output-error methods

Consider the system described by (80) without a specific model for the noise (i.e. $p(t) = w(t)$), except that it is independent of the external excitation and has a spectrum $\phi_w(\omega)$. The closed-loop output-error algorithm (CLOE) uses the following predictor for the closed-loop system:

$$\hat{y}(t) = G(\hat{\theta})\hat{u}(t), \quad (86)$$

where $\hat{u}(t) = \hat{S}_{yp}r_u(t)$ and $\hat{S}_{yp}$ is the sensitivity function in the designed closed loop. Then the prediction error is given by

$$\varepsilon_{CL}(t) = S_{yp}\{[G - G(\hat{\theta})]\hat{u}(t) + w(t)\}. \quad (87)$$

Now substituting $\hat{u}(t)$ by $\hat{S}_{yp}r_u(t)$, the resulting bias distribution will be

$$\begin{aligned} \hat{\theta}^* = \arg \min_{\hat{\theta}} \int_{-\pi}^{\pi} |S_{yp}|^2 \{ |G - G(\hat{\theta})|^2 |\hat{S}_{yp}|^2 \phi_{r_u}(\omega) \\ + \phi_w(\omega) \} d\omega. \end{aligned} \quad (88)$$

This expression shows that

- the estimation of the plant model parameters is unbiased when G is in the model set;
- the bias distribution is not affected by the spectrum of the noise (which is the case for the filtered open-loop method);
- the bias distribution is not only weighted by the sensitivity function but is further weighted by the estimated sensitivity function.

Two other algorithms of this family, F-CLOE and AF-CLOE, have exactly the same bias distribution, because they use the same predictor.

For the algorithms in which the noise model is also identified (X-CLOE and G-CLOE), the optimal predictor for the closed loop system is

$$\hat{y}(t) = G(\hat{\theta})\hat{u}(t) + \bar{W}(\hat{\theta})\varepsilon_{CL}(t), \quad (89)$$

where

$$\bar{W}(\hat{\theta}) = W(\hat{\theta}) - \hat{S}_{yp}^{-1}.$$

Remark. For an ARMAX model we have $W(\hat{\theta}) = \hat{C}/\hat{A}$, and therefore

$$\begin{aligned} \hat{y}(t) = G(\hat{\theta})\hat{u}(t) \\ + \frac{\hat{C}S - \hat{A}S - q^{-d}\hat{B}R}{\hat{A}S} \varepsilon_{CL}(t), \end{aligned} \quad (90)$$

which is the predictor used in the X-CLOE method (see (32) and (34)). The prediction error then can be computed as follows (for more details see Karimi and Landau, 1997):

$$\begin{aligned} \varepsilon_{CL}(t) = [1 + \bar{W}(\hat{\theta})\hat{S}_{yp}]^{-1} \{ S_{yp}[G - G(\hat{\theta})]\hat{S}_{yp}r_u(t) \\ + WS_{yp}e(t) - [1 + \bar{W}(\hat{\theta})\hat{S}_{yp}]e(t) \} \\ + e(t), \end{aligned} \quad (91)$$

which leads to the following bias distribution (denoting $\bar{W} = W - S_{yp}^{-1}$):

$$\begin{aligned} \hat{\theta}^* = \arg \min_{\hat{\theta}} \int_{-\pi}^{\pi} \{ [1 + \bar{W}(\hat{\theta})\hat{S}_{yp}]^{-1} \}^2 \\ \times [|S_{yp}|^2 |G - G(\hat{\theta})|^2 |\hat{S}_{yp}|^2 \phi_{r_u}(\omega) \\ + |\bar{W}S_{yp} - \bar{W}(\hat{\theta})\hat{S}_{yp}|^2 \phi_e(\omega)] d\omega. \end{aligned} \quad (92)$$

The following can be noted.

- This method gives an unbiased estimation of the plant and noise model when they are in the model set.

- It has the same asymptotic properties as the direct identification method in terms of the bias distribution. In fact, if in (92) we replace $\tilde{W}$ and $\tilde{W}(\hat{\theta})$ respectively by $W - S_{yp}^{-1}$ and $W(\hat{\theta}) - \hat{S}_{yp}^{-1}$ then (82) (with $L = 1$) is obtained.

The conclusion of this analysis is that, with respect to the bias distribution, if the objective is to enhance the accuracy of the model in the critical frequency regions for design then CLOE, F-CLOE and AF-CLOE are the most suitable methods.

7. VALIDATION OF MODELS IDENTIFIED IN CLOSED LOOP

The first remark to be made is that, independently of the criterion used, the model validation will be controller-dependent. The objective of the validation will be to find the plant model that gives the best prediction for the output of the closed-loop system with the controller used for identification. Several validation procedures could be defined: statistical validation, pole closeness validation and time-domain validation.

7.1. Statistical validation

Statistical validation tests can be developed in a similar way as in the open-loop identification (Ljung, 1987; Landau, 1990), with the difference that a closed-loop predictor incorporating the plant model will be used to obtain the residual closed-loop prediction error. Two types of tests can be done.

Uncorrelation test. Using the 'output-error' configuration shown in Fig. 1, where the predictor is given by (8)–(11), the uncorrelation between the closed-loop output error $\varepsilon_{CL}(t+1)$ and the components of the observation vector of $\phi(t)(\hat{y}(t), \hat{u}(t-d))$ and their delayed values) will be tested together with the covariance of the residual closed-loop output prediction error. This type of test is motivated on the one hand by the fact that uncorrelation between the observations (components of $\phi(t)$) and the closed-loop output prediction error leads to unbiased parameter estimates and on the other hand by the fact that this uncorrelation also implies uncorrelation between the prediction error and the external excitation (i.e. the residual prediction error does not contain any information depending upon the external excitation, and therefore all correlations between the external excitation and the output of the closed loop system are represented by the predictor).

One defines

$$R_{xz}(i) = \frac{1}{N} \sum_{t=1}^N x(t)z(t-i), \quad (93)$$

and computes:

$$RN_{ex}(i) = \frac{R_{ex}(i)}{[R_{xx}(0)R_{\varepsilon\varepsilon}(0)]^{1/2}}, \quad x = \hat{y}, \hat{u}, \quad (94)$$

$$i = 1, 2, \dots, \max(n_A, n_B + d).$$

As a confidence test, one can use the criterion

$$|RN_{ex}(i)| \leq \frac{\alpha}{\sqrt{N}}, \quad (95)$$

where α is the confidence level (a typical value is 2.17; Landau, 1990) and N is the number of data.

