On a Conjecture of Webb

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Let p be a prime divisor of the order of a finite group G and let $S_p(G)$ be the poset of non-trivial p-subgroups of G. Associated to this poset, we consider the simplicial complex $\Delta(S_p(G))$ of chains of non-trivial p-subgroups, called Brown's complex (in honour of K.S. Brown who first introduced it). The group G acts by conjugation on $S_p(G)$, hence also on $\Delta(S_p(G))$, and this natural action of G can be used for handling p-local information in the theory of p-modular representations and mod p cohomology of the group G, as for instance in [W1], [W2], [W3], [KR], [Bo], [Th]. We refer to Webb's survey paper [W2] for more details about Brown's complex and applications.

In connection with these recent developments, P.J. Webb stated the following conjecture.

CONJECTURE (Webb [W2, 4.2]). *Let $|\Delta(S_p(G))|$ be the geometric realization of the simplicial complex $\Delta(S_p(G))$. Then the orbit space $|\Delta(S_p(G))|/G$ is contractible.*

By passing to the barycentric subdivision of $\Delta(S_p(G))$, one can view $|\Delta(S_p(G))|/G$ as the geometric realization of the simplicial complex associated with a suitable poset. Thus we can again work with a poset.

Webb first realized that the Euler characteristic of $|\Delta(S_p(G))|/G$ is trivial [W1, 8.2] (see also [Th, 4.4] for another proof), and then more generally he proved that $|\Delta(S_p(G))|/G$ is mod p acyclic [W3, 2.6]. This is the main evidence for the conjecture. Note also that if G is a group of Lie type in characteristic p, then $\Delta(S_p(G))$ is G-homotopy equivalent to the Tits building of G (see [TW, 2.4] for details), and the conjecture holds because the orbit complex of the building is a single simplex, hence contractible.

The purpose of this note is to prove the conjecture in a few cases, as follows.

THEOREM A. *Webb's conjecture holds for a p-solvable group G.*

THEOREM B. *Webb's conjecture holds if a Sylow p-subgroup of G is either abelian, or generalized quaternion, or TI.*

Recall that a Sylow p-subgroup P is called *TI* if $P^g \cap P = 1$ for all $g \notin N_G(P)$. In other words every p-subgroup is contained in a unique Sylow p-subgroup.

Both theorems easily follow from rather elementary results on p-subgroups and control of fusion which we describe in the first section. The proofs appear in the second section.

1. The poset of orbits

The orbit set $\Delta(S_p(G))/G$ is not a simplicial complex, because in general there are several orbits of simplices with given orbits of vertices. Thus we pass to the barycentric subdivision, that is, we consider the poset $\Sigma_p(G)$ of (non empty) chains in $S_p(G)$. The associated simplicial complex $\Delta(\Sigma_p(G))$ consists of chains of chains of non-trivial p-subgroups, but we concentrate on the poset $\Sigma_p(G)$ itself. An element $\sigma \in \Sigma_p(G)$ will be written

$$\sigma = (P_0 < P_1 < \ldots < P_n),$$

where $P_i \in S_p(G)$ and $n = \dim(\sigma)$. It is immediate to check that the poset $\Sigma_p(G)$ satisfies the crucial condition :

$$(1.1) \qquad \tau \leq \sigma \quad \text{and} \quad \tau^g \leq \sigma \quad \text{imply that} \quad \tau = \tau^g \qquad (\tau, \sigma \in \Sigma_p(G)\,,\ g \in G\,).$$

Thus $\Sigma_p(G)$ is a *regular* G-poset in the sense of [CR, §66].

Now write $[\sigma]_G$ for the orbit of σ under G. The orbit set $\Sigma_p(G)/G$ is a poset, with the order relation $[\tau]_G \leq [\sigma]_G$ provided that $\tau' \leq \sigma'$ for some $\tau' \in [\tau]_G$ and $\sigma' \in [\sigma]_G$. An easy consequence of the regularity condition (1.1) (see [CR, 66.6] for details) is that $\Delta(\Sigma_p(G))/G$ is a simplicial complex and is isomorphic to $\Delta(\Sigma_p(G)/G)$, the simplicial complex associated with the poset $\Sigma_p(G)/G$. Moreover by [CR, 66.1 and 66.8], the orbit space $|\Delta(S_p(G))|/G$ (which we are interested in) is homeomorphic to the space associated with the poset $\Sigma_p(G)/G$. Thus we only have to work with this poset.

Let $f : H \to G$ be a homomorphism of finite groups. Since $S_p(G)$ and $\Sigma_p(G)$ do not involve the trivial subgroup, f can induce a map of posets $f_* : \Sigma_p(H) \to \Sigma_p(G)$ only if no p-subgroup is contained in $K = \mathrm{Ker}(f)$, that is, if p does not divide $|K|$. If this condition is satisfied, it is clear that f_* exists, is order preserving and preserves the length of chains of p-subgroups. Moreover if $\sigma, \tau \in \Sigma_p(H)$ are H-conjugate, then $f_*(\sigma)$ and $f_*(\tau)$ are G-conjugate, and therefore f_* induces a map $\overline{f}_* : \Sigma_p(H)/H \to \Sigma_p(G)/G$ which is order preserving.

(1.2) PROPOSITION. *Let* K *be a normal subgroup of* G *of order prime to* p *and let* $Q = G/K$. *Then the canonical group homomorphism* $f : G \to Q$ *induces an isomorphism of posets* $\overline{f}_* : \Sigma_p(G)/G \to \Sigma_p(Q)/Q$.

Proof. Let $\sigma = (P_0 < \ldots < P_n) \in \Sigma_p(Q)$ and let H be the inverse image of P_n in G. Let Q_n be a Sylow p-subgroup of H (so that $H = KQ_n$). Then f induces an isomorphism $Q_n \cong P_n$ and so there is a chain in $\Sigma_p(G)$ (with maximal element Q_n) mapping onto σ. It follows in particular that $\overline{f}_* : \Sigma_p(G)/G \to \Sigma_p(Q)/Q$ is surjective.

Let $\sigma = (P_0 < \ldots < P_n) \in \Sigma_p(G)$ and $\tau = (Q_0 < \ldots < Q_n) \in \Sigma_p(G)$. Assume that $[f(\sigma)]_Q = [f(\tau)]_Q$, so that there exists $g \in G$ with $f(\sigma) = f(\tau)^{f(g)}$. We have to show that $[\sigma]_G = [\tau]_G$, so replacing τ by τ^g, we can assume that $f(\sigma) = f(\tau)$. Then $f(P_n) = f(Q_n)$, that is, $KP_n = KQ_n$. Since P_n and Q_n are Sylow p-subgroups of KP_n, there exists $k \in K$ such that $P_n = Q_n^k$. Replacing τ by τ^k, we can assume further that $P_n = Q_n$. Now f induces an isomorphism $P_n \cong f(P_n)$ and the images of the subgroups P_i and Q_i are equal. Therefore $P_i = Q_i$ for all i, and $\sigma = \tau$. This proves the injectivity of $\overline{f}_*$. □

If $f : H \to G$ is a group homomorphism, the proposition reduces the study of the induced map $\overline{f}_* : \Sigma_p(H)/H \to \Sigma_p(G)/G$ to the case of the inclusion $\mathrm{Im}(f) \to G$. Thus we can assume that f is an inclusion. Recall that a subgroup H of G *controls* p*-fusion in* G if some Sylow p-subgroup of G is contained in H (i.e. $|G : H|$ is prime to p) and the following condition is satisfied: whenever Q is a p-subgroup of H and $Q^g \leq H$ for some $g \in G$, then $g = ch$ with $c \in C_G(Q)$ and $h \in H$. Here $C_G(Q)$ denotes the centralizer of Q in G.

(1.3) PROPOSITION. *Let* H *be a subgroup of* G *which controls* p*-fusion in* G. *Then the inclusion* $i : H \to G$ *induces an isomorphism of posets* $\overline{i}_* : \Sigma_p(H)/H \to \Sigma_p(G)/G$.

Proof. The surjectivity is easy: if P_n is the maximal subgroup of $\sigma \in \Sigma_p(G)$, then since $|G : H|$ is prime to p, there exists $g \in G$ such that $P_n^g \leq H$; then $\sigma^g \in \Sigma_p(H)$ and $\overline{i}_*([\sigma^g]_H) = [\sigma^g]_G = [\sigma]_G$, as required.

Now let $\sigma, \tau \in \Sigma_p(H)$ such that $[\sigma]_G = [\tau]_G$, that is, $\tau = \sigma^g$ for some $g \in G$. Let P_n be the maximal subgroup of σ. Since P_n^g is the maximal subgroup of τ which is a chain in H, we have $P_n^g \leq H$. By control of fusion, it follows that $g = ch$ with $c \in C_G(P_n)$ and $h \in H$. Since c stabilizes the whole chain σ, we obtain $\tau = \sigma^h$, proving the injectivity of $\overline{i}_*$. □

(1.4) REMARK. It is possible to express the results above in a categorical setting as in [Pu], where Puig introduces the Frobenius category. We also introduce a weak form of the Frobenius category. Let $\mathrm{Frob}_p(G)$ (respectively $\mathrm{Weak}_p(G)$) be the category whose objects are all p-subgroups of G (including 1) and whose set of morphisms between the p-subgroups P and Q is equal to $C_G(P)\backslash T_G(P,Q)$ (respectively $N_G(P)\backslash T_G(P,Q)$), where $T_G(P,Q) = \{ g \in G \mid P^g \leq Q \}$. Similarly we define $\mathrm{CFrob}_p(G)$ (respectively $\mathrm{CWeak}_p(G)$) to be the category whose objects are all chains of p-subgroups of G (which may include 1) and whose set of morphisms between the chains σ and τ is equal to $C_G(\sigma)\backslash T_G(\sigma,\tau)$ (respectively $N_G(\sigma)\backslash T_G(\sigma,\tau)$), where $T_G(\sigma,\tau)$ is defined as follows: if $\sigma = (P_0 < \ldots < P_n)$ and $\tau = (Q_0 < \ldots < Q_m)$, then $T_G(\sigma,\tau)$ is the set of all $g \in G$ such that $P_i^g \leq Q_{\phi(i)}$ for all i, for some order-preserving injective map $\phi : \{0,\ldots,n\} \to \{0,\ldots,m\}$.

Given a group homomorphism $f : H \to G$, we want to understand when f induces an equivalence of categories $f_* : \mathrm{Frob}_p(H) \xrightarrow{\sim} \mathrm{Frob}_p(G)$, and similarly for the other categories. It is not hard to prove that $f_* : \mathrm{CWeak}_p(H) \xrightarrow{\sim} \mathrm{CWeak}_p(G)$ if and only if the map of posets $\overline{f}_* : \Sigma_p(H)/H \to \Sigma_p(G)/G$ is an isomorphism. On the other hand it is well-known that $f_* : \mathrm{Frob}_p(H) \xrightarrow{\sim} \mathrm{Frob}_p(G)$ if and only if p does not divide $|\mathrm{Ker}(f)|$ and $\mathrm{Im}(f)$ controls p-fusion in G. Therefore the two propositions above say that

$$f_* : \mathrm{Frob}_p(H) \xrightarrow{\sim} \mathrm{Frob}_p(G), \quad \Longrightarrow \quad f_* : \mathrm{CWeak}_p(H) \xrightarrow{\sim} \mathrm{CWeak}_p(G)\,.$$

For completeness, we also mention that

$$f_* : \mathrm{Frob}_p(H) \xrightarrow{\sim} \mathrm{Frob}_p(G) \quad \Longleftrightarrow \quad f_* : \mathrm{CFrob}_p(H) \xrightarrow{\sim} \mathrm{CFrob}_p(G)\,,$$

and that

$$f_* : \mathrm{CWeak}_p(H) \xrightarrow{\sim} \mathrm{CWeak}_p(G) \quad \Longrightarrow \quad f_* : \mathrm{Weak}_p(H) \xrightarrow{\sim} \mathrm{Weak}_p(G)\,.$$

We do not elaborate further on this. Indeed only $\mathrm{Frob}_p(G)$ seems to play an important role in group theory, for instance in questions of classification (see [Pu]) and in group cohomology (see [Mi]).

2. Webb's conjecture

We start with the well-known observation that Webb's conjecture holds if $O_p(G) \neq 1$. As usual, $O_p(G)$ denotes the largest normal p-subgroup of G.

(2.1) LEMMA. *If $P = O_p(G)$ is non-trivial, then $\Sigma_p(G)/G$ is contractible.*

Proof. It is here more convenient to work with $S_p(G)$. By Section 1, $|\Delta(\Sigma_p(G)/G)|$ is homeomorphic to $|\Delta(S_p(G))|/G$. Now the orbit space X/G of a G-space X is contractible if X is G-contractible, because a G-equivariant homotopy $F : X \times [0,1] \to X$ between id_X and the constant map (onto a G-fixed point) induces a homotopy $F : X/G \times [0,1] \to X/G$. Thus it suffices to prove that $|\Delta(S_p(G))|$ is G-contractible. Quillen [Qu, 2.4] proved that it is contractible via the maps

$$Q \mapsto QP \mapsto P \qquad (Q \in S_p(G)\,)$$

and this contraction is obviously G-equivariant (see also [TW, 1.2]). □

Our strategy is now simply to use the results of Section 1 to reduce to the situation where $O_p(G) \neq 1$. We first consider Theorem A of the introduction.

(2.2) PROPOSITION. *If G is p-solvable, then $\Sigma_p(G)/G$ is contractible.*

Proof. Let $K = O_{p'}(G)$, the largest normal subgroup of G of order prime to p, and let $Q = G/K$. By Proposition 1.2, $\Sigma_p(G)/G$ is isomorphic to $\Sigma_p(Q)/Q$. But by definition of a p-solvable group, the group Q has a non-trivial normal p-subgroup. Therefore by Lemma 2.1, $\Sigma_p(Q)/Q$ is contractible. □

Next we prove Theorem B of the introduction.

(2.3) PROPOSITION. *Let P be a Sylow p-subgroup of G and assume that P is either abelian, or generalized quaternion, or TI. Then $\Sigma_p(G)/G$ is contractible.*

Proof. If P is abelian, then it was already known to Burnside that $H = N_G(P)$ controls p-fusion in G. Therefore by Proposition 1.3, $\Sigma_p(G)/G \cong \Sigma_p(H)/H$ and the latter is contractible by Lemma 2.1. For the reader's convenience, we recall Burnside's argument. If $Q, Q^g \leq N_G(P)$, then $Q, Q^g \leq P$, and so $Q \leq P$ and $Q \leq P^{g^{-1}}$. Since P is abelian, P and $P^{g^{-1}}$ are Sylow p-subgroups of $C_G(Q)$, hence conjugate by some $c \in C_G(Q)$. It follows that $g^{-1}c \in N_G(P)$, as required.

If P is a generalized quaternion 2-group, then P has a unique subgroup Z of order 2. But $N_G(Z)$ controls 2-fusion in G and we conclude again by Proposition 1.3 and Lemma 2.1. We also recall why $N_G(Z)$ controls 2-fusion. If $Q, Q^g \leq N_G(Z)$, then $Z, Z^g \leq N_G(Z)$. But since Z is the unique subgroup of $N_G(Z)$ of order 2, we have $Z = Z^g$, hence $g \in N_G(Z)$.

If P is *TI*, then $N_G(P)$ controls p-fusion in G and we conclude again by Proposition 1.3 and Lemma 2.1. The proof that $N_G(P)$ controls fusion is a straightforward consequence of the fact that every p-subgroup Q is contained in a unique Sylow p-subgroup. □

REMARKS. (a) The Brauer-Suzuki theorem on groups G with a generalized quaternion Sylow 2-subgroup asserts that $G/O_{p'}(G)$ has a central subgroup of order 2. Thus the result above is also a consequence of Proposition 1.2.

(b) It is clear that the argument of Proposition 2.3 says that if G has a subgroup H which controls p-fusion and which has a normal p-subgroup, then $\Sigma_p(G)/G$ is contractible.

We end this paper with another application of Proposition 1.3.

(2.4) PROPOSITION. *If Webb's conjecture (for the prime p) holds for the symmetric group S_{pn}, then it also holds for each symmetric group S_{pn+k} where $0 \leq k \leq p-1$.*

Proof. It is well-known that S_{m-1} controls p-fusion in S_m if p does not divide m. The proof is left to the reader. Repeated applications of this property lead to the result, thanks to Proposition 1.3. □

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