Pressure driven long wavelength MHD instabilities in an axisymmetric toroidal resistive plasma

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A general set of equations that govern global resistive interchange, resistive internal kink and resistive infernal modes in a toroidal axisymmetric equilibrium are systematically derived in detail. Tractable equations are developed such that resistive effects on the fundamental rational surface can be treated together with resistive effects on the rational surfaces of the sidebands. Resistivity introduces coupling of pressure driven toroidal instabilities with ion acoustic waves, while compression introduces flute-like flows and damping of instabilities, enhanced by toroidal effects. It is shown under which equilibrium conditions global interchange, internal kink modes or infernal modes occur. The $m = 1$ internal kink is derived for the first time from higher order infernal mode equations, and new resistive infernal modes resonant at the $q = 1$ surface are reduced analytically. Of particular interest are the competing effects of resistive corrections on the rational surfaces of the fundamental harmonic and on the sidebands, which in this paper is investigated for standard profiles developed for the $m = 1$ internal kink problem.

I. INTRODUCTION

Pressure driven instabilities have unique and counter intuitive properties in laboratory plasmas. The fact that toroidal and full electromagnetic effects are leading order of importance implies that pressure driven instabilities should be weak. But nevertheless they often determine the operating limits of experimental plasma scenarios.

The control of MHD oscillations such as tearing modes and Edge Localised Modes (ELMs) has become critical for developing scenarios for tokamak reactors such as ITER and European DEMO. Experiments in JET show that tearing modes enhance heavy impurity transport [1], while infrequent large ELMs are dangerous for plasma facing components, and also enhance the risk of heavy impurity ingress. Consequently, experimental effort has been focussed on constructing equilibria that can avoid sawteeth (which can seed tearing modes) and also avoid ELMs. A common feature of these scenarios can be an extended region of low magnetic shear, something that is particularly pronounced in so called hybrid scenarios, which typically have the safety factor $q \approx 1$ in the core region. Pressure driven instabilities in equilibria with such q-profiles are sometimes studied analytically with so called infernal mode equations [2, 3], these equations having been extended to include a vacuum region (so called exfernal equations [4]) for modelling [5] of edge harmonic oscillations (EHOs) which occur during ELM free QH-modes. Nevertheless, these model equations are not always appropriate for modelling experiments if the safety factor evolves so that it is resonant with the main harmonic of the pressure driven long wavelength instability, or its sidebands. An analytic model [6] has been developed for resistive effects on the sidebands of infernal modes, but not for resistive effects of the main harmonic, or both. One of the topics treated in this paper is to address these deficiencies.

The main aim of this paper is to develop a unified set of equations which can treat long wavelength pressure driven internal instabilities with resistive corrections. The equations will also describe short wavelength interchange modes. The only internal modes out of scope are short wavelength ballooning modes. Hence, the system of equations will be shown to reproduce toroidal internal kink modes (including $m = 1$ modes [7]) with resistive corrections [8], resistive interchange (twisting) modes [9–11] (as well as current driven tearing modes), and new resonant resistive infernal modes. From the novel set of equations it will be shown that new classes of pressure driven internal kink modes can be described analytically (though approximately), via both ideal and resistive descriptions. Other more complicated modes are identified, but their solution is left for future numerical analysis. The derivation of these equations is quite involved, but it is not too difficult to convey providing enough detail is given. Indeed, justification of the lengthy derivation provided here might readily be accepted given that the $m = 1$ internal kink problem is developed fully, something that is beyond almost all papers and all textbooks. The system of equations developed here also provides the essentials for a code which aims to efficiently solve for long wavelength pressure driven instabilities in a torus, with or without resistive effects.

The paper is organised as follows. In section II the coupled toroidal resistive eigenvalue equations are derived in detail. Following that section III summarises the eigenvalue equations, discusses appropriate boundary conditions for ideal and resistive problems, and defines simplified equations for equilibria with ultra-low magnetic shear. Section IV derives the $m = 1$ ideal internal kink mode in a torus, recovering the results of Bussac *et. al.* [7]. Section V identifies the conditions under which the resistive infernal equations describe resistive interchange modes. Section VI investigates infernal modes with and without rational surfaces. We identify new classes of resonant infernal modes, and as an example we investigate in detail the competing effects of resistivity on the main harmonic and sideband of the $m = n = 1$ internal kink mode in an equilibrium with low magnetic shear in the core. A summary of the paper and a list of future related work is presented in section VII.

II. DEVELOPMENT OF A GLOBAL LINEAR SYSTEM OF RESISTIVE MHD EQUATIONS FOR LONG WAVELENGTH PRESSURE DRIVEN INSTABILITIES IN A TORUS

This section derives the global linear resistive MHD equations for long wavelength internal instabilities in a torus. It is undertaken in detail, which we hope will be valuable both for understanding the results of this paper, and as a reference. In this paper we do not treat instabilities fundamentally driven by corrections associated with magnetic fluctuations extending beyond the plasma (e.g. external kink modes), nor those strongly affected by them (e.g. EHOs). Such modes, which may be treated in future work, are not internal modes for which all fluctuations vanish at the plasma-vacuum interface.

A. Convenient displacement variables in a resistive plasma

The following linearised momentum equation in a static MHD plasma is valid also in a resistive plasma:

$$\boldsymbol{X} \equiv -\rho\gamma^2\boldsymbol{\xi}+(\boldsymbol{\delta B}\cdot\boldsymbol{\nabla})\boldsymbol{B}+(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\delta B}-\boldsymbol{\nabla}(\boldsymbol{\delta B}\cdot\boldsymbol{B})-\boldsymbol{\nabla}\delta P = 0, \tag{1}$$

where $(\boldsymbol{\delta B}\cdot\boldsymbol{\nabla})\boldsymbol{B}+(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\delta B}-\boldsymbol{\nabla}(\boldsymbol{\delta B}\cdot\boldsymbol{B})$ is perturbed $\boldsymbol{j}\times\boldsymbol{B}$, and δP is the total perturbed pressure:

$$\delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi}, \tag{2}$$

(Γ the adiabatic index). In the ideal MHD model $\partial\boldsymbol{\xi}_\perp/\partial t$ is the perpendicular flow associated with E-cross-B flow, i.e. $\partial\boldsymbol{\xi}_\perp/\partial t = \boldsymbol{\delta E}\times\boldsymbol{B}/B^2$. But, in the resistive MHD model, resistive Ohm's law $\boldsymbol{\delta E}+\partial\boldsymbol{\xi}/\partial t\times\boldsymbol{B} = \eta\boldsymbol{\delta j}$ applies, so that,

$$\frac{\partial\boldsymbol{\xi}_\perp}{\partial t} = -\frac{\eta(\boldsymbol{\nabla}\times\boldsymbol{\delta B})\times\boldsymbol{B}}{B^2} + \frac{\boldsymbol{\delta E}\times\boldsymbol{B}}{B^2}.$$

For resistive MHD problems it can be convenient to retain $\boldsymbol{\xi}$ as an eigenfunction variable (in particular the radial component of $\boldsymbol{\xi}$). We can develop resistive Ohm's law, using Faraday's law, to yield,

$$\frac{\partial\boldsymbol{\delta B}}{\partial t} = \boldsymbol{\nabla}\times\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\times\boldsymbol{B}\right) - \eta\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\delta B}),$$

so that the expected ideal result is obtained in the limit $\eta = 0$. Letting $\partial/\partial t = \gamma$, as expected for modes of type $\delta B \sim \exp(\gamma t)$, one may write

$$\boldsymbol{\delta B} = \boldsymbol{\delta B}_I + \boldsymbol{\Delta B} \tag{3}$$

where the ideal field $\boldsymbol{\delta B}_I$ and resistive correction $\boldsymbol{\Delta B}$ are respectively

$$\boldsymbol{\delta B}_I = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}) \tag{4}$$

$$\boldsymbol{\Delta B} = -\frac{\eta}{\gamma}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\delta B}) = \frac{\eta}{\gamma}\nabla^2(\boldsymbol{\delta B}). \tag{5}$$

Now for the first novelty of this paper. We are free to choose a gauge where the vector potential parallel to the equilibrium magnetic field is zero. Thus, letting $\boldsymbol{\delta A} = \boldsymbol{\xi}_R\times\boldsymbol{B}$ one may write the perturbed magnetic field in terms of a displacement $\boldsymbol{\xi}_R$ that accounts for ideal and resistive motion,

$$\boldsymbol{\delta B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}_R\times\boldsymbol{B}). \tag{6}$$

Equating Eq. (6) with Eq. (3), and using (4) one easily sees that the resistive correction to the magnetic field is

$$\boldsymbol{\Delta B} = -\boldsymbol{\nabla}\times(\boldsymbol{\Delta\xi}\times\boldsymbol{B}) \tag{7}$$

where the following resistive corrected 'displacement' variable will be used:

$$\boldsymbol{\Delta\xi} = \boldsymbol{\xi} - \boldsymbol{\xi}_R. \tag{8}$$

Using all the above results one may write down the differential equations for the resistive correction $\boldsymbol{\Delta\xi}$:

$$\boldsymbol{\Delta B}(\boldsymbol{\Delta\xi}) = \frac{\eta}{\gamma}\nabla^2[\boldsymbol{\delta B}_I(\boldsymbol{\xi}) + \boldsymbol{\Delta B}(\boldsymbol{\Delta\xi})], \tag{9}$$

with $\nabla^2[\boldsymbol{X}]$ the vector-Laplacian, operating on vector $\boldsymbol{X}$. The radial component of Eq. (9) will ultimately be the differential equation relating the radial components of $\boldsymbol{\xi}$ and $\Delta\xi$, which must be solved in concert with the main eigenvalue equation. The toroidal component of Eq. (9) establishes the relations between the radial and poloidal components of $\boldsymbol{\xi}$ to relevant order, together with consideration of the radial vorticity.

B. Inverse aspect ratio expansion of the equilibrium

Concerning the equilibrium, the magnetic field for the right handed flux coordinate system (r, θ, ϕ) is,

$$\boldsymbol{B} = F(r)\boldsymbol{\nabla}\phi + \frac{d\psi}{dr}\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}r,$$

so that $\boldsymbol{B}\cdot\boldsymbol{\nabla}\phi = F/R^2$. In the above form of the field,

$$\frac{d\psi}{dr} = \frac{rF(r)}{q(r)R_0},$$

where $R_0 = R(r=0)$ and we will later have $B_0 = B(r = 0)$. The equilibrium and linearised stability equations are expanded in local inverse aspect ratio $\epsilon = r/R_0$ and $q(r)$ is the safety factor. The equilibrium will adopt $\beta \equiv 2P/B^2 \sim \epsilon^2$. With this conventional beta ordering, one obtains $F(r) = F_0 + F_2(r)$ where $F_0 = R_0B_0$ is a constant and $F_2(r)/F_0 \sim \epsilon^2$ is a small radially varying correction,

$$\frac{dF_2}{dr} = B_0\left(\frac{\alpha}{2q^2} - \frac{\epsilon}{q^2}(2-s)\right),$$

where we recognise the important parameters for ballooning modes and interchange modes, $s = (r/q)dq/dr$ and $\alpha = -2R_0q^2(dP/dr)/B_0^2$. Expansion of the equilibrium also defines the Shafranov shift $\Delta(r)$, where

$$\frac{d\Delta(r)}{dr} = \epsilon\left[\frac{q(r)^2}{r^4}\int_0^r dr\frac{r^3}{q(r)^2} + \beta_p(r)\right],$$
$$\beta_p(r) = -\frac{2q(r)^2}{B_0^2\epsilon^2r^2}\int_0^r dr\, r^2\frac{dP}{dr}.$$

Ultimately the equilibrium is defined in terms of a poloidal coordinate associated with Jacobian:

$$\mathcal{J} = \frac{1}{\boldsymbol{\nabla}r\cdot(\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\phi)} = \frac{rR(r,\theta)^2}{R_0}. \tag{10}$$

For the sake of brevity we do not outline the equilibrium expansion in terms of this straight field line coordinate in further detail.

C. Inverse aspect ratio expansion of perturbations

The recent paper on parallel magnetic fluctuations [12] provides an explanation of the ϵ expansion of magnetic and fluid fluctuations. A purpose of the current paper is to generalise the results to include resistive contributions to magnetic field fluctuations. It is a slightly awkward problem. The reason being that the flute (fundamental harmonic) component of the ideal radial perturbed field $\boldsymbol{\delta B}_I\cdot\boldsymbol{\nabla}r$ vanishes on the rational surface, while the resistive correction to the perturbed field ($\boldsymbol{\Delta B}\cdot\boldsymbol{\nabla}r$) does not. As will be seen, this means that $\boldsymbol{\Delta\xi}\cdot\boldsymbol{\nabla}r$ is singular on the rational surface. It nevertheless remains convenient to use $\boldsymbol{\Delta\xi}$ and $\boldsymbol{\xi}_R$ as variables because solution of the linear problem turns out to involve operations on $\Delta q\Delta\xi^r$, which is not singular (here $\Delta q = q - m/n$). It is noted that $\Delta B^r \propto \Delta q\Delta\xi^r$, which is of course not singular on the rational, except under the special case approaching marginal stability, where all variables are singular.

In this work we attempt a general approach which encompasses solutions compatible with resistive interchange and resistive infernal modes in a torus. Resistive interchange modes occur in the limit of large magnetic shear in the region around $q_s = m/n$, or large mode numbers, while resistive infernal modes introduce important corrections over the interchange description if the magnetic shear s is weak. Both can be unstable in plasmas with standard tokamak pressure gradients. We will apply our work to cases where strong ballooning corrections are not important, though crucially we keep weak ballooning effects associated with infernal mode corrections. This is achieved by assuming long wavelength instabilities (small toroidal and poloidal mode numbers), for which it will be necessary to carry only nearest neighbour poloidal harmonics that satellite the main harmonic m in the assumed nearly circular cross section geometry. It is recalled [9] that resistive interchange modes can be unstable for cases with large magnetic shear because the addition of resistivity tends to nullify the large magnetic field line bending stabilisation occurring in an ideal plasma. In the core of the plasma where the magnetic shear is expected to be weak, a relatively weak ballooning parameter $\alpha \sim \epsilon$ will drive infernal modes.

The following concerning spatial components and inverse aspect ratio ordering applies equally to $\boldsymbol{\xi}_R$, $\boldsymbol{\xi}$, and $\boldsymbol{\Delta\xi}$ (given the relative ordering assumption described just above), though in what follows directly below it is chosen to apply the discussion to $\boldsymbol{\xi}$ (which reduces notation). We follow [7] and allow

$$\boldsymbol{\xi} = \boldsymbol{\xi}_B + \zeta\boldsymbol{B}, \tag{11}$$

where $\boldsymbol{\xi}_B$ is chosen to have the property $\boldsymbol{\xi}_B\cdot\boldsymbol{\nabla}\phi = 0$. With this assumption, $\boldsymbol{\xi}_B$ has two covariant components. The contravariant variables are defined as

$$\xi^r = (F/F_0)\boldsymbol{\xi}_B\cdot\boldsymbol{\nabla}r \;\text{ and }\; \xi^\theta = (F/F_0)r\boldsymbol{\xi}_B\cdot\boldsymbol{\nabla}\theta.$$

Now ξ^r and $\Delta\xi^r$ are chosen to be the fundamental variables for which the eigenvalue equations are to be solved. On considering a (Shafranov) shifted near-circular equilibrium, it can be shown that only the fundamental component and the directly neighbouring poloidal sidebands of ξ^r (and ξ^r_R, $\Delta\xi^r$) contribute to the stability problem to relevant order in the expansion of the equation of motion. No other components are required (unless the plasma cross section is manifestly non-circular), i.e.

$$\xi^r = \xi^{r(m)} + \xi^{r(m+1)} + \xi^{r(m-1)} + O(\epsilon^4) \tag{12}$$

where

$$\xi^{r(l)} = \hat{\xi}^{r(l)}(r)\exp(in\phi - il\theta + \gamma t).$$

We now consider the above in the context of an expansion in the local inverse aspect ratio $\epsilon = r/R_0$. We expand the radial eigenfunctions as follows:

$$\xi^r = \xi_0^r + \epsilon\xi_1^r + \epsilon^2\xi_2^r + \epsilon^3\xi_3^r,$$

where ϵ above is a tag used simply to identify the ordering of the terms. It can be shown that the sidebands are order ϵ smaller than the fundamental harmonic in the region where they coexist. An expansion that satisfies Eq. (12) to relevant order is

$$\xi_0^r = \xi_0^{r(m)} = \hat{\xi}_0^{r(m)} \exp(in\phi - im\theta + \gamma t)$$
$$\xi_1^r = \xi_1^{r(m+1)} + \xi_1^{r(m-1)}, \quad \xi_1^{r(l)} = \hat{\xi}_1^{r(l)} \exp(in\phi - il\theta + \gamma t)$$
$$\xi_2^r = 0,$$

and it can be shown that $\boldsymbol{\xi}_3$ doesn't enter the stability problem to the order calculated in this work (this will be discussed later). Freely setting $\epsilon^2\xi_2^r = 0$ in the expansion of the equations that follow applies equally to the resistive variable $\Delta\xi^r$. In particular, with this choice, the fundamental displacement is given by ξ_0^r at all relevant orders of accuracy. When expanding the equations to highest order, the necessary corrections to the ideal and resistive displacement variables ($\boldsymbol{\xi}$ and $\boldsymbol{\Delta\xi}$) are taken up by the poloidal component of the displacements. Hence it is seen that,

$$\xi^\theta = \xi_0^\theta + \epsilon\xi_1^\theta + \epsilon^2\xi_2^\theta + \epsilon^3\xi_3^\theta.$$

It turns out the equations that involve ξ_2^θ are the same for a cylindrical and toroidal equilibrium assumption. Since eigenfunctions have a unique poloidal mode number in a cylinder, it is clear that ξ_2^θ has fundamental harmonic only. One thus has:

$$\xi_0^\theta = \xi_0^{\theta(m)} = \hat{\xi}_0^{\theta(m)} \exp(in\phi - im\theta + \gamma t)$$
$$\xi_1^\theta = \xi_1^{\theta(m+1)} + \xi_1^{\theta(m-1)}, \quad \xi_1^{\theta(l)} = \hat{\xi}_1^{\theta(l)} \exp(in\phi - il\theta + \gamma t)$$
$$\xi_2^\theta = \xi_2^{\theta(m)} = \hat{\xi}_2^{\theta(m)} \exp(in\phi - im\theta + \gamma t).$$

Hence, the role of ξ_2^θ is to correct the flute contribution to $\boldsymbol{\xi}_B$ obtained at lower order in the governing equations. As mentioned earlier, all that has been written here applies to ideal and resistive displacements. Finally, ξ_3^θ and ξ_3^r will be considered in more detail later.

The parallel component of the fluid displacement ζ can be similarly Fourier analysed, i.e,

$$\zeta = \zeta^{(m)} + \zeta^{(m+1)} + \zeta^{(m-1)}, \quad \zeta^l = \hat{\zeta}^{(l)}(r)\exp[in\phi - il\theta + \gamma t].$$

But, care needs to be taken with the expansion since it is **not** found that $\zeta^{(m\pm1)} \sim \epsilon\hat{\zeta}^{(m)}$. For ideal MHD, it can be shown that the flute component $\hat{\zeta}^{(m)} = 0$, i.e. dominant parallel displacements arrive from the sidebands. For resistive MHD, a novel result is that $\hat{\zeta}^{(m)} \sim \zeta^{(m\pm1)}$, apparently indicating important flute-like inertia contributions to the growth rate. Due to the rather different nature of the ordering in ϵ encountered for ζ, the subscript notation denoting the ordering in ϵ is not adopted for ζ.

Finally, as will seen in the forthcoming sections, it is convenient to split $\boldsymbol{X}$ defined in Eq. (1) as follows,

$$\boldsymbol{X} = \boldsymbol{X}_P + \boldsymbol{X}_I, \quad \text{with,} \tag{13}$$
$$\boldsymbol{X}_P = (\boldsymbol{\delta B}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\delta B} - \boldsymbol{\nabla}(\boldsymbol{\delta B}\cdot\boldsymbol{B}) + \boldsymbol{\nabla}(\boldsymbol{\xi}_R\cdot\boldsymbol{\nabla}P) + \boldsymbol{\nabla}(\boldsymbol{\Delta\xi}\cdot\boldsymbol{\nabla}P)^{(m)} + \boldsymbol{\nabla}(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m)} \tag{14}$$
$$\boldsymbol{X}_I = -\rho\gamma^2\boldsymbol{\xi} + \boldsymbol{\nabla}(\boldsymbol{\Delta\xi}\cdot\boldsymbol{\nabla}P)^{(m+1)} + \boldsymbol{\nabla}(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m+1)} + \boldsymbol{\nabla}(\boldsymbol{\Delta\xi}\cdot\boldsymbol{\nabla}P)^{(m-1)} + \boldsymbol{\nabla}(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m-1)}. \tag{15}$$

Subscript I denotes inertia, and P potential. The result is general providing only one upper and one lower sideband is required (which is valid for circular cross section to relevant order in inverse aspect ratio). The breaking up of $\boldsymbol{X}$ in this way is convenient when performing vorticity operations, since near the rational surface of the fundamental mode, the resistive corrections of $(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m\pm1)}$ exactly cancel with $(\boldsymbol{\Delta\xi}\cdot\boldsymbol{\nabla}P)^{(m\pm1)}$, both being inertia-like. The remaining inertia terms (i.e. those involving the growth rate γ) in $\boldsymbol{X}_I$ are much simplified. Finally, the fundamental harmonic of $\boldsymbol{\nabla}(\boldsymbol{\Delta\xi}\cdot\boldsymbol{\nabla}P) + \boldsymbol{\nabla}(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})$ provides important contributions to $\boldsymbol{X}_P$ that are not connected to inertia.

D. Plasma compression: resistive effects and magnetic shear

The purpose of this section is to identify the limitations of the problem investigated, and to examine some fundamental properties. This section is redundant in the ideal limit. It is first noted that the contravarient definitions of the magnetic fields are defined exactly in terms of the full resistive displacement:

$$\delta B^r \equiv \boldsymbol{\delta B}\cdot\boldsymbol{\nabla}r = \boldsymbol{B}\cdot\boldsymbol{\nabla}\xi_R^r = \frac{F_0}{R^2}\left[\frac{1}{q}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\right]\xi_R^r, \tag{16}$$

$$\delta B^\theta \equiv r\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}\theta = -\frac{F_0}{R^2}\left[\frac{\partial}{\partial r}\left(\frac{r\xi_R^r}{q}\right) - \frac{\partial\xi_R^\theta}{\partial\phi}\right], \tag{17}$$

$$\delta B^\phi \equiv R\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}\phi = -\frac{F_0}{rR}\left[\frac{\partial\xi_R^\theta}{\partial\theta} + \frac{\partial(r\xi_R^r)}{\partial r}\right], \tag{18}$$

with $\partial/\partial\phi = in$. Also note that the magnetic operator $\boldsymbol{B}\cdot\boldsymbol{\nabla}$ has been defined above. The three components of the field are given by two components of the resistive displacement. The properties of these equations will be examined in detail in the next section. Similarly

$$\Delta B^r \equiv \boldsymbol{\Delta B}\cdot\boldsymbol{\nabla}r = -\boldsymbol{B}\cdot\boldsymbol{\nabla}\Delta\xi^r = -\frac{F_0}{R^2}\left[\frac{1}{q}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\right]\Delta\xi^r \tag{19}$$

$$\Delta B^\theta \equiv r\boldsymbol{\Delta B}\cdot\boldsymbol{\nabla}\theta = \frac{F_0}{R^2}\left[\frac{\partial}{\partial r}\left(\frac{r\Delta\xi^r}{q}\right) - \frac{\partial\Delta\xi^\theta}{\partial\phi}\right] \tag{20}$$

$$\Delta B^\phi \equiv R\boldsymbol{\Delta B}\cdot\boldsymbol{\nabla}\phi = \frac{F_0}{rR}\left[\frac{\partial\Delta\xi^\theta}{\partial\theta} + \frac{\partial(r\Delta\xi^r)}{\partial r}\right], \tag{21}$$

where it is recalled that $\boldsymbol{\delta B} = \boldsymbol{\delta B}_I + \boldsymbol{\Delta B}$.

The toroidal component of Ohm's law Eq. (5) is examined next:

$$\Delta B^\phi = \frac{\eta}{\gamma}\left[\nabla^2 \boldsymbol{\delta B}\right]\cdot e_\phi.$$

It is instructive to briefly consider perturbations in an (R, Z, ϕ) coordinate system. The equilibrium can be expanded easily in this orthogonal coordinate system. A covariant expansion of the perturbed field is,

$$\boldsymbol{\delta B} = \boldsymbol{\delta B}_R \boldsymbol{e}_R + \boldsymbol{\delta B}_Z \boldsymbol{e}_Z + \boldsymbol{\delta B}_\phi \boldsymbol{e}_\phi$$

where $\boldsymbol{e}_Z$, $\boldsymbol{e}_Z$ and $\boldsymbol{e}_\phi$ are unit vectors and $\boldsymbol{\delta B}_\phi = \boldsymbol{\delta B}^\phi$, i.e. Eq. (18) applies to the toroidal covariant and contravariant forms. With this coordinate system the vector Laplacian operating on $\boldsymbol{\delta B}$ is exactly,

$$\nabla^2 \boldsymbol{\delta B} = \left[\nabla^2 \delta B_R - \frac{1}{R^2}\delta B_R - \frac{2}{R^2}\frac{\partial \delta B_\phi}{\partial \phi}\right]\boldsymbol{e}_R + \left[\nabla^2 \delta B_Z\right]\boldsymbol{e}_Z + \left[\nabla^2 \delta B_\phi - \frac{1}{R^2}\delta B_\phi + \frac{2}{R^2}\frac{\partial \delta B_R}{\partial \phi}\right]\boldsymbol{e}_\phi.$$

The toroidal component of resistive Ohm's law Eq. (5) is therefore,

$$\Delta B^\phi = \left(\frac{\eta}{\gamma R_0^2}\right)\frac{R_0^2}{r^2}\left[\left(r^2\nabla^2 - \left(\frac{r}{R}\right)^2\right)\delta B^\phi + 2\left(\frac{r}{R}\right)^2 in\delta B_R\right]$$

δB_R can be constructed from the contravariant components δB^r and δB^θ defined in Eqs. (16) and (17). From inspection of the dependence of δB^r, δB^θ and δB^ϕ on $\boldsymbol{\xi}_R$ in Eqs. (16) - (18) it is clear that formally $\delta B^r \sim \epsilon\delta B^\phi$ and $\delta B^\theta \sim \epsilon\delta B^\phi$. It therefore follows that formally $\delta B_R \sim \epsilon\delta B^\phi$ and thus, using $\Delta B^\phi = \delta B^\phi - \delta B_I^\phi$ gives

$$\delta B_I^\phi = \delta B^\phi - \frac{\eta}{\gamma}\left[\nabla^2 - \frac{1}{R_0^2}\right]\delta B^\phi + O(\epsilon\delta B^\phi).$$

We drop the $1/R_0^2$ term in the expansion above, since $\nabla^2 \sim \epsilon^{-2}/R_0^2$ or faster. Hence,

$$\delta B_I^\phi = \delta B^\phi - \frac{\eta}{\gamma}\nabla^2\delta B^\phi + O(\epsilon\delta B^\phi). \tag{22}$$

We may now attempt to develop an evolution equation for the perturbed pressure Eq. (2), i.e. $\delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$. The first job is to consider $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$. From Eq. (11), and $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_B + \boldsymbol{B}\cdot\boldsymbol{\nabla}\zeta. \tag{23}$$

Now using $\boldsymbol{\xi}_B\cdot\boldsymbol{\nabla}\phi = 0$ the following properties hold:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_B =& \frac{1}{\mathcal{J}}\left[\frac{\partial}{\partial r}\left(\mathcal{J}\boldsymbol{\xi}_B\cdot\boldsymbol{\nabla}r\right) + \frac{\partial}{\partial\theta}\left(\mathcal{J}\boldsymbol{\xi}_B\cdot\boldsymbol{\nabla}\theta\right)\right]\\ =& \frac{1}{\mathcal{J}}\left[\frac{\partial}{\partial r}\left(\frac{\mathcal{J}F_0}{F}\xi^r\right) + \frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}F_0}{rF}\xi^\theta\right)\right]\\ =& \frac{1}{\mathcal{J}}\left[r\xi^r\frac{\partial}{\partial r}\left(\frac{\mathcal{J}F_0}{rF}\right) + \xi^\theta\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}F_0}{rF}\right)\right.\\ &\left. + \underbrace{\left(\frac{\partial(r\xi^r)}{\partial r} + \frac{\partial\xi^\theta}{\partial\theta}\right)}_{-\frac{R_0\mathcal{J}}{F_0 R}\delta B_I^\phi}\frac{\mathcal{J}F_0}{rF}\right].\end{aligned} \tag{24}$$

We let

$$K = \frac{1}{\mathcal{J}}\left[r\xi^r\frac{\partial}{\partial r}\left(\frac{\mathcal{J}F_0}{rF}\right) + \xi^\theta\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}F_0}{rF}\right)\right]$$

and we note that K is related to a projection of $\boldsymbol{\xi}_B$ with the magnetic curvature, in particular:

$$K = -2\boldsymbol{\xi}_B\cdot\boldsymbol{\kappa}[1 + O(\epsilon)].$$

It follows that,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = -\frac{R}{F}\delta B_I^\phi + K + \boldsymbol{B}\cdot\boldsymbol{\nabla}\zeta$$

and therefore from Eq. (2),

$$\delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \Gamma P\left[K + \boldsymbol{B}\cdot\boldsymbol{\nabla}\zeta - \frac{R}{F}\delta B_I^\phi\right].$$

On substituting Eq. (22), we obtain to relevant order,

$$\delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \Gamma P\left[K + \boldsymbol{B}\cdot\boldsymbol{\nabla}\zeta - \frac{R}{F}\left(\delta B^\phi - \frac{\eta}{\gamma}\nabla^2\delta B^\phi\right)\right]$$

To develop an approximate equation for the evolution of δP, we recall that flute dominated modes conserve the magnetic curvature (see e.g. Ref. [12], which for an isotropic plasma requires that $\delta P = -B\delta B_\parallel$). Since $\delta B_\parallel = \delta B^\phi[1 + O(\epsilon)]$, we may adopt $\delta B^\phi = -\delta P/B$ in the above, to yield to relevant order,

$$\delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \Gamma P\left[K + \boldsymbol{B}\cdot\boldsymbol{\nabla}\zeta - \frac{\eta}{\gamma B^2}\nabla^2\delta P\right], \tag{25}$$

where we keep, for now, the $\nabla^2\delta P$ term, despite it being order β^2 smaller than the other terms, in case radial derivatives are very strong (which, as will be seen later, will manifest itself as large magnetic shear). In the above, we recognise each term in Eq. (6d) of [10]. In particular, we recognise the last term, which is related to cross field classical transport. The differential equation for δP is therefore,

$$\left[1 - \frac{\Gamma P}{B^2}\frac{\eta}{\gamma}\nabla^2\right]\delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \Gamma P\left[K + \boldsymbol{B}\cdot\boldsymbol{\nabla}\zeta\right].$$

The resistive term on the left hand side introduces considerable complexity. The main results section of GGJ [9] dropped this term, but it was retained as a major line of research in Refs. [11] and [10] in order to investigate the stablisation of resistive interchange and resistive ballooning modes by compression in a strongly sheared plasma. It is instructive at this point to write down a schematic of the terms that comprise the toroidal vorticity operation on the equation of motion $V_\phi^{(m)}$ (see definition later) for the resistive infernal mode problem studied here. We heuristically include contributions we expect from the inclusion of cross field classical transport, as indicated in the governing stability criteria equations of Refs. [11] and

[10]. In particular where ideal modes are stable, we expect, assuming a parabolic type pressure profile, and an exact resonance at $q(r_s) = q_s$:

$$\frac{V_\phi^{(m)}}{\xi^r} \sim s_R^2 + \epsilon\alpha\left[1 - \frac{1}{q_s^2}\right] + s\epsilon\alpha - \frac{1}{m}\frac{q_a}{q_s}\left(\frac{r_s}{a}\right)^2\alpha^2 + s^2\beta\Gamma(1+2q_s^2), \tag{26}$$

marginal stability being $V_\phi^{(m)} = 0$. The first term in $V_\phi^{(m)}$ above corresponds to field line bending stabilisation associated with resistive modes. Here we use subscript R to indicate diminished effective shear by resistivity. It is well known that field line bending stabilisation caused by magnetic shear is strongly impaired for cases where ideal modes are stable but resistive modes are unstable, hence the effective shear s_R has the property $s_R \ll s$. The second term is the standard Mercier-interchange contribution. The third term is a correction to the interchange contribution due to compression and resistivity. We derive it and include its effects in this work, but it has been seen before in Ref. [10], e.g. the content of the square bracket in Eq. (40h) in Ref. [10]. The fourth term is the drive from infernal modes. It is written here in terms of the ideal drive, where the scaling compared to the Mercier term can be seen directly in Eq. (33) of Ref. [12] (note the scaling is obtained via a particular choice of q-profile). The last term is the stabilising effect of classical transport on compression. It is directly associated with the last term in Eq. (25).

It is clear that the classical transport term can compete with the Mercier term and the compression effect in the third term only if $s \sim 1$. And, it can compete with the infernal mode drive only if $s \sim m^{-1}(\alpha/\epsilon)^{1/2}$, where we take $(r_s/a)^2 q_a/q_s \sim 1$. We recall that our main interest is for infernal modes, where for core modes typically $s \sim \epsilon$ and α ranges from $\alpha \sim \epsilon$ to $\alpha \sim 1$ at r_s. Near the edge where we might use infernal mode physics to describe edge harmonic oscillations, we may indeed have $s \sim 1$, but we would have $\alpha \sim 1$ and again $\beta \sim \epsilon^2$. It therefore appears that there are no obvious regimes of interest in a tokamak where the classical resistive diffusion contribution is leading order for long wavelength pressure driven resistive instabilities. Mathematically, it is clear that we may neglect classical transport correction providing that $s < \alpha$, or at most, $s \sim \alpha$. We will later treat ϵ and s as separate expansion parameters, adopting $\alpha \sim \epsilon$, and in the final collection of terms $s \sim \epsilon$. This is done to ease the analytical calculations, but the final equations can in principle treat applications where $\alpha \sim 1$ with $\beta \sim \epsilon^2$, together with $s \sim \epsilon$ or $s \sim 1$ (e.g. for edge harmonic oscillations).

Treating problems for which $s \sim \alpha$, or less, thereby legitimately dropping the classical transport contribution in all the follows, we henceforth adopt

$$\delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \Gamma P\left[K + \boldsymbol{B}\cdot\boldsymbol{\nabla}\zeta\right]. \tag{27}$$

Hence, we deploy $\delta P = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P - \Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$, with

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \frac{1}{\mathcal{J}}\left[r\xi^r\frac{\partial}{\partial r}\left(\frac{\mathcal{J}F_0}{rF}\right) + \xi^\theta\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}F_0}{rF}\right)\right] + \boldsymbol{B}\cdot\boldsymbol{\nabla}\zeta. \tag{28}$$

In addition, for convenience we take the magnetic shear to be a small expansion parameter, and legitimately, keep only linear terms in s for all contributions except the field line bending terms, where clearly quadratic terms are needed. This will be explained in more detail later.

E. Elimination of poloidal displacements

One can either solve the radial vorticity equation, or the poloidal projection of the equation of motion, to yield the results of this section. In either case, operations are on $\boldsymbol{X}$ defined in Eq. (1). We choose the radial vorticity operation on $\boldsymbol{X} = 0$:

$$V_r^{(l)}(\boldsymbol{X}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\frac{\exp(il\theta)}{il}\mathcal{J}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{X}\times\boldsymbol{\nabla}r}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\phi}\right) = 0. \tag{29}$$

The effects of finite inertia are not necessary for the calculations in this section, whose purpose is to identify the poloidal components of the displacement in terms of the radial displacements. The effects of inertia appear at an order in ϵ higher than is relevant for this section. Hence our purposes are then met by solving:

$$V_r^{(l)}(\boldsymbol{X}_P) = 0.$$

Noting $P = P(r)$ one then solves the following for $l = m-1$, $l = m$, $l = m+1$:

$$0 = V_r^{(l)}\left[(\boldsymbol{\delta B}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\delta B} - \boldsymbol{\nabla}(\boldsymbol{\delta B}\cdot\boldsymbol{B}) + \boldsymbol{\nabla}\left\{(\xi_{R0}^r + \Delta\xi_0^r + \Delta\xi_{\Gamma 0}^r)\frac{dP}{dr}\right\}\right], \tag{30}$$

with $\boldsymbol{\delta B}$ given in terms of $\boldsymbol{\xi}_R$ according to Eq. (6) (here we drop for convenience the (m) notation for leading order displacements). Since $\beta \sim \epsilon^2$, the displacements inside the $\{\}$ brackets are required only to leading order, and it is sufficient to take $F = F_0$ inside the $\{\}$ brackets. Concerning $\Delta\xi_{\Gamma 0}^r$ in Eq. (30), the section that solves the parallel momentum equation obtains $(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m)}$ in terms of the convenient variable $\Delta\xi_{\Gamma 0}^r$ (see Eq. (48)). It will be shown that on adopting the definition of $\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ from Eq. (28), one has

$$(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m)} = \Delta\xi_{\Gamma 0}^r\frac{dP}{dr}, \quad \text{with}$$
$$\Delta\xi_{\Gamma 0}^r = -\Delta\xi_0^r\left(\frac{\omega_s^2(nq-m)^2}{\omega_s^2(nq-m)^2 + \gamma^2 q^2}\right),$$

where we note already the connection with sound waves which have dispersion relation $\omega^2 = k_\parallel^2 v_s^2 = (nq-m)^2\omega_s^2/q^2$, with $\omega_s = V_s/R = \sqrt{\Gamma P/(\rho R_0^2)}$ the sound frequency. Note that $\Delta\xi^r_{\Gamma 0}$ will play a role in the vicinity of the rational surface unless $\gamma \gg (n - m/q)\omega_s$, which will not be the case except very close to the rational surface (note that we may expect $\gamma \lessapprox \epsilon\omega_s$. This convenient representation for $(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m)}$ would not be possible if we had included the classical transport contribution to δP.

1. Resistive displacements variables

The contravarient definitions of the magnetic fields are defined exactly in terms of the full resistive displacement, as given in Eqs. (16) - (18). Covariant forms of $\boldsymbol{\delta B}$ are also easily obtained in terms of $\boldsymbol{\xi}_R$, as required in Eq. (30). Those properties help fulfill an objective of this section, which is to obtain ξ^θ_R in terms of ξ^r_R, $\Delta\xi^r_0$ and $\Delta\xi^r_{\Gamma 0}$ for each order in ϵ. The result of the expansion of the radial vorticity is:

$$O(\epsilon^0) : V_r^{(m)}(\boldsymbol{X}) = 0 \quad \longrightarrow \quad \xi^\theta_{R0} = -i\frac{[r\xi^r_{R0}]'}{m} \tag{31}$$

$$O(\epsilon^1) : V_r^{(m\pm1)}(\boldsymbol{X}) = 0 \quad \longrightarrow \quad \xi^{\theta(m\pm1)}_{R1} = -i\frac{\left[r\xi^{r(m\pm1)}_{R1}\right]'}{m\pm1} \tag{32}$$

$$O(\epsilon^2) : V_r^{(m)}(\boldsymbol{X}) = 0 \quad \longrightarrow \xi^\theta_{R2} = i\frac{\epsilon}{m}\left\{\frac{\alpha}{2q^2}\left(\xi^r_0 + \Delta\xi^r_{\Gamma 0}\right) + \epsilon\frac{n}{mq}\left[\left(\frac{n}{m}+\frac{2}{q}\right)\Delta q\xi^r_{R0} + \frac{n}{m}r\left(\Delta q\xi^r_{R0}\right)'\right]\right\} \tag{33}$$

$$O(\epsilon^3) : V_r^{(m)}(\boldsymbol{X}) = 0 \quad \longrightarrow \quad \xi^{\theta(m)}_{R3} = 0 \tag{34}$$

where we have used $\xi^r_0 = \xi^r_{R0} + \Delta\xi^r_0$, $X' = \partial X/\partial r$. Substitution of the above results into Eq. (18) gives that δB^ϕ is zero to order ϵ^0 and ϵ^1. In addition, δB^ϕ to order ϵ^2 is obtained by substituting the solution ξ^θ_{R2} given above, and $\xi^r_{R2} = 0$, into Eq. (18):

$$\begin{aligned}\delta B^\phi &= -\frac{B_0}{R}\left(\frac{i\epsilon}{m}\right)^{-1}\xi^\theta_{R2} \\ &= -\frac{B_0}{R}\left\{\frac{\alpha}{2q^2}\left(\xi^r_0 + \Delta\xi^r_{\Gamma 0}\right) + \right. \\ &\left.\epsilon\frac{n}{mq}\left[\left(\frac{n}{m}+\frac{2}{q}\right)\Delta q\xi^r_{R0} + \frac{n}{m}r\left(\Delta q\xi^r_{R0}\right)'\right]\right\}.\end{aligned}$$

Finally, as we see above, $O(\epsilon^3) : V_r^{(m)}(\boldsymbol{X}) = 0$ yields that $\xi^{\theta(m)}_{R3} = 0$. One can also obtain from $O(\epsilon^3) : V_r^{(m\pm1)}(\boldsymbol{X}) = 0$ relationships between $\xi^{\theta(m\pm1)}_{R3}$ and $\xi^{r(m\pm1)}_{R1}$ and ξ^r_{R0}, but neither these harmonics, nor higher harmonics, contribute to the stability problem at relevant order.

2. Fluid displacement variables

We now obtain the important relations between the fluid displacement components ξ^r and ξ^θ (note, so far we have obtained the relations for the resistive displacements ξ^r_R, ξ^θ_R). This is undertaken via the relation of Eq. (22), i.e. $\delta B^\phi_I = \delta B^\phi - (\eta/\gamma)\nabla^2\delta B^\phi$. We recall it was argued that it is appropriate to drop the classical transport term in the pressure evolution equation (25). Doing so clearly requires that

$$\frac{\eta}{\gamma}\nabla^2 \sim \epsilon^{-1}$$

or smaller, on assuming $\beta \sim \epsilon^2$ and $\Gamma \sim 1$. Since we have obtained that $\delta B^\phi = \delta B^\phi_2$, i.e. $\delta B^\phi_0 = 0$ and $\delta B^\phi_1 = 0$, it follows that

$$\delta B^\phi_{I0} \equiv -\frac{F_0}{rR}\left[\frac{\partial\xi^\theta_0}{\partial\theta} + \frac{\partial(r\xi^r_0)}{\partial r}\right] = 0, \quad \longrightarrow \quad \xi^\theta_0 = -i\frac{[r\xi^r_0]'}{m}. \tag{35}$$

Had we retained the classical diffusion effect via assuming an ordering $\beta(\eta/\gamma)\nabla^2 \sim 1$, we would obtain a leading order correction in the relationship between ξ^θ_0 and ξ^r_0. Presumably this explains partly why the associated contribution manifests itself in Refs. [11] and [10] with the toroidal inertia enhancement factor $1+2q^2$. Finally, it will be seen later that we do not require a relationship between ξ^r and ξ^θ to the next order in ϵ. However, for solving the global problem, we may wish to take inertia into account in the sideband equations, and this will require a relationship between $\xi^{\theta(m\pm1)}_1$ and $\xi^{r(m\pm1)}_1$. Importantly, inertia for the sidebands matters only in the high shear region, where the sidebands are resonant, and in this region clearly classical transport associated with the main mode is negligible. In addition, any classical transport effects associated directly with the sidebands is negligible to the required order of accuracy. Thus $\delta B^\phi_{I1} = \delta B^\phi_1 = 0$, so that where required, we may safely adopt,

$$\xi^{\theta(m\pm1)}_1 = -i\frac{\left[r\xi^{r(m\pm1)}_1\right]'}{m\pm1}. \tag{36}$$

Equation (35) will be required in the solution for the parallel momentum equation, and in all the inertia contributions to the toroidal vorticity.

It is useful to point out some additional properties that follow from the results of this section, and the last one. Since $\boldsymbol{\nabla}\cdot\boldsymbol{\delta B} = 0$, it is seen that

$$\frac{\partial}{\partial r}[\mathcal{J}\delta B^r] + \frac{\partial}{\partial\theta}[\mathcal{J}(\delta B^\theta/r)] + \frac{\partial}{\partial\phi}[\mathcal{J}(\delta B^\phi/R)] = 0.$$

Since $\delta B^\phi = 0$ to order ϵ:

$$\frac{\partial}{\partial r}[\mathcal{J}\delta B^r] + \frac{\partial}{\partial\theta}[\mathcal{J}(\delta B^\theta/r)] + O(\epsilon^2) = 0, \tag{37}$$

which can easily be verified on inspection of Eqs. (16) - (18), recalling that $\mathcal{J} = R^2 r/R_0$ and setting $\delta B^\phi = 0$.

The poloidal component of the field can be Fourier analysed in terms of the radial component of the field if needed (the perturbed magnetic fields will be looked at in detail later when considering the radial component of Ohm's law and the closure of the resistive MHD equations).

