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It is still an unending source of surprise
for me to see how a few scribbles on a sheet
of paper could change the course of human affairs.

— Stanislaw Ulam

Abstract

In part I, we address the issue of existence of solutions for Cauchy problems involving nonlinear hyperbolic equations for initial data in Sobolev spaces with scaling subcritical regularity. In particular, we analyse nonlinear estimates for null-forms in the context of wave Sobolev spaces $H^{s,b}$, first in a flat background, then we generalize to more general curved backgrounds. We provide the foundations to show that the Yang-Mills equation in $\mathbb{R}^{1+3}$ are globally well-posedness for small weighted $H^{3/4+} \times H^{-1/4+}$ initial data, matching the minimal regularity obtained by Tao [106]. Our method, inspired from [14], combines the classical Penrose compactification of Minkowski space-time with a null-form estimates for second order hyperbolic operators with variable coefficients. The proof of the null-form appearing in the Yang-Mills equation will be provided in a subsequent work. As a consequence of our argument, we shall obtain sharp pointwise decay bounds.

In part II, we show that the finite time type II blow-up solutions for the energy critical nonlinear wave equation

$$\Box u = -u^5$$

on $\mathbb{R}^{3+1}$ constructed in [62], [61] are stable along a co-dimension one Lipschitz manifold of data perturbations in a suitable topology, provided the scaling parameter $\lambda(t) = t^{-1-\nu}$ is sufficiently close to the self-similar rate, i. e. $\nu > 0$ is sufficiently small. This result is qualitatively optimal in light of the result of [56], it builds on the analysis of [49] and it is joint work with my thesis advisor Prof. J. Krieger.

Key words: critical wave equation ; blowup ; Yang-Mills equation ; nonlinear waves ; null structures ; space-time compactification ; Penrose transform.

Contents

Introduction

As already mentioned in the abstract, the objective of the current work is twofold. Part I concerns the existence of solutions for a wide class of nonlinearity whereas in part II we analyse the obstructions to the existence of solutions, i.e. blow-up solutions.

Part I: Low regularity theory for scaling subcritical equations

In the first chapter we investigate the existence of local-in-time solutions to Cauchy problems associated to a class nonlinear wave equations in flat space-time. We address the issue of finding the minimal assumptions on the regularity of initial data such that a unique local solution exists. Finding such low regularity thresholds is important for several reasons: the obvious one is to obtain a solution even for very rough initial data. Secondly, for some equations, conservation laws of L^2 and H^1 can be easily obtained, therefore extending a local existence result to a global existence one can be easier if we are working at low regularities than high regularities. All the results presented below are perturbative in the sense that they are obtained via a contraction argument in a suitable Banach space.

We consider a class of semilinear wave equations with quadratic nonlinearities called geometric wave equations, such a class includes Wave maps, Maxwell-Klein-Gordon, and Yang-Mills equation:

$$\Box u = \Gamma(u) N_0(\partial u, \partial u) \tag{N_0}$$

$$\Box u = \Gamma(u) N_{\alpha\beta}(\partial v, \partial v) \tag{$N_{\alpha\beta}$}$$

$$\begin{cases} \Box u = D^{-1} N(\partial v, \partial v) \\ \Box v = N(D^{-1}\partial u, \partial v) \end{cases} \tag{MKG type}$$

$$\Box u = D^{-1} N(\partial u, \partial u) + N(D^{-1}\partial u, \partial u) \tag{YM type}$$

$$\Box u = B(\partial u, \partial u) \tag{GQ}$$

where $N_0(\partial u, \partial u) = \partial^\alpha u \partial_\alpha u$ and $N_{\alpha\beta}(\partial u, \partial u) = \partial_\alpha u \partial_\beta u - \partial_\beta u \partial_\alpha u$ are called *null-structures*, and N is a linear combination of N_{ij} null-structure. Moreover $B(\partial u, \partial u) = b^{\alpha\beta}(u)\partial_\alpha u \partial_\beta u$, and (GQ) stands for a general quadratic nonlinearity. For a complete introduction, the reader should refer to [91], [45] [118], [29] and [13].

Let us couple one of the previously mentioned equations with initial data of s-regularity given on a time-slice $t = 0$:

$$(u, u_t)|_{t=0} = (u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$$

We aim to determine the optimal exponent s such that the Cauchy problem is locally well-posed. The first breakthrough in this direction was achieved by Klainerman and Machedon, in [38] they show local well-posedness in $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ for the N_0 and the $N_{\alpha\beta}$ equations. Thus they were able to gain an extra $1/2$ in regularity compared with the energy method. However, the scaling critical exponent is $s_c = 3/2$ in dimension $n = 3$, therefore there is another $1/2$ room of regularity to explore. The previous result was improved by the same authors in [35] and [36] to $n \geq 3$ and $s > n/2$, reaching the critical scaling exponent. More precisely, in [35] and [36] only the case $n = 3$ is considered. However the argument presented there extends to $n \geq 3$ without major differences. In [35], Klainerman and Machedon consider equations involving only the N_0 null-forms and in the subsequent paper [36] they were able to extend the result for $N_{\alpha\beta}$ null-forms. Subsequently, Klainerman and Selberg [44] extended the local well-posedness theory for N_0 null-forms in the harder case $n = 2$ and $s > 1$ completing the subcritical theory for wave maps equations. The $n = 1$ case was analyzed by Keel and Tao in [34]. The corresponding result for the $N_{\alpha\beta}$ null-form in dimension $n = 2$ holds only for $s > 1 + 1/4$, hence there is a $1/4$ gap between N_0 and $N_{\alpha\beta}$ null-forms in dimension $n = 2$. However for $n \geq 3$ the local well-posedness results obtained are the same.

The next part of the theory led to the study of Maxwell-Klein-Gordon (MKG), and Yang-Mills (YM) equations. In fact, in [39] Klainerman and Machedon proved that MKG type equation when $n = 3$ is locally well-posed in the non-optimal range $s \geq 1$. The previous result was extended to $n \geq 4$ spatial dimensions in [42] giving local well-posedness for the optimal $s > n/2 - 1$. Turning back to the case of $n = 3$ we have to mention that the lower bound on the exponent to assure local well-posedness was improved by Cuccagna [12] to $s > 3/4$, which also prove that for the MKG type problem this is the optimal exponent. However Machedon and Sterbenz [70] proved that the *full* MKG equation is locally well-posed for $s > 1/2$ and $n = 3$, reaching the optimal result. For the YM type equation the situation is similar: in [40] Klainerman and Machedon studied the $n = 3$ case proving local well-posedness for $s \geq 1$, their result was improved by Tao [106] to $s > 3/4$. The optimal well-posedness result up to $s > 1/2$ for the *full* YM equation is still open. On the other hand, for $n \geq 4$ spatial dimensions Klainerman and Tataru [46] proved the optimal local well-posedness result for $s > n/2 - 1$.

In this chapter we take a pedagogical approach. First, we start to show how Sobolev embedding leads to a series of existence results for very regular initial data. The argument applied here is often referred in the literature as the energy method. Next, we describe how Strichartz estimates and hyperbolic Sobolev spaces lead to better lower bounds on the regularity of the initial data for a class of problems involving nonlinearities which are multilinear forms of the space-time gradient of the unknown. A detailed analysis is carried out for the extensively studied null-forms nonlinearities.

Scaling critical problems

While in this first part of the thesis we consider only scaling subcritical problems we shall briefly mention here few results concerning scaling critical equations. Here one tries to prove global existence for data having small critical $s = s_c$ homogeneous Sobolev norm. In the literature, this is referred to *global regularity for small data*. In the pivotal work [112], Tataru showed that the wave maps problem admits a global solution if the initial data have small $\dot{B}^{n/2}_{2,1} \times \dot{B}^{n/2-1}_{2,1}$ norms in $n = 2, 3$ dimensions. This result extended the previous work [114] which treated dimensions $n \geq 4$. Furthermore, replace Besov norm with the Critical Sobolev norm $\dot{H}^{n/2}$ was a non-trivial issue. Tao was able to improve Tataru result and to show the existence of global solutions of the wave maps equation with sphere target if the initial data have small critical $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$ norms. Tao first work [103] settled the high dimensional $n \geq 5$ case, and in a subsequent paper [104] extend the result to low dimensions $n \geq 2$. Subsequently Krieger enlarge Tao result to include maps with target the hyperbolic plane, global existence with small $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$ data was proved in $n = 3$ dimensions [47], and in $n = 2$ dimensions [48]. The general case of *any* target manifold was solved subsequently by Tataru in [116]. Moreover the work by Shatah and Struwe [93] provide an alternative proof, based on the Hodge system, of global regularity for small critical Sobolev data in high dimensions $n \geq 4$ and for general manifolds (parallelizzable and bounded curvature). See also the related work by Nahmod, Stefanov, and Uhlenbeck, [76] and Klainerman and Rodnianski [43] for similar results in high-dimensional setting.

For the Maxwell-Klein-Gordon type equation, global regularity for small $\dot{H}^{n/2-1}(\mathbb{R}^n) \times \dot{H}^{n/2-2}(\mathbb{R}^n)$ data and high dimensions, $n \geq 6$, was proven by Tao and Rodnianski [90]. This result was improved to include dimensions up to $n \geq 4$ by Krieger, Sterbenz, and Tataru [64]. The corresponding results for the Yang-Mills type equation are due to Krieger and Sterbenz [63] in the high dimensional setting $n \geq 6$ and Krieger and Tatatu [65] in the optimal $n \geq 4$ case. For the Yang-Mills type equation global regularity for small critical Besov *radial* data was solved by Sterbenz [98] for $n \geq 4$ and by Stefanov [96] in $n = 5$ dimensions. The global regularity problem for small critical Sobolev spaces for both Maxwell-Klein-Gordon and Yang-Mills type equations in dimension $n = 3$ is still open.

For general quadratic nonlinearities, i.e. without relying on any null-structure, global regularity for small $\dot{B}^{n/2}_{2,1} \times \dot{B}^{n/2-1}_{2,1}$ and high dimensions $n \geq 6$ was show by Sterbenz in [97]. An open problem is to prove global regularity for general quadratic nonlinearities with data having small $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$ norms in high dimensions. Hence to extend Sterbenz result to critical Soblev spaces in the same spirit that Tao, Krieger and Tataru extended to critical Soblev spaces the global regularity result for wave maps obtained earlier by Tataru for critical Besov spaces.

There are few works that threat scaling critical problems with large data. First of all, large data results are only proved in energy critical problems, that is when the energy controls the critical Sobolev norm. Hence for wave maps we consider the domain to be the Minkowski space $\mathbb{R}^{1+2}$, for Maxwell-Klein-Gordon and Yang-Mills we consider $n = 4$ spatial dimensions. Loosely speaking *global regularity for large data* means that the Cauchy problem is global well-posed for any initial data in $H^{s_c}(\mathbb{R}^{n_c}) \times H^{s_c-1}(\mathbb{R}^{n_c})$ whose energy is less than the minimal energy required to have a stationary solution.

For the wave maps equation, we can isolate three major contributions. The impressive series of work, [107][108][109][110][111] Tao solved the global regularity for large data problems for wave maps with H^m target. The second major contribution is the book by Krieger and Schalg [58] where wave maps with H^2 target are considered. Global existence is proved using a concentration compactness argument. The third major contribution comes from the works of Sterbenz and Tataru [100] [99] and it primary addresses the case when the target N is a compact manifold.

The global regularity for large data problem is solved for both Maxwell-Klein-Gordon and Yang-Mills equation. For the Maxwell-Klein-Gordon equation, two similar results are proved by Oh and Tataru [82][78][77] and by Krieger and Luhrmann [51]. The first extends the previous work of Tataru for the wave maps equation and the second is a continuation of the Krieger and Schlag's book, it does apply the concentration compactness argument to the Maxwell-Klein-Gordon system. Concentration compactness is not yet applied to the Yang-Mills equation, however, Oh and Tataru, in a series of works [79][83][81][80], where able to push their techniques further and show global regularity for large data for the Yang-Mills equation.

Finally, for general quadratic nonlinearities the global regularity for large data problem seems to be open, here, nonetheless one should first understand what the corresponding energy critical exponent is.

Yang-Mills equation in $\mathbb{R}^{1+3}$ Minkowski space-time

In the second chapter of part I, we analyse in more details the Yang-Mills equation. Let G be a Lie group and $(\mathfrak{g}, [\cdot])$ its associated Lie algebra. The unknown of the Yang-Mills equation is $A = A_\alpha dx^\alpha$: a connection 1-form on the Minkowski space-time $\mathbb{R}^{1+3}$ with value in $\mathfrak{g}$. Let

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$

be the correspondent curvature 2-form. The Yang-Mills equation $D_\alpha F^{\alpha\beta} = 0$ are obtained as the Euler-Lagrange equations of the Yang-Mills Lagrangian

$$\mathscr{L}(A) = -\frac{1}{4}\int_{\mathbb{R}^{1+3}} \langle F^{\alpha\beta}, F_{\alpha\beta}\rangle \, dt dx.$$

Here $D_\alpha = \partial_\alpha + [A_\alpha, \cdot]$ is the covariant derivative. To obtain a more familiar formulation from a PDE perspective one can expand the Yang-Mills equation in term of the connection 1-form. Set the initial data $A_0 \in H^s(\mathbb{R}^3)$ and $A_1 \in H^{s-1}(\mathbb{R}^3)$ on the time slice $t = 0$, and consider the following initial value problem for the Yang-Mills equation:

$$\begin{cases} \Box A_\beta - \partial_\beta \partial^\alpha A_\alpha = -2[A_\alpha, \partial^\alpha A_\beta] + [A_\beta, \partial^\alpha A_\alpha] + [A_\alpha, \partial_\beta A^\alpha] + [A^\alpha, [A_\beta, A_\alpha]] \\ A(0,\cdot) = A_0 \\ \partial_0 A(0,\cdot) = A_1 \end{cases} \tag{1}$$

We are interested in proving global well-posedness for the Cauchy problem (1) with small H^s data. By this we mean that for any given initial data $(A_0, A_1) \in H^s \times H^{s-1}$, there exists some $\varepsilon > 0$ such that if $\|A_0\|_{H^s} + \|A_1\|_{H^{s-1}} < \varepsilon$ then a unique global solution of (1) which lies in $C^0(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$ exists. The primary aims of our research is to prove a new global well-posedness result on the Minkowski space-time $\mathbb{R}^{1+3}$ for small weighted $H^{3/4+} \times H^{-1/4+}$ data. This result will match the minimal regularity assumption available for the local theory [106].

Our technique requires the use of the *Penrose compactification* of Minkowski space-time, which allows us to transfer the Cauchy problem on the flat Minkowski space-time (1) into a Cauchy problem on a pre-compact manifold with curved metric (M, g). Since the Penrose map is a confomorphism, the Yang-Mills equation on (M, g) take the form

$$D_\alpha F^{\alpha\beta} = \nabla_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0 \tag{2}$$

where A_α and $F^{\alpha\beta}$ are respectively the components of a connection 1-form and of the curvature 2-form. That is $A_\alpha, F^{\alpha\beta} : M \to \mathfrak{g}$ are defined on the pre-compact manifold with curved metric (M, g) with value in $\mathfrak{g}$. When we expand equation (2) in term of the connection we obtain the following PDE:

$$\nabla_\alpha \nabla^\alpha A^\beta - \nabla_\alpha \nabla^\beta A^\alpha + 2[A_\alpha, \nabla^\alpha A^\beta] - [A_\alpha, \nabla^\beta A^\alpha] + [\nabla_\alpha A^\alpha, A^\beta] + [A_\alpha, [A^\alpha, A^\beta]] = 0 \tag{3}$$

which is exactly the first equation in (1) where derivatives have been replaced by covariant derivatives.

We can reduce the equation (3) further by *fixing the gauge*. Following the idea of Tao [106] for flat space-time, we choose to work under the temporal gauge to obtain a semilinear wave equation on curved background with a well-behaved nonlinearity, called Q_{ij} *null structure*[1]. Let us divide the connection A in its temporal and spatial components $A_\alpha = (A_0, A)$ where $A = (A_1, A_2, A_3)$, and let us fix the connection to lie in the temporal gauge, thus $A_0 = 0$. We decompose further the spatial component A of the connection in its divergence-free part A^{df} and curl-free part A^{cf}, then the Yang-Mills equation (3) simplifies to

$$\begin{cases} \triangle(\partial_0 A^{cf}) = \nabla(A\cdot\partial_0 A + \mathfrak{E}_0(A) + \mathfrak{E}_0(\partial A) + \mathscr{E}_0(A,A)) \\ \Box_g A^{df} = |\nabla|^{-1} Q(A^{df}, A^{df}) + Q(|\nabla|^{-1}A^{df}, A^{df}) + \mathfrak{M}(A^{df}, \partial A^{cf}) + \mathfrak{M}(A^{cf}, \partial A^{df}) \\ \qquad + \mathfrak{M}(A^{cf}, \partial A^{cf}) - \mathfrak{M}(A,A,A) - \mathfrak{E}(A) - \mathfrak{E}(\partial A) - \mathfrak{E}(A,A) \end{cases}$$

where $\mathfrak{E}_0$, $\mathfrak{M}$, and $\mathfrak{E}$, are nonlinear functions and Q is a linear combination with constant coefficients of Q_{ij}-null-forms. Hence, if we ignore the well-behaved elliptic equation for the curl-free part A^{cf} and the high-order nonlinearities in the hyperbolic equation for the divergence-free part A^{df}, the resulting *model equation* for the Yang-Mills system is

$$\Box_g A = |\nabla|^{-1} Q(A,A) + Q(|\nabla|^{-1}A, A).$$

The aimed result will be reached via a fixed point argument in the $X^{s,\theta}$ spaces introduced by Geba-Tataru [30]. This is the key step of the proof since the extension of such hyperbolic Sobolev spaces, used extensively in the 90s by Klainerman and Machedon [40] for flat metrics, to the curved setting, developed by Geba-Tataru [30] and Geba [28], does not include Q_{ij} nonlinearities. Thus we prove the following novel bound for the Q_{ij} null structure in the context of a curved background metric: let $n = 3$, $3/4 < \theta < 1$, and $s - 1 > \theta$, then

$$\|Q_{ij}(u,v)\|_{X^{s-1,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}} \tag{4}$$

The Q_{ij} estimate above shall generalize to include the corresponding estimates for the Yang-Mills null-forms: let $n = 3$, $3/4 < \theta < 1$, and $s > \theta$, then

$$\||\nabla|^{-1} Q(u,v)\|_{X^{s-1,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}}, \tag{5}$$

$$\|Q(|\nabla|^{-1}u,v)\|_{X^{s-1,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}}. \tag{6}$$

The proof of estimates (5) and (6) will be deferred to a subsequent work. Here we shall prove only (4). Finally, translating this result back to the Minkowski space $\mathbb{R}^{1+3}$ will lead to small data global well-posedness in a weighed Sobolev space.

The potential of $X^{s,\theta}$ spaces for non-flat metrics is difficult to overestimate. Looking back to the development in the past 30 years of the low-regularity theory for geometric hyperbolic equations, such as Wave maps, Maxwell-Klein-Gordon, Yang-Mills, and Einstein's equations,

[1] The Q_{ij} null structure is defined as $Q_{ij}(u,v) = \partial_i u \partial_j v - \partial_j u \partial_i v$

the $X^{s,\theta}$ spaces play a central role. The careful analysis of the extension to non-flat metrics, carried out in our work, is the starting point for many different research programs. The most ambitious of all will be to adapt these spaces to study quasilinear problems. In the author modest opinion, this represents a far-reaching goal, the ultimate application for such techniques. More accessible are the questions of existence of local solution of Wave maps, Maxwell-Klein-Gordon or of equations with general quadratic forms, in cured space-time and in subcritical regime. Once the subcritical theory is completed we can aim to attach the more difficult critical problems, mimicking the program already developed in the context of a flat metric. The application of non-flat $X^{s,\theta}$ spaces to semilinear problems certainly represents a mayor area of development that will be the training ground to then assault quasilinear problems.

Part II: focusing energy critical wave equation

The understanding of long time dynamics for critical nonlinear dispersive equations has attracted a lot of attention in recent years. Roughly speaking a nonlinearity is called critical if it is as strong as the linear part of the PDE. In critical setting the dichotomy between blow-up and scattering is delicate to settle since the linear part, which forces the solution to scatter at infinity, and the nonlinearity, which push the solution to blow-up, have the same strength.

We focus our analysis on one of the most studied critical dispersive equations: the quintic focusing semilinear wave equation in $\mathbb{R}^{1+3}$:

$$\begin{cases} \Box u = -u^5 \quad \text{in } \mathbb{R}^{1+3}, \\ (u, u_t)\Big|_{t=0} = (u_0, u_1), \end{cases} \tag{7}$$

where $\Box = -\partial_t^2 + \triangle$ is the d'Alembert operator, $u_0 \in H^1(\mathbb{R}^3)$ the initial position and $u_1 \in L^2(\mathbb{R}^3)$ the initial velocity. By Strichartz estimates, one can show that problem (7) is locally well posed in $H^1 \times L^2$. However, the equation is focusing, that is the nonlinearity represents an attracting force; hence one can construct solutions which blow-up in finite time. One divides such blow-up solutions into two classes: we say that a solution is type I if

$$\sup_{t\in I} \|\nabla_{t,x} u(t,\cdot)\|_{L^2_x} = \infty,$$

where the open interval I is the maximal interval of existence in the sense of Shatah-Struwe. On the other hand, u is called a type II solution iff

$$\sup_{t\in I} \|\nabla_{t,x} u(t,\cdot)\|_{L^2_x} < \infty.$$

Radial type II solutions were classified by Duyckaerts-Kenig-Merle [23] as a finite sum of traveling waves, called solitons, plus a small radiation term. In parallel, radial one-soliton solutions have been constructed explicitly by Krieger-Shlag-Tataru [62] using renormalization and distorted Fourier transform techniques. The construction was improved to all possible blow-up speeds by the first two authors in the subsequent work [59]. Such blow-up solutions are formed by a bulk term plus a small high-oscillating radiation term:

$$u_\nu(t,r) = W_{\lambda(t)}(r) + \eta(t,r).$$

Here $W(x) = (1+|x|^2/3)^{-1/2}$ is a stationary solution of (7), also called Talenti-Aubin functions from its geometric origins, and η is a small error term. We have defined the rescaling of W as $W_{\lambda(t)}(r) = \lambda^{1/2}(t)W(\lambda(t)r)$, where $\lambda(t) = t^{-1-\nu}$ and $\nu > 0$ is a fixed constant representing the blow-up speed.

The stability properties of type II solutions have been a conundrum due to the presence of a negative eigenvalue in the spectrum of the linearized operator. In [56] Krieger-Nakanishi-Schlag show that there exist a co-dimension one Lipschitz manifold Σ lying in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ such that if we take $(u_0, u_1) \in \Sigma$ then the solution of

$$\begin{cases} \Box u = -u^5 \quad \text{in } (0, t_0] \times \mathbb{R}^3 \\ (u, \partial_t u)\Big|_{t=t_0} = (u_\nu, \partial_t u_\nu)\Big|_{t=t_0} + (u_0, u_1) + \gamma(\phi_d, 0) \end{cases}$$

blows up in finite time if $\gamma > 0$ or scatter towards zero if $\gamma < 0$. Here the initial time t_0 is positive and chosen sufficiently small. This leaves the following question open: what are the dynamics on the manifold Σ? Since for small enough ν the functions u_ν will approach a stable type I blow-up, one would expect a positive answer for ν in some positive neighborhood of the origin. Indeed, in [49] Krieger showed there exists a co-dimension 2 Lipschitz hyper surface $\Sigma_0 \subset H_\perp^{3/2+} \times H_\perp^{1/2+}$ such that if we take $(u_0, u_1) \in \Sigma_0$ small enough in an appropriate topology then the solution of

$$\begin{cases} \Box u = -u^5 \quad \text{in } (0, t_0] \times \mathbb{R}^3 \\ (u, \partial_t u)\Big|_{t=t_0} = (u_\nu, \partial_t u_\nu)\Big|_{t=t_0} + (u_0, u_1) + \gamma(\phi_d, 0) + \phi_d\big(\gamma_1(u_0, u_1, \gamma), \gamma_2(u_0, u_1, \gamma)\big) \end{cases}$$

is a type II blow-up solution exactly of the type constructed in [62] and [59]. Here $|\gamma| \leq \delta$ is chosen small enough, and $\gamma_{1,2}$ are suitable Lipschitz functions.

The aim of the second part of this thesis is to present the joint work with Krieger [7] in which the extra co-dimension 2 conditions is removed, thus leading to the optimal co-dimension 1 stability result: set $(u_0, u_1) \in H_\perp^{3/2+} \times H_\perp^{1/2+}$ small enough and $|\gamma| \leq \delta$, then the solution of

$$\begin{cases} \Box u = -u^5 \quad \text{in } (0, t_0] \times \mathbb{R}^3 \\ (u, \partial_t u)\Big|_{t=t_0} = (u_\nu, \partial_t u_\nu)\Big|_{t=t_0} + (u_0, u_1) + \gamma(\phi_d, 0) \end{cases}$$

is a type II blow-up solution of the type constructed in [62] and [59] provided that $|\gamma| \leq \delta$ is chosen small enough.

Low regularity theory Part I

1 Low-regularity local well-posedness theory in flat spacetime

After a brief outline of the energy method in §1.1 for general nonlinearities, we focus the discussion on one particular class of nonlinearities which is intensively studied in the literature, namely the ones which involve products of first-order derivatives of the unknown such as

$$N(u,\partial u) = q^{\alpha_1 \dots \alpha_l}(u)\partial_{\alpha_1} u \dots \partial_{\alpha_l} u$$

where $\alpha_i \in \mathbb{N}^n$ and Einstein summation convention is in force. In §1.3 we show that Strichartz estimates allow us to obtain a local well-posedness theory for which the minimal regularity one must impose on the initial data is below the one required by the energy method. In addition, for high-oder nonlinearities ($l \geq 3$) Strichartz estimates allow already to reach the sharp result, this is the subject of §1.4. Furthermore, we carefully analyse quadratic nonlinearities ($l = 2$) and its subclasses of null-forms; we introduce in §1.5 and §1.6 more sophisticated spaces which go under the name of hyperbolic Sobolev spaces. Originally introduced by Bourgain in the context of Schrödinger equation [4] and Korteweg-de Vries (KdV) equation [5], hyperbolic Sobolev spaces are associated to a hyperbolic operator in the same way the classical Sobolev spaces are associated to the Laplacian. Implicitly, these spaces were also present in previous works by Rauch-Reed [89] and Beals [2]. The extension to wave equation was carried out subsequently by Klainerman and Machedon in [35]. In the last sections of this chapter, we shall explain how hyperbolic Sobolev spaces enable us to obtain better local well-posedness results than the one obtained by Strichartz estimates.

1.1 Energy methods

In this section we prove classical results concerning the existence of a unique local solution for a wide class of semilinear and quasilinear wave equations via a fixed point argument. The method used below also goes under the name of energy method since its main ingredients are the energy inequality and Sobolev embeddings. Notice that the use of Sobolev embeddings such as $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ imposes the restriction $s > n/2$. As a consequence, energy method are very powerful techniques because they impose little assumptions on the nonlinearity;

nevertheless they fail to provide a sharp lower bound on the regularity exponent of the initial data.

Semilinear problems spit naturally into two major subclasses depending on whether the nonlinearity depends only on the unknown function or also on its first derivatives, whereas for quasilinear problems the distinction hinges on the dependence of the metric just on the unknown function or also on its first derivatives. Therefore we are led naturally to the following four theorems.

Theorem 1 (Semilinear). *Let $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}(\mathbb{R}^n)$, and $N \in C^\infty(\mathbb{R}^{n+1})$ such that $N(0) = 0$. Let $s > \frac{n}{2}$, then there exist a $T > 0$ and an unique $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ solution to*

$$\begin{cases} \Box u = N(u) & \text{in } (0,T) \times \mathbb{R}^n \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases}$$

Theorem 2 (D-Semilinear). *Let $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}(\mathbb{R}^n)$, and $N \in C^\infty(\mathbb{R}^{n+2})$ such that $N(0) = 0$. Let $s > \frac{n}{2} + 1$, then there exist a $T > 0$ and an unique $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ solution to*

$$\begin{cases} \Box u = N(u, \partial u) & \text{in } (0,T) \times \mathbb{R}^n \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases}$$

Theorem 3 (Quasilinear). *Let $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}(\mathbb{R}^n)$, and $N, g \in C^\infty(\mathbb{R}^{n+1})$ such that $g(0) = N(0) = 0$. Suppose that $\sum_{\alpha\beta} |\eta^{\alpha\beta} - g^{\alpha\beta}| \leq 1/2$, where η is the Minkowski metric, furthermore the matrix (g^{ij}) is positive definite and $g^{00} \leq 0$. Let $s > \frac{n}{2} + 1$, then there exist a $T > 0$ and an unique $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ solution to*

$$\begin{cases} \Box_{g(u)} u = N(u, \partial u) & \text{in } (0,T) \times \mathbb{R}^n \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases}$$

Theorem 4 (D-Quasilinear). *Let $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}(\mathbb{R}^n)$, and $N, g \in C^\infty(\mathbb{R}^{n+1})$ such that $g(0) = N(0) = 0$. Suppose that $\sum_{\alpha\beta} |\eta^{\alpha\beta} - g^{\alpha\beta}| \leq 1/2$, where η is the Minkowski metric, furthermore the matrix (g^{ij}) is positive definite and $g^{00} \leq 0$. Let $s > \frac{n}{2} + 2$, then there exist a $T > 0$ and an unique $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ solution to*

$$\begin{cases} \Box_{g(u,\partial u)} u = N(u, \partial u) & \text{in } (0,T) \times \mathbb{R}^n \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases}$$

Below we outline the main steps of the proof of these theorems, see [95] for a complete argument. In general, for initial data lying in an inhomogeneous Sobolev spaces one has to restrict the time interval to a finite time slice. Thus, for a constant $T > 0$ that will be fixed later, we define the space

$$X_T^s = C([0,T], H^s) \cap C^1([0,T], H^{s-1})$$

and endow it with its natural norm

$$\|u\|_{X_T^s} = \|\partial u\|_{L^\infty((0,T),H^{s-1}(\mathbb{R}^n))} \approx \sup_{t\in[0,T]} \sum_{\alpha\le 1} \|\partial_{t,x}^\alpha u(t,\cdot)\|_{H^{s-1}}$$

In order to avoid cumbersome notation we shall hereafter denote $L^\infty H^{s-1}(S_T)$ the space $L^\infty((0,T),H^{s-1}(\mathbb{R}^n))$, and $\partial = \partial_{t,x}$ will denote the space-time gradient.

The homogeneous and inhomogeneous solution maps are defined as

$$\mathcal{H}_g(u_0,u_1) = u \quad \Longleftrightarrow \quad \{\Box_g u = 0,\ u(0) = u_0,\ \partial_t u(0) = u_1\}$$

$$\Box_g^{-1} F = u \quad \Longleftrightarrow \quad \{\Box_g u = F,\ u(0) = 0,\ \partial_t u(0) = 0\}$$

We simply write $\mathcal{H} = \mathcal{H}_\eta$ and $\Box^{-1} = \Box_\eta^{-1}$ if the metric η is the Minkowski metric. Recall that

$$\widehat{\mathcal{H}(u_0,u_1)}(t,\xi) = \cos(t|\xi|)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi), \qquad \widehat{\Box^{-1}F}(t,\xi) = \int_0^t \frac{\sin((t-s)|\xi|)}{|\xi|}\widehat{F}(s,\xi)\,ds.$$

The solution to the inhomogeneous wave equation with forcing term F and initial data u_0 and u_1 is given by the solution map $\mathcal{S}_g u = \mathcal{H}_g(u_0,u_1) + \Box_g^{-1}F$. From the energy inequality we can deduce the following linear properties of the homogeneous and inhomogeneous solution operators.

Proposition 5 (Linear estimates, [95]). *For any $s \in \mathbb{R}$, we have that $\mathcal{H}_g \in \mathcal{L}(H^s \times H^{s-1}, X_T^s)$, and $\Box_g^{-1} \in \mathcal{L}(X_T^{s-1}, X_T^s)$ with $\|\mathcal{H}_g\|, \|\Box_g^{-1}\| \lesssim \langle T\rangle \exp\Big(\|\partial g(u,\partial u)\|_{L^1L^\infty(S_T)}\Big)$. Precisely we have*

$$\|\partial u(t)\|_{H^{s-1}} \lesssim \langle t\rangle \Big(\|\partial u(0)\|_{H^{s-1}} + \int_0^t \|\Box_g u(s)\|_{H^{s-1}} ds\Big) \exp\Big(\frac{1}{4}\int_0^t \|\partial g(s)\|_\infty ds\Big)$$

Notice that, when $g = \eta$ the constants in the homogeneous estimate and inhomogeneous estimate depends only on T, whereas when $g \neq \eta$ one needs to consider the L^1L^∞ norm of the metric as well. Moreover, observe that form the energy inequality one only has $\Box^{-1} \in \mathcal{L}(L^1H^{s-1}(S_T), X_T^s)$, however clearly $X_T^{s-1} \subset L^1H^{s-1}(S_T)$. To prove well-posedness results via energy method it is sufficient to consider $\Box^{-1} \in \mathcal{L}(X_T^{s-1}, X_T^s)$. The corresponding nonlinear estimates needed to prove the fixed point theorem follows from Moser inequality and Sobolev embeddings.

Proposition 6 (Nonlinear estimates). *Let $s > n/2+1$, then there exists a continuous positive function $c_\mathcal{N}$ such that $c_\mathcal{N}(0) = 0$ and*

$$\|N(u,\partial u)\|_{X_T^{s-1}} \le c_\mathcal{N}(\|u\|_{X_T^s})\|u\|_{X_T^s}.$$

Moreover, let $s > n/2$ then there exists a continuous positive function $c_\mathcal{N}$ such that $c_\mathcal{N}(0) = 0$ and

$$\|N(u)\|_{X_T^{s-1}} \le c_\mathcal{N}(\|u\|_{X_T^s})\|u\|_{X_T^s}.$$

Proof. Moser inequality, see [74], implies that there exists a continuous function $\phi_s : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi_s(0) = 0$ and

$$\|N(u,\partial u)(t)\|_{H^{s-1}} \leq \phi_s\big(\|u(t)\|_{L^\infty} + \|\partial u(t)\|_{L^\infty}\big)(\|u(t)\|_{H^{s-1}} + \|\partial u(t)\|_{H^{s-1}})$$

$$\|\partial_t N(u,\partial u)(t)\|_{H^{s-2}} \leq \phi_s\big(\|u(t)\|_{L^\infty} + \|\partial u(t)\|_{L^\infty}\big)(\|u(t)\|_{H^{s-2}} + \|\partial u(t)\|_{H^{s-2}})$$

Moreover Sobolev embedding implies $\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u(t)\|_{H^s(\mathbb{R}^n)}$ for $s > n/2$ and $\|\partial u(t)\|_{L^\infty(\mathbb{R}^n)} \leq \|\partial u(t)\|_{H^{s-1}(\mathbb{R}^n)} \leq \|u\|_{X^s_T}$ for $s > n/2+1$. Hence we obtain the desired frist estimate $\|N(u,\partial u)\|_{X^{s-1}_T} \leq c_{\mathcal{N}}(\|u\|_{X^s_T})\|u\|_{X^s_T}$, where $c_{\mathcal{N}}$ is a continuous positive function.

In this case where the nonlinearity depends only on the unknown Moser inequality reduced to

$$\|N(u)(t)\|_{H^{s-1}} \leq \phi_s\big(\|u(t)\|_{L^\infty}\big)\|u(t)\|_{H^{s-1}}$$

$$\|\partial_t N(u)(t)\|_{H^{s-2}} \leq \phi_s\big(\|u(t)\|_{L^\infty}\big)\|u(t)\|_{H^{s-2}}$$

But here Sobolev embedding implies $\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u(t)\|_{H^s(\mathbb{R}^n)} \leq \|\partial u\|_{L^\infty H^{s-1}} = \|u\|_{X^s_T}$ for the larger range $s > n/2$. □

To close the fixed-point argument and to complete the proof of Theorems 1 and 2 observe that the ball $B(0,R) \subset X^s_T$ is mapped via $\mathcal{S}$ into the ball $B(0,R') \subset X^s_T$ where $R' = \langle T\rangle\|(u_0,u_1)\|_{H^s\times H^{s-1}} + \langle T\rangle c_{\mathcal{N}}(R)R$. In fact, combining the linear with the nonlinear estimates we obtain

$$\begin{aligned} \|\mathcal{S}u\|_{X^s_T} &\leq \|\mathcal{H}(u_0,u_1)\|_{X^s_T} + \|\Box^{-1}N(u,\partial u)\|_{X^s_T} \\ &\leq \langle T\rangle\|(u_0,u_1)\|_{H^s\times H^{s-1}} + \langle T\rangle c_{\mathcal{I}}\|N(u,\partial u)\|_{X^{s-1}_T} \\ &\leq \langle T\rangle\|(u_0,u_1)\|_{H^s\times H^{s-1}} + \langle T\rangle c_{\mathcal{N}}(\|u\|_{X^s_T})\|u\|_{X^s_T}. \end{aligned}$$

In order to obtain that $\mathcal{S}$ maps the ball of radius R into itself estimates we need to choose the time interval of local existence $I = [0,T]$ so that $\langle T\rangle\|(u_0,u_1)\|_{H^s\times H^{s-1}} + \langle T\rangle c_{\mathcal{N}}(R)R < R$ holds for some $R > 0$. Under which conditions does $\mathcal{S}$ has a fixed-point in $B(0,R)$? The Mean value theorem allow us to conclude that for every $u, v \in B(0,R)$ we have

$$\begin{aligned} \|\mathcal{S}u - \mathcal{S}v\|_{X^s_T} &= \|\Box^{-1}(N(u,\partial u) - N(v,\partial v))\|_{X^s_T} \\ &\leq \langle T\rangle\|N(u,\partial u) - N(v,\partial v)\|_{X^{s-1}_T} \\ &\leq \langle T\rangle c_{\mathcal{D}}(\|u\|_X, \|v\|_{X^s_T})\|u - v\|_{X^s_T} \end{aligned}$$

where $c_{\mathcal{D}}$ is a continuous positive function such that $c_{\mathcal{D}}(0) = 0$. Thus $\mathcal{S}$ is a contraction if we choose T small enough so that $\langle T\rangle c_{\mathcal{N}}(R) < 1$. Therefore a contraction argument implies the existence of a local solution if the time of existence is chosen sufficiently small. This conclude the proof of Theorem 1 and Theorem 2.

We now turn to the proof of Theorem 4. To bound the norm of the solution map we need to control $\|\partial g(u,\partial u)\|_{L^1L^\infty(S_T)}$. Here there is no need to invoke Moser inequality, indeed by the chain rule and Sobolev embedding we obtain:

$$\begin{aligned}\|\partial g(u,\partial u)\|_{L^1L^\infty(S_T)} &\le \|\partial_u g(u,\partial u)\|_{L^\infty L^\infty(S_T)}\|\partial u\|_{L^1L^\infty(S_T)} + \|\partial_{\partial u} g(u,\partial u)\|_{L^\infty L^\infty(S_T)}\|\partial^2 u\|_{L^1L^\infty(S_T)}\\ &\le \|\partial u\|_{L^\infty H^{s-1}(S_T)} + \|\partial^2 u\|_{L^\infty H^{s-2}(S_T)}\\ &\le \|u\|_{X^s_T}\end{aligned}$$

since, by hypothesis, the metric g and its derivatives are bounded. Notice that the two inequalities obtained in the second line follows form Sobolev embedding, the first one holds for $s > n/2+1$ while the second holds for $s > n/2+2$. For the sake of completeness, let us outline the contraction argument here. Recall that solution map for the quasilinear problem is $\mathscr{S}_g(u) = \mathscr{H}_g(u_0,u_1) + \Box_g^{-1}\mathscr{N}(u,\partial u)$. Then

$$\|\mathscr{S}_g(u)\|_{X^s_T} \lesssim_T \exp\Big(\|\partial g(u,\partial u)\|_{L^1L^\infty(S_T)}\Big)\Big(\|u_0,u_1\|_{H^s\times H^{s-1}} + \|N(u,\partial u)\|_{X^{s-1}_T}\Big)$$

By Moser inequality we can bound the nonlinear term $\|N(u,\partial u)\|_{X^{s-1}_T} \lesssim c_{\mathscr{N}}(\|u\|_{X^s_T})$, and by the chain rule and Sobolev embedding theorem we can control $\|\partial g(u,\partial u)\|_{L^1L^\infty(S_T)} \lesssim_T \|u\|_{X^s_T}$. Therefore

$$\|S_g(u)\|_{X^s_T} \lesssim_T \exp\Big(\|u\|_{X^s_T}\Big)\Big(\|u_0,u_1\|_{H^s\times H^{s-1}} + c_{\mathscr{N}}(\|u\|_{X^s_T})\Big)$$

The rest of the argument follows verbatim the one given in the semilinear case. This conclude the proof of Theorem 4.

In order to prove Theorem 3 we modify slightly the previous argument. Suppose that the metric g does not depend on derivatives of the unknown, then by the chain rule and Sobolev embedding we obtain the estimate

$$\|\partial g(u)\|_{L^1L^\infty(S_T)} \le \|\partial_u g(u)\|_{L^\infty L^\infty(S_T)}\|\partial u\|_{L^1L^\infty(S_T)} \le \|\partial u\|_{L^\infty H^{s-1}(S_T)} \le \|u\|_{X^s_T}$$

which holds for $s > n/2+1$ instead.

Remark (On global well-posedness). Notice that the arguments presented so far are strictly tight to the local theory and in general they fail when proving the existence of a global solution due to the fact that we lose the control over the constant in the contraction estimates. However, if we consider small enough initial data in homogeneous Sobolev spaces, then a unique global solution exists, see [95].

1.2 Beyond the energy method

In the rest of the chapter, the main objective will be how to weaken the regularity assumptions imposed on the initial data by the energy method to ensure the existence of a local solution. First let us understand that there exists a natural lower bound on such Sobolev exponent which is imposed by scaling considerations. Precisely define $u_\lambda(t,x) = \lambda^e u(\lambda t, \lambda x)$ where $\lambda, e \in \mathbb{R}$ are constants. The constant β will be chosen to that if u solves a nonlinear wave equation, then also u_λ will solves the same nonlinear wave equation (suppose that the nonlinearity is homogeneous). One then defines the *critical Sobolev regularity exponent* s_c as the unique $s \in \mathbb{R}$ such that the homogeneous data space $\dot{H}^s(\mathbb{R}^n) \times \dot{H}^{s-1}(\mathbb{R}^n)$ remains invariant under scaling. Klainerman highlights the importance of the critical Sobolev regularity exponent in [37] where he proposed the following well-posedness conjecture: for all classical field theories the Cauchy problem is:

i. locally well-posed in the subcritical range $s > s_c$,

ii. globally well-posed in the critical case $s = s_c$ for smooth initial data with small critical $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm,

(iii) ill-posed in the supercritical range $s < s_c$.

To illustrate Klainerman's conjecture consider for instance the following two homogeneous nonlinearities:

$$\mathcal{N}(u) = u^p, \text{ and } \mathcal{N}(u,\partial u) = u^p|\partial u|^q$$

Then the solutions of the equations $\Box u = \mathcal{N}(u)$ and $\Box u = \mathcal{N}(u,\partial u)$ are invariant under the scaling $u_\lambda(t,x) = \lambda^c u(\lambda t, \lambda x)$, where respectively

$$c(p) = \frac{2}{p-1}, \text{ and } c(p,q) = \frac{2-q}{p+q-1}$$

Therefore the critical Sobolev regularity exponent are respectively

$$s_c = \frac{n}{2} - \frac{2}{p-1}, \text{ and } s_c = \frac{n}{2} - \frac{2-q}{p+q-1}$$

Below the critical scaling exponent s_c the problems are supposed to be ill-posed. Whereas if $s > s_e$, where respectively

$$s_e = \frac{n}{2}, \text{ and } s_e = \frac{n}{2} + 1$$

then the the local well-posedness follows form the energy method. The question we address in the subsequent sections is to establish local well-posedness or disprove it in the strip $s_c \leq s \leq s_e$. We will argue that, in general, scaling predicts the sharp local well-posedness result in higher dimensions. However, in low dimensions, namely $n = 2,3,4$, we need to impose more regularity then the one predicted by the critical Sobolev exponent to obtain a local solution. The phenomenon responsible for this fact is the concentration along light ray, see [67], [68].

1.3 General quadratic nonlinearities

In this section we focus on nonlinearities which are bilinear forms on the space-time gradient $B(\partial u, \partial u) = b^{\alpha\beta}\partial_\alpha u \partial_\beta u$. The prototypical nonlinearity of this type is $(\partial_t u)^2$. Observe that for such nonlinearities we obtain $s_c = n/2$. The goal of this section is to prove the following theorem, which is taken from [29].

