Improving Accuracy of Viscous Fluid Simulation Using Finite Particle Method

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***Abstract*—The present paper reports the development and application of Finite Particle Method for viscous fluid simulation. The main effort in present study is focused on increase of accuracy. To improve accuracy two approaches are studied. First of all, consistency of standard SPH restored using FPM. Secondly, automatic adaptivity and clustering of particles is remedied by applying shifting algorithm. Moreover new shifting appropriate for free surface flows introduced. Finally new algorithm validated and verified with 2D and 3D benchmarks.**

I. INTRODUTION

The Smoothed Particle Hydrodynamics (SPH) method is well suited for simulating complex fluid dynamics. Although, recent studies [l] [2] [3] [4] results in great improvement of viscous particle-based simulations, comparing to grid-based approaches, particle methods are still suffering from insufficient accuracy. Existing approaches mainly suggests two ways to increase the accuracy.

First of all, accuracy improvement is achieved by restoring consistency of standard SPH. To improve consistency of SPH substantial works has been done. For example symmetrisation formulations[12], corrective smoothed particle method (CSPM) [13], moving least square particle hydrodynamics (MLSPH) [14], the integration kernel correction [15], the reproducing kernel particle method (RKPM) [16] and finally finite particle method (FPM)[1] [17] [8].

Secondly, automatic adaptivity and clustering of particles is remedied by using re-meshing algorithms [5] or by shifting particles [4]. In re-meshing technique particles are reinitialised at regular intervals by interpolation onto a regular grid. For moving particle computations in this work we propose and demonstrate a formulation for multidimensional non-Lagrangian motion, henceforth referred to as particle motion correction

Shifting idea arose up from finite volume particle method (FVPM) where Schick [18] introduced non-Lagrangian particle motion to maintain adequate particle spacing for a 1-D problem. Furthermore Nestor et al. [3] and recently Xu et al. [4] used similar idea for more complicated problems.

In present study we try to get the benefit of both approaches simultaneously. To do this, FPM as a powerful model to restore consistency, combined with particle shifting approach.

In next section we briefly denote governing equation for fluid simulation. In Section III lack of consistency in particle approximation is addressed and FPM technique is explained by detail. Section IV mentioned a totally conservative SPH discretization adopted for FPM. We next speak about shifting technique in Section V and solution algorithm in section VI. In section VII some simulations are presented for verification and validation of proposed algorithm and .finally we have conclusion in last section.

II. GOVERNING EQUATION

To simulate viscous fluid behaviour, Navier-Stokes equations are often used as governing equations. Lagrangian description of these equations is:

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{C} \tag{1}$$

$$\rho\frac{D\vec{C}}{Dt} = \vec{\nabla}\cdot\overline{\overline{\tau}} + \vec{f} \tag{2}$$

Where (1) and (2) are mass and momentum equations respectively. Moreover, position of infinitesimal fluid elements is governed by:

$$\frac{D\vec{X}}{Dt} = \vec{C} \tag{3}$$

In (1), (2) and (3), D/Dt denotes substantial derivative; ρ is density; $\vec{C}$ is velocity vector; $\overline{\overline{\tau}}$ is stress tensor; $\vec{f}$ is external body force and $\vec{X}$ is position vector.

For weakly compressible Newtonian fluid, $\overline{\overline{\tau}}$ is defined as:

$$\bar{\bar{\tau}} = \left(-p - \frac{2}{3}\mu\vec{\nabla}\cdot\vec{C}\right)\bar{\bar{I}} + 2\mu\bar{\bar{D}} \quad (4)$$

Where p is static pressure; μ is dynamic viscosity; $\bar{\bar{I}}$ is unit tensor and $\bar{\bar{D}}$ is strain rate tensor which is defined as:

$$\bar{\bar{D}} = \frac{1}{2}\left(\nabla\vec{C} + \left(\nabla\vec{C}\right)^T\right) \quad (5)$$

To close system of equation one notes equation of state. Tait equation of state is usually used for water:

$$p = k_0\left(\left(\frac{\rho}{\rho_0}\right)^\gamma - 1\right) \quad (6)$$

Where $\gamma = 7$ and k_0 is a parameter which depends on sound speed. It is normally chosen to impose Mach number of flow below 0.1.

III. Restoring Consistency

A. SPH

In SPH function and the function derivatives usually approximated as:

$$f^{(i)} = \sum_{j=1}^{N}\frac{m^{(j)}}{\rho^{(j)}} f^{(j)} W^{(ij)} \quad (7)$$

$$f_\alpha^{(i)} = \sum_{j=1}^{N}\frac{m^{(j)}}{\rho^{(j)}}\left(f^{(j)} - f^{(i)}\right) W_\alpha^{(ij)} \quad (8)$$

Where $\rho^{(j)}, m^{(j)}$ are density and mass of particle j and N denotes to number of particles around particle i. Moreover, $f^{(i)} = f\left(\vec{X}^{(i)}\right)$, $f_\alpha^{(i)} = \frac{\partial f\left(\vec{X}^{(i)}\right)}{\partial X_\alpha}$, $W^{(ij)} = W\left(\vec{X}^{(i)} - \vec{X}^{(j)}, h\right)$, and $W_\alpha^{(ij)} = \frac{\partial W\left(\vec{X}^{(i)} - \vec{X}^{(j)}, h\right)}{\partial X_\alpha}$.

Function value and its derivative computed from (7) and (8), are called particle approximation. It is well-known that particle approximation is not first order consistent for arbitrary distribution of particles [2]. As mentioned before several approaches exist to improve particle approximation. In present study, Finite Particle Method is used as a base to restore consistency of particle approximation to first order.

B. Finite Particle Method

Chen and Beraun [6], Liu et al. [1], proposed a set of correction formulas for both the SPH kernel function and the derivatives of the SPH kernel function, which they called finite particle method.. In continue, a brief description of the finite particle method is coming.

Assuming function f is sufficiently smooth at point $\vec{X}^{(i)}$. Taylor series expansion to first order of derivatives for f around point $\vec{X}^{(i)}$ is:

$$f\left(\vec{X}\right) = f\left(\vec{X}^{(i)}\right) + \left(\vec{X} - \vec{X}^{(i)}\right)\cdot\vec{\nabla}f\left(\vec{X}^{(i)}\right) + O(\Delta X^2) \quad (9)$$

Multiplying both side of (9) with set of functions (i.e. SPH kernel and kernel derivatives) and integrating over volume yields:

$$\int_V f\left(\vec{X}\right)\Phi dV = f^{(i)}\int_V \Phi dV + f_\alpha^{(i)}\int_V\left(X_\alpha - X_\alpha^{(i)}\right)\Phi dV \quad (10)$$

$$\Phi = \begin{bmatrix} W \\ W_\beta \end{bmatrix} \quad (11)$$

Where, $W = W\left(\vec{X}^{(i)} - \vec{X}, h\right)$, $W_\beta = \frac{\partial W\left(\vec{X}^{(i)} - \vec{X}, h\right)}{\partial X_\beta}$.

Substituting integrals in (10) with SPH particle approximation result in:

$$\begin{aligned}\sum_{j=1}^{N} f^{(j)} W^{(ij)} V^{(j)} &= f^{(i)}\sum_{j=1}^{N} W^{(ij)} V^{(j)} + f_\alpha^{(i)}\sum_{j=1}^{N} X_\alpha^{(ij)} W^{(ij)} V^{(j)} \\ \sum_{j=1}^{N} f^{(j)} W_\beta^{(ij)} V^{(j)} &= f^{(i)}\sum_{j=1}^{N} W_\beta^{(ij)} V^{(j)} + f_\alpha^{(i)}\sum_{j=1}^{N} X_\alpha^{(ij)} W_\beta^{(ij)} V^{(j)}\end{aligned} \quad (12)$$

Where, $V^{(j)} = \frac{m^{(j)}}{\rho^{(j)}}$ and $X_\alpha^{(ij)} = X_\alpha^{(j)} - X_\alpha^{(i)}$.

As Fang et al. [2] did, we assume following quality to be able to extract corrected kernel and gradients.

$$\begin{aligned} f^{(i)} &= \sum_{j=1}^{N} f^{(j)}\tilde{W}^{(ij)} V^{(j)} \\ f_\alpha^{(i)} &= \sum_{j=1}^{N} f^{(j)}\tilde{W}_\beta^{(ij)} V^{(j)}\end{aligned} \quad (13)$$

Where $\tilde{W}^{(ij)}$ and $\tilde{W}_\beta^{(ij)}$ are considered as FPM kernel and kernel derivatives.

Substituting (13) to (12), FPM kernel and kernel derivatives are computed as:

$$\begin{bmatrix}\tilde{W}^{(ij)} \\ \tilde{W}_{\beta}^{(ij)}\end{bmatrix} = \begin{bmatrix}\sum_j W^{(ij)}V^{(j)} & \sum_j X_{\alpha}^{(ij)}W^{(ij)}V^{(j)} \\ \sum_j W_{\beta}^{(ij)}V^{(j)} & \sum_j X_{\alpha}^{(ij)}W_{\beta}^{(ij)}V^{(j)}\end{bmatrix}^{-1}\begin{bmatrix}W^{(ij)} \\ W_{\beta}^{(ij)}\end{bmatrix} \quad (14)$$

Now FPM kernel computed from (14) is first order consistent [1].

IV. Discretization

A. Governing Equations

Discretization of governing equations has an important role in stability and convergence of solution algorithm.

Fang et al. [2] have applied an energy-based framework to derive a general set of discrete hydrodynamics equations which conserves the total linear momentum for any particle approximations and the total angular momentum for particle approximations of first-order consistency (e.g. FPM).

In present study we apply this method. Final discretized equations for mass and momentum are as below:

$$\frac{D\rho^{(i)}}{Dt} = \rho^{(i)}\sum_{j=1}^{N}\frac{m^{(j)}}{\rho^{(j)}}\left(C_{\beta}^{(i)} - C_{\beta}^{(j)}\right)\tilde{W}_{\beta}^{(ij)} \quad (15)$$

$$\frac{DC_{\alpha}^{(i)}}{Dt} = \sum_{j=1}^{N} m^{(j)}\left(\frac{\tau_{\alpha\beta}^{(i)}}{\rho^{(i)}\rho^{(j)}}\tilde{W}_{\beta}^{(ij)} - \frac{\tau_{\alpha\beta}^{(j)}}{\rho^{(i)}\rho^{(j)}}\tilde{W}_{\beta}^{(ji)}\right) \quad (16)$$

$$\frac{D\vec{X}^{(i)}}{Dt} = \vec{C}^{(i)} \quad (17)$$

To model viscous effects, we follow Moris et al [21].

$$\vec{F}^{(ij)} = \frac{\left(\mu^{(i)} + \mu^{(j)}\right)}{\rho^{(i)}\rho^{(j)}}\left(\frac{\frac{1}{2}\left(\vec{\nabla}\tilde{W}^{(ij)} + \vec{\nabla}\tilde{W}^{(ij)}\right)\cdot\vec{X}^{(ij)}}{\left|\vec{X}^{(ij)}\right|^2 + 0.01h^2}\right)\vec{C}^{(ij)} \quad (18)$$

Where, $\vec{F}^{(ij)}$ denotes viscous force acting for pair of particles. Finally (16) is rewritten as:

$$\frac{DC_{\alpha}^{(i)}}{Dt} = \sum_{j=1}^{N} m^{(j)}\left(\frac{P^{(i)}}{\rho^{(i)}\rho^{(j)}}\tilde{W}_{\alpha}^{(ij)} - \frac{P^{(j)}}{\rho^{(i)}\rho^{(j)}}\tilde{W}_{\alpha}^{(ji)} + F_{\alpha}^{(ij)}\right) \quad (19)$$

B. Boundary Condition

Utilizing FPM to discretize continuity equation, result in effective treatment of particle deficiency problem near free-surface boundaries.

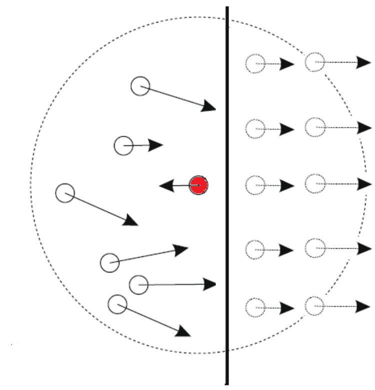


Figure 1. Velocity extrapolation for dummy particles

In present study, no-slip wall boundary condition enforced by dummy particles. In this way, three layers of stationary particles called dummy particle, are generated in opposite side of wall and their spacing is same as initial fluid particle spacing.

Values of variables for dummy particles are changing respect to fluid particle properties. One simple way is that all dummy particles contributing for a fluid particle have same density and pressure and equal to fluid particle one.

To enforce no-slip condition, velocity of dummy particle, would be extrapolated from fluid particle and wall position (Fig. 1).

V. Particle Shift

In SPH methods because of Lagrangian formulation of equations, computational elements are adopting respect to map dictated by flow. In high distortion flow if the streamlines direct particles toward each others, particles clusters locally. Moreover tensile instability [19] may occurs and infect particle distribution more. Increasing irregularity in particle distribution leads to increase of spatial discretization error. In restoring consistency methods like FPM this may leads to singularity of correction matrix.

To improve results against particle clustering, Monaghan [19] introduces artificial repulsion stresses in pressure term. The effective alternative of re-meshing on a uniform grid was first introduced by Chaniotis et al.[5]. Recently Xu et al. [4] improving particle distribution by shifting particles based on correction velocity introduced by Nestor et al. [3]. In present study we adopt shifting method [4] for free-surface problems.

Reference [4] introduces shifting term as:

$$\delta\vec{X}^{(i)} = K\alpha\vec{R}^{(i)} \quad (20)$$

Where K is a constant, set as 0.01-0.1, α is shifting magnitude which is equal to maximum particle convection distance $C_{max}\,dt$ with C_{max} as maximum particle velocity and dt as time step. $\vec{R}^{(i)}$ is the shifting vector, and reads:

$$\vec{R}^{(i)} = \sum_{j=1}^{N} \left(\frac{\overline{r}^{(i)}}{r^{(ij)}} \right)^2 \vec{n}^{(ij)} \quad (21)$$

Where N is number of neighbour particles around particle i; $r^{(ij)}$ is the distance between particle i and j; $\overline{r}^{(i)}$ is particle average spacing $\overline{r}^{(i)} = \frac{1}{N}\sum_{j=1}^{N} r^{(ij)}$ and $\vec{n}^{(ij)}$ is the unit distance vector between particle i and particle j.

Use of (21) for set of particles exposed to free-surface result in artificial spreading of particles through the void domain. To avoid this behaviour we suggest shifting vector defined as:

$$\vec{R}^{(i)} = \sum_{j=1}^{N} \left(\frac{\Delta}{r^{(ij)}} \right)^2 H\left(\Delta - r^{(ij)}\right) \vec{n}^{(ij)} \quad (22)$$

Where, Δ is initial spacing and H is defined as:

$$H(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases} \quad (23)$$

VI. SOLOUTION ALGORITHM

In present study, first order Euler scheme is used for temporal integration.

For summarization dicretized equation are rewritten in form of $\frac{D\vec{C}^{(i)}}{Dt} = \vec{F}^{(i)}$, $\frac{D\rho^{(i)}}{Dt} = E^{(i)}$ and $\frac{D\vec{X}^{(i)}}{Dt} = \vec{C}^{(i)}$.

Solution algorithm is as below:

1. $\left(\rho^{(i)}\right)^* = \left(\rho^{(i)}\right)^n + \delta t\left(E^{(i)}\right)^n$
2. $\left(\vec{C}^{(i)}\right)^* = \left(\vec{C}^{(i)}\right)^n + \delta t\left(\vec{F}^{(i)}\right)^n$
3. $\left(\vec{X}^{(i)}\right)^* = \left(\vec{X}^{(i)}\right)^n + \delta t\left(\vec{C}^{(i)}\right)^n$
4. $\left(\vec{X}^{(i)}\right)^{n+1} = \left(\vec{X}^{(i)}\right)^* + \left(\delta\vec{X}^{(i)}\right)^*$
5. $\left(\rho^{(i)}\right)^{n+1} = \left(\rho^{(i)}\right)^* + \left(\delta\vec{X}^{(i)}\right)^* \cdot \left(\vec{\nabla}\rho^{(i)}\right)^*$
6. $\left(\vec{C}^{(i)}\right)^{n+1} = \left(\vec{C}^{(i)}\right)^* + \left(\delta\vec{X}^{(i)}\right)^* \cdot \left(\vec{\nabla}\vec{C}^{(i)}\right)^*$

Where superscript n, (*) and (n+1) indicate respectively the actual time instant, the predicted one and corrected new time instants.

VII. RESULTS

For all simulation presented in this section, cubic spline kernel is used and smoothing length is constant and indicated regarding to *h/d=1.3*. Moreover for all simulation CFL is set to 0.2.

A. *Rotating Square Patch of Fluid*

This test case was first proposed and solved by Colagrossi [9]. Due to centrifugal forces, the corners of this initial square are stretched and finally change into four fluid arms, while the square size decreases. This case results in large free boundary deformations, which are responsible for the occurrence of strong instabilities.

Reference [10] mentioned that the Lagrangian particle movements and the use of a Cartesian grid as an initial particle setting result in the occurrence of line structures or particle clumping. These progressively degrade the interpolation procedure, contaminating the entire simulation. Thus a highly accurate and stable scheme is required.

Initial condition for square patch problem is depicted in Figure 2. Initial condition for pressure is computed using series which details are in [9] . Furthermore in present simulation, artificial viscosity terms are applied as presented in [11].

In this part three algorithms are tested as below:

- Algorithm without shifting procedure (NOSHIFT)
- Algorithm with shifting procedure (19) (SHIFT-1)
- Algorithm with shifting procedure (20) (SHIFT-2)

Six snapshots of rotating square patch for different algorithm and different time instants are shown in Figures 3 to 5. Figure 3, belongs to NOSHIFT, Figure 4, belongs to SHIFT-1 and Figure 5, belongs belong to SHIFT-2.

It is obvious that shifting procedure improve results by prevention of particle clamping. Moreover, comparing Figure 4 and Figure 5, indicates that SHIFT-2 algorithm was successful to preserve free-surface in contrast with SHIFT-1.

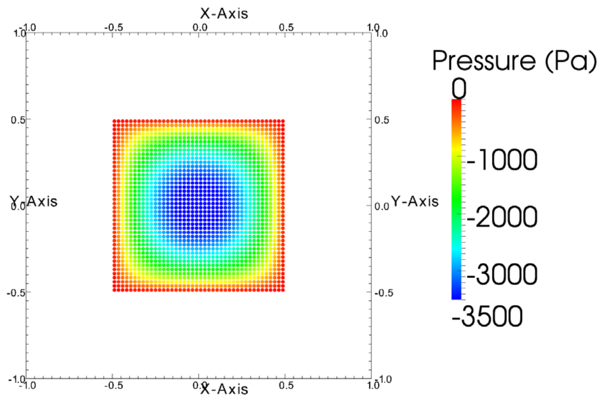
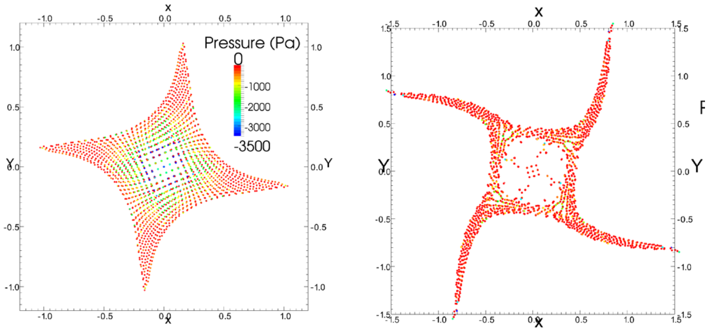
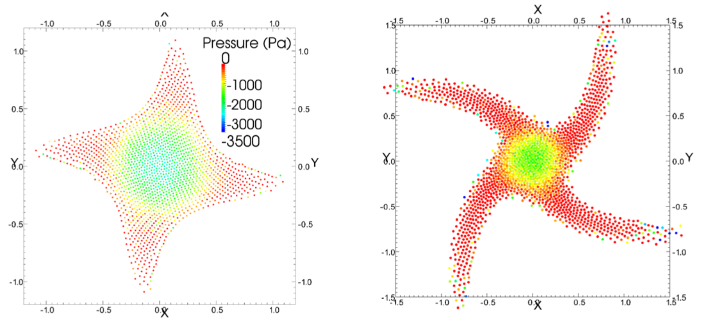


Figure 2. Initial rotating velocity $\omega = 5$ rad s^{-1}, $\rho = 1000$ kg m^{-3}



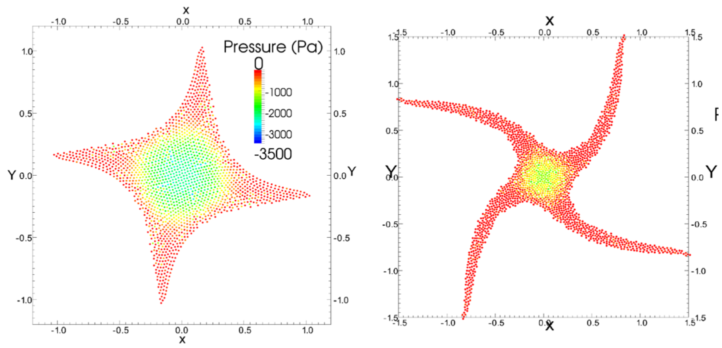
(a) (b)

Figure 3. Free surface and pressure NOSHIFT a) t=0.245 s, b) t=0.49



(a) (b)

Figure 4. Free surface and pressure SHIFT-1 a) t=0.245 s, b) t=0.49



(a) (b)

Figure 5. Free surface and pressure SHIFT-2 a) t=0.245 s, b) t=0.49

B. *Cavity Flow*

Cavity flow as a convenient test case is studied here for 2D and 3D cases. In this problem flow in square or cubic cavity would be formed because of moving top wall along horizontal axis. Figure 6 shows geometry and parameters used in this study. According to Figure 6, Reynolds number is defined as

$$\mathrm{Re} = \frac{\rho C_{ref} L}{\mu} \tag{24}$$

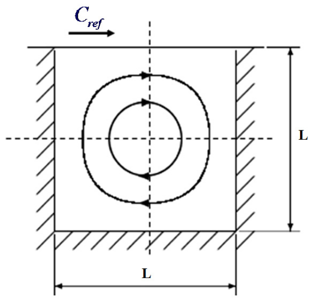


Figure 6. Geometry and parameters for 2D cavity flow

Figure 7 and Figure 8 shows horizontal velocity along vertical midline for 2D and 3D cavity at Re = 1000.

For 2D case, comparing to similar studies, significant improvement in results even with lower number of particles is considerable.

For 3D case (Figure 8), increasing number of particles resulted in convergent behaviour. Although computed velocity are not in good concordance with benchmark, it should be noted that benchmark data were obtained using adapted grids near boundaries which is not easily achievable in particle methods.

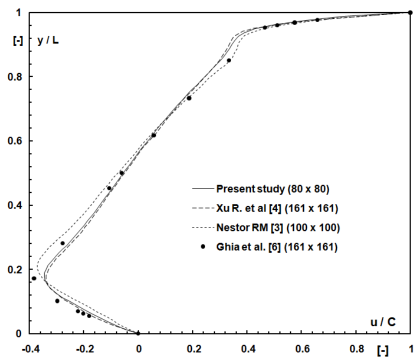


Figure 7. Comparison of horizontal velocity at mid-line of 2D cavity (Re=1000)

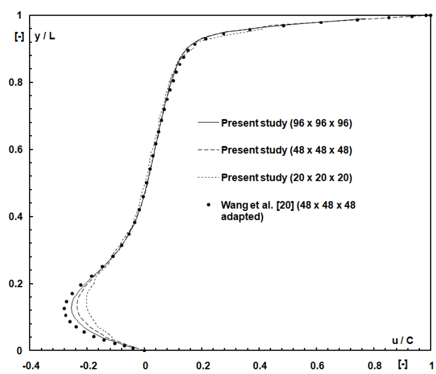


Figure 8. Comparison of horizontal velocity at mid-line of 3D cavity (Re=1000)

VIII. CONCLUSION

We proposed a method consist of 1st order consistent FPM and particle shifting procedure. Results proof that significant improvement were obtained. Moreover it showed that particle shifting in free-surface flow required special treatment to avoid spread of particle. We successfully test a new shift term which did not affect free-surface.

REFERENCES

[1] MB Liu, WP Xie, and Liu GR, "Modeling incompressible flows using a finite particle method," Appl. Math. Model, vol. 29, pp. 1252-1270, 2005.

[2] JN Fang, A Parriaux, M Rentschler, and C Ancey, "Improved SPH methods for simulating free surface flows of viscous fluids," Appl. Numer. Math., vol. 59, pp. 251–271, 2009.

[3] RM Nestor, M Basa, M Lastiwka, and NJ Quinlan, "Extension of the finite volume particle method to viscous flow," J. Comput. Phys., vol. 228, pp. 1733–1749, 2009.

[4] R Xu, P Stansby, and L Laurence, "Accuracy and stability in incompressible SPH (ISPH) based on the projection method and a new approach," J. Comput. Phys., vol. 228, pp. 6703–6725, 2009.

[5] AK Chaniotis, D Poulikakos, and P Koumoutsakos, "Remeshed smoothed particle hydrodynamics for the simulation of viscous and heat conducting flows," J. Comput. Phys., vol. 182, pp. 67–90, 2002.

[6] U Ghia, K Ghia, and C Shin, "High-Re solutions for incompressible flow using the Navier–Stokes equations and a multigrid method," J. Comput. Phys., vol. 48, pp. 387–411, 1982.

[7] JK Chen and JE Beraun, "A generalized smoothed particle hydrodynamics method for nonlinear dynamic problems," Comput. Methods Appl. Mech. Engrg.,vol. 190, pp, 225–239, 2000

[8] GM Zhang, RC Batra, "Modified smoothed particle hydrodynamics method and its applications to transient problems," Comput. Mech, vol. 34, pp. 137–146, 2004

[9] A Colagrossi, "A meshless Lagrangian method for free-surface and interface flows with fragmentation," Ph.D. thesis, Università di Roma La Sapienza, 2005.

[10] G. Oger, M. Doring, B. Alessandrini and P. Ferrant, "An Improved SPH Method : Towards Full Second Order Convergence," J. Comput. Phys., vol. 225, pp 1472-1492, 2007

[11] D Molteni, A. Colagrossi, "A simple procedure to improve the pressure evaluation in hydrodynamic context using the SPH," Computer Physics Communications, vol, 180, pp. 861–872, 2009

[12] JJ Monaghan "Smooth particle hydrodynamics," Annu Rev Astron Astrophys, vol 30, pp, 543–574, 1992

[13] JK Chen, JE Beraun and TC Carney, "A corrective smoothed particle method for boundary value problems in heat conduction," Int. J. Numer. Methods Eng., vol. 46, pp. 231–252, 1999

[14] GA Dilts, "Moving-Least-Squares-particle hydrodynamics. I: Consistency and stability,". Int. J. Numer. Methods Eng., vol. 44, pp.:1115– 1155, 1999

[15] J Bonet and S Kulasegaram "Correction and stabilization of smooth particle hydrodynamics methods with applications in metal forming simulations, " Int, J, Numer. Methods Eng., vol. 47, pp. 1189–1214, 2000

[16] WK Liu,Y Chen, S Jun, JS Chen, T Belytschko, C Pan, RA Uras and CT Chang, "Overview and applications of the reproducing kernel particle methods," Arch. Comput. Methods. Eng., vol 3, pp. 3–80, 1996

[17] MB Liu and GR Liu, "Restoring particle consistency in smoothed particle hydrodynamics," Appl. Numer. Math., vol 56, pp.19–36, 2006

[18] C. Schick, "Anisotropic smoothing kernels for particle methods in fluid flow," Master's thesis, University of Kaiserslautern, 2000.

[19] JJ Monaghan, "SPH without a tensile instability," J. Comput. Phys., vol. 159, pp. 290–311, 2000

[20] KL Wong and AJ Baker, "A 3D incompressible Navier–Stokes velocity–vorticity weak form finite element algorithm," International Journal for Numerical Methods in Fluids, vol. 38, pp. 99 –123, 2002

[21] JP Morris , PJ Fox and YJ Zhu, "Modelling low Reynolds number incompressible flows using SPH," J. Comput. Phys., vol 136, pp. 214–26, 1997