3. *Special case* $m = 1$

Note for the $m = 1$ special case (e.g. quasi-interchange mode for the case $m = n = 1$), Eq. (32) does not hold for the lower sideband displacement $m - 1 = 0$. The vorticity definition $V_r^{(m-1)}$ is not useful, since the fact that $\partial/\partial\theta = 0$ for lower sideband perturbations of $m = 1$ modes means that the radial covariant component of $\boldsymbol{X}$ does not enter $V_r^{(m-1)}$. We perform instead

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \exp[i(m-1)\theta)] \left(\frac{\boldsymbol{X}_P \cdot \boldsymbol{e}_r}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\phi}\right) = 0$$

with $m = 1$, where we note that $\boldsymbol{X}_P \cdot \boldsymbol{e}_r = 0$, due to there being no inertial layer for this special case (unless $q = 0$ exists in the plasma). At the required order, we may interchange between a radial contravarient or radial covariant representation X_{Pr}. The result is

$$r^2 \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\xi_{R1}^{r(0)}\right)\right] = 0$$

which is simply

$$\frac{d\,\delta B_1^{\phi(0)}}{dr} = 0$$

on noting that $\partial\xi^{\theta(0)}\partial\theta = 0$ for $m = 1$. The general solution is $\xi_{R1}^{r(0)} = c_1 r^{-1} + c_2 r$. Noting that this is the solution in the whole plasma, boundary conditions at $r = 0$ and $r = a$ are met only for $c_1 = c_2 = 0$. Hence the result of this section is that

$$\xi_{R1}^{r(0)} = 0 \tag{38}$$

so that the development of equations involving the lower sibeband for the $m = 1$ case will involve $\xi_{R1}^{\theta(0)}$, which will ultimately be analytically eliminated from the governing eigenvalue equations. Regarding the fluid displacement for $m = 1$ we find that $\delta B_{I1}^{\phi(0)} = 0$, and

$$\xi_1^{r(0)} = 0.$$

F. Parallel flow in a resistive plasma, inertia $\boldsymbol{X}_I$ and plasma compression

In this section we eliminate the parallel displacement in favour of the radial displacement. In so doing, we can obtain tractable forms for the inertia $\boldsymbol{X}_I$ and the effect of compression on the perturbed pressure. This is undertaken most conveniently by taking the dot product of the full momentum equation with the full magnetic field $\boldsymbol{B} + \boldsymbol{\delta B}$, then linearising afterwards, i.e.

$$\begin{aligned} \text{Linearisation}\left\{\left[-\rho\gamma^2\boldsymbol{\xi} + (\boldsymbol{j}+\boldsymbol{\delta j})\times(\boldsymbol{B}+\boldsymbol{\delta B}) - \boldsymbol{\nabla}(P+\delta P)\right]\cdot(\boldsymbol{B}+\boldsymbol{\delta B})\right\} = 0 \\ -\rho\gamma^2\boldsymbol{\xi}\cdot\boldsymbol{B} - \boldsymbol{\delta B}\cdot\boldsymbol{\nabla}P - \boldsymbol{B}\cdot\boldsymbol{\nabla}\delta P = 0 \end{aligned} \tag{39}$$

where $\boldsymbol{B}$ and P are considered equilibrium quantities in the above. The parallel momentum equation of Eq. (39) will also be Fourier analysed, with both the sidebands and main harmonic playing a role. It is important to highlight the contribution $-\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}P$, which vanishes for ideal MHD on the rational surface (i.e. where the inertia is relevant), but fundamentally does not vanish for resistive MHD.

Adopting Eq. (11) for the full plasma displacement $\boldsymbol{\xi}$, the parallel momentum equation of Eq. (39) becomes, on setting $\boldsymbol{B}\cdot\boldsymbol{\xi} = \zeta B^2$ (noting that $\boldsymbol{B}\cdot\boldsymbol{\xi}_B \sim \epsilon\zeta B^2$ and thus neglecting it),

$$\rho\gamma^2 B^2\zeta = -\frac{dP}{dr}\Delta B^r + \Gamma P\boldsymbol{B}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right) \tag{40}$$

where the following has been used:

$$\delta B^r = \delta B_I^r + \Delta B^r \quad \text{and} \quad \boldsymbol{B}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P\right) = \frac{dP}{dr}\delta B_I^r,$$

and where

$$\begin{aligned} \delta B_I^r &= (\boldsymbol{B}\cdot\boldsymbol{\nabla})(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}r) = \frac{F_0}{F}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi^r, \\ \Delta B^r &= -(\boldsymbol{B}\cdot\boldsymbol{\nabla})(\boldsymbol{\Delta\xi}\cdot\boldsymbol{\nabla}r) = -\frac{F_0}{F}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\Delta\xi^r. \end{aligned}$$

The objective of this section is to solve Eq. (40) for ζ (noting that the last term in Eq. (40) is dependent on ζ), thus ultimately enabling the construction of $\boldsymbol{X}_I$ required for the toroidal vorticity calculation. The plasma is not incompressible. It is therefore necessary to examine $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$, for which we may refer to Eq. (28), i.e.

$$\nabla\cdot\boldsymbol{\xi} = \frac{1}{\mathcal{J}}\left[r\xi^r\frac{\partial}{\partial r}\left(\frac{\mathcal{J}F_0}{rF}\right) + \xi^\theta\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}F_0}{rF}\right)\right] + \boldsymbol{B}\cdot\nabla\zeta.$$

It is only necessary to obtain $\nabla\cdot\boldsymbol{\xi}$ directly in terms of leading order fluid displacement. From Eq. (35), and in addition using $\mathcal{J}(r,\theta) = rR_0[1+2(r/R_0)\cos\theta + O(\epsilon^2)]$ and $F = F_0(1+O(\epsilon^2))$, easily yields the leading order result,

$$\nabla\cdot\boldsymbol{\xi}_B = \frac{2R_0}{R^2}\left[\xi_0^r\cos\theta - \xi_0^\theta\sin\theta\right], \quad \text{with} \quad \xi_0^\theta = -i\frac{[r\xi_0^r]'}{m}. \tag{41}$$

It is now possible to solve for ζ, which is written in the general convenient form:

$$\zeta(r,\theta,\phi,t) = \hat{\zeta}(r,\theta)\exp(in\phi - im\theta + \gamma t).$$

Since

$$\boldsymbol{B}\cdot\nabla = \frac{F}{qR^2}\left(q(r)\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}\right)$$

(where $\psi' = rF/(qR_0)$ has been used) one obtains

$$\boldsymbol{B}\cdot\nabla\zeta = \frac{F_0}{qR^2}\left[\frac{\partial\hat{\zeta}}{\partial\theta} + \hat{\zeta}i(nq-m)\right]\exp(in\phi - im\theta + \gamma t). \tag{42}$$

Hence

$$\begin{aligned}\boldsymbol{B}\cdot\nabla\left(\nabla\cdot\boldsymbol{\xi}\right) =& \frac{F_0}{q^2R_0R^2}\left\{B_0\left[\frac{\partial^2\hat{\zeta}}{\partial\theta^2} + i(nq-m)\frac{\partial\hat{\zeta}}{\partial\theta}\right.\right.\\ &\left.\left. -(nq-m)^2\hat{\zeta}\right] - 2q\left[\hat{\xi}_0^r(r)\cos\theta - \hat{\xi}_0^\theta(r)\sin\theta\right]\right\}\\ &\exp(in\phi - im\theta + \gamma t).\end{aligned}$$

Substituting these results into Eq. (40) and noting the sound frequency $\omega_s = \Gamma P/(\rho R_0^2)$ yields,

$$\begin{aligned}\gamma^2q^2\left(\frac{RB}{R_0B_0}\right)^2 B_0\hat{\zeta} =& \frac{\omega_s^2}{\Gamma}\left(\frac{r}{P}\frac{dP}{dr}\right)\left(\frac{iq^2\Delta\chi}{\epsilon}\right)\exp(-in\phi + im\theta - \gamma t)\\ &+\omega_s^2\left\{B_0\left[\frac{\partial^2\hat{\zeta}}{\partial\theta^2} + i2(nq-m)\frac{\partial\hat{\zeta}}{\partial\theta} - (nq-m)^2\hat{\zeta}\right] - 2q\omega_s^2\left[\hat{\xi}_0^r(r)\sin\theta - \hat{\xi}_0^\theta(r)\cos\theta\right]\right\},\end{aligned} \tag{43}$$

where

$$\Delta\chi = i\frac{R^2\Delta B^r}{F_0} \equiv -i\left(\frac{1}{q}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\right)\Delta\xi^r.$$

Equation (19) has been used in the last identity. We solve Eq. (43) for ζ to significant order in ϵ. We may replace $(RB/(R_0B_0))^2$ with unity since corrections occur at order ϵ^2. We will check after Fourier analysing ζ that it was consistent to drop $\boldsymbol{\xi}_B\cdot\boldsymbol{b}$ on the left hand side of the parallel momentum equation, and to drop higher order terms in Eq. (41). The following Fourier expansion is assumed:

$$\hat{\zeta}(r,\theta) = \hat{\zeta}^{(m-1)}(r)\exp(i\theta) + \hat{\zeta}^{(m)}(r) + \hat{\zeta}^{(m+1)}(r)\exp(-i\theta).$$

Similarly,

$$\begin{aligned}\Delta\chi(r,\theta,\phi) =& \left[\hat{\Delta\chi}^{(m-1)}(r)\exp(i\theta) + \hat{\Delta\chi}^{(m)}(r)\right.\\ &\left.+\hat{\Delta\chi}^{(m+1)}(r)\exp(-i\theta)\right]\exp(in\phi - im\theta + \gamma t)\end{aligned}$$

so that,

$$\begin{aligned}q\hat{\Delta\chi}^{(m-1)}(r) &= [nq(r) - (m-1)]\hat{\Delta\xi}^{r(m-1)}(r),\\ q\hat{\Delta\chi}^{(m)}(r) &= [nq(r) - m]\hat{\Delta\xi}^{r(m)}(r)\\ q\hat{\Delta\chi}^{(m+1)}(r) &= [nq(r) - (m+1)]\hat{\Delta\xi}^{r(m+1)}(r).\end{aligned}$$

It is therefore seen that,

$$\gamma^2 q^2 B_0 \left[\hat{\zeta}^{(m-1)}\exp(i\theta)+\hat{\zeta}^{(m)}+\hat{\zeta}^{(m+1)}\exp(-i\theta)\right] =$$
$$\frac{\omega_s^2}{\Gamma}\left(\frac{r}{P}\frac{dP}{dr}\right)\frac{iq}{\epsilon}\left[q\hat{\Delta\chi}^{(m-1)}\exp(i\theta)+q\hat{\Delta\chi}^{(m)}+q\hat{\Delta\chi}^{(m+1)}\exp(-i\theta)\right]$$
$$-\omega_s^2 B_0\left\{\hat{\zeta}^{(m-1)}\exp(i\theta)+\hat{\zeta}^{(m+1)}\exp(-i\theta)+2(nq-m)\left[\hat{\zeta}^{(m-1)}\exp(i\theta)-\hat{\zeta}^{(m+1)}\exp(-i\theta)\right]\right.$$
$$\left.+(nq-m)^2\left[\hat{\zeta}^{(m-1)}\exp(i\theta)+\hat{\zeta}^{(m)}+\hat{\zeta}^{(m+1)}\exp(-i\theta)\right]\right\}$$
$$+\omega_s^2 q\left[i\hat{\xi}_0^r(\exp(i\theta)-\exp(-i\theta))-\hat{\xi}_0^\theta(\exp(i\theta)+\exp(-i\theta))\right].$$

One then obtains the Fourier coefficients:

$$B_0\hat{\zeta}^{(m-1)} = q\left[\left(\frac{r}{P}\frac{dP}{dr}\right)\left(\frac{iq\hat{\Delta\chi}^{(m-1)}}{\Gamma\epsilon}\right)-(\hat{\xi}_0^\theta - i\hat{\xi}_0^r)\right]\left[\frac{\omega_s^2}{\omega_s^2(1+(nq-m))^2+\gamma^2q^2}\right] \tag{44}$$

$$B_0\hat{\zeta}^{(m)} = q\left(\frac{r}{P}\frac{dP}{dr}\right)\left(\frac{iq\hat{\Delta\chi}^{(m)}}{\Gamma\epsilon}\right)\left(\frac{\omega_s^2}{(nq-m)^2\omega_s^2+\gamma^2q^2}\right) \tag{45}$$

$$B_0\hat{\zeta}^{(m+1)} = q\left[\left(\frac{r}{P}\frac{dP}{dr}\right)\left(\frac{iq\hat{\Delta\chi}^{(m+1)}}{\Gamma\epsilon}\right)-(\hat{\xi}_0^\theta + i\hat{\xi}_0^r)\right]\left[\frac{\omega_s^2}{\omega_s^2(1+(nq-m))^2+\gamma^2q^2}\right], \tag{46}$$

where of course Eq. (35) is used to eliminate $\hat{\xi}_0^\theta$ in favour of $\hat{\xi}_0^r$. Resistive corrections are those involving the harmonics of $\Delta\chi$. In standard ideal MHD the flute harmonic is zero at the order in which the ideal sidebands appear. This justifies the approximation adopted for the derivatives of the Jacobian, and the neglect of $\boldsymbol{b}\cdot\boldsymbol{\xi}_B$ in $\boldsymbol{b}\cdot\boldsymbol{\xi}$. However, resistivity introduces apparently potentially large flute flows, associated with the radial magnetic perturbation on the rational surface (the fluctuation that causes the magnetic island). For locally large magnetic shear, the flute corrections can be expected to be as large as the sideband flows, which generate the usual ideal MHD enhanced inertia factor. The effect of the new flute flow will need to be carefully established. Although $\zeta^{(m)}$ appears to be singular on the $q = m/n$ surface (at the ideal MHD accumulation point $\gamma^2 = 0$), the inertia $\gamma^2 B_0\zeta^{(m)}$ has no singularity. The parallel flow is clearly connected to sound waves, which have dispersion relation $\omega^2 = k_\parallel^2 V_s^2$, with $V_s = \omega_s R$ the sound velocity as mentioned earlier.

Resistive corrections to the sidebands in the parallel displacement are also considered. However, it will now be seen that the effect of the resistive corrections to sidebands cancel in the inertia contribution to $\boldsymbol{X}$, i.e. in Eq. (15). It is seen that $\boldsymbol{X}_I$ and $\boldsymbol{X}$ requires calculation of $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$. Noting Eqs. (28), (41) and Eq. (42), and using the above solutions for the ζ harmonics yields,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \frac{1}{R_0}\left(\frac{\exp(in\phi - im\theta+\gamma t)}{\omega_s^2(1+(nq-m))^2+\gamma^2q^2}\right)\left[\omega_s^2\left(\frac{r}{P}\frac{dP}{dr}\right)\left(\frac{q\hat{\Delta\chi}^{(m+1)}\exp(-i\theta)-q\hat{\Delta\chi}^{(m-1)}\exp(i\theta)}{\Gamma\epsilon}\right)\right.$$
$$+2q^2\gamma^2\left(\hat{\xi}_0^r\cos\theta-\hat{\xi}_0^\theta\sin\theta\right)$$
$$\left.-\omega_s^2\left(\frac{r}{P}\frac{dP}{dr}\right)\frac{q\hat{\Delta\chi}^{(m)}}{\Gamma\epsilon}\left(\frac{(nq-m)[\omega_s^2(1+(nq-m))^2+\gamma^2q^2]}{(nq-m)^2\omega_s^2+\gamma^2q^2}\right)\right]. \tag{47}$$

Concerning the contribution $(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m)}$ in $\boldsymbol{X}_P$, using $q\Delta\chi^{(m)} = (nq-m)\Delta\xi^r$, it is clearly seen that,

$$(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m)} = -\frac{dP}{dr}(nq-m)q\Delta\chi^{(m)}\left(\frac{\omega_s^2}{(nq-m)^2\omega_s^2+\gamma^2q^2}\right) = -\frac{dP}{dr}(nq-m)^2\Delta\xi^{r(m)}\left(\frac{\omega_s^2}{(nq-m)^2\omega_s^2+\gamma^2q^2}\right).$$

Hence adopting the convenient notation employed in Eq. (30):

$$(\Gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m)} = \Delta\xi^r_{\Gamma 0}\frac{dP}{dr}$$

we clearly obtain

$$\Delta\xi^r_{\Gamma 0} = -\Delta\xi^r_0 \left(\frac{\omega_s^2 (nq-m)^2}{(nq-m)^2\omega_s^2 + \gamma^2 q^2} \right). \tag{48}$$

Moving to the inertia component of the momentum equation, $\boldsymbol{X}_I$, we note that $q\hat{\Delta\chi}^{(m+1)} = -(1 - n\Delta q)\hat{\Delta\xi}^{r(m+1)}$ and $q\hat{\Delta\chi}^{(m-1)} = (1 + n\Delta q)\hat{\Delta\xi}^{r(m-1)}$. Adopting $\gamma^2 q^2 \ll \omega_s^2$, and dropping terms of type $n\Delta q \Delta\xi^{(m\pm1)}$ which vanish for $\Delta q \to 0$ (unlike $\Delta q \Delta\xi^{(m)}$ which does not vanish for $\Delta q \to 0$) we easily obtain, for Eq. (15), to leading relevant order in Δq and ϵ:

$$\boldsymbol{X}_I = -\rho\gamma^2 \Big\{ \boldsymbol{\xi}_B^{(m)} + B_0 \zeta^{(m)} \boldsymbol{b} \\ -R_0 \boldsymbol{\nabla} \left[2q^2 \left(\xi^r_0 \cos\theta - \xi^\theta_0 \sin\theta \right) \right] \Big\}, \tag{49}$$

with $B_0\zeta^{(m)}$ given by Eq. (45). The resistive sidebands associated with $\Delta\chi^{(m\pm1)}$ contained in $(\Gamma P \boldsymbol{\nabla}\cdot\boldsymbol{\xi})^{(m\pm1)}$ cancel with terms of type $(\boldsymbol{\Delta\xi}\cdot\boldsymbol{\nabla}P)^{(m\pm1)}$ in $\boldsymbol{X}_I$ (to leading order in Δq), which justifies the convenient definition of $\boldsymbol{X}_I$ in Eq. (15).

G. Ohm's Law

This section obtains the required relation between ξ^r and $\Delta\xi^r$, which together with the toroidal vorticity operation on $\boldsymbol{X}$, closes the system of equations. The governing eigenvalue equation will depend only on the main harmonic of $\Delta\xi^r$ and ξ^r, and thus only this harmonic is required in radial Ohm's law. Note in this section that subscript zero is removed from the displacements (all expressions are in terms of leading order (in ϵ) displacements). The radial component of resistive Ohm's law Eq. (5) is

$$\Delta B^r = \frac{\eta}{\gamma} \left(\nabla^2 \boldsymbol{\delta B} \right)\cdot \boldsymbol{e}_r \quad \text{with} \quad \boldsymbol{\delta B} = \boldsymbol{\delta B}_I + \boldsymbol{\Delta B}.$$

Notice that the relation involves the vector-Laplacian. It turns out that it is sufficient to neglect toroidicity to the required accuracy. Thus, adopting (r,θ,z) cylindrical geometry, and using $\partial/\partial z = (1/R)\partial/\partial\phi$, we have that,

$$\left(\nabla^2 \boldsymbol{\delta B}\right)\cdot \boldsymbol{e}_r = \nabla^2 \delta B^r - \frac{1}{r^2}\delta B^r - \frac{2}{r^2}\frac{\partial \delta B^\theta}{\partial\theta}, \quad \text{with} \\ \nabla^2 \delta B^r = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \delta B^r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \delta B^r}{\partial\theta^2} + \frac{1}{R^2}\frac{\partial^2 \delta B^r}{\partial\phi^2}$$

and $\delta B^r = \boldsymbol{\delta B}\cdot\boldsymbol{\nabla} r = \boldsymbol{\delta B}\cdot\boldsymbol{e}_r$, and $\delta B^\theta = \boldsymbol{\delta B}\cdot\boldsymbol{e}_\theta = r\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}\theta$. We may eliminate δB^θ via Eq. (37), i.e. to required order in ϵ:

$$\frac{\partial}{\partial\theta}\delta B^\theta_0 = -\frac{\partial}{\partial r}\left(r\delta B^r_0 \right).$$

For the main harmonic we have $\partial^2 \delta B^r_0/\partial\theta^2 = -m^2 \delta B^r_0$, and to relevant order $R^{-2}\partial^2/\partial\phi^2 = 0$. Thus,

$$\left(\nabla^2 \boldsymbol{\delta B}_0\right)\cdot \boldsymbol{e}_r = \nabla^2 \delta B^r_0 - \frac{1}{r^2}\delta B^r_0 + \frac{2}{r^2}\frac{\partial}{\partial r}\left(r\delta B^r_0 \right), \quad \text{with} \\ \nabla^2 \delta B^r_0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta B^r_0}{\partial r}\right) - m^2 \frac{\delta B^r_0}{r^2} \\ = \frac{1}{r^3}\left[\frac{\partial}{\partial r}\left(r^3 \frac{\partial \delta B^r_0}{\partial r}\right) + r(1-m^2)\delta B^r_0 \right].$$

Substituting these results into radial resistive Ohm's law gives:

$$\Delta B^r_0 = \frac{\eta}{\gamma}\left(\nabla^2 \boldsymbol{\delta B_0}\right)\cdot\boldsymbol{e}_r \\ = \frac{\eta}{\gamma}\frac{1}{r^3}\left[\frac{\partial}{\partial r}\left(r^3 \frac{\partial \delta B^r_0}{\partial r}\right) + r(1-m^2)\delta B^r_0 \right]$$

Writing the latter equation in terms of the magnetic operator and radial displacements (Eqs. (16) and (19)) gives for the main harmonic:

$$-\boldsymbol{B}\cdot\boldsymbol{\nabla}\Delta\xi^{r(m)} = \frac{\eta}{\gamma}\frac{1}{r^3}\left[\frac{\partial}{\partial r}\left(r^3 \frac{\partial}{\partial r}\left\{ \boldsymbol{B}\cdot\boldsymbol{\nabla}\xi_{R0}\right\}\right) + r(1-m^2)\boldsymbol{B}\cdot\boldsymbol{\nabla}\xi_R^{r(m)} \right] \\ = \frac{\eta}{\gamma}\frac{1}{r^3}\left[\frac{\partial}{\partial r}\left(r^3 \frac{\partial}{\partial r}\left\{ \boldsymbol{B}\cdot\boldsymbol{\nabla}(\xi^{r(m)} - \Delta\xi^{r(m)})\right\}\right) + r(1-m^2)\boldsymbol{B}\cdot\boldsymbol{\nabla}(\xi^{r(m)} - \Delta\xi^{r(m)}) \right].$$

This equation can be written conveniently in terms of the variable $\Delta\chi^{(m)} = q^{-1}(nq-m)\Delta\xi^{r(m)}$ used in the parallel momentum equation above. Taking the leading order identity for the magnetic operator, and the leading order perturbations (fundamental harmonic), one obtains

$$\chi^{(m)} = \frac{\eta}{\gamma}\frac{1}{r^3}\left[\frac{d}{dr}\left(r^3 \frac{d}{dr}\left\{ q_s \left(\frac{1}{q} - \frac{1}{q_s}\right)\xi^{r(m)} + \chi^{(m)} \right\}\right) + r(1-m^2)\left\{ q_s\left(\frac{1}{q} - \frac{1}{q_s}\right)\xi^{r(m)} + \chi^{(m)}\right\}\right] \tag{50}$$

where we identify the variable,

$$\chi^{(m)} \equiv \frac{\Delta\chi^{(m)}}{n} = \frac{\Delta q \Delta\xi^{r(m)}}{q}. \tag{51}$$

As will be seen, similar expressions to (50) can be identified for the sideband variables providing that the main harmonic is not dominant near the rational surfaces of the sidebands (this is usually the case). Such an equation will be required where resistive effects are included on all relevant rational surfaces.

It is useful to clearly write down the full radial magnetic field in various ways in terms of the variables we have used in this paper. The standard ideal and resistive corrected component in terms of these variables should be clear in the following:

$$\begin{aligned}\delta B^r &= \delta B^r_I + \Delta B^r \\ &= \boldsymbol{B}\cdot\boldsymbol{\nabla}\xi^{r(m)} - \boldsymbol{B}\cdot\boldsymbol{\nabla}\Delta\xi^{r(m)} \\ &= -\frac{B_0}{R_0} im \left[\left(\frac{1}{q}-\frac{n}{m}\right)\xi^{r(m)} - \left(\frac{1}{q}-\frac{n}{m}\right)\Delta\xi^{r(m)}\right] \\ &= -\frac{B_0}{R_0} in \left[q_s\left(\frac{1}{q}-\frac{n}{m}\right)\xi^{r(m)} + \Delta q\frac{\Delta\xi^{r(m)}}{q}\right] \\ &= -\frac{B_0}{R_0} in \left[q_s\left(\frac{1}{q}-\frac{n}{m}\right)\xi^{r(m)} + \chi^{(m)}\right]. \end{aligned} \quad (52)$$

Or, in terms of the magnetic flux $\delta\psi$ and the resistive displacement,

$$\delta B^r = \frac{im}{r}\delta\psi = -\frac{B_0}{R_0} inq_s\left(\frac{1}{q}-\frac{n}{m}\right)\xi^{r(m)}_R. \quad (53)$$

H. Resolution of eigenvalue equations via toroidal vorticity

Having obtained the parallel and poloidal components of the displacements in terms of the radial displacements (resistive and ideal), we can now use the toroidal vorticity operation to yield an eigenvalue equation in terms of radial component of the displacements. The toroidal vorticity operating on $\boldsymbol{X}$ is

$$V^{(l)}_\phi(\boldsymbol{X}) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{\exp(il\theta)}{il}\frac{R_0\mathcal{J}}{B_0}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{X}\times\boldsymbol{\nabla}\phi}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\phi}\right). \quad (54)$$

First we solve for $V^{(m)}_\phi(\boldsymbol{X}_P)$, then for $V^{(m)}_\phi(\boldsymbol{X}_I)$, and then sum them together $V^{(m)}_\phi(\boldsymbol{X}_P) + V^{(m)}_\phi(\boldsymbol{X}_I) = 0$, thus giving the main eigenvalue equation. The main eigenvalue equation depends on the sidebands of the fundamental displacement. The latter are eliminated by solving the sideband harmonics of the toroidal vorticity: $V^{(m\pm1)}_\phi(\boldsymbol{X}) = 0$. It is first necessary to explain that a double expansion is performed, in ϵ and in shear s. Hence the main vorticity equation is expanded as:

$$V_\phi = \epsilon^0 V_{\phi 0} + \epsilon^1 V_{\phi 1} + \epsilon^1 V_{\phi 1} + \epsilon^2 V_{\phi 2} + \epsilon^3 V_{\phi 3}$$

where

$$V_{\phi i} = s^0 V_{\phi i,0} + s^1 V_{\phi i,1} + s^2 V_{\phi i,2} + s^3 V_{\phi i,3}$$

and $\epsilon^i s^j$ appearing in front of coefficients $V_{\phi i,j}$ are just tags, which are used to indicate the above expansion, but are not included in the expansion that follows below.

1. Solving for $V^{(m)}_\phi(\boldsymbol{X}_P)$

We now solve the main harmonic $V^{(m)}_\phi(\boldsymbol{X}_P)$. We do this using the variables ξ^r, $\Delta\xi^r$ and $\Delta\xi^r_\Gamma$. Hence we write ξ^r_R in terms of ξ^r, $\Delta\xi^r$, and we use the properties from the earlier sections to eliminate poloidal displacement components etc (note for $m = 1$ case the radial component of the lower sideband is eliminated instead). The main harmonic displacements that appear in the following expressions are the leading order (in ϵ), and for convenience we henceforth drop the subscript zero denoting the ordering of the displacement. The lowest order expression for $V^{(m)}_\phi(\boldsymbol{X}_P) = 0$ in ϵ, to all orders in s is:

$$\begin{aligned}V^{(m)}_{\phi 0}(\boldsymbol{X}_P) =& \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{r(m)}_R}{dr}\right] \\ & -(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi^{r(m)}_R. \end{aligned} \quad (55)$$

This expression can be written in terms of the $\Delta q\Delta\xi^{r(m)}$ (and hence in $\chi^{(m)} = \Delta q\Delta\xi^{r(m)}/q$, which is not singular on the rational surface):

$$\begin{aligned}V^{(m)}_{\phi 0}(\boldsymbol{X}_P) =& \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{r(m)}}{dr}\right] \\ &+\frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d}{dr}\left(\frac{\Delta q\Delta\xi^{r(m)}}{q}\right)\right. \\ &\left.-\frac{r^3}{q_s}\left(\frac{\Delta q\Delta\xi^r_0}{q}\right)\frac{d}{dr}\left(\frac{1}{q}\right)\right] \\ &-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi^{r(m)} \\ &-(m^2-1)\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\left(\frac{\Delta q\Delta\xi^{r(m)}}{q}\right) \end{aligned} \quad (56)$$

where we have also used the identity,

$$\frac{1}{q}-\frac{1}{q_s} = -\frac{\Delta q}{qq_s}.$$

The expression $V^{(m)}_{\phi 0}(\boldsymbol{X}_P) \sim \epsilon^0 s^2\xi^r$ near the rational surface in an ideal plasma. But, in a resistive plasma, where ideal modes are stable, it can be seen from inspection of Ohm's law that $\chi^{(m)} \equiv \Delta q\Delta\xi^{r(m)}/q \approx \Delta q\xi^{r(m)}/q$. As a consequence, field line bending is much diminished in the resistive regime. We may thus write that $V^{(m)}_{\phi 0}(\boldsymbol{X}_P) \sim \epsilon^0 s_R^2\xi^r$, where s_R is an effective shear that is diminished by resistivity, i.e. $s_R^2 \ll s^2$.

Moving now to the next order in ϵ it is found that $V^{(m)}_{\phi 1} = 0$. To the next order in ϵ it is noted that it is not only convenient, but necessary to first exploit Eq. (32) (or Eq. (38) for the case of $m = 1$). Not doing so would require unnecessary higher order calculation of the metric. Hence using the properties from the section on radial vorticity (or equivalent for the $m=1$ case), we obtain the following expansion in the shear:

$$\begin{aligned}
V_{\phi 2}^{(m)}(\boldsymbol{X}_P) =& V_{\phi 2,0}^{(m)}(\boldsymbol{X}_P) + V_{\phi 2,1}^{(m)}(\boldsymbol{X}_P) + O(\epsilon^2, s^2) \\
V_{\phi 2,0}^{(m)}(\boldsymbol{X}_P) =& \frac{\alpha}{q_s^2}\left[\epsilon\left(\frac{1}{q_s^2}-1\right)-\frac{\alpha}{2}\right]\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right)+\frac{\alpha}{2q_s^2}\left[Z_{1,0}^{(m+1)}+Z_{1,0}^{(m-1)}\right], \qquad (57)\\
V_{\phi 2,1}^{(m)}(\boldsymbol{X}_P) =& \frac{\alpha}{2q_s^2}\left[Z_{1,1}^{(m+1)}+Z_{1,1}^{(m-1)}\right]-\frac{1}{q_s^3}\left(\frac{\epsilon\alpha}{2q_s^2}\right)r\frac{d}{dr}\left[\Delta q\left(\Delta\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right)\right]-\frac{\alpha\Delta'}{q_s^3}r\frac{d}{dr}\left[\Delta q(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)})\right]\\
&-\frac{\Delta q}{q_s^3}\left[2(1+m)\epsilon+(2+m)\alpha-(4+3m)\Delta'\right]Z_{1,0}^{(m+1)}\\
&-\frac{\Delta q}{q_s^3}\left[2(1-m)\epsilon+(2-m)\alpha-(4-3m)\Delta'\right]Z_{1,0}^{(m-1)}\\
&-\frac{\Delta q}{q_s^3}\Delta'\left\{r\frac{d}{dr}\left(Z_{1,0}^{(m+1)}+Z_{1,0}^{(m-1)}\right)-r\frac{d}{dr}\left[\alpha\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right)\right]\right\}\\
&+\frac{\Delta q}{q_s^3}\left[\epsilon+\alpha-4\Delta'\right]\left[(2+m)\xi_{R1,0}^{r(m+1)}+(2-m)\xi_{R1,0}^{r(m-1)}\right]\\
&+\frac{\Delta q}{q_s^3}\left[8\epsilon^2+\alpha(5\epsilon+2\alpha)-\Delta'(6\epsilon+7\alpha)+12(\Delta')^2-\frac{2\epsilon}{q_s^2}(2\epsilon+\alpha)\right]\left(\xi^{r(m)}-\Delta\xi^{r(m)}\right)\\
&+\frac{\Delta q}{q_s^3}\left[\alpha^2+2\epsilon\alpha-\frac{\epsilon}{q_s^2}\left(4\alpha+\frac{1}{2}r\frac{d\alpha}{dr}\right)-\Delta' r\frac{d\alpha}{dr}\right]\left(\Delta\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right). \qquad (58)
\end{aligned}$$

(where we have used $dq/dr = d\Delta q/dr$), with $Z_{1,j}^{(m\pm1)}$ the expansion (in components ϵ and s) of $Z_1^{(m\pm1)}$, where

$$Z_1^{(0)} = -i\xi_{R1}^{\theta(0)} \qquad (59)$$

$$Z_1^{(m\pm1)} = \frac{r^{-(1\pm m)}}{1\pm m}\frac{d}{dr}\left(\xi_{R1}^{r(m\pm1)}r^{2\pm m}\right), \qquad (60)$$

where the second expression is not valid for $(m-1)=(0)$ (the first expression should be used). The equations for the sidebands $\xi_R^{r(m\pm1)}$ will be identified from the sideband components of the toroidal vorticity. It is in fact quite difficult to fully identify $Z^{(m\pm1)}$ in a convenient analytic expression to order s^1 as will be seen. Even explicitly identifying $Z^{(m\pm1)}$ to order s^0 is delicate, but crucial, due to constants of integration in the equation for $\xi_R^{r(m\pm1)}$, which must be calculated by matching procedure if an analytic treatment is pursued. In the above,

$$\begin{aligned}
Z_{1,0}^{(m\pm1)} &= \frac{r^{-(1\pm m)}}{1\pm m}\frac{d}{dr}\left(\xi_{R1,0}^{r(m\pm1)}r^{2\pm m}\right),\\
Z_{1,1}^{(m\pm1)} &= \frac{r^{-(1\pm m)}}{1\pm m}\frac{d}{dr}\left(\xi_{R1,1}^{r(m\pm1)}r^{2\pm m}\right),
\end{aligned}$$

except for $(m-1)=0$, for which, clearly,

$$Z_{1,0}^{(0)} = -i\xi_{R1,0}^{\theta(0)}, \quad Z_{1,1}^{(0)} = -i\xi_{R1,1}^{\theta(0)}.$$

Finally it can be shown that $V_{\phi3}^{(m)}=0$. Vorticity to this order depends on $\boldsymbol{\xi}_3$. One can obtain the neighbouring sidebands $\xi_3^{\theta(m\pm1)}$ in terms of $\xi_1^{r(m\pm1)}$ and $\xi_0^{r(m)}$ via the radial vorticity, but their effects integrate to zero in $V_{\phi3}^{(m)}=0$, as do arbitrary higher harmonics of $\boldsymbol{\xi}_3$. The flute component $\xi_3^{\theta(m)}$ is obtained to be zero in the radial vorticity (see Eq. (34) and associated discussion). The effects of $\boldsymbol{\xi}_0$, $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are also shown to be zero in $V_{\phi3}^{(m)}$. The fact that $V_{\phi3}^{(m)}=0$ means that the problem outlined in this document is valid mathematically if $\Delta q\sim s\sim\epsilon$, in particular it is valid to include Δq corrections in $V_{\phi2}^{(m)}$, i.e. with contributions $V_{\phi2,1}^{(m)}\sim\epsilon^2\Delta q\xi^{(m)}$ and Δq corrections in the low shear region of $V_{\phi1}^{(m\pm1)}$, in particular $V_{\phi1,1}^{(m\pm1)}\sim\epsilon\Delta q\xi^{r(m)}$. These higher order corrections are important for recovering and even extending the correct results for some instabilities with an exact rational surface, e.g. for the $m=1$ internal kink mode. Another ordering which is easier to handle analytically is considered later, in particular where $\Delta q\sim\alpha\sim\epsilon^2$.

2. Equations for $V_\phi^{(m\pm1)}(\boldsymbol{X}_P)$ and reduction of $V_\phi^{(m)}(\boldsymbol{X}_P)$ for $\boldsymbol{m\neq1}$

This section is valid for all cases except for the lower sideband of a fundamental harmonic having $m=1$, where, as stated earlier, we require a different treatment of the lower sideband. So, provided $(m-1)\neq0$ we obtain the following expansion for $V_\phi^{(m\pm1)}(\boldsymbol{X_P})$:

$$V_\phi^{(m\pm1)}(\boldsymbol{X_P}) = \epsilon^0 V_{\phi,0}^{(m\pm1)}(\boldsymbol{X_P}) + \epsilon^1 V_{\phi,1}^{(m\pm1)}(\boldsymbol{X_P})$$

where ϵ^i are tags, which are henceforth dropped. The first contribution $V_{\phi,0}^{(m\pm1)}(\boldsymbol{X_P}) = 0$ (note that $\xi^{r(m\pm1)} \sim \epsilon\xi^r$). Solving to the next order in ϵ, zeroth order in s, yields, on using Eq. (32):

$$\begin{aligned}
V_{\phi1,0}^{(m\pm1)}(\boldsymbol{X}_P) =& \left\{\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{n}{m\pm1}\right)^2\frac{d\xi_{R1,0}^{r(m\pm1)}}{dr}\right] - m(m\pm2)\left(\frac{1}{q}-\frac{n}{m\pm1}\right)^2\xi_{R1,0}^{r(m\pm1)}\right\}_{q\to q_s}\\
&-\frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left\{\frac{\alpha}{2r^{\pm m}}\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right)\right\}\\
=&\frac{r^{1\pm m}}{q_s^2(m\pm1)^2}\frac{d}{dr}\left[r^{-(1\pm2m)}\frac{d}{dr}\left(r^{2\pm m}\xi_{R1,0}^{r(m\pm1)}\right)\right] - \frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left[\frac{\alpha}{2r^{\pm m}}\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right)\right], \qquad (61)
\end{aligned}$$

where in the low shear region, essentially, $\xi_R^{r(m\pm1)} = \xi^{r(m\pm1)}$, but we keep the resistive notation in case we wish to add resistive effects for the sidebands (relevant on their own singular surfaces) later on. Note that, in the low shear region, the expression for $Z_{1,0}^{m\pm1}$ is obtained by neglecting sideband inertia (setting $\boldsymbol{X}_I$ =0 in $V_\phi^{(m\pm1)}(\boldsymbol{X})$). So, direct integration of Eq. (61) yields:

$$Z_{1,0}^{m\pm1} = \frac{\alpha}{2}\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right) + C^\pm r^{\pm m}, \qquad (62)$$

where the constants of integration comprise the crucial drive for infernal modes. Notice from Eq. (62) that the fourth line of $V_{\phi2,1}^{(m)}$ in Eq. (58) cancels except for the integration constant associated with infernal mode drive.

Identifying $V_{\phi1}^{(m\pm1)}(\boldsymbol{X}_P)$ to the next order in s, we obtain,

$$\begin{aligned}
V_{\phi1,1}^{(m\pm1)}(\boldsymbol{X}_P) =& \left\{\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{n}{m\pm1}\right)^2\frac{d\xi_{R1,0}^{r(m\pm1)}}{dr}\right] - m(m\pm2)\left(\frac{1}{q}-\frac{n}{m\pm1}\right)^2\xi_{R1,0}^{r(m\pm1)}\right\}_{\text{Linear in } s}\\
&+\frac{r^{1\pm m}}{q_s^2(m\pm1)^2}\frac{d}{dr}\left[r^{-(1\pm2m)}\frac{d}{dr}\left(r^{2\pm m}\xi_{R1,1}^{r(m\pm1)}\right)\right]\\
&-\frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left\{\frac{\Delta'}{r^{\pm m}}r\frac{d}{dr}\left(\frac{\Delta q\xi^{r(m)}}{q_s}-\frac{\Delta q\Delta\xi^{r(m)}}{q_s}\right)\right\}\\
&+\frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left\{\left[(1\pm2m)\epsilon+(1\pm m)(\alpha-3\Delta')\right]\frac{\Delta q\xi_R^{r(m)}}{q_s r^{\pm m}}+\frac{\alpha}{r^{\pm m}}\left(\frac{\Delta q\Delta\xi^{r(m)}}{q_s}+\frac{\Delta q\Delta\xi_\Gamma^{r(m)}}{q_s}\right)\right\}\\
&+(2\pm m)\left(\epsilon+\alpha-4\Delta'\right)\frac{\Delta q\xi_R^{r(m)}}{q_s^3}. \qquad (63)
\end{aligned}$$

This equation can be written in terms of the new variable $\overline{\xi}_{R1,1}^{r(m\pm1)}$ defined as :

$$\begin{aligned}
\overline{Z}_{1,1}^{(m\pm1)} =& \frac{r^{-(1\pm m)}}{1\pm m}\frac{d}{dr}\left(r^{2\pm m}\overline{\xi}_{R1,1}^{r(m\pm1)}\right)\\
=&\frac{r^{-(1\pm m)}}{1\pm m}\frac{d}{dr}\left(r^{2\pm m}\xi_{R1,1}^{r(m\pm1)}\right) - \Delta' r\frac{d}{dr}\left(\frac{\Delta q\xi_R^{r(m)}}{q_s}\right)\\
&+\left[(1\pm2m)\epsilon+(1\pm m)(\alpha-3\Delta')\right]\frac{\Delta q\xi_R^{r(m)}}{q_s}+\alpha\left(\frac{\Delta q\Delta\xi^{r(m)}}{q_s}+\frac{\Delta q\Delta\xi_\Gamma^{r(m)}}{q_s}\right)\\
=&Z_{1,1}^{(m\pm1)} - \Delta' r\frac{d}{dr}\left(\frac{\Delta q\xi_R^{r(m)}}{q_s}\right) + \left[(1\pm2m)\epsilon+(1\pm m)(\alpha-3\Delta')\right]\frac{\Delta q\xi_R^{r(m)}}{q_s}+\alpha\left(\frac{\Delta q\Delta\xi^{r(m)}}{q_s}+\frac{\Delta q\Delta\xi_\Gamma^{r(m)}}{q_s}\right).
\end{aligned} \qquad (64)$$

In terms of this variable, we have for the following results for the linear shear contributions to the main vorticity and the sideband vorticty:

$$V_{\phi 2,1}^{(m)}(\boldsymbol{X}_P) = \frac{\alpha}{2q_s^2}\left[\overline{Z}_{1,1}^{(m+1)} + \overline{Z}_{1,1}^{(m-1)}\right] - \frac{1}{q_s^3}\left(\frac{\epsilon\alpha}{2q_s^2}\right) r\frac{d}{dr}\left[\Delta q\left(\Delta\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right)\right] - \frac{\alpha\Delta'}{q_s^3} r\frac{d}{dr}\left[\Delta q(\Delta\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)})\right] + \tilde{R}^{(m)} \tag{65}$$

$$V_{\phi 1,1}^{(m\pm1)}(\boldsymbol{X}_P) = \left\{\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q} - \frac{n}{m\pm1}\right)^2 \frac{d\xi_{R1,0}^{r(m\pm1)}}{dr}\right] - m(m\pm2)\left(\frac{1}{q} - \frac{n}{m\pm1}\right)^2 \xi_{R1,0}^{r(m\pm1)}\right\}_{\text{Linear in } s} + \frac{r^{1\pm m}}{q_s^2(m\pm1)^2}\frac{d}{dr}\left[r^{-(1\pm2m)}\frac{d}{dr}\left(r^{2\pm m}\overline{\xi}_{R1,1}^{r(m\pm1)}\right)\right] + (2\pm m)\left(\epsilon + \alpha - 4\Delta'\right)\frac{\Delta q\xi_R^{r(m)}}{q_s^3}. \tag{66}$$

where

$$\begin{aligned}
\tilde{R}^{(m)} = &- \frac{\Delta q}{q_s^3}\left[2(1+m)\epsilon + (2+m)\alpha - (4+3m)\Delta'\right] Z_{1,0}^{(m+1)} \\
&- \frac{\Delta q}{q_s^3}\left[2(1-m)\epsilon + (2-m)\alpha - (4-3m)\Delta'\right] Z_{1,0}^{(m-1)} \\
&- \frac{\Delta q}{q_s^3}\Delta'\left\{r\frac{d}{dr}\left(Z_{1,0}^{(m+1)} + Z_{1,0}^{(m-1)}\right) - r\frac{d}{dr}\left[\alpha\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right)\right]\right\} \\
&+ \frac{\Delta q}{q_s^3}\left[\epsilon + \alpha - 4\Delta'\right]\left[(2+m)\xi_{R1,0}^{r(m+1)} + (2-m)\xi_{R1,0}^{r(m-1)}\right] \\
&+ \frac{\Delta q}{q_s^3}\left[8\epsilon^2 + \alpha(4\epsilon+\alpha) - \Delta'\left(6\epsilon + 4\alpha\right) + 12(\Delta')^2 - \frac{2\epsilon}{q_s^2}(2\epsilon+\alpha)\right]\left(\xi^{r(m)} - \Delta\xi^{r(m)}\right) \\
&+ \frac{\Delta q}{q_s^3}\left[2\epsilon\alpha - \frac{\epsilon}{q_s^2}\left(4\alpha + \frac{1}{2}r\frac{d\alpha}{dr}\right) - \Delta' r\frac{d\alpha}{dr}\right]\left(\Delta\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right).
\end{aligned} \tag{67}$$

We obtain, on summing Eqs. (57) and (65), and also Eqs. (61) and (66),

$$V_{\phi 2}^{(m)}(\boldsymbol{X}_P) = \frac{\alpha}{q_s^2}\left[\epsilon\left(\frac{1}{q_s^2} - 1\right) - \frac{\alpha}{2}\right]\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right) + \frac{\alpha}{2q_s^2}\left[\left(Z_{1,0}^{(m+1)} + \overline{Z}_{1,1}^{(m+1)}\right) + \left(Z_{1,0}^{(m-1)} + \overline{Z}_{1,1}^{(m-1)}\right)\right] - \frac{1}{q_s^3}\left(\frac{\epsilon\alpha}{2q_s^2}\right) r\frac{d}{dr}\left[\Delta q\left(\Delta\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right)\right] - \frac{\alpha\Delta'}{q_s^3} r\frac{d}{dr}\left[\Delta q(\Delta\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)})\right] + \tilde{R}^{(m)} \tag{68}$$

$$V_{\phi 1}^{(m\pm1)}(\boldsymbol{X}_P) = \left\{\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q} - \frac{n}{m\pm1}\right)^2 \frac{d\left(\xi_{R1,0}^{r(m\pm1)} + \overline{\xi}_{R1,1}^{r(m\pm1)}\right)}{dr}\right] - m(m\pm2)\left(\frac{1}{q} - \frac{n}{m\pm1}\right)^2\left(\xi_{R1,0}^{r(m\pm1)} + \overline{\xi}_{R1,1}^{r(m\pm1)}\right)\right\}_{q\to q_s} + \left\{\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q} - \frac{n}{m\pm1}\right)^2 \frac{d\xi_{R1,0}^{r(m\pm1)}}{dr}\right] - m(m\pm2)\left(\frac{1}{q} - \frac{n}{m\pm1}\right)^2 \xi_{R1,0}^{r(m\pm1)}\right\}_{\text{linear in } s} - \frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left\{\frac{\alpha}{2r^{\pm m}}\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right)\right\} + (2\pm m)\left(\epsilon + \alpha - 4\Delta'\right)\frac{\Delta q\xi_R^{r(m)}}{q_s^3}. \tag{69}$$

These two latter equations are simplified by defining:

$$\xi_{R1,0}^{r(m\pm1)} + \overline{\xi}_{R1,1}^{r(m\pm1)} = \overline{\xi}_{R1}^{r(m\pm1)}$$

and correspondingly,

$$Z_{1,0}^{(m+1)} + \overline{Z}_{1,1}^{(m+1)} = \overline{Z}_1^{(m+1)}.$$

It is of course difficult to calculate the linear s contribution in $V_\phi^{(m\pm1)}(\boldsymbol{X}_P)$ above. And it is not necessary, nor convenient to try. Even though negligible close to the main resonance, it is convenient to include quadratic contributions together with the linear contributions in s in such a way that the sideband equations are valid in both the low and high shear regions. The result is evidently,

$$V_{\phi 1}^{(m\pm1)}(\boldsymbol{X}_P) = \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{n}{m\pm1}\right)^2\frac{d\overline{\xi}_{R1}^{r(m\pm1)}}{dr}\right] - m(m\pm2)\left(\frac{1}{q}-\frac{n}{m\pm1}\right)^2\overline{\xi}_{R1}^{r(m\pm1)}$$
$$-\frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left\{\frac{\alpha}{2r^{\pm m}}\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right)\right\} + (2\pm m)\left(\epsilon+\alpha-4\Delta'\right)\frac{\Delta q\xi_R^{r(m)}}{q_s^3}. \tag{70}$$

The standard sideband equations in the high shear region coincide with Eq. (70) on setting $\xi^{r(m)}=0$ and $\Delta\xi_\Gamma^{r(m)}=0$, which is indeed the case in the high shear region due to strong field line bending damping of the main harmonic. It is noted that

$$\overline{Z}_1^{(m\pm1)} = \frac{r^{-(1\pm m)}}{1\pm m}\frac{d}{dr}\left(\overline{\xi}_{R1}^{r(m\pm1)}r^{2\pm m}\right)$$

is obtained in full from the solution for $\overline{\xi}_{R1}^{r(m\pm1)}$ of the equation $V_\phi^{(m\pm1)}(\boldsymbol{X}_P)+V_\phi^{(m\pm1)}(\boldsymbol{X}_I)=0$, with $V_\phi^{(m\pm1)}(\boldsymbol{X}_P)$ given in general by Eq. (70), and $V_\phi^{(m\pm1)}(\boldsymbol{X}_I)$ will be obtained in the next section. Boundary conditions would be applied at $r=0$ and $r=a$. One may choose instead to solve $V_\phi^{(m\pm1)}(\boldsymbol{X_P})=0$, with boundary conditions applied in the high shear region where $\xi^{r(m)}$ is negligible. For the latter approach, the boundary conditions require careful matching of $\overline{\xi}_R^{r(m\pm1)}$ across the transition. Including the field line bending terms of Eq. (56), $\overline{Z}_1^{(m\pm1)}$ enters into the main equation as follows:

$$V_\phi^{(m)}(\boldsymbol{X}_P) = \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{r(m)}}{dr}\right] + \frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d}{dr}\left(\frac{\Delta q\Delta\xi^{r(m)}}{q}\right) - \frac{r^3}{q_s}\left(\frac{\Delta q\Delta\xi_0^r}{q}\right)\frac{d}{dr}\left(\frac{1}{q}\right)\right]$$
$$-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi^{r(m)} - (m^2-1)\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\left(\frac{\Delta q\Delta\xi^{r(m)}}{q}\right)$$
$$\frac{\alpha}{q_s^2}\left[\epsilon\left(\frac{1}{q_s^2}-1\right)-\frac{\alpha}{2}\right]\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right) + \frac{\alpha}{2q_s^2}\left[\overline{Z}_1^{(m+1)}+\overline{Z}_1^{(m-1)}\right]$$
$$-\frac{1}{q_s^3}\left(\frac{\epsilon\alpha}{2q_s^2}\right)r\frac{d}{dr}\left[\Delta q\left(\Delta\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right)\right] - \frac{\alpha\Delta'}{q_s^3}r\frac{d}{dr}\left[\Delta q(\Delta\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)})\right] + \tilde{R}^{(m)}. \tag{71}$$

It is finally noted that Eqs. (70) and (71) are valid continuously in the whole plasma domain, so that when added to global inertia contributions, we can expect to develop global continuous solutions to the eigenvalue problem.

3. *Equations for $V_\phi^{(m\pm1)}(\boldsymbol{X}_P)$ and reduction of $V_\phi^{(m)}(\boldsymbol{X}_P)$ for $\boldsymbol{m=1}$*

This section treats the lower sideband equation for the case where $m=1$. The equations for the upper sideband, e.g. Eq. (70), remain valid for $m=1$. We could address the problem once again with the toroidal vorticity, though with renormalisation to remove the singularity, i.e. we calculate $ilV_\phi^{(l)}(\boldsymbol{X})=0$, defined by Eq. (54). For the case $l=0$, it turns out that the contribution from the radial covariant component of $\boldsymbol{X}_P$ in $ilV_\phi^{(l)}(\boldsymbol{X}_P)$ does not appear. Neglecting the inertia contribution $\boldsymbol{X}_I$ at this order it can be shown that

$$ilV_\phi^{(l)}(\boldsymbol{X}) \propto \frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\exp(il\theta)\frac{\partial}{\partial r}\left(\frac{X_{P\theta}}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\phi}\right)=0, \quad \text{with} \quad l=m-1=0,$$

where $X_{P\theta}$ is the poloidal covariant component of $\boldsymbol{X}_P$. The solution, on using Eq. (38), is

$$\xi_1^{\theta(m-1)} = i\frac{\alpha}{2}\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right) + i\Delta' r\frac{d}{dr}\left[\frac{\Delta q}{q_s}\xi_R^{r(m)}\right] + i\epsilon\frac{\Delta q}{q_s}\xi_R^{r(m)} - i\alpha\frac{\Delta q}{q_s}\left(\Delta\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right) + Cr, \quad \text{for} \quad m=1,$$

with C an integration constant. Again, we note that the above solution is valid for all r. Since $\xi_1^{\theta(m-1)}(a)=0$, $\xi^{r(m)}(a)=0$, $\Delta\xi^{r(m)}(a)=0$ and $\Delta\xi_R^{r(m)}(a)=0$, the constant of integration is zero. Hence, from Eq. (59) we have

$$Z_{1,0}^{(0)} = \frac{\alpha}{2}\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right) \tag{72}$$
$$Z_{1,1}^{(0)} = \Delta' r\frac{d}{dr}\left[\frac{\Delta q}{q_s}\left(\xi^{r(m)}-\Delta\xi^{r(m)}\right)\right] + \epsilon\frac{\Delta q}{q_s}\left(\xi^{r(m)}-\Delta\xi^{r(m)}\right) - \alpha\frac{\Delta q}{q_s}\left(\Delta\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right). \tag{73}$$

The modified variable defined by Eq. (64) is then

$$\overline{Z}_1^{(0)} = \frac{\alpha}{2}\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right)$$

which together with $\xi_R^{(m-1)} = 0$ for $m = 1$ can be substituted directly into Eq. (71) (note some terms don't appear for the case of $m^2 - 1 = 0$). The upper sideband is treated by solving Eq. (70) for $\xi_R^{(m+1)}$ and $\overline{Z}_1^{(m+1)}$ for $m = 1$, and substitution into Eq. (71).

4. The inertia contribution $V_\phi^{(m)}(\boldsymbol{X}_I)$

It remains to evaluate the $V_\phi^{(m)}(\boldsymbol{X}_I)$ so that when added to Eq. (71) we can establish the governing equation $V_\phi^{(m)}(\boldsymbol{X}) = 0$. Recall $\boldsymbol{X}_I$ given by Eq. (49). The calculation is so straightforward that computer algebra simplification is not required. Via the toroidal vorticity definition of Eq. (29) and following the approach given by Eqs. (20) and (21) in Ref. [13] by writing $\boldsymbol{X}_I = \boldsymbol{Y} + \boldsymbol{\nabla} W$, we have

$$V_\phi^{(m)}(\boldsymbol{X}_I) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \left(\frac{R_0}{B_0}\right)\left\{\frac{1}{im}\frac{\partial}{\partial r}\left[\left(\frac{R^2}{F}\right)\hat{Y}_\theta - \hat{W}\frac{\partial}{\partial\theta}\left(\frac{R^2}{F}\right)\right] + \left[\left(\frac{R^2}{F}\right)\hat{Y}_r - \hat{W}\frac{\partial}{\partial r}\left(\frac{R^2}{F}\right)\right]\right\}, \tag{74}$$

with $W = (\rho\gamma^2 R_0)\left[2q^2\left(\xi_0^r\cos\theta - \xi_0^\theta\sin\theta\right)\right]$ and $\boldsymbol{Y} = -\rho\gamma^2(\boldsymbol{\xi}_B + \zeta^{(m)}\boldsymbol{B})$. We require the covariant form $\boldsymbol{Y} = Y_r\boldsymbol{\nabla}r + Y_\theta\boldsymbol{\nabla}\theta + Y_\phi\boldsymbol{\nabla}\phi$, where

$$Y_r \approx |\boldsymbol{\nabla}r|^{-2}\left[-\rho\gamma^2(\boldsymbol{\xi}_B^{(m)} + \zeta^{(m)}\boldsymbol{B})\right]\cdot\boldsymbol{\nabla}r \approx -\rho\gamma^2\xi_0^r$$
$$Y_\theta \approx |\boldsymbol{\nabla}\theta|^{-2}\left[-\rho\gamma^2(\boldsymbol{\xi}_B^{(m)} + \zeta^{(m)}\boldsymbol{B})\right]\cdot\boldsymbol{\nabla}\theta \approx -\rho\gamma^2 r\left(\xi_0^\theta + \frac{\epsilon}{q}B_0\zeta^{(m)}\right).$$

The problem can be solved trivially. Note Eq. (45) written in terms of $\chi^{(m)}$ is

$$B_0\zeta^{(m)} = inq^2\chi^{(m)}\frac{R_0}{\Gamma P}\frac{dP}{dr}\left(\frac{\omega_s^2}{(nq-m)^2\omega_s^2 + \gamma^2q^2}\right) = -in\frac{\alpha}{2}\chi^{(m)}\left(\frac{\omega_A^2}{(nq-m)^2\omega_s^2 + \gamma^2q^2}\right),$$

where $\omega_s^2 = \omega_A^2\Gamma P/B_0^2$ has been used, where $\omega_A^2 = B_0^2/(\rho R_0^2)$ is the square of the Alfvén frequency. Using also Eq. (35) for eliminating $\xi_0^{\theta(m)}$ in favour of $\xi_0^{r(m)}$ gives

$$\begin{aligned} V_\phi^{(m)}(\boldsymbol{X}_I) =& \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2}\right)\frac{d\xi^{r(m)}}{dr}\right] - (m^2-1)\left(\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2}\right)\xi^{r(m)} \\ &+ \frac{1}{2q_s^2}\left(2\epsilon\alpha + \epsilon r\frac{d\alpha}{dr}\right)\left(\frac{\gamma^2\chi^{(m)}}{(nq-m)^2\omega_s^2 + \gamma^2q^2}\right) + \frac{\epsilon\alpha}{2q_s^2}r\frac{d}{dr}\left(\frac{\gamma^2\chi^{(m)}}{(nq-m)^2\omega_s^2 + \gamma^2q^2}\right). \end{aligned} \tag{75}$$

Equation (75) can be written in more convenient form by noting that

$$\frac{\chi^{(m)}\gamma^2q^2}{(nq-m)^2\omega_s^2 + \gamma^2q^2} = \chi^{(m)} + \chi_\Gamma^{(m)} \quad \text{with} \quad \chi_\Gamma^{(m)} = \frac{\Delta q\Delta\xi_\Gamma^{(m)}}{q} = -\chi^{(m)}\left(\frac{\omega_s^2(nq-m)^2}{(nq-m)^2\omega_s^2 + \gamma^2q^2}\right). \tag{76}$$

Hence, we obtain

$$\begin{aligned} V_\phi^{(m)}(\boldsymbol{X}_I) =& \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2}\right)\frac{d\xi^{r(m)}}{dr}\right] - (m^2-1)\left(\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2}\right)\xi^{r(m)} \\ &+ \frac{1}{2q_s^4}\left(2\epsilon\alpha + \epsilon r\frac{d\alpha}{dr}\right)\left(\chi^{(m)} + \chi_\Gamma^{(m)}\right) + \left(\frac{\epsilon\alpha}{2q_s^4}\right)r\frac{d(\chi^{(m)} + \chi_\Gamma^{(m)})}{dr}. \end{aligned} \tag{77}$$

5. The inertia contribution $V_\phi^{(m\pm1)}(\boldsymbol{X}_I)$

We may follow the results of the last subsection, and the results for the parallel momentum equation, in order to obtain the results needed here. We note that inertia matters for the sidebands only close to their own rational surfaces. In those regions, the main harmonic is usually negligible. The two sidebands therefore decouple from the main harmonic, and also decouple from one another. Hence, for constructing $\boldsymbol{X}_I$, we therefore identify Eq. (49) as the relevant inertia, but with $m \to m \pm 1$. In addition, for calculation to relevant order in ϵ in the main eigenvalue equation, we set $\zeta^{(m\pm1)} = 0$. Hence,

$$\boldsymbol{X}_I = -\rho\gamma^2 \left\{ \boldsymbol{\xi}_B^{(m\pm1)} - R_0 \boldsymbol{\nabla} \left[2q^2 \left(\xi^{r(m\pm1)} \cos\theta - \xi^{\theta(m\pm1)} \sin\theta \right) \right] \right\}. \tag{78}$$

Following the last subsection which constructed $V_\phi^{(m)}(\boldsymbol{X}_I)$, and noting Eq. (36), we clearly obtain,

$$V_\phi^{(m\pm1)}(\boldsymbol{X}_I) = \frac{1}{r}\frac{d}{dr}\left[r^3 \left(\frac{\gamma^2(1+2q^2)}{(m\pm1)^2\omega_A^2} \right) \frac{d\xi^{r(m\pm1)}}{dr} \right] - m(m\pm2) \left(\frac{\gamma^2(1+2q^2)}{(m\pm1)^2\omega_A^2} \right) \xi^{r(m\pm1)}, \tag{79}$$

valid for all cases except $(m-1) = (0)$, i.e. the lower sideband of the $m = 1$ case. An important point to note is that since the inertia is important on the rational surface of the sideband, the value of q in Eq. (79) will be essentially $q_s^{(m\pm1)} = (m \pm 1)/n$. Therefore, for the upper sideband, the toroidal inertia enhancement $1 \to 1 + q^2$ is larger than for the main harmonic.

III. GOVERNING EIGENVALUE EQUATIONS

We now consider $V_\phi^{(m)}(\boldsymbol{X}) = 0$ by summing Eqs. (71) and (77) and setting the result to zero. In Eq. (71) we will adopt the resistive variable $\chi^{(m)} \equiv \frac{\Delta\chi^{(m)}}{n} = \frac{\Delta q \Delta\xi^{r(m)}}{q}$ as defined in Eq. (51), which will cleanly define the field line bending terms. In the terms that are not associated with field line bending, we are free to replace $\Delta q \Delta\xi^{r(m)}/q_s$ with $\Delta q \Delta\xi^{r(m)}/q = \chi^{(m)}$, since the corresponding corrections to Eq. (71) would appear at the next order in s. Similarly we are free to replace $\Delta q \Delta\xi_\Gamma^{r(m)}/q_s$ with $\Delta q \Delta\xi_\Gamma^{r(m)}/q = \Delta\xi_\Gamma^{r(m)}$ in Eq. (71). Noting the cancelation of the first term on the last line of Eq. (71) with the last term of Eq. (77), and dropping over-line notation (and numerical subscripts) in $Z^{(m\pm1)}$ and $\xi_R^{r(m\pm1)}$, we have

$$\begin{aligned}
0 =& \frac{1}{r}\frac{d}{dr}\left\{ r^3 \left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q} - \frac{1}{q_s} \right)^2 \right] \frac{d\xi^{r(m)}}{dr} \right\} - (m^2-1) \left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q} - \frac{1}{q_s} \right)^2 \right] \xi^{r(m)} \\
&+ \frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q} - \frac{1}{q_s} \right) \frac{d\chi^{(m)}}{dr} - \frac{r^3}{q_s}\chi^{(m)} \frac{d}{dr}\left(\frac{1}{q} \right) \right] - (m^2-1)\frac{1}{q_s}\left(\frac{1}{q} - \frac{1}{q_s} \right)\chi^{(m)} \\
&+ \frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2} - 1 \right)\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)} \right) + \frac{\alpha}{2q_s^2}\left[Z^{(m+1)} + Z^{(m-1)} - \alpha\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)} \right) \right] - \frac{\alpha\Delta'}{q_s^2} r \frac{d}{dr}\left[\chi^{(m)} + \chi_\Gamma^{(m)} \right] \\
&+ R^{(m)}
\end{aligned} \tag{80}$$

with,

$$\begin{aligned}
R^{(m)} =& -\frac{\Delta q}{q_s^3}\left[2(1+m)\epsilon + (2+m)\alpha - (4+3m)\Delta' \right] Z_{1,0}^{(m+1)} \\
&- \frac{\Delta q}{q_s^3}\left[2(1-m)\epsilon + (2-m)\alpha - (4-3m)\Delta' \right] Z_{1,0}^{(m-1)} \\
&- \frac{\Delta q}{q_s^3}\Delta' r \frac{d}{dr}\left[Z_{1,0}^{(m+1)} + Z_{1,0}^{(m-1)} - \alpha\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)} \right) \right] \\
&+ \frac{\Delta q}{q_s^3}\left[\epsilon + \alpha - 4\Delta' \right]\left[(2+m)\xi_{R1,0}^{r(m+1)} + (2-m)\xi_{R1,0}^{r(m-1)} \right] \\
&+ \frac{1}{q_s^2}\left[8\epsilon^2 + \alpha(4\epsilon+\alpha) - \Delta'(6\epsilon+4\alpha) + 12(\Delta')^2 - \frac{2\epsilon}{q_s^2}(2\epsilon+\alpha) \right]\left(\frac{\Delta q \xi^{r(m)}}{q_s} - \chi^{(m)} \right) \\
&+ \frac{1}{q_s^2}\left[\epsilon\alpha\left(2 - \frac{3}{q_s^2} \right) - \Delta' r \frac{d\alpha}{dr} \right]\left(\chi^{(m)} + \chi_\Gamma^{(m)} \right).
\end{aligned} \tag{81}$$

and

$$\Delta\xi_\Gamma^{r(m)} = \frac{q\chi_\Gamma^{(m)}}{q-q_s}, \quad \text{and} \quad \chi_\Gamma^{(m)} = -\chi^{(m)}\left(\frac{\omega_s^2(q-q_s)^2}{(q-q_s)^2\omega_s^2 + (\gamma^2/m^2)q_s^2q^2}\right).$$

A. Governing sideband equations

We develop the general sideband equations $V_\phi^{(m\pm1)}(\boldsymbol{X}) = 0$ by summing Eqs. (70) and (79), giving on dropping the overline notation,

$$\begin{aligned}
0 =& \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{\gamma^2(1+2q^2)}{(m\pm1)^2\omega_A^2}\right)\frac{d\xi^{r(m\pm1)}}{dr}\right] - m(m\pm2)\left(\frac{\gamma^2(1+2q^2)}{(m\pm1)^2\omega_A^2}\right)\xi^{r(m\pm1)} \\
&+ \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{n}{m\pm1}\right)^2\frac{d\xi_R^{r(m\pm1)}}{dr}\right] - m(m\pm2)\left(\frac{1}{q}-\frac{n}{m\pm1}\right)^2\xi_R^{r(m\pm1)} \\
&- \frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left\{\frac{\alpha}{2r^{\pm m}}\left(\xi^{r(m)}+\Delta\xi_\Gamma^{r(m)}\right)\right\} + (2\pm m)\left(\epsilon+\alpha-4\Delta'\right)\frac{1}{q_s^2}\left(\frac{\Delta q\xi^{r(m)}}{q_s}-\chi^{(m)}\right).
\end{aligned} \tag{82}$$

Solution of this equation defines $\xi_R^{r(m\pm1)}$ in $R^{(m)}$ (if needed) and

$$Z^{(m\pm1)} = r^{-(1\pm m)}\frac{d}{dr}\left(\xi_R^{r(m\pm1)}r^{2\pm m}\right)$$

which is required in Eq. (80). Note, for the special case $m = 1$, we use, instead of the above,

$$Z^{(m-1)} = \frac{\alpha}{2}\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right), \quad \xi_R^{r(m-1)} = 0, \quad \text{for} \quad m = 1. \tag{83}$$

Inertia and resistivity is not required for the calculation of $Z^{(0)}$ providing that $q = 0$ does not exist somewhere in the plasma. Hence a different treatment would be required for the interesting case of the edge of a reverse field pinch.

B. Resistive (radial) Ohm's law

Clearly an equation is required which relates $\chi^{(l)}$ to $\xi^{r(l)}$ in the main harmonic and sideband equations. Assuming that the main harmonic displacement is not larger than sideband displacement on the sideband rational surface, radial Ohm's law (Eq. (50) for the main harmonic) applies for each harmonic separately, i.e.

$$\begin{aligned}
\chi^{(l)} =& \frac{\eta}{\gamma}\frac{1}{r^3}\left[\frac{d}{dr}\left(r^3\frac{d}{dr}\left\{q_s\left(\frac{1}{q}-\frac{1}{q_s}\right)\xi^{r(l)}\right.\right.\right. \\
&\left.\left.\left. +\chi^{(l)}\right\}\right) + r(1-l^2)\left\{q_s\left(\frac{1}{q}-\frac{1}{q_s}\right)\xi^{r(l)} + \chi^{(l)}\right\}\right],
\end{aligned}$$

with $q_s = l/n$ and $l = m-1, m, m+1$. In the sideband equation Eq. (82) $\xi_R^{r(l)} = \xi^{r(l)} - \Delta\xi^{r(l)}$, with $\chi^{(l)} = (q - l/n)\Delta\xi^{r(l)}/q$. It is thus possible to write Eq. (82) in terms of $\xi^{r(m\pm1)}$ and $\chi^{(m\pm1)}$. In this paper either resistive effects are ignored on the rational surfaces of the sidebands, or analytic solutions can be used for treating the narrow layer. Notice that there is no explicit toroidal coupling in these equations.

C. Boundary conditions and associated approximations

The general internal plasma problem is solved with Dirichlet boundary conditions (BCs) for all variables placed at $r = 0$ and the plasma edge $r = a$. We now consider the consequences and potential remedies for having rational surfaces associated with the sidebands. If the fully global problem is attempted, with BCs applied at $r = 0$ and $r = a$, one requires inertia in the sideband equations, and if resistivity is considered for the sidebands, then Ohm's law for the sidebands has to be solved too.

1. *Ideal sidebands*

First, if we wish to consider the fully global problem with ideal sidebands, we note that $\xi_R^{r(m\pm1)} = \xi^{r(m\pm1)}$, and resistive Ohm's law isn't required for the sidebands. We will nevertheless require inertia in the sideband equations if the sideband has a rational inside the plasma.

We now show how we may reduce the size of the problem, and avoid explicit inclusion of sideband inertia (note inertia is always required for the main harmonic, even if an exact rational for the main harmonic does not exist). It is always assumed that a rational for the main mode exists, or there is an extremum in q where the field line bending stabilisation is minimised. For both cases in this subsection the associated radius is denoted r_s. There might in fact be two main mode rational surfaces, but it is assumed here that the sidebands do not have rational surfaces in between the main mode rational surfaces. If there are two rational surfaces, the below assumes that r_s is the closest rational surface to the rational of the sideband considered.

Consider now the region $0 < r < r_s$. If there exists a rational $r_s^{(m\pm1)} < r_s$ for which $q(r_s^{(m\pm1)}) = (m \pm 1)/n$ then for that variable, one may apply a Neumann BC:

$$\lim_{\delta\to 0} \left.\frac{d\xi_R^{r(m\pm1)}}{dr}\right|_{r_s^{(m\pm1)}+\delta} = 0$$

while adopting Dirichlet BCs for the other variables. If no sideband rationals exist in $0 < r < r_s$, then Dirichlet BC's are deployed for all variables at $r = 0$. Note that what is written here does not hold for the $m = 1$ mode, where Neumann BC's must be placed at $r = 0$ for the main harmonic, Dirichlet BC for the upper sideband, and the lower sideband isn't required. In addition, for the $m = 2$ mode, Neumann BC's will be applied for the lower sideband, at $r = 0$ if it has no rational in $0 < r_s^{(m-1)} < r_s$, or at $r_s^{(m-1)}$ if it does have a rational in $0 < r_s^{(m-1)} < r_s$.

For the region $r_s < r < a$, if there exists a rational $r_s^{(m\pm1)} > r_s$ for which $q(r_s^{(m\pm1)}) = (m \pm 1)/n$ then for that variable, one may apply a Neumann BC:

$$\lim_{\delta\to 0} \left.\frac{d\xi_R^{r(m\pm1)}}{dr}\right|_{r_s^{(m\pm1)}-\delta} = 0 \tag{84}$$

while adopting Dirichlet BCs for the other variables. If no sideband rationals exist in $r_s < r < a$, then Dirichlet BC's are deployed for all variables at $r = a$. BC's at $r = a$ should be modified if magnetic perturbations are allowed to propagate into the vacuum region $r > a$. This is required for the study of Edge Harmonic Oscillations [4].

2. *Resistive sidebands*

The separation of regions described in the last subsection hold, but the BC's for the resonant sideband is adjusted to an appropriate Robin BC in order to take into account the effect of resistivity. We follow the derivation given in [6]. In the below we assume a sideband rational $r_s^{(m+1)} > r_s$, but it can easily be applied to other cases too, using the logic of the last subsection. We write the resistive displacement for the upper sideband in terms of its associated magnetic flux,

$$\xi_R^{(m+1)} = -\frac{R_0}{B_0}\left(\frac{1}{q} - \frac{n}{m+1}\right)^{-1}\frac{\psi}{r}. \tag{85}$$

In the region $r \gg r_s$ we may neglect the main harmonic in the sideband equation of Eq. (82). Avoiding the inertial region of the sideband too, the associated sideband equation in terms of $\delta\psi = \psi$ (Eq. (53)),

$$\left(\frac{1}{q} - \frac{n}{m+1}\right)\left[\frac{d}{dr}\left(r\frac{d}{dr}\psi\right) - (m+1)^2\frac{\psi}{r}\right] - \frac{\psi}{r^2}\frac{d}{dr}\left[r^3\frac{d}{dr}\left(\frac{1}{q}\right)\right] = 0, \tag{86}$$

which can be reconciled with the usual cylindrical tearing mode equation in the 'outer' region, noting that $(R_0/B_0)dJ_\phi/dr = (1/r^2)(d/dr)[r^3(d/dr)(1/q)]$.

We construct ψ on the basis of large and small solutions ψ_s and ψ_L. Here ψ_L is chosen to be continuous across the rational, while ψ_s provides the jump in the derivative associated with Δ'_R. Both a jump in the derivative, and a non-zero value in ψ are required at the rational for tearing. Here, Δ'_R is written explicitly in terms of the growth rate (the standard 'inner' region result with toroidal inertia corrections):

$$\Delta'_R = \left(\frac{2\pi\Gamma(3/4)}{\Gamma(1/4)}\right)\frac{(1+2q^2)^{1/4}(\gamma/\omega_A)^{5/4}S^{3/4}}{r_s^{(m+1)}[(m+1)s/q]^{1/2}}, \tag{87}$$

where q and shear s etc are evaluated at the rational surface of the sideband. Here $S = \tau_R\omega_A$ with $\tau_R = (r_s^{(m+1)})^2/\eta$.

The advantage of composing $\psi(r)$ in terms of $\psi_L(r)$ and $\psi_s(r)$ is that they can both be established in the region $r_s < r < r_s^{(m+1)}$ in advance of the eigenvalue calculation. Both $\psi_s(r)$ and $\psi_L(r)$ for $r_s < r < r_s^{(m+1)}$ are obtained by solving Eq. (86) with boundary conditions applied at $r_s^{(m+1)} - \delta$. The boundary conditions are

$$\psi_s(r_s^{(m+1)} - \delta) = 0, \quad \left.\frac{d\psi_s}{dr}\right|_{r_s^{(m+1)}-\delta} = (\psi_s^-)'$$
$$\psi_L(r_s^{(m+1)} - \delta) = \psi_L^-, \quad \left.\frac{d\psi_L}{dr}\right|_{r_s^{(m+1)}-\delta} = (\psi_L^-)'.$$

Note that one may freely choose e.g. $(\psi_s^-)' = -1$. However, we have to establish values of ψ_L^- and $(\psi_L^-)'$. This is achieved by solving Eq. (86) with BC's $\psi_L(a) = 0$ and $d\psi_L/dr|_a = -1$, shooting inwards towards $r_s^{(m+1)}$ from a, calculating

$$\psi_L(r_s^{(m+1)} + \delta), \quad \text{and} \quad \left.\frac{d\psi_L}{dr}\right|_{r_s^{(m+1)}+\delta}$$

which are identified with, and therefore define respectively, ψ_L^- and $(\psi_L^-)'$.

From the above results we may obtain the full radial dependence of ψ in region $r_s < r < r_s^{(m+1)} - \delta$. We construct ψ as a linear sum of the two solutions,

$$\psi(r) = \psi_L(r) + \Lambda\psi_s(s).$$

From the definition of Δ'_R, and from the characteristics of $\psi_s(r)$ and $\psi_L(r)$ across the rational surface, it is clear that

$$\Lambda = -\Delta'_R \frac{\psi_L^-}{(\psi_s^-)'}.$$

A bit of elementary algebra gives,

$$\begin{aligned}\frac{r}{\psi}\frac{d\psi}{dr} =& \frac{r}{\psi_L(r)+\Lambda\psi_s(r)}\frac{d}{dr}(\psi_L(r)+\Lambda\psi_s(r)) \\ =& \left(\frac{1}{1+A(r)}\right)\left[\frac{r}{\psi_L(r)}\frac{d\psi_L(r)}{dr} + A(r)\frac{r}{\psi_s(r)}\frac{d\psi_s(r)}{dr}\right]\end{aligned}$$

with $A(r) = \Lambda\psi_s(r)/\psi_L(r)$. Hence, in terms of Δ'_R:

$$A(r) = -\Delta'_R \frac{\psi_L^-}{(\psi_s^-)'}\frac{\psi_s(r)}{\psi_L(r)},$$

which for an analytic approach is identified in terms of eigenvalue γ via Eq. (87).

Finally, we obtain the Robin boundary condition for $\xi_R^{(m+1)}$ upon consideration of Eq. (85), where in the region $r_s < r < r_s^{(m+1)} - \delta$ we have

$$\frac{r}{\xi_R^{(m+1)}}\frac{d\xi_R^{(m+1)}}{dr} = \left(\frac{1}{q} - \frac{n}{m+1}\right)^{-1}\frac{r}{q}\frac{dq}{dr} + \frac{r}{\psi}\frac{d\psi}{dr} - 1. \tag{88}$$

This final quantity will be evaluated at a location $r < r_s^{(m+1)}$ where Dirichlet BC's may safely be applied to the main harmonic (and lower sideband), noting that $\xi_R^{(m+1)}$ is singular exactly at $r_s^{(m+1)}$, so proximity to the upper rational needs to be avoided. An analytic treatment evaluates Eq. (88) at r_b (or r_a depending on the problem considered - see definitions of r_a and r_b below). Note that for the ideal limit $S \to \infty$, $\Delta'_R \to \infty$ and $A \to -\infty$, so that $\psi \to \psi_s$, which indeed recovers ideal results for the upper sideband displacement when substituted into Eq. (88).

3. Solution for $Z_{1,0}^{(m\pm1)}$

To make analytic progress on some problems, it is useful to be able to calculate $Z_{1,0}$. That is to evaluate $Z_1^{(m\pm1}$ to lowest order in Δq. Referring to Eq. (62) this requires calculating the constants of integration $C^\pm$. We define radial positions $r_a < r_s$ and $r_b > r_s$, whereby $|\Delta q/q| \gg \epsilon$ for $r < r_a$ and $r > r_b$, and $|\Delta q/q| \sim \epsilon$ for $r_a < r < r_b$. For this lowest order problem, we impose Dirichlet BC's for the main harmonic at r_a and r_b. The solutions for $C^\pm$ in $Z_{1,0}^{r(m\pm1)}$ are easily shown [14] to be,

$$C^+ = -\left[\frac{(1+m)(2+m+b_{m+1})(2+m+c_{m+1})}{(m-b_{m+1})(2+m+c_{m+1})r_a^{2+2m} - (m-c_{m+1})(2+m+b_{m+1})r_b^{2+2m}}\right]\int_{r_a}^{r_b} dr\, r^{1+m}\alpha\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right) \tag{89}$$

$$C^- = -\left[\frac{(1-m)(2-m+b_{m-1})(2-m+c_{m-1})}{(m+c_{m-1})(2-m+b_{m-1})r_b^{2-2m} - (m+b_{m-1})(2-m+c_{m-1})r_a^{2-2m}}\right]\int_{r_a}^{r_b} dr\, r^{1-m}\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right), \tag{90}$$

where

$$b_{m\pm1} = \lim_{r\to r_a^-}\frac{r}{\xi_R^{(m\pm1)}}\frac{d\xi_R^{(m\pm1)}}{dr},$$

$$c_{m\pm1} = \lim_{r\to r_b^+}\frac{r}{\xi_R^{(m\pm1)}}\frac{d\xi_R^{(m\pm1)}}{dr}.$$

The limit in $b_{m\pm1}$ is taken to mean r approaching r_a from within the range $0 \le r < r_a$, and the limit in $c_{m\pm1}$ is taken to mean r approaching r_b from within the range $r_b < r \le a$. Note that $\xi_R^{(m\pm1)}$ are obtained in these two regions by solving the sideband equations (82) in the absence of the main harmonic and inertia. Those equations are solved using the boundary conditions described in the previous two subsections. Note, if we wish to include resistive effects on upper sideband for example, we would reconcile $c_{m\pm1}$ with Eq. (88) and preceding equations, evaluating the latter at r_b.

D. Equations for very low shear and pressure gradients

Here we address the comment in Ref. [3] that there is an alternative ordering problem for $\alpha \sim \epsilon^2$ and $\Delta q \sim \epsilon^2$. Waelbroeck [2] considered this ordering for the case $q_s = 1$, and separately for $m = 1$. We again treat a double expansion in ϵ and s:

$$V_\phi = \epsilon^0 V_{\phi0} + \epsilon^2 V_{\phi2} + \epsilon^4 V_{\phi4}$$

where

$$V_{\phi 0} = s^2 V_{\phi 0,2}$$
$$V_{\phi 2} = \epsilon^2 s V_{\phi 2,1}$$
$$V_{\phi 4} = \epsilon^4 V_{\phi 4,0}.$$

All of these terms are formally the same order if $s \sim \Delta q \sim \epsilon^2$. Clearly, we already have terms of type $V_{\phi 0}$ and $V_{\phi 4}$ in Eq. (80), which are respectively the field line bending contributions and the infernal (and Mercier) contributions to the pressure. In the development of Eq. (80) we kept some terms of type $V_{\phi 2}$, in particular the contribution of Eq. (58). But, we did not specifically calculate $R^{(m)}$, which is defined by Eq. (81). With the modified ordering, terms in Eq. (81) are of leading order. In particular those terms proportional to $\Delta q \epsilon^2$, $\Delta q \Delta' \epsilon$ and $\Delta q (\Delta')^2$. The other terms can be ignored. In the limit of very low magnetic shear we have $\Delta' = \epsilon/4$, giving,

$$R^{(m)} = \frac{\epsilon^2 \Delta q \xi_R^{(m)}}{q_s^3} \left[\frac{13}{4} - 4 \left(\frac{1}{q_s^2} - 1 \right) \right]. \qquad (91)$$

Terms in $R^{(m)}$ related to $Z^{(m\pm1)}$ do not enter because $Z^{(m\pm1)} \sim \alpha \xi^{(m)}$ with $\alpha \sim \epsilon^2$. Terms in $R^{(m)}$ related $\xi_{R1,0}^{(m\pm1)}$ cancel to relevant order for the very low shear case (since $\epsilon + \alpha - 4\Delta' \sim O(\epsilon^2)$). The sideband equations that govern $Z^{(m\pm1)}$ can neglect the last term in Eq. (82) for the same reason.

We now write the equations for the specific case of very low shear and pressure gradients by depositing Eq. (91) for $R^{(m)}$ in Eq. (80),

$$\begin{aligned} 0 =& \frac{1}{r}\frac{d}{dr}\left\{ r^3 \left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q} - \frac{1}{q_s}\right)^2 \right] \frac{d\xi^{r(m)}}{dr} \right\} \\ &- (m^2-1)\left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q} - \frac{1}{q_s}\right)^2 \right] \xi^{r(m)} \\ &+ \frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q} - \frac{1}{q_s}\right)\frac{d\chi^{(m)}}{dr} - \frac{r^3}{q_s}\chi^{(m)}\frac{d}{dr}\left(\frac{1}{q}\right) \right] \\ &- (m^2-1)\frac{1}{q_s}\left(\frac{1}{q} - \frac{1}{q_s}\right)\chi^{(m)} \\ &+ \frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2} - 1\right)\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right) \\ &+ \frac{\alpha}{2q_s^2}\left[Z^{(m+1)} + Z^{(m-1)} - \alpha\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)}\right) \right] \\ &- \frac{\alpha\Delta'}{q_s^2} r \frac{d}{dr}\left[\chi^{(m)} + \chi_\Gamma^{(m)} \right] \\ &- \frac{\epsilon^2}{q_s^2}\left[\frac{13}{4} - 4\left(\frac{1}{q_s^2} - 1\right) \right]\left[q_s\left(\frac{1}{q} - \frac{1}{q_s}\right)\xi^{r(m)} + \chi^{(m)} \right]. \end{aligned} \qquad (92)$$

The sideband equation is, on setting $\Delta' = \epsilon/4 + O(\epsilon^2)$:

$$\begin{aligned} 0 =& \frac{1}{r}\frac{d}{dr}\left[r^3 \left(\frac{\gamma^2(1+2q^2)}{(m\pm1)^2\omega_A^2} \right) \frac{d\xi^{r(m\pm1)}}{dr} \right] \\ &- m(m\pm2)\left(\frac{\gamma^2(1+2q^2)}{(m\pm1)^2\omega_A^2} \right) \xi^{r(m\pm1)} \\ &+ \frac{1}{r}\frac{d}{dr}\left[r^3 \left(\frac{1}{q} - \frac{n}{m\pm1} \right)^2 \frac{d\xi_R^{r(m\pm1)}}{dr} \right] \\ &- m(m\pm2)\left(\frac{1}{q} - \frac{n}{m\pm1} \right)^2 \xi_R^{r(m\pm1)} \\ &- \frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left\{ \frac{\alpha}{2r^{\pm m}}\left(\xi^{r(m)} + \Delta\xi_\Gamma^{r(m)} \right) \right\}. \end{aligned} \qquad (93)$$

As will be seen, for the special case $m = 1$, Eq. (92), together with the lowest order solution (62) of Eq. (93), is valid also for $\alpha \sim \epsilon$ and $\Delta q \sim \epsilon$. After demonstrating that in the next section, we thus use Eqs. (92) and (62) for the rest of this paper, neglecting error imposed for cases with $m \neq 1$, hoping that these are small. It is expected in any case that the results will be qualitatively correct, probably nearly correct quantitatively. We will assess this in future work.

IV. RECOVERY OF IDEAL $m = 1$ TOROIDAL INTERNAL KINK FROM INFERNAL EQUATIONS

We now solve the $m = 1$ ideal problem analytically, assuming that there is an exact rational surface at $q(r_s) = m/n$. It will be shown later that near marginal stability, for arbitrarily low shear, an infernal mode with an exact rational surface will have leading order radial displacement of the form $\xi^{r(m)} = \xi_0 (r/r_s)^{m-1} \mathrm{H}(r - r_s)$, where $\mathrm{H}(r - r_s)$ is unity for $r < r_s$ and zero for $r > r_s$, where in the following we assume that $\Delta q \sim \epsilon$ in the region $r < r_s$. Following Eqs. (83) and (62) gives

$$\begin{aligned} Z^{(m-1)} &= \frac{\alpha}{2}\xi_0, \qquad \xi_R^{r(m-1)} = 0 \\ Z_{1,0}^{(m+1)} &= \frac{\alpha}{2}\xi_0 + C^+ r, \qquad \xi_{1,0}^{r(m+1)} = \xi_0\left(\Delta' - \frac{r}{4R_0} \right) + \frac{C^+ r}{2} \end{aligned} \qquad (94)$$

for $m = 1$ in the low shear region, where we note that $\xi_{1,0}^{r(m+1)}$ satisfies the definition of $Z_{1,0}^{(m+1)}$ having used $r\Delta'' = \alpha + r/R_0 - 3r^2\Delta'$ to leading order in Δq. Substituting these results into Eq. (81) gives

$$\begin{aligned} R^{(m)} =& \frac{\Delta q}{q_s^q}\left[\frac{13}{4}\epsilon^2 - 4\epsilon^2\left(\frac{1}{q_s^2} - 1\right) + \frac{\alpha}{4}(12\Delta' - 4\alpha - 3\epsilon) \right]\xi_0 \\ &+ O(\epsilon \Delta q C^+) \end{aligned}$$

for $m = 1$ to necessary order. We still require $Z_{1,1}^{(m+1)}$ in the low shear region for the definition of $Z^{(m+1)}$ in the main harmonic equation. From Eq. (66) we may obtain in the low shear region,

$$Z_{1,1}^{(m\pm1)} = 2\frac{\Delta q}{q_s} Z_{1,0}^{(m\pm1)} - 2(2\pm m) r^{\pm m}\int dr\,\frac{\xi_{1,0}^{(m\pm1)}}{r^{\pm m}}\frac{d}{dr}\left[\frac{\Delta q}{q_s}\right] - (2\pm m)(1\pm m) r^{\pm m}\int dr\, r^{-(1\pm m)}\frac{\Delta q \xi^{r(m)}}{q_s} r^2\frac{d}{dr}\left[\frac{\Delta'}{r}\right],$$

where we have used $\epsilon + \alpha - 4\Delta' = r^2 d/dr(\Delta'/r)$. Hence, for the case at hand $m = 1$, using above expressions for $Z_{1,0}^{(m+1)}$ and $\xi_{1,0}^{r(m+1)}$ we obtain,

$$Z_{1,1}^{(m\pm1)} = -\frac{\Delta q}{q_s}\frac{\alpha}{2}(12\Delta' - 4\alpha - 3\epsilon)\xi_0 + O(\epsilon \Delta q C^+)$$

in the low shear region for $m = 1$. Substituting these results into the governing equation (80) we obtain

$$\begin{aligned} 0 =& \frac{1}{r}\frac{d}{dr}\left\{ r^3\left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q}-\frac{1}{q_s}\right)^2\right]\frac{d\xi^{r(m)}}{dr}\right\} \\ &- (m^2-1)\left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q}-\frac{1}{q_s}\right)^2\right]\xi^{r(m)} \\ &+ \frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2}-1\right)\xi^{r(m)} + \frac{\alpha}{2q_s^2}\left[C^+ r^m\left(1+O(\Delta q)\right)\right] \\ &- \frac{\epsilon^2}{q_s^2}\left[\frac{13}{4} - 4\left(\frac{1}{q_s^2}-1\right)\right] q_s\left(\frac{1}{q}-\frac{1}{q_s}\right)\xi^{r(m)}. \end{aligned} \tag{95}$$

in the low shear region for $m = 1$. We now see why we did not need to calculate the $O(\epsilon\Delta q C^+)$ corrections. Notice that this is the result that would also have been obtained from the ultra-low shear and α equations of (92) together with the lowest order solution for $Z^{(m+1)}$ given by Eq. (62).

For the case at hand, the last two lines of Eq. (95) will adopt the Heaviside step function, but the inertia term and field line bending terms do not, as these contributions require finer resolution. We now integrate Eq. (95) from 0 to r_s, giving,

$$\frac{1}{\xi_0}\left\{ r^m \Delta Q^2 \frac{d\left(r^{1-m}\xi^{r(m)}\right)}{dr}\right\}\Bigg|_{r=r_s} = \delta\hat{W} \tag{96}$$

for $m = 1$, where

$$\Delta Q^2 = \frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q}-\frac{1}{q_s}\right)^2. \tag{97}$$

and

$$\delta\hat{W} = \int_0^{r_s}\frac{r\,dr}{r_s^2}\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\left[\frac{13}{4} - 4\left(\frac{1}{q_s^2}-1\right)\right]\left(\frac{r}{R_0}\right)^2 - \int_0^{r_s}\frac{dr\,r}{r_s^2}\frac{r\alpha}{R_0}\frac{1}{q_s^2}\left(\frac{1}{q_s^2}-1\right) - \frac{1}{q_s^2}\left[\frac{3+c}{1-c}\right]\left[\int_0^{r_s}\frac{dr}{r_s}\frac{\alpha r^2}{r_s^2}\right]^2. \tag{98}$$

In $\delta\hat{W}$ we have set $c = c_{m+1}$ in the definition of C^+ given by Eq. (89) for $m = 1$. In addition, we have taken $r_a = 0$ which gives $b_{m+1} = 0$ and we have set $r_b = r_s$ which is a valid choice for the resonant problem (as will be seen later). Also, as mentioned above, we have taken $\xi^{r(m)} = \xi_0 H[r - r_s]$ in the terms in $\delta\hat{W}$. The result can be written in the form,

$$\delta\hat{W} = (1-q_s^2)\delta\hat{W}_C + q_s^2\delta\hat{W}_T$$

with

$$\delta\hat{W}_C = -\frac{1}{q_s^2}\left\{\frac{\epsilon_s}{q_s^2}\int_0^{r_s}\frac{dr\,r^2}{r_s^3}\alpha + 4\epsilon_s^2\int_0^{r_s}\frac{dr\,r^3}{r_s^4}\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\right\} \tag{99}$$

$$\delta\hat{W}_T = -\frac{1}{q_s^4}\left\{\left[\frac{3+c}{1-c}\right]\left[\int_0^{r_s}\frac{dr\,r^2}{r_s^3}\alpha\right]^2 - \frac{13}{4}\epsilon_s^2\int_0^{r_s}\frac{dr\,r^3}{r_s^4}q_s\left(\frac{1}{q}-\frac{1}{q_s}\right)\right\}, \tag{100}$$

where $\epsilon_s = r_s/R_0$. The growth rate can be easily established for the case where resistive effects at r_s are ignored by careful calculation of the left hand side of Eq. (96), giving

$$\frac{\gamma}{\omega_A} = -\frac{m\pi}{q_s s(r_s)\sqrt{1+2q_s^2}} q_s^2\delta\hat{W}, \tag{101}$$

where it is reminded that $m = 1$.

A. Bussac form for internal kink mode and agreement with infernal mode description for small $\Delta q/q_s$

The normalisation for $\delta \hat{W}$ used in this paper differs from $\delta \tilde{W}$ in Ref. [7] by a factor $\epsilon_s^2 q_s^{-4}$. Hence multiplying the relations in Ref. [7] by $\epsilon_s^2 n^4 = \epsilon_s^2/q_s^4$ (for $m=1$) the internal kink mode in a torus with circular cross section, assuming $\Delta q/q_s \sim 1$, gives:

$$\delta \hat{W} = (1-q_s^2)\delta \hat{W}_C + q_s^2 \delta \hat{W}_T$$

with

$$\delta \hat{W}_C = -\frac{\epsilon_s^2}{q_s^2}\left\{\frac{\beta_p}{q_s^2} + \int_0^{r_s} \frac{dr\, r^3}{r_s^4}\left(\frac{3}{q_s}+\frac{1}{q}\right)\left(\frac{1}{q}-\frac{1}{q_s}\right)\right\}, \quad \beta_p = -2\frac{q(r_s)^2}{B_0^2\epsilon_s^2}\int_0^{r_s}\frac{dr\, r^2}{r_s^2}\frac{dP}{dr} \tag{102}$$

$$\delta \hat{W}_T = \frac{\epsilon_s^2}{q_s^4}\left\{\frac{32(b-c)\sigma + 9(b-1)(1-c)}{64(b-c)} - \frac{3(b-1)(c+3)}{8(b-c)}(\beta_p+\sigma) - \frac{(c+3)(b+3)}{4(b-c)}(\beta_p+\sigma)^2\right\}, \tag{103}$$

where

$$\sigma = \int_0^{r_s}\frac{dr}{r_s}\left(\frac{r}{r_s}\right)^3\left(\frac{q_s^2}{q^2}-1\right), \tag{104}$$

and

$$b = \frac{r}{\xi^{r(m+1)}}\frac{d\xi^{r(m+1)}}{dr}\bigg|_{r_s-\delta}, \quad c = \frac{r}{\xi^{r(m+1)}}\frac{d\xi^{r(m+1)}}{dr}\bigg|_{r_s+\delta},$$

with $m=1$, and we note c has the same definition as for infernal modes. We may define,

$$b = 1+\Delta b, \quad c = -3+\Delta c$$

where for an unsheared q-profile $\Delta b = \Delta c = 0$. For the infernal mode applications considered here, where $r_a = 0$, we have $\Delta c \sim 1$ and $\Delta b \sim \Delta q$. Noting also that $\sigma \sim \Delta q$, we obtain the following expansion:

$$\frac{q_s^4}{\epsilon_s^2}\delta \hat{W}_T = \frac{1}{2}\sigma + \left[\frac{9}{64} - \frac{3}{8}\left(\frac{3+c}{1-c}\right)\beta_p\right]\Delta b + \left(\frac{3+c}{1-c}\right)\beta_p\left(\beta_p+2\sigma\right) + O(\Delta q^2)$$

It is shown in the appendix that $\Delta b = 8\sigma[1+O(\Delta q/q_s)]$ for arbitrary r_s, and we may use,

$$\sigma = -2\int_0^{r_s}\frac{dr}{r_s}\left(\frac{r}{r_s}\right)^3\frac{\Delta q(r)}{q_s} + O(\Delta q^2/q_s^2) = 2\int_0^{r_s}\frac{dr}{r_s}\left(\frac{r}{r_s}\right)^3 q_s\left(\frac{1}{q}-\frac{1}{q_s}\right) + O(\Delta q^2/q_s^2),$$

Expanding $\delta \hat{W}_C$ in Δq we may finally write the Bussac expressions in expanded form:

$$\delta \hat{W}_C = -\frac{\epsilon_s^2}{q_s^2}\left\{\frac{\beta_p}{q_s^2} + 4\int_0^{r_s}\frac{dr\, r^3}{r_s^4}\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right) + O(\Delta q^2)\right\} \tag{105}$$

$$\delta \hat{W}_T = -\frac{\epsilon_s^2}{q_s^4}\left\{\left[\left(\frac{3+c}{1-c}\right)\beta_p^2 - \frac{13}{4}\int_0^{r_s}\frac{dr}{r_s}\left(\frac{r}{r_s}\right)^3 q_s\left(\frac{1}{q}-\frac{1}{q_s}\right)\right] + O(\Delta q^2, \Delta q\beta_p)\right\}. \tag{106}$$

The corrections $O(\Delta q^2, \Delta q\beta_p)$ cannot be ignored for the standard ordering used in reference Ref. [7] ($\beta_p \sim 1$, $\Delta q/q_s \sim 1$). But for infernal modes, we have that $\Delta q \sim \epsilon, \beta_p \sim 1$, so that we can neglect $O(\Delta q^2, \Delta q\beta_p)$ terms in the results from Bussac. Thus the above expressions expanded from Bussac are identical to those derived from the infernal mode approach, i.e. Eqs. (99) and (100). This point is further emphasised by noting that the $O(\Delta q^2, \Delta q\beta_p)$ terms do not appear in the well known internal kink result for the specific safety factor profile

$$q = q_s + \Delta[(r/r_s)^\nu - 1], \tag{107}$$

$q_s = 1$, in particular the expression for δW_T on page 1641 in Ref. [7] (here c is solved analytically in the limit of small r_s, i.e $c = -3 + 12\nu\Delta/(4-\nu)$). We recover these well known results with the higher order infernal mode equations of this paper, i.e. for $n=1$ and $q_s = 1$, substituting $c = -3 + 12\nu\Delta/(4-\nu)$ and $1/q - 1/q_s = \Delta[1-(r/r_s)^\nu]$ into Eq. (100) or Eq. (106) we obtain,

$$\delta\hat{W}_T = -\epsilon_s^2\nu\Delta\left\{\frac{3}{4-\nu}\beta_p^2 - \frac{13}{16(4+\nu)}\right\}. \qquad (108)$$

This agrees exactly with Eq. (7) of Ref. [17], and indeed with the results at the end of Ref. [7], i.e. marginal poloidal beta $\beta_p = [13(4-\nu)/(48(4+\nu))]^{1/2}$, which for the quadratic q-profile $\nu = 2$ gives the well know marginal value $\beta_p \approx 0.3$ (which is $\alpha(r_s) = 1.2\epsilon_s$ for a parabolic pressure profile for which $\beta_p = \alpha/(4\epsilon)$).

V. RESISTIVE INTERCHANGE MODES

The previous section on the $m = 1$ internal kink mode has demonstrated that the low-shear, low-beta main harmonic equation (92) together with the lowest order solution (62) in the ideal limit recovered the ideal internal kink result of Ref. [7]. We therefore adopt (92) together with (62) for the rest of the analytic calculations in this paper, also for resistive studies. We choose to drop the resistive-compression contribution $\Delta\xi_\Gamma$ from (62) and in the definitions of C^+ and C^- given by Eqs. (90) and (89). Dropping this compression effect will increase the growth rate by a factor, this factor vanishing where resistive effects are negligible. We also drop terms in (92) that involve χ, except those involving derivatives in χ, since we know that χ is localised near the rational surface. The consequences of the approximations mentioned here will be addressed in future work where the full governing equations will be solved numerically. Setting $\Delta\xi_\Gamma^{r(m)} = 0$ is expected to be the worst approximation in the forthcoming results in this paper, which is based on solutions of:

$$\begin{aligned}0 =& \frac{1}{r}\frac{d}{dr}\left\{r^3\left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q}-\frac{1}{q_s}\right)^2\right]\frac{d\xi^{r(m)}}{dr}\right\}\\ &-(m^2-1)\left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q}-\frac{1}{q_s}\right)^2\right]\xi^{r(m)}\\ &+\frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d\chi^{(m)}}{dr} - \frac{r^3}{q_s}\chi^{(m)}\frac{d}{dr}\left(\frac{1}{q}\right)\right]\\ &+\frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2}-1\right)\xi^{r(m)} - \frac{\alpha\Delta'}{q_s^2}r\frac{d}{dr}\left[\chi^{(m)}+\chi_\Gamma^{(m)}\right]\\ &+\frac{\alpha}{2q_s^2}\left[C^+r^m + C^-r^{-m}\right]\\ &-\frac{\epsilon^2}{q_s^2}\left[\frac{13}{4} - 4\left(\frac{1}{q_s^2}-1\right)\right]q_s\left(\frac{1}{q}-\frac{1}{q_s}\right)\xi^{r(m)}, \qquad (109)\end{aligned}$$

with C^+ and C^- defined by Eqs. (90) and (89) in the limit $\Delta\xi_\Gamma^{r(m)} \to 0$.

We consider now interchange modes. These occur in the limit when the radial scale length of the instability is short, i.e. when the magnetic shear is large, and/or when m is large. Since this paper deals with $\Delta q/q_s \sim \epsilon$, we thus consider that interchange modes occur where $m \gg 1$. Under such conditions, the infernal mode corrections associated with C^+ and C^- are insignificant. This can be seen by inspection of Eq. (26), the last term on the first line being due to C^+, and it is seen [15] that it scales as $\sim 1/m$. As mentioned, interchange modes are highly localised, so the last term in Eq. (109) can also be neglected (since it vanishes at the rational surface). Hence, resistive interchange modes are governed by,

$$\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left\{r^3\left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q}-\frac{1}{q_s}\right)^2\right]\frac{d\xi^{r(m)}}{dr}\right\}\\ &-(m^2-1)\left[\frac{\gamma^2(1+2q^2)}{m^2\omega_A^2} + \left(\frac{1}{q}-\frac{1}{q_s}\right)^2\right]\xi^{r(m)}\\ &+\frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d\chi^{(m)}}{dr} - \frac{r^3}{q_s}\chi^{(m)}\frac{d}{dr}\left(\frac{1}{q}\right)\right]\\ &= -\frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2}-1\right)\xi^{r(m)} + \frac{\alpha\Delta'}{q_s^2}r\frac{d}{dr}\left[\chi^{(m)}+\chi_\Gamma^{(m)}\right],\end{aligned} \qquad (110)$$

the left hand side being the field line bending and inertia terms.

1. *Ideal Interchange*

In the ideal limit we set $\chi = \chi_\Gamma = 0$. The left hand side of Eq. (110) providing field line bending stabilisation, and the right hand side causing potentially the drive for interchange instability[16]. One obtains ideal instability if

$$\frac{s^2}{4} < \epsilon\alpha\left(\frac{1}{q_s^2}-1\right)$$

and under those conditions, the growth rate normalised to the Alfvén frequency is given by

$$\frac{\gamma}{\omega_A} = \frac{16\, s\, \exp\{\pi\left[\epsilon\alpha(1/q_s^2-1)/s^2 - 1/4\right]^{-1/2} - C + \pi/2\}}{q_s\sqrt{1+2q_s}},$$

where $C = 0.577..$ is the Euler-Mascheroni constant.

2. *Resistive Interchange*

We now investigate the modification of the ideal stability criteria and growth rates for interchange modes given above due to resistive effects. At marginal stability resistive Ohm's law of Eq. (50) dictates that magnetic field line bending at the rational surface vanishes, and we recall that interchange modes are essentially confined to the rational surface at marginal stability. In particular that,

$$q_s\left(\frac{1}{q}-\frac{1}{q_s}\right)\xi^{r(m)} + \chi^{(m)} \approx 0 \qquad (111)$$

at marginal stability, and thus $\Delta q \xi_R^{r(m)} = 0$, so that the field line bending terms on the left hand side of Eq. (110) vanish. While the $s^2/(4q_s^2)$ stabilising term in the ideal interchange criteria is lost due to resistivity, a new contribution is gained from the last term on the right hand side of Eq. (110). Evaluating $d/dr(\chi^{(m)} + \chi_\Gamma^{(m)})$ at $r = r_s$ for small (but non-zero γ), using Eq. (76) for $\chi^{(m)} + \chi_\Gamma^{(m)}$ and Eq. (111) we arrive at

$$\frac{d}{dr}\left[\chi^{(m)} + \chi_\Gamma^{(m)}\right]\Big|_{r_s} \approx \xi^{r(m)}\frac{dq}{dr}$$

near the rational surface. Hence with the addition of this new contribution, and the loss of the field line bending contribution, resistive interchange are unstable for,

$$0 < s^2 D_R, \quad s^2 D_R = \epsilon\alpha\left(\frac{1}{q_s^2} - 1\right) - s^2 H, \quad H = \frac{\alpha\Delta'}{s}. \tag{112}$$

Ordinarily in a tokamak, with monotonically decreasing pressure, and positive magnetic shear, H is stabilising. This definition of H is important for internal kink modes, and so is λ, the growth rate normalised as follows in a resistive plasma,

$$\lambda = \gamma\tau_H^{2/3}\tau_R^{1/3}, \quad \tau_H = \frac{\tau_A q_s\sqrt{1+2q_s^2}}{sm}, \quad \tau_R = \frac{r^2}{\eta}, \tag{113}$$

where $\tau_A = 1/\omega_A$. Following the dispersion relation of Ref. [9],

$$\Delta_R' = \frac{2\pi\Gamma(3/4)\lambda^{5/4}}{\Gamma(1/4)L_R}\left(1 - \frac{\pi D_R}{4\lambda^{3/2}}\right), \quad L_R = r\tau_H^{1/3}\tau_R^{-1/3},$$

(with L_R the resistive interchange layer width) the resistive interchange growth rate is obtained with the approximation $\Delta_R' = 0$, i.e. the dispersion relation is

$$\lambda^{3/2} = \frac{\pi}{4}D_R, \tag{114}$$

all evaluated at r_s. In terms of Lundquist number $S = \tau_R/\tau_A$,

$$\lambda = S^{1/3}\frac{\gamma}{\omega_A}\left(\frac{q_s\sqrt{1+2q_s^2}}{sm}\right)^{2/3}$$

so that,

$$\frac{\gamma}{\omega_A} = S^{-1/3}\left(\frac{\pi sm}{4q_s\sqrt{1+2q_s^2}}\right)^{2/3}\left(\frac{\pi}{4}D_R\right)^{2/3}.$$

Important here to notice is the relatively slow scaling of growth rate on Lundquist number (compared to standard current driven tearing modes where $\gamma \sim S^{-3/5}$), and that the growth rate increases with reducing magnetic shear, $\gamma \sim s^{-2/3}$ (compared to standard current driven tearing modes $\gamma \sim s^{2/5}$). This clearly motivates the study on resistive infernal modes, which is the correct treatment for long wavelength modes where the shear is weak.

VI. ANALYTICAL RESISTIVE INFERNAL MODES WITH $q_s = 1$

Having investigated resistive interchange modes from the infernal mode description, we now investigate resistive infernal modes and resistive kink modes. We choose to simplify the problem by setting $q_s = 1$, which removes the interchange drive in an axisymmetric toroidal equilibrium with circular cross section. Combined infernal and interchange drive have been investigated previously in Refs. [12, 15] in the ideal limit, but equilibria with $q_s \neq 1$ are out of scope for the resistive studies investigated in the rest of this paper. We also set $r_a = 0$ so that the low shear region extends to the axis, assuming a monotonically increasing q-profile, ensuring that the lower sideband is not resonant. In order to look at the nature of such resistive infernal modes, we first need to consider ideal infernal modes, with and without exact resonance of the fundamental harmonic.

A. Ideal problem in absence of $q(r_s) = 1$ rational surface, assuming $\Delta q \sim \epsilon$ or larger

We start from Eq. (109), simplifying the term involving $(\Delta Q^2)'$ (neglecting spatial variation in the inertia contribution in the $(1-m)(\Delta Q^2)'$ term) and treating the ideal problem ($\chi^{(m)} \to 0$, $\chi_\Gamma^{(m)} \to 0$) for $q_s = 1$. Thus

$$r^{-m}\frac{d}{dr}\left[r^{2m+1}\Delta Q^2\frac{d\left(r^{1-m}\xi^{r(m)}\right)}{dr}\right] = \left[\frac{13}{4}\left(\frac{r}{R_0}\right)^2 + 2(m-1)s(r)\right]\left(\frac{1}{q} - 1\right)\xi^{r(m)} - \frac{\alpha C^+}{2}r^m, \tag{115}$$

where $s = (r/q)dq/dr$. Integrating from 0 to r, and using BC at $r = 0$ yields,

$$r^{2m+1}\Delta Q^2\frac{d\left(r^{1-m}\xi^{r(m)}\right)}{dr} = \int_0^r dt\left[\frac{13}{4}\left(\frac{t}{R_0}\right)^2 + 2(m-1)s(t)\right]\left(\frac{1}{q} - 1\right)t^m\xi^{r(m)} - \int_0^r dt\,\frac{\alpha C^+}{2}t^{2m}. \tag{116}$$

If there is no rational surface, the left hand side of Eq. (116) is not singular even as $\gamma \to 0$. The left hand side of Eq. (116) provides stabilising field line bending effects at all radial positions, and it permits the establishment of marginal stability conditions, where the destabilising effect comes predominantly from the last term on the right. As we will see, if on the other hand an exact rational surface exists, the only means of obtaining marginal stability conditions will be if we retain the first term on the right of Eq. (116), since the left hand side will be singular. By avoiding the rational, the pressure must be reasonably large in order for a mode to be unstable (since the field line bending term on the left hand side of Eq. (116) produces a strong stabilising contribution). We may therefore argue that it could be reasonable to drop the first integral on the right hand side of Eq. (116) providing that an exact rational is avoided, the shear is weak (so $(m-1)s$ isn't large), and Δq isn't too small. We will address this approximation in the next subsection, but currently adopting the approximation and noting that by avoiding the rational we can always neglect resistive effects in the core (so we can interchange $\xi_R^{r(m)}$ and $\xi^{r(m)}$), we now have,

$$r^{2m+1}\Delta Q^2 \frac{d\left(r^{1-m}\xi_R^{r(m)}\right)}{dr} = -\frac{C^+}{2}\int_0^r dt\,\alpha(t)t^{2m}.$$

This equation can be integrated again, and we apply boundary conditions at $r=0$, and $r=r_b$. We note that the outer boundary condition at r_b will better defined if there is a rational, but if there is not, we nevertheless apply $\xi^{r(m)}(r_b)=0$. It can be shown that the eigenvalue equation is

$$\begin{aligned} 1 =& \frac{1+m}{2} r_b^{-2-2m}\left[\frac{m+2+c}{m-c}\right] \\ &\times \int_0^{r_b} dr\,\frac{r^{-2m-1}}{\Delta Q^2(r,\gamma)}\left(\int_0^r dv\,v^{2m}\alpha(v)\right)^2. \end{aligned} \quad (117)$$

This yields γ through the definition of ΔQ^2, and from knowledge of γ, the eigenvector can be obtained:

$$\begin{aligned} \xi^{r(m)}(r) &= \xi_0 r^{m-1}[1-I(r)/I(r_b)], \\ I(r) &= \int_0^r dt\,\frac{t^{-2m-1}}{\Delta Q^2(t)}\left[\int_0^t dv\,v^{2m}\alpha(v)\right] \end{aligned} \quad (118)$$

where the constant

$$\xi_0 = \frac{C^+}{2}I(r_b).$$

The critical point to notice here is that if there is an exact resonance at $r=r_s$, $r_s=r_b$, then as we approach marginal stability we have $\Delta Q^2(r_s)\to 0$, and hence

$$\lim_{\gamma\to 0}\xi^{r(m)}(r) = \xi_0(r/r_s)^{m-1}H(r-r_s), \quad (119)$$

where we recognise the Heaviside step function $H(r-r_s)=1$ for $r<r_s$ and $H(r-r_s)=0$ for $r>r_s$. And, if there is a rational surface, the marginal stability criterion $\gamma\to 0$ occurs only for $\alpha\to 0$. Hence, if there is an exact rational, under the approximation we have made in this section (dropping the first term on the RHS of (116)), physical marginal stability criteria cannot be obtained.

B. Ideal problem with exact rational surface $q(r_s)=1$

We have seen that near marginal stability, for the case with an exact rational, the eigenfunction has a discontinuity at the rational, but is otherwise smooth, with the eigenfunction given approximately by Eq. (119). We now use this knowledge in a more refined ideal model with an exact rational surface. The first step is to evaluate Eq. (116) at $r=r_s$. The right hand side of Eq. (116) simply yields Eq. (96) but now with

$$\begin{aligned} \delta\hat{W} =& \int_0^{r_s} dr\left[\frac{13}{4}\left(\frac{r}{R_0}\right)^2 + 2(m-1)s(r)\right]\left[\frac{1}{q}-1\right] \\ &\times \frac{r^m\xi^{r(m)}(r)}{r_s^{m+1}\xi_0} - \int_0^{r_s} dr\,\frac{C^+\alpha r^{2m}}{2r_s^{m+1}\xi_0}. \end{aligned}$$

We note that only the left hand side of Eq. (96) is sensitive to the details of $\xi^{r(m)}$ near $r=r_s$. In $\delta\hat{W}$ we may neglect the effects of finite inertia, and hence in $\delta\hat{W}$ we adopt Eq. (119) for $\xi^{r(m)}(r)$. Using also $r_b=r_s$ in C^+, and thus also $\xi_0=(C^+/2)I(r_s)$ we obtain,

$$\begin{aligned} \delta\hat{W} =& \int_0^{r_s} dr\,\frac{r^{2m-1}}{r_s^{2m}}\left[\frac{1}{q}-1\right]\left[\frac{13}{4}\left(\frac{r}{R_0}\right)^2 + 2(m-1)s(r)\right] \\ &-\left(\frac{1+m}{2}\right)\left[\frac{m+2+c}{m-c}\right]\left[\int_0^{r_s} dr\,\frac{\alpha r^{2m}}{r_s^{2m+1}}\right]^2, \end{aligned} \quad (120)$$

also with c evaluated at r_s. The first line in Eq. (120) is stabilising (assuming monotonically increasing q in $0<r<r_s$), while the second line is the destabilising effect of infernal drive. The left hand side of Eq. (96) for the ideal case has already been treated earlier.

The ideal dispersion relation for modes where the generalised Heaviside step function (119) applies is Eq. (101). With the new expression of $\delta\hat{W}$ given by Eq. (120) we have generalised Eq. (98) for arbitrary m (but with $q_s=1$). Corresponding growth rates (101) are plotted in Fig. 1 for a parabolic pressure profile for which $\alpha(r_s)=4\epsilon_s\beta_p$. Figure 1 plots the growth rate with respect to $\alpha(r_s)$ for different poloidal mode number m on choosing the q-profile of Eq. (107) for $\nu=4$, $\Delta=0.01$, $r_s=0.25$, $a=1$ and $\epsilon_s=0.25/3$. For $m=1$ one can use Eq. (100) or Eq. (122), with c given by Eq. (12), or more accurately, Eq. (43) of Ref. [17]. For $m>1$ we solve c numerically, dropping inertia and the last line of Eq. (82), and applying the boundary condition (84).

As expected, these resonant ideal modes are increasingly stabilised by magnetic shear as m is increased. Such modes form the basis for resistive infernal modes, the main features of which will be investigated next.

1. Resistive Infernal Modes

When resistive effects are to be included on the main rational surface, we require a resistive treatment of the left hand side of Eq. (96). This has been undertaken for $m=1$ in Ref. [8] in the cylindrical limit, and a generalisation with toroidal effects in the layer is reported in Ref.

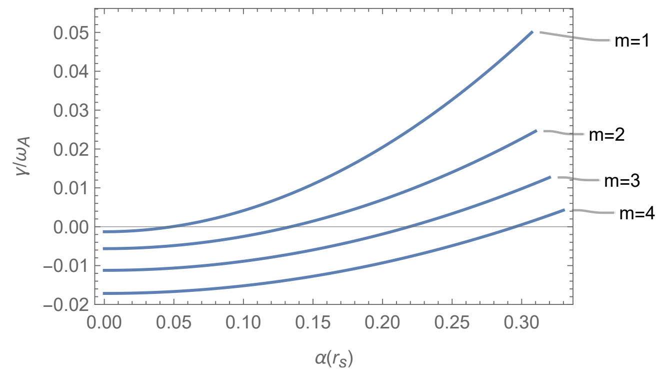

FIG. 1: The growth rate of resonant ideal $m = n$ modes, plotted with respect to $\alpha(r_s)$ for different m. The plots assume a parabolic pressure profile, and q-profile given by Eq. (107) with $\nu = 4$, $\Delta = 0.01$ and $r_s = 0.25$.

[18]. The toroidal correction involves H of Eq. (114), i.e. the effect of $d/dr(\chi^{(m)} + \chi_\Gamma^{(m)})$ has to be included on the left hand side of Eq. (96). The resistive dispersion relation generalised for arbitrary m but $q_s = 1$ can be shown to be approximately:

$$\frac{1}{\gamma_I} = \frac{\lambda^{9/4}}{8\gamma} \frac{\left(\lambda^{3/2} + H\right)\left(\lambda^{3/2} - H\right)}{\lambda^3} \times \frac{\Gamma\left[\frac{\left(\lambda^{3/2} - H\right)\left(\lambda^{3/2} + H - 1\right)}{4\lambda^{3/2}}\right]}{\Gamma\left[\frac{\left(\lambda^{3/2} + H\right)\left(\lambda^{3/2} - H + 1\right)}{4\lambda^{3/2}} + 1\right]} \tag{121}$$

where Γ is the Gamma function, and γ_I is the ideal growth rate related to $\delta\hat{W}$ by Eq. (101), i.e.

$$\frac{\gamma_I}{\omega_A} = -\frac{m\pi}{s(r_s)\sqrt{3}} \delta\hat{W},$$

for $q_s = 1$, where again $\delta\hat{W}$ is given by Eq. (120) and λ is given by Eq. (113). If we wish to include resistive effects on the $m + 1$ upper sideband, we reconcile c in $\delta\hat{W}$ with Eq. (88), evaluating the latter at r_s. The ideal limit, i.e. $\gamma = \gamma_I$ in Eq. (121), is obtained in the limit $\lambda \gg 1$ (note that $H \sim \epsilon$ for $s \sim \epsilon$). The toroidal resistive kink-infernal mode is obtained for the case where $\gamma_I = 0$ (at marginal ideal MHD stability conditions), which occurs when the gamma function on the numerator of Eq. (121) is infinite, and thus the argument of the gamma function on the numerator vanishes. The relevant root is $\lambda^{3/2} + H - 1 = 0$. The tearing-infernal mode is obtained for $\lambda \ll 1$, though the inclusion of H makes the derivation of convenient analytic forms (by taking suitable limits of the gamma functions) slightly awkward. Nevertheless, it is noted that H yields a stabilising effect in all regimes providing $\alpha/s > 0$. Either hollow pressure profiles or hollow q-profiles instead produce a destabilising effect via H.

In this section we will separately examine the effects of resistivity on the main rational surface and the upper sideband. We choose to investigate $m = 1$ resistive modes, with coupling to the $m + 1 = 2$ tearing mode. Cases with $m > 1$ are no more difficult to treat analytically than $m = 1$ now that the ideal resonant problem has been explored, and shown in Fig. 1. We choose to highlight specifically $m = 1$ to make contact with the resistive toroidal coupling problem investigated previously in Ref. [19] and subsequent papers. The infernal mode approach developed in the present paper provides the advantage of an arguably more transparent and simple calculation than that of Ref. [19]. The simplification in the present work is due to the ordering of the magnetic shear at the outset, which essentially means that logarithmic b contributions, which are explicit in the general toroidal contribution of Eq. (103), are algebraically handled due to $\Delta q \sim s \sim \epsilon$. In addition, due to the ordering expansion developed at the start, the main harmonic displacement functions, and magnetic fields (including the island contributions) are an order (in ϵ) larger than the sidebands, and resultingly the sideband equations are those of the cylinder in the sideband-resonance region. The self-consistent resistive infernal mode approach developed here offers hope for inclusion of further physics to be added in future work.

We choose the q-profile deployed in Ref. [7], i.e. Eq. (107) for $q_s = 1$. For arbitrary m we would use Eq. (120), which for $m = 1$ is (see also Eq. (100)),

$$\delta\hat{W} = -\epsilon_s^2 \left\{ \left[\frac{3 + c}{1 - c}\right] \beta_p^2 - \frac{13\nu\Delta}{16(4 + \nu)} \right\} \tag{122}$$

with c calculated numerically via Eq. (88) and preceding equations if resistive effects are required on the upper sideband rational surface. If the upper rational surface is treated in the ideal limit we choose to solve c numerically, dropping inertia and the last line of Eq. (82), and applying the boundary condition (84). To make contact with the most well know ideal results, in Fig. 2 we take $\nu = 2$, and we choose $\Delta = 0.1$ and $r_s = 0.25$. In that plot we adopt a parabolic pressure profile, and Lundquist number $S = 10^7$ on both rational surfaces. The variation in $\alpha(r_s) = 4\epsilon_s \beta_p$, and the parameters and profiles just mentioned, fully defines the equilibrium, including H. The results of three physical models are visible in Fig. 2, which plots normalised growth rate obtained from the roots of Eq. (121) verses $\alpha(r_s)$. If the main harmonic is treated in the ideal limit, we solve Eq. (101) (which is more convenient than solving Eq. (121) in the limit $S \to \infty$), but we may again handle the upper sideband in the ideal or resistive limit by calculating c appropriately, as described above.

The most stable case in Fig. 2 is where both the main rational surface and its sideband are treated in the ideal limit. Slightly more unstable is the case where the main mode is treated in the ideal limit while the upper sideband includes resistive effects on its own rational surface; this model is stable for small $\alpha(r_s)$ because, for the chosen equilibrium, the uncoupled $m + 1$ mode is stable (to current gradients). This can be verified by solving Eq.

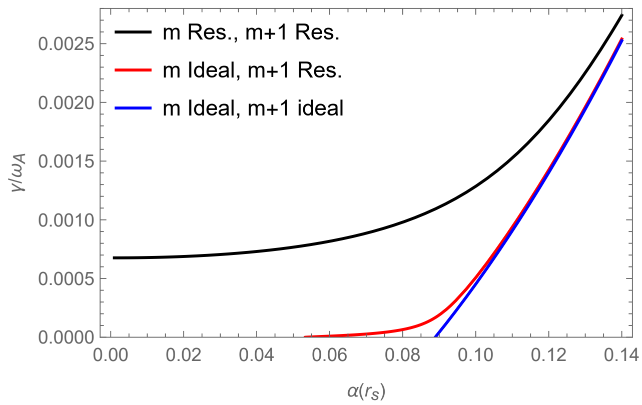

FIG. 2: The growth rate of $m = 1$ resonant infernal modes plotted as a function of $\alpha(r_s)$, for parabolic pressure profile and for a q-profile given by Eq. (107) with $\nu = 2$, $\Delta = 0.1$ and $r_s = 0.25$. Different curves show results were the main harmonic and upper sideband are treated in different combinations of ideal or resistive limits.

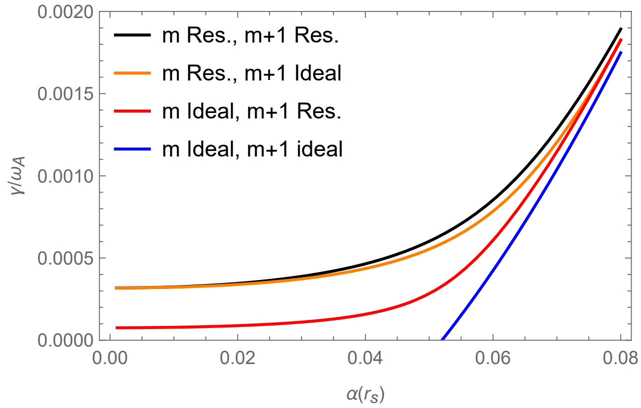

FIG. 3: The growth rate of $m = 1$ resonant infernal modes plotted as a function of $\alpha(r_s)$, for parabolic pressure profile and for a q-profile given by Eq. (107) with $\nu = 4$, $\Delta = 0.02$ and $r_s = 0.25$. Different curves show results were the main harmonic and upper sideband are treated in different combinations of ideal or resistive limits.

(86) and evaluating the sign of Δ'_R (negative if stable). The next most unstable case is where the main harmonic is treated in the resistive limit, but the upper sideband is treated in the ideal limit. For the particular equilibrium deployed in Fig. 2, that case is indistinguishable from the most unstable case where both surfaces are treated in the resistive limit. It is verified that near ideal marginal stability the growth rate of the most unstable case scales with S as $\gamma \sim S^{-1/3}$ and for small $\alpha(r_s)$ one obtains $\gamma \sim S^{-3/5}$.

The results shown in Fig. 3 differ from Fig. 2 only because a different q-profile has been selected. In particular, we assume Eq. (107) once again in Fig. 3 but this time with $\nu = 4$, and we choose $\Delta = 0.02$ and $r_s = 0.25$. The q-profile is similar to that used for the ideal calculations of Fig. 1, and indeed the must stable curve shown in Fig. 3 is seen to be similar to the $m = 1$ curve of Fig. 1. This most stable case is of course where both rational surfaces are taken as ideal. The next most stable case (the red curve) shown in Fig. 3 is where the main harmonic is taken as ideal, but the upper sideband is treated resistively. Unlike in Fig. 2, this latter case is unstable for all $\alpha(r_s)$. The reason for this is that the chosen q-profile drives the $m + 1$ mode unstable via the current gradient. In fact, the growth rate for the red curve in the limit $\alpha(r_s) \to 0$ is identical to that obtained by solving the uncoupled cylindrical equation for the upper sideband, Eq. (86), evaluating Δ'_R from that equation, and substituting the result into the left hand side of Eq. (87), and rearranging for γ/ω_A. The next most unstable case is where the main mode is treated in the resistive limit, but the upper sideband is treated in the ideal limit. Due to the fact that $\Delta'_R > 0$ for the upper sideband, the most unstable case (where both surfaces are treated in the resistive limit) is this time visibly more unstable than the case where the main mode is resistive and the sideband is ideal. It is important to point out that the four cases would most likely be more distinct if the stabilising effect associated with $\Delta\xi_\Gamma^{r(m)}$ had been self consistently included. The effects of compression on resistive infernal modes, and the coupling to sound waves, will be investigated in future work.

Finally it is pointed out that it is possible to calculate $m > 1$ cases, and make plots similar to those shown in Figs. 2 and 3. The curves would be shifted to the right and stretched, relative to the $m = 1$ cases shown here, as expected from the ideal cases shown in Fig. 1, but the qualitative features of such simulations would be similar to those seen in Figs. 2 and 3. Of course, such results are made possible by the novel analytic identification of ideal $m \geq 1$ resonant infernal modes.

VII. CONCLUSIONS

This paper attempts to treat the fundamentals of internal long wavelength pressure driven MHD instabilities in a resistive, axisymmetric, toroidal plasma equilibrium. We use the term ‘long wavelength’ to distinguish from ballooning instabilities, which are the only pressure driven modes not covered in this derivation (though infernal modes do have certain ballooning features, and indeed we treat interchange modes which are seen by some as a short wavelength, secular, limit of ballooning modes). From a general global set of equations we have been able to recover the known analytic descriptions of interchange modes, internal kink instabilities, non-resonant infernal modes, as well as identify new resonant infernal mode instabilities. Concerning these instabilities, resistive effects have been taken into account on the rational surfaces of the coupled harmonics. The derivation relies on an expansion which assumes the magnetic shear is quite small, and it relies on an inverse aspect ratio expansion of generalised perturbed displacement variables which are valid in a resistive plasma. A great deal of detail is given, which is justified as follows: the derivation of pressure driven plasma instabilities in a torus, either ideal or resistive, is very involved. The step by step guide in this paper fills, we think, a significant gap in papers and textbooks, for example we provide a *simple* complete derivation of pressure driven internal kink modes in a torus. But the higher-level set of equations are shown to describe also interchange instabilities and infernal modes. The latter are classified as non-resonant and resonant infernal modes, the properties of which are explored and exploited for a dedicated study into fully resistive infernal modes. Potential cascades of such modes are likely to be relevant for sawteeth, together with other non-linear processes [20]. This will likely be explored in the future, advanced further by inclusion of modes with faster radial oscillations.

While an objective of this paper has been to derive a broad set of known, and new, analytic results, the equations are presented in a general form for the future establishment of an efficient resistive eigenvalue solver. This will be required in order to investigate items that were out of scope in this paper, such as treating infernal modes with $q_s > 1$ or $q_s < 1$ where the drive or damping from interchange physics changes the nature of the eigenfunctions [15]. In particular it would enable investigation into the weakened stability properties of internal transport barriers by resistive effects, should fundamental rational surfaces exist at $q_s > 1$. Also to be explored in more detail in future work will be the effects of compression and the coupling of sound waves with resistive infernal modes. Collecting these improvements in the linear theory of resistive infernal modes will establish a platform for a non-linear treatment that will investigate the potential seeding [21, 22] of neoclassical tearing modes by resistive infernal modes. The robust theoretical platform presented here, with its consistent expansions in inverse aspect ratio and magnetic shear, should enable the inclusion of further physics attributes in future work, such as large equilibrium shear flows and coupling with zonal modes [13]. Inclusion of such advanced physics is likely intractable in previously presented coupled toroidal problems where the magnetic shear is taken to be large everywhere from the outset [19]. We therefore hope that the present contribution will be a useful detailed reference for long wavelength ideal and resistive instabilities in a torus, and a platform for deeper studies.

VIII. APPENDIX

Here we show that $\Delta b = 8\sigma[1 + O(\Delta q/q_s)]$ for arbitrary r_s and q-profile. This is a necessary result for reproducing in this paper the $m = 1$ internal kink problem of Ref. [7]. We allow the q profile to adopt the form $q(r) = q_s[1 - \Delta f(r)]$ where Δ is a constant such that $\Delta/q_s \ll 1$. $f(r)$ has the property $f(0) = 1$ and $f(r_s) = 0$ but is otherwise arbitrary, and it is seen that $\Delta q(r) = -q_s\Delta f(r)$. Hence the magnetic shear is $s = -\Delta r df/dr$ to leading order in Δ. Also, σ, defined by Eq. (104), is to leading order in Δ:

$$\sigma = 2\Delta \int_0^{r_s} \frac{dr}{r_s}\left(\frac{r}{r_s}\right)^3 f(r). \tag{123}$$

To obtain b we require the solution of

$$r^2\frac{d^2\xi^+}{dr^2} + 3r\frac{d\xi^+}{dr} - 3\xi^+ = -4r^2\frac{df}{dr}\Delta\frac{d\xi^+}{dr} \tag{124}$$

which is the upper sideband equation of (82) for $m = 1$ in the absence of inertia and the main harmonic (i.e. the homogeneous form). We write the solution $\xi^+ = \xi_0^+ + \Delta\,\xi_1^+$, where $\xi_1^+/\xi_0^1 \sim O(1)$. The boundary conditions are $\xi_0^+(0) = 0$ and $\xi_1^+(0) = 0$. Clearly $\xi_0^+(r) = \overline{\xi}_0^+ r/r_s$, with $\overline{\xi}_0^+$ a constant, since this satisfies Eq. (124) with the right hand side being zero at order Δ^0. We now consider Eq. (124) at the next order, multiplying by r^2 we may write it in the form,

$$\Delta\frac{d}{dr}\left[r^5\frac{d}{dr}\left(\frac{\xi_1^+}{r}\right)\right] = -4r^4\frac{df}{dr}\Delta\frac{d\xi_0^+}{dr}.$$

Integrating from 0 to r_s, substituting $\xi_0^+ = \overline{\xi}_0^+ r/r_s$ and Eq. (123), applying the boundary conditions, and integrating by parts gives:

$$\frac{r_s}{\overline{\xi}_0^+}\Delta\frac{d\xi_1^+}{dr}\bigg|_{r_s} - \frac{\Delta}{\overline{\xi}_0^+}\xi_1^+(r_s) = 8\sigma.$$

The left hand side of the last equation can be seen to be Δb, where $b = 1 + \Delta b$. This is shown via the definition of b:

$$\begin{aligned} b &= \frac{r}{\xi^+}\frac{d\xi^+}{dr}\bigg|_{r_s-\delta} \\ &= \frac{r}{\xi_0^+}\frac{d\xi_0^+}{dr}\bigg|_{r_s-\delta} + \frac{r}{\xi_0^+}\Delta\frac{d\xi_1^+}{dr}\bigg|_{r_s-\delta} - \frac{r\Delta\xi_1^+}{\left(\xi_0^+\right)^2}\frac{d\xi_0^+}{dr}\bigg|_{r_s-\delta} + O(\Delta^2) \\ &= 1 + \frac{r_s}{\overline{\xi}_0^+}\Delta\frac{d\xi_1}{dr}\bigg|_{r_s-\delta} - \frac{\Delta}{\overline{\xi}_0^+}\xi_1^+(r_s-\delta) + O(\Delta^2). \end{aligned}$$

Hence, it is seen that in the limit $\delta \to 0$,

$$\Delta b = 8\sigma$$

independent of r_s and the q-profile, providing that Δ is small, i.e. $\Delta q/q_s \sim \epsilon$. The fact that the result is independent of $q(r)$ and r_s is a reason that the infernal mode equations are obtained in such a convenient and general form.

Acknowledgements

We thank Daniele Brunetti for helpful discussion. This work has been carried out within the framework of the EUROfusion consortium and has received funding from the Euratom research and training programme 2014-2018 and 2019-2020 under grant agreement No 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

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