Theorem 7. *Let* $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}(\mathbb{R}^n)$, *and* $b^{\alpha\beta} \in C^\infty(\mathbb{R})$ *with all derivatives bounded and* $b^{\alpha\beta}(0) = 0$. *Let*

$$s > \max\Big\{\frac{n}{2} + \frac{1}{2}, \frac{n+5}{4}\Big\} = \begin{cases} \frac{7}{4} & \text{if } n = 2 \\ \frac{n}{2} + \frac{1}{2} & \text{if } n \geq 3 \end{cases},$$

then there exist a $T > 0$ *and an unique solution* $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ *of*

$$\begin{cases} \Box u = b^{\alpha\beta}(u)\partial_\alpha u \partial_\beta u & \text{in } (0,T) \times \mathbb{R}^n \\ (u, u_t)(0) = (u_0, u_1) \end{cases} \tag{1.1}$$

The proof relies on Strichartz estimates; the different bounds on s when $n = 2$ or when $n \geq 3$ are due to smaller range of Strichartz estimates available in low dimensions. Observe that this result is sharp only in dimensions $n = 2, 3$. In fact, the sharp result

$$s > \max\Big\{\frac{n}{2}, \frac{n+5}{4}\Big\}, \qquad n \geq 2$$

was proved Tataru [113] using hyperbolic Sobolev spaces. In order to prove Theorem 7 we recall the following four estimates:

i. Energy inequality:

$$\|\partial u\|_{L^\infty H^{s-1}(S_T)} \lesssim_T \|(u_0, u_1)\|_{H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)} + \|\Box u\|_{L^1 H^{s-1}(S_T)}$$

that is $\mathscr{H} \in \mathscr{L}(H^s \times H^{s-1}, \partial^{-1} L^\infty H^{s-1})$, and $\Box^{-1} \in \mathscr{L}(L^1 H^{s-1}, \partial^{-1} L^\infty H^{s-1})$ where $\|\mathscr{H}\|_{\mathscr{L}} \approx T$ and $\|\Box^{-1}\|_{\mathscr{L}} \approx T$.

ii. ∂-Strichartz inequality: let (p, q, σ) a wave admissible triplet [1] and $s > \sigma + 1$ then

$$\|\partial u\|_{L^p L^q(S_T)} \lesssim \|(u_0, u_1)\|_{H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)} + \|\Box u\|_{L^1 H^{s-1}(S_T)}$$

that means $\mathscr{H} \in \mathscr{L}(H^s \times H^{s-1}, \partial^{-1} L^p L^q)$, and $\Box^{-1} \in \mathscr{L}(L^1 H^{s-1}, \partial^{-1} L^p L^q)$ where $\|\mathscr{H}\|_{\mathscr{L}} \approx 1$ and $\|\Box^{-1}\|_{\mathscr{L}} \approx 1$. We refer to the Appendix A for a proof.

iii. Calculus inequality: let $s \geq 0$ then

$$\|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}$$

iv. Moser inequality: suppose that $F \in C^\infty(\mathbb{R})$ with all derivatives bounded and $F(0) = 0$.

[1] Recall that (p, q, σ) a wave admissible triplet if $2 \leq p, q < \infty$, $\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}$, $\sigma = \frac{n}{2} - \frac{1}{p} - \frac{n}{q}$, and $(p, q, \sigma) \neq (2, \infty, 1)$.

Then for any $s \geq 0$ there exists a continuous function $\phi_s : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|F(u)(t)\|_{H^s} \leq \phi_s(\|u(t)\|_{L^\infty})\|u(t)\|_{H^s}$$

Define the iteration space to be $X_T = \partial^{-1}L^\infty H^{s-1}(S_T) \cap \partial^{-1}L^p L^\infty(S_T)$, endowed with norm

$$\|u\|_{X_T} = \|\partial u\|_{L^\infty H^{s-1}(S_T)} + \|\partial u\|_{L^p L^\infty(S_T)}.$$

Notice that since $q = \infty$ to have a wave admissible triplet we must choose

$$p \begin{cases} = 4 & \text{if } n = 2 \\ > 2 & \text{if } n = 3 \\ = 2 & \text{if } n \geq 4 \end{cases}$$

and $\sigma = n/2 - 1/p$, then clearly

$$s > \frac{n}{2} - \frac{1}{p} + 1 > \max\{\frac{n}{2} + \frac{1}{2}, \frac{n+5}{4}\}.$$

By energy and Strichartz inequalities we obtain the linear estimates needed to apply a fixed point argument: $\mathscr{H} \in \mathscr{L}(H^s \times H^{s-1}, X_T)$, and $\Box^{-1} \in \mathscr{L}(L^1 H^{s-1}, X_T)$, where $\|\mathscr{H}\|_{\mathscr{L}} \approx T$ and $\|\Box^{-1}\|_{\mathscr{L}} \approx T$. Therefore to close the perturbative argument we need to prove the following nonlinear estimates.

Proposition 8 (Nonlinear estimates)**.** *Let $B(\partial u, \partial u) = b^{\alpha\beta}(u)\partial_\alpha u \partial_\beta u$, where $b^{\alpha\beta}$ satisfies the hypothesis of Theorem 7. If $s > \max\{\frac{n}{2} + \frac{1}{2}, \frac{n+5}{4}\}$ then*

(i) *B has the good mapping properties:*

$$\|B(\partial u, \partial u)\|_{L^1 H^{s-1}(S_T)} \lesssim c_{\mathscr{N}}(\|u\|_{X_T})\|u\|_{X_T}$$

(ii) *B is a contraction:*

$$\|B(\partial u, \partial u) - B(\partial v, \partial v)\|_{L^1 H^{s-1}(S_T)} \lesssim c_{\mathscr{D}}(\|u\|_{X_T}, \|v\|_{X_T})\|u - v\|_{X_T}$$

where $c_{\mathscr{N}}$ and $c_{\mathscr{D}}$ are positive continuous functions such that $c_{\mathscr{N}}(0) = c_{\mathscr{D}}(0) = 0$.

Proof. From the calculus inequality and the Lebesgue nesting $L^\infty(0,T) \subset L^p(0,T) \subset L^1(0,T)$ for $1 \leq p \leq \infty$ we obtain the estimate

$$\begin{aligned} &\|B(\partial u, \partial u)\|_{L^1 H^{s-1}(S_T)} \\ \lesssim\ &\|b^{\alpha\beta}(u)\|_{L^\infty L^\infty(S_T)}\|\partial_\alpha u \partial_\beta u\|_{L^1 H^{s-1}(S_T)} + \|b^{\alpha\beta}(u)\|_{L^\infty H^{s-1}(S_T)}\|\partial_\alpha u \partial_\beta u\|_{L^1 L^\infty(S_T)} \\ \lesssim\ &\|b^{\alpha\beta}(u)\|_{L^\infty L^\infty(S_T)}\|\partial_\alpha u\|_{L^1 L^\infty(S_T)}\|\partial_\beta u\|_{L^\infty H^{s-1}(S_T)} + \|b^{\alpha\beta}(u)\|_{L^\infty H^{s-1}(S_T)}\|\partial_\alpha u\|_{L^2 L^\infty(S_T)}\|\partial_\beta u\|_{L^2 L^\infty(S_T)} \\ \lesssim\ &\|b^{\alpha\beta}(u)\|_{L^\infty L^\infty(S_T)}\|\partial_\alpha u\|_{L^p L^\infty(S_T)}\|\partial_\beta u\|_{L^\infty H^{s-1}(S_T)} + \|b^{\alpha\beta}(u)\|_{L^\infty H^{s-1}(S_T)}\|\partial_\alpha u\|_{L^p L^\infty(S_T)}\|\partial_\beta u\|_{L^p L^\infty(S_T)} \end{aligned}$$

To estimate the terms containing $b^{\alpha\beta}$ we use Sobolev embedding and Moser inequality: assume $s > n/2$ then

$$\|b^{\alpha\beta}(u)\|_{L^\infty L^\infty(S_T)} \lesssim \|b^{\alpha\beta}(u)\|_{L^\infty H^s(S_T)} \lesssim \phi_s(\|u\|_{L^\infty L^\infty(S_T)})\|u\|_{L^\infty H^s(S_T)} \lesssim \phi_s(\|u\|_{X_T})\|u\|_{X_T}$$

and

$$\|b^{\alpha\beta}(u)\|_{L^\infty H^{s-1}(S_T)} \lesssim \phi_s(\|u\|_{L^\infty L^\infty(S_T)})\|u\|_{L^\infty H^{s-1}(S_T)} \lesssim \phi_s(\|u\|_{X_T})\|u\|_{X_T}$$

where ϕ_s is a continuous positive function such that $\phi_s(0) = 0$. Moreover, the norms of terms involving $\partial_\alpha u$ and $\partial_\beta u$ are included in the definition of X_T norm. This proves (i). To prove (ii) observe that we can write the difference $B(\partial u, \partial u) - B(\partial v, \partial v)$ as the sum of the three terms:

$$B(\partial u, \partial u) - B(\partial v, \partial v) = [b^{\alpha\beta}(u) - b^{\alpha\beta}(v)]\partial_\alpha u \partial_\beta u + b^{\alpha\beta}(v)\partial_\alpha(u-v)\partial_\beta u + b^{\alpha\beta}(v)\partial_\alpha(u-v)\partial_\beta v$$

and each term can be estimated as in (i). □

Remark. Notice that the key point where we used the fact that we are working with quadratic nonlinearities and not with general higher-order nonlinearities is where we estimated

$$\begin{aligned} \|\partial_\alpha u \partial_\beta u\|_{L^1 L^\infty(S_T)} &\lesssim \int_0^T \|\partial_\alpha u(\tau)\|_{L^\infty} \|\partial_\beta u(\tau)\|_{L^\infty} d\tau \\ &= \|\partial_\alpha u\|_{L^2 L^\infty(S_T)} \|\partial_\beta u\|_{L^2 L^\infty(S_T)} \end{aligned}$$

For general higher-order nonlinearities this estimates does not hold. Next section is devoted to show the existence of a local solution for problem involving general higher-order nonlinearities.

1.4 General multi-linear nonlinearities

In this section we present the result of Ponce and Sideris [86] for the $n = 3$ case and its generalisation to any dimensions $n \geq 2$ by Fang and Wang [25] and Ye [119]. Let us consider the following Cauchy problem:

$$\begin{cases} \Box u = b^\alpha(u)(\partial u)_\alpha & \text{in } (0,T) \times \mathbb{R}^n \\ (u, u_t)(0) = (u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n) \end{cases} \tag{1.2}$$

where $\alpha \in \mathbb{N}^l$ is a multi-index. Notice that $b^\alpha(u)(\partial u)_\alpha = q^{\alpha_1 \dots \alpha_l}(u)\,\partial_{\alpha_0} u \partial_{\alpha_1} u \dots \partial_{\alpha_l} u$, where $\alpha_i \in \{0, \dots, n\}$, contains all the possible combinations of l-derivatives. For simplicity we will consider the nonlinearity to be of the form $b^\alpha(u)(\partial u)_\alpha = b(u)|\partial u|^l$ but the result extend easily to general multi-linear nonlinearities considered above. We prove the following local well-posedness result:

Theorem 9. *Let $u_0 \in H^s(\mathbb{R}^n)$, $u_1 \in H^{s-1}(\mathbb{R}^n)$, and $b \in C^\infty(\mathbb{R})$ with all derivatives bounded and* *such that* $b(0) = 0$. *Let*

$$s > \max\left\{\frac{n+5}{4}, \frac{n}{2} + \frac{1}{2}, \frac{n}{2} + \frac{l-2}{l-1}\right\} \qquad \textit{if } n \geq 2.$$

then there exist a $T > 0$ *and an unique solution* $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ *of* (1.2).

Remark. Notice that we can rewrite the lower bounds for s in the following way:

$$s > \frac{7}{4} \qquad \text{if } n = 2 \text{ and } 2 \leq l \leq 5,$$

$$s > 1 + \frac{l-2}{l-1} \qquad \text{if } n = 2 \text{ and } l \geq 5,$$

$$s > \frac{n}{2} + \frac{1}{2} \qquad \text{if } n \geq 3 \text{ and } l = 2,$$

$$s > \frac{n}{2} + \frac{l-2}{l-1} \qquad \text{if } n \geq 3 \text{ and } l \geq 3,$$

Moreover the scaling for equation (1.2) gives $s_c = \frac{n}{2} + \frac{l-2}{l-1}$, therefore when $n = 2$ and $l \geq 5$ or $n \geq 3$ and $l \geq 3$ the lower bound on s given in Theorem 9 reach the critical scaling exponent, thus the result is sharp. Moreover, due to counterexamples by Lindblad [67], and [68] the lower bound in Theorem 9 is sharp in lower dimensions $n = 2$ and $n = 3$. On the other hand in higher dimensions and quadratic nonlinearities ($n \geq 4$ and $l = 2$) we obtain $s_c = \frac{n}{2}$ thus there is still a $1/2$ gap to explore.

To prove Theorem 9 we have to rely on Strichartz estimates, see Appendix A. Let us recall here the generalized Leibniz rule and a Strichartz-type estimates:

(i) Generalized Leibniz rule: let $s \geq 0$ and $2 \leq q_i \leq \infty$ and $1/2 = 1/p_i + 1/q_i$ then

$$\|D^s(fg)\|_{L^2(\mathbb{R}^n)} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^s g\|_{L^{q_2}}$$

The inhomogeneous counterpart, where we replace D^s by $\langle D\rangle^s$ holds as well.

(ii) $D^{\gamma-1}\partial$ Strichartz estimate: let u be the solution of a Cauchy problem for wave equation and (p, q, σ), $(\widetilde{p}, \widetilde{q}, \widetilde{\sigma})$ wave admissible triplets, and $\gamma \in \mathbb{R}$ then

$$\|D^{\gamma-1}\partial u\|_{L^p L^q(S_T)} \lesssim \|(u_0, u_1)\|_{H^{\gamma+\sigma} \times H^{\gamma+\sigma-1}} + \|D^{\gamma+\widetilde{\sigma}+\sigma-1}\Box u\|_{L^{\widetilde{p}'} L^{\widetilde{q}'}(S_T)}$$

Observe that if we choose the energy triplet $(\widetilde{\sigma}, \widetilde{p}, \widetilde{q}) = (0, 2, \infty)$, we obtain

$$\|D^{\gamma-1}\partial u\|_{L^p L^q(S_T)} \lesssim \|(u_0, u_1)\|_{H^{\gamma+\sigma} \times H^{\gamma+\sigma-1}} + \|D^{\gamma+\sigma-1}\Box u\|_{L^1 L^2(S_T)}$$

Moreover if we choose $s \geq \gamma + \sigma$ then

$$\|D^{\gamma-1}\partial u\|_{L^p L^q(S_T)} \lesssim \|(u_0, u_1)\|_{H^s \times H^{s-1}} + \|\Box u\|_{L^1 H^{s-1}(S_T)}$$

We set the iteration space norm to be

$$\|u\|_{X_T} = \|\partial u\|_{L^\infty H^{s-1}(S_T)} + \|\partial u\|_{L^p H^{\gamma-1,q}(S_T)} = \|\partial u\|_{L^\infty H^{s-1}(S_T)} + \|D^{\gamma-1}\partial u\|_{L^p L^q(S_T)} \tag{1.3}$$

The linear estimates

$$\|u\|_{X_T} \lesssim_T \|(u_0, u_1)\|_{H^s \times H^{s-1}} + \|\Box u\|_{L^1 H^{s-1}(S_T)}$$

follows form the energy inequality and $D^{\gamma-1}\partial$ Strichartz estimate. Therefore it suffices to prove the nonlinear estimates.

Proposition 10 (Nonlinear estimates)**.** *Let $F(u) = b(u)|\partial u|^l$, where b and s satisfies the hypothesis of Theorem 9. Then*

(i) *F has the good mapping properties:*

$$\|F(u)\|_{L^1 H^{s-1}(S_T)} \lesssim c_{\mathcal{N}}(\|u\|_{X_T})\|u\|_{X_T}$$

(ii) *F is a contraction:*

$$\|F(u) - F(v)\|_{L^1 H^{s-1}(S_T)} \lesssim c_{\mathcal{D}}(\|u\|_{X_T}, \|v\|_{X_T})\|u - v\|_{X_T}$$

where $c_{\mathcal{N}}$ and $c_{\mathcal{D}}$ are positive continuous functions such that $c_{\mathcal{N}}(0) = c_{\mathcal{D}}(0) = 0$.

Proof. Observe that s the initial regularity, l the strength of the nonlinearity, and n the spatial dimension, are given. Hence the norm defined in (1.3) depends only on 3 parameters: p, q, and γ. Moreover up to now the only constraint on this parameters is that (p, q, σ) is a wave admissible triplet and $\gamma \in \mathbb{R}$ satisfies $\gamma \leq s - \sigma$. Recall that σ is determined by p, q and n via $\sigma = n/2 - 1/p - n/q$. Let us suppose $\gamma > \frac{n}{q} + 1$ and

$$\frac{1}{p} \begin{cases} = \min\{\frac{1}{4}, \frac{1}{l-1}\} & \text{if } n = 2 \\ < \min\{\frac{1}{2}, \frac{1}{l-1}\} & \text{if } n = 3 \\ = \min\{\frac{1}{2}, \frac{1}{l-1}\} & \text{if } n \geq 4 \end{cases}$$

and q is determined via the sharp wave admissible condition

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}$$

then

$$s \geq \gamma + \sigma > \frac{n}{q} + 1 + \frac{n}{2} - \frac{1}{p} - \frac{n}{q} = \frac{n}{2} - \frac{1}{p} + 1$$

Furthermore if $l = 2$ we obtain

$$s > \frac{n}{2} - \frac{1}{p} + 1 = \max\{\frac{n}{2} + \frac{1}{2}, \frac{n+5}{4}\}$$

while if $3 \le l \le 4$ we obtain

$$s > \frac{n}{2} - \frac{1}{p} + 1 = \max\{\frac{n}{2} + \frac{l-2}{l-1}, \frac{n+5}{4}\}$$

and if $l \ge 5$ we have

$$s > \frac{n}{2} - \frac{1}{p} + 1 = \frac{n}{2} + \frac{l-2}{l-1}.$$

The reason why we have choose such a special Strichartz triplet will become clear in the estimates below.

As we already notice, we can't use the calculus inequality: $\|fg\|_{H^s} \le \|f\|_{L^\infty}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{L^\infty}$ since we have no control over the $\|(\partial u)^l\|_{L^1L^\infty(S_T)}$ term. However, we can use the generalised Leibniz rule before applying the calculus inequality. Let $1/2 = 1/p + 1/\tilde{p}$, from the generalized Leibniz rule we obtain the estimate

$$\|F(u)\|_{L^1H^{s-1}(S_T)} \lesssim \|b(u)\|_{L^\infty L^\infty(S_T)}\||\partial u|^l\|_{L^1H^{s-1}(S_T)} + \|b(u)\|_{L^\infty H^{s-1,p}(S_T)}\||\partial u|^l\|_{L^1L^{\tilde{p}}(S_T)}$$

Now suppose that $1/2 - 1/n \le 1/p \le 1/2$, then $H^s \subset H^{s-1,p}$. Moreover let $1/2-(s-1)/n \le 1/\tilde{p} \le 1/2$, then $H^{s-1} \subset L^{\tilde{p}}$. Since $s > n/2$ we can obtain such a pair $(p, \tilde{p})$ so that $1/2 = 1/p + 1/\tilde{p}$. Furthermore from the assumption that $s > n/2$ we obtain

$$\begin{aligned} \|F(u)\|_{L^1H^{s-1}(S_T)} &\lesssim \|b(u)\|_{L^\infty H^s(S_T)}\|(\partial u)^l\|_{L^1H^{s-1}(S_T)} \\ &\lesssim \|b(u)\|_{L^\infty H^s(S_T)}\|\partial u\|_{L^\infty H^{s-1}(S_T)}\|\partial u\|^{l-1}_{L^{l-1}L^\infty(S_T)} \end{aligned}$$

Furthermore, by the Lebesgue nesting $L^\infty(0,T) \subset L^p(0,T) \subset L^1(0,T)$ for $1 \le p \le \infty$, we can bound $\|\partial u\|^{l-1}_{L^{l-1}L^\infty(S_T)} \le \|\partial u\|^{l-1}_{L^pL^\infty(S_T)}$ since $p \ge l-1$. Moreover, since $(\gamma-1)q > n$ from the Sobolev embedding theorem we have $H^{\gamma-1,q} \subset L^\infty$ thus

$$\|\partial u\|^{l-1}_{L^pL^\infty(S_T)} \lesssim \|\partial u\|^{l-1}_{L^pH^{\gamma-1,q}(S_T)} \le \|\partial u\|^{l-1}_{X_T}$$

To estimate the term containing the coefficients b we use Sobolev embedding and Moser inequality:

$$\|b(u)\|_{L^\infty H^s(S_T)} \lesssim \phi_s(\|u\|_{L^\infty L^\infty(S_T)})\|u\|_{L^\infty H^s(S_T)} \lesssim \phi_s(\|u\|_{X_T})\|u\|_{X_T}$$

Hence (i) is proved. To prove (ii) observe that we can write the difference $F(u) - F(v)$ as the sum of the three terms:

$$F(u) - F(v) = [b(u) - b(v)](\partial u)^l + b(v)\partial(u-v)\sum_{a+b=l-1}(\partial u)^a(\partial v)^b$$

and each term can be estimated as in i.. □

Remark. Notice that the bounds needed in this proof are $(\gamma-1)q > n$, $s > n/2$ for Sobolev embedding and $p \ge l-1$ for the Lebesgue nesting. Moreover notice that $1/\tilde{p} \ge 1/2 - (s-1)/n$

is equivalent to

$$\frac{1}{p} \leq \frac{s-1}{n}$$

and $1/2 - 1/n \leq (s-1)/n$ therefore one should include the bound

$$s \geq \frac{n}{p} + 1$$

However this bound is weaker then $s > n/2 - 1/p + 1$ if

$$\frac{1}{p} \geq \frac{n}{2(n+1)}$$

Hence we should add this lower bound on $1/p$ which is harmless in the computation of the lowest s possible since such an s is reach by the upper bound of $1/p$.

1.5 Hyperbolic Sobolev spaces

The material from this section is based on the works [91] and [29]. The best way to introduce hyperbolic Sobolev spaces is for first order equations in time; thus before considering wave equation let us fist study the following Cauchy problem

$$\begin{cases} \partial_t u - i\phi(D)u = F \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^n) \end{cases}$$

where $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ is a unknown function and u_0 is a given initial data, and $\phi(D)$ is a Fourier multiplier:

$$\phi(D)u = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \phi(\xi)\widehat{u}(t,\xi)d\xi$$

and the function ϕ is called *dispersion relation* of the equation. For the Schrödinger equation we have $\phi(\xi) = |\xi|^2$ and for the linearized KdV equation we have $\phi(\xi) = i\xi^3$. The solution of the homogeneous $F = 0$ equation can be written in term of Fourier transform:

$$S(t)u_0 = \mathscr{F}_x^{-1}[e^{it\phi(\xi)}\widehat{u}_0(\xi)]$$

Moreover Duhamel's principle allow us to write the solution to the inhomogeneous problem as the following sum:

$$u = S(t)u_0 + \int_0^t S(t-s)F(s)ds$$

We examine the following question: given some integrability and differentiability properties of the initial data u_0 and the inhomogeneous force F, can we infer some integrability and differentiability properties of the solution u? To mesure these integrability and differentiability properties we introduce the following norm.

Definition. Let $s, b \in \mathbb{R}$. Define the space $H^{s,b}_\phi(\mathbb{R}^{1+n})$ to be the closure of the Schwartz functions $\mathscr{S}(\mathbb{R}^{1+n})$ under the norm

$$\|u\|_{H^{s,b}_\phi} = \|\langle\xi\rangle^s\langle\tau-\phi(\xi)\rangle^b\widetilde{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi}$$

where $\widetilde{u}$ is the space-time Fourier transform of u.

Notice that by the change of variable $\tau = \tau - \phi(\xi)$ we obtain:

$$\begin{aligned}\|u\|_{H^{s,b}_\phi} &= \|\langle\xi\rangle^s\langle\tau\rangle^b\widetilde{u}(\tau+\phi(\xi),\xi)\|_{L^2_\xi L^2_\tau} = \|\langle\xi\rangle^s\langle\tau\rangle^b\mathscr{F}_t[e^{-it\phi(\xi)}\widehat{u}(t,\xi)]\|_{L^2_\xi L^2_\tau}\\ &= \|\langle\xi\rangle^s e^{-it\phi(\xi)}\widehat{u}(t,\xi)\|_{L^2_\xi H^b_t}\end{aligned}$$

We cannot transform the $H^{s,b}_\phi$ into a pure $H^b_t H^s_\xi$ norm because of the factor $e^{-it\phi(\xi)}$. However we can use the multiplicative properties of Sobolev spaces: recall that $H^b(\mathbb{R})$ is an algebra for $b>1/2$, then we have the following lemma.

Lemma 11. *Assume $s\in\mathbb{R}$, $b>1/2$, $0<T<1$, and $\psi\in\mathscr{S}(\mathbb{R})$ then*

$$\|\psi_T(t)u\|_{H^{s,b}_\phi} \lesssim T^{1/2-b}\|u\|_{H^{s,b}_\phi}$$

here $\psi_T(t)=\psi(t/T)$ is a rescaling of ψ.

This results holds for function ψ which depends uniquely on time; for function which depends also on space multiplicative properties of $H^{s,b}_\phi$ spaces holds as well, but they are much harder to prove.

Proof. Use the previous characterization of the $H^{s,b}_\phi$ norm to obtain

$$\|\psi_T(t)u\|_{H^{s,b}_\phi} = \|\langle\xi\rangle^s e^{-it\phi(\xi)}\psi_T(t)\widehat{u}(t,\xi)\|_{L^2_\xi H^b_t}$$

It is easy to see that this is a product of two time-dependent functions. Then the multiplicative property of $H^b(\mathbb{R})$, which holds since $b>1/2$, yield to

$$\|\psi_T(t)u\|_{H^{s,b}_\phi} \le \|\psi_T(t)\|_{H^b_t}\|\langle\xi\rangle^s e^{-it\phi(\xi)}\widehat{u}(t,\xi)\|_{L^2_\xi H^b_t} \le T^{b-1/2}\|\psi\|_{H^b}\|u\|_{H^{s,b}_\phi}$$

since the scaling for the Sobolev spaces reads $\|\psi(t/T)\|_{H^b_t}\le T^{1/2-b}\|\psi(t)\|_{H^b_t}$, and for $T<1$ we have $\langle t/T\rangle \le T^{-1}\langle t\rangle$. The lemma then follows form the bound $\|\psi\|_{H^b}<\infty$. □

The reason why we have introduced $H^{s,b}_\phi$ spaces is to prove local well-posedness result for linear and nonlinear evolution equation such as KdV and Schrödinger equations. Therefore we will need the following embedding property $H^{s,b}_\phi \subset C(\mathbb{R},H^s)$. We will see that the previous embedding holds for $b>1/2$, but before let us prove the following proposition which gives a sufficient condition to obtain an embedding.

Proposition 12 (Transfer Principle). *Let Y be a Banach space of space-time functions on $\mathbb{R}^{1+n}$ with the property that, for any $\tau_0 \in \mathbb{R}$ and for every $u_0 \in L^2(\mathbb{R}^n)$ we have*

$$\| e^{it\tau_0} S(t) u_0 \|_{Y_{t,x}} \lesssim \| u_0 \|_{L^2(\mathbb{R}^n)}$$

Assume $b > 1/2$, then $H^{0,b} \subseteq Y$.

Proof. We need to show that $\|u\|_Y \lesssim \|u\|_{H^{0,b}}$ for any $u \in H^{0,b}$. Observe that the hypothesis can be written as

$$\| e^{it\tau_0} \int e^{ix\xi + it\phi(\xi)} \widehat{u}_0(\xi) d\xi \|_{Y_{t,x}} \lesssim \| \widehat{u}_0 \|_{L^2(\mathbb{R}^n)}$$

Furthermore form the inverse Fourier transform theorem and a simple change of variable we obtain

$$\begin{aligned} u(t,x) &= \iint e^{it\tau + ix\xi} \widetilde{u}(\tau,\xi)\, d\xi\, d\tau \\ &= \int e^{it\tau} \int e^{ix\xi + it\phi(\xi)} \widetilde{u}(\tau + \phi(\xi), \xi)\, d\xi\, d\tau \end{aligned}$$

Therefore Minkowski inequality and Cauchy-Schwarz yield to

$$\begin{aligned} \|u\|_{Y_{t,x}} &\lesssim \int \| e^{it\tau} \int e^{ix\xi + it\phi(\xi)} \widetilde{u}(\tau + \phi(\xi), \xi) d\xi \|_{Y_{t,x}} d\tau \\ &\lesssim \int \langle\tau\rangle^{-b} \langle\tau\rangle^{b} \| \widetilde{u}(\tau + \phi(\xi), \xi) \|_{L^2_\xi} d\tau \\ &\lesssim \| \langle\tau\rangle^{b} \widetilde{u}(\tau + \phi(\xi), \xi) \|_{L^2_\tau L^2_\xi} \end{aligned}$$

since $\|\langle\tau\rangle^{-b}\|_{L^2_\tau} < \infty$ if $b > 1/2$. □

Notice that a similar version of the transfer principle hold in the case $s \neq 0$. In fact we can show by a similar argument that the bound $\|e^{it\tau_0} S(t) u_0\|_{Y_{t,x}} \lesssim \|u_0\|_{H^s}$ implies $H^{s,b}_\phi \subseteq Y$, for $b > 1/2$. We are now ready to prove the aforementioned embedding into the solutions space.

Corollary 13. *Let $s \in \mathbb{R}$ and $b > 1/2$ then $H^{s,b}_\phi \subset C(\mathbb{R}, H^s)$.*

Proof. By the transfer principle it suffices to show that $\|e^{it(\tau_0 + \phi(D))} u_0\|_{C(\mathbb{R},H^s)} \lesssim \|u_0\|_{H^s}$ for every $u_0 \in H^s(\mathbb{R}^n)$ and $\tau_0 \in \mathbb{R}$. By the modulation invariance of the Lebesgue norms we obtain

$$\| \langle\xi\rangle^s e^{it(\tau_0 + \phi(\xi))} \widehat{u}_0(\xi) \|_{L^\infty_t L^2_\xi} \leq \| \langle\xi\rangle^s \widehat{u}_0(\xi) \|_{L^2_\xi} = \|u_0\|_{H^s(\mathbb{R}^n)}$$

□

Next, we prove the energy-type inequality in the context of $H^{s,b}_\phi$ spaces. Notice that since we have constructed the $H^{s,b}_\phi$ spaces to solve a first order PDE we do not loose any elliptic regularity exponent for the inhomogeneous estimate. On the other hand, we will see later that for second order problems, namely for wave equations, we will loose one degree of s in the inhomogeneous estimate.

Proposition 14. *Assume $s, b \in \mathbb{R}$, and $\psi \in \mathscr{S}(\mathbb{R})$.*

(i) *The following homogeneous estimate hold: let $0 < T < 1$ then*

$$\|\psi_T(t)S(t)u_0\|_{H_\phi^{s,b}} \lesssim T^{1/2-b}\|u_0\|_{H^s}$$

(ii) *Let $b > 1/2$ and $0 < T < 1$ then the following inhomogeneous estimate hold:*

$$\|\psi_T(t)\int_0^t S(t-s)f(s)ds\|_{H_\phi^{s,b}} \lesssim T^{1/2-b}\|f\|_{H_\phi^{s,b-1}}$$

Proof. We first show (i), recall that $\mathscr{F}_x[S(t)u_0] = e^{it\phi(\xi)}\widehat{u}_0(\xi)$, therefore

$$\|\psi_T(t)S(t)u_0\|_{H_\phi^{s,b}} = \|\langle\xi\rangle^s\langle\tau-\phi(\xi)\rangle^b\mathscr{F}_t[\psi_T(t)e^{it\phi(\xi)}](\tau)\widehat{u}_0(\xi)\|_{L_\xi^2 L_\tau^2}$$

However $\mathscr{F}_t[\psi_T(t)e^{it\phi(\xi)}](\tau) = \widehat{\psi_T}(\tau-\phi(\xi))$, thus if we perform the change of variable as before to eliminate the ξ dependence from the function ψ_T, we obtain

$$\|\psi_T(t)S(t)u_0\|_{H_\phi^{s,b}} = \|\psi_T\|_{H^b}\|u_0\|_{H^s} \lesssim T^{1/2-b}\|u_0\|_{H^s}$$

For (ii), by the definition of $H_\phi^{s,b}$ norm it follows that the left-hand side of (ii) equals to

$$\|\langle\xi\rangle^s\psi_T(t)\int_0^t e^{-is\phi(\xi)}\widehat{f}(s,\xi)ds\|_{L_\xi^2 H_t^b}$$

Thus to prove (ii) it suffices to prove that for any $g \in H^{b-1}(\mathbb{R})$

$$\|\psi_T(t)\int_0^t g(s)ds\|_{H_t^b} \lesssim T^{1/2-b}\|g\|_{H^{b-1}}$$

To eliminate the integral form 0 to t we apply the Fourier inversion theorem to obtain $\int_0^t g(s)ds = \int\int_0^t e^{is\sigma}\widehat{g}(\sigma)ds\,d\sigma = \int\frac{e^{it\sigma}-1}{i\sigma}\widehat{g}(\sigma)d\sigma$. We split the argument into two parts: suppose that $|\sigma| < 1$ in dσ-integral, then from Taylor's expansion we obtain

$$\begin{aligned}
\|\psi_T(t)\int_{|\sigma|<1}\frac{e^{it\sigma}-1}{i\sigma}\widehat{g}(\sigma)d\sigma\|_{H_t^b} &\leq \sum_{n=1}^{\infty}\|\psi_T(t)\int_{|\sigma|<1}\frac{(it\sigma)^n}{i\sigma n!}\widehat{g}(\sigma)d\sigma\|_{H_t^b} \\
&\leq \sum_{n=1}^{\infty}\left\|\frac{t^n}{n!}\psi_T(t)\right\|_{H_t^b}\left|\int_{|\sigma|<1}(i\sigma)^{n-1}\widehat{g}(\sigma)d\sigma\right| \\
&\leq T^{1/2-b}\left|\int_{|\sigma|<1}\frac{1}{1-i\sigma}\widehat{g}(\sigma)d\sigma\right| \\
&\leq T^{1/2-b}\|(1-i\sigma)^{-1}\langle\sigma\rangle^{1-b}\|_{L^2_{|\sigma|<1}}\|\langle\sigma\rangle^{b-1}\widehat{g}(\sigma)\|_{L^2} \\
&\leq T^{1/2-b}\|g\|_{H^{b-1}}
\end{aligned}$$

We have used Cauchy-Schwarz in the second to last line. Moreover notice that

$$\left\|\frac{t^n}{n!}\psi_T(t)\right\|_{H_t^b} = \frac{T^{1/2-b+n}}{n!}\|t^n\psi(t)\|_{H_t^b} \lesssim T^{1/2-b}$$

since $t^n\psi(t) \in \mathscr{S}(\mathbb{R})$, and $\|(1-i\sigma)^{-1}\langle\sigma\rangle^{1-b}\|_{L^2_{|\sigma|<1}} < \infty$. Notice that this estimate holds even if $b > 0$. Next, we need to control the region where $|\sigma| \geq 1$, we split the argument further into two parts:

$$\|\psi_T(t)\int_{|\sigma|\geq 1}\frac{e^{it\sigma}-1}{i\sigma}\widehat{g}(\sigma)d\sigma\|_{H_t^b} \leq \|\psi_T(t)\int_{|\sigma|\geq 1}\frac{\widehat{g}(\sigma)}{i\sigma}d\sigma\|_{H_t^b} + \|\psi_T(t)\int_{|\sigma|\geq 1}\frac{e^{it\sigma}}{i\sigma}\widehat{g}(\sigma)d\sigma\|_{H_t^b} =: I + II$$

For the I term we have by Cauchy-Schwarz inequality

$$I \leq \|\psi_T(t)\|_{H_t^b}\left|\int_{|\sigma|\geq 1}\frac{\widehat{g}(\sigma)}{i\sigma}d\sigma\right| \lesssim T^{1/2-b}\|g\|_{H^{b-1}}\|\sigma^{-1}\langle\sigma\rangle^{1-b}\|_{L^2_{|\sigma|\geq 1}}$$

Observe that $\|\langle\sigma\rangle^{-b}\|_{L^2_{|\sigma|\geq 1}}$ is bounded since $b > 1/2$. For the II term we have by the multiplicative properties of Sobolev spaces

$$II = \|\psi_T(t)\mathscr{F}_t^{-1}[\chi_{|\sigma|\geq 1}\frac{\widehat{g}(\sigma)}{i\sigma}]\|_{H_t^b} \leq \|\psi_T(t)\|_{H_t^b}\|\langle\sigma\rangle^b\chi_{|\sigma|\geq 1}\frac{\widehat{g}(\sigma)}{i\sigma}\|_{L^2_\sigma} \lesssim T^{1/2-b}\|g\|_{H^{b-1}}$$

The proof is then complete. □

We now adapt the hyperbolic Sobolev spaces introduced previously to a second order in time (i.e. wave equation) Cauchy problem such as

$$\begin{cases} \partial_{tt}u - \phi(D)u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n) \end{cases}$$

where $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is a unknown function and u_0, u_1 are given initial data and $\phi(D)$ is a Fourier multiplier. We set the *dispersion relation* of the equation to be $\phi(\xi) = |\xi|^2$ so that $\phi(D) = \triangle$ is the Laplacian operator.

Definition. Define the spacetime Fourier symbols

$$\begin{aligned} w_+(\tau,\xi) &= 1+||\tau|+|\xi|| \approx \langle|\tau|+|\xi|\rangle \approx \sqrt{1+\tau^2+|\xi|^2} \\ w_-(\tau,\xi) &= 1+||\tau|-|\xi|| \approx \langle|\tau|-|\xi|\rangle \approx 1+\frac{\tau^2-|\xi|^2}{1+\tau^2+|\xi|^2} \end{aligned}$$

and the associated spacetime Fourier multipliers $\widetilde{\Lambda^{\theta}_{\pm} u}(\tau,\xi) = w^{\theta}_{\pm}(\tau,\xi)\widetilde{u}(\tau,\xi)$. To keep the notation homogeneous we define $\widehat{\Lambda^s u}(\xi) = \langle\xi\rangle^s\widehat{u}(\xi)$.

In the literature different variations of hyperbolic Sobolev spaces have been proposed. For example Klainerman and Selberg in [44] and [45] used the equivalent norms

$$\|u\|_{H^{s,\theta}_1} = \|\Lambda^s\Lambda^{\theta}_{-}u\|_{L^2} + \|\Lambda^{s-1}\Lambda^{\theta}_{-}\partial_t u\|_{L^2}, \text{ and } \|u\|_{H^{s,\theta}_2} = \|\Lambda^{s-1}\Lambda_{+}\Lambda^{\theta}_{-}u\|_{L^2}$$

The fact that this two norms are equivalent follows from

$$\begin{aligned} \|u\|_{H^{s,\theta}_1} &\approx \int \langle\xi\rangle^{2s}\langle|\tau|-|\xi|\rangle^{2\theta}|\widetilde{u}(\tau,\xi)|^2 d\xi + \int \langle\xi\rangle^{2s-2}\langle|\tau|-|\xi|\rangle^{2\theta}|\widetilde{\partial_t u}(\tau,\xi)|^2 d\xi \\ &= \int \langle\xi\rangle^{2s-2}\big(|\tau|^2+\langle\xi\rangle^2\big)\langle|\tau|-|\xi|\rangle^{2\theta}|\widetilde{u}(\tau,\xi)|^2 d\xi \\ &\approx \int \langle\xi\rangle^{2s-2}\langle|\tau|+|\xi|\rangle^{2}\langle|\tau|-|\xi|\rangle^{2\theta}|\widetilde{u}(\tau,\xi)|^2 d\xi = \|u\|_{H^{s,\theta}_2} \end{aligned}$$

On the other hand Geba and Grillakis [29] use a stronger version of hyperbolic Sobolev spaces:

$$\|u\|_{H^{s,\theta}_3} = \|\Lambda^s_{+}\Lambda^{\theta}_{-}u\|_{L^2}$$

clearly we have $\|u\|_{H^{s,\theta}_2} \leq \|u\|_{H^{s,\theta}_3}$, therefore we obtain the following embedding $H^{s,\theta}_3 \subset H^{s,\theta}_2 \approx H^{s,\theta}_1$. In the context of wave equation we will work with the following definition:

Definition. Let

$$H^{s,\theta}(\mathbb{R}^{1+n}) = \{u \in \mathscr{S}(\mathbb{R}^{1+n}) : \Lambda^s\Lambda^{\theta}_{-}u \in L^2(\mathbb{R}^{1+n})\}$$

and we endow it with the natural norm

$$\|u\|_{H^{s,\theta}} = \|\Lambda^s\Lambda^{\theta}_{-}u\|_{L^2} = \|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^{\theta}\widetilde{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi}$$

In the case $\theta = 0$ the $H^{s,0}$ space is simply $L^2_t H^s_x$, and if both $s = \theta = 0$ then $H^{0,0} = L^2(\mathbb{R}^{1+n})$. Moreover we have the trivial nesting $H^{s_1,\theta_1} \subseteq H^{s_2,\theta_2}$ if $s_1 \geq s_2$ and $\theta_1 \geq \theta_2$, as usual the bigger the exponent is the smaller the space will be. Notice the similarity with the $H^{s,b}_{\phi}$ space defined previously, however here we have the absolute value of τ which complicate things a bit. Moreover observe that any $u \in H^{s,\theta}$ has a unique decomposition $u = u_- + u_+$ where $u_-, u_+ \in H^{s,\theta}$ and $\operatorname{supp}\widetilde{u}_{\pm} \subseteq \mathbb{R}^{\pm}\times\mathbb{R}^n$. Such decomposition is obtained by multiplying on the Fourier side

u by a smooth cutoff function in time supported in the desired region: $\widetilde{u}_\pm(\tau,\xi) = \chi_{\pm\tau\geq 0}\widetilde{u}(\tau,\xi)$. We have the following trivial property of the wave-Sobolev norm: $\|u\|_{H^{s,\theta}} = \|u_+\|_{H^{s,\theta}} + \|u_-\|_{H^{s,\theta}}$ and

$$\begin{aligned} \|u_\pm\|_{H^{s,\theta}} &= \|\langle\xi\rangle^s\langle\tau\mp|\xi|\rangle^\theta\widetilde{u}_\pm(\tau,\xi)\|_{L^2_\tau L^2_\xi} = \|\langle\xi\rangle^s\langle\tau\rangle^\theta\widetilde{u}_\pm(\tau\pm|\xi|,\xi)\|_{L^2_\xi L^2_\tau} \\ &= \|\langle\xi\rangle^s\langle\tau\rangle^\theta\widetilde{u}_\pm(\tau\pm|\xi|,\xi)\|_{L^2_\xi L^2_\tau} = \|\langle\xi\rangle^s e^{\pm it|\xi|}\widehat{u}_\pm(t,\xi)\|_{H^\theta_t L^2_\xi} \end{aligned}$$

Notice that the decomposition into u_-, u_+ allow us to eliminate the absolute value on τ and to perform a change of variable in the same spirit as in the previous section.

Remark (Why such spaces are called wave-Sobolev space?). Recall that Sobolev spaces are well-behaved with respect to the laplacian, precisely if $u \in H^s(\mathbb{R}^n)$ then $\triangle u \in H^{s-2}(\mathbb{R}^n)$. Furthermore, in a standard Sobolev space H^s one can differentiate using the elliptic derivative Λ^s s-times and still remain square integrable. For the wave-Sobolev space $H^{s,\theta}$ one can differentiate using s-times the elliptic derivative Λ^s and θ-times the dispersive derivative Λ_-^θ and still remain square integrable. Furthermore the d'Alembertian operator maps the space $H_3^{s,\theta}$ into $H_3^{s-1,\theta-1}$, hence we say that the d'Alembertian operator make us loose an elliptic derivative and an hyperbolic derivative. That is

$$\|\Box u\|_{H_3^{s-1,\theta-1}} = \|\Lambda_+^{s-1}\Lambda_-^{\theta-1}(\tau^2-|\xi|^2)\widehat{u}\|_{L^2} \leq \|\Lambda_+^{s-1}\Lambda_-^{\theta-1}\Lambda_-\Lambda_+\widehat{u}\|_{L^2} = \|u\|_{H_3^{s,\theta}}$$

It is not obvious that the inverse of the d'Alembertian maps $H^{s-1,\theta-1}$ into $H^{s,\theta}$. We will show below that this indeed is the case.

There is a remarkably connection between the wave-Sobolev spaces $H^{s,\theta}$ and the space of solutions of the homogeneous wave equations with initial data in H^s. In effect every element of $H^{s,\theta}$ may be thought of as superposition of half-waves with initial data in H^s. Next proposition clarifies this claim.