In many practical situations, from one set of data several models can be identified using the various methods (also a model identified in open loop may be available). Then a comparative validation has to be done, and useful comparison indicators are provided by $R_{\varepsilon\varepsilon}(0)$ and $\max |RN_{ex}|$ for each model (however, other comparison criteria can be considered).

Whiteness test. If information upon the noise model affecting the output of the plant is available, through the use of appropriate identification algorithms, a whiteness test on the residual closed-loop prediction error can be done. A predictor of the form given by (32) has to be used. If the input-output plant model and the noise model are known, the closed-loop prediction error is a white noise. The whiteness of the residual prediction error implies that

- unbiased parameter estimates can be obtained;
- the identified model allows the optimal predictor for the closed-loop system to be obtained;
- there is no correlation between the external excitation and the residual closed-loop prediction error.

One computes

$$RN_{\varepsilon}(i) = \frac{R_{\varepsilon\varepsilon}(i)}{R_{\varepsilon\varepsilon}(0)}, \quad (96)$$

$$i = 1, 2, \dots, \max(n_A, n_B + d).$$

As a confidence test, one can use the criterion (95). For comparative validation of several models $R_{\varepsilon\varepsilon}(0)$ and $\max |RN_{\varepsilon}(i)|$ are good indicators.

7.2. Pole closeness validation

The idea behind this test is the following. If the model identified allows a good predictor of the closed-loop system to be constructed for the

given controller used during identification, this also implies that the poles of the true closed-loop system and those of the closed-loop predictor are close (assuming the richness of the excitation). Therefore the closeness of the closed-loop predictor poles (which can be computed) and the poles of the true closed-loop system (which can be identified) will give an indication of the quality of the identified model.

Closeness criteria in the z plane can be associated with this validation procedure, but simply visual pole chart examination allows various models to be compared clearly.

Extensive simulation and experimental results have shown that statistical validation and pole closeness validation are two coherent procedures for model validation in closed loop, and allow a clear comparison of the identified plant models.

The identification of the closed-loop poles is done by simply identifying the closed loop (between the external excitation and the plant output) using standard open-loop identification and validation techniques. Note that this identification can be carried out using the data that are also used for plant model identification in closed loop.

7.3. Time-domain validation

While this approach seems very natural (comparing the time response of the true closed-loop system with that of the closed-loop predictor), our experience shows that it does not always allow the comparative quality of various models to be easily quantified. However, it was observed that good pole closeness implies excellent superposition of simulated and real-time responses.

8. COMPARATIVE EVALUATION OF THE VARIOUS ALGORITHMS

This section deals with the evaluation of the various algorithms in simulation as well as on real-time applications via different validation tests.

8.1. Simulation results

In order to compare different closed-loop identification methods and study the behaviour of these algorithms, four simulation examples are considered. In the first example the parameters of the system and controller are chosen such that S/P is a positive-real transfer function and the influence of positive-realness of $1/C$ on the convergence of various algorithm is studied. Several model validation tests are also examined on the identified models. A non-positive-real function for S/P is considered in

the second example, and the convergence of various methods is studied. In the third example a narrow-band filter is chosen as the noise model, and the X-CLOE and G-CLOE methods are compared. In the fourth example the robustness of the algorithms with respect to unmodeled dynamic is demonstrated.

A Monte Carlo simulation with 100 runs is carried out in order to study the statistical behaviour of various algorithms with respect to different realizations of the noise. The mean value of the parametric error, standard deviation and the variance of the estimated parameters are used for evaluating the different methods. These criteria are defined as follows:

$$\begin{aligned}\bar{\theta} &= \frac{1}{n_{\theta}} \sum_{j=1}^{n_{\theta}} |\hat{\theta}(j) - \theta(j)|, & \bar{\sigma} &= \frac{1}{n_{\theta}} \sum_{j=1}^{n_{\theta}} \sigma(j), \\ \bar{v} &= \frac{1}{n_{\theta}} \sum_{j=1}^{n_{\theta}} \sigma^2(j),\end{aligned}\quad (97)$$

where $n_{\theta} = n_A + n_B$ is the number of parameters, $\sigma(j)$ and $\sigma^2(j)$ are respectively the standard deviation and variance of the j th parameter. $\hat{\theta}$ is the mean value of the estimated parameter vector, defined as:

$$\hat{\theta} = \frac{1}{100} \sum_{i=1}^{100} \hat{\theta}_i, \quad (98)$$

where $\hat{\theta}_i$ is the estimated parameter vector in the i th simulation.

Example 1. Figure 2 shows the block diagram of the simulation system, where B/A is the plant model, R/S is the controller, C/A is the noise model and $e(t)$ is a zero-mean value equally distributed white noise. The parameters of the system are chosen as follows:

$$A(z^{-1}) = 1 - 1.5z^{-1} + 0.7z^{-2}, \quad (99)$$

$$B(z^{-1}) = z^{-1}(1 + 0.5z^{-1}). \quad (100)$$

A controller using the pole-placement technique is computed such that S/P is positive-real (the controller contains an integrator, i.e. S contains a

Fig. 2. Block diagram of the simulation system (Example 1).

term $1 - z^{-1}$ that vanishes at zero frequency). The parameters of the controller and the closed-loop characteristic polynomial are

$$R(z^{-1}) = 0.8659 - 1.2763z^{-1} + 0.5204z^{-2}, \quad (101)$$

$$S(z^{-1}) = 1 - 0.6283z^{-1} - 0.3717z^{-2}, \quad (102)$$

$$P(z^{-1}) = 1 - 1.2624z^{-1} - 0.4274z^{-2}. \quad (103)$$

For identification of the plant model in closed loop, a pseudorandom binary sequence (PRBS) generated by a 7-bit shift register and a clock frequency of $\frac{1}{2}f_s$ (the sampling frequency $f_s = 1$) is considered as reference signal. The noise signal ratio at the output of the closed-loop system is about 10% in terms of variance. Two different cases are considered for the noise model:

case (a): $1/C$ is strictly positive-real, i.e. $C(z^{-1}) = 1 + 0.5z^{-1} + 0.5z^{-2}$;

case (b): $1/C$ is not positive-real, i.e. $C(z^{-1}) = 1 + 1.6z^{-1} + 0.9z^{-2}$.

The extended recursive least-squares algorithm with filtered data has been used as FOL identification method.

For case (a) the mean value of parametric error, standard deviation and variance for 100 simulations are compared for different methods in Table 2. It can be seen that while all of the closed-loop identification algorithms lead to a small parametric error and variance, the FOL method gives a parametric error about 10 times larger than those obtained with CLOE-type methods. The parametric distance, defined by

$$D(t) = \left\{ \sum_{i=1}^{n_A} [a_i - \hat{a}_i(t)]^2 + \sum_{i=0}^{n_B} [b_i - \hat{b}_i(t)]^2 \right\}^{1/2}, \quad (104)$$

may also be used as a criterion for comparing different methods in one run. Since we are only interested in the identification of the process, the criterion involves only the parameters of the plant and their estimates. Evolutions of this criterion for case (a) (i.e. $1/C$ is strictly positive-real) using CLOE, F-CLOE, AF-CLOE, X-CLOE and a filtered open-loop identification method (FOL) are illustrated in

Fig. 3. In this simulation, data have been filtered by the exact value of S/P (instead of an estimated value of P) for the FOL method as well as for the F-CLOE method.

Figure 3 shows that an unbiased estimation of the parameters of the plant is obtained using various recursive closed-loop identification methods, while the parameter estimation for the FOL method is biased. This is because a finite-order polynomial $\hat{C}$ is used to estimate the parameters of the noise model CS/P , while theoretically an infinite-order polynomial for $\hat{C}$ has to be used. Increasing the order of $\hat{C}$ in this case might improve the results, but it will also augment the number of parameters to be estimated, which consequently reduces the convergence speed.

It can be observed that the convergence speed is slightly slower in the X-CLOE method, which has more parameters to estimate than the CLOE and F-CLOE methods. The evolution of the estimated parameters of the plant using the CLOE method is shown in Fig. 4.

The estimated parameters of the plant model for one run after 1024 samples using the various methods and the corresponding statistical validation tests (uncorrelation test and whiteness test) are given in Table 3.

The magnitudes of the frequency responses of the plant models identified in closed loop are compared with that of the true model in Fig. 5. The true model and closed-loop-identified models are almost superimposed in the frequency domain for CLOE, F-CLOE and X-CLOE methods. This is not the case for the FOL method (particularly in high frequencies), which did not converge to the true parameters of the system.

Figure 6 shows the closed-loop poles computed on the basis of the plant model identified by the various methods and the true closed-loop poles. A coherence between this test and the statistical validation test is clearly observed.

Evolutions of the parametric distance for case (b) (i.e. $1/C$ is not positive-real) are illustrated in Fig. 7 (for one run). This figure shows that the positive-realness of $1/C$ does not affect the convergence of the CLOE and F-CLOE methods, but the parameter estimation is biased for the X-CLOE method. The results of the Monte Carlo simulation (Table 4) also show that the parametric error, standard deviation and variance for the X-CLOE method are greater than those of case (a).

Example 2. Figure 8 shows the block diagram of the simulation system for this example, where B/A is the plant model, C/A is the model of

Table 2. Results of the Monte Carlo simulation with 100 runs (Example 1, case (a))

	$\bar{\theta}$	$\bar{\sigma}$	$\bar{v}$
CLOE	0.0136	± 0.0450	0.0022
F-CLOE	0.0059	± 0.0454	0.0023
AF-CLOE	0.0112	± 0.0641	0.0052
X-CLOE	0.0138	± 0.0358	0.0013
FOL	0.0987	± 0.0540	0.0031

Fig. 3. Evolution of the parametric distance. (Example 1, case (a), one run).

noise and $e(t)$ is a zero-mean value uniformly distributed white noise. The parameters of the system are chosen as follows:

$$A = 1 - 2.5q^{-1} + 0.2q^{-2}, \quad (105)$$

$$B = q^{-1}(1 + 0.5q^{-1}), \quad (106)$$

$$C = 1 + 0.5q^{-1} + 0.5q^{-2}. \quad (107)$$

The open-loop system, which in fact is unstable, can be stabilized using a unit feedback

($R = S = 1$). The characteristic polynomial of the closed-loop system is given by

$$P = 1 - 1.5q^{-1} + 0.7q^{-2}, \quad (108)$$

which leads to a non-positive-real discrete-time transfer function for S/P , while $1/C - \frac{1}{2}$ is strictly positive real. The same PRBS as in Example 1 is used in order to identify the plant model in closed loop. The variance of

Fig. 4. Evolution of the plant estimated parameters (CLOE method, Example 1, case (a)).

Table 3. Estimated parameters of the plant model and statistical validation tests after 1024 samples (Example 1, case (a))

	Estimated parameters				Statistical validation			
	a_1	a_2	b_1	b_2	Uncorrelation		Whiteness	
					$R_e(0)$	$\max_i RN_{ep}(i) $	$R_e(0)$	$\max_i RN_e(i) $
Nominal	-1.5	0.7	1.0	0.5	—	—	—	—
CLOE	-1.481	0.672	0.953	0.490	0.030	0.026	—	—
F-CLOE	-1.476	0.675	0.954	0.504	0.030	0.011	—	—
AF-CLOE	-1.485	0.682	0.968	0.472	0.030	0.017	—	—
X-CLOE	-1.447	0.649	0.955	0.510	0.030	0.029	0.007	0.001
FOL	-1.407	0.629	1.139	0.777	0.031	0.025	0.010	0.211

noise/signal at the output of the system is about 10%. The results of the Monte Carlo simulation with 100 runs are given in Table 5. The minimum value for the average of the variance and standard deviation is obtained by the X-CLOE method. Figure 9 shows the evolutions of the parametric distance for various methods and for one particular realization of the noise. One can observe that although the CLOE method does not converge with a non-positive-real transfer function for S/P for this particular noise realization, it gives a very small parametric error on average for 100 simulations. The results of F-CLOE and X-CLOE in the Monte Carlo simulation as well as in terms of parametric distance for one run are satisfactory. This example clearly illustrates that the positive-realness condition on S/P can be relaxed using the F-CLOE, AF-CLOE or X-CLOE methods.

One also sees that the FOL method cannot be a reliable tool for plant model identification in closed loop in this case.

The estimated parameters of the plant model for one run after 1024 samples using the various methods and the corresponding statistical validation tests (uncorrelation test and whiteness test) are given in Table 6. One can clearly see that the statistical tests are coherent with the magnitude of the parameter errors associated with each method.

Example 3. The previous example is again considered, with a different noise model. The noise model is $1/DA$, where $1/D$ is a narrow-band filter. The polynomial $D(z^{-1}) = 1 + 0.497z^{-1} + 0.686z^{-2}$ is used, which in fact corresponds to a second-order continuous system with $\omega_n = 0.25f/f_s$ and $\xi = 0.1$.

Fig. 5. Frequency response of the plant (simulation): —, true plant model; ---, plant model identified in closed loop using different methods.

Fig. 6. Pole closeness validation test (simulation): $\circ$, true closed-loop poles; $\times$, computed closed-loop poles using the closed-loop identified plant model and the controller.

The same PRBS as in the previous examples is used, and the variance of noise/signal at the output of the system is about 5%. The mean value of parametric error, standard deviation and variance computed from 100 simulation are given in Table 7. It is observed that for this type of disturbance there is no significant difference between the G-CLOE, which is dedicated to this type of noise, and X-CLOE methods. This is

because in closed-loop operation the noise acting at the output of the system is filtered by the sensitivity function AS/P , which changes the frequency characteristic of the noise. Therefore, with this type of noise model, we cannot expect in general an improvement of the parameter estimates using G-CLOE instead of X-CLOE.

Example 4. In order to demonstrate the

Fig. 7. Evolution of the parametric distance for one run (Example 1, case (b)).

Table 4. Results of the Monte Carlo simulation with 100 runs (Example 1, case (b))

	$\bar{\theta}$	$\bar{\sigma}$	$\bar{v}$
CLOE	0.0058	± 0.0480	0.0024
F-CLOE	0.0056	± 0.0412	0.0018
AF-CLOE	0.0056	± 0.0658	0.0053
X-CLOE	0.0390	± 0.0538	0.0030
FOL	0.1122	± 0.0851	0.0081

robustness of the algorithms with respect to unmodeled dynamics, an additional pole is considered in the plant model of Example 1, and then the plant model is identified with a second-order model. It is supposed that there is no noise at the output of the plant, in order to have only the effect of unmodeled dynamics. The simulation shows that for a pole in high frequencies the frequency response of the identified model and of the *true* are almost superimposed whatever algorithm is used. The situation is different when the additional pole is near to the dominant poles of the plant. The frequency response of the identified second-order model is compared with the *true* third-order model, which has an additional pole at 0.5, in Fig. 10. It can be seen that the effect of unmodeled dynamics on the identified models using the family of closed-loop output-error identification methods is not very different from that of the filtered open-loop identification method. It should be noted that in this case the transfer function S/P_t (where P_t is the true closed-loop polynomial) is not positive-real, and this does not affect the stability and the convergence of the CLOE algorithm whenever S/P computed using the reduced-order model remains positive-real.

8.2. Experimental results

The experimental device is depicted in Fig. 11. It consists of a flexible transmission formed by a set of three pulleys coupled by two very elastic belts. The system is controlled by a PC via an I/O board. The sampling frequency is 20 Hz.

The system identification is carried out in open

Fig. 8. Block diagram of the simulation system (Example 2).

Table 5. Results of the Monte Carlo simulation with 100 runs (Example 2)

	$\bar{\theta}$	$\bar{\sigma}$	$\bar{v}$
CLOE	0.0025	± 0.1036	0.0118
F-CLOE	0.0284	± 0.0884	0.0082
AF-CLOE	0.0114	± 0.1118	0.0131
X-CLOE	0.0032	± 0.0215	0.0005
FOL	0.3166	± 0.1602	0.0268

loop with a PC using WinPIM identification software (Adaptech, 1996). The output error with extended prediction model algorithm provided the best results in terms of statistical validation (Landau, 1990). The model obtained in open loop, which passes the validation test, is

$$A = 1 - 1.3528q^{-1} + 1.5502q^{-2} - 1.2798q^{-3} + 0.9115q^{-4},$$

$$B = 0.4116q^{-1} + 0.524q^{-2}, \quad d = 2.$$

The main characteristics of the system are two very oscillatory modes, an unstable zero and a time delay of two sampling periods. A controller for this system is computed by the pole-placement method with WinREG software (Adaptech, 1996). The controller is designed in order to obtain two dominant poles with the same frequency of the first mode of the open-loop model, but with a damping factor of 0.8. The precompensator $T(q^{-1})$ is chosen to obtain unit closed-loop gain. The parameters of the RST controller are as follows:

$$R(q^{-1}) = 0.4526 - 0.4564q^{-1} - 0.6857q^{-2} + 1.0955q^{-3} - 0.1449q^{-4},$$

$$S(q^{-1}) = 1 + 0.2345q^{-1} - 0.8704q^{-2} - 0.4474q^{-3} + 0.0833q^{-4},$$

$$T(q^{-1}) = 0.2612.$$

Implementing the above controller on the real platform using WinTRAC software (Adaptech, 1996), the identification of the plant in closed loop is carried out using various methods. A PRBS generated by a 7-bit shift register and a clock frequency of $\frac{1}{2}f_s$ is considered as reference signal.

The frequency responses of the models identified in closed loop using various methods are compared with that of the open-loop model in Fig. 12. The first oscillatory identified mode is almost the same as that of the open-loop model, while the second identified mode is a bit different.

In order to validate the identified models, the real achieved closed-loop poles (which can be obtained by identification of the whole closed-loop system with standard open-loop identifica-

Fig. 9. Evolution of the parametric distance for one run (Example 2).

tion methods) and the computed ones (using the plant model identified by various methods and the *RST* controller) are given in Fig. 13. It is observed that the computed closed-loop poles using the different models identified in closed loop and the real ones are almost superimposed, particularly at low frequencies, whereas the computed poles using the open-loop-identified model are very far from the real closed-loop poles. The results of the statistical validation test are also given in Table 8 for comparison.

It is clearly seen that the plant models identified in closed loop give better results than the open-loop-identified model. It is also observed that the F-CLOE method gives the minimum value of the uncorrelation test, while the other methods give more or less good results. It should be noted that the output error with extended prediction model (OEEPM) is used to identify the plant model in the FOL method. An

estimation of $\hat{P}$ based on the open-loop-identified model is used in the filters for the FOL and F-CLOE methods.

9. HINTS FOR PRACTICAL USE OF THE ALGORITHMS

Two major reasons for the use of these algorithms are as follows:

- (i) identification in closed loop when a controller exists already or the plant cannot be operated in open loop;
- (ii) improvement of the open-loop-identified model for control purpose (control redesign).

One first distinguishes several situations:

- a first estimation of a plant model is available (either from an open-loop identification or a previous closed-loop identification);

Table 6. Estimated parameters of the plant model after 1024 samples (Example 2)

	Estimated parameters				Statistical validation			
					Uncorrelation		Whiteness	
	a_1	a_2	b_1	b_2	$R_e(0)$	$\max_i RN_{ey}(i) $	$R_e(0)$	$\max_i RN_e(i) $
Nominal	-2.5	0.2	1.0	0.5	—	—	—	—
CLOE	-2.604	0.590	1.043	0.157	3.970	0.150	—	—
F-CLOE	-2.449	0.153	0.960	0.538	3.556	0.007	—	—
AF-CLOE	-2.482	0.195	0.989	0.500	3.555	0.010	—	—
X-CLOE	-2.473	0.187	0.984	0.501	3.560	0.039	0.111	0.022
FOL	-2.748	0.555	1.230	0.155	3.685	0.097	0.789	0.294

Table 7. Results of the Monte Carlo simulation with 100 runs (Example 3)

	$\bar{\theta}$	$\bar{\sigma}$	$\bar{v}$
CLOE	0.0098	± 0.0999	0.0112
F-CLOE	0.0095	± 0.0587	0.0035
AF-CLOE	0.0228	± 0.1380	0.0196
X-CLOE	0.0022	± 0.0479	0.0024
FOL	0.0761	± 0.1354	0.0189
G-CLOE	0.0064	± 0.0483	0.0024

- the controller is known;
- the controller is unknown;
- the plant model order is unknown.

When the controller is known and a first estimation of the plant model is available, all the algorithms presented can be used straightforwardly. However, as in open-loop identification, there is no unique algorithm that will give the best results in all practical situations. Therefore the various algorithms should be used, and only the validation methods described in Section 7 will allow one to decide what the best model is. This comparative validation should also include the a priori available model.

If the controller is known but an a priori model of the plant is not available, one can use the CLOE, X-CLOE or AF-CLOE methods in order to obtain a model; then, using this model, the other methods that require an a priori estimation of the model can be used. Another alternative is to simply use the open-loop

Fig. 11. Block diagram of the flexible transmission.

methods without any data filtering in order to get a first plant model estimation and then use all the other methods that require a priori information upon the plant model. The third alternative is to identify the closed-loop poles from the data and build the required filters. Note, however, that all these operations require just one set of data.

If the controller is unknown, one should use a two-step method similar to that proposed in van den Hof and Schrama (1993) as follows:

- identify the sensitivity function $S_{yp} = AS/P$;
- filter the reference signal by this sensitivity function in order to generate an instrumental variable $\hat{u}(t)$;

Fig. 10. Frequency response of the plant (Example 4): — true plant model; ---, plant model identified in closed loop using different methods.

Fig. 12. Frequency response of flexible transmission system models: —, plant model identified in open loop; ---, plant model identified in closed loop using different methods.

- (iii) filter $\hat{u}(t)$ and $y(t)$ through the identified sensitivity function;
- (iv) use an open-loop output-error identification in order to identify the plant model.

A controller can then be redesigned on the basis of the identified model, and a new closed-loop

identification eventually using all the methods can be carried out.

If the model order is unknown, it can be estimated using the preliminary data obtained either from open-loop operation, if possible, or from closed-loop operation. Since the input of the plant is correlated with the noise in the

Fig. 13. Closed-loop pole charts for flexible transmission system: O, real achieved closed-loop poles; x, computed closed-loop poles using the closed-loop identified plant model and the controller; *, computed closed-loop poles using the open-loop model and the controller.

Table 8. Statistical-model validation test for the flexible transmission system

	$R_e(0)$	$\max_i R_{ey}(i) $	$\max_i RN_{ey}(i) $
Open loop	0.0367	0.0652	0.5758
CLOE	0.0019	0.0047	0.1978
F-CLOE	0.0009	0.0022	0.1353
X-CLOE	0.0017	0.0041	0.1840
FOL	0.0012	0.0039	0.2055

closed-loop case, it should be replaced by an instrumental variable. A suitable choice for this instrumental variable is the reference signal (input of the closed-loop system) filtered by an appropriate sensitivity function (i.e. A/P), which can be identified using the open-loop identification methods (van den Hof and Schrama, 1993). Then the classical order estimation methods explained in Ljung (1987) and Söderström and Stoica (1989) or a method based on instrumental variables introduced in Duong and Landau (1996) can be utilized in order to estimate the structure of the plant model.

10. CONCLUSION

A unified framework for recursive plant model identification in closed loop has been provided.

A methodology for validation of the models identified in closed loop has been proposed and used successfully for comparative evaluation of the various identified models on simulation and real experimental data.

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APPENDIX A—STABILITY OF PARAMETER ADAPTATION ALGORITHMS

Theorem A.1 (Landau (1980).) Assume that the following parameter adaptation algorithm for updating a parameter vector $\hat{\theta}(t)$ is used:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F(t)\phi(t)v(t+1), \quad (\text{A.1})$$

$$F^{-1}(t+1) = \lambda_1(t)F^{-1}(t) + \lambda_2(t)\phi(t)\phi^T(t), \quad (\text{A.2})$$

$$F(0) > 0, \quad 0 < \lambda_1 \leq 1, \quad 0 \leq \lambda_2 < 2,$$

$$F^{-1}(t) > \alpha F^{-1}(0) \quad 0 < \alpha < \infty.$$

Assume that the relation between $v(t+1)$ and $\hat{\theta}(t+1)$ is given by

$$v(t+1) = H(q^{-1})\phi^T(t)[\theta - \hat{\theta}(t+1)], \quad (\text{A.3})$$

where $\phi(t)$ is a bounded or unbounded vector sequence, $H(z^{-1})$ is a rational discrete transfer function (ratio of monic polynomials multiplied eventually by a positive gain) and θ is a constant vector. Then, if

$$H'(z^{-1}) = H(z^{-1}) - \frac{\lambda}{2} \quad (\text{A.4})$$

is strictly positive-real, where

$$\max_t [\lambda_2(t)] \leq \lambda < 2, \quad (\text{A.5})$$

one has for any bounded $v(0)$ and $\hat{\theta}(0)$,

$$\lim_{t \rightarrow \infty} v(t+1) = 0, \quad (\text{A.6})$$

$$\lim_{t_1 \rightarrow \infty} \sum_{t=0}^{t_1} v^2(t+1) < C_1(v(0), \hat{\theta}(0)), \quad 0 < C_1 < \infty, \quad (\text{A.7})$$

$$\|\hat{\theta}(t)\| \leq C_2, \quad 0 < C_2 < \infty. \quad (\text{A.8})$$

Remark. The result (A.6) corresponds to equation (2.8) of Theorem 2.1 from Landau (1980). The result (A.7) is a direct consequence of equation (A.6) from the same reference, since $v(t+1)$ is the output of the system characterized by the transfer function $H(z^{-1})$ (state $x(t)$) and that has as input $[\theta - \hat{\theta}(t+1)]^T \phi(t)$.

APPENDIX B—ODE CONVERGENCE ANALYSIS

Theorem B.1. (Dugard and Landau (1980).) Consider the parameter-adaptation algorithm

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F(t)\phi(t)v(t+1), \quad (\text{B.1})$$

$$F^{-1}(t+1) = F^{-1}(t) + \lambda_2 \phi(t)\phi^T(t), \quad 0 \leq \lambda_2 < 2. \quad (\text{B.2})$$

- Assume that the stationary process $\phi(t, \hat{\theta})$ and $v(t+1, \hat{\theta})$ can be defined for $\hat{\theta}(t) = \hat{\theta}$.
- Assume that $\hat{\theta}(t)$ generated by the algorithm belongs infinitely often to the domain (D_s) for which the stationary processes $\phi(t, \hat{\theta})$ and $v(t+1, \hat{\theta})$ can be defined.
- Assume that $v(t+1, \hat{\theta})$ is given by an equation of the form

$$v(t+1, \hat{\theta}) = H(q^{-1})\phi^T(t, \hat{\theta})[\theta - \hat{\theta}] + \omega(t+1, \hat{\theta}), \quad (\text{B.3})$$

where $H(q^{-1})$ is a discrete transfer-function operator, a ratio of monic polynomials.

- Assume that either $\omega(t+1, \hat{\theta})$ is a sequence of independent equally distributed normal random variables $(0, \sigma)$ or $E\{\phi(t, \hat{\theta}), \omega(t+1, \hat{\theta})\} = 0$.
- Define the convergence domain D_c as

$$D_c = \{\hat{\theta} : \phi^T(t, \hat{\theta})[\theta - \hat{\theta}] = 0\}. \quad (\text{B.4})$$

Then, if

$$H'(z^{-1}) = H(z^{-1}) - \frac{\lambda_2}{2} \quad (\text{B.5})$$

is a strictly positive-real discrete transfer function, one has

$$\text{Prob} \left\{ \lim_{t \rightarrow \infty} \hat{\theta}(t) \in D_c \right\} = 1. \quad (\text{B.6})$$

Corollary B.1. If

$$\phi^T(t, \hat{\theta})(\theta - \hat{\theta}) = 0 \quad (\text{B.7})$$

has a unique solution $\hat{\theta} = \theta$, the condition that $H'(z^{-1})$ given by (B.5) be positive-real implies that

$$\text{Prob} \left\{ \lim_{t \rightarrow \infty} \hat{\theta}(t) = \theta \right\} = 1. \quad (\text{B.8})$$

APPENDIX C—MARTINGALE CONVERGENCE ANALYSIS

Theorem C.1. (Landau (1982).) Consider the parameter-adaptation algorithm

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F(t)\phi(t)v(t+1), \quad (\text{C.1})$$

$$F^{-1}(t+1) = F^{-1}(t) + \lambda_2(t)\phi(t)\phi^T(t), \quad 0 \leq \lambda_2(t) < 2, \quad (\text{C.2})$$

$$v(t+1) = \frac{v^o(t+1)}{1 + \phi^T(t)F(t)\phi(t)}. \quad (\text{C.3})$$

- Assume that there exists a value of the parameter vector θ

such that for $\hat{\theta}(t) = \theta$ the adaptation error $v(t+1)$ is equal to $e(t+1)$, i.e.

$$\exists \theta: v(t+1)|_{\hat{\theta}(t)=\theta} = e(t+1), \quad (\text{C.4})$$

where $\{e(t)\}$ is a martingale difference sequence defined on a probability space $(\Omega, \mathcal{A}, P)$ adapted to the sequence of increasing σ -algebras $\mathcal{F}_t$ generated by observations up to and including time t .

- Assume that the martingale difference sequence satisfies the following equations:

$$E\{e(t+1) | \mathcal{F}_t\} = 0, \quad (\text{C.5})$$

$$E\{e^2(t+1) | \mathcal{F}_t\} = \sigma^2, \quad (\text{C.6})$$

$$\lim_{N \rightarrow \infty} \sup \frac{1}{N} \sum_{t=1}^N e^2(t) < \infty. \quad (\text{C.7})$$

- Assume that the residual error $v(t+1)$ satisfies

$$v(t+1) = H(q^{-1})[\theta - \hat{\theta}(t+1)]^T \phi(t) + e(t+1), \quad (\text{C.8})$$

where $H(q^{-1}) = H_1(q^{-1})/H_2(q^{-1})$ is a ratio of monic polynomials ($H'_i(q^{-1}) = 1 + q^{-1}H_i^*(q^{-1})$).

- Assume that the a priori adaptation error $v^o(t+1)$ satisfies

$$\begin{aligned} v^o(t+1) = & [\theta - \hat{\theta}(t)]^T \phi(t) + H_1^*(q^{-1})[\theta - \hat{\theta}(t)]^T \phi(t-1) \\ & - H_2^*(q^{-1})v(t) + H_2^*(q^{-1})e(t) + e(t+1). \end{aligned} \quad (\text{C.9})$$

- Assume that $\phi(t)$ is $\mathcal{F}_t$ -measurable.

If

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \phi^T(t)\phi(t) < \infty \quad (\text{C.10})$$

and there exists λ_2 with $\max_t \lambda_2(t) \leq \lambda_2 < 2$, such that

$$H'(z^{-1}) = H(z^{-1}) - \frac{\lambda_2}{2} \quad (\text{C.11})$$

is strictly positive-real, then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N [v(t) - e(t)]^2 = 0 \quad \text{a.s.}, \quad (\text{C.12})$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N [v(t)]^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N [e(t)]^2 \quad \text{a.s.}, \quad (\text{C.13})$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \{[\theta - \hat{\theta}(t)]^T \phi(t-1)\}^2 = 0 \quad \text{a.s.}, \quad (\text{C.14})$$

$$\lim_{N \rightarrow \infty} [\theta - \hat{\theta}(N)]^T \left(\frac{F^{-1}(N)}{N} \right) [\theta - \hat{\theta}(N)] = 0 \quad \text{a.s.} \quad (\text{C.15})$$

If, in additions,

$$\lim_{N \rightarrow \infty} \frac{1}{N} F^{-1}(N) > 0 \quad \text{a.s.} \quad (\text{C.16})$$

then

$$\lim_{t \rightarrow \infty} \hat{\theta}(t) = \theta \quad \text{a.s.} \quad (\text{C.17})$$

Lemma C.1. (Landau (1982).) For the parameter adaptation algorithm (C.1)–(C.3) with $\lambda_2(t) = 1$, if

$$\frac{1}{N} \sum_{t=1}^N |\phi_i(t)|^2 \leq K_1 + \frac{K_2}{N} \sum_{t=1}^N v^2(t-1), \quad 0 < K_1, K_2 < \infty, \quad (\text{C.18})$$

where

$$\phi^T(t) = [\phi_1(t) \quad \dots \quad \phi_n(t)] \quad (\text{C.19})$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N [\hat{\theta}^T(t)\phi(t) + v^o(t+1)]^2 < \infty, \quad (\text{C.20})$$

then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N v^2(t) < \infty, \quad (\text{C.21})$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \phi^T(t)\phi(t) < \infty. \quad (\text{C.22})$$

APPENDIX D—PROOF OF THEOREM 4.1

The proof uses the following lemmas.

Lemma D.1. (Popov (1973).) For a linear system (input $u(t)$ and output $y(t)$) characterized by a strictly positive-real transfer function $H(z^{-1})$, one has

$$\sum_{t=0}^N y(t)u(t) \geq -\gamma^2 + \delta \sum_{t=0}^N u^2(t), \quad \gamma^2 < \infty, \quad \delta > 0. \quad (\text{D.1})$$

Lemma D.2. (Goodwin and Sin (1984).) For a linear system (input $u(t)$ and output $y(t)$) characterized by an asymptotically stable transfer function $H(z^{-1})$ (i.e. one that has all its poles in $|z| < 1$), one has

$$\sum_{t=0}^N y^2(t) \leq m_1 \sum_{t=0}^N u^2(t) + m_2, \quad 0 < m_1 < \infty, \quad 0 \leq m_2 < \infty. \quad (\text{D.2})$$

Lemma D.3. For the parameter adaptation algorithm (13)–(14) with $\lambda_1(t) = 1$, one has

$$\sum_{t=0}^N \tilde{\theta}^T(t+1)\phi(t)\mu(t+1) \geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0) \quad (\text{D.3})$$

for all $N > 0$, where

$$\tilde{\theta}(t+1) = \hat{\theta}(t+1) - \theta, \quad (\text{D.4})$$

$$\mu(t+1) = v(t+1) + \frac{\lambda}{2}\tilde{\theta}^T(t+1)\phi(t), \quad \lambda \geq \sup_t \lambda_2(t). \quad (\text{D.5})$$

Proof of Lemma D.3. From (13) and (14), with $\lambda_1(t) = 1$, one has

$$\begin{aligned} & \tilde{\theta}^T(t+1)F^{-1}(t+1)\tilde{\theta}(t+1) \\ &= \tilde{\theta}^T(t+1)[F^{-1}(t) + \lambda_2(t)\phi(t)\phi^T(t)]\tilde{\theta}(t+1) \\ &= \tilde{\theta}^T(t+1)F^{-1}(t)\tilde{\theta}(t+1) + \lambda_2(t)[\tilde{\theta}^T(t+1)\phi(t)]^2, \end{aligned}$$

but

$$\begin{aligned} & \tilde{\theta}^T(t+1)F^{-1}(t)\tilde{\theta}(t+1) \\ &= [\tilde{\theta}(t) + F(t)\phi(t)v(t+1)]^T F^{-1}(t) [\tilde{\theta}(t) + F(t)\phi(t)v(t+1)] \\ &= \tilde{\theta}^T(t)F^{-1}(t)\tilde{\theta}(t) + \phi^T(t)F(t)\phi(t)v^2(t+1) \\ &\quad + 2\tilde{\theta}^T(t)\phi(t)v(t+1) \\ &= \tilde{\theta}^T(t)F^{-1}(t)\tilde{\theta}(t) + 2\tilde{\theta}^T(t+1)\phi(t)v(t+1) \\ &\quad - \phi^T(t)F(t)\phi(t)v^2(t+1). \end{aligned}$$

Therefore

$$\begin{aligned} & \tilde{\theta}^T(t+1)F^{-1}(t+1)\tilde{\theta}(t+1) \\ &= \tilde{\theta}^T(t)F^{-1}(t)\tilde{\theta}(t) + 2\tilde{\theta}^T(t+1)\phi(t)v(t+1) \\ &\quad + \lambda_2(t)[\tilde{\theta}^T(t+1)\phi(t)]^2 - \phi^T(t)F(t)\phi(t)v^2(t+1), \end{aligned}$$

from which one obtains

$$\begin{aligned} & \tilde{\theta}^T(t+1)\phi(t)v(t+1) = \frac{1}{2}\tilde{\theta}^T(t+1)F^{-1}(t+1)\tilde{\theta}(t+1) \\ &\quad - \frac{1}{2}\tilde{\theta}^T(t)F^{-1}(t)\tilde{\theta}(t) \\ &\quad - \frac{\lambda_2(t)}{2}[\tilde{\theta}^T(t+1)\phi(t)]^2 \\ &\quad + \frac{1}{2}\phi^T(t)F(t)\phi(t)v^2(t+1). \end{aligned}$$

Summing from $t = 0$ to $t = N$, one gets

$$\begin{aligned} \sum_{t=0}^N \tilde{\theta}^T(t+1)\phi(t)v(t+1) &= \frac{1}{2}\tilde{\theta}^T(N+1)F^{-1}(N+1)\tilde{\theta}(N+1) \\ &\quad - \frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0) \\ &\quad - \frac{1}{2}\sum_{t=0}^N \lambda_2(t)[\tilde{\theta}^T(t+1)\phi(t)]^2 \\ &\quad + \frac{1}{2}\sum_{t=0}^N \phi^T(t)F(t)\phi(t)v^2(t+1). \end{aligned} \quad (\text{D.6})$$

Since the first and fourth terms of (D.6) are greater than or equal to zero, one obtains the following inequality:

$$\begin{aligned} \sum_{t=0}^N \tilde{\theta}^T(t+1)\phi(t)v(t+1) &\geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}(0)\tilde{\theta}(0) \\ &\quad - \frac{1}{2}\sum_{t=0}^N \lambda_2(t)[\tilde{\theta}^T(t+1)\phi(t)]^2. \end{aligned} \quad (\text{D.7})$$

Therefore, using (D.5), one gets

$$\begin{aligned} & \sum_{t=0}^N \tilde{\theta}^T(t+1)\phi(t)\mu(t+1) \\ &\geq -\frac{1}{2}\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0) + \frac{1}{2}\sum_{t=0}^N [\lambda - \lambda_2(t)][\tilde{\theta}^T(t+1)\phi(t)]^2. \end{aligned} \quad (\text{D.8})$$

Since the second term on the right-hand side of the (D.8) is positive, one obtains the inequality of Lemma D.3. $\square$

Proof of Theorem 4.1. The proof is carried out for the CLOE and F-CLOE methods. The a posteriori closed-loop prediction error for the system described by (59) is given by

$$v(t+1) = H(z^{-1})[\theta - \hat{\theta}(t+1)]^T \phi(t) + \frac{AS}{P} \eta(t+1). \quad (\text{D.9})$$

Then, using (D.4) and (D.5), one obtains

$$-\hat{\theta}^T(t+1)\phi(t) = \frac{1}{H'(z^{-1})}\mu(t+1) - \bar{\eta}(t+1), \quad (\text{D.10})$$

where

$$\bar{\eta}(t+1) = \frac{AS}{P} \frac{1}{H'(z^{-1})} \eta(t+1). \quad (\text{D.11})$$

Since the inverse of an SPR transfer function is also SPR, Lemma D.1 can be used to obtain the following inequality:

$$\begin{aligned} -\sum_{t=0}^N \tilde{\theta}^T(t+1)\phi(t)\mu(t+1) &\geq -\gamma^2 + \delta \sum_{t=0}^N \mu^2(t+1) \\ &\quad - \sum_{t=0}^N \bar{\eta}(t+1)\mu(t+1). \end{aligned} \quad (\text{D.12})$$

It can be shown that for $\beta > 0$ one gets

$$\begin{aligned} -\sum_{t=0}^N \bar{\eta}(t+1)\mu(t+1) &\geq -\frac{\beta}{2}\sum_{t=0}^N \mu^2(t+1) \\ &\quad - \frac{1}{2\beta}\sum_{t=0}^N \bar{\eta}^2(t+1) \end{aligned} \quad (\text{D.13})$$

(this is obtained by expanding $-\beta\mu(t+1) - \bar{\eta}(t+1)]^2 \leq 0$ and summing from $t=0$ up to N). Using the inequalities (D.12) and (D.13) and Lemma D.3, the following inequality is obtained:

$$\left(\delta - \frac{\beta}{2}\right) \sum_{t=0}^N \mu^2(t+1) \leq \gamma^2 + \frac{1}{2\beta} \sum_{t=0}^N \bar{\eta}^2(t+1) + \frac{1}{2} \bar{\theta}^T(0) F^{-1}(0) \bar{\theta}(0). \quad (\text{D.14})$$

From the norm-boundedness of $\eta(t+1)$ and Lemma D.2, there results the norm-boundedness of $\bar{\eta}(t+1)$. Therefore, with $\beta < 2\delta$, the inequality (D.14) implies that $\mu(t+1)$ is also norm-bounded. Now it remains to show that $v(t+1)$ and $\phi(t)$ are norm-bounded. From (D.5) and (D.10), one gets

$$\left[1 + \frac{\lambda/2}{H'(z^{-1})}\right] \mu(t+1) = v(t+1) + \frac{\lambda}{2} \bar{\eta}(t+1). \quad (\text{D.15})$$

Using the Schwartz inequality and summing from $t=0$ to N ,

one obtains

$$\sum_{t=0}^N v^2(t+1) \leq 2 \sum_{t=0}^N \bar{\mu}^2(t+1) + \frac{2\lambda^2}{4} \sum_{t=0}^N \bar{\eta}^2(t+1), \quad (\text{D.16})$$

where

$$\bar{\mu}(t+1) = \left[1 + \frac{\lambda/2}{H'(z^{-1})}\right] \mu(t+1) \quad (\text{D.17})$$

is norm-bounded (applying Lemma D.2). This, together with the norm-boundedness of $\bar{\eta}(t+1)$, implies the norm-boundedness of $v(t+1)$. It is now easy to show that $\hat{y}(t+1) = y(t+1) - v(t+1)$ is also norm-bounded (because of the boundedness of $v(t+1)$ and $y(t+1)$). Assuming asymptotic stability of the controller and the boundedness of the excitation signal $r(t)$, $\hat{u}(t)$ is also norm-bounded. Finally, this implies the boundedness of $\phi(t)$, which contains $\hat{y}(t-i)$ ($i=0, \dots, n_A$) and $\hat{u}(t-d-j)$ ($j=0, \dots, n_B$).

The proof is similar for the X-CLOE method, except that the term AS/P in (D.9) and (D.11) is replaced by A . $\square$