Proposition 15 (Integral representation)**.** *Let $u \in H^{s,\theta}$, then $u = u_+ + u_-$ where $\widetilde{u_\pm}$ is supported in $\mathbb{R}^\pm \times \mathbb{R}^n$ and there exist $f_+, f_- \in L^2(\mathbb{R}, H^s)$ such that*

$$u_\pm(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{it(\rho\pm D)}f_\pm(\rho)}{(1+|\rho|)^\theta}d\rho \tag{1.4}$$

Notice that $f_\pm : \rho \in \mathbb{R} \to f(\rho,\cdot) \in H^s(\mathbb{R}^n)$ and $\widehat{f}_\pm(\rho,\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}f_\pm(\rho,x)dx$. Form the proof will follow that $\widehat{f}_+(\rho,\xi) = 0$ for $|\xi|+\rho<0$, and $\widehat{f}_-(\rho,\xi) = 0$ for $|\xi|+\rho>0$. Moreover $\|u\|_{H^{s,\theta}(\mathbb{R}^{1+n})} \leq \|f_+\|^2_{L^2(\mathbb{R},H^s)} + \|f_-\|^2_{L^2(\mathbb{R},H^s)}$.

Proof. First recall that any $u \in H^{s,\theta}$ has a unique decomposition $u = u_- + u_+$ where $u_-, u_+ \in$

$H^{s,\theta}$ and supp $\widetilde{u}_\pm \subseteq \mathbb{R}^\pm \times \mathbb{R}^n$. Define the functions $\widehat{f}_\pm(\rho,\xi) = \langle\rho\rangle^\theta \widetilde{u}_\pm(\rho \pm |\xi|,\xi)$, that is

$$f_\pm(\rho,x) = \int_{\mathbb{R}^n} e^{i\xi\cdot x}\langle\rho\rangle^\theta \widetilde{u}_\pm(\rho\pm|\xi|,\xi)\,d\xi$$

We must prove that $f_\pm \in L^2(\mathbb{R},H^s)$ and the integral representation (1.4) holds. Observe that

$$\begin{aligned}\|f_\pm\|^2_{L^2(\mathbb{R},H^s)} &= \|\langle\xi\rangle^s \widehat{f}_\pm(\rho,\xi)\|_{L^2_\rho L^2_\xi} = \|\langle\xi\rangle^s\langle\rho\rangle^\theta \widetilde{u}_\pm(\rho\pm|\xi|,\xi)\|_{L^2_\rho L^2_\xi}\\ &= \|\langle\xi\rangle^s\langle\tau\mp|\xi|\rangle^\theta \widetilde{u}_\pm(\tau,\xi)\|_{L^2_\tau L^2_\xi} \le \|u_\pm\|_{H^{s,\theta}} < \infty\end{aligned}$$

here we just have perform the change of variable $\tau = \rho \pm |\xi|$ in the $d\tau$ integral and use the fact that supp $\widetilde{u}_\pm \subseteq \mathbb{R}^\pm \times \mathbb{R}^n$, therefore $\langle\tau\mp|\xi|\rangle^\theta \widetilde{u}_\pm = \langle|\tau|-|\xi|\rangle^\theta \widetilde{u}_\pm$. Since $\langle\tau-|\xi|\rangle^\theta \widetilde{u}_+$ has support in $\tau > 0$, then $\langle\tau-|\xi|\rangle^\theta \widetilde{u}_+ = \langle|\tau|-|\xi|\rangle^\theta \widetilde{u}_+$. Moreover $\langle\tau+|\xi|\rangle^\theta \widetilde{u}_-$ has support in $\tau < 0$, thus $\langle\tau+|\xi|\rangle^\theta \widetilde{u}_- = \langle-|\tau|+|\xi|\rangle^\theta \widetilde{u}_-$, hence $f_\pm \in L^2(\mathbb{R},H^s)$. Furthermore notice that

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{it(\rho\pm|\xi|)}\widehat{f}_\pm(\rho,\xi)}{(1+|\rho|)^\theta}d\rho &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{it(\rho\pm|\xi|)}\widetilde{u}_\pm(\rho\pm|\xi|,\xi)d\rho\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau}\widetilde{u}_\pm(\tau,\xi)d\tau = \widehat{u}(t,\xi)\end{aligned}$$

by the same change of variable $\tau = \rho \pm |\xi|$. Finally notice that

$$\|u\|^2_{H^{s,\theta}} \le \|u_+\|^2_{H^{s,\theta}} + \|u_-\|^2_{H^{s,\theta}} \le \|f_+\|^2_{L^2(\mathbb{R},H^s)} + \|f_-\|^2_{L^2(\mathbb{R},H^s)}$$

□

An immediate corollary of the integral representation is that every function in $H^{s,\theta}$ can be seen as superposition of half-waves.

Corollary 16 (Superposition principle, [45]). *Let $u \in H^{s,\theta}$. Then*

$$u = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i\rho t}u_\rho}{\langle\rho\rangle^\theta}d\rho$$

where $\{u_\rho\}_{\rho\in\mathbb{R}}$ is a one-parameter family of solutions of the Cauchy initial value problem

$$\begin{cases}\Box u_\rho = 0\\ (u_\rho,\partial_t u_\rho)(0) = (f_\rho,0)\end{cases}$$

and $\rho \in \mathbb{R} \to f_\rho \in H^s(\mathbb{R}^n)$

Proof. Recall that the solution of an homogeneous wave equation with data $(f_\rho,0)$ is given by a combination of half waves. Moreover by the integral representation, Proposition 15 we can take $(f_\rho, 0) = (f_\pm(\rho), 0)$ as initial data for the homogeneous wave equation. Then $u_\rho = e^{\pm itD} f_\rho$. □

An important tool to prove embedding of wave-Sobolev spaces is the *transfer principle*. Loosely speaking the transfer principle state that every multilinear spacetime estimate for solution of the homogeneous wave equation with initial data $(u_0, 0)$ with $u_0 \in H^s$, implies a corresponding embedding for $H^{s,\theta}$ spaces.

Proposition 17 (Transfer principle)**.** *Let Y be a Banach space of functions on $\mathbb{R}^{1+n}$ with the property that, for every $\tau_0 \in \mathbb{R}$ and for every $u_0 \in H^s$ we have $\|e^{it(\tau_0 \pm D)} u_0\|_{Y_{t,x}} \lesssim \|u_0\|_{H^s}$. If $\theta > 1/2$, then $H^{s,\theta} \subseteq Y$, that is we have $\|u\|_{Y_{t,x}} \lesssim \|u\|_{H^{s,\theta}}$ for every $u \in Y$.*

Notice that if we define the half wave propagator $S_\pm(t)u_0 = \mathscr{F}_\xi^{-1}[e^{\pm it|\xi|}\widehat{u}_0(\xi)]$, then the transfer principle resembles closely the transfer principle for first order equation, Proposition 12.

Proof. Observe that the hypothesis can be written as

$$\|e^{it\tau_0} \int e^{ix\xi \pm it|\xi|} \widehat{u}_0(\xi)\, d\xi\|_{Y_{t,x}} \lesssim \|\langle\xi\rangle^s \widehat{u}_0(\xi)\|_{L^2_\xi(\mathbb{R}^n)}$$

Furthermore form the inverse Fourier transform theorem and change of variable we have

$$\begin{aligned} u(t,x) &= \iint e^{it\tau + ix\xi} \big(\widetilde{u}_+(\tau,\xi) + \widetilde{u}_-(\tau,\xi)\big)\, d\xi\, d\tau \\ &= \int e^{it\tau} \int e^{ix\xi + it|\xi|} \widetilde{u}_+(\tau + |\xi|, \xi)\, d\xi\, d\tau + \int e^{it\tau} \int e^{ix\xi - it|\xi|} \widetilde{u}_-(\tau - |\xi|, \xi)\, d\xi\, d\tau \end{aligned}$$

here we have used the decoupling $\widetilde{u}(\tau,\xi) = \widetilde{u}_+(\tau,\xi) + \widetilde{u}_-(\tau,\xi) = \chi_{\tau\geq 0}\widetilde{u}(\tau,\xi) + \chi_{\tau<0}\widetilde{u}(\tau,\xi)$. Therefore Minkowski inequality and Cauchy-Schwarz yield to

$$\begin{aligned} \|u\|_{Y_{t,x}} &\lesssim \int \langle\tau\rangle^{-\theta}\langle\tau\rangle^{\theta} \|\langle\xi\rangle^s \widetilde{u}_+(\tau + |\xi|, \xi)\|_{L^2_\xi}\, d\tau + \int \langle\tau\rangle^{-\theta}\langle\tau\rangle^{\theta} \|\langle\xi\rangle^s \widetilde{u}_-(\tau - |\xi|, \xi)\|_{L^2_\xi}\, d\tau \\ &\lesssim \|\langle\xi\rangle^s \langle\tau\rangle^{\theta} \widetilde{u}_+(\tau + |\xi|, \xi)\|_{L^2_\tau L^2_\xi} + \|\langle\xi\rangle^s \langle\tau\rangle^{\theta} \widetilde{u}_-(\tau - |\xi|, \xi)\|_{L^2_\tau L^2_\xi} \\ &= \|\langle\xi\rangle^s \langle\tau - |\xi|\rangle^{\theta} \widetilde{u}_+(\tau, \xi)\|_{L^2_\tau L^2_\xi} + \|\langle\xi\rangle^s \langle\tau + |\xi|\rangle^{\theta} \widetilde{u}_-(\tau, \xi)\|_{L^2_\tau L^2_\xi} \\ &\lesssim \|\langle\xi\rangle^s \langle|\tau| - |\xi|\rangle^{\theta} \widetilde{u}(\tau, \xi)\|_{L^2_\tau L^2_\xi} \end{aligned}$$

since $\|\langle\tau\rangle^{-\theta}\|_{L^2_\tau} < \infty$ for $\theta > 1/2$. □

Observe that if we take $Y = C_b(\mathbb{R}, H^s)$, and $\|u\|_Y = \sup_{t\in\mathbb{R}} \|u(t)\|_{H^s}$. Then we have the estimate

$$\|e^{it(\tau_0 \pm D)} u_0\|_Y = \sup_{t\in\mathbb{R}} \|\langle\xi\rangle^s e^{it(\tau_0 \pm |\xi|)} \widehat{u}_0(\xi)\|_{L^2(\mathbb{R}^n)} \leq \sup_{t\in\mathbb{R}} \|\langle\xi\rangle^s \widehat{u}_0(\xi)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{H^s(\mathbb{R}^n)}$$

for every $\tau_0 \in \mathbb{R}$. Hence Proposition 17 implies $H^{s,\theta} \subset C_b(\mathbb{R}, H^s)$ for $\theta > 1/2$. Moreover since

$$\|u\|_{H_1^{s,\theta}} = \|u\|_{H^{s,\theta}} + \|\partial_t u\|_{H^{s-1,\theta}}$$

then we also have the embedding into the solution space $H_1^{s,\theta} \subset C_b(\mathbb{R}, H^s) \cap C_b^1(\mathbb{R}, H^{s-1})$ for $\theta > 1/2$. Furthermore observing that the space $L_t^q L_x^r$ are invariant under modulation, multiplication by phases, one conclude that

$$\|e^{it(\tau_0 \pm D)} u_0\|_{L_t^q L_x^r} = \|e^{it\tau_0} \mathscr{F}_\xi^{-1}(e^{it|\xi|}\widehat{u}_0(\xi))\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L_t^q L_x^r}$$

for every $\tau_0 \in \mathbb{R}$. Therefore we have proved the following:

Corollary 18 (Embedding Properties)**.** *If* $\theta > 1/2$ *then*

(i) $H^{s,\theta} \subset C_b(\mathbb{R}, H^s)$,

(ii) $H_3^{s,\theta} \subset H_2^{s,\theta} \approx H_1^{s,\theta} \subset C_b(\mathbb{R}, H^s) \cap C_b^1(\mathbb{R}, H^{s-1})$

(ii) $H^{s,\theta} \subset L^q L^r$*, for every wave admissible Strichartz triplet* (q, r, s).

We know that the solution to the homogeneous Cauchy problem for the wave equation is given by

$$\mathscr{H}(u_0, u_1) = \frac{1}{2}(e^{itD} + e^{-itD})u_0 + \frac{1}{2i}(e^{itD} - e^{-itD})D^{-1}u_1$$

where $D = \sqrt{-\triangle}$. Since the Fourier transform of the exponential $h(t) = e^{\pm it\rho}$, with $\rho \in \mathbb{R}$ is the delta measure $\widehat{h}(\tau) = 2\pi\delta(\tau \mp \rho)d\tau$, we conclude that the space-time Fourier transform of $\mathscr{H}(u_0, u_1)$ is the measure

$$\mathscr{F}[\mathscr{H}(u_0, u_1)](\tau, \xi) = \pi\delta(\tau - |\xi|)\big(\widehat{u}_0(\xi) - i|\xi|^{-1}\widehat{u}_1(\xi)\big)d\tau d\xi + \pi\delta(\tau + |\xi|)\big(\widehat{u}_0(\xi) + i|\xi|^{-1}\widehat{u}_1(\xi)\big)d\tau d\xi$$

Next proposition shows that if one localise in time then one obtains $\mathscr{H}(u_0, u_1) \in H^{s,\theta}$, for $(u_0, u_1) \in H^s \times H^{s-1}$. Define $H_T^{s,\theta}(\mathbb{R}^{1+n}) = \{\chi_T u : u \in H^{s,\theta}(\mathbb{R}^{1+n})\}$, where $\chi \in C_0^\infty(\mathbb{R})$ is a smooth cutoff with compact support and $\chi_T(t) = \chi(t/T)$. Clearly $\|u\|_{H_T^{s,\theta}} = \|\chi_T u\|_{H^{s,\theta}}$.

Proposition 19 (Homogeneous $H^{s,\theta}$ estimates)**.** *Let* $f \in H^s(\mathbb{R}^n)$*,* $g \in H^{s-1}(\mathbb{R}^n)$*, and* $\theta > 0$ *then*

$$\|\mathscr{H}(u_0, u_1)\|_{H_T^{s,\theta}(\mathbb{R}^{1+n})} \lesssim T^{\frac{1}{2}-\theta}\|(u_0, u_1)\|_{H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)}$$

Proof. Recall that the Fourier transform of $k(t) = \chi_T(t)e^{\pm it\rho}$, where $\rho \in \mathbb{R}$, is the Schwartz function $\widehat{k}(\tau) = 2\pi T\widehat{\chi_{1/T}}(\tau \mp \rho)$. Therefore the space-time Fourier transform of $\chi_T(t)e^{\pm itD}f$ is

$2\pi T\widehat{\chi_{1/T}}(\tau \mp |\xi|)\widehat{f}(\xi)$, thus when $\rho = |\xi|$ we obtain

$$\begin{aligned}
\|\chi_T(t)e^{\pm itD}f\|_{H^{s,\theta}} &\approx T\|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^\theta\widehat{\chi_{1/T}}(\tau\mp|\xi|)\widehat{f}(\xi)\|_{L^2_\tau L^2_\xi} \\
&\lesssim T\|\langle\xi\rangle^s\langle\tau\mp|\xi|\rangle^\theta\widehat{\chi_{1/T}}(\tau\mp|\xi|)\widehat{f}(\xi)\|_{L^2_\tau L^2_\xi} \\
&\le T\|\langle\xi\rangle^s\langle\tau\rangle^\theta\widehat{\chi_{1/T}}(\tau)\widehat{f}(\xi)\|_{L^2_\tau L^2_\xi} \\
&\le T\|\chi_{1/T}\|_{H^\theta}\|f\|_{H^s}
\end{aligned}$$

Notice that $\|\chi_{1/T}\|_{H^\theta} = T^{-\frac{1}{2}-\theta}\|\chi\|_{H^\theta} \lesssim T^{-\frac{1}{2}-\theta}$ since $\widehat{\chi} \in \mathscr{S}(\mathbb{R})$. Therefore clearly $\|\chi_T(t)\cos(tD)f\|_{H^{s,\theta}} \lesssim T^{\frac{1}{2}-\theta}\|f\|_{H^s}$.

Next, we show that $\|\chi_T(t)D^{-1}\sin(tD)g\|_{H^{s,\theta}} \lesssim T^{\frac{1}{2}-\theta}\|g\|_{H^{s-1}}$. We split the argument into two parts: first assume $|\xi| \ge 1$, then $\langle\xi\rangle \approx |\xi|$ therefore applying the previous estimate we obtain

$$\|\chi_T(t)e^{\pm itD}|D|^{-1}g\|_{H^{s,\theta}} \le T\|\chi_{1/T}\|_{H^\theta}\|g\|_{H^{s-1}}$$

Let consider the remaining case of low frequencies, $|\xi| \le 1$; this case is worst then the previous one because of the factor $|\xi|$ in the denominator which makes the symbol explode as ξ tends to zero. However one can write the symbols as

$$\chi_T(t)D^{-1}\sin(tD) = \frac{1}{2}\int_{-1}^{1} t\chi_T(t)e^{it\rho D}\,d\rho$$

and recall that the space-time Fourier transform of $\chi_T(t)e^{it\rho D}g$ is $T\widehat{\chi_{1/T}}(\tau-\rho|\xi|)\widehat{g}(\xi)$. However, integrating this with respect to $d\rho$ will lead to an elliptic integral, thus we first perform the integrals $d\tau$ and $d\xi$. Since $\widehat{\chi} \in \mathscr{S}(\mathbb{R})$ we obtain

$$\begin{aligned}
\|\chi_T(t)D^{-1}\sin(tD)g\|_{H^{s,\theta}} &\lesssim T\int_{-1}^{1}\|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^\theta\widehat{\chi_{1/T}}(\tau-\rho|\xi|)\widehat{g}(\xi)\|_{L^2_\tau L^2_\xi}d\rho \\
&\lesssim T\sup_{|\rho|\le 1}\|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^\theta\widehat{\chi_{1/T}}(\tau-\rho|\xi|)\widehat{g}(\xi)\|_{L^2_\tau L^2_\xi} \\
&\lesssim T\|\chi_{1/T}\|_{H^\theta}\|g\|_{H^{s-1}}
\end{aligned}$$

Above we have used the bound $\langle|\tau|-|\xi|\rangle \lesssim \langle\tau-\rho|\xi|\rangle$ which holds for $\rho \in [-1,1]$ and $|\xi| < 1$. Observe that we have perform a change of variable $\widetilde{\tau} = \tau - \rho|\xi|$ in the $d\tau$ integral killing the ρ and ξ dependence of the norm. Moreover, in this case we trow away a power of $\langle\xi\rangle$ to infer the desired bound since for $|\xi| \le 1$ we have $\langle\xi\rangle \lesssim 1$. This conclude the proof. □

We now turn to the inhomogeneous solution map:

$$\Box^{-1}F = \frac{1}{2i}\int_0^t (e^{itD}-e^{-itD})D^{-1}F(s)ds$$

Notice that the symbol of the wave operator vanishes on the light cone in Fourier space-time thus if one naively takes the space-time Fourier transform of $\Box u = F$, one obtain

$$\widetilde{\Box^{-1}F}(\tau,\xi) = (|\tau|^2 - |\xi|^2)^{-1}\widetilde{F}(\tau,\xi)$$

If $\widetilde{F}$ is nonzero and continuous on the light cone then $\widetilde{\Box^{-1}F}$ is evidently not tempered. Therefore, we will split the inhomogeneity in two parts F_1 and F_2 such that $F = \phi(\Lambda_-)F + \phi(1-\Lambda_-)F =: F_1 + F_2$, where $\phi \in C_0^\infty(\mathbb{R})$ is a smooth cutoff with compact support included in $[-1,1]$ and $\phi = 1$ on $[-1/2, 1/2]$. Note that by definition $\widetilde{F}_1 = \phi(w_-)\widetilde{F}$, and $\widetilde{F}_2 = (1-w_-))\widetilde{F}$. Hence the support of $\widetilde{F}_1$ is concentrated near the light cone while the support of $\widetilde{F}_2$ is dispersed far from the light cone. More precisely define the neighborhood of the light cone in the Fourier side as

$$\mathcal{N} = \{(\tau,\xi) \in \mathbb{R}^{1+n} \; : \; ||\tau| - |\xi|| \lesssim 1\}$$

then $\text{supp}\widetilde{F}_1 \subseteq \mathcal{N}$ and $\text{supp}\widetilde{F}_2 \subseteq \mathbb{R}^{1+n} \setminus \mathcal{N}$. The solution of the inhomogeneous problem is given by the sum of two function u_1 and u_2, such that

$$\begin{cases} \Box u_1 = F_1 & \text{in } \mathbb{R}^n \\ (u_1, \partial_t u_1)\Big|_{t=0} = (0,0) \end{cases} \qquad \begin{cases} \Box u_2 = F_2 & \text{in } \mathbb{R}^n \\ (u_2, \partial_t u_2)\Big|_{t=0} = (0,0) \end{cases}$$

As the next proposition show, on can easily invert the piece of the wave operator when the support of $\widetilde{F}$ is dispersed far from the light cone.

Proposition 20 (Invertible-inhomogeneous estimates)**.** *Let $F \in H^{s-1,\theta-1}(\mathbb{R}^{1+n})$, with supp$\widetilde{F} \subseteq \mathbb{R}^{1+n} \setminus \mathcal{N}$ then*

$$\|\Box^{-1}F\|_{H^{s,\theta}} \lesssim \|F\|_{H^{s-1,\theta-1}}$$

Or equivalently, the operator $\Box^{-1}(1-\phi(\Lambda_-)) : H^{s-1,\theta-1} \to H^{s,\theta}$ is bounded.

Proof. By straightforward calculations we obtain

$$\begin{aligned} \|\Box^{-1}F\|_{H^{s,\theta}} &= \|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^\theta(\tau^2 - |\xi|^2)^{-1}\widetilde{F}(\tau,\xi)\|_{L^2_\tau L^2_\xi} \\ &\lesssim \|\langle\xi\rangle^{s-1}\langle|\tau|-|\xi|\rangle^{\theta-1}\widetilde{F}(\tau,\xi)\|_{L^2_\tau L^2_\xi} \\ &= \|F\|_{H^{s-1,\theta-1}} \end{aligned}$$

□

Observe that we don't have to cutoff in a time slice for this term of the solution. The operator $\Box^{-1}(1-\phi(\Lambda_-))$ may be though of as the invertible component of the wave operator. What is remarkable is that a similar result holds for the inhomogeneous part which support is concentrated near the light cone. However, in this case we must cutoff in a time slice, see [91].

Proposition 21 (Not-invertible-inhomogeneous estimates)**.** *Let $F \in H^{s-1,\theta-1}(\mathbb{R}^{1+n})$ with supp$\widetilde{F} \subseteq$*

$\mathcal{N}$, and $\theta > 0$ then

$$\|\Box^{-1}F\|_{H^{s,\theta}_T} \lesssim T^{\frac{1}{2}-\theta}\|F\|_{H^{s-1,\theta}}$$

Or equivalently, the operator $\chi_T\Box^{-1}\phi(\Lambda_-) : H^{s-1,\theta} \to H^{s,\theta}$ is bounded.

Proof. Observe that since the support of $\widetilde{F}(\tau,\xi)$ is included in the set $||\tau|-|\xi|| \lesssim 1$, then $w_-(\tau,\xi) \approx 1$, this implies that $\|F\|_{H^{s-1,0}} \approx \|F\|_{H^{s-1,\theta-1}}$. Consider for now *large frequencies*: $|\xi| \geq 1$, then $\langle D\rangle \approx |D|$, moreover notice that $|F(s)| \leq \int |\check{F}(\sigma)|d\sigma$. Hence

$$\begin{aligned}
\chi_T(t)\Box^{-1}F(t) &= \int\int_0^t \chi_T(t)\langle D\rangle^{-1}\sin((t-s)D)|\check{F}(\sigma)|\,ds\,d\sigma \\
&= \int \chi_T(t)\langle D\rangle^{-2}[1-\cos(tD)]|\check{F}(\sigma)|d\sigma
\end{aligned}$$

We need to recall that $\mathscr{F}_t[\chi_T(t)\cos(tD)] = \pi T[\widehat{\chi}_{1/T}(\tau-D)+\widehat{\chi}_{1/T}(\tau+D)]$. Then we compute $\|\chi_T(t)\Box^{-1}F\|_{H^{s,\theta}}$, which lead to estimate the follow three terms:

$$\begin{aligned}
I+II+III &= \int \|\langle\xi\rangle^{s-2}\langle|\tau|-|\xi|\rangle^{\theta}\langle\tau\rangle^{-N}\widetilde{F}(\sigma,\xi)\|_{L^2_\tau L^2_\xi}d\sigma \\
&+ \int \|\langle\xi\rangle^{s-2}\langle|\tau|-|\xi|\rangle^{\theta}\langle\tau-|\xi|\rangle^{-N}\widetilde{F}(\sigma,\xi)\|_{L^2_\tau L^2_\xi}d\sigma \\
&+ \int \|\langle\xi\rangle^{s-2}\langle|\tau|-|\xi|\rangle^{\theta}\langle\tau+|\xi|\rangle^{-N}\widetilde{F}(\sigma,\xi)\|_{L^2_\tau L^2_\xi}d\sigma
\end{aligned}$$

To get rid of the $d\sigma$-integral in the second and third terms we multiply and divide by $\langle\sigma\rangle$ and apply Cauchy-Schwarz inequality. Moreover we use the bound $\langle|\tau|-|\xi|\rangle \leq \langle\tau-|\xi|\rangle$ to control the Fourier multipliers. This leads us to the estimate

$$II+III \lesssim \|\langle\xi\rangle^{s-2}\langle\tau\pm|\xi|\rangle^{\theta-N}\langle\sigma\rangle\widetilde{F}(\sigma,\xi)\|_{L^2_\sigma L^2_\tau L^2_\xi}$$

Notice that $\langle\sigma\rangle \lesssim \langle|\sigma|-|\xi|\rangle\langle\xi\rangle \lesssim \langle\xi\rangle$ since the support of $\widetilde{F}(\sigma,\xi)$ is inside the set defined via $||\sigma|-|\xi|| \lesssim 1$. Therefore if we make the change of variable $\widetilde{\tau} = \tau \pm |\xi|$ in $d\tau$-integral we obtain

$$II+III \lesssim \|\langle\xi\rangle^{s-1}\widetilde{F}(\sigma,\xi)\|_{L^2_\sigma L^2_\xi} = \|F\|_{H^{s-1,0}}$$

To control I we proceed as above to obtain

$$I \lesssim \|\langle\xi\rangle^{s-1}\langle\tau-|\xi|\rangle^{\theta-N}\langle\tau\rangle^{-N}\widetilde{F}(\sigma,\xi)\|_{L^2_\sigma L^2_\tau L^2_\xi}$$

Suppose $2\tau > |\xi|$, then if we perform the change of variable $\widetilde{\tau} = \tau - |\xi|$ in $d\tau$ integral we get $\langle\widetilde{\tau}\rangle^{\theta-N}\langle\widetilde{\tau}+|\xi|\rangle^{-N} \lesssim \langle\widetilde{\tau}\rangle^{\theta-2N} \in L^2_{\widetilde{\tau}}$ since we have $\widetilde{\tau} > -1/2|\xi|$ which implies $\langle\widetilde{\tau}+|\xi|\rangle \geq \langle\widetilde{\tau}\rangle$. On the other hand if $2\tau \leq |\xi|$ then we use the bound $\langle\tau-|\xi|\rangle \gtrsim \langle\tau\rangle$ to obtain $\langle\tau-|\xi|\rangle^{\theta-N}\langle\tau\rangle^{-N} \lesssim \langle\tau\rangle^{\theta-2N} \in L^2_\tau$. Therefore

$$I \lesssim \|\langle\xi\rangle^{s-1}\widetilde{F}(\sigma,\xi)\|_{L^2_\sigma L^2_\xi} = \|F\|_{H^{s-1,0}}$$

Let us now study the *low frequencies* regime $|\xi| \le 1$, this case is worst then the previous one because of the factor $|\xi|$ in the denominator which makes the symbol explode as ξ tends to zero. However as in the proof of the homogeneous estimate one can rewrite the symbol as

$$D^{-1}\sin((t-s)D) = \frac{1}{2}\int_{-1}^{1}(t-s)e^{i(t-s)\rho D}\,d\rho$$

Moreover by the Fourier inversion formula $F(s) = \frac{1}{2\pi}\int e^{is\sigma}\check{F}(\sigma)d\sigma$ we infer

$$\begin{aligned}\chi_T(t)\Box^{-1}F(t) = &= \frac{1}{4\pi}\int_{-1}^{1}\int\Big(\Big(\int_0^t (t-s)e^{is(\sigma-\rho D)}ds\Big)\chi(t)e^{it\rho D}\check{F}(\sigma)d\sigma d\rho\\ &= \frac{1}{4\pi}\int_{-1}^{1}\int\Big(\frac{\chi_T(t)e^{it\rho D}}{(\sigma-\rho D)^2} - \frac{\chi_T(t)e^{it\sigma}}{(\sigma-\rho D)^2} - \frac{t\chi_T(t)e^{it\rho D}}{i(\sigma-\rho D)}\Big)\check{F}(\sigma)d\sigma d\rho\end{aligned}$$

Next we compute the Fourier transform. Recall that the time Fourier transform of $\chi_T(t)e^{\pm it\rho}$ is $2\pi T\widehat{\chi}_{1/T}(\tau\mp\rho) \lesssim_T \langle\tau\mp\rho\rangle^{-N}$, for a sufficiently large $N\in\mathbb{N}$. Thus

$$\mathscr{F}_t[\chi_T(t)\Box^{-1}F(t)](\tau) \lesssim_T \int_{-1}^{1}\int\Big(\frac{\langle\tau-\rho D\rangle^{-N}}{(\sigma-\rho D)^2} - \frac{\langle\tau-\sigma\rangle^{-N}}{(\sigma-\rho D)^2} - \frac{\langle\tau-\rho D\rangle^{1-N}}{i(\sigma-\rho D)}\Big)\check{F}(\sigma)d\sigma d\rho$$

When we compute $\|\chi_T(t)\Box^{-1}F\|_{H^{s,\theta}}$ we are lead to estimate the follow three terms:

$$\begin{aligned}&I+II+III=\\ &\int_{-1}^{1}\int \|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^\theta\langle\tau-\rho|\xi|\rangle^{-N}|\sigma-\rho|\xi||^{-2}\widetilde{F}(\sigma,\xi)\|_{L^2_\tau L^2_\xi}d\sigma d\rho\\ &+\int_{-1}^{1}\int \|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^\theta\langle\tau-\sigma\rangle^{-N}|\sigma-\rho|\xi||^{-2}\widetilde{F}(\sigma,\xi)\|_{L^2_\tau L^2_\xi}d\sigma d\rho\\ &+\int_{-1}^{1}\int \|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^\theta\langle\tau-\rho|\xi|\rangle^{1-N}|\sigma-\rho|\xi||^{-1}\widetilde{F}(\sigma,\xi)\|_{L^2_\tau L^2_\xi}d\sigma d\rho\end{aligned}$$

The first and the third terms are similar. The key fact here is that the support of $\widetilde{F}(\sigma,\xi)$ is included in the set $||\sigma|-|\xi|| \lesssim 1$, therefore $|\sigma-\rho|\xi||$ is bounded. In fact $|\sigma-\rho|\xi|| \le |\sigma|+|\xi| \le ||\sigma|-|\xi||+2|\xi| \lesssim 1$, since $|\rho|\le 1$ and $|\xi|\le 1$. Moreover to estimate the first and the third terms we need the bound $\langle|\tau|\pm|\xi|\rangle \lesssim \langle\tau-\rho|\xi|\rangle$ which holds for $\rho\in[-1,1]$ and $|\xi|\le 1$. Hence we have a bound on the multipliers:

$$\langle|\tau|-|\xi|\rangle^\theta\langle\tau-\rho|\xi|\rangle^{-N} \lesssim \langle\tau-\rho|\xi|\rangle^{\theta-N}$$

We perform the change of variable $\widetilde{\tau} = \tau-\rho|\xi|$ in the $d\tau$ integral which is bounded provided that we choose $N \in \mathbb{N}$ big enough. Therefore we obtain

$$\begin{aligned} I + III &\lesssim \int_{-1}^{1} \int \| \langle \xi \rangle^{s-1} |\sigma - \rho|\xi||^{-2} \widetilde{F}(\sigma,\xi) \|_{L^2_\xi} d\sigma d\rho \\ &+ \int_{-1}^{1} \int \| \langle \xi \rangle^{s-1} |\sigma - \rho|\xi||^{-1} \widetilde{F}(\sigma,\xi) \|_{L^2_\xi} d\sigma d\rho \end{aligned}$$

Notice that Cauchy-Schwarz inequality in $d\sigma$ integral yield to $I+III \lesssim \|F\|_{H^{s-1,0}}$ since $\sup_{|\xi|<1} \sup_{|\rho|<1} \int |\sigma - \rho|\xi||^{-2} d\sigma < \infty$, recall $|\sigma - \rho|\xi||$ is bounded and has enough decay when σ is large to assure the convergence of the integral. This settles I and III. To bound II observe that $\langle |\tau| - |\xi| \rangle^\theta \lesssim \langle \tau - \rho|\xi| \rangle^\theta$ which holds for $\rho \in [-1,1]$ and $|\xi| \le 1$. Moreover $\langle \tau - \rho|\xi| \rangle^\theta \le \langle \tau - \sigma \rangle^\theta \langle \sigma - \rho|\xi| \rangle^\theta$, simply because $\langle a \pm b \rangle \le \langle a \rangle \langle b \rangle$. Hence we have a bound on the multiplier:

$$\langle |\tau| - |\xi| \rangle^\theta \langle \tau - \sigma \rangle^{-N} \lesssim \langle \tau - \sigma \rangle^{\theta - N} \langle \sigma - \rho|\xi| \rangle^\theta$$

Furthermore recall that $\langle \sigma - \rho|\xi| \rangle$ is bounded since $|\rho| \le 1$ and $|\xi| \le 1$. Therefore

$$\begin{aligned} II &\lesssim \int_{-1}^{1} \int \| \langle \xi \rangle^{s-1} \langle \tau - \sigma \rangle^{\theta - N} |\sigma - \rho|\xi||^{-2} \widetilde{F}(\sigma,\xi) \|_{L^2_\tau L^2_\xi} d\sigma d\rho \\ &\lesssim \int_{-1}^{1} \int \| \langle \xi \rangle^{s-1} |\sigma - \rho|\xi||^{-2} \widetilde{F}(\sigma,\xi) \|_{L^2_\xi} d\sigma d\rho \\ &\lesssim \|F\|_{H^{s-1,0}} \end{aligned}$$

Here we have performed the change of variable $\widetilde{\tau} = \tau - \sigma$ in the $d\tau$ integral and we have used the Cauchy-Schwarz inequality in $d\sigma$ integral. □

1.6 Littlewood-Paley decomposition of hyperbolic Sobolev spaces

In this section we extend the previous theory of hyperbolic Sobolev spaces to the case $\theta = 1/2$. Recall that the embedding of hyperbolic Sobolev spaces into the solution space, namely Corollary 13, required θ to be strictly bigger than $1/2$. However, as we shall see, to be able to prove local well-posedness theorem at the scaling critical regality one needs a linear theory up to $\theta = 1/2$. The results below are based on Chapter 5 of [118].

Consider the following first order in time Cauchy problem

$$\begin{cases} \partial_t u - i\phi(D) u = F) \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^n) \end{cases} \tag{1.5}$$

We define H_k a dyadic localized $H^{s,b}$ type space as

$$H_k = \left\{ f \in L^2(\mathbb{R}^{1+n}) : \operatorname{supp} f \subset \{(\tau,\xi) : 2^{k-1} \le |\xi| \le 2^{k+1}\} \right\}$$

And we endowed it with Besov l^1 type norm

$$\|f\|_{H_k} = \sum_{j=0}^{\infty} 2^{j/2} \|\varphi_j(\tau - \phi(\xi)) f(\tau,\xi)\|_{L^2_\tau L^2_\xi}$$

The space H_k contains all space-time L^2 function with support localised in the strip $|\xi| \approx 2^k$. Infact denote $I_0 = [-2,2]$ and $I_k = [2^{k-1}, 2^{k+1}]$, for $k \in \mathbb{N}$. Assume (φ_k) to be a sequence of function for the non-homogeneous dyadic decomposition. Then we resemble this space in a Littlewood-Paley manner: we define the inhomogeneous Besov-type $H^{s,1/2}$ norm as

$$\|u\|^2_{F^s} = \sum_{k=0}^{\infty} \big(2^{sk} \|\varphi_k(\xi)\widetilde{u}(\tau,\xi)\|_{H_k}\big)^2$$

It is easy to see that F^s is a good substitute for $H^{s,1/2}$, since $H^{s,1/2+} \subset F^s \subset H^{s,1/2}$. This is a consequence that in the second dyadic decomposition, or *modulation*, where we cut at fixed distances to the characteristic surface and we sum up in l^1 sense. Then an easy application of the embedding $l^1 \subset l^2$ implies

$$\begin{aligned}
\|u\|^2_{H^{s,1/2}} &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \big(2^{sk}2^{j/2}\|\varphi_k(\xi)\varphi_j(\tau-\phi(\xi))\widetilde{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi}\big)^2 \\
&\le \sum_{k=0}^{\infty} \big(2^{sk}\sum_{j=0}^{\infty} 2^{j/2}\|\varphi_k(\xi)\varphi_j(\tau-\phi(\xi))\widetilde{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi}\big)^2 \\
&= \sum_{k=0}^{\infty} \big(2^{sk}\|\varphi_k(\xi)\widetilde{u}(\tau,\xi)\|_{H_k}\big)^2
\end{aligned}$$

By Cauchy-Schwarz we obtain the second inequality:

$$\begin{aligned}
\|u\|^2_{F^s} &= \sum_{k=0}^{\infty} \big(2^{sk}\sum_{j=0}^{\infty} 2^{j/2}\|\varphi_k(\xi)\varphi_j(\tau-\phi(\xi))\widetilde{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi}\big)^2 \\
&\le \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \big(2^{sk}2^{j/2+}\|\varphi_k(\xi)\varphi_j(\tau-\phi(\xi))\widetilde{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi}\big)^2 \sum_{j=0}^{\infty}(2^{-j})^2 \\
&\lesssim \|u\|^2_{H^{2,1/2+}}
\end{aligned}$$

Moreover let us define the corresponding $H^{s,b-1}$ space, the space where the nonlinear terms of our Cauchy problem lies, as

$$\begin{aligned}
\|f\|^2_{N^s} &= \sum_{k=0}^{\infty} \big(2^{sk}\|\varphi_k(\xi)\langle\tau-\phi(\xi)\rangle^{-1}\widetilde{f}(\tau,\xi)\|_{H_k}\big)^2 \\
&\approx \sum_{k=0}^{\infty} \big(2^{sk}\sum_{j=0}^{\infty} 2^{-j/2}\|\varphi_k(\xi)\varphi_j(\tau-\phi(\xi))\widetilde{f}(\tau,\xi)\|_{L^2_\tau L^2_\xi}\big)^2
\end{aligned}$$

Remark (Characterisation of F^s norm). Observe that by making the usual change of variable and using the properties of Fourier transform we can rewrite the F^s-norm in the following way

$$\begin{aligned}
\|u\|_{F^s}^2 &= \sum_{k=0}^{\infty}\big(2^{sk}\sum_{j=0}^{\infty}2^{j/2}\|\varphi_k(\xi)\varphi_j(\tau)\widetilde{u}(\tau+\phi(\xi),\xi)\|_{L^2_\tau L^2_\xi}\big)^2\\
&= \sum_{k=0}^{\infty}\big(2^{sk}\sum_{j=0}^{\infty}2^{j/2}\|\varphi_k(\xi)\varphi_j(\tau)\mathscr{F}_t[e^{-it\phi(\xi)}\widehat{u}(t,\xi)]\|_{L^2_\tau L^2_\xi}\big)^2\\
&= \sum_{k=0}^{\infty}\big(2^{sk}\|\varphi_k(\xi)e^{-it\phi(\xi)}\widehat{u}(t,\xi)\|_{L^2(\mathbb{R}^n_\xi)B^{1/2}_{2,1}(\mathbb{R}_t)}\big)^2
\end{aligned}$$

The previous characterisation allow us to prove a first multiplicative estimate for F^s space when one of the two functions depends only on time.

Lemma 22. *Assume $s\in\mathbb{R}$, $0<T<1$, and $\psi\in\mathscr{S}(\mathbb{R})$ then*

$$\|\psi_T(t)u\|_{F^s}\lesssim_T\|u\|_{F^s}$$

here $\psi_T(t)=\psi(t/T)$ is a rescaling of ψ.

Proof. By direct computation we obtain

$$\begin{aligned}
\|\psi_T(t)u\|_{F^s}^2 &= \sum_{k=0}^{\infty}\big(2^{sk}\|\varphi_k(\xi)\varphi_j(\tau)e^{-it\phi(\xi)}\psi_T(t)\widehat{u}(t,\xi)\|_{L^2(\mathbb{R}^n_\xi)B^{1/2}_{2,1}(\mathbb{R}_t)}\big)^2\\
&\le \|\psi_T(t)\|^2_{B^{1/2}_{2,1}(\mathbb{R}_t)}\sum_{k=0}^{\infty}\big(2^{sk}\|\varphi_k(\xi)\varphi_j(\tau)e^{-it\phi(\xi)}\widehat{u}(t,\xi)\|_{L^2(\mathbb{R}^n_\xi)B^{1/2}_{2,1}(\mathbb{R}_t)}\big)^2
\end{aligned}$$

since $B^{1/2}_{2,1}(\mathbb{R}_t)$ is an algebra. □

We are interested in the local well-posedness properties of the Cauchy problem (1.5). Therefore it is natural to ask that $F^s\subset C(\mathbb{R},H^s)$. The purpose of the next Proposition is to prove the bound $\|u\|_{L^\infty H^s}\lesssim\|u\|_{F^s}$. As in the previous section we rely on the transfer principle which is analogous to the case $b>1/2$.

Proposition 23 (Transfer principle)**.** *Let Y be a Banach space of space-time functions on $\mathbb{R}^{1+n}$ with the property that, for any $\tau_0\in\mathbb{R}$ and for every $u_0\in L^2(\mathbb{R}^n)$, we have*

$$\|e^{it\tau_0}S(t)u_0\|_{Y_{t,x}}\lesssim\|u_0\|_{L^2(\mathbb{R}^n)}$$

Then for any $k\in\mathbb{N}$ and $u\in F^0$ we obtain

$$\|P_k u\|_{Y_{t,x}}\lesssim\|\varphi_k(\xi)\widetilde{u}(\tau,\xi)\|_{H_k}$$

where P_k is the Littlewood-Paley cutoff at frequencies $|\xi| \approx 2^k$, precisely $P_k u(t,x) = \mathscr{F}_x^{-1}[\varphi_k(\xi)\widehat{u}(t,\xi)] = \mathscr{F}_{t,x}^{-1}[\varphi_k(\xi)\widetilde{u}(\tau,\xi)]$.

Corollary 24. *Let $s \in \mathbb{R}$ then $F^s \subset C(\mathbb{R}, H^s)$.*

Proof. The result follow easily form the transfer principle since the bound $\|e^{it(\tau_0+\phi(D))}u_0\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{L^2}$ holds for every $u_0 \in L^2(\mathbb{R}^n)$ and $\tau_0 \in \mathbb{R}$. Indeed Littlewood-Paley theory yield to

$$\|u\|_{L^\infty H^s}^2 \leq \sum_{k=0}^{\infty}(2^{ks}\|P_k u\|_{L_t^\infty L_x^2})^2 \leq \sum_{k=0}^{\infty}(2^{ks}\|\varphi_k(\xi)\widetilde{u}(\tau,\xi))\|_{H_k})^2 = \|u\|_{F^s}^2$$

□

Proof of Proposition 23. The proof follows the argument given in the $b > 1/2$ case. Observe that the hypothesis can be written as

$$\|e^{it\tau_0}\int e^{ix\xi+it\phi(\xi)}\widehat{u}_0(\xi)\,d\xi\|_{Y_{t,x}} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}$$

Furthermore form the inverse Fourier transform and change of variable we have

$$\begin{aligned} P_k u(t,x) &= \iint e^{it\tau+ix\xi}\,\varphi_k(\xi)\widetilde{u}(\tau,\xi)\,d\xi\,d\tau \\ &= \int e^{it\tau}\int e^{ix\xi+it\phi(\xi)}\,\varphi_k(\xi)\widetilde{u}(\tau+\phi(\xi),\xi)\,d\xi\,d\tau \end{aligned}$$

Therefore Minkowski inequality yield to

$$\begin{aligned} \|P_k u\|_{Y_{t,x}} &\lesssim \int \|e^{it\tau}\int e^{ix\xi+it\phi(\xi)}\varphi_k(\xi)\widetilde{u}(\tau+\phi(\xi),\xi)\,d\xi\|_{Y_{t,x}}\,d\tau \\ &\lesssim \int \|\varphi_k(\xi)\widetilde{u}(\tau+\phi(\xi),\xi)\|_{L_\xi^2}\,d\tau \end{aligned}$$

To close the argument let us cut the integral in $d\tau$ into a sum of dyadic pieces, using the fact that $\sum_{j=0}^{\infty}\varphi_j(\tau) = 1$, then by Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \int \|\varphi_k(\xi)\widetilde{u}(\tau+\phi(\xi),\xi)\|_{L_\xi^2}\,d\tau &= \sum_{j=0}^{\infty}\int_{2^{j-1}}^{2^{j+1}}\varphi_j(\tau)\|\varphi_k(\xi)\widetilde{u}(\tau+\phi(\xi),\xi)\|_{L_\xi^2}\,d\tau \\ &\leq \sum_{j=0}^{\infty}\|\varphi_k(\xi)\varphi_j(\tau)\widetilde{u}(\tau+\phi(\xi),\xi)\|_{L_\tau^2 L_\xi^2}\Big(\int_{2^{j-1}}^{2^{j+1}}d\tau\Big)^{1/2} \\ &\lesssim \sum_{j=0}^{\infty}2^{j/2}\|\varphi_k(\xi)\varphi_j(\tau-\phi(\xi))\widetilde{u}(\tau,\xi)\|_{L_\tau^2 L_\xi^2} \end{aligned}$$

□

Next Proposition gives us a complete understanding of the local solution of the linear problem (1.5).

Proposition 25. *Assume $s \in \mathbb{R}$, and $\psi \in \mathscr{S}(\mathbb{R})$, then*

(i) The following homogeneous estimate hold

$$\|\psi_T(t)S(t)u_0\|_{F^s} \lesssim_T \|u_0\|_{H^s}$$

(ii) The following inhomogeneous estimate hold

$$\|\psi_T(t)\int_0^t S(t-s)f(s)ds\|_{F^s} \lesssim_T \|f\|_{N^s}$$

Proof. Notice that (i) follows form $\|\varphi_k(\xi)\mathscr{F}_{t,x}[\psi_T(t)S(t)u_0]\|_{H_k} \lesssim \|\varphi_k(\xi)u_0(\xi)\|_{L^2_\xi}$. By definition of H_k we get

$$\|\varphi_k(\xi)\mathscr{F}_{t,x}[\psi_T(t)S(t)u_0]\|_{H_k} = \sum_{j=0}^{\infty} 2^{j/2}\|\varphi_j(\tau-\phi(\xi))\varphi_k(\xi)\mathscr{F}_{t,x}[\psi_T(t)S(t)u_0]\|_{L^2_\tau L^2_\xi}$$

We would like to get rid of the L^2_τ integral, we accomplish that by a change of variable. From the definition of $S(t)$ and the fact that the time-Fourier transform of $e^{it\rho}\psi_T(t)$ is $\widehat{\psi}_T(\tau-\rho)$, we obtain $\mathscr{F}_{t,x}[\psi_T(t)S(t)u_0] = \widehat{\psi}_T(\tau-\phi(\xi))\widehat{u}_0(\xi)$. Therefore if we set $\tau = \tau - \phi(\xi)$ in dτ integral we can split the integral as follows

$$\begin{aligned} \|\varphi_j(\tau-\phi(\xi))\varphi_k(\xi)\mathscr{F}_{t,x}[\psi_T(t)S(t)u_0]\|_{L^2_\tau L^2_\xi} &= \|\varphi_j(\tau)\varphi_k(\xi)\widehat{\psi}_T(\tau)\widehat{u}_0(\xi)\|_{L^2_\tau L^2_\xi} \\ &= \|\varphi_j(\tau)\widehat{\psi}_T(\tau)\|_{L^2_\tau}\|\varphi_k(\xi)\widehat{u}_0(\xi)\|_{L^2_\xi} \end{aligned}$$

Summing over j gives the desired bound: $\|\varphi_k(\xi)\mathscr{F}_{t,x}[\psi_T(t)S(t)u_0]\|_{H_k} \le \|\psi_T\|_{B^{1/2}_{2,1}(\mathbb{R})}\|\varphi_k(\xi)u_0(\xi)\|_{L^2_\xi}$ since $\psi_T \in \mathscr{S}(\mathbb{R})$.

In order to prove (ii) it suffices to show that the corresponding estimate at the level of H_k holds:

$$\Big\|\varphi_k(\xi)\mathscr{F}_{t,x}\big[\psi_T(t)\int_0^t S(t-s)f(s)ds\big]\Big\|_{H_k} \lesssim \|\varphi_k(\xi)\langle\tau-\phi(\xi)\rangle^{-1}\widetilde{f}(\tau,\xi)\|_{H_k}$$

To compute the space-time Fourier transform of $\psi_T(t)\int_0^t S(t-s)f(s)ds$ first apply the space-Fourier transform to obtain $\mathscr{F}_t[\psi_T(t)\int_0^t e^{i(t-s)\phi(\xi)}\widehat{f}(s,\xi)ds]$. Then let us use the Fourier inversion theorem and write $\widehat{f}(s,\xi) = \int_{\mathbb{R}} e^{is\sigma}\widetilde{f}(\sigma,\xi)d\sigma$. Computing the integral in ds yield to

$$\mathscr{F}_{t,x}\big[\psi_T(t)\int_0^t S(t-s)f(s)ds\big](\tau,\xi) = \int_{\mathbb{R}} \frac{\widehat{\psi}_T(\tau-\sigma)-\widehat{\psi}_T(\tau-\phi(\xi))}{i(\sigma-\phi(\xi))}\widetilde{f}(\sigma,\xi)\,d\sigma$$

Define $g_k(\tau,\xi) = \varphi_k(\xi)\langle\tau-\phi(\xi)\rangle^{-1}\widetilde{f}(\tau,\xi)$, then it suffices to show that the operator

$$Tg_k(\tau,\xi) = \int_{\mathbb{R}} \frac{\widehat{\psi}_T(\tau-\sigma)-\widehat{\psi}_T(\tau-\phi(\xi))}{i(\sigma-\phi(\xi))}\langle\sigma-\phi(\xi)\rangle g_k(\sigma,\xi)\,d\sigma$$

is bounded in H_k: this means that $\|Tg_k\|_{H_k} \lesssim \|g_k\|_{H_k}$ holds uniformly for $k \geq 0$. By definition of H_k and a change of variable we have

$$\|Tg_k\|_{H_k} = \sum_{j=0}^{\infty} 2^{j/2}\|\varphi_j(\tau)Tg_k(\tau+\phi(\xi),\xi)\|_{L^2_\tau L^2_\xi}$$

Notice that by a similar change of variable in $d\sigma$ integral we have

$$Tg_k(\tau+\phi(\xi),\xi) = \int_{\mathbb{R}} \frac{\widehat{\psi}_T(\tau-\sigma)-\widehat{\psi}_T(\tau)}{i\sigma}\langle\sigma\rangle g_k(\sigma+\phi(\xi),\xi)\,d\sigma$$

It is easy to see that

$$\Big|\frac{\widehat{\psi}_T(\tau-\sigma)-\widehat{\psi}_T(\tau)}{i\sigma}\langle\sigma\rangle\Big| \lesssim (1+|\tau|)^{-4}+(1+|\tau-\sigma|)^{-4}$$

Since the H_k norm of g_k contains the L^2 norm is the time-frequency variable we need to apply Cauchy-Schwarz in the dσ integral, however the term σ^{-1} will make thinks explode. Therefore we cut the integral in $d\sigma$ into a sum of dyadic pieces, using the fact that $\sum_{l=0}^{\infty}\varphi_l(\sigma) = 1$. We have

$$|Tg_k(\tau+\phi(\xi),\xi)| \lesssim \sum_{l=0}^{\infty}\int_{I_l}\varphi_l(\sigma)[(1+|\tau|)^{-4}+(1+|\tau-\sigma|)^{-4}]g_k(\sigma+\phi(\xi),\xi)\,d\sigma =: I + II$$

For the term I, we apply Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
I &\lesssim (1+|\tau|)^{-4}\sum_{l=0}^{\infty}\int_{I_l}\varphi_l(\sigma)g_k(\sigma+\phi(\xi),\xi)\,d\sigma = \\
&\lesssim (1+|\tau|)^{-4}\sum_{l=0}^{\infty}\Big(\int_{I_l}|\varphi_l(\sigma)g_k(\sigma+\phi(\xi),\xi)|^2\,d\sigma\Big)^{1/2}\Big(\int_{I_l}d\sigma\Big)^{1/2} \\
&\lesssim (1+|\tau|)^{-4}\sum_{l=0}^{\infty}2^{l/2}\|\varphi_l(\sigma-\phi(\xi))g_k(\sigma,\xi)\|_{L^2_\sigma}
\end{aligned}$$

where $I_l = [2^{l-1},2^{l+1}]$ is the dyadic interval where φ_l is supported. Hence finally

$$\sum_{j=0}^{\infty}2^{j/2}\|\varphi_j(\tau)I\|_{L^2_\tau L^2_\xi} \lesssim \Big(\sum_{j=0}^{\infty}2^{j/2}\|\varphi_j(\tau)(1+|\tau|)^{-4}\|_{L^2_\tau}\Big)\|g_k\|_{H_k}$$

For the term II, the situation is quite different: it doesn't suffices to multiply only by $\sum_{m=0}^{\infty}\varphi_m(\sigma) =$

1, but one needs to multiply also by $\sum_{l=0}^{\infty}\varphi_l(\tau-\sigma)=1$. This way we obtain the triple series

$$\sum_{j=0}^{\infty} 2^{j/2}\|\varphi_j(\tau)II\|_{L^2_\tau L^2_\xi} = \sum_{j,l,m=0}^{\infty} 2^{j/2}\|\varphi_j(\tau)\int \varphi_l(\tau-\sigma)(1+|\tau-\sigma|)^{-4}\varphi_m(\sigma)g_k(\sigma+\phi(\xi),\xi)\,d\sigma\|_{L^2_\tau L^2_\xi}$$

This has some resemblance with paraproduct. In fact if we define $f(\tau)=(1+|\tau|)^{-4}$ and $h(\tau)=\|g_k(\tau+\phi(\xi),\xi)\|_{L^2_\xi}$ then we can write

$$\sum_{j=0}^{\infty} 2^{j/2}\|\varphi_j(\tau)II\|_{L^2_\tau L^2_\xi} \le \sum_{j,l,m=0}^{\infty} 2^{j/2}\|\varphi_j(\tau)(\varphi_l f * \varphi_m h)(\tau)\|_{L^2_\tau}$$

We know form paraproduct estimate that the term $\varphi_j(\tau)\varphi_m(\sigma)\varphi_l(\tau-\sigma)$ is non zero only in the following three cases:

i. $m\approx j$ and $l \ll m$

ii. $m \approx l$ and $l \gg j$

iii. $l \approx j$ and $m \ll l$

Therefore the triple series over j,l,m is reduced to a single serie plus a finite sum. The argument to prove the bound in the case (1) and (2) are quite similar, while (3) requires extra care.

i. In this case we first use the trivial bound $|\varphi_m(\tau)|\le 1$ and then apply Young's inequality for convolution to obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{l=0}^{m-3} 2^{m/2}\|\varphi_m(\tau)(\varphi_l f * \varphi_m h)(\tau)\|_{L^2_\tau} &\le \sum_{m=0}^{\infty}\sum_{l=0}^{m-3} 2^{m/2}\|\varphi_l f\|_{L^\infty_\tau}\|\varphi_m h\|_{L^2_\tau}\\ &\le \sum_{m=0}^{\infty} 2^{m/2}\|\varphi_m h\|_{L^2_\tau}\end{aligned}$$

since $\sum_{l=0}^{m-3}\|\varphi_l f\|_{L^\infty_\tau}\lesssim 1$. Notice that by our definition

$$\sum_{m=0}^{\infty} 2^{m/2}\|\varphi_m h\|_{L^2_\tau} = \sum_{m=0}^{\infty} 2^{m/2}\|\varphi_m(\tau)g_k(\tau+\phi(\xi),\xi)\|_{L^2_\tau L^2_\xi} = \|g_k\|_{H_k}$$

ii. Here again use the trivial bound $|\varphi_k(\tau)|\le 1$ and then apply Young's inequality for convolution to obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{j=0}^{m-3} 2^{j/2}\|\varphi_j(\tau)(\varphi_m f * \varphi_m h)(\tau)\|_{L^2_\tau} &\le \sum_{m=0}^{\infty}\sum_{j=0}^{m-3} 2^{j/2}\|\varphi_m f\|_{L^\infty_\tau}\|\varphi_m h\|_{L^2_\tau}\\ &\lesssim \sum_{m=0}^{\infty} 2^{m/2}\|\varphi_m h\|_{L^2_\tau}\end{aligned}$$

since $\sum_{j=0}^{m-3} 2^{j/2}\|\varphi_m f\|_{L^\infty_\tau} \le \sum_{l=0}^{m-3} 2^{j/2} \lesssim 2^{m/2}$.

iii. In this case we can't use directly the trivial bound $|\varphi_k(\tau)| \le 1$ but we first use Hölder inequality and then Young's inequality for convolution. We have

$$\begin{aligned}
\sum_{l=0}^{\infty}\sum_{m=0}^{l-3} 2^{l/2}\|\varphi_l(\tau)(\varphi_l f * \varphi_m h)(\tau)\|_{L^2_\tau} &\le \sum_{l=0}^{\infty}\sum_{m=0}^{l-3} 2^{l/2}\|\varphi_l\|_{L^2_\tau}\|\varphi_l f * \varphi_m h\|_{L^\infty_\tau} \\
&\lesssim \sum_{l=0}^{\infty}\sum_{m=0}^{l-3} 2^{l}\|\varphi_l f\|_{L^2_\tau}\|\varphi_m h\|_{L^2_\tau} \\
&\le \Big(\sum_{l=0}^{\infty} 2^{l}\|\varphi_l(\tau)(1+|\tau|)^{-4}\|_{L^2_\tau}\Big)\sum_{m=0}^{\infty} 2^{m/2}\|\varphi_m h\|_{L^2_\tau}
\end{aligned}$$

Therefore we obtain

$$\|Tg_k\|_{H_k} \lesssim \sum_{j=0}^{\infty} 2^{j/2}\|\varphi_j(\tau) I\|_{L^2_\tau L^2_\xi} + \sum_{j=0}^{\infty} 2^{j/2}\|\varphi_j(\tau) II\|_{L^2_\tau L^2_\xi} \lesssim \|g_k\|_{H_k}$$

□

This conclude the study of the linear Cauchy problem (1.5).

Wave-Sobolev spaces

In what follow, we consider a second order in time Cauchy problem, such as the one for a linear wave equation:

$$\begin{cases} \partial_{tt} u - \phi(D) u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n) \end{cases}$$

As before we would like to extend the previous linear theory in hyperbolic Sobolev spaces, developed in §1.5, to include the case $\theta = 1/2$. Inspired by the definition of F^s space we define the spaces $H^{s,\theta}_{p,q}$ where we exploit different ways to sum over frequency and modulation localized pieces. Let us define Q_d is a space-time multiplier with symbol $q_d(\tau,\xi) = \varphi_d(|\tau| - |\xi|)$, where φ_d truncates smoothly on a annulus of radii $d/2$ and $2d$. Here $d = 2^j$ is a dyadic number. That is $Q_d f(t,x) = \mathscr{F}^{-1}_{t,x}[\varphi_d(|\tau| - |\xi|)\widetilde{f}(\tau,\xi)]$. Here we are using inhomogeneous decomposition: $\operatorname{supp}\phi_1 \subset B(0,1)$ the ball of radius 2 entered in the origin, and $\phi_1 = 1$ on $B(0,1/2)$.

Definition (q-Besov type $H^{s,\theta}$ Spaces)**.** For any $1 \le q \le \infty$, $s,\theta \in \mathbb{R}$, we define the following space

$$H^{\theta,q}_k = \{f \in L^2(\mathbb{R}^{1+n}) : \operatorname{supp}\widetilde{f} \subset \{(\tau,\xi) : 2^{k-1} \le |\xi| \le 2^{k+1}\}\}$$

and we endow it with a q-Besov norm on L^2 base:

$$\begin{aligned}\|f\|_{H_k^{\theta,q}} &= \Big[\sum_{j=0}^{\infty}\big(2^{j\theta}\|Q_j f\|_2\big)^q\Big]^{1/q}\\ &= \Big[\sum_{j=0}^{\infty}\big(2^{j\theta}\|\varphi_j(|\tau|-|\xi|)\widetilde{f}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}^{n+1})}\big)^q\Big]^{1/q}\end{aligned}$$

Furthermore, for any $1 \le p \le \infty$, define the space $H^{s,\theta}_{p,q}$ as a inhomogeneous p-Besov space with base $H_k^{\theta,q}$:

$$\|u\|_{H^{s,\theta}_{p,q}} = \Big[\sum_{k=0}^{\infty}(2^{sk}\|P_k u\|_{H_k^{\theta,q}})^p\Big]^{1/p} = \Big[\sum_{k=0}^{\infty}\Big[\sum_{j=0}^{\infty}(2^{ks}2^{j\theta}\|P_k Q_j u\|_{L^2L^2})^q\Big]^{p/q}\Big]^{1/p}$$

where P_λ is the Littlewood-Paley cutoff at frequencies $|\xi| \approx \lambda$, precisely $P_\lambda u(t,x) = \mathscr{F}_x^{-1}[\varphi_\lambda(\xi)\widehat{u}(t,\xi)] = \mathscr{F}_{t,x}^{-1}[\varphi_\lambda(\xi)\widetilde{u}(\tau,\xi)]$. Here again $\lambda = 2^k$ is a dyadic number. It is also useful to define the space $H_k^{s,\theta,q} = 2^{-ks}H_k^{\theta,q}$ so that

$$\|u\|_{H^{s,\theta}_{p,q}} = \Big[\sum_{k=0}^{\infty}(\|P_k u\|_{H_k^{s,\theta,q}})^p\Big]^{1/p}$$

Observe that, when $p = q = 2$ we recover the classical $H^{s,\theta}$ wave-Sobolev norm:

$$\|u\|_{H^{s,\theta}_{2,2}} \approx \Big[\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\infty}(\lambda^s d^\theta\|P_\lambda Q_d u\|_2)^2\Big]^{1/2}.$$

Therefore when both Besov indices p and q are equal to 2 we write $H^{s,\theta} = H^{s,\theta}_{2,2}$. From the inclusion of $l^{p_1} \subset l^{p_2}$ when $p_1 \le p_2$, we obtain the corresponding inclusion of $H^{s,\theta}_{p,q}$ spaces:

$$H^{s,\theta}_{p_1,q_1} \subset H^{s,\theta}_{p_2,q_2}$$

for $p_1 \le p_2$ and $q_1 \le q_2$. In particular $H^{s,\theta}_{1,1}$, $H^{s,\theta}_{2,1}$, $H^{s,\theta}_{1,2} \subset H^{s,\theta}$. Moreover notice the difference between the three Littlewood-Paley cutoff type: S_λ, P_λ, and Q_d.

$$\text{supp }(S_\lambda u) = \{(\tau,\xi) \in \mathbb{R}^{1+n} : \lambda/2 \le ||\tau| + |\xi|| \le 2\lambda\}$$

$$\text{supp }(P_\lambda u) = \{(\tau,\xi) \in \mathbb{R}^{1+n} : \frac{\lambda}{2} \le |\xi| \le 2\lambda\}$$

$$\text{supp }(Q_d u) = \{(\tau,\xi) \in \mathbb{R}^{1+n} : \frac{d}{2} \le ||\tau| - |\xi|| \le 2d\}$$

Moreover notice that supp $(P_\lambda Q_d u) = \emptyset$ if $d \ge \lambda/4$, therefore in the second dyadic decomposition the sum over d can be restricted to the range $0 \le d \le \lambda/4$.

Why are such spaces relevant? They can be used to prove local well-posedness with initial data in Besov spaces. In order to extend the linear estimate to this Besov variant spaces, we prove below a transfer principle which resemble the one of the previous section.

Proposition 26 (Transfer Principle)**.** *Let Y be a Banach space of space-time functions on $\mathbb{R}^{1+n}$ with the property that, for any $\tau_0 \in \mathbb{R}$ and for every $u_0 \in L^2(\mathbb{R}^n)$ we have*

$$\|e^{it(\tau_0 \pm D)} u_0\|_{Y_{t,x}} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}$$

If $\theta > 1/2$ and $q \geq 1$ or if $\theta = 1/2$ and $q = 1$, then we have for any $k \in \mathbb{N}$

$$\|P_k u\|_{Y_{t,x}} \lesssim \|P_k u\|_{H_k^{\theta,q}}$$

Proof. Observe that the hypothesis can be written as $\|e^{it\tau_0} \int e^{ix\xi \pm it|\xi|} \widehat{u}_0(\xi) d\xi\|_{Y_{t,x}} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}$. Denote $\widetilde{u} = \chi_{\tau \geq 0} \widetilde{u} + \chi_{\tau < 0} \widetilde{u} := \widetilde{u}_+ + \widetilde{u}_-$. Furthermore form the inverse Fourier transform and change of variable we have

$$\begin{aligned}
P_k u(t,x) &= \iint e^{it\tau + ix\xi} \varphi_k(\xi) \big(\widetilde{u}_+(\tau,\xi) + \widetilde{u}_-(\tau,\xi)\big) \, d\xi \, d\tau \\
&= \int e^{it\tau} \int \varphi_k(\xi) \big(e^{ix\xi + it|\xi|} \widetilde{u}_+(\tau + |\xi|, \xi) + e^{ix\xi - it|\xi|} \widetilde{u}_-(\tau - |\xi|, \xi)\big) \, d\xi \, d\tau
\end{aligned}$$

Therefore Minkowski inequality and the hypothesis yield to

$$\|P_k u\|_{Y_{t,x}} \lesssim \int \|\varphi_k(\xi) \widetilde{u}_+(\tau + |\xi|, \xi)\|_{L^2_\xi} d\tau + \int \|\varphi_k(\xi) \widetilde{u}_-(\tau - |\xi|), \xi)\|_{L^2_\xi} d\tau$$

Let us cut the integral in $d\tau$ into a sum of dyadic pieces, using the fact that $\sum_{j=0}^\infty \varphi_j(\tau) = 1$. By Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\int \|\varphi_k(\xi) \widetilde{u}_\pm(\tau \pm |\xi|, \xi)\|_{L^2_\xi} d\tau &= \sum_{j=0}^\infty \int_{2^{j-1}}^{2^{j+1}} \varphi_j(\tau) \|\varphi_k(\xi) \widetilde{u}_\pm(\tau \pm |\xi|, \xi)\|_{L^2_\xi} d\tau \\
&\leq \sum_{j=0}^\infty \|\varphi_k(\xi) \varphi_j(\tau) \widetilde{u}_\pm(\tau \pm |\xi|, \xi)\|_{L^2_\tau L^2_\xi} \Big(\int_{2^{j-1}}^{2^{j+1}} d\tau\Big)^{1/2} \\
&\lesssim \sum_{j=0}^\infty 2^{j/2} \|\varphi_k(\xi) \varphi_j(\tau \mp |\xi|) \widetilde{u}_\pm(\tau, \xi)\|_{L^2_\tau L^2_\xi} \\
&\lesssim \sum_{j=0}^\infty 2^{j/2} \|\varphi_k(\xi) \varphi_j(|\tau| - |\xi|) \widetilde{u}(\tau, \xi)\|_{L^2_\tau L^2_\xi}
\end{aligned}$$

This prove the Proposition in the case $\theta = 1/2$ and $q = 1$. On the other hand if $\theta > 1/2$ then we can apply Holder inequality to obtain

$$\sum_{j=0}^\infty 2^{j/2} \|P_k Q_j u\|_{L^2 L^2} \lesssim \Big[\sum_{j=0}^\infty (2^{j\theta} \|P_k Q_j u\|_{L^2 L^2})^q\Big]^{1/q} \Big[\sum_{j=0}^\infty 2^{j(1/2 - \theta) q'}\Big]^{1/q'}$$

The second sum converges since $\theta > 1/2$. □

The transfer principle allow us to easily prove the following embeddings.

Corollary 27. *Let $\theta > 1/2$ and $q \geq 1$ or $\theta = 1/2$ and $q = 1$, then $H^{s,\theta}_{p,q} \subset C_b(\mathbb{R}, B^s_{2,p}(\mathbb{R}^n))$.*

Proof. Littlewood-Paley theory gives

$$\|u\|_{C_b(\mathbb{R},B^s_{2,p})} \leq \Big[\sum_{k=0}^{\infty}(2^{ks}\|P_k u\|_{L^\infty_t L^2_x})^p\Big]^{1/p}$$

Therefore it suffices to prove that $\|P_k u\|_{L^\infty_t L^2_x} \lesssim \|P_k u\|_{H^{\theta,q}_k}$ for any $k \in \mathbb{N}$. However by the transfer principle the previous inequality holds since $\|e^{it(\tau_0 \pm D)} u_0\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2}$ for any $u_0 \in L^2(\mathbb{R}^n)$. □

Corollary 28. *Let $\theta > 1/2$ or let $\theta = 1/2$ and $q = 1$, then $H^{s+,\theta}_{p,q} \subset L^{\tilde{p}} L^{\tilde{q}}$, where $(\tilde{p}, \tilde{q}, s)$ is a Strichartz triplet. Moreover if $1 \leq p \leq 2$ then $H^{s,\theta}_{p,q} \subset L^{\tilde{p}} L^{\tilde{q}}$.*

Proof. By Littlewood-Paley theory and Holder inequality we obtain

$$\|u\|_{L^{\tilde{p}}L^{\tilde{q}}} \leq \sum_{k=0}^{\infty}\|P_k u\|_{L^{\tilde{p}}L^{\tilde{q}}} \leq \Big[\sum_{k=0}^{\infty}(2^{k\delta_0}\|P_k u\|_{L^{\tilde{p}}L^{\tilde{q}}})^p\Big]^{1/p}\Big[\sum_{k=0}^{\infty}2^{-k\delta_0 p'}\Big]^{1/p'}$$

The second integral in convergent. To close we apply the transfer principle: the frequency-localized Strichartz estimates for the half-wave propagator implies $\|e^{it(\tau_0 \pm D)} P_k u_0\|_{L^{\tilde{p}}L^{\tilde{q}}} \lesssim 2^{ks}\|P_k u_0\|_{L^2}$ for any $u_0 \in L^2(\mathbb{R}^n)$.

On the other hand if $1 \leq p \leq 2$ then one can use Littlewood-Paley inequality and the embedding $l^p \subset l^2$ to avoid using Holder inequality. Precisely one obtains

$$\|u\|_{L^{\tilde{p}}L^{\tilde{q}}} \lesssim \Big[\sum_{k=0}^{\infty}(\|P_k u\|_{L^{\tilde{p}}L^{\tilde{q}}})^2\Big]^{1/2} \leq \Big[\sum_{k=0}^{\infty}(\|P_k u\|_{L^{\tilde{p}}L^{\tilde{q}}})^p\Big]^{1/p}$$

Then by the previous argument it follows that $H^{s,\theta}_{p,q} \subset L^{\tilde{p}} L^{\tilde{q}}$. □

We continue our analysis of $H^{s,\theta}_{p,q}$ spaces and study the connection with the linear wave equation. In the next lemma we present a first multiplicative estimate for $H^{s,\theta}_{p,q}$ when one of the two functions depend only on time.

Lemma 29. *Assume $s \in \mathbb{R}$, $\theta > 1/2$ and $q \geq 1$, or assume $\theta = 1/2$ and $q = 1$, and $\chi \in \mathscr{S}(\mathbb{R})$ then*

$$\|\chi_T(t)u\|_{H^{s,\theta}_{p,q}} \lesssim T^{\frac{1}{2}-\theta}\|u\|_{H^{s,\theta}_{p,q}}$$

here $\chi_T(t) = \chi(t/T)$ is a rescaling of χ.

Proof. By the definition of $H^{s,\theta}_{p,q}$ space it is sufficient to show the claim at the level of the $H^{\theta,q}_k$ spaces. In fact if we show that $\|\chi_T(t)P_k u\|_{H^{\theta,q}_k} \lesssim \|P_k u\|_{H^{\theta,q}_k}$ holds for every $k \in \mathbb{N}$ then the proof is completed. Notice that one can split the $H^{\theta,q}_k$ as $\|u\|_{H^{\theta,q}_k} \le \|u^+\|_{H^{\theta,q}_{k,+}} + \|u^-\|_{H^{\theta,q}_{k,-}}$, where $\widetilde{u}^\pm = \chi_{\pm\tau>0}\widetilde{u}$ and

$$\|u\|_{H^{\theta,q}_{k,\pm}} = \Big[\sum_{j=0}^{\infty}\big(2^{j\theta}\|\varphi_j(\tau\mp|\xi|)\widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}^{n+1})}\big)^q\Big]^{1/q}$$

Then we can perform a change of variable and conclude that $\|u\|_{H^{\theta,q}_{k,\pm}} = \|e^{\mp it|\xi|}\widehat{u}(t,\xi)\|_{L^2(\mathbb{R}^n_\xi)B^\theta_{2,q}(R_t)}$. Therefore the Lemma follows form the multiplicative properties of Besov spaces. □

We are now ready to prove the key property of this section.

Proposition 30 (Linear estimates)**.** *Let $\theta > 1/2$ and $q \ge 1$ or let $\theta = 1/2$ and $q = 1$, then local in time solutions to the homogeneous wave equation with initial data in $B^s_{2,p} \times B^{s-1}_{2,p}$ belong to $H^{s,\theta}_{p,q}$:*

$$\|\chi_T\mathcal{H}(u_0,u_1)\|_{H^{s,\theta}_{p,q}} \lesssim T^{\frac{1}{2}-\theta}\|(u_0,u_1)\|_{B^s_{2,p}\times B^{s-1}_{2,p}} \tag{1.6}$$

where $\chi \in C^\infty_0(\mathbb{R})$ is a bump function. Moreover, the local solution to the inhomogeneous wave equation with zero initial data and inhomogeneous term in $H^{s-1,\theta-1}_{p,q}$ belong to $H^{s,\theta}_{p,q}$:

$$\|\chi_T\Box^{-1}F\|_{H^{s,\theta}_{p,q}} \lesssim T^{\frac{1}{2}-\theta}\|F\|_{H^{s-1,\theta-1}_{p,q}} \tag{1.7}$$

Proof. Recall that the half-wave principle tell us that the homogeneous solution map is a linear combination of exponentials, therefore to prove (1.6) it suffices to show that $\|\chi_T(t)P_k e^{\pm itD}f\|_{H^{\theta,q}_k} \lesssim \|P_k f\|_{L^2}$ for any $f \in L^2(\mathbb{R}^n)$ and $k \in \mathbb{N}$. Recall that $\mathcal{F}_{t,x}[\chi_T(t)e^{\pm itD}f] \approx T\widehat{\chi}(T(\tau\mp|\xi|))\widehat{f}(\xi)$. Moreover notice that $\widehat{\chi}(T(\tau\mp|\xi|)) \lesssim \langle T(\tau\mp|\xi|)\rangle^{-N} \lesssim \langle T(|\tau|\mp|\xi|)\rangle^{-N}$ for a big enough $N \in \mathbb{N}$. Therefore

$$\begin{aligned}
\|\chi_T(t)P_k e^{\pm itD}f\|_{H^{\theta,q}_k} &\lesssim T\Big[\sum_{j=0}^{\infty}(2^{j\theta}\|\varphi_k(\xi)\varphi_j(|\tau|-|\xi|)\langle T(|\tau|-|\xi|)\rangle^{-N}\widehat{f}(\xi)\|_{L^2_{\tau,\xi}})^q\Big]^{1/q}\\
&\lesssim T\Big[\sum_{j=0}^{\infty}(2^{j\theta}\|\varphi_j(\tau)\langle T\tau\rangle^{-N}\|_{L^2_\tau})^q\Big]^{1/q}\|\varphi_k(\xi)\widehat{f}(\xi)\|_{L^2_\xi}\\
&\lesssim T^{\frac{1}{2}-\theta}\|\varphi_k(\xi)\widehat{f}(\xi)\|_{L^2_\xi}
\end{aligned}$$

Hence (1.6) holds. Now let us turn to the inhomogeneous estimate (1.7), it suffices to show that the estimate $\|P_k\chi_T\Box^{-1}F\|_{H^{\theta,q}_k} \lesssim T^{\frac{1}{2}-\theta}2^{-k}\|P_kF\|_{H^{\theta-1,q}_k}$ holds uniformly for $k \in \mathbb{N}$. Recall that the inhomogeneous solution map is defined via Duhamel's principle by $\Box^{-1}F = \int_0^t D^{-1}\sin((t-$

$s)D)F(s)ds$, hence by Fourier inversion theorem $\widehat{F}(s) = \int_{\mathbb{R}} e^{is\sigma}\widetilde{F}(\sigma)d\sigma$ we can write

$$\begin{aligned}\widehat{\Box^{-1}F}(t,\xi) &= \int\int_0^t \frac{\sin((t-s)|\xi|)}{|\xi|} e^{is\sigma}\widetilde{F}(\sigma,\xi)ds\,d\sigma \\ &= \frac{1}{2i|\xi|}\int\Big(\frac{e^{it\sigma}-e^{it|\xi|}}{i(\sigma-|\xi|)} - \frac{e^{it\sigma}-e^{-it|\xi|}}{i(\sigma+|\xi|)}\Big)\widetilde{F}(\sigma,\xi)d\sigma\end{aligned}$$

Now we have to take the Fourier transform with respect to time, and recall that $\mathscr{F}_t[e^{it\rho}\chi_T(t)](\tau) \approx_T \widehat{\chi}_{1/T}(\tau-\rho)$. Thus we obtain

$$\mathscr{F}_{t,x}[\chi_T\Box^{-1}F](\tau,\xi) \approx_T \frac{1}{2i|\xi|}\int\Big(\frac{\widehat{\chi}_{1/T}(\tau-\sigma)-\widehat{\chi}_{1/T}(\tau-|\xi|)}{i(\sigma-|\xi|)} - \frac{\widehat{\chi}_{1/T}(\tau-\sigma)-\widehat{\chi}_{1/T}(\tau+|\xi|)}{i(\sigma+|\xi|)}\Big)\widetilde{F}(\sigma,\xi)d\sigma$$

Therefore performing the usual decomposition and change of variables we obtain

$$\begin{aligned}\|P_kQ_j\chi_T\Box^{-1}F\|_{L^2L^2} &\leq \|\varphi_k(\xi)\varphi_j(\tau)\mathscr{F}_{t,x}[\chi_T\Box^{-1}F]_\pm(\tau\pm|\xi|,\xi)\|_{L^2_\tau L^2_\xi} \\ &\approx_T 2^{-k}\|\varphi_k(\xi)\varphi_j(\tau)\int\frac{\widehat{\chi}_{1/T}(\tau-\sigma)-\widehat{\chi}_{1/T}(\tau)}{\sigma}\widetilde{F}(\sigma\pm|\xi|,\xi)d\sigma\|_{L^2_\tau L^2_\xi} \\ &=: 2^{-k}(I+II)\end{aligned}$$

Define $F_k(\sigma,\xi) = \varphi_k(\xi)\widetilde{F}(\sigma\pm|\xi|,\xi)$ and consider the two terms separately. Notice that

$$\begin{aligned}\|P_k\chi_T\Box^{-1}F\|_{H_k^{\theta,q}} &= \Big[\sum_{j=0}^{\infty}(2^{j\theta}\|P_kQ_j\chi_T\Box^{-1}F\|_{L^2L^2})^q\Big]^{1/q} \\ &\lesssim_T 2^{-k}\Big[\sum_{j=0}^{\infty}(2^{j\theta}(I+II))^q\Big]^{1/q}\end{aligned}$$

The second term can be estimated once we multiply by $\sum_{l=0}^{\infty}\varphi_l(\sigma)=1$. In fact by Cauchy-Schwarz we have

$$\begin{aligned}II &\leq \sum_{l=0}^{\infty}2^{-l}\|\varphi_j(\tau)\widehat{\chi}_{1/T}(\tau)\int\varphi_l(\sigma)F_k(\sigma,\xi)d\sigma\|_{L^2_\tau L^2_\xi} \\ &\leq \|\varphi_j(\tau)\widehat{\chi}_{1/T}(\tau)\|_{L^2_\tau}\sum_{l=0}^{\infty}2^{-l/2}\|\varphi_l(\sigma)F_k(\sigma,\xi)\|_{L^2_\sigma L^2_\xi}\end{aligned}$$

Thus

$$\Big[\sum_{j=0}^{\infty}(2^{j\theta}(II)^q\Big]^{1/q} \leq \|\chi_T\|_{B^\theta_{2,q}}\sum_{l=0}^{\infty}2^{-l/2}\|\varphi_l(\sigma)F_k(\sigma,\xi)\|_{L^2_\sigma L^2_\xi}$$

This proves the case $q=1$ and $\theta=1/2$. In the hypothesis of $\theta>1/2$ and $q\geq 1$ we apply Hölder inequality to obtain

$$\Big[\sum_{j=0}^{\infty}(2^{j\theta}(II)^q\Big]^{1/q} \quad\leq\quad \|\chi_T\|_{B^{\theta}_{2,q}}\Big[\sum_{l=0}^{\infty}(2^{l(\theta-1)}\|\varphi_l(\sigma)F_k(\sigma,\xi)\|_{L^2_\sigma L^2_\xi})^q\Big]^{1/q}\Big[\sum_{l=0}^{\infty}2^{l(1/2-\theta)q'}\Big]^{1/q'}$$

Hence the estimate for II holds. To estimate the fist term we need to multiply with $\sum_{m=0}^{\infty}\varphi_m(\sigma)=1$ and $\sum_{l=0}^{\infty}\varphi_l(\tau-\sigma)=1$. This way we obtain

$$\begin{aligned}\Big[\sum_{j=0}^{\infty}(2^{j\theta}(I)^q\Big]^{1/q} &\leq \Big[\sum_{j=0}^{\infty}\Big[\sum_{l,m=0}^{\infty}(2^{j\theta}2^{-m}\|\varphi_j(\tau)\int\varphi_l(\tau-\sigma)\widehat{\chi}_{1/T}(\tau-\sigma)\varphi_m(\sigma)F_k(\sigma,\xi)d\sigma\|_{L^2_\tau L^2_\xi})^q\Big]^{1/q}\Big]\\ &\leq \sum_{j,l,m=0}^{\infty}2^{j\theta}2^{-m}\|\varphi_j(\tau)\int\varphi_l(\tau-\sigma)\widehat{\chi}_{1/T}(\tau-\sigma)\varphi_m(\sigma)F_k(\sigma,\xi)d\sigma\|_{L^2_\tau L^2_\xi}\end{aligned}$$

since $l^1\subset l^q$ for $q\geq 1$. We proceed as above. We know form paraproduct estimate that the term $\varphi_j(\tau)\varphi_m(\sigma)\varphi_l(\tau-\sigma)$ is non zero only in the following three cases:

i. $m\approx j$ and $l\ll m$

ii. $m\approx l$ and $l\gg j$

iii. $l\approx j$ and $m\ll l$

Therefore the triple series over j,l,m is reduced to a single serie plus a finite sum. The argument to prove the bound in the case (1) and (2) are quite similar, while (3) requires extra care.

i. In this case we first use the trivial bound $|\varphi_m(\tau)|\leq 1$ and then apply Young's inequality for convolution to obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{l=0}^{m-3}2^{m(\theta-1)}\|\varphi_m(\tau)(\varphi_l\widehat{\chi}_{1/T}*\varphi_mF_k)(\tau)\|_{L^2_\tau L^2_\xi} &\leq \sum_{m=0}^{\infty}\sum_{l=0}^{m-3}2^{m(\theta-1)}\|\varphi_l\widehat{\chi}_{1/T}\|_{L^\infty_\tau}\|\varphi_mF_k\|_{L^2_\tau L^2_\xi}\\ &\leq \sum_{m=0}^{\infty}2^{m(\theta-1)}\|\varphi_mF_k\|_{L^2_\tau L^2_\xi}\end{aligned}$$

since $\sum_{l=0}^{m-3}\|\varphi_l\widehat{\chi}_{1/T}\|_{L^\infty_\tau}\lesssim 1$.

ii. Here again use the trivial bound $|\varphi_k(\tau)|\leq 1$ and then apply Young's inequality for convolution to obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{j=0}^{m-3}2^{j\theta-m}\|\varphi_j(\tau)(\varphi_m\widehat{\chi}_{1/T}*\varphi_mF_k)(\tau)\|_{L^2_\tau L^2_\xi} &\leq \sum_{m=0}^{\infty}\sum_{j=0}^{m-3}2^{j\theta-m}\|\varphi_m\widehat{\chi}_{1/T}\|_{L^\infty_\tau}\|\varphi_mF_k\|_{L^2_\tau L^2_\xi}\\ &\lesssim \sum_{m=0}^{\infty}2^{m(\theta-1)}\|\varphi_mF_k\|_{L^2_\tau L^2_\xi}\end{aligned}$$

since $\sum_{j=0}^{m-3} 2^{j\theta - m}\|\varphi_m \widehat{\chi}_{1/T}\|_{L^\infty_\tau} \lesssim 2^{-m}\sum_{l=0}^{m-3} 2^{j\theta} \lesssim 2^{m(\theta-1)}$.

iii. In this case we can't use directly the trivial bound $|\varphi_k(\tau)| \le 1$ but we first use Hölder inequality and then Young's inequality for convolution. We have

$$\begin{aligned}\sum_{l=0}^{\infty}\sum_{m=0}^{l-3} 2^{l\theta-m}\|\varphi_l(\tau)(\varphi_l\widehat{\chi}_{1/T} * \varphi_m F_k)(\tau)\|_{L^2_\tau L^2_\xi} &\le \sum_{l=0}^{\infty}\sum_{m=0}^{l-3} 2^{l\theta-m}\|\varphi_l\|_{L^2_\tau}\|\varphi_l\widehat{\chi}_{1/T} * \varphi_m F_k\|_{L^\infty_\tau L^2_\xi} \\ &\lesssim \sum_{l=0}^{\infty}\sum_{m=0}^{l-3} 2^{l(\theta+1/2)-m}\|\varphi_l\widehat{\chi}_{1/T}\|_{L^2_\tau}\|\varphi_m F_k\|_{L^2_\tau L^2_\xi} \\ &\le \|\psi_{1/T}\|_{B^{\theta+1/2}_{2,1}(\mathbb{R})}\sum_{m=0}^{\infty} 2^{-m/2}\|\varphi_m F_k\|_{L^2_\tau L^2_\xi}\end{aligned}$$

and $\|\chi_{1/T}\|_{B^{\theta+1/2}_{2,1}(\mathbb{R})} < \infty$ since $\chi \in \mathscr{S}(\mathbb{R})$. Therefore the proposition holds in the case $q = 1$ and $\theta = 1/2$. In the case $\theta > 1/2$ and $q \ge 1$ proceed as for the II term. First assume $\theta = 1/2$ in the above computation, then multiply by $2^{m\theta}2^{-m\theta}$, and apply Hölder inequality. □

In the last part of this chapter we will outline how the $H^{s,\theta}_{p,q}$ spaces are used to improve the low regularity theory for a wide class of nonlinear wave equations.

1.7 Wave maps equation

Using a contraction argument in wave-Sobolev spaces, in this section we are able to a prove sharp local well-posedness result for subcritical wave maps. Consider the Cauchy problem:

$$\begin{cases} \Box u = \Gamma(u)\partial^\alpha u \partial_\alpha u \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \tag{1.8}$$

where $i, j, k = 1, \ldots, N$. Notice that $u : \mathbb{R}^{1+n} \to \mathbb{R}^N$ is vector valued and the expression $\Gamma(u)\partial^\alpha u\partial_\alpha u$ stands for $\Gamma^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k$, where $u^i : \mathbb{R}^{1+n} \to \mathbb{R}$ are the component of the vector u. The following theorem is based on the works [35], [105] and [114].

Theorem 31. *Let $n \ge 3$, $p \ge 1$, and $s > n/2$, then there exist an unique local solution $u \in C([0,T], B^s_{2,p}(\mathbb{R}^n)) \cap C^1([0,T], B^{s-1}_{2,p}(\mathbb{R}^n))$ to the Cauchy problem* (1.8) *with initial data in $B^s_{2,p}(\mathbb{R}^n) \times B^{s-1}_{2,p}(\mathbb{R}^n)$. Moreover if $n \ge 4$ and $p = 1$ one can take $s = n/2$.*

The result is sharp in view of the ill-posedness result of D'Ancona and Georgiev [13]. The proof given here follows the argument given in [29], extending it to initial data lying in a Besov space. Notice that we restrict the discussion to $n \ge 3$ since the $n = 2$ case require some modification of the $H^{s,\theta}_{p,q}$ norm. We list below the key nonlinear estimates that are needed in the proof Theorem 31. The first one is a Moser-type result:

Theorem 32 ([91])**.** *Assume that $\Gamma \in C^\infty(\mathbb{R}^N)$ with all derivatives bounded and $\Gamma(0) = 0$. Let $n \ge 2$ and $1/2 < \theta \le s - \frac{n-1}{2}$, then there exists a continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$g(0)=0$ *and*

$$\|\Gamma(u)\|_{H^{s,\theta}_{p,q}} \leq g\Big(\|u\|_{H^{\frac{n}{2}+\theta-\frac{1}{2},\theta}_{p,q}}\Big)\|u\|_{H^{s,\theta}_{p,q}}$$

Notice that since $s \geq n/2+\theta-1/2$ we have $H^{s,\theta}(\mathbb{R}^{1+n}) \subset H^{\frac{n}{2}+\theta-\frac{1}{2},\theta}(\mathbb{R}^{1+n})$, thus

$$\|\Gamma(u)\|_{H^{s,\theta}_{p,q}} \leq g\Big(\|u\|_{H^{s,\theta}_{p,q}}\Big)\|u\|_{H^{s,\theta}_{p,q}}$$

The proof of Theorem 32 follows from the proof of Theorem 7.7 in [45] and it will not be presented here. The second result needed in the proof of Theorem 32 is the key estimate for the N_0 null-forms.

Proposition 33 (N_0 Product Estimate). *Let $N_0(u,v)=\partial_\alpha u\partial^\alpha v$, then the estimate*

$$\|N_0(u,v)\|_{H^{s-1,\theta-1}_{p,q}} \lesssim \|u\|_{H^{s,\theta}_{p,q}}\|v\|_{H^{s,\theta}_{p,q}}$$

holds if one of the three assumptions form Table 1.1 below is satisfied.

$n \geq 3$,	$p>1$,	$q>1$,	and $s-(n-1)/2>\theta>1/2$
$n \geq 3$,	$p>1$,	$q=1$,	and $s-(n-1)/2>\theta=1/2$
$n \geq 4$,	$p=1$,	$q>1$,	and $s-(n-1)/2=\theta>1/2$

Table 1.1: Assumptions on the exponents

We will see below that the estimate in Proposition 33 is a consequence of the following two multiplicative properties of $H^{s,\theta}_{p,q}$ spaces.

Proposition 34. *Suppose that one of three hypothesis from Table 1.1 holds, then the space $H^{s,\theta}_{p,q}$ is an algebra, that is*

$$\|uv\|_{H^{s,\theta}_{p,q}} \lesssim \|u\|_{H^{s,\theta}_{p,q}}\|v\|_{H^{s,\theta}_{p,q}}$$

Proposition 35. *Suppose that one of three hypothesis from Table 1.1 holds, then we have the asymmetric multiplicative estimate*

$$\|uv\|_{H^{s-1,\theta-1}_{p,q}} \lesssim \|u\|_{H^{s,\theta}_{p,q}}\|v\|_{H^{s-1,\theta-1}_{p,q}}$$

Proof of Proposition 33. Recall the relationship between the N_0 null-form and the d'Alembert operator: $N_0(u,v)=\frac{1}{2}[\Box(uv)-v\Box u-u\Box v]$. Therefore Propositions 34 and 35 yield to

$$\begin{aligned}
\|N_0(u,v)\|_{H^{s-1,\theta-1}_{p,q}} &\lesssim \|\Box(uv)\|_{H^{s-1,\theta-1}_{p,q}}+\|v\Box u\|_{H^{s-1,\theta-1}_{p,q}}+\|u\Box v\|_{H^{s-1,\theta-1}_{p,q}}\\
&\lesssim \|uv\|_{H^{s,\theta}_{p,q}}+\|v\|_{H^{s,\theta}_{p,q}}\|\Box u\|_{H^{s-1,\theta-1}_{p,q}}+\|u\|_{H^{s,\theta}_{p,q}}\|\Box v\|_{H^{s-1,\theta-1}_{p,q}}\\
&\lesssim \|u\|_{H^{s,\theta}_{p,q}}\|v\|_{H^{s,\theta}_{p,q}}
\end{aligned}$$

□

We postpone the proof of Propositions 34 and 35 for the moment, and we show how Theorem 32 and Proposition 33 imply Theorem 31.

Proof of Theorem 31. Define the null-form nonlinearity $N(u,\partial u) = \Gamma(u)\partial^\alpha u \partial_\alpha u$. The linear theory developed in §1.5 can be extended to vector valued functions: if $\theta > 1/2$ and $q \geq 1$ or if $\theta = 1/2$ and $q = 1$ we have $\chi_T \mathcal{H} \in \mathcal{L}(B^s_{2,p} \times B^{s-1}_{2,p}, H^{s,\theta}_{p,q})$ with $\|\mathcal{H}\|_{\mathcal{L}} \approx T^{\frac{1}{2}-\theta}$, and $\chi_T \Box^{-1} \in \mathcal{L}(H^{s-1,\theta-1}_{p,q}, H^{s-1,\theta-1}_{p,q})$ with $\|\Box^{-1}\|_{\mathcal{L}} \approx T^{\frac{1}{2}-\theta}$; moreover we have $H^{s,\theta} \subset C_b(\mathbb{R}, B^s_{2,p})$. In view of these linear results the thesis follows once we prove the following nonlinear estimates:

(i) $\|N(u,\partial u)\|_{H^{s-1,\theta-1}_{p,q}} \lesssim C_{\mathcal{N}}(\|u\|_{H^{s,\theta}_{p,q}})$

(ii) $\|N(u,\partial u) - N(v,\partial v)\|_{H^{s-1,\theta-1}_{p,q}} \lesssim C_{\mathcal{D}}(\|u\|_{H^{s,\theta}_{p,q}}, \|v\|_{H^{s,\theta}_{p,q}})\|u-v\|_{H^{s,\theta}_{p,q}}$

where $C_{\mathcal{N}}$ and $C_{\mathcal{D}}$ are positive continuous functions such that $C_{\mathcal{N}}(0) = C_{\mathcal{D}}(0) = 0$. The asymmetric multiplicative estimate from Proposition 35, together with Theorem 32, and the N_0 null-form estimate of Proposition 33 yield to

$$\|N(u,\partial u)\|_{H^{s-1,\theta-1}_{p,q}} \leq \|\Gamma(u)\|_{H^{s,\theta}_{p,q}} \|N_0(\partial u,\partial u)\|_{H^{s-1,\theta-1}_{p,q}} \lesssim C_{\mathcal{N}}(\|u\|_{H^{s,\theta}_{p,q}})$$

This proves (i). Now let us show (ii), to estimate the difference $N(u,\partial u) - N(v,\partial v)$ observe that we can write

$$\begin{aligned}
N(\partial u) - N(\partial v) &= \frac{1}{2}\Gamma(u)[\Box(u^2) - 2u\Box u] - \frac{1}{2}\Gamma(v)[\Box(v^2) - 2v\Box v] \\
&= \frac{1}{2}\left[\Gamma(u)\Box(u^2) - \Gamma(v)\Box(v^2)\right] + \Gamma(v)v\Box v - \Gamma(u)u\Box u \\
&= \frac{1}{2}\Big[\big(\Gamma(u) - \Gamma(v)\big)\Box(u^2) + \Gamma(v)\Box(u^2 - v^2)\Big] \\
&+ \big(\Gamma(v) - \Gamma(u)\big)v\Box v + \Gamma(u)v\Box(v-u) + \Gamma(u)(v-u)\Box u
\end{aligned}$$

Therefore we need to estimate five terms:

$$\begin{aligned}
\|\big(\Gamma(u) - \Gamma(v)\big)\Box(u^2)\|_{H^{s-1,\theta-1}_{p,q}} &\leq \|\Gamma(u) - \Gamma(v)\|_{H^{s,\theta}_{p,q}} \|\Box(u^2)\|_{H^{s-1,\theta-1}_{p,q}} \\
&\leq g(\|u\|_{H^{s,\theta}_{p,q}}, \|v\|_{H^{s,\theta}_{p,q}})\|u-v\|_{H^{s,\theta}_{p,q}} \|u^2\|_{H^{s,\theta}_{p,q}} \\
&\leq C_{\mathcal{D}}(\|u\|_{H^{s,\theta}_{p,q}}, \|v\|_{H^{s,\theta}_{p,q}})\|u-v\|_{H^{s,\theta}_{p,q}}
\end{aligned}$$

$$\begin{aligned}
\|\Gamma(v)\Box(u^2 - v^2)\|_{H^{s-1,\theta-1}_{p,q}} &\leq \|\Gamma(v)\|_{H^{s,\theta}_{p,q}} \|\Box(u^2 - v^2)\|_{H^{s-1,\theta-1}_{p,q}} \\
&\leq g(\|v\|_{H^{s,\theta}_{p,q}})\|u+v\|_{H^{s,\theta}_{p,q}} \|u-v\|_{H^{s,\theta}_{p,q}} \\
&\leq C_{\mathcal{D}}(\|u\|_{H^{s,\theta}_{p,q}}, \|v\|_{H^{s,\theta}_{p,q}})\|u-v\|_{H^{s,\theta}_{p,q}}
\end{aligned}$$

$$\begin{aligned}\|\big(\Gamma(v)-\Gamma(u)\big)v\Box v\|_{H^{s-1,\theta-1}_{p,q}} &\leq \|\Gamma(v)-\Gamma(u)\|_{H^{s,\theta}_{p,q}}\|v\Box v\|_{H^{s-1,\theta-1}_{p,q}}\\ &\leq g(\|u\|_{H^{s,\theta}_{p,q}},\|v\|_{H^{s,\theta}_{p,q}})\|u-v\|_{H^{s,\theta}_{p,q}}\|v\|_{H^{s,\theta}_{p,q}}\|\Box v\|_{H^{s-1,\theta-1}_{p,q}}\\ &\leq C_{\mathscr{D}}(\|u\|_{H^{s,\theta}_{p,q}},\|v\|_{H^{s,\theta}_{p,q}})\|u-v\|_{H^{s,\theta}_{p,q}}\end{aligned}$$

$$\begin{aligned}\|\Gamma(u)v\Box(v-u)\|_{H^{s-1,\theta-1}_{p,q}} &\leq \|\Gamma(u)\|_{H^{s,\theta}}\|v\Box(v-u)\|_{H^{s-1,\theta-1}_{p,q}}\\ &\leq g(\|u\|_{H^{s,\theta}_{p,q}})\|v\|_{H^{s,\theta}_{p,q}}\|\Box(v-u)\|_{H^{s-1,\theta-1}_{p,q}}\\ &\leq C_{\mathscr{D}}(\|u\|_{H^{s,\theta}_{p,q}},\|v\|_{H^{s,\theta}_{p,q}})\|u-v\|_{H^{s,\theta}_{p,q}}\end{aligned}$$

$$\begin{aligned}\|\Gamma(u)(v-u)\Box u\|_{H^{s-1,\theta-1}_{p,q}} &\leq \|\Gamma(u)\|_{H^{s,\theta}_{p,q}}\|(v-u)\Box u\|_{H^{s-1,\theta-1}_{p,q}}\\ &\leq g(\|u\|_{H^{s,\theta}_{p,q}})\|\Box u\|_{H^{s-1,\theta-1}_{p,q}}\|v-u\|_{H^{s,\theta}_{p,q}}\\ &\leq C_{\mathscr{D}}(\|u\|_{H^{s,\theta}_{p,q}},\|v\|_{H^{s,\theta}_{p,q}})\|u-v\|_{H^{s,\theta}_{p,q}}\end{aligned}$$

The constants in the nonlinear estimates do not depend on time. However, since the constants in the linear estimates do depend on time, the usual fixed point argument apply: by choosing the time of existence small enough we can assure the existence of a unique local solution. □

In the last part of this section we apply Littlewood-Paley techniques to prove the multiplicative properties of $H^{s,\theta}_{p,q}$ spaces: Propositions 34 and 35. Let us start by derive the corresponding Strichartz estimate in the context of $H^{s,\theta}_{p,q}$ spaces that will be used in the sequel. Recall from Appendix A the frequency-localized Strichartz estimates for half-wave propagator: let (p,q,s) a Strichartz admissible triple, then for every dyadic number $\lambda \in 2^{\mathbb{Z}}$:

$$\|e^{\pm itD}P_\lambda f\|_{L^pL^q} \lesssim \lambda^s\|P_\lambda f\|_2$$

Lemma 36 (Strichartz inequality for modulation cutoffs)**.** *Let (p,q,s) a Strichartz admissible triple then for every $j,k\in\mathbb{Z}$ we have*

$$\|P_kQ_ju\|_{L^pL^q} \lesssim 2^{ks}2^{j/2}\|P_kQ_ju\|_{L^2L^2}$$

Notice that $s = n/2-1/2-n/q$ and it is zero only when $(p,q)=(2,\infty)$ the energy couple, when $(p,q)=(\infty,\infty)$ one loses the maximum in the previous bound since $s=n/2$. On the other hand when $p=2$ and $\frac{2n-2}{n-3}\leq q<\infty$ ones loses a factor of $s=n/2-1/2-n/q$. Finally if (p,q) are sharp, i.e. $2/p+(n-1)/2=(n-1)/2$, one loses $s=(n/2+1/2)(1/2-1/q)$ for $2\leq q\leq\frac{2n-2}{n-3}$.

Proof. By definition of frequency and modulation cutoffs we have

$$\begin{aligned}
P_kQ_ju(t,x) &= \iint e^{it\tau+ix\cdot\xi}\varphi_k(\xi)\varphi_j(|\tau|-|\xi|)\widetilde{u}(\tau,\xi)\;d\xi d\tau\\
&= \iint e^{it\tau+ix\cdot\xi}\varphi_k(\xi)\left[\varphi_j(\tau-|\xi|)\widetilde{u}_+(\tau,\xi)+\varphi_j(\tau+|\xi|)\widetilde{u}_-(\tau,\xi)\right]\;d\xi d\tau\\
&= \int e^{it\tau}\int e^{ix\cdot\xi+it|\xi|}\varphi_k(\xi)\varphi_j(\tau)\widetilde{u}_+(\tau+|\xi|,\xi)\;d\xi d\tau\\
&+ \int e^{it\tau}\int e^{ix\cdot\xi-it|\xi|}\varphi_k(\xi)\varphi_j(\tau)\widetilde{u}_-(\tau-|\xi|,\xi)\;d\xi d\tau\\
&= \int e^{it\tau}(e^{-itD}P_kf_{\tau,j,+})(x)\;d\tau+\int e^{it\tau}(e^{-itD}P_kf_{\tau,j,-})(x)\;d\tau
\end{aligned}$$

where $\widetilde{u}_\pm(\tau,\xi)=\chi_{\{\pm\tau\geq 0\}}\widetilde{u}(\tau,\xi)$ and $\widehat{f}_{\tau,j,\pm}(\xi)=\varphi_j(\tau)\widetilde{u}_\pm(\tau\pm|\xi|,\xi)$. Therefore Minkowski and Cauchy Schwarz inequalities yield to

$$\begin{aligned}
\|P_kQ_ju\|_{L^p_tL^q_x} &\lesssim \int \|e^{\pm itD}P_kf_{\tau,j,\pm}\|_{L^p_tL^q_x}\;d\tau\\
&\lesssim 2^{ks}\int\|P_kf_{\tau,j,\pm}\|_{L^2_x}d\tau\\
&= 2^{ks}\int\|\varphi_k(\xi)\varphi_j(\tau)\widetilde{u}_\pm(\tau\pm|\xi|,\xi)\|_{L^2_\xi}d\tau\\
&\lesssim 2^{ks}\|\widetilde{\widetilde{\varphi}}_j(\tau)\|_{L^2_\tau}\|\varphi_k(\xi)\varphi_j(\tau)\widetilde{u}_\pm(\tau\pm|\xi|,\xi)\|_{L^2_\xi L^2_\tau}\\
&\lesssim 2^{ks}2^{j/2}\|P_kQ_ju\|_{L^2L^2}
\end{aligned}$$

where $\widetilde{\widetilde{\varphi}}_j$ is such that $\widetilde{\widetilde{\varphi}}_j\varphi_j=\varphi_j$. □

Corollary 37 (Strichartz inequality for $H^{\theta,q}_k$ spaces)**.** *Let (p,q,s) a Strichartz admissible triple and $\theta>1/2$ for $q\geq 1$ or $\theta=1/2$ for $q=1$. Then for every $k\in\mathbb{Z}$ we have*

$$\|P_ku\|_{L^pL^q}\lesssim 2^{ks}\|P_ku\|_{H^{\theta,q}_k}$$

Moreover, let (p,q,s) a Strichartz admissible triple and $\sigma,\theta\in\mathbb{R}$, we have

$$\|P_kQ_ju\|_{L^pL^q}\lesssim 2^{k(s-\sigma)}2^{j(1/2-\theta)}\|P_kQ_ju\|_{H^{\sigma,\theta,q}_k}$$

Proof. The proof follow from an application of Hölder inequality and Strichartz inequality for modulation cutoffs. We have

$$\begin{aligned}
\|P_k u\|_{L^pL^q} &\lesssim \Big[\sum_{j=0}^{\infty} \|P_k Q_j u\|^2_{L^pL^q}\Big]^{1/2} \\
&\lesssim 2^{ks}\Big[\sum_{j=0}^{\infty} 2^j \|P_k Q_j u\|^2_{L^2L^2}\Big]^{1/2} \\
&\lesssim 2^{ks}\Big[\sum_{j=0}^{\infty} (2^{j\theta}\|P_k Q_j u\|_{L^2L^2})^q\Big]^{1/q}\Big[\sum_{j=0}^{\infty} 2^{j(1-2\theta)r}\Big]^{1/r}
\end{aligned}$$

where $1/2 = 1/q + 1/r$, the second sum converges since $\theta > 1/2$. This conclude the proof in the case $\theta > 1/2$ and $q \geq 1$. If $q = 1$ we can simply used the embedding $l^1 \subset l^2$ instead for Cauchy-Schwarz to obtain

$$2^{ks}\Big[\sum_{j=0}^{\infty} (2^{j/2}\|P_k Q_j u\|_{L^2L^2})^2\Big]^{1/2} \lesssim 2^{ks}\sum_{j=0}^{\infty} 2^{j/2}\|P_k Q_j u\|_{L^2L^2}$$

□

In the proof of the multiplicative properties of $H^{s,\theta}_{p,q}$ spaces we shall need the following fact about the $P_k Q_j$ multipliers.

Lemma 38. *For every k_1, k_2, k_3 and j_1, j_2, j_3 we have*

$$P_{k_3} Q_{j_3} (P_{k_1} Q_{j_1} u P_{k_2} Q_{j_2} v) = 0$$

unless one of the following cases is satisfied:

i. LHH - low modulations: $k_1 \ll k_2 \approx k_3$, $j^{123}_{\max} \precsim k_1$.

ii. LHH - med modulations: $k_1 \ll k_2 \approx k_3$, $j^{12}_{\min} \approx j^{12}_{\max} \gg k_1$ and $j_3 \precsim j^{12}_{\max}$.

iii. LHH - high modulations: $k_1 \ll k_2 \approx k_3$, $j^{12}_{\max} \gg \max\{j^{12}_{\min}, k_1\}$ and $j_3 \approx j^{12}_{\max}$.

iv. HLH - low modulations: $k_2 \ll k_1 \approx k_3$ and $j^{123}_{\max} \precsim k_2$.

v. HLH - med modulations: $k_2 \ll k_1 \approx k_3$, $j^{12}_{\min} \approx j^{12}_{\max} \gg k_2$ and $j_3 \precsim j^{12}_{\max}$.

vi. HLH - high modulations: $k_2 \ll k_1 \approx k_3$, $j^{12}_{\max} \gg \max\{j^{12}_{\min}, k_2\}$ and $j_3 \approx j^{12}_{\max}$.

vii. HHL - low modulations: $k_1 \approx k_2 \succsim k_3$ and $j^{123}_{\max} \precsim k^{12}_{\min}$.

viii. HHL - med modulations: $k_1 \approx k_2 \succsim k_3$, $j^{12}_{\min} \approx j^{12}_{\max} \gg k^{12}_{\max}$ and $j_3 \precsim j^{12}_{\max}$.

ix. HHL - high modulations: $k_1 \approx k_2 \succsim k_3$, $j^{12}_{\max} \gg \max\{j^{12}_{\min}, k^{12}_{\max}\}$ and $j_3 \approx j^{12}_{\max}$.

The cases $i.$, $iii.$ and $vii.$ in the literature are sometimes referred to as hyperbolic regime, while the remaining cases are called to elliptic regime.

Proof. Observe that

$$\begin{aligned}
&\operatorname{supp} \mathscr{F}_{\tau,\xi}(P_{k_1}Q_{j_1}u P_{k_2}Q_{j_2}v) \\
&= \operatorname{supp}\widetilde{P_{k_1}Q_{j_1}u} + \operatorname{supp}\widetilde{P_{k_2}Q_{j_2}v} \\
&= \{(\tau,\xi) \in \mathbb{R}^{1+n} : \tau = \tau_1 + \tau_2,\ \xi = \xi_1 + \xi_2,\ 2^{k_i-1} \le |\xi_i| \le 2^{k_i+1}, 2^{j_i-1} \le ||\tau_i| - |\xi_i|| \le 2^{j_i+1}\}.
\end{aligned}$$

Moreover we have $|\xi^{12}_{\max}| - |\xi^{12}_{\min}| \le |\xi| \le |\xi^{12}_{\max}| + |\xi^{12}_{\min}|$, where $\xi^{12}_{\max} = \max\{\xi_1, \xi_2\}$ and $\xi^{12}_{\min} = \min\{\xi_1, \xi_2\}$. Hence

$$2^{k^{12}_{\max}-1} - 2^{k^{12}_{\min}+1} \le |\xi| \le 2^{k^{12}_{\max}+1} + 2^{k^{12}_{\min}+1}$$

Suppose there is some separation between the two inputs frequencies: $k^{12}_{\min} \le k^{12}_{\max} - 4$, then clearly

$$2^{k^{12}_{\max}}(2^{-1} - 2^{-3}) \le |\xi| \le 2^{k^{12}_{\max}}(2^{-3} + 2)$$

which implies $k_3 \approx k^{12}_{\max}$. On the other hand if there is no separation between the two input frequencies: $k^{12}_{\max} - 3 \le k^{12}_{\min} \le k^{12}_{\max}$ then the lower bound on the sum is lost and we obtain:

$$|\xi| \le 2^{k^{12}_{\max}}(2^{-3} + 2)$$

which implies $k_3 \le k^{12}_{\max} + O(1)$.

Furthermore, let us analyze the sum of the modulations. A similar bound as for ξ holds for τ, indeed we have

$$2^{j_i+1} \ge ||\tau_i| - |\xi_i|| \ge |\tau_i| - |\xi_i| \ge |\tau_i| - 2^{k_i+1}$$

and

$$2^{j_i+1} \ge ||\tau_i| + |\xi_i|| \ge -|\tau_i| + |\xi_i| \ge -|\tau_i| + 2^{k_i-1}.$$

Hence

$$2^{k_i-1} - 2^{j_i+1} \le |\tau_i| \le 2^{k_i+1} + 2^{j_i+1}$$

We can control the outer modulation from above by the following argument. If $|\tau| > |\xi|$ then

$$||\tau| - |\xi|| = |\tau| - |\xi| \le |\tau_{\max}| + |\tau_{\min}| - |\xi_{\max}| + |\xi_{\min}| \le 2^{j_{\max}+1} + 2^{j_{\min}+1} + 2^{k_{\min}+2}.$$

Moreover, if $|\tau| < |\xi|$ then

$$||\tau| - |\xi|| = -|\tau| + |\xi| \le -|\tau_{\max}| + |\tau_{\min}| + |\xi_{\max}| + |\xi_{\min}| \le 2^{j_{\max}+1} + 2^{j_{\min}+1} + 2^{k_{\min}+2}.$$

Therefore by using the previous bounds we obtain

$$||\tau| - |\xi|| \le 2^{j_{\max}+1} + 2^{j_{\min}+1} + 2^{k_{\min}+2}$$

To obtain a lower bound on j_3 we proceed as follows. If $|\tau^{12}_{\max}| > |\xi^{12}_{\max}|$ observe that

$$\begin{aligned} ||\tau| - |\xi|| \geq |\tau| - |\xi| &\geq |\tau^{12}_{\max}| - |\tau^{12}_{\min}| - |\xi^{12}_{\max}| - |\xi^{12}_{\min}| \\ &= ||\tau^{12}_{\max}| - |\xi^{12}_{\max}|| - |\tau^{12}_{\min}| - |\xi^{12}_{\min}| \\ &\geq 2^{j^{12}_{\max}-1} - 2^{j^{12}_{\min}+1} - 2^{k^{12}_{\min}+2}. \end{aligned}$$

On the other hand if $|\tau^{12}_{\max}| < |\xi^{12}_{\max}|$ a similar argument yield to the same lower bound:

$$\begin{aligned} ||\tau| - |\xi|| \geq -|\tau| + |\xi| &\geq -|\tau^{12}_{\max}| - |\tau^{12}_{\min}| + |\xi^{12}_{\max}| - |\xi^{12}_{\min}| \\ &= ||\tau^{12}_{\max}| - |\xi^{12}_{\max}|| - |\tau^{12}_{\min}| - |\xi^{12}_{\min}| \\ &\geq 2^{j^{12}_{\max}-1} - 2^{j^{12}_{\min}+1} - 2^{k^{12}_{\min}+2}. \end{aligned}$$

Therefore we have obtained the needed control over the outer modulation:

$$2^{j^{12}_{\max}-1} - 2^{j^{12}_{\min}+1} - 2^{k^{12}_{\min}+2} \leq ||\tau| - |\xi|| \leq 2^{j_{\max}+1} + 2^{j_{\min}+1} + 2^{k_{\min}+2}.$$

From the previous estimate it is easy to see that the lower bound is negative if $j^{12}_{max} \leq k^{12}_{\min} + O(1)$. This correspond to cases $i.$, $iv.$ and $vii.$ where we only obtain the upper bound on the output modulation $j_3 \leq k^{12}_{\min} + O(1)$.

Moreover, if we suppose that $j^{12}_{max} \geq k^{12}_{\min} + O(1)$, then we shall split further into two sub-cases. Firstly, if there is no separation between the inputs modulations, that is if $j^{12}_{\max} \approx j^{12}_{\min}$, then the two modulations can cancel out and give a much smaller outer modulation. Hence in this case, which correspond to cases $ii.$, v and $viii.$, we have the bound $j_3 \leq j^{12}_{max} + O(1)$. Secondly, if the two inputs modulations are separated, i.e. $j^{12}_{\min} \ll j^{12}_{\max}$, then we finally obtain a lower bound on the output modulation:

$$j^{12}_{\max} + O(1) \leq j_3 \leq j^{12}_{\max} + O(1)$$

which implies $iii.$, $vi.$ and $ix.$. □

We are now ready to prove the algebra property of $H^{s,\theta}_{p,q}$ spaces.

Proof of Proposition 34. Recall that one can view $H^{s,\theta}_{p,q}$ as an l^p Besov space with base the space $H^{s,\theta,q}_k$ therefore it suffices to prove the following two frequency localized estimates:

(High-output) Let $k_1 \ll k_3$, if $p > 1$ and $\alpha < 0$, or if $p = 1$ and $\alpha \leq 0$, we have

$$\|P_{k_3}(P_{k_1} u P_{k_3} v)\|_{H^{s,\theta,q}_{k_3}} \lesssim 2^{k_1 \alpha} \|P_{k_1} u\|_{H^{s,\theta,q}_{k_1}} \|P_{k_3} v\|_{H^{s,\theta,q}_{k_3}}$$

(Low-output) Let $k_1 \gg k_3$, $\beta > 0$, and $\alpha + \beta \le 0$, then we have

$$\|P_{k_3}(P_{k_1}uP_{k_1}v)\|_{H^{s,\theta,q}_{k_3}} \lesssim 2^{k_3\beta}2^{k_1\alpha}\|P_{k_1}u\|_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|_{H^{s,\theta,q}_{k_1}}$$

Let us start by proving the *low output estimate*. By splitting further $u = \sum_{j=0}^{\infty} Q_j u$ and $v = \sum_{j=0}^{\infty} Q_j v$ we obtain three different terms: one in which the three modulations are smaller then the lowest inner frequency and the other two which the maximum inner modulation is higher then the lowest inner frequency. By Lemma 38 we obtain

$$\begin{aligned}
\|P_{k_3}(P_{k_1}uP_{k_1}v)\|^q_{H^{s,\theta,q}_{k_3}} &= 2^{qk_3s}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3\theta}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_2}v)\|_2\Big)^q \\
&+ 2^{qk_3s}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3\theta}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_1}v)\|_2\Big)^q \\
&+ 2^{qk_3s}\sum_{j_3\ge k_1}\Big(\sum_{j_1}2^{j_3\theta}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_3}v)\|_2\Big)^q \\
&=: HHL_I + HHL_{II} + HHL_{III}
\end{aligned}$$

Observe that in the HHL_{II} and HHL_{III} terms we have suppose without loosing generality that $j_2 = \max\{j_1, j_2\}$. The the low modulations term HHL_I is slightly harder to estimate and requires the application of Hölder and Strichartz inequalities in the following clever way:

$$\begin{aligned}
\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_2}v)\|_2 &\lesssim \|P_{k_1}Q_{j_1}u\|_{L^{2+}L^\infty}\|P_{k_1}Q_{j_2}v\|_{L^\infty L^2} \\
&\lesssim 2^{k_1(n/2-1/2+)}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2
\end{aligned} \tag{1.9}$$

Therefore if $q = 1$ and $\theta = 1/2$ the j_1 and j_2 sums decouples and we simply have

$$\begin{aligned}
HHL_I &\lesssim 2^{k_3s}\Big(\sum_{j_3\le k_1}2^{j_3\theta}\Big)2^{k_1(-2s+n/2-1/2+)}\|P_{k_1}u\|_{H^{s,1/2,1}_{k_1}}\|P_{k_1}v\|_{H^{s,1/2,1}_{k_1}} \\
&\lesssim 2^{k_3s}2^{k_1(-2s+\theta+n/2-1/2+)}\|P_{k_1}u\|_{H^{s,1/2,1}_{k_1}}\|P_{k_1}v\|_{H^{s,1/2,1}_{k_1}}
\end{aligned}$$

Therefore $\alpha = -2s + \theta + n/2 - 1/2+$ and $\beta = s$. On the other hand if $q > 1$ we must use Cauchy-Schwarz in j_1 and j_2 sums and to have convergent reminder we must impose $\theta > 1/2$. In this case we obtain

$$\begin{aligned}
HHL_I &\lesssim 2^{qk_3s}\Big(\sum_{j_3\le k_1}2^{j_3\theta q}\Big)2^{qk_1(-2s+n/2-1/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}} \\
&\lesssim 2^{qk_3s}2^{qk_1(-2s+\theta+n/2-1/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}}
\end{aligned}$$

This conclude the estimate for HHL_I, notice that it is sharp in terms of s and θ. Contrary, as we shall see, the estimates for the high-outer-modulation terms HHL_{II} and HHL_{III} are not sharp: they require just $s > n/2$. Indeed, to estimate HHL_{II} we put the term $P_{k_1}Q_{j_1}u$ into L^∞

and the term $P_{k_1}Q_{j_1}v$ into L^2, then Strichartz estimates and Cauchy-Schwarz inequality yield to:

$$\begin{aligned} HHL_{II} &\lesssim 2^{qk_3s}2^{qk_1n/2}\sum_{j_3}(2^{-\epsilon j_3 q})\Big(\sum_{j_1\geq j_3}2^{j_1/2}2^{j_3(\theta+\epsilon)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_1}v\|_2\Big)^q \\ &\lesssim 2^{qk_3s}2^{qk_1(-2s+n/2)}\sum_{j_3}\big(2^{-\epsilon j_3 q}\big)\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}} \end{aligned}$$

Observe that the j_3 sum converges and we have controlled $2^{j_1/2}2^{j_3(\theta+\epsilon)}\leq 2^{j_1\theta}2^{j_1\theta}$ since $\theta>1/2$ and $j_3\leq j_1$. Therefore this estimate holds for just $s>n/2$. To estimate the HHL_{III} term we proceed in a similar way: we put the lower-inner-modulation term $P_{k_1}Q_{j_1}u$ into L^∞ and the higher-inner-modulation term $P_{k_1}Q_{j_3}v$ into L^2, then we apply Strichartz estimates and Cauchy-Schwarz inequality. This yield to:

$$\begin{aligned} HHL_{III} &\lesssim 2^{qk_3s}2^{qk_1n/2}\sum_{j_3}(2^{j_3\theta}\|P_{k_1}Q_{j_3}v\|_2)^q\Big(\sum_{j_1}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\Big)^q \\ &\lesssim 2^{qk_3s}2^{qk_1(-2s+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}} \end{aligned}$$

Observe that the j_3 and j_1 sums decouples, moreover in the last line if $q>1$ we use Cauchy-Schwarz in j_1-sum and the hypothesis $\theta>1/2$ to control the reminder, whereas if $q=1$ and $\theta=1/2$ there is no need to use Cauchy-Schwarz. This conclude the proof of the low-output case.

Consider now the *high output interaction.* Thanks to symmetry it suffices to prove the LHH interaction case. Moreover, Lemma 38 allow us to split the LHH term into three cases:

$$\begin{aligned} \|P_{k_3}(P_{k_1}uP_{k_3}v)\|^q_{H^{s,\theta,q}_{k_3}} &= 2^{qk_3s}\sum_{j_3\leq k_1}\Big(\sum_{j_1\leq k_1}\sum_{j_2\leq k_1}2^{j_3\theta}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_3}Q_{j_2}v)\|_2\Big)^q \\ &+ 2^{qk_3s}\sum_{j_3}\Big(\sum_{j_1\geq\max\{k_1,j_3\}}2^{j_3\theta}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_3}Q_{j_1}v)\|_2\Big)^q \\ &+ 2^{qk_3s}\sum_{j_3\geq k_1}\Big(\sum_{j^{12}_{\max}\approx j_3}2^{j_3\theta}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_3}Q_{j_2}v)\|_2\Big)^q \\ &=: LHH_I+LHH_{II}+LHH_{III} \end{aligned}$$

Here we cannot suppose that $j_2=\max\{j_1,j_2\}$ since the two inner frequencies are not the same anymore. To estimate the low modulations term LHH_I we put the low frequency term $P_{k_1}Q_{j_1}u$

into $L^{2+}L^{\infty}$ and the high frequency term $P_{k_3}Q_{j_2}v$ into $L^{\infty}L^2$, then Strichartz estimates yield to

$$\begin{aligned} LHH_I &\lesssim 2^{qk_3s}\sum_{j_3\leq k_1}\Big(\sum_{j_1\leq k_1}\sum_{j_2\leq k_1}2^{k_1(n/2-1/2+)}2^{j_3\theta}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim \Big(\sum_{j_3\leq k_1}2^{j_3\theta q}\Big)2^{qk_1(-s+n/2-1/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \\ &\lesssim 2^{qk_1(-s+\theta+n/2-1/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

We have used the ubiquitous Cauchy-Schwarz inequality in j_1 and j_2 sums when $q>1$. Next, to control the middle modulations term LHH_{II} we place $P_{k_3}Q_{j_1}v$ into L^2 and $P_{k_1}Q_{j_1}u$ into L^{∞}. We then obtain

$$\begin{aligned} LHH_{II} &\lesssim 2^{qk_3s}\sum_{j_3}\Big(\sum_{j_1\geq j_3}2^{k_1n/2}2^{j_3\theta}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_1}v\|_2\Big)^q \\ &\lesssim 2^{qk_1(-s+n/2)}\sum_{j_3}\big(2^{-\epsilon j_3 q}\big)\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

Here we have used the fact that $2^{j_1/2}2^{j_3(\theta+\epsilon)}\leq 2^{j_1\theta}2^{j_1\theta}$. Next, we prove the estimate for LHH_{III} we split into two cases: firstly suppose that $j^{12}_{\max}=j_1\approx j_3$, then we place the low frequency term $P_{k_1}Q_{j_3}u$ into L^2L^{∞} and use Bernstein inequality. The high frequency term $P_{k_3}Q_{j_2}v$ goes into the energy space $L^{\infty}L^2$ and here we use Strichartz inequality. This yield to

$$\begin{aligned} LHH_{III} &\lesssim 2^{qk_3s}\sum_{j_3\geq k_1}\Big(\sum_{j_2}2^{k_1n/2}2^{j_3\theta}2^{j_2/2}\|P_{k_1}Q_{j_3}u\|_2\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_3s}2^{qk_1n/2}\sum_{j_3}(2^{j_3\theta}\|P_{k_1}Q_{j_3}u\|_2)^q\Big(\sum_{j_2}2^{j_2/2}\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_1(-s+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

Secondly, if $j^{12}_{\max}=j_2\approx j_3$ then we place the low frequency term $P_{k_1}Q_{j_1}u$ into $L^{\infty}L^{\infty}$, use Strichartz and proceed as in the previous case.

We summarize below the different α and β obtained in the estimates above:

	HHL_I	HHL_{II}	HHL_{III}	LHH_I	LHH_{II}	LHH_{II}
α	$-2s+\theta+(n-1)/2+$	$-2s+n/2$	$-s+n/2$	$-s+(n-1)/2+\theta+$	$-s+n/2$	$-s+n/2$
β	s	s				

Notice that the $+\epsilon$ in the estimates above comes form the fact that we have used the $L^{2+}L^{\infty}$ Strichartz estimate, hence it can be removed if $n\geq 4$ since the pair $(2,\infty)$ is Strichartz admissible in higher dimensions. To control the HHL_I and HHL_{II} terms $-s+(n-1)/2+\theta+\leq 0$ suffices; however the bounds for LHH_I, LHH_I and LHH_{III} requires, if $p>1$, the strict inequality $-s+(n-1)/2+\theta+<0$, and only if $p=1$ (and $n\geq 4$) we can relax it to $-s+(n-1)/2+\theta=0$. This is the reason why when $n\geq 4$ and both $p=q=1$ we obtain $s-(n-1)/2\geq\theta\geq 1/2$ which gives $s\geq n/2$. This conclude the proof of the algebra property. □

We now prove the second fundamental asymmetric multiplicative property of the $H^{s,\theta}$ spaces: Proposition 35. In the argument used in the proof it is necessary, to estimate low modulation terms, to replace Strichartz estimate by the following weaker Bernstein inequality.

Lemma 39 (Bernstein inequality)**.** *Let $p,q\geq 2$ and $j,k\in\mathbb{Z}$, then*

$$\|P_kQ_ju\|_{L^pL^q}\lesssim\max\{2^j,2^k\}^{\frac{1}{2}-\frac{1}{p}}2^{nk(\frac{1}{2}-\frac{1}{q})}\|P_kQ_ju\|_{L^2L^2}$$

Proof. We shall distinguish between two cases: modulation lower then the frequency, i.e. $j\leq k$, and modulation higher than the frequency, i.e. $j\geq k$. Let us introduce the following cutoff in time $T_{\leq k}u(t,x)=\mathscr{F}_t^{-1}[\varphi_{\leq k}(\tau)\mathscr{F}_tu(\tau,x)]$. If $j\leq k$ then $P_kQ_j=T_{\leq k+2}P_kQ_j$ since

$$\begin{aligned}\operatorname{supp}\widetilde{P_kQ_ju} &= \{(\tau,\xi):2^{k-1}\leq|\xi|\leq 2^{k+1},\ 2^{j-1}\leq||\tau|-|\xi||\leq 2^{j+1}\}\\ &\subset \{(\tau,\xi):2^{k-1}\leq|\xi|\leq 2^{k+1},\ |\tau|\leq 2^{k+2}\}\end{aligned}$$

Therefore Bernstein inequality in time and space yield to the desired estimate

$$\|P_kQ_ju\|_{L^pL^q}\lesssim 2^{k(\frac{1}{2}-\frac{1}{p})}2^{nk(\frac{1}{2}-\frac{1}{q})}\|P_kQ_ju\|_{L^2L^2}$$

On the other hand if $j\geq k$ then $P_kQ_j=T_{\leq j+2}P_kQ_j$ since

$$\begin{aligned}\operatorname{supp}\widetilde{P_kQ_ju} &= \{(\tau,\xi):2^{k-1}\leq|\xi|\leq 2^{k+1},\ 2^{j-1}\leq||\tau|-|\xi||\leq 2^{j+1}\}\\ &\subset \{(\tau,\xi):2^{k-1}\leq|\xi|\leq 2^{k+1},\ |\tau|\leq 2^{j+2}\}\end{aligned}$$

Thus we obtain the bound

$$\|P_kQ_ju\|_{L^pL^q}\lesssim 2^{j(\frac{1}{2}-\frac{1}{p})}2^{nk(\frac{1}{2}-\frac{1}{q})}\|P_kQ_ju\|_{L^2L^2}$$

□

The Bernstein inequality proved above is useful when one wants to control a frequency and modulation localized function in mixed Lebesgue spaces with high (greater then two) exponents, with the L^2L^2 norm. On the other hand, a cleaver combination of Bernstein and Minkowski inequalities allow one to control the L^2L^2 norm of a modulation localized function with its L^pL^2 norm, where $1\leq p<2$.

Lemma 40 (L^2-L^p Bernstein inequality)**.** *Let $1\leq p<2$ and $j\in\mathbb{Z}$, then*

$$\|Q_ju\|_{L^2L^2}\lesssim 2^{j\left(\frac{1}{p}-\frac{1}{2}\right)}\|Q_ju\|_{L^pL^2}$$

Proof. By Plancherel we can write

$$\|Q_j u\|_{L^2_t L^2_x} = \|\varphi_j(|\tau| - |\xi|)\widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}^{n+1})} = \|\|\varphi_j(|\tau| - |\xi|)\widetilde{u}(\tau,\xi)\|_{L^2_\tau}\|_{L^2_\xi}$$

If we define $\widetilde{u}_\pm(\tau,\xi) = \chi_{\pm\tau>0}\widetilde{u}(\tau,\xi)$ we have

$$\begin{aligned}
\|\varphi_j(|\tau| - |\xi|)\widetilde{u}(\tau,\xi)\|_{L^2_\tau} &\leq \|\varphi_j(\tau - |\xi|)\widetilde{u}_+(\tau,\xi)\|_{L^2_\tau} + \|\varphi_j(\tau + |\xi|)\widetilde{u}_-(\tau,\xi)\|_{L^2_\tau} \\
&= \|\varphi_j(\tau)\widetilde{u}_+(\tau + |\xi|,\xi)\|_{L^2_\tau} + \|\varphi_j(\tau)\widetilde{u}_-(\tau - |\xi|,\xi)\|_{L^2_\tau}
\end{aligned}$$

Define the functions $\widetilde{f}_\pm(\tau,\xi) = \widetilde{u}_\pm(\tau \pm |\xi|,\xi)$, then Bernstein inequality yield to the bound

$$\|\varphi_j(|\tau| - |\xi|)\widetilde{u}(\tau,\xi)\|_{L^2_\tau} \leq \|T_j\widehat{f}_\pm(\cdot,\xi)\|_{L^2_t} \lesssim 2^{j\left(\frac{1}{p}-\frac{1}{2}\right)}|T_j\widehat{f}_\pm(\cdot,\xi)\|_{L^p_t}$$

Therefore if we estimate its L^2_ξ norm, by Plancherel and Minkowski inequality we obtain

$$\|Q_j u\|_{L^2_t L^2_x} \lesssim 2^{j\left(\frac{1}{p}-\frac{1}{2}\right)}\|\|T_j\widehat{f}_\pm(\cdot,\xi)\|_{L^p_t}\|_{L^2_\xi} \lesssim 2^{j\left(\frac{1}{p}-\frac{1}{2}\right)}\|\|Q_j u\|_{L^p_t}\|_{L^2_x} \lesssim 2^{j\left(\frac{1}{p}-\frac{1}{2}\right)}\|\|Q_j u\|_{L^2_x}\|_{L^p_t}$$

□

Notice that Strichartz inequality implies Bernstein inequality in the low modulation regime and at high modulation the situation is reversed. In fact we have

$$\max\{2^j, 2^k\}^{\frac{1}{2}-\frac{1}{p}}2^{nk(\frac{1}{2}-\frac{1}{q})} = \begin{cases} 2^{k(\frac{n}{2}-\frac{1}{p}-\frac{n}{q}+\frac{1}{2})} & \text{if } j \leq k \\ 2^{j(\frac{1}{2}-\frac{1}{p})}2^{k(\frac{n}{2}-\frac{n}{q})} & \text{if } j \geq k \end{cases}$$

and if $j \leq k$ we have

$$2^{j/2}2^{ks} = 2^{j/2}2^{k(\frac{n}{2}-\frac{1}{p}-\frac{n}{q})} \leq 2^{k(\frac{n}{2}-\frac{1}{p}-\frac{n}{q}+\frac{1}{2})}$$

while if $j \geq k$ we obtain

$$2^{j/2}2^{ks} = 2^{j/2}2^{k(\frac{n}{2}-\frac{1}{p}-\frac{n}{q})} \geq 2^{j(\frac{1}{2}-\frac{1}{p})}2^{k(\frac{n}{2}-\frac{n}{q})}$$

Proof of Proposition 35. Due to lack of symmetry here we must prove following three frequency localized estimates:

(LHH) Let $k_1 \ll k_3$, if $p > 1$ and $\alpha < 0$, or if $p = 1$ and $\alpha = 0$, we have

$$\|P_{k_3}(P_{k_1}uP_{k_3}v)\|_{H^{s-1,\theta-1,q}_{k_3}} \lesssim 2^{k_1\alpha}\|P_{k_1}u\|_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|_{H^{s-1,\theta-1,q}_{k_3}}$$

(HLH) Let $k_2 \ll k_3$, if $p > 1$ and $\alpha < 0$, or if $p = 1$ and $\alpha = 0$, we have

$$\|P_{k_3}(P_{k_3}uP_{k_2}v)\|_{H^{s-1,\theta-1,q}_{k_3}} \lesssim 2^{k_2\alpha}\|P_{k_3}u\|_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|_{H^{s-1,\theta-1,q}_{k_2}}$$

(HHL) Let $k_1 \gg k_3$, $\beta > 0$, and $\alpha + \beta \le 0$ then we have

$$\|P_{k_3}(P_{k_1}uP_{k_1}v)\|_{H^{s-1,\theta-1,q}_{k_3}} \lesssim 2^{k_3\beta}2^{k_1\alpha}\|P_{k_1}u\|_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|_{H^{s-1,\theta-1,q}_{k_1}}$$

Let us begin by proving the high-high-low interaction case.

i. High-high-low interaction: based on the proof of the support property of the P_kQ_j multiplier of Lemma 38 we split further into three cases:

$$\begin{aligned}
\|P_{k_3}(P_{k_1}uP_{k_1}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_2}v)\|_2\Big)^q \\
&+ 2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_1}v)\|_2\Big)^q \\
&+ 2^{qk_3(s-1)}\sum_{j_3\ge k_1}\Big(\sum_{j^{12}_{\max}\approx j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_2}v)\|_2\Big)^q \\
&=: HHL_I + HHL_{II} + HHL_{III}
\end{aligned}$$

To estimate the low modulations term HHL_I we have to be careful since the exponent of the outer modulation j_3 is negative we cannot gain some smallness form the sum $\sum_{j_3\le k_1}2^{j_3(\theta-1)}$. However here we invoke Lemma 40 to obtain a factor $2^{j_3/2}$ which make the j_3 exponent positive:

$$\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_2}v)\|_2 \lesssim 2^{j_3/2}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_2}v)\|_{L^1L^2}$$

Now we apply Strichartz inequality mimicking the proof of the algebra property:

$$\begin{aligned}
\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_2}v)\|_2 &\lesssim 2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^{2+}L^\infty}\|P_{k_1}Q_{j_2}v\|_{L^2L^2} \\
&\lesssim 2^{k_1(n/2-1/2+)}2^{j_1/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2
\end{aligned} \tag{1.10}$$

We have to place the term $P_{k_1}Q_{j_2}v$ into $H^{s-1,\theta-1,q}_{k_1}$ therefore we need the factor $2^{-j_2/2}$ inside the square sum in order to apply Cauchy-Schwarz, since we have the bound $j_2 \le k_1$, we can obtain it by paying a factor $2^{k_1/2}$. We obtain

$$HHL_I \lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+)}\sum_{j_3\le k_1}\Big(2^{qj_3(\theta-1/2)}\Big)\Big(\sum_{j_1,j_2\le k_1}2^{j_1/2}2^{-j_2/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2\Big)^q$$

Hence Cauchy-Schwarz in j_1 and j_2 sums yield to

$$HHL_I \lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+\theta+n/2+1/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s-1,\theta-1,q}_{k_1}}$$

which is exactly the HHL estimate we wanted where $\alpha = -2s+\theta+n/2+1/2+$ and $\beta = s-1$. Next, to estimate the middle modulation term HHL_{II} we shall use a similar procedure as above except that we replace Strichartz inequality by Bernstein inequality in (1.10):

$$\begin{aligned}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_1}Q_{j_1}v)\|_2 &\lesssim 2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2L^\infty}\|P_{k_1}Q_{j_1}v\|_{L^2L^2}\\ &\lesssim 2^{k_1n/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_1}v\|_2\end{aligned}$$

Therefore

$$\begin{aligned}HHL_{II} &\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\sum_{j_3}\left(2^{-\epsilon j_3 q}\right)\Big(\sum_{j_1\geq j_3}2^{j_3(\theta-1/2+\epsilon)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_1}v\|_2\Big)^q\\ &\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\Big(\sum_{j_1}2^{j_1\theta}2^{j_1(\theta-1)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_1}v\|_2\Big)^q\\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s-1,\theta-1,q}_{k_1}}\end{aligned}$$

Next, to estimate the high modulations term HHL_{III} we first suppose that $j_1\leq j_2$ which implies $j_2\approx j_3$, then we proceed as follows: we place the low inner modulation term $P_{k_1}Q_{j_1}u$ into L^∞ and the higher inner modulation term $P_{k_1}Q_{j_3}v$ into L^2, then Strichartz estimates yield to

$$\begin{aligned}HHL_{III} &\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\sum_{j_3}(2^{j_3(\theta-1)}\|P_{k_1}Q_{j_3}v\|_2)^q\Big(\sum_{j_1}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\Big)^q\\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+1+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s-1,\theta-1,q}_{k_1}}\end{aligned}$$

Observe that in the last line if $q>1$ we use Cauchy-Schwarz and the hypothesis $\theta>1/2$, whereas if $q=1$ and $\theta=1/2$ there is no need to use Cauchy-Schwarz. In the case $j_2\leq j_1$, which implies $j_1\approx j_3$, we place $P_{k_1}Q_{j_3}u$ into L^∞ and $P_{k_1}Q_{j_2}v$ into L^2, therefore we obtain

$$\begin{aligned}HHL_{III} &\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\sum_{j_3\geq k_1}\Big(\sum_{j_2}2^{j_3(\theta-1)}2^{j_3/2}\|P_{k_1}Q_{j_3}u\|_2\|P_{k_1}Q_{j_2}v\|_2\Big)^q\\ &\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\sum_{j_3}(2^{j_3\theta}\|P_{k_1}Q_{j_3}u\|_2)^q\Big(\sum_{j_2}2^{-j_2/2}\|P_{k_1}Q_{j_2}v\|_2\Big)^q\\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+1+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s-1,\theta-1,q}_{k_1}}\end{aligned}$$

We have used the fact that $j_2\leq j_1\approx j_3$ to bound the extra $2^{-j_3/2}$ term by $2^{-j_2/2}$. Notice that in the last line we have used Cauchy-Schwarz inequality in j_2 sum. This conclude the proof of the low output case.

ii. *Low-high-high interaction*: let $k_1\ll k_3$. As in the previous interaction, Lemma 38 allow us to split further into three cases:

$$\begin{aligned}
\|P_{k_3}(P_{k_1}uP_{k_3}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_3}Q_{j_2}v)\|_2\Big)^q\\
&+ 2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_3}Q_{j_1}v)\|_2\Big)^q\\
&+ 2^{qk_3(s-1)}\sum_{j_3\ge k_1}\Big(\sum_{j^{12}_{\max}\approx j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_3}Q_{j_2}v)\|_2\Big)^q\\
&=: LHH_I+LHH_{II}+LHH_{III}
\end{aligned}$$

To estimate the low modulations term LHH_I we follow the argument use to bound HHL_I. Lemma 40 and Strichartz inequality yield to

$$\begin{aligned}
\|P_{k_3}Q_{j_3}(P_{k_1}Q_{j_1}uP_{k_3}Q_{j_2}v)\|_2 &\lesssim 2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^{2+}L^\infty}\|P_{k_3}Q_{j_2}v\|_{L^2L^2}\\
&\lesssim 2^{k_1(n/2-1/2+)}2^{j_1/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_2}v\|_2
\end{aligned}$$

Moreover, the restriction $j_2\le k_1$ and Cauchy-Schwarz in j_1 and j_2 sums give

$$\begin{aligned}
LHH_I &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+)}\sum_{j_3\le k_1}\Big(2^{qj_3(\theta-1/2)}\Big)\Big(\sum_{j_1,j_2\le k_1}2^{j_1/2}2^{-j_2/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_2}v\|_2\Big)^q\\
&\lesssim 2^{qk_1(-s+\theta+n/2-1/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s-1,\theta-1,q}_{k_3}}
\end{aligned}$$

Next, we estimate the middle modulation term LHH_{II}. We shall proceed as in HHL_{II}, that is using lemma 40 we shall place $P_{k_1}Q_{j_1}u$ into L^2L^∞ and apply Bernstein inequality. Then we obtain

$$\begin{aligned}
LHH_{II} &\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\sum_{j_3}\big(2^{-\epsilon j_3q}\big)\Big(\sum_{j_1\ge j_3}2^{j_3(\theta-1/2+\epsilon)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_1}v\|_2\Big)^q\\
&\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\Big(\sum_{j_1}2^{j_1\theta}2^{j_1(\theta-1)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_1}v\|_2\Big)^q\\
&\lesssim 2^{qk_1(-s+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s-1,\theta-1,q}_{k_3}}
\end{aligned}$$

Next, to estimate the high modulations term LHH_{III} we split further: first suppose that $j_1\le j_2$ which implies $j_2\approx j_3$, let us place $P_{k_1}Q_{j_1}u$ into L^∞ and $P_{k_3}Q_{j_3}v$ into L^2, then we obtain

$$\begin{aligned}
LHH_{III} &\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\sum_{j_3\ge k_1}\Big(\sum_{j_1}2^{j_3(\theta-1)}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_3}v\|_2\Big)^q\\
&\lesssim 2^{qk_3(s-1)}2^{qk_1n/2}\sum_{j_3}(2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}v\|_2)^q\Big(\sum_{j_1}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\Big)^q\\
&\lesssim 2^{qk_1(-s+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s-1,\theta-1,q}_{k_3}}
\end{aligned}$$

On the other hand if $j_2 \le j_1$ which implies $j_1 \approx j_3$, let us place $P_{k_1} Q_{j_3} u$ into L^∞ and $P_{k_3} Q_{j_2} v$ into L^2, then we obtain

$$\begin{aligned} LHH_{III} &\lesssim 2^{qk_3(s-1)} 2^{qk_1 n/2} \sum_{j_3 \ge k_1} \Big(\sum_{j_2} 2^{j_3(\theta-1)} 2^{j_3/2} \| P_{k_1} Q_{j_3} u \|_2 \| P_{k_3} Q_{j_2} v \|_2 \Big)^q \\ &\lesssim 2^{qk_3(s-1)} 2^{qk_1 n/2} \sum_{j_3} (2^{j_3\theta} \| P_{k_1} Q_{j_3} u \|_2)^q \Big(\sum_{j_2} 2^{-j_2/2} \| P_{k_3} Q_{j_2} v \|_2 \Big)^q \\ &\lesssim 2^{qk_1(-s+n/2)} \| P_{k_1} u \|^q_{H^{s,\theta,q}_{k_1}} \| P_{k_3} v \|^q_{H^{s-1,\theta-1,q}_{k_3}} \end{aligned}$$

We have used the fact that $j_2 \le j_1 \approx j_3$ to bound the extra $2^{-j_3}/2$ term by $2^{-j_2/2}$. This conclude the proof of the low-high-high case.

iii. High-low-high interaction: let $k_2 \ll k_3$. As in the previous interaction, Lemma 38 allow us to split further into three cases:

$$\begin{aligned} \| P_{k_3}(P_{k_3} u P_{k_2} v) \|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)} \sum_{j_3 \le k_2} \Big(\sum_{j_1 \le k_2} \sum_{j_2 \le k_2} 2^{j_3(\theta-1)} \| P_{k_3} Q_{j_3} (P_{k_3} Q_{j_1} u P_{k_2} Q_{j_2} v) \|_2 \Big)^q \\ &+ 2^{qk_3(s-1)} \sum_{j_3} \Big(\sum_{j_1 \ge \max\{k_2, j_3\}} 2^{j_3(\theta-1)} \| P_{k_3} Q_{j_3} (P_{k_3} Q_{j_1} u P_{k_2} Q_{j_1} v) \|_2 \Big)^q \\ &+ 2^{qk_3(s-1)} \sum_{j_3 \ge k_2} \Big(\sum_{j^{12}_{\max} \approx j_3} 2^{j_3(\theta-1)} \| P_{k_3} Q_{j_3} (P_{k_3} Q_{j_1} u P_{k_2} Q_{j_2} v) \|_2 \Big)^q \\ &=: HLH_I + HLH_{II} + HLH_{III} \end{aligned}$$

To estimate the low modulations term HLH_I we follow the arguments use to bound HHL_I and LHH_I: suppose $n \ge 4$ then Lemma 40, Strichartz and Bernstein inequality yield to

$$\begin{aligned} \| P_{k_3} Q_{j_3} (P_{k_3} Q_{j_1} u P_{k_2} Q_{j_2} v) \|_2 &\lesssim 2^{j_3/2} \| P_{k_3} Q_{j_1} u P_{k_2} Q_{j_2} v \|_{L^1 L^2} \\ &\lesssim 2^{j_3/2} \| P_{k_3} Q_{j_1} u \|_{L^2 L^{2n/(n-3)}} \| P_{k_2} Q_{j_2} v \|_{L^2 L^{2n/3}} \\ &\lesssim 2^{k_3} 2^{k_2(n/2-3/2)} 2^{j_1/2} 2^{j_3/2} \| P_{k_3} Q_{j_1} u \|_2 \| P_{k_2} Q_{j_2} v \|_2 \end{aligned}$$

Observe that the pair $(2, 2n/(n-3))$ is Strichartz admissible for $n \ge 4$. Thus we insert this bound into V and apply Cauchy-Schwarz in j_1 and j_2 sums, we then obtain

$$\begin{aligned} HLH_I &\lesssim 2^{qk_3 s} 2^{qk_2(n/2-3/2)} \sum_{j_3 \le k_2} \Big(2^{qj_3(\theta-1/2)} \Big) \Big(\sum_{j_1 \le k_2} \sum_{j_2 \le k_2} 2^{j_1/2} \| P_{k_3} Q_{j_1} u \|_2 \| P_{k_2} Q_{j_2} v \|_2 \Big)^q \\ &\lesssim 2^{qk_2(\theta+n/2-3/2)} \| P_{k_3} u \|^q_{H^{s,\theta,q}_{k_3}} \Big(\sum_{j_2 \le k_2} 2^{-j_2/2} \| P_{k_2} Q_{j_2} v \|_2 \Big)^q \\ &\lesssim 2^{qk_2(-s+\theta+n/2-1/2)} \| P_{k_3} u \|^q_{H^{s,\theta,q}_{k_3}} \| P_{k_2} v \|^q_{H^{s-1,\theta-1,q}_{k_2}} \end{aligned}$$

Thus estimate for HLH_I holds for $n \ge 4$. Let us consider the case $n = 3$. Here we modify slightly the Strichartz pair used. From Hölder inequality we have

$$\begin{aligned}
\|P_{k_3}Q_{j_1}uP_{k_2}Q_{j_2}v\|_{L^1L^2} &\lesssim \|P_{k_3}Q_{j_1}\|_{L^2L^{\infty-}}\|P_{k_2}Q_{j_2}v\|_{L^2L^{2+}}\\
&\lesssim \|P_{k_3}Q_{j_1}\|_{L^{2+}L^{\infty-}}\|P_{k_2}Q_{j_2}v\|_{L^2L^{2+}}
\end{aligned}$$

where $\frac{1}{2+} := \frac{1}{2}-\varepsilon$ and $\frac{1}{\infty-} := \varepsilon$. Observe that the pair $(2+,\infty-)$ is Strichartz admissible in dimension $n=3$, therefore

$$\|P_{k_3}Q_{j_1}uP_{k_2}Q_{j_2}v\|_{L^1L^2} \lesssim 2^{k_3(1-2\varepsilon)}2^{3k_2\varepsilon}2^{j_1/2}\|P_{k_3}Q_{j_1}\|_{L^2L^2}\|P_{k_2}Q_{j_2}v\|_{L^2L^2}$$

Thus if we insert this estimate into HLH_I we obtain

$$\begin{aligned}
HLH_I &\lesssim 2^{qk_3(s-2\varepsilon)}2^{qk_2(3\varepsilon)}\sum_{j_3\le k_2}\Big(2^{qj_3(\theta-1/2)}\Big)\Big(\sum_{j_1\le k_2}\sum_{j_2\le k_2}2^{j_1/2}\|P_{k_3}Q_{j_1}u\|_2\|P_{k_2}Q_{j_2}v\|_2\Big)^q\\
&\lesssim 2^{qk_2(\theta+\varepsilon)}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\Big(\sum_{j_2\le k_2}2^{-j_2/2}\|P_{k_2}Q_{j_2}v\|_2\Big)^q\\
&\lesssim 2^{qk_2(-s+\theta+1+\varepsilon)}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|^q_{H^{s-1,\theta-1,q}_{k_2}}
\end{aligned}$$

Since in dimension $n=3$ our hypothesis gives $-s+\theta+1+\varepsilon\le 0$, the estimate of term HLH_I is completed.

Next, we estimate the easier HLH_{II} and HLH_{III} cases. To control the former we proceed exactly as in HHL_{II} and LHH_{II}, by applying lemma 40 and placing $P_{k_2}Q_{j_1}v$ into L^2L^∞ and apply Bernstein inequality. Then we obtain

$$\begin{aligned}
HHL_{II} &\lesssim 2^{qk_3(s-1)}2^{qk_2n/2}\sum_{j_3}\big(2^{-\varepsilon j_3q}\big)\Big(\sum_{j_1\ge j_3}2^{j_3(\theta-1/2+\varepsilon)}\|P_{k_3}Q_{j_1}u\|_2\|P_{k_2}Q_{j_1}v\|_2\Big)^q\\
&\lesssim 2^{qk_3(s-1)}2^{qk_2n/2}\Big(\sum_{j_1}2^{j_1\theta}2^{j_1(\theta-1)}\|P_{k_3}Q_{j_1}u\|_2\|P_{k_2}Q_{j_1}v\|_2\Big)^q\\
&\lesssim 2^{qk_2(-s+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_2}v\|^q_{H^{s-1,\theta-1,q}_{k_2}}
\end{aligned}$$

The estimate of the high modulations term HLH_{III} is reminiscent to LHH_{III}, let us split in two sub-cases: $j_1\le j_2$ and $j_2\le j_1$. Consider the latter, then $j_1\approx j_3$, let us place $P_{k_3}Q_{j_3}u$ into L^2 and $P_{k_2}Q_{j_2}v$ into L^∞, then we obtain

$$\begin{aligned}
HLH_{III} &\lesssim 2^{qk_3(s-1)}2^{qk_2n/2}\sum_{j_3\ge k_2}\Big(\sum_{j_2}2^{j_3(\theta-1)}2^{j_2/2}\|P_{k_3}Q_{j_3}u\|_2\|P_{k_2}Q_{j_2}v\|_2\Big)^q\\
&\lesssim 2^{qk_3(s-1)}2^{qk_2n/2}\sum_{j_3}(2^{j_3\theta}\|P_{k_3}Q_{j_3}u\|_2)^q\Big(\sum_{j_2}2^{-j_2/2}\|P_{k_2}Q_{j_2}v\|_2\Big)^q\\
&\lesssim 2^{qk_2(-s+n/2)}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|^q_{H^{s-1,\theta-1,q}_{k_2}}
\end{aligned}$$

since $k_2 \ll k_3$. On the other hand if $j_1 \leq j_2$, then $j_2 \approx j_3$ and we obtain

$$HLH_{III} \lesssim 2^{qk_3(s-1)} \sum_{j_3 \geq k_2} \Big(\sum_{j_1} 2^{j_3(\theta-1)} \| P_{k_3} Q_{j_3} (P_{k_3} Q_{j_1} u P_{k_2} Q_{j_3} v) \|_2 \Big)^q$$

Here we have to be careful since we shall avoid to raise the power of the high modulation j_3 and the high frequency k_3. Thus we cannot use Strichartz inequality to both terms as we did previously, instead we will rely on Bernstein inequality. We obtain

$$\begin{aligned} \| P_{k_3} Q_{j_3} (P_{k_3} Q_{j_1} u P_{k_2} Q_{j_3} v) \|_2 &\lesssim \| P_{k_3} Q_{j_1} u \|_{L^\infty L^2} \| P_{k_2} Q_{j_3} v \|_{L^2 L^\infty} \\ &\lesssim 2^{k_2(n/2)} \| P_{k_3} Q_{j_1} u \|_{L^\infty L^2} \| P_{k_2} Q_{j_3} v \|_{L^2 L^2} \\ &\lesssim 2^{k_2(n/2)} 2^{j_1/2} \| P_{k_3} Q_{j_1} u \|_2 \| P_{k_2} Q_{j_3} v \|_2 \end{aligned}$$

Therefore

$$\begin{aligned} HLH_{III} &\lesssim 2^{qk_3(s-1)} 2^{qk_2(n/2)} \sum_{j_3 \geq k_2} \Big(\sum_{j_1 \leq j_3} 2^{j_3(\theta-1)} 2^{j_1/2} \| P_{k_3} Q_{j_1} u \|_2 \| P_{k_2} Q_{j_3} v \|_2 \Big)^q \\ &\lesssim 2^{qk_3(s-1)} 2^{qk_2(n/2)} \sum_{j_3} (2^{j_3(\theta-1)} \| P_{k_2} Q_{j_3} v \|_2)^q \Big(\sum_{j_1} 2^{j_1/2} \| P_{k_3} Q_{j_1} u \|_2 \Big)^q \\ &\lesssim 2^{qk_2(-s+n/2)} \| P_{k_3} u \|^q_{H^{s,\theta,q}_{k_3}} \| P_{k_2} v \|^q_{H^{s-1,\theta-1,q}_{k_2}} \end{aligned}$$

since $k_2 \ll k_3$. This conclude the proof of the high-low-high interaction case, and thus the proof is completed. In the table below we summarize the different α and β obtained in the previous nine estimates:

	HHL_I	HHL_{II}, HHL_{III}	HLH_I, LHH_I	HLH_{II}, HLH_{III} LHH_{II}, LHH_{III}
α	$-2s+\theta+n/2+1/2+$	$-2s+1+n/2$	$-s+(n-1)/2+\theta+$	$-s+n/2$
β	$s-1$	$s-1$		

□

1.8 Maxwell-Klein-Gordon and Yang-Mills equation

This section is devoted to the proof of the following local well-posedness result for Maxwell-Klein-Gordon and Yang-Mills equation, which is based on [45] and [105]. See also [97] and [98] for global results in critical Besov spaces.

Theorem 41. *Let $n \geq 3$, $p \geq 1$, and $s > n/2 - 3/4$, then there exist an unique local solution*

$u \in C([0,T], B^s_{2,p}(\mathbb{R}^n)) \cap C^1([0,T], B^{s-1}_{2,p}(\mathbb{R}^n))$ *to the Cauchy problem*

$$\begin{cases} \Box u = \widetilde{N}(\partial u, \partial u) \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in B^s_{2,p}(\mathbb{R}^n) \times B^{s-1}_{2,p}(\mathbb{R}^n) \end{cases}$$

where $\widetilde{N}(\partial u, \partial u)$ *is a linear combinations of* $\widetilde{N}_{ij}(u,v) = R_i u \partial_j v - R_j u \partial_i v$ *null-forms. Moreover if* $n \geq 4$ *and* $p = 1$ *one can take* $s = n/2 - 3/4$.

Notice that we can combine the two null-forms $D^{-1}N_{ij}(\partial u, \partial u)$ and $N_{ij}(D^{-1}\partial u, \partial u)$ present in the Maxwell-Klein-Gordon and Yang-Mills type equations into one by means of the Riesz transform, in fact

$$N_{ij}(D^{-1}u, v) = R_i u \partial_j v - R_j u \partial_i v = \widetilde{N}_{ij}(u,v)$$

and

$$D^{-1}N_{ij}(u,v) = R_i u \partial_j v - R_j u \partial_i v + R_j v \partial_i u - R_i v \partial_j u = \widetilde{N}_{ij}(u,v) - \widetilde{N}_{ij}(v,u)$$

Therefore, as far the schematic form of Maxwell-Klein-Gordon and Yang-Mills equation are concern, it suffices to prove an estimate for $\widetilde{N}$, which is defined as a linear combination with constant coefficients of $\widetilde{N}_{ij}$ null-forms. The only inconvenient is that $\widetilde{N}$ is not symmetric anymore. The result of Theorem 41 is not optimal in the sense that it does not reach the scaling critical threshold $s = n/2 - 1$, nonetheless it is the best result available if one doesn't modify the $H^{s,\theta}_{p,q}$ norm [46]. Moreover in view of [70], Theorem 41 is optimal in $n = 3$. The proof of Theorem 41 follows from a contraction argument. From the linear theory developed in §1.5 it suffices to prove the following nonlinear estimate.

Proposition 42 (MKG and YM multiplicative estimate)**.** *The multiplicative estimate*

$$\|\widetilde{N}(u,v)\|_{H^{s-1,\theta-1}_{p,q}} \lesssim \|u\|_{H^{s,\theta}_{p,q}} \|v\|_{H^{s,\theta}_{p,q}}$$

holds in the following cases:

$n \geq 3$,	$p > 1$,	$q > 1$,	$1/2 < \theta < 1$,	$s - (n-3)/2 > \theta$,	*and* $s + \theta > n/2$
$n \geq 3$,	$p > 1$,	$q = 1$,	$\theta = 1/2$,	$s - (n-3)/2 > \theta$,	*and* $s + \theta > n/2$
$n \geq 4$,	$p = 1$,	$q > 1$,	$1/2 < \theta < 1$,	$s - (n-3)/2 = \theta$,	*and* $s + \theta = n/2$

The two conditions $s > n/2 - 3/2 + \theta$ and $s + \theta > n/2$ are both satisfied in the region highlighted in Figure 1.1. It follows the bound $s > n/2 - 3/4$ for $\theta = 3/4$.

Before proving Proposition 42 let us consider pure N_{ij} null-forms without Riesz transform. We have the following:

Proposition 43 (N multiplicative estimate)**.** *Let* N *be a linear combination with constant coefficients of* N_{ij} *null-forms, then the multiplicative estimate*

$$\|N(u,v)\|_{H^{s-1,\theta-1}_{p,q}} \lesssim \|u\|_{H^{s,\theta}_{p,q}} \|v\|_{H^{s,\theta}_{p,q}}$$

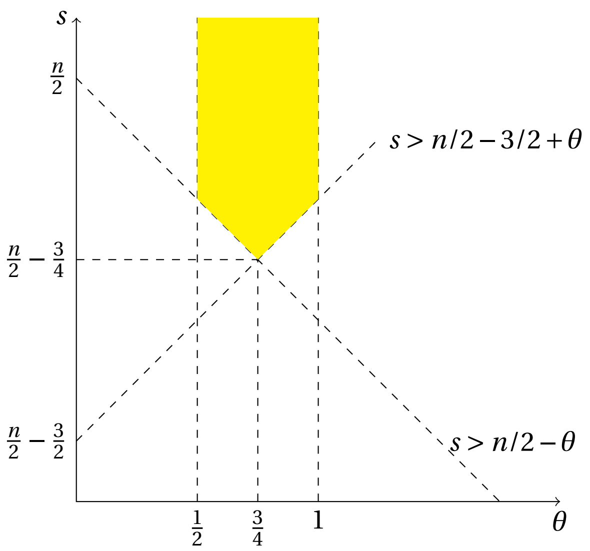


Figure 1.1: Maxwell-Klein-Gordon and Yang-Mills null-forms

holds in the following cases:

$n \geq 3,$	$p > 1,$	$q > 1,$	$1/2 < \theta < 1,$	$s - (n-1)/2 > \theta,$	*and* $s + \theta > n/2 + 1$
$n \geq 3,$	$p > 1,$	$q = 1,$	$\theta = 1/2,$	$s - (n-1)/2 > \theta,$	*and* $s + \theta > n/2 + 1$
$n \geq 4,$	$p = 1,$	$q > 1,$	$1/2 < \theta < 1,$	$s - (n-1)/2 = \theta,$	*and* $s + \theta = n/2 + 1$

In Figure 1.2 we highlight the region where both conditions $s > n/2 - 1/2 + \theta$ and $s + \theta > n/2 + 1$ holds. It is clear then that $s > \frac{n}{2} + \frac{1}{4}$ when $\theta = 3/4$, which gives an improvement of $1/4$ over Theorem 7.

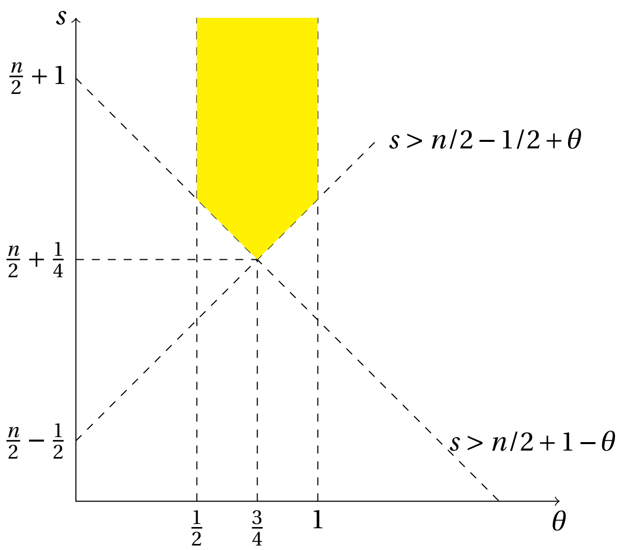


Figure 1.2: N null-form

In the proof of Proposition 43 we will need to decompose further the support of the frequency localizer P_k into angular regions. This method has been used in various forms for some time now, see Bourgain's appendix [41] and Krieger [48], Tataru [114], and Tao work [103], [104] on critical wave maps. Precisely consider the Littlewood-Paley frequency cutoff

$$P_k u = \mathscr{F}^{-1}\big(m_k(\xi)\,\widehat{u}(\xi)\big)$$

where $m_k(\xi)$ has support in the annulus $\mathscr{A} = \{2^{k-1} \le |\xi| \le 2^{k+1}\}$. Let us fix a real positive number α, called angular scale, and decompose $\mathscr{A}$ into a finite number of overlapping spherical caps of angular size 2^{α}. Fix $\omega_i \in \mathbb{S}^{n-1}$, where $i \in \Omega_\alpha$ and $|\Omega_\alpha| \lesssim 2^{\alpha(n-1)}$, such that the spherical caps centered at w_i with angular size 2^{α}: $K_\alpha^{w_i} = \{\omega \in \mathbb{S}^{n-1} : \angle(\omega,\omega_i) \lesssim 2^{\alpha}\}$ form an partition of the sphere, $\mathbb{S}^{n-1} = \cup_{i\in\Omega_\alpha} K_\alpha^{w_i}$, with the property that $K_\alpha^{w_i} \cap K_\alpha^{w_j} \neq \emptyset$ only if w_i and w_j are relatively close to each other. Now define the sets

$$K_{k,\alpha}^{w_i} = \{\xi \in \mathbb{R}^n : 2^{k-1} \le |\xi| \le 2^{k+1}, \frac{\xi}{|\xi|} \in K_\alpha^{w_i}\} = \{\xi \in \mathbb{R}^n : 2^{k-1} \le |\xi| \le 2^{k+1}, \angle(\frac{\xi}{|\xi|},\omega_i) \lesssim \alpha\}$$

which form a partition of $\mathscr{A}$. Next define the Littlewood-Paley cutoffs

$$S_{k,\alpha}^{\omega_i} u = \mathscr{F}^{-1}\big(m_{k,\alpha}^{\omega_i}(\xi)\,\widehat{u}(\xi)\big)$$

where $m_{k,\alpha}^{\omega_i}(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that supp $m_{k,\alpha}^{\omega_i} = 2K_{k,\alpha}^{w_i}$ and $m_{k,\alpha}^{\omega_i} = 1$ on $K_{k,\alpha}^{w_i}$. Then, for every angular size α, one can decompose the operator P_k radially:

$$P_k = \sum_{i\in\Omega_\alpha} S_{k,\alpha}^{\omega_i}$$

In the sequel we will need the following Lemma, that is reminiscent to Lemma 38, for a similar version of this lemma see Lemma 4.1 in [58] or Lemma 2.1 [92].

Lemma 44. *For every frequencies k_1, k_2, k_3, modulations j_1, j_2, j_3, angular scale α, and angular directions ω_i, ω_j we have*

$$P_{k_3} Q_{j_3}(S_{k_1,\alpha}^{\omega_i} Q_{j_1} u\, S_{k_2,\alpha}^{\omega_j} Q_{j_2} v) = 0$$

unless one of the following cases is satisfied:

i. LHH - low modulations: $k_1 \ll k_2 \approx k_3$, $j_{\max}^{123} \precsim k_1$ and $\alpha \approx (j_{\max}^{123} - k_1)/2$.

ii. LHH - med modulations: $k_1 \ll k_2 \approx k_3$, $j_{\min}^{12} \approx j_{\max}^{12} \gg k_1$ and $j_3 \precsim j_{\max}^{12}$.

iii. LHH - high modulations: $k_1 \ll k_2 \approx k_3$, $j_{\max}^{12} \gg \max\{j_{\min}^{12}, k_1\}$ and $j_3 \approx j_{\max}^{12}$.

iv. HLH - low modulations: $k_2 \ll k_1 \approx k_3$, $j_{\max}^{123} \precsim k_2$ and $\alpha \approx (j_{\max}^{123} - k_2)/2$.

v. HLH - med modulations: $k_2 \ll k_1 \approx k_3$, $j_{\min}^{12} \approx j_{\max}^{12} \gg k_2$ and $j_3 \precsim j_{\max}^{12}$.

vi. HLH - high modulations: $k_2 \ll k_1 \approx k_3$, $j_{\max}^{12} \gg \max\{j_{\min}^{12}, k_2\}$ and $j_3 \approx j_{\max}^{12}$.

vii. *HHL - low modulations:* $k_1 \approx k_2 \gtrsim k_3$, $j^{123}_{\max} \lesssim k^{12}_{\min}$ *and* $\alpha \approx (j^{123}_{\max} + k_3 - k_1 - k_2)/2$.

viii. *HHL - med modulations:* $k_1 \approx k_2 \gtrsim k_3$, $j^{12}_{\min} \approx j^{12}_{\max} \gg k^{12}_{\max}$ *and* $j_3 \lesssim j^{12}_{\max}$.

ix. *HHL - high modulations:* $k_1 \approx k_2 \gtrsim k_3$, $j^{12}_{\max} \gg \max\{j^{12}_{\min}, k^{12}_{\max}\}$ *and* $j_3 \approx j^{12}_{\max}$.

Proof. Observe that

$$\begin{aligned}&\operatorname{supp} \mathscr{F}_{\tau,\xi}(S^{\omega_i}_{k_1,\alpha} Q_{j_1} u\, S^{\omega_j}_{k_2,\alpha} Q_{j_2} v) = \operatorname{supp}\widetilde{S^{\omega_i}_{k_1,\alpha} Q_{j_1}} u + \operatorname{supp}\widetilde{S^{\omega_j}_{k_2,\alpha} Q_{j_2}} v\\ &= \{(\tau_3,\xi_3) : \tau_3 = \tau_1 + \tau_2,\ \xi_3 = \xi_1 + \xi_2,\ 2^{(k_i-1)} \le |\xi_i| \le 2^{(k_i+1)},\ 2^{(j_i-1)} \le ||\tau_i| - |\xi_i|| \le 2^{(j_i+1)},\\ &\angle(\xi_1,\omega_i) \le 2^{\alpha}, \angle(\xi_2,\omega_j) \le 2^{\alpha}\}\end{aligned}$$

Define $h_i = -\tau_i + \operatorname{sign}(\tau_i)|\xi_i|$, and notice that $||\tau_i| - |\xi_i|| = |h_i|$. We have

$$h_3 - h_1 - h_2 = \operatorname{sign}(\tau_3)|\xi_3| - \operatorname{sign}(\tau_1)|\xi_1| - \operatorname{sign}(\tau_2)|\xi_2|$$

First consider the case $\operatorname{sign}(\tau_1) = \operatorname{sign}(\tau_2)$, the so called $(+,+)$ or $(-,-)$ cases. If τ_1, τ_2 are both positive we obtain

$$h_3 - h_1 - h_2 = |\xi_3| - |\xi_1| - |\xi_2|$$

on the other hand if τ_1, τ_2 are both negative we have

$$h_3 - h_1 - h_2 = -|\xi_3| + |\xi_1| + |\xi_2|$$

If we use the following relationship

$$|\xi_1| + |\xi_2| - |\xi_1 + \xi_2| \approx |\xi^{12}_{\min}| \angle^2(\xi_1,\xi_2) \tag{1.11}$$

which follows form equation (26) in [92], we then obtain that

$$|h^{123}_{\max}| \gg |\xi^{12}_{\min}| \angle^2(\xi_1,\xi_2)$$

Furthermore, if we suppose that $|h^{123}_{max}| \gg |h^{123}_{med}|$ we that obtain a lower bound as well, which gives

$$|h^{123}_{\max}| \approx |\xi^{12}_{\min}| \angle^2(\xi_1,\xi_2)$$

In particular $j^{123}_{\max} \le k^{123}_{\min} + O(1)$. Now consider the opposite cases, when $\operatorname{sign}(\tau_1) \ne \operatorname{sign}(\tau_2)$, the so called $(+,-)$ or $(-,+)$ cases. If $\tau_1 > 0$ and $\tau_2 < 0$ we obtain

$$h_3 - h_1 - h_2 = \operatorname{sign}(\tau_3)|\xi_3| - |\xi_1| + |\xi_2|$$

on the other hand if $\tau_1 < 0$ and $\tau_2 > 0$ we have

$$h_3 - h_1 - h_2 = \operatorname{sign}(\tau_3)|\xi_3| + |\xi_1| - |\xi_2|$$

Here we use the fact that

$$|\xi_3| - ||\xi_1| - |\xi_2|| \approx \frac{|\xi_1||\xi_2|}{|\xi_3|} \angle^2(\xi_1, \xi_2)$$

to obtain a lower bound on the maximum modulation

$$|h^{123}_{\max}| \gg \frac{|\xi_1||\xi_2|}{|\xi_3|} \angle^2(\xi_1, \xi_2)$$

Furthermore, if we suppose that $|h^{123}_{max}| \gg |h^{123}_{med}|$ we that obtain a lower bound as well, which gives

$$|h^{123}_{\max}| \approx \frac{|\xi_1||\xi_2|}{|\xi_3|} \angle^2(\xi_1, \xi_2)$$

In particular $j^{123}_{\max} \leq k^{123}_{\min} + O(1)$. Finally, notice that in the HHL case ($|\xi_3| \ll |\xi_1| \approx |\xi_2|$), from (1.11) we obtain $\angle^2(\xi_1, \xi_2) \approx 1$, which gives *vi.* □

Notice that since the multiplier of the N_{ij} null-form do not depend neither on τ_1 nor on τ_2 we can avoid the Fourier transform with respect to the time variable. In fact we can write

$$N_{ij}(u, v) = \int_{\mathbb{R}^{n+1}} e^{ix\cdot(\xi_1+\xi_2)} m_{ij}(\xi_1, \xi_2)\ \widehat{u}(t, \xi_1)\widehat{v}(t, \xi_2)\ d\xi_1 d\xi_2$$

Consider N to be a linear combination with constant coefficients of N_{ij} null-forms, then denoting by m its symbol we have the bound

$$|m(\xi_1, \xi_2)| \lesssim |\xi_1||\xi_2||\sin(\angle(\xi_1, \xi_2))|$$

Therefore if we feed N with frequency and angular localized functions we obtain the following estimate

$$\|P_{k_3} Q_{j_3} N(S^{w_i}_{k_1,\alpha} Q_{j_1} u,\ S^{w_j}_{k_2,\alpha} Q_{j_2} u)\|_2 \lesssim 2^{k_1} 2^{k_2} 2^{\alpha} \|P_{k_3} Q_{j_3} (S^{\omega_i}_{k_1,\alpha} Q_{j_1} u S^{\omega_j}_{k_2,\alpha} Q_{j_2} u)\|_2 \qquad (1.12)$$

holds uniformly for any frequencies k_1, k_2, k_3, modulations j_1, j_2, j_3, angular scale α and angular directions ω_i, ω_j. In the proof of Proposition 43 we will need the following Lemma which involve some key bilinear L^2 estimates for null-forms via angular localization.

Lemma 45 (Angular localization improvement)**.** *We have the following estimates:*

(i) *For LHH frequencies ($k_1 \ll k_2 \approx k_3$), low modulations ($j_1, j_2, j_3 \ll k_1$), and maximum modulation coupled with low frequency ($j_1 = j^{123}_{\max}$):*

$$\|P_{k_3} Q_{j_3} N(P_{k_1} Q_{j_1} u, P_{k_3} Q_{j_2} v)\|_{L^2_{t,x}} \lesssim 2^{k_1 \frac{n+3}{4}} 2^{k_3} 2^{j_1 \frac{n+1}{4}} 2^{j_2/2} \|P_{k_1} Q_{j_1} u\|_{L^2_{t,x}} \|P_{k_3} Q_{j_2} v\|_{L^2_{t,x}}$$

(ii) *For LHH frequencies ($k_1 \ll k_2 \approx k_3$), low modulations ($j_1, j_2, j_3 \ll k_1$), and maximum* *modulation not coupled with low frequency ($j_1 \neq j^{123}_{\max}$):*

$$\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u, P_{k_3}Q_{j_2}v)\|_{L^2_{t,x}} \lesssim 2^{k_1(n/2+)}2^{k_3}2^{j_1/2}2^{j_2/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}$$

(iii) *For HHL frequencies ($k_3 \ll k_1 \approx k_2$), low modulations ($j_1, j_2, j_3 \ll k_1$)*

$$\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u, P_{k_1}Q_{j_2}v)\|_{L^2_{t,x}} \lesssim 2^{k_1(n/2+1/2+)}2^{k_3/2}2^{j_1/2}2^{j_2/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_1}Q_{j_2}v\|_{L^2_{t,x}}$$

Notice that similar estimates to the LHH case hold in the HLH case, here k_1 is replaced by k_2. Precisely we have if $k_2 \ll k_1 \approx k_3$, $j_1, j_2, j_3 \ll k_2$, and maximum modulation coupled with low frequency $j_2 = j^{123}_{\max}$:

$$\|P_{k_3}Q_{j_3}N(P_{k_3}Q_{j_1}u, P_{k_2}Q_{j_2}v)\|_{L^2_{t,x}} \lesssim 2^{k_2\frac{n+3}{4}}2^{k_3}2^{j_1\frac{n+1}{4}}2^{j_2/2}\|P_{k_3}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_2}Q_{j_2}v\|_{L^2_{t,x}}$$

Moreover in the HLH case $k_2 \ll k_1 \approx k_3$, low modulations $j_1, j_2, j_3 \ll k_2$, and maximum modulation not coupled with low frequency $j_2 \neq j^{123}_{\max}$:

$$\|P_{k_3}Q_{j_3}N(P_{k_3}Q_{j_1}u, P_{k_2}Q_{j_2}v)\|_{L^2_{t,x}} \lesssim 2^{k_2(n/2+)}2^{k_3}2^{j_1/2}2^{j_2/2}2^{j_3/2}\|P_{k_3}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_2}Q_{j_2}v\|_{L^2_{t,x}}$$

Proof. In what follows we simplify the notations a bit: let us denote by κ a spherical cap of $\mathbb{S}^{n-1}$ with angular size 2^{α}, then we simply write

$$P_k = \sum_{\kappa} P_{k,\kappa}$$

Moreover notice that in the LHH cases the angular separation is controlled by $2^{(j^{123}_{\max}-k^{12}_{min})/2}$.

i. This is the most delicate case since we must place $P_{k_3,\kappa_2}Q_{j_2}v$ into $L^\infty_t L^2_x$ to avoid any extra power of the high frequency k_3. This force us to place $P_{k_1,\kappa_1}Q_{j_1}u$ into $L^2_t L^\infty_x$, and to recover the $L^2_t L^2_x$ norm we use Bernstein inequality, notice that the support on the Fourier side has size $2^{(|\kappa_1|+k_1)(n-1)}2^{k_1} = 2^{|\kappa_1|(n-1)}2^{k_1 n}$. Therefore we obtain

$$\begin{aligned}
&\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u, P_{k_3}Q_{j_2}v)\|_{L^2_{t,x}}\\
&\lesssim \sum_{\kappa_1\approx\kappa_2}\|P_{k_3}Q_{j_3}N(P_{k_1,\kappa_1}Q_{j_1}u, P_{k_3,\kappa_2}Q_{j_2}v)\|_{L^2_{t,x}}\\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+k_1+k_3}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_t L^\infty_x}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^\infty_t L^2_x}\\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+k_1+k_3}2^{|\kappa|(n-1)/2+k_1(n/2)}2^{j_2/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^2_{t,x}}\\
&\lesssim 2^{k_1\frac{n+3}{4}}2^{k_3}2^{j_1\frac{n+1}{4}}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}
\end{aligned}$$

ii. Here we split the argument in two parts: if $j_2 = j^{123}_{\max}$ we use the $L^2 - L^1$ Bernstein inequality of Lemma 40 to obtain a factor $2^{j_3/2}$, this yield to the estimate

$$\begin{aligned}
&\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u, P_{k_3}Q_{j_2}v)\|_{L^2_{t,x}} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{j_3/2}\|P_{k_3}Q_{j_3}N(P_{k_1,\kappa_1}Q_{j_1}u, P_{k_3,\kappa_2}Q_{j_2}v)\|_{L^1_tL^2_x} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+k_1+k_3}2^{j_3/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^{2+}_tL^\infty_x}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^2_{t,x}} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+k_1+k_3}2^{k_1(n/2-1/2+)}2^{j_1/2}2^{j_3/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^2_{t,x}} \\
&\lesssim 2^{k_1(n/2+)}2^{k_3}2^{j_1/2}2^{j_2/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}
\end{aligned}$$

On the other hand if $j_3 = j^{123}_{\max}$ we don't need to use the Lemma 40 since the $2^{j_3/2}$ comes from the bound on the size of the spherical cap, we just proceed as usual placing the low frequency term in $L^{2+}_tL^\infty_x$ and the high frequency term in $L^\infty_tL^2_x$. Thus we obtain the same estimate as the $j_2 = j^{123}_{\max}$ case but via a different argument:

$$\begin{aligned}
&\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u, P_{k_3}Q_{j_2}v)\|_{L^2_{t,x}} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} \|P_{k_3}Q_{j_3}N(P_{k_1,\kappa_1}Q_{j_1}u, P_{k_3,\kappa_2}Q_{j_2}v)\|_{L^2_{t,x}} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+k_1+k_3}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^{2+}_tL^\infty_x}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^\infty_tL^2_x} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+k_1+k_3}2^{k_1(n/2-1/2+)}2^{j_1/2}2^{j_2/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^2_{t,x}} \\
&\lesssim 2^{k_1(n/2+)}2^{k_3}2^{j_1/2}2^{j_2/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}
\end{aligned}$$

iii. Recall that in the HHL low modulation case the angular separation is smaller then the corresponding LHH case, in fact it is controlled by $2^{(j^{123}_{\max}+k_3-2k^{12}_{min})/2}$. First consider the case $j_3 = j^{123}_{\max}$, then we obtain

$$\begin{aligned}
&\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u, P_{k_1}Q_{j_2}v)\|_{L^2_{t,x}} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} \|P_{k_3}Q_{j_3}N(P_{k_1,\kappa_1}Q_{j_1}u, P_{k_1,\kappa_2}Q_{j_2}v)\|_{L^2_{t,x}} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+2k_1}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^{2+}_tL^\infty_x}\|P_{k_1,\kappa_2}Q_{j_2}v\|_{L^\infty_tL^2_x} \\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+2k_1}2^{k_1(n/2-1/2+)}2^{j_1/2}2^{j_2/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_1,\kappa_2}Q_{j_2}v\|_{L^2_{t,x}} \\
&\lesssim 2^{k_1(n/2+1/2+)}2^{k_3/2}2^{j_1/2}2^{j_2/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_1}Q_{j_2}v\|_{L^2_{t,x}}
\end{aligned}$$

If now the maximum modulation is not couple with the lowest frequency, i.e. $j_3 \neq j^{123}_{\max}$, we rely on the $L^2 - L^1$ Bernstein Lemma 40 to get a factor of $2^{j_3/2}$. Furthermore we place the inner lowest modulation term into $L^{2+}L^\infty$ and use Strichartz. For example if $j_1 = j^{123}_{\max}$ then we obtain

$$\begin{aligned}
&\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_2}v)\|_{L^2_{t,x}}\\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{j_3/2}\|P_{k_3}Q_{j_3}N(P_{k_1,\kappa_1}Q_{j_1}u,P_{k_1,\kappa_2}Q_{j_2}v)\|_{L^1_tL^2_x}\\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+2k_1}2^{j_3/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_tL^2_x}\|P_{k_1,\kappa_2}Q_{j_2}v\|_{L^{2+}_tL^\infty_x}\\
&\lesssim \sum_{\kappa_1\approx\kappa_2} 2^{|\kappa|+2k_1}2^{k_1(n/2-1/2+)}2^{j_2/2}2^{j_3/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_1,\kappa_2}Q_{j_2}v\|_{L^2_{t,x}}\\
&\lesssim 2^{k_1(n/2+1/2+)}2^{k_3/2}2^{j_1/2}2^{j_2/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_1}Q_{j_2}v\|_{L^2_{t,x}}
\end{aligned}$$

On the other hand if $j_2=j^{123}_{\max}$ then we place $P_{k_1,\kappa_2}Q_{j_2}v$ into $L^2_{t,x}$ and $P_{k_1,\kappa_1}Q_{j_1}u$ into $L^{2+}_tL^\infty_x$, this yield to the same bound.

□

We are now ready to prove multiplicative estimates involving the N_{ij} null-form.

Proof of Proposition 43. Since the estimate is symmetric it suffices to prove the two frequency localized estimates:

- *(High-output)* Let $k_1\ll k_3$, if $p>1$ and $\alpha<0$, or if $p=1$ and $\alpha=0$, we have

$$\|P_{k_3}N(P_{k_1}u,P_{k_3}v)\|_{H^{s-1,\theta-1,q}_{k_3}}\lesssim 2^{k_1\alpha}\|P_{k_1}u\|_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|_{H^{s,\theta,q}_{k_3}}$$

- *(Low-output)* Let $k_1\gg k_3$, $\beta>0$, and $\alpha+\beta\le 0$ then we have

$$\|P_{k_3}N(P_{k_1}u,P_{k_1}v)\|_{H^{s-1,\theta-1,q}_{k_3}}\lesssim 2^{k_3\beta}2^{k_1\alpha}\|P_{k_1}u\|_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|_{H^{s,\theta,q}_{k_1}}$$

We begin by proving the easier *low-output estimate.* As in the proof of the algebra property we split further into low modulations, med modulations, and high modulations:

$$\begin{aligned}
\|P_{k_3}N(P_{k_1}u,P_{k_1}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_2}v)\|_2\Big)^q\\
&+ 2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_1}v)\|_2\Big)^q\\
&+ 2^{qk_3(s-1)}\sum_{j_3\ge k_1}\Big(\sum_{j_1\le j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_3}v)\|_2\Big)^q\\
&=: HHL_I+HHL_{II}+HHL_{III}
\end{aligned}$$

Notice that in the second and third terms we have supposed, without loosing generality, that $\max\{j_1,j_2\}=j_2$. In order to estimate the *low modulations term HHL_I* we use angular localization and the Bilinear L^2 estimates for null-forms. From Lemma 45 iii. we obtain

$$\begin{aligned} HHL_I &\lesssim 2^{qk_3(s-1/2)}2^{qk_1(n/2+1/2+)}\sum_{j_3\leq k_1}\Big(\sum_{j_1\leq k_1}\sum_{j_2\leq k_1}2^{j_3(\theta-1/2)}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_3(s-1/2)}2^{qk_1(-2s+n/2+\theta+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}} \end{aligned}$$

Next, to estimate the middle modulation term HHL_{II} we shall not need the precise structure of the null-form, hence we don't need any angular localization. The argument presented below works for any general bilinear form in space gradients. For such forms we loose a factor of 2^{2k_1} since both input are frequency localized at 2^{k_1}. Let us use the standard Bernstein estimate:

$$\begin{aligned} \|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u, P_{k_1}Q_{j_1}v)\|_{L^2_{t,x}} &\lesssim 2^{j_3/2}2^{2k_1}\|P_{k_1}Q_{j_1}u\|_{L^2L^\infty}\|P_{k_1}Q_{j_1}v\|_{L^2_{t,x}} \\ &\lesssim 2^{k_1(n/2+2)}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2L^2}\|P_{k_1}Q_{j_1}v\|_{L^2L^2} \end{aligned}$$

Then

$$\begin{aligned} HHL_{II} &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+2)}\big(\sum_{j_3}2^{-\epsilon j_3 q}\big)\Big(\sum_{j_1\geq \max\{k_1,j_3\}}2^{j_3(\theta-1/2+)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_1}v\|_2\Big)^q \\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+n/2+1)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}} \end{aligned}$$

Here we have used the upper bound on the j_3 sum to extract a factor of 2^{-k_1}: $2^{j_3(\theta-1/2+)}\leq 2^{j_1(\theta-1/2+)}\leq 2^{-k_1}2^{j_1\theta}2^{j_1\theta}$. Observe that we only need to impose $s>n/2$ to obtain convergence.

It remains to control the *high modulations term* HHL_{III}. Here, as for the previous term, we wont need to use the special null structure. In this case we place $P_{k_1}Q_{j_1}u$ in $L^{2+}L^\infty$ and $P_{k_1}Q_{j_3}v$ in $L^\infty L^2$ and use the fact that the j_2 sum collapse. We proceed as follows

$$\begin{aligned} HHL_{III} &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+3/2+)}\sum_{j_3\geq k_1}\Big(\sum_{j_1}2^{j_3(\theta-1)}2^{j_1/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_3}v\|_2\Big)^q \\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+1+)}\sum_{j_3}(2^{j_3\theta}\|P_{k_1}Q_{j_3}v\|_2)^q\Big(\sum_{j_1}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\Big)^q \\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+1+n/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}} \end{aligned}$$

Here we have used the high modulation hypothesis to control $2^{j_3(\theta-1/2)}\leq 2^{j_3\theta}2^{-k_1/2}$. Moreover notice that $\beta=s-1$ is positive and $\alpha+\beta=-s+n/2+$.

Next we prove the more difficult *high-output estimate*. As usual, due to Lemma 44 we split further into low, med and high modulations:

$$\begin{aligned}\|P_{k_3}N(P_{k_1}u,P_{k_3}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_2\Big)^q\\ &+ 2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_1}v)\|_2\Big)^q\\ &+ 2^{qk_3(s-1)}\sum_{j_3\ge k_1}\Big(\sum_{j^{12}_{\max}\approx j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_2\Big)^q\\ &=: LHH_I+LHH_{II}+LHH_{III}\end{aligned}$$

To estimate the most delicate term LHH_I term involving *low modulations* we must invoke the angular decomposition and use Lemma 45 i. and ii.. Thus we split further into $j_1=j^{123}_{max}$ and $j_1\neq j^{123}_{max}$ cases. When $j_1=j^{123}_{max}$ from i. we obtain

$$\begin{aligned}LHH_I &\lesssim \sum_{j_3\le k_1}\Big(\sum_{j_3\le j_1\le k_1}\sum_{j_2\le j_1}2^{k_3s}2^{j_3(\theta-1)}2^{k_1\frac{n+3}{4}}2^{j_1\frac{n+1}{4}}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q\\ &\lesssim \big(\sum_{j_3}2^{qj_3(\theta-1)}\big)\big(\sum_{j_1\le k_1}2^{q'j_1(-\theta+\frac{n+1}{4})}\big)^{q/q'}2^{qk_1(-s+\frac{n+3}{4})}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}}\\ &\lesssim 2^{qk_1(-s-\theta+n/2+1)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}}\end{aligned}$$

Notice that the j_3 sum converge since $\theta<1$. Moreover we need to impose $s+\theta>n/2+1$ to obtain convergence. On the other hand if $j_1\neq j^{123}_{max}$ then Lemma 45 ii. yield to

$$\begin{aligned}LHH_I &\lesssim \sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{k_3s}2^{j_3(\theta-1/2)}2^{k_1(n/2+)}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q\\ &\lesssim 2^{qk_1(-s+\theta+n/2-1/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}}\end{aligned}$$

Thus in this case $-s+\theta+n/2-1/2<0$ suffices. This conclude the proof for the LHH low modulations case.

Next, we estimate the easier *med modulation term* LHH_{II}. We shall consider a general bilinear form, and we use the standard Bernstein estimate:

$$\begin{aligned}\|P_{k_3}Q_{j_3}N(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_1}v)\|_{L^2_{t,x}} &\lesssim 2^{j_3/2}2^{k_1}2^{k_3}\|P_{k_1}Q_{j_1}u\|_{L^2L^\infty}\|P_{k_3}Q_{j_1}v\|_{L^2_{t,x}}\\ &\lesssim 2^{k_3}2^{k_1(n/2+1)}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2L^2}\|P_{k_3}Q_{j_1}v\|_{L^2L^2}\end{aligned}$$

Then

$$\begin{aligned}LHH_{II} &\lesssim 2^{qk_3s}2^{qk_1(n/2+1)}\big(\sum_{j_3}2^{-\epsilon j_3q}\big)\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1/2+)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_1}v\|_2\Big)^q\\ &\lesssim 2^{qk_1(-s+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}}\end{aligned}$$

Here we have used the upper bound on j_3 to extract a factor of 2^{-k_1}: $2^{j_3(\theta-1/2+)} \le 2^{j_1(\theta-1/2+)} \le 2^{-k_1}2^{j_1\theta}2^{j_1\theta}$. Observe that we only need to impose $s > n/2$ to obtain convergence.

Next, let us estimate the easier *high modulation term* LHH_{III}. As for the bound of HHL_{III} the following argument it is not restricted to the particular type of null-form but holds for any general type of bilinear term. Thus in what follows we shall not make use of angular decomposition. Let us follow the argument used to estimate HHL_{III}. If $j^{12}_{\max} = j_1$ let us place $P_{k_1}Q_{j_3}u$ in $L^{2+}L^\infty$ and $P_{k_3}Q_{j_2}u$ in $L^\infty L^2$ we obtain

$$\begin{aligned} LHH_{III} &\lesssim 2^{qk_3s}2^{qk_1(n/2+1/2+)}\sum_{j_3\ge k_1}\Big(\sum_{j_2}2^{j_3(\theta-1/2)}2^{j_2/2}\|P_{k_1}Q_{j_3}u\|_2\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_3s}2^{qk_1(n/2+)}\sum_{j_3}(2^{j_3\theta}\|P_{k_1}Q_{j_3}u\|_2)^q\Big(\sum_{j_2}2^{j_2/2}\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_1(-s+n/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

On the other hand if $j^{12}_{\max} = j_2$ we place $P_{k_1}Q_{j_1}u$ in $L^\infty L^\infty$ and $P_{k_3}Q_{j_3}u$ in L^2L^2. Thus we obtain

$$\begin{aligned} LHH_{III} &\lesssim 2^{qk_3s}2^{qk_1(n/2+1)}\sum_{j_3\ge k_1}\Big(\sum_{j_1}2^{j_3(\theta-1)}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_3}v\|_2\Big)^q \\ &\lesssim 2^{qk_3s}2^{qk_1(n/2)}\sum_{j_3}(2^{j_3\theta}\|P_{k_3}Q_{j_3}v\|_2)^q\Big(\sum_{j_1}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\Big)^q \\ &\lesssim 2^{qk_1(-s+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

This conclude the proof of the high-output modulation estimate. We summarize in the table below where the different conditions needed to assure convergence were used in the proof.

Cases	low modulations		med modulations	high modulations
HHL	$s > n/2 - 1/2 + \theta$		$s > n/2$	$s > n/2$
LHH/HLH	$s + \theta > n/2 + 1$	if $j_1 = j^{123}_{\max}$	$s > n/2$	$s > n/2$
	$s > n/2 - 1/2 + \theta$	if $j_1 \ne j^{123}_{\max}$		

From this is clear that all the high modulations cases holds for a larger set of exponent, and the worst case is the LHH/HLH when the maximum modulation is coupled with the minimum frequency. □

We conclude this section with the proof of the multiplicative estimate for Maxwell-Klein-Gordon and Yang-Mills null-forms.

Proof of Proposition 42. We shall prove three frequency localized estimates:

(LHH) Let $k_1 \ll k_3$, if $p > 1$ and $\alpha < 0$, or if $p = 1$ and $\alpha = 0$, we have

$$\|P_{k_3}\widetilde{N}(P_{k_1}u, P_{k_3}v)\|_{H^{s-1,\theta-1,q}_{k_3}} \lesssim 2^{k_1\alpha}\|P_{k_1}u\|_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|_{H^{s,\theta,q}_{k_3}}$$

(HLH) Let $k_2 \ll k_3$, if $p > 1$ and $\alpha < 0$, or if $p = 1$ and $\alpha = 0$, we have

$$\|P_{k_3}\widetilde{N}(P_{k_3}u, P_{k_2}v)\|_{H^{s-1,\theta-1,q}_{k_3}} \lesssim 2^{k_2\alpha}\|P_{k_3}u\|_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|_{H^{s,\theta,q}_{k_2}}$$

(HHL) Let $k_1 \gg k_3$, $\beta > 0$, and $\alpha + \beta < 0$ then we have

$$\|P_{k_3}\widetilde{N}(P_{k_1}u, P_{k_1}v)\|_{H^{s-1,\theta-1,q}_{k_3}} \lesssim 2^{k_3\beta}2^{k_1\alpha}\|P_{k_1}u\|_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|_{H^{s,\theta,q}_{k_1}}$$

We start by proving the HHL case: let us split further into low, med and high modulations:

$$\begin{aligned}
\|P_{k_3}\widetilde{N}(P_{k_1}u, P_{k_1}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u, P_{k_1}Q_{j_2}v)\|_2\Big)^q \\
&+ 2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u, P_{k_1}Q_{j_1}v)\|_2\Big)^q \\
&+ 2^{qk_3(s-1)}\sum_{j_3\ge k_1}\Big(\sum_{j_1\le j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u, P_{k_1}Q_{j_3}v)\|_2\Big)^q \\
&=: HHL_I + HHL_{II} + HHL_{III}
\end{aligned}$$

In order to estimate the *low modulations term* HHL_I we use angular localization and the Bilinear L^2 estimates for null-forms. From a similar argument of Lemma 45 $iii.$, where N is replaced by $\widetilde{N}$ and thus we gain a k_1 factor, we obtain

$$\begin{aligned}
HHL_I &\lesssim 2^{qk_3(s-1/2)}2^{qk_1(n/2-1/2+)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1/2)}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2\Big)^q \\
&\lesssim 2^{qk_3(s-1/2)}2^{qk_1(-2s+\theta+n/2-1+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}}
\end{aligned}$$

Notice that $\beta = s+\theta-1$ is positive and $\alpha+\beta = -s+\theta+n/2-3/2+$.

Next, we estimate the easier *med modulation term* LHH_{II}. We shall consider a general bilinear form and use the standard Bernstein estimate:

$$\begin{aligned}
\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u, P_{k_1}Q_{j_1}v)\|_2 &\lesssim 2^{k_1}2^{k_3(n/2)}\|P_{k_1}Q_{j_1}uP_{k_1}Q_{j_1}v\|_{L^2_tL^1_x} \\
&\lesssim 2^{k_3(n/2)}2^{k_1}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_1}v\|_2
\end{aligned}$$

Then

$$\begin{aligned} HHL_{II} &\lesssim 2^{qk_3(s-1+n/2)}2^{qk_1}\big(\sum_{j_3}2^{-\epsilon j_3 q}\big)\Big(\sum_{j_1\geq\max\{k_1,j_3\}}2^{j_3(\theta-1/2+)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_1}v\|_2\Big)^q \\ &\lesssim 2^{qk_3(s-1+n/2)}2^{qk_1(-2s)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}} \end{aligned}$$

Here we have used the upper bound on j_3 to extract a factor of 2^{-k_1}: $2^{j_3(\theta-1/2+)}\leq 2^{j_1(\theta-1/2+)}\leq 2^{-k_1}2^{j_1\theta}2^{j_1\theta}$. Observe that we only need to impose $s>n/2-1$ to obtain convergence.

To control the *high modulations term* HHL_{III} we do not need to use the special null structure, in fact we present the estimate for any general bilinear form. Here the argument is slightly more involved than the other high-modulation cases since we need to gain some powers of k_3. By Bernstein inequality we obtain

$$\begin{aligned} \|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_2}v)\|_2 &\lesssim 2^{k_1}2^{k_3(n/2)}\|P_{k_1}Q_{j_1}uP_{k_1}Q_{j_2}v\|_{L^2_tL^1_x} \\ &\lesssim 2^{k_3(n/2)}2^{k_1}2^{j^{12}_{\min}/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2 \end{aligned}$$

by placing the low-modulation term into $L^\infty L^2$ and the high modulation term into L^2L^2. Therefore

$$\begin{aligned} HHL_{III} &\lesssim 2^{qk_3(s-1+n/2)}2^{qk_1}\sum_{j_3\geq k_1}\Big(\sum_{j^{12}_{\max}\approx j_3}2^{j_3(\theta-1)}2^{j^{12}_{\min}/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_3(s-1+n/2)}2^{qk_1(-2s)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}} \end{aligned}$$

since $j_3(\theta-1)\leq j_3\theta-k_1$. Notice that this case requires $s>n/2-1$ only.

We now focus on the more difficult *LHH case*. As usual we split further into low, med and high modulations:

$$\begin{aligned} \|P_{k_3}\widetilde{N}(P_{k_1}u,P_{k_3}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)}\sum_{j_3\leq k_1}\Big(\sum_{j_1\leq k_1}\sum_{j_2\leq k_1}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_2\Big)^q \\ &+ 2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\geq\max\{k_1,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_1}v)\|_2\Big)^q \\ &+ 2^{qk_3(s-1)}\sum_{j_3\geq k_1}\Big(\sum_{j^{12}_{\max}\approx j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_2\Big)^q \\ &=: LHH_I+LHH_{II}+LHH_{III} \end{aligned}$$

To estimate the most delicate term LHH_I involving *low modulations* we must invoke the angular decomposition and use Lemma 45 i. and ii., where again N is replaced by $\widetilde{N}$. Thus we split further into $j_1 = j^{123}_{max}$ and $j_1 \neq j^{123}_{max}$ cases. When $j_1 = j^{123}_{max}$ from i. we obtain

$$\begin{aligned} LHH_I &\lesssim \sum_{j_3\le k_1}\Big(\sum_{j_3\le j_1\le k_1}\sum_{j_2\le j_1} 2^{k_3 s}2^{j_3(\theta-1)}2^{k_1\frac{n-1}{4}}2^{j_1\frac{n+1}{4}}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q \\ &\lesssim (\sum_{j_3}2^{qj_3(\theta-1)})(\sum_{j_1\le k_1}2^{q'j_1(-\theta+\frac{n+1}{4})})^{q/q'}2^{qk_1(-s+\frac{n-1}{4})}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \\ &\lesssim 2^{qk_1(-s-\theta+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

Notice that the j_3 sum converge since $\theta < 1$. Moreover we need to impose $s+\theta > n/2$ to obtain convergence. On the other hand if $j_1 \neq j^{123}_{max}$ then Lemma 45 ii. yield to

$$\begin{aligned} LHH_I &\lesssim \sum_{j_3\le k_1}\Big(\sum_{j_3\le j_1\le k_1}\sum_{j_2\le j_1} 2^{k_3 s}2^{j_3(\theta-1/2)}2^{k_1(n/2-1+)}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q \\ &\lesssim 2^{qk_1(-s+\theta+n/2-3/2+)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

Thus in this case $-s+\theta+n/2-3/2 < 0$ suffices. This conclude the proof for the LHH_I , the low modulations case.

Next, we estimate the easier *med modulation term* LHH_{II}. We shall consider a general bilinear form, and we use the standard Bernstein estimate:

$$\begin{aligned} \|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u, P_{k_3}Q_{j_1}v)\|_{L^2_{t,x}} &\lesssim 2^{j_3/2}2^{k_3}\|P_{k_1}Q_{j_1}u\|_{L^2L^\infty}\|P_{k_3}Q_{j_1}v\|_{L^2_{t,x}} \\ &\lesssim 2^{k_3}2^{k_1(n/2)}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2L^2}\|P_{k_3}Q_{j_1}v\|_{L^2L^2} \end{aligned}$$

Then

$$\begin{aligned} LHH_{II} &\lesssim 2^{qk_3 s}2^{qk_1(n/2)}\big(\sum_{j_3}2^{-\epsilon j_3 q}\big)\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1/2+)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_1}v\|_2\Big)^q \\ &\lesssim 2^{qk_1(-s+n/2-1)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

Here we have used the upper bound on j_3 to extract a factor of 2^{-k_1}: $2^{j_3(\theta-1/2+)} \le 2^{j_1(\theta-1/2+)} \le 2^{-k_1}2^{j_1\theta}2^{j_1\theta}$. Observe that we only need to impose $s > n/2-1$ to obtain convergence.

Next let us estimate the easier *high modulation term* LHH_{III}. The following argument it is not restricted to the particular type of null-form but holds for any general type of bilinear term. Thus in what follows we do not make use of angular decomposition. If $j^{12}_{\max} = j_1$ let us place $P_{k_1}Q_{j_3}u$ in $L^{2+}L^\infty$ and $P_{k_3}Q_{j_2}u$ in $L^\infty L^2$ so that

$$\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_3}u, P_{k_3}Q_{j_2}v)\|_2 \lesssim 2^{k_3}2^{k_1(n/2-1/2+)}2^{j_3/2}2^{j_2/2}\|P_{k_1}Q_{j_3}u\|_2\|P_{k_3}Q_{j_2}v\|_2$$

Therefore we obtain

$$\begin{aligned} LHH_{III} &\lesssim 2^{qk_3 s}2^{qk_1(n/2-1/2+)}\sum_{j_3\ge k_1}\Big(\sum_{j_2\le j_1}2^{j_3(\theta-1/2)}2^{j_2/2}\|P_{k_1}Q_{j_3}u\|_2\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_3 s}2^{qk_1(n/2-1+)}\sum_{j_3}(2^{j_3\theta}\|P_{k_1}Q_{j_3}u\|_2)^q\Big(\sum_{j_2}2^{j_2/2}\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_1(-s+n/2-1)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

On the other hand if $j^{12}_{\max}=j_2$ we place $P_{k_1}Q_{j_1}u$ in $L^\infty L^\infty$ and $P_{k_3}Q_{j_3}u$ in L^2L^2 so that

$$\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_3}v)\|_2\lesssim 2^{k_3}2^{k_1(n/2)}2^{j_1/2}\|P_{k_1}Q_{j_3}u\|_2\|P_{k_3}Q_{j_2}v\|_2$$

Thus we obtain

$$\begin{aligned} LHH_{III} &\lesssim 2^{qk_3 s}2^{qk_1(n/2)}\sum_{j_3\ge k_1}\Big(\sum_{j_1\le j_2}2^{j_3(\theta-1)}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_3}Q_{j_3}v\|_2\Big)^q \\ &\lesssim 2^{qk_3 s}2^{qk_1(n/2-1)}\sum_{j_3}(2^{j_3\theta}\|P_{k_3}Q_{j_3}v\|_2)^q\Big(\sum_{j_1}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\Big)^q \\ &\lesssim 2^{qk_1(-s+n/2-1)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

This conclude the proof of the LHH estimate.

It remains to prove the HLH interaction case. As usual we split further into low, med and high modulations:

$$\begin{aligned} &\|P_{k_3}\widetilde{N}(P_{k_3}u,P_{k_2}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} \\ &=2^{qk_3(s-1)}\sum_{j_3\le k_2}\Big(\sum_{j_1\le k_2}\sum_{j_2\le k_2}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_3}Q_{j_1}u,P_{k_2}Q_{j_2}v)\|_2\Big)^q \\ &+2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_2,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}B(P_{k_3}Q_{j_1}u,P_{k_2}Q_{j_1}v)\|_2\Big)^q \\ &+2^{qk_3(s-1)}\sum_{j_3\ge k_2}\Big(\sum_{j^{12}_{\max}\approx j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_3}Q_{j_1}u,P_{k_2}Q_{j_2}v)\|_2\Big)^q \\ &=: HLH_I+HLH_{II}+HLH_{III} \end{aligned}$$

To estimate the delicate term HLH_I involving *low modulations* we must invoke the angular decomposition and use Lemma 45 i. and ii., see Remark after Lemma 45. Thus we split further into $j_2=j^{123}_{max}$ and $j_2\neq j^{123}_{max}$ cases. When $j_2=j^{123}_{max}$ from i., where N is replaced by $\widetilde{N}$ and thus we gain a k_3 factor, we obtain

$$
\begin{aligned}
HLH_I &\lesssim \sum_{j_3\le k_2}\Big(\sum_{j_1\le j_2}\sum_{j_3\le j_2\le k_2} 2^{k_3(s-1)}2^{j_3(\theta-1)}2^{k_2\frac{n+3}{4}}2^{j_1\frac{n+1}{4}}2^{j_2/2}\|P_{k_3}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_2}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q\\
&\lesssim (\sum_{j_3}2^{qj_3(\theta-1)})(\sum_{j_1\le k_2}2^{q'j_1(-\theta+\frac{n+1}{4})})^{q/q'}2^{qk_2(-s+\frac{n-1}{4})}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|^q_{H^{s,\theta,q}_{k_2}}\\
&\lesssim 2^{qk_2(-s-\theta+n/2)}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|^q_{H^{s,\theta,q}_{k_2}}
\end{aligned}
$$

Notice that the j_3 sum converge since $\theta<1$. Moreover we need to impose $s+\theta>n/2$ to obtain convergence. On the other hand if $j_2\ne j^{123}_{max}$ then Lemma 45 $ii.$ (see Remark after Lemma 45) where N is replaced by $\widetilde{N}$, yield to

$$
\begin{aligned}
HLH_I &\lesssim \sum_{j_3\le k_1}\Big(\sum_{j_1\le j_2}\sum_{j_3\le j_2\le k_2} 2^{k_3(s-1)}2^{j_3(\theta-1/2)}2^{k_1(n/2+)}2^{j_1/2}2^{j_2/2}\|P_{k_3}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_2}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q\\
&\lesssim 2^{qk_2(-s+\theta+n/2-3/2+)}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|^q_{H^{s,\theta,q}_{k_2}}
\end{aligned}
$$

Thus in this case $-s+\theta+n/2-3/2<0$ suffices. This conclude the proof for the HLH low modulations case. Notice that this case is easier then the corresponding LHH case since in the HLH case we gain a factor of the high frequency k_3.

Next, we estimate the easier *med modulation term* HLH_{II}. We shall consider a general bilinear form, and we use the standard Bernstein estimate:

$$
\begin{aligned}
\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_3}Q_{j_1}u,P_{k_2}Q_{j_1}v)\|_{L^2_{t,x}} &\lesssim 2^{j_3/2}2^{k_3}\|P_{k_3}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_2}Q_{j_1}v\|_{L^2L^\infty}\\
&\lesssim 2^{k_3}2^{k_2(n/2)}2^{j_3/2}\|P_{k_3}Q_{j_1}u\|_{L^2L^2}\|P_{k_2}Q_{j_1}v\|_{L^2L^2}
\end{aligned}
$$

Then

$$
\begin{aligned}
LHH_{II} &\lesssim 2^{qk_3s}2^{qk_2(n/2)}\big(\sum_{j_3}2^{-\epsilon j_3q}\big)\Big(\sum_{j_1\ge\max\{k_2,j_3\}}2^{j_3(\theta-1/2+)}\|P_{k_3}Q_{j_1}u\|_2\|P_{k_2}Q_{j_1}v\|_2\Big)^q\\
&\lesssim 2^{qk_2(-s+n/2-1)}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|^q_{H^{s,\theta,q}_{k_2}}
\end{aligned}
$$

Here we have used the upper bound on j_3 to extract a factor of 2^{-k_2}: $2^{j_3(\theta-1/2+)}\le 2^{j_1(\theta-1/2+)}\le 2^{-k_2}2^{j_1\theta}2^{j_1\theta}$. Observe that we only need to impose $s>n/2-1$ to obtain convergence.

Next, let us estimate the easier *high modulation term* HLH_{III}. The following argument it is not restricted to the particular type of null-form but holds for any general type of bilinear term. Thus in what follows we do not make use of angular decomposition. If $j^{12}_{\max}=j_1$ let us place $P_{k_3}Q_{j_3}u$ in $L^\infty L^2$ and $P_{k_2}Q_{j_2}u$ in $L^{2+}L^\infty$ so that

$$
\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_3}Q_{j_3}u,P_{k_2}Q_{j_2}v)\|_2\lesssim 2^{k_2(n/2+1/2+)}2^{j_3/2}2^{j_2/2}\|P_{k_3}Q_{j_3}u\|_2\|P_{k_2}Q_{j_2}v\|_2
$$

Therefore we obtain

$$\begin{aligned} HLH_{III} &\lesssim 2^{qk_3(s-1)}2^{qk_2(n/2+1/2+)}\sum_{j_3\ge k_2}\Big(\sum_{j_2}2^{j_3(\theta-1/2)}2^{j_2/2}\|P_{k_1}Q_{j_3}u\|_2\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_3(s-1)}2^{qk_2(n/2+)}\sum_{j_3}(2^{j_3\theta}\|P_{k_3}Q_{j_3}u\|_2)^q\Big(\sum_{j_2}2^{j_2/2}\|P_{k_3}Q_{j_2}v\|_2\Big)^q \\ &\lesssim 2^{qk_1(-s+n/2-1)}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|^q_{H^{s,\theta,q}_{k_2}} \end{aligned}$$

On the other hand if $j^{12}_{\max}=j_2$ we place $P_{k_3}Q_{j_1}u$ in $L^\infty L^2$ and $P_{k_2}Q_{j_3}u$ in $L^{2+}L^\infty$ so that

$$\|P_{k_3}Q_{j_3}\widetilde{N}(P_{k_3}Q_{j_1}u,P_{k_2}Q_{j_3}v)\|_2\lesssim 2^{k_2(n/2+1/2+)}2^{j_1/2}2^{j_3/2}\|P_{k_3}Q_{j_1}u\|_2\|P_{k_2}Q_{j_3}v\|_2$$

Thus we obtain

$$\begin{aligned} HLH_{III} &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+1/2+)}\sum_{j_3\ge k_2}\Big(\sum_{j_1}2^{j_3(\theta-1/2)}2^{j_1/2}\|P_{k_3}Q_{j_1}u\|_2\|P_{k_2}Q_{j_3}v\|_2\Big)^q \\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+)}\sum_{j_3}(2^{j_3\theta}\|P_{k_2}Q_{j_3}v\|_2)^q\Big(\sum_{j_1}2^{j_1/2}\|P_{k_3}Q_{j_1}u\|_2\Big)^q \\ &\lesssim 2^{qk_2(-s+n/2-1+)}\|P_{k_3}u\|^q_{H^{s,\theta,q}_{k_3}}\|P_{k_2}v\|^q_{H^{s,\theta,q}_{k_2}} \end{aligned}$$

This conclude the proof of the HLH interaction case. In the table below we summarize the different conditions needed in the different cases.

	low modulations		med modulations	high modulations
HHL	$s>n/2-3/2+\theta$		$s>n/2-1$	$s>n/2-1$
LHH	$s+\theta>n/2$	if $j_1=j^{123}_{\max}$	$s>n/2-1$	$s>n/2-1$
	$s>n/2-3/2+\theta$	if $j_1\neq j^{123}_{\max}$		
HLH	$s+\theta>n/2$	if $j_2=j^{123}_{\max}$	$s>n/2-1$	$s>n/2-1$
	$s>n/2-3/2+\theta$	if $j_2\neq j^{123}_{\max}$		

□

1.9 General quadratic nonlinearities (revisited)

In this section we shall prove a multiplicative estimate for general quadratic form without any null structure of the type we already encountered in §1.3: $B(u,v)=b^{\alpha\beta}\partial_\alpha u\partial_\beta v$. What might be surprising is that we obtain the same lower bound on s, namely $s>n/2+1/4$, as the N_{ij} null-form estimate of Proposition 43. Here however we are working at $n\ge 4$ spatial dimensions and the lack of null structure is compensated by the higher dimension. Furthermore, Theorem 7 already settled the low dimensional $n=2,3$ cases, where the optimal result ($s>\frac{n+5}{4}$) is reached by Strichartz estimates. The following theorem, based on [113], gives an improvement of $1/4$ or of $1/2$ over the result obtained in Theorem 7 for dimension $n = 4$, or $n \geq 5$ respectively.

Theorem 46. *Let $n \geq 4$, $p \geq 1$, and $s > \max\{\frac{n}{2} + \frac{1}{4}, \frac{n+5}{4}\}$, then there exist an unique local solution $u \in C([0,T], B^s_{2,p}(\mathbb{R}^n)) \cap C^1([0,T], B^{s-1}_{2,p}(\mathbb{R}^n)$ to the Cauchy problem*

$$\begin{cases} \Box u = B(u,u) \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in B^s_{2,p}(\mathbb{R}^n) \times B^{s-1}_{2,p}(\mathbb{R}^n) \end{cases}$$

Moreover if $p = 1$ one can take $s = \max\{\frac{n}{2} + \frac{1}{4}, \frac{n+5}{4}\}$.

Notice that for $n \geq 4$, we clearly have $\max\{\frac{n}{2} + \frac{1}{4}, \frac{n+5}{4}\} = \frac{n}{2} + \frac{1}{4}$. However we have kept this form to highlight the similarities with the Maxwell-Klein-Gordon and Yang-Mills nonlinearities considered in the previous section. The proof of Theorem 46 reduces to the proof of the following general bilinear estimate.

Proposition 47. *The multiplicative estimate*

$$\|B(u,v)\|_{H^{s-1,\theta-1}_{p,q}} \lesssim \|u\|_{H^{s,\theta}_{p,q}} \|v\|_{H^{s,\theta}_{p,q}}$$

holds in the following cases:

$n \geq 4$	$p > 1$	$q > 1$	$1/2 < \theta < 1$	$s - (n-1)/2 > \theta$	$s + \theta > n/2 + 1$
$n \geq 4$	$p > 1$	$q = 1$	$\theta = 1/2$	$s - (n-1)/2 > \theta$	$s + \theta > n/2 + 1$
$n \geq 4$	$p = 1$	$q > 1$	$1/2 < \theta < 1$	$s - (n-1)/2 = \theta$	$s + \theta = n/2 + 1$

Proof. Since the estimate is symmetric it suffices to prove the two frequency localized estimates:

- *(High-output)* Let $k_1 \ll k_3$, if $p > 1$ and $\alpha < 0$, or if $p = 1$ and $\alpha = 0$, we have

$$\|P_{k_3} B(P_{k_1} u, P_{k_3} v)\|_{H^{s-1,\theta-1,q}_{k_3}} \lesssim 2^{k_1 \alpha} \|P_{k_1} u\|_{H^{s,\theta,q}_{k_1}} \|P_{k_3} v\|_{H^{s,\theta,q}_{k_3}}$$

- *(Low-output)* Let $k_1 \gg k_3$, $\beta > 0$, and $\alpha + \beta \leq 0$ then we have

$$\|P_{k_3} B(P_{k_1} u, P_{k_1} v)\|_{H^{s-1,\theta-1,q}_{k_3}} \lesssim 2^{k_3 \beta} 2^{k_1 \alpha} \|P_{k_1} u\|_{H^{s,\theta,q}_{k_1}} \|P_{k_1} v\|_{H^{s,\theta,q}_{k_1}}$$

We begin by proving the easier *low-output estimate*. As in the proof of the algebra property we split further into low modulations, med modulations, and high modulations:

$$\begin{aligned}
\|P_{k_3}B(P_{k_1}u,P_{k_1}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_2}v)\|_2\Big)^q\\
&+ 2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_1}v)\|_2\Big)^q\\
&+ 2^{qk_3(s-1)}\sum_{j_3\ge k_1}\Big(\sum_{j_1\le j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_3}v)\|_2\Big)^q\\
&=: HHL_I+HHL_{II}+HHL_{III}
\end{aligned}$$

Notice that in the third term we have supposed, without loosing generality, that $\max\{j_1,j_2\}=j_2$. In order to estimate the *low modulations term* HHL_I we split the argument into two parts. First suppose that $n\ge 5$, then the L^2L^4 Strichartz pair is admissible hence the L^2-L^1 Bernstein inequality yield to

$$\begin{aligned}
\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_2}v)\|_{L^2_{t,x}} &\lesssim 2^{j_3/2}\|B(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_2}v)\|_{L^1_tL^2_x}\\
&\lesssim 2^{j_3/2}2^{2k_1}\|P_{k_1}Q_{j_1}u\|_{L^2L^4}\|P_{k_1}Q_{j_2}v\|_{L^2L^4}\\
&\lesssim 2^{k_1(n/2+1)}2^{j_1/2}2^{j_2/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2L^2}\|P_{k_1}Q_{j_2}v\|_{L^2L^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
HHL_I &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+1)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1/2)}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2\Big)^q\\
&\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+\theta+n/2+1/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}}
\end{aligned}$$

To close this case it suffices that the sum of the two exponents is negative, that is $s>n/2-1/2+\theta$. On the other hand if $n=4$ the L^2L^4 Strichartz pair is not admissible anymore. However one can find two Strichartz admissible pair (p_1,q_1), and (p_2,q_2) so that $L^{p_1}L^{q_1}\cdot L^{p_2}L^{q_2}\subset L^{4/3}L^2$, hence

$$\begin{aligned}
\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_2}v)\|_{L^2_{t,x}} &\lesssim 2^{j_3/4}\|B(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_2}v)\|_{L^{4/3}_tL^2_x}\\
&\lesssim 2^{j_3/4}2^{2k_1}\|P_{k_1}Q_{j_1}u\|_{L^{p_1}L^{q_1}}\|P_{k_1}Q_{j_2}v\|_{L^{p_2}L^{q_2}}\\
&\lesssim 2^{k_1(n/2+5/4)}2^{j_1/2}2^{j_2/2}2^{j_3/4}\|P_{k_1}Q_{j_1}u\|_{L^2L^2}\|P_{k_1}Q_{j_2}v\|_{L^2L^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
HHL_I &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+5/4)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-3/4)}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_2}v\|_2\Big)^q\\
&\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+n/2+5/4)}\max\{2^{k_1(\theta-3/4)},1\}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}}
\end{aligned}$$

Notice that if $\theta > 3/4$ we obtain $k_3(s-1)k_1(-2s+\theta+n/2+1/2)$, to gain smallness and close the argument we must have $-s+\theta+n/2-1/2<0$. On the other hand if $1/2<\theta<3/4$ then we obtain $k_3(s-1)k_1(-2s+n/2+5/4)$, which lead us to the condition $s>n/2+1/4$.

Next, to estimate the middle modulation term HHL_{II}, let us use the standard Bernstein estimate:

$$\begin{aligned}\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_1}Q_{j_1}v)\|_{L^2_{t,x}} &\lesssim 2^{j_3/2}2^{2k_1}\|P_{k_1}Q_{j_1}u\|_{L^2L^\infty}\|P_{k_1}Q_{j_1}v\|_{L^2_{t,x}}\\ &\lesssim 2^{k_1(n/2+2)}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2L^2}\|P_{k_1}Q_{j_1}v\|_{L^2L^2}\end{aligned}$$

Then

$$\begin{aligned}HHL_{II} &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+2)}\big(\sum_{j_3}2^{-\epsilon j_3 q}\big)\Big(\sum_{j_1\geq\max\{k_1,j_3\}}2^{j_3(\theta-1/2+)}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_1}v\|_2\Big)^q\\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+n/2+1)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}}\end{aligned}$$

Here we have used the upper bound on the j_3 sum to extract a factor of 2^{-k_1}: $2^{j_3(\theta-1/2+)}\leq 2^{j_1(\theta-1/2+)}\leq 2^{-k_1}2^{j_1\theta}2^{j_1\theta}$. Observe that we only need to impose $s>n/2$ to obtain convergence and that the lower bound on the dimension is not needed here.

It remains to control the *high modulations term* HHL_{III}. Here the argument follow closely the one used for the N null-form. We place $P_{k_1}Q_{j_1}u$ in L^2L^∞ and $P_{k_1}Q_{j_3}v$ in $L^\infty L^2$ and use the fact that the j_2 sum collapse. We proceed as follows

$$\begin{aligned}HHL_{III} &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+3/2)}\sum_{j_3\geq k_3}\Big(\sum_{j_1}2^{j_3(\theta-1)}2^{j_1/2}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_2\|P_{k_1}Q_{j_3}v\|_2\Big)^q\\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(n/2+1)}\sum_{j_3}(2^{j_3\theta}\|P_{k_1}Q_{j_3}v\|_2)^q\Big(\sum_{j_1}2^{j_1/2}\|P_{k_1}Q_{j_1}u\|_2\Big)^q\\ &\lesssim 2^{qk_3(s-1)}2^{qk_1(-2s+1+n/2)}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_1}v\|^q_{H^{s,\theta,q}_{k_1}}\end{aligned}$$

Here we have used the high modulation hypothesis to control $2^{j_3(\theta-1/2)}\leq 2^{j_3\theta}2^{-k_1/2}$. Moreover notice that to close this estimate we only need $s>n/2$.

Next we prove the more difficult *high-output estimate.* As usual, due to Lemma 44 we split further into low, med and high modulations:

$$\begin{aligned}
\|P_{k_3}B(P_{k_1}u,P_{k_3}v)\|^q_{H^{s-1,\theta-1,q}_{k_3}} &= 2^{qk_3(s-1)}\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_2\Big)^q\\
&+ 2^{qk_3(s-1)}\sum_{j_3}\Big(\sum_{j_1\ge\max\{k_1,j_3\}}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_1}v)\|_2\Big)^q\\
&+ 2^{qk_3(s-1)}\sum_{j_3\ge k_1}\Big(\sum_{j^{12}_{\max}\approx j_3}2^{j_3(\theta-1)}\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_2\Big)^q\\
&=: LHH_I+LHH_{II}+LHH_{III}
\end{aligned}$$

To estimate the most delicate term LHH_I term involving *low modulations* we must invoke the angular decomposition and use Lemma 45 i. and ii.. However, since we are working with a general bilinear form, we wont get any smallness form the angular separation between the two inputs. Let us split further into three cases based on the maximum modulation. When $j_1=j^{123}_{max}$ following i., by Bernstein and Strichartz inequalities, we obtain

$$\begin{aligned}
&\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_{L^2_{t,x}}\\
&\lesssim\sum_{\kappa_1\approx\kappa_2}\|P_{k_3}Q_{j_3}B(P_{k_1,\kappa_1}Q_{j_1}u,P_{k_3,\kappa_2}Q_{j_2}v)\|_{L^2_{t,x}}\\
&\lesssim\sum_{\kappa_1\approx\kappa_2}2^{k_1+k_3}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_tL^\infty_x}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^\infty_tL^2_x}\\
&\lesssim\sum_{\kappa_1\approx\kappa_2}2^{k_1+k_3}2^{|\kappa|(n-1)/2+k_1(n/2)}2^{j_2/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^2_{t,x}}\\
&\lesssim 2^{k_1\frac{n+5}{4}}2^{k_3}2^{j_1\frac{n-1}{4}}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}
\end{aligned}$$

Then

$$\begin{aligned}
LHH_I&\lesssim\sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{k_3s}2^{j_3(\theta-1)}2^{k_1\frac{n+5}{4}}2^{j_1\frac{n-1}{4}}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q\\
&\lesssim(\sum_{j_3}2^{qj_3(\theta-1)})(\sum_{j_1\le k_1}2^{q'j_1(-\theta+\frac{n-1}{4})})^{q/q'}2^{qk_1(-s+\frac{n+5}{4})}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}}\\
&\lesssim 2^{qk_1(-s+\frac{n+5}{4})}\max\{2^{k_1(-\theta+\frac{n-1}{4})},1\}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}}
\end{aligned}$$

Notice that the j_3 sum converge since $\theta<1$. Moreover the j_1 exponent is negative only if $n=4$ and $\theta>3/4$, in this case we can bound the j_1 sum without loosing any k_1 factor. Thus to close this case we have to impose $s>(n+5)/4$ when $n=4$ and $\theta>3/4$, and $s+\theta>n/2+1$ in the reminding cases ($n=4$ and $\theta<3/4$, or $n\ge5$). On the other hand if $j^{123}_{\max}=j_3$ we take advantage that we are working in $n\ge4$ space dimensions so we have access to a larger class of Strichartz estimates. In this case we still have to place $P_{k_3,\kappa_2}Q_{j_2}v$ into $L^\infty L^2$ to avoid any extra power of the higher frequency k_3, thus $P_{k_1,\kappa_1}Q_{j_1}u$ is placed into L^2L^∞. To estimate the latter we use a combination of Bernstein and Strichartz estimates: first we use Bernstein to reach the Pecher pair L^2L^p, where $1/p=(n-3)/(2n-2)$, then we use Strichartz to reach the L^2L^2 norm. We obtain

$$\begin{aligned}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2L^\infty} &\lesssim 2^{(|\kappa|(n-1)+k_1n)/p}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2L^p}\\ &\lesssim 2^{(|\kappa|(n-1)+k_1n)/p}2^{k_1(n/2-1/2-n/p)}2^{j_1/2}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2L^2}\\ &\lesssim 2^{k_1\frac{n+1}{4}}2^{j_1/2}2^{j_3\frac{n-3}{4}}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2L^2}\end{aligned}$$

Then we obtain the following estimate for the inner $L^2_{t,x}$ norm:

$$\begin{aligned}&\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_{L^2_{t,x}}\\ &\lesssim \sum_{\kappa_1\approx\kappa_2}\|P_{k_3}Q_{j_3}B(P_{k_1,\kappa_1}Q_{j_1}u,P_{k_3,\kappa_2}Q_{j_2}v)\|_{L^2_{t,x}}\\ &\lesssim \sum_{\kappa_1\approx\kappa_2}2^{k_1+k_3}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_tL^\infty_x}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^\infty_tL^2_x}\\ &\lesssim \sum_{\kappa_1\approx\kappa_2}2^{k_1+k_3}2^{k_1\frac{n+1}{4}}2^{j_1/2}2^{j_2/2}2^{j_3\frac{n-3}{4}}\|P_{k_1,\kappa_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3,\kappa_2}Q_{j_2}v\|_{L^2_{t,x}}\\ &\lesssim 2^{k_1\frac{n+5}{4}}2^{k_3}2^{j_1/2}2^{j_2/2}2^{j_3\frac{n-3}{4}}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}\end{aligned}$$

Therefore plugging the previous estimate into LHH_I yield to

$$\begin{aligned}LHH_I &\lesssim \sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_2\le k_1}2^{k_3s}2^{j_3(\theta+\frac{n-7}{4})}2^{k_1\frac{n+5}{4}}2^{j_1/2}2^{j_2/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q\\ &\lesssim 2^{qk_1(-s+\frac{n+5}{4})}\max\{2^{k_1(\theta+\frac{n-7}{4})},1\}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}}\end{aligned}$$

Then if $n\ge 5$ or $n=4$ and $\theta>3/4$ we must have $-s+\theta+n/2-1/2<0$. On the other hand if $n=4$ and $\theta>3/4$ then $s>\frac{n+5}{4}$ suffices. Next, if $j_2=j^{123}_{max}$ then we use a similar argument as in the previous case, the minor modification is that we use the L^2-L^1 Bernstein to obtain a factor of $2^{j_3/2}$ and this allow us to place $P_{k_3,\kappa_2}Q_{j_2}v$ into $L^2_{t,x}$, thus we obtain

$$\|P_{k_3}Q_{j_3}B(P_{k_1}Q_{j_1}u,P_{k_3}Q_{j_2}v)\|_{L^2_{t,x}}\lesssim 2^{k_1\frac{n+5}{4}}2^{k_3}2^{j_1/2}2^{j_2\frac{n-3}{4}}2^{j_3/2}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}$$

Therefore we obtain

$$\begin{aligned}LHH_I &\lesssim \sum_{j_3\le k_1}\Big(\sum_{j_1\le k_1}\sum_{j_3\le j_2\le k_1}2^{k_3s}2^{j_3(\theta-1/2)}2^{k_1\frac{n+5}{4}}2^{j_1/2}2^{j_2\frac{n-3}{4}}\|P_{k_1}Q_{j_1}u\|_{L^2_{t,x}}\|P_{k_3}Q_{j_2}v\|_{L^2_{t,x}}\Big)^q\\ &\lesssim 2^{qk_1(-s+\frac{n+5}{4})}\sum_{j_3\le k_1}2^{qj_3(\theta-1/2)}\Big(\sum_{j_3\le j_2\le k_1}2^{q'j_2(-\theta+\frac{n-3}{4})}\Big)^{q/q'}\|P_{k_1}u\|^q_{H^{s,\theta,q}_{k_1}}\|P_{k_3}v\|^q_{H^{s,\theta,q}_{k_3}}\end{aligned}$$

Notice that if $\theta<\frac{n-3}{4}$ then

$$\sum_{j_3\le k_1}2^{qj_3(\theta-1/2)}\Big(\sum_{j_3\le j_2\le k_1}2^{q'j_2(-\theta+\frac{n-3}{4})}\Big)^{q/q'}\lesssim 2^{k_1\frac{n-5}{4}}$$

and if $\theta < \frac{n-3}{4}$ we obtain

$$\sum_{j_3 \le k_1} 2^{qj_3(\theta-1/2)} \big(\sum_{j_3 \le j_2 \le k_1} 2^{q' j_2(-\theta + \frac{n-3}{4})} \big)^{q/q'} \lesssim \sum_{j_3 \le k_1} 2^{j_3 \frac{n-5}{4}}$$

To close this case we have to impose the bounds $s > n/2$ for $n \ge 5$ and $s > \frac{n+5}{4}$ for $n = 4$. This conclude the proof for the LHH_I, the low modulations case.

Next, we estimate the easier *mid modulation term* LHH_{II}. We shall use the standard Bernstein estimate:

$$\begin{aligned} \|P_{k_3} Q_{j_3} B(P_{k_1} Q_{j_1} u, P_{k_3} Q_{j_1} v)\|_{L^2_{t,x}} &\lesssim 2^{j_3/2} 2^{k_1} 2^{k_3} \|P_{k_1} Q_{j_1} u\|_{L^2 L^\infty} \|P_{k_3} Q_{j_1} v\|_{L^2_{t,x}} \\ &\lesssim 2^{k_3} 2^{k_1(n/2+1)} 2^{j_3/2} \|P_{k_1} Q_{j_1} u\|_{L^2 L^2} \|P_{k_3} Q_{j_1} v\|_{L^2 L^2} \end{aligned}$$

Then

$$\begin{aligned} LHH_{II} &\lesssim 2^{qk_3 s} 2^{qk_1(n/2+1)} \big(\sum_{j_3} 2^{-\epsilon j_3 q} \big) \Big(\sum_{j_1 \ge \max\{k_1, j_3\}} 2^{j_3(\theta-1/2+)} \|P_{k_1} Q_{j_1} u\|_2 \|P_{k_3} Q_{j_1} v\|_2 \Big)^q \\ &\lesssim 2^{qk_1(-s+n/2)} \|P_{k_1} u\|^q_{H^{s,\theta,q}_{k_1}} \|P_{k_3} v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

Here we have used the upper bound on j_3 to extract a factor of 2^{-k_1}: $2^{j_3(\theta-1/2+)} \le 2^{j_1(\theta-1/2+)} \le 2^{-k_1} 2^{j_1\theta} 2^{j_1\theta}$. Observe that we only need to impose $s > n/2$ to obtain convergence and that the lower bound on the dimension is not needed here.

Let us estimate the easier *high modulation term* LHH_{III}. Let us follow the argument used to estimate III. If $j^{12}_{\max} = j_1$ let us place $P_{k_1} Q_{j_3} u$ in $L^2 L^\infty$, $P_{k_3} Q_{j_2} u$ in $L^\infty L^2$ and use Strichartz inequalities. We obtain

$$\begin{aligned} LHH_{III} &\lesssim 2^{qk_3 s} 2^{qk_1(n/2+1/2)} \sum_{j_3 \ge k_1} \Big(\sum_{j_2} 2^{j_3(\theta-1/2)} 2^{j_2/2} \|P_{k_1} Q_{j_3} u\|_2 \|P_{k_3} Q_{j_2} v\|_2 \Big)^q \\ &\lesssim 2^{qk_3 s} 2^{qk_1 n/2} \sum_{j_3} (2^{j_3\theta} \|P_{k_1} Q_{j_3} u\|_2)^q \Big(\sum_{j_2} 2^{j_2/2} \|P_{k_3} Q_{j_2} v\|_2 \Big)^q \\ &\lesssim 2^{qk_1(-s+n/2)} \|P_{k_1} u\|^q_{H^{s,\theta,q}_{k_1}} \|P_{k_3} v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

On the other hand if $j^{12}_{\max} = j_2$ we place $P_{k_1} Q_{j_1} u$ in $L^\infty L^\infty$, $P_{k_3} Q_{j_3} u$ in $L^2 L^2$ and use Strichartz inequalities. Thus we obtain

$$\begin{aligned} LHH_{III} &\lesssim 2^{qk_3 s} 2^{qk_1(n/2+1)} \sum_{j_3 \geq k_1} \Big(\sum_{j_1} 2^{j_3(\theta-1)} 2^{j_1/2} \|P_{k_1} Q_{j_1} u\|_2 \|P_{k_3} Q_{j_3} v\|_2 \Big)^q \\ &\lesssim 2^{qk_3 s} 2^{qk_1 n/2} \sum_{j_3} (2^{j_3\theta} \|P_{k_3} Q_{j_3} v\|_2)^q \Big(\sum_{j_1} 2^{j_1/2} \|P_{k_1} Q_{j_1} u\|_2 \Big)^q \\ &\lesssim 2^{qk_1(-s+n/2)} \|P_{k_1} u\|^q_{H^{s,\theta,q}_{k_1}} \|P_{k_3} v\|^q_{H^{s,\theta,q}_{k_3}} \end{aligned}$$

This conclude the proof of the high-output modulation estimate.

We summarize below where the different conditions needed to assure convergence are used in the proof. We remark that in the $n = 4$ dimensional case the worst case is in the LHH case when $j_1 = j^{123}_{\max}$.

$n=4$	low modulations		med modulations	high modulations
HHL	$s > n/2 + 1/4$	if $1/2 < \theta < 3/4$	$s > n/2$	$s > n/2$
	$s > n/2 - 1/2 + \theta$	if $3/4 < \theta < 1$		
LHH	$s + \theta > n/2 + 1$	if $j_1 = j^{123}_{\max}$, $1/2 < \theta < 3/4$	$s > n/2$	$s > n/2$
	$s > n/4 + 5/4$	if $j_1 = j^{123}_{\max}$, $3/4 < \theta < 1$		
	$s > n/4 + 5/4$	if $j_2 = j^{123}_{\max}$, $1/2 < \theta < 3/4$		
	$s > n/2 - 1/2 + \theta$	if $j_2 = j^{123}_{\max}$, $3/4 < \theta < 1$		
	$s > n/4 + 5/4$	if $j_3 = j^{123}_{\max}$, $1/2 < \theta < 3/4$		
	$s > n/2 - 1/2 + \theta$	if $j_3 = j^{123}_{\max}$, $3/4 < \theta < 1$		

$n \geq 5$	low modulations		med modulations	high modulations
HHL	$s > n/2 - 1/2 + \theta$		$s > n/2$	$s > n/2$
LHH	$s + \theta > n/2 + 1$	if $j_1 = j^{123}_{\max}$	$s > n/2$	$s > n/2$
	$s > n/2 - 1/2 + \theta$	if $j_2 = j^{123}_{\max}$		
	$s > n/2 - 1/2 + \theta$	if $j_3 = j^{123}_{\max}$		

□

2 Global regularity for Yang-Mills equation below the energy norm in $\mathbb{R}^{1+3}$

This chapter is devoted to the analysis of long-time behavior of solutions to Cauchy initial value problem for the hyperbolic Yang-Mills equation in $\mathbb{R}^{1+3}$ space-time. In particular, we lay down the foundations to show that such a Cauchy problem is globally well-posed for small weighted $H^{3/4+}(\mathbb{R}^3) \times H^{-1/4+}(\mathbb{R}^3)$ initial data, thus matching the regularity in the original work of Tao [106].

Our technique uses the Penrose compactification of Minkowski space-time, which allows us to transfer the original Cauchy problem on the flat Minkowski space-time into a Cauchy problem on a precompact manifold with a curved metric. Such a technique can be traced back to the pivotal work by Christodoulou [10] where the existence of global solutions quasilinear systems of hyperbolic partial differential equations was settled. Recently, Dasgupta, Gao, and Krieger [14] applied a similar argument to the wave map equation. The drawback of such a procedure is that we are forced to work with a curved version of hyperbolic Sobolev spaces which was introduced, in the context of the wave equation, by Geba and Tataru [30], see also [27] for a refinement of such spaces. Despite its non-trivial definition and properties, the author strongly believes that such spaces will play a central role in the further development of the field. The main novelty is contained in §2.10 where we provide a proof of a key estimate involving a null-form in the context of curved hyperbolic Sobolev spaces. The argument provided here will serve as a guide in the proof the corresponding estimate for the Yang-Mills null-form which is current work in progress and will be addressed in a subsequent paper.

This work is somewhat in the line with the sequence of works generating from the studies of Klainerman and Machedon at Princeton during the 90s. In a series of papers, Klainerman and Machedon studied the optimal local well-posedness problem for a class of quasilinear problem with quadratic nonlinearities. At this point in time, the subcritical well posedness theory for the Yang-Mills equation is well established in high dimensions: if $n \geq 4$ spatial dimensions Klainerman and Tataru proved in [46] the optimal local well-posedness result for $s > n/2 - 1$. However, in low dimensions there are still challenging open problems to be solved. For the $n = 3$ problem, in [40] Klainerman and Machedon proved a local well-posedness result for $s \geq 1$ in the Coulomb gauge, their result improved the classical one of Eardley and Moncrief

[24] which concerned only smooth solutions. Subsequently, by working in the temporal gauge, Tao was able to prove local existence [106] for initial data with regularity $s > 3/4$, thus going below the energy norm. Recently Pecher [85] generalized Tao's result to the more general Yang-Mills-Higgs system and to general dimensions $n \geq 3$. See also the work of Chrusciel and Shatah [11] for a global well-posedness result for the Yang-Mills equation on curved manifold and $s \geq 2$. The optimal well-posedness result up to $s > 1/2$ for the *full* YM equation is still open.

We now introduce the Yang-Mills equation. Let G be a semi-simple Lie group and $(\mathfrak{g}, [\cdot])$ its associated Lie algebra. We denote by $ad(X)Y = [X,Y]$ the Lie bracket on $\mathfrak{g}$ and by $\langle X, Y\rangle = tr(ad(X)ad(Y))$ its associated non-degenerate Killing form. The unknown of the Yang-Mills equation is a connection 1-form $A = A_\alpha dx^\alpha$ on the Minkowski space-time $\mathbb{R}^{1+3}$ with value in $\mathfrak{g}$. Let $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$ be the correspondent curvature 2-form, then the Yang-Mills equation

$$D_\alpha F^{\alpha\beta} = 0$$

are obtained as the Euler-Lagrange equations of the Yang-Mills Lagrangian

$$\mathscr{L}(A) = -\frac{1}{4}\int_{\mathbb{R}^{1+3}} \langle F^{\alpha\beta}, F_{\alpha\beta}\rangle \, dt dx.$$

Here $D_\alpha = \partial_\alpha + [A_\alpha, \cdot]$ is the covariant derivative. To obtain a more familiar formulation from a partial differential equations perspective we can expand the Yang-Mills equation in term of the connection 1-form coefficients and obtain the following system of hyperbolic equations

$$\Box A_\beta - \partial_\beta \partial^\alpha A_\alpha + 2[A_\alpha, \partial^\alpha A_\beta] - [A_\alpha, \partial_\beta A^\alpha] + [\partial^\alpha A_\alpha, A_\beta] + [A^\alpha, [A_\beta, A_\alpha]] = 0 \tag{2.1}$$

where $\Box = \partial_\alpha \partial^\alpha = -\partial_t^2 + \triangle$ is the d'Alembertian. To obtain a well-posed problem we need to fix the gauge. Let us divide the connection $A_\alpha = (A_0, A)$ in its temporal and spatial components where $A = (A_1, A_2, A_3)$, then let us impose the connection to lie in the temporal gauge, that is $A_0 = 0$. Then the Yang-Mills equation simplifies to the following mixed hyperbolic/elliptic system:

$$\begin{cases} \partial_0(\mathrm{div} A) + [A^i, \partial_0 A_i] = 0 \\ \Box A_j - \partial_j(\mathrm{div} A) + 2[A^i, \partial_i A_j] - [A^i, \partial_j A_i] + [\mathrm{div} A, A_j] + [A^i, [A_i, A_j]] = 0. \end{cases}$$

Setting the initial data pair (A_0, A_1) on the time slice $t = 0$, we consider the following initial value problem for the Yang-Mills equation:

$$\begin{cases} \partial_0(\mathrm{div} A) + [A^i, \partial_0 A_i] = 0 \\ \Box A_j - \partial_j(\mathrm{div} A) + 2[A^i, \partial_i A_j] - [A^i, \partial_j A_i] + [\mathrm{div} A, A_j] + [A^i, [A_i, A_j]] = 0 \\ A(0, \cdot) = A_0, \qquad \partial_0 A(0, \cdot) = A_1, \end{cases} \tag{2.2}$$

where the initial data must satisfy the compatibility condition

$$\mathrm{div} A_1 + [A_0{}^i, A_{1,i}] = 0. \tag{2.3}$$

The primary aim of our work is to provide the basis and pave the way to prove a novel global well-posedness result for the Cauchy problem (2.2) with small initial data lying in the weighted $H^{3/4+}(\mathbb{R}^3) \times H^{-1/4+}(\mathbb{R}^3)$ space, matching the minimal regularity assumption available for the local theory [106].

Conjecture 48. *Let $s > 3/4$, then the initial value problem for the Yang-Mills equation in the temporal gauge* (2.2) *is globally well-posed for data $(A_0, A_1) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ satisfying the compatibility condition* (2.3) *and the smallness condition*

$$\|\langle\cdot\rangle^{2s-1} A_0\|_{H^s(\mathbb{R}^3)} + \|\langle\cdot\rangle^{2s-1} A_1\|_{H^{s-1}(\mathbb{R}^3)} < \epsilon.$$

for some $\epsilon > 0$. Moreover, the global solution satisfies the point wise decay bounds

$$|A(t,x)| \leq C(1 + ||t| - |x||)^{-1}(1 + ||t| + |x||)^{-1}$$

The full proof of Conjecture 48 will not be provided here. However, we shall provide a fairly complete outline of its argument. As already mentioned above, the proof consists of two steps: first we rely on the Penrose compactification which we briefly introduce in the next section. Then, the problem has been transferred to the Einstein cylinder a contraction argument is employed. The proof of the nonlinear estimates used in the contraction argument will be addressed in a subsequent paper. In this work we shall prove a nonlinear estimate involving a pure N_{ij} null-form, which will be used as a starting point to prove the nonlinear estimates required in the fixed point argument which involves a slightly different type of null-form.

2.1 Penrose compactification of Minkowski spacetime

The Penrose map is a conformal map from the $(1+n)$ Minkowski spacetime to an open bounded set of the Einstein cylinder $\Sigma^{n+1} = \mathbb{R} \times S^n$. We shall give a detailed description below, for more details the reader should consult [32].

Parameterize $\mathbb{R}^n$ by spherical coordinates $r, \theta^1, \ldots, \theta^{n-1}$, and use $x^\mu = (t, r, \theta^1, \ldots, \theta^{n-1})$ as local coordinates on the Minkowski space-time $\mathbb{R}^{1+n}$. Moreover, denote $R, \Theta^1, \ldots, \Theta^{n-1}$ the pseudo-angular coordinates on the n-dimensional unit sphere S^n, that is $R \in (0,\pi)$, and (Θ^i) angular coordinates of S^{n-1}: observe that we can parameterize S^n except the two antipodal points via

$$(R, \Theta) \in (0,\pi) \times S^{n-1} \longrightarrow (\cos R, \Theta \sin R) \in S^n.$$

Subsets R equal constant are spheres S^{n-1}, and notice that when $R = 0$ and $R = \pi$ the subsets reduce to a point, called North pole and South pole respectively. We take $\tilde{x}^\mu = (T, R, \Theta^1, \ldots, \Theta^{n-1})$

as local coordinates on Σ^{n+1}, the Penrose map P is given by

$$\begin{aligned} P : (\mathbb{R}^{1+n}, \eta) &\to (\widetilde{\Sigma}^{1+n}, \widetilde{\gamma}), \\ (t, r, \theta^1, \dots, \theta^{n-1}) &\mapsto (T, R, \Theta^1, \dots, \Theta^{n-1}), \end{aligned} \tag{2.4}$$

where

$$T = \arctan(t+r) + \arctan(t-r), \quad R = \arctan(t+r) - \arctan(t-r), \quad \theta^i = \Theta^i.$$

The metrics on the Minkowski spacetime and on the Einstein cylinder are respectively defined by

$$\eta = -dt^2 + dr^2 + r^2(dS^{n-1})^2 \quad \text{and} \quad \widetilde{\gamma} = -dT^2 + dR^2 + \sin^2 R(dS^{n-1})^2.$$

Notice that the range of the Penrose map is the open bounded set $\widetilde{\Sigma}^{1+n} \subset \Sigma^{1+n}$ defined by

$$\widetilde{\Sigma}^{1+n} = \{(T, R, \Theta^1, \dots, \Theta^{n-1}) \in \Sigma^{1+n} : \ |T| + |R| < \pi, \ (\Theta^i) \in S^{n-1}\}$$

We now show that the Penrose map is a confomorphism, in the sense of the following definition.

Definition. A diffeomorphism $\psi : (M, g) \to (\widetilde{M}, \widetilde{g})$ between Lorentzian manifolds is called a *confomoprhism* if $\psi^* \widetilde{g} = \Omega^2 g$, where the *conformal factor* Ω is a positive scalar function. Denote by $(x^\mu)_{\mu=0,\dots,n}$ the local coordinates of $U \subset M$, and by $(\widetilde{x}^\mu)_{\mu=0,\dots,n}$ the local coordinates of $\widetilde{U} = \psi(U) \subset \widetilde{M}$. Denote the transformation ψ as $\widetilde{x}^\mu = \widetilde{x}^\mu(x)$ and its inverse by $x^\mu = x^\mu(\widetilde{x})$. The conformal relationship in local coordinates is expressed by the formulae

$$\Omega^2 g_{\mu\nu} = \frac{\partial \widetilde{x}^\alpha}{\partial x^\mu} \frac{\partial \widetilde{x}^\beta}{\partial x^\nu} \widetilde{g}_{\alpha\beta}, \quad \Omega^{-2} g^{\mu\nu} = \frac{\partial x^\mu}{\partial \widetilde{x}^\alpha} \frac{\partial x^\nu}{\partial \widetilde{x}^\beta} \widetilde{g}^{\alpha\beta}$$

or equivalently

$$\Omega^{-2} \widetilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \widetilde{x}^\mu} \frac{\partial x^\beta}{\partial \widetilde{x}^\nu} g_{\alpha\beta}, \quad \Omega^2 \widetilde{g}^{\mu\nu} = \frac{\partial \widetilde{x}^\mu}{\partial x^\alpha} \frac{\partial \widetilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}.$$

It is important to notice that for confomoprhism the classical transformation laws of covectors and contravariant vectors are violated. We have to face a choice: either we allow covectors to transform as usual and contravariant vectors to be rescaled by the conformal factor, or vice versa. We employ the former:

$$\widetilde{\partial}_\mu = \frac{\partial x^\alpha}{\partial \widetilde{x}^\mu} \partial_\alpha, \qquad \widetilde{\partial}^\mu = \Omega^{-2} \frac{\partial \widetilde{x}^\mu}{\partial x^\alpha} \partial^\alpha$$

$$\partial_\mu = \frac{\partial \widetilde{x}^\alpha}{\partial x^\mu} \widetilde{\partial}_\alpha, \qquad \partial^\mu = \Omega^2 \frac{\partial x^\mu}{\partial \widetilde{x}^\alpha} \widetilde{\partial}^\alpha$$

An important class of confomoprhisms is given by *Weyl rescalings* which are defined as a simultaneous point-wise rescaling of the metric: $\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. Notice that under Weyl rescalings covectors transform as identities $\partial_\alpha = \widetilde{\partial}_\alpha$, on the other hand contravariant vectors are rescaled $\partial^\alpha = \Omega^2 \widetilde{\partial}^\alpha$.

Proposition 49. *The Penrose map P is a confomorphism, form $\mathbb{R}^{1+n}$ and $\widetilde{\Sigma}^{1+n}$. Moreover, it holds that $P^*\widetilde{\gamma} = \Omega^2\eta$, with the conformal factor given by $\Omega = \cos T + \cos R$, hence the Penrose map is a Weyl rescalings.*

Observe that the South pole $i_0 = (0, \pi, \Theta^1, \dots, \Theta^{n-1})$ of the sphere $\{T = 0\} \times S^n$ is not included in $\widetilde{\Sigma}^{1+n}$ hence it has no pre-image on $\mathbb{R}^{1+n}$. On the other hand, the North pole $i_{T_0} = (T_0, 0, \Theta^1, \dots, \Theta^{n-1})$ of the sphere $\{T = T_0\} \times S^n$ has pre-image the point $I_{T_0} = (\tan(T_0/2), 0, \Theta^1, \dots, \Theta^{n-1})$ on the time axis.

To better express the inverse of the Penrose map we may use Cartesian coordinates, $x^\mu = (t, x_1, \dots, x_n)$ on $\mathbb{R}^{1+n}$ and $\widetilde{x}^\mu = (T, X_0, X_1, \dots, X_n)$ on Σ^{n+1}, instead of using spherical coordinates and pseudo-angular coordinates. We have embedded sphere S^n into $\mathbb{R}^{1+n}$ in the canonical way. In Cartesian coordinates the inverse of the Penrose map is given by

$$\begin{aligned} P^{-1} : (\widetilde{\Sigma}^{1+n}, \widetilde{\gamma}) &\to (\mathbb{R}^{1+n}, \eta) \\ (T, X_0, X_1, \dots, X_n) &\mapsto (t, x_1, \dots, x_n) \end{aligned}$$

where

$$t = \frac{\sin T}{\cos T + X_0}, \quad x_i = \frac{X_i}{\cos T + X_0}$$

Observe that with respect to Cartesian coordinates we have $\widetilde{\Sigma}^{1+n} = \{(T, X_0, X_1, \dots, X_n) : |T| < \pi, \ \cos T + X_0 > 0\}$, the conformal factor is

$$\Omega^2 = \frac{4}{(1 + (t + |x|)^2)(1 + (t - |x|)^2)}$$

and the metrics are $\eta = -dt^2 + d\vec{x}^2$, and $\gamma = -dT^2 + d\vec{X}^2$.

In the study of partial differential equations, it is useful to consider the composition of the Penrose map with the classical stereographic projections, in oder to have an Euclidean space $\mathbb{R}^{1+n}$ on both sides. Recall that the n-sphere can be seen as an hypersurface embedded in $\mathbb{R}^{1+n}$:

$$S^n = \{(X_0, X_1, \dots, X_n) \in \mathbb{R}^{1+n} : \ X_0^2 + X_1^2 + \dots + X_n^2 = 1\}.$$

The stereographic projections $St_\pm$ from the South pole $(-1, \vec{0})$ and from the North pole $(1, \vec{0})$ are defined respectively as

$$\begin{aligned} St_\pm : S^n \setminus (\pm 1, \vec{0}) &\to \mathbb{R}^n \\ (X_0, X_1, \dots, X_n) &\mapsto (Y_1, \dots, Y_n) \end{aligned}$$

where $Y_i = \frac{X_i}{1\mp X_0}$. The inverse transformations, for the North projection and the South projection, are given respectively by:

$$\begin{cases} X_0 &= \mp\left(\frac{1-|Y|^2}{1+|Y|^2}\right) \\ X_i &= \frac{2Y_i}{1+|Y|^2} \end{cases}$$

Proposition 50. *The stereographic projections $St_\pm$ are Weyl rescaling from $S^n \setminus (\pm 1, \vec{0})$ to $\mathbb{R}^n$. In fact, it holds that $St_\pm^* \gamma_n = \Omega^2 \gamma_{n+1}$, where $\Omega = \frac{1}{1\mp X_0}$. The metric γ_m is the euclidean metric on $\mathbb{R}^m$:*

$$\gamma_{n+1} = d\vec{X}^2 = dX_0^2 + dX_1^2 + \cdots + dX_n^2 \quad \textit{and} \quad \gamma_n = d\vec{Y}^2 = dY_1^2 + \cdots + dY_n^2$$

There is an equivalent way to define stereographic projections. Let us consider S^n minus the North and South poles parametrized by the pseudo-angular coordinates: $R, \Theta^1, \ldots, \Theta^{n-1}$, where $R \in (0,\pi)$, and (Θ^i) angular coordinates of S^{n-1}. Moreover, parametrize $\mathbb{R}^n$ by means of classical spherical coordinates $\alpha, \omega^1, \ldots, \omega^{n-1}$, where $\alpha \in \mathbb{R}^+$ and (Θ^i) angular coordinates of S^{n-1}. Then an equivalent definition of the stereographic projection from the North pole is

$$\begin{aligned} St_+ : S^n \setminus (0, \vec{\Theta}) &\to \mathbb{R}^n, \\ (R, \Theta^1, \ldots, \Theta^{n-1}) &\mapsto (\alpha, \omega^1, \ldots, \omega^{n-1}), \end{aligned} \tag{2.5}$$

where $\alpha = \cot\left(\frac{R}{2}\right)$ and $\omega^i = \Theta^i$. Analogously we define the stereographic projection from the South pole

$$\begin{aligned} St_- : S^n \setminus (\pi, \vec{\Theta}) &\to \mathbb{R}^n \\ (R, \Theta^1, \ldots, \Theta^{n-1}) &\mapsto (\alpha, \omega^1, \ldots, \omega^{n-1}) \end{aligned} \tag{2.6}$$

where $\alpha = \tan\left(\frac{R}{2}\right)$ and $\omega^i = \Theta^i$. To check that these definitions are equivalent to the classical definitions of the Stereographic projection given above look at Figure 2.1 below and use the law of sines.

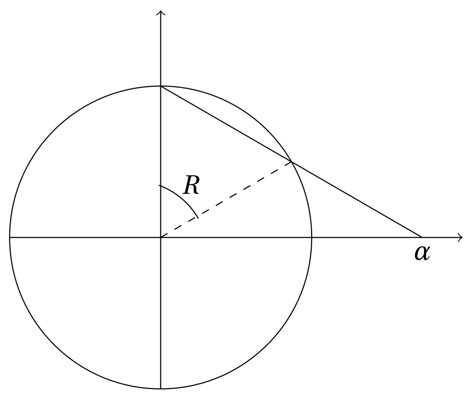


Figure 2.1: Stereographic Projection St_+

Next, let us analyse the composition of the Penrose transformation (2.4) with the stereographic projection from the North or South poles defined in (2.5) and (2.6).

Definition. We define the two maps $\psi_\pm = (\mathbb{R}^{1+n},\eta) \to ((-\pi,\pi)\times\mathbb{R}^n,\widetilde{g})$ that maps the Minkowski space-time into a Lorentzian manifold with bounded temporal coordinate:

$$\begin{array}{rcl}\psi_\pm = f_\pm \circ P : (\mathbb{R}^{1+n},\eta) & \to & ((-\pi,\pi)\times\mathbb{R}^n,\widetilde{g}) \\ (t,r,\theta^1,\dots,\theta^{n-1}) & \mapsto & (T,\alpha,\omega^1,\dots,\omega^{n-1})\end{array}$$

where $f_\pm = (Id, St_\pm) : \mathbb{R}\times S^n \to \mathbb{R}^{1+n}$ denote the stereographic projection on the sphere and the identity in the time variable.

Clearly $\psi_\pm$ are conformal maps being the composition of two conformal maps. Moreover, the inverse function are given by the formulae

$$t = \frac{(\alpha^2+1)\sin T}{(\alpha^2+1)\cos T \pm (\alpha^2-1)}, \qquad r = \frac{2\alpha}{(\alpha^2+1)\cos T \pm (\alpha^2-1)}, \qquad \theta^i = \omega^i.$$

If instead of using spherical coordinates we use Cartesian coordinates $x^\mu = (t,x_1,\dots,x_n)$ on the domain $\mathbb{R}^{1+n}$ and $\widetilde{y}^\mu = (T,Y_1,\dots,Y_n)$ on the codomain $\mathbb{R}^{1+n}$. From the composition of pullbacks of the metrics we find $P^*(f_\pm^*\widetilde{g}) = \Omega^2\eta$ where

$$\widetilde{g}_{\alpha\beta} = \mathrm{diag}\Big(-1,\Big(\frac{2}{1+|Y|^2}\Big)^2,\dots,\Big(\frac{2}{1+|Y|^2}\Big)^2\Big)$$

and $\Omega = \cos T \mp \frac{1-|Y|^2}{1+|Y|^2}$. Thus $\psi_\pm$ are Weyl conformal rescalings. Observe that we have compactified the time variable, the price we have to pay is that the flat Minkowski metric is transformed into a curved diagonal metric $\widetilde{g}_{\alpha\beta}$.

In order to compute the pushforward of the vector fields in $\mathbb{R}^{1+n}$ by the inverse of $\psi_\pm$ we need to calculate the partial derivatives of the map

$$\frac{\partial\widetilde{y}^\mu}{\partial x^\nu}(\widetilde{y}) = \begin{cases} 1 \mp \cos T\frac{1-|Y|^2}{1+|Y|^2} & \text{if } \mu=0,\nu=0, \\ -Y^i\sin T & \text{if } \mu=i,\nu=0, \\ -\frac{2Y_i}{1+|Y|^2}\sin T & \text{if } \mu=0,\nu=i, \\ (1-\cos T)Y_iY^j + \frac{(|Y|^2+1)\cos T\mp(|Y|^2-1)}{2}\delta_i^j & \text{if } \mu=j,\nu=i. \end{cases}$$

From this formula it is not hard to check that the conformal relationship for the metrics in local coordinates $\Omega^2\eta_{\mu\nu} = \frac{\partial\widetilde{y}^\alpha}{\partial x^\mu}\frac{\partial\widetilde{y}^\beta}{\partial x^\nu}\widetilde{g}_{\alpha\beta}$ holds. Moreover we shall also need the formula

$$\partial_\nu\Omega = \frac{\partial\widetilde{y}^\mu}{\partial x^\nu}\widetilde{\partial}_\mu\Omega = \begin{cases} -\Omega\sin T\frac{1-|Y|^2}{1+|Y|^2} & \text{if } \nu=0, \\ -\Omega\cos T\frac{2Y_i}{1+|Y|^2} & \text{if } \nu=i. \end{cases}$$

Remark. The definition 2.1 is loosely stated because when we compose the Penrose transformation with the stereographic projection from the North pole the time axis $\{x = 0\}$ of Minkowski spacetime is mapped by the Penrose transformation into the set $(-\pi,\pi) \times P_N$ which has no image through the stereographic projection from the North pole. Therefore we have to restrict the domain of ψ_+ to the set $\mathbb{R}^{1+n} \setminus \{x = 0\}$. Observe that for ψ_- we do not have to restrict the domain since $(-\pi,\pi) \times P_S \notin \widetilde{\Sigma}^{1+n}$.

2.2 A conformal method for hyperbolic equations

In this section we analyse how the Penrose compactification map is used in the context of Cauchy problems for hyperbolic partial differential equations. Consider a general nonlinear initial value problem on Minkowski spacetime $(\mathbb{R}^{1+n},\eta)$:

$$\begin{cases} \Box u = N(u), \\ (u, u_t)|_{t=0} = (u_0, u_1). \end{cases} \tag{2.7}$$

The Penrose transform $P : (\mathbb{R}^{1+n},\eta) \to (\widetilde{\Sigma}^{1+n},\widetilde{\gamma})$ is well behaved with respect to initial value problem since it maps the submanifold $\{t = 0\} \times \mathbb{R}^n$ of $\mathbb{R}^{1+n}$ into $\{T = 0\} \times S^n$ on the Einstein side. Therefore we can translate an initial value problem on Minkowski spacetime into an initial value problem on $\widetilde{\Sigma}^{1+n}$. Recall that, if $\psi : (M, g) \to (\widetilde{M},\widetilde{g})$ is any general confomorphism and $1 + n$ be the dimension of the manifolds, then the general conformal law between wave operators with respect to two conformal metrics reads

$$\Omega^{-(n+3)/2}\Big(\Box_g - R(g)\frac{n-1}{4n}\Big)u = \Big(\widetilde{\Box}_{\widetilde{g}} - R(\widetilde{g})\frac{n-1}{4n}\Big)\widetilde{u}$$

where $\widetilde{u} = (\Omega^{-(n-1)/2}u) \circ \psi^{-1}$, in local coordinates we have $u(x) = \Omega^{(n-1)/2}(\widetilde{x})\widetilde{u}(\widetilde{x})$. Observe that, in our case, the scalar curvatures of $(\mathbb{R}^{1+n},\eta)$ and $(\widetilde{\Sigma}^{1+n},\widetilde{\gamma})$ are respectively $R(\eta) = 0$ and $R(\widetilde{g}) = -n(n-1)$, hence we can translate the Cauchy problem (2.7) into the following one

$$\begin{cases} \Big(\widetilde{\Box}_{\widetilde{\gamma}} + \big(\frac{n-1}{2}\big)^2\Big)\widetilde{u} = \widetilde{N}(\widetilde{u}) \\ (u, u_t)|_{T=0} = (\widetilde{u}_0, \widetilde{u}_1) \end{cases}$$

where $\widetilde{u} = (\Omega^{-(n-1)/2}u)\circ P^{-1}$ and the nonlinearity is rescaled by a power of the conformal factor: $\widetilde{N}(\widetilde{u}) = \Omega^{-(n+3)/2}N(\Omega^{(n-1)/2}\widetilde{u})$. Furthermore, if we consider the case of $1+3$ dimensional Minkowski space-time, we recover the initial valued problem on $(\widetilde{\Sigma}^{1+3},\widetilde{\gamma})$:

$$\begin{cases} (\widetilde{\Box}_{\widetilde{\gamma}} + 1)\widetilde{u} = \widetilde{N}(\widetilde{u}) \\ (u, u_t)|_{T=0} = (\widetilde{u}_0, \widetilde{u}_1) \end{cases}$$

where $u(x) = \Omega(\widetilde{x})\widetilde{u}(\widetilde{x})$ and $\widetilde{N}(\widetilde{u}) = \Omega^{-3}N(\Omega\widetilde{u})$.

The next step is to reduce the initial value problem on $\widetilde{\Sigma}^{1+n}$ into two initial value problems on $(-\pi,\pi) \times \mathbb{R}^n$. For that purpose define an open cover of the sphere S^n constitute by two sets U_+

and U_-, with U_+ containing the South pole and U_- containing the North pole. We denote $\chi_\pm$ a smooth partition of unity subordinate to the cover $\{U_+, U_-\}$, and we localize the function $\widetilde{u}$ to one of the stereographic coordinate charts, by setting $\widetilde{u}_\pm = \chi_\pm \widetilde{u}$. Therefore by localizing and projecting through the stereographic projection we can translate the initial value problem on $(\widetilde{\Sigma}^{1+n}, \widetilde{\gamma})$ into two initial value problems

$$\begin{cases} \left(\widetilde{\Box}_{\widetilde{g}} + \left(\frac{n-1}{2}\right)^2\right) \widetilde{u}^I_\pm = \widetilde{N}(\widetilde{u}_\pm) \\ (u_\pm, \partial_t u_\pm)|_{T=0} = (\widetilde{u}_{0\pm}, \widetilde{u}_{1\pm}) \end{cases} \tag{2.8}$$

on the Euclidean space $\big((-\pi,\pi) \times \mathbb{R}^n, \widetilde{g}\big)$, where $u_{0\pm} = \chi_\pm u_0$ and $u_{1\pm} = \chi_\pm u_1$. We will refer to (2.8) as the initial value problem in stereographic coordinates.

The biggest drawback of this procedure is that we end up with a non-flat metric $\widetilde{g}$, that the nonlinearity is multiplied by a power of the conformal factor depending on the dimension, and that a mass term is added to the equation. Despite this disadvantages this method has the enormous advantage to confined the time variable to the finite interval $(-\pi, \pi)$.

2.3 The Yang-Mills equation in stereographic coordinates

In the following section, we apply the procedure outline in the previous section to the Yang-Mills equation. We first translate the original Cauchy problem on Minkowski space-time to a Cauchy problem on the Lorentzian manifold $((-\pi,\pi) \times \mathbb{R}^n, \widetilde{g})$, then we impose the connection to lie in the temporal gauge, and by means of splitting into divergence free and curl free components we highlight the null structure present in the quadratic terms.

First, let us show that the Yang-Mills equation on manifold of dimensions $1+3$ is invariant under Weyl conformal transformations, here the fact that we are working in $1+3$ dimensions is crucial.

Proposition 51. *Let $\psi : (M, g) \to (\widetilde{M}, \widetilde{g})$ be a general conformism between two Lorentzian manifolds of dimensions $1+3$. If the curvature $F^{\alpha\beta} : (M, g) \to \mathfrak{g}$ satisfies the Yang-Mills equation on (M, g), then $\widetilde{F}_{\alpha\beta} = F_{\alpha\beta} \circ \psi^{-1} : \widetilde{M} \to \mathfrak{g}$ satisfies the analogous Yang-Mills equation on $(\widetilde{M}, \widetilde{g})$.*

Let (M, g) be a Lorentzian manifold with non-flat metric and let $A_\alpha, F_{\alpha\beta} : M \to \mathfrak{g}$ be the connection and the curvature tensors on (M, g) with value in the Lie algebra. We define the curvature in term of the connection as

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta] = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$

by symmetry of Christoffel symbols. The Yang-Mills equation on (M, g) are

$$D_\alpha F^{\alpha\beta} = \nabla_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0$$

Notice that we are raising and lowering the indices with respect to the non-flat metric g, hence we obtain the identity $F^{\alpha\beta} = \nabla^\alpha A^\beta - \nabla^\beta A^\alpha + [A^\alpha, A^\beta] = \partial^\alpha A^\beta - \partial^\beta A^\alpha + (\Gamma^{\alpha\beta}_\gamma - \Gamma^{\beta\alpha}_\gamma)A^\gamma + [A^\alpha, A^\beta]$, where $\Gamma^{\alpha\beta}_\gamma = g^{\alpha\lambda}\Gamma^\beta_{\lambda\gamma}$ is obtained by raising the indices of the Christoffel symbols. When expanded the Yang-Mills equation in term of the connection are

$$\nabla_\alpha \nabla^\alpha A^\beta - \nabla_\alpha \nabla^\beta A^\alpha + 2[A_\alpha, \nabla^\alpha A^\beta] - [A_\alpha, \nabla^\beta A^\alpha] + [\nabla_\alpha A^\alpha, A^\beta] + [A_\alpha, [A^\alpha, A^\beta]] = 0$$

which is exactly equation (2.1) where derivative have been replaced by covariant derivative. We can expand the covariant derivative further to obtain the system of nonlinear wave equations

$$\begin{aligned}
&\Box_g A^\beta - \partial^\beta \partial_\alpha A^\alpha + 2[A_\alpha, \partial^\alpha A^\beta] - [A_\alpha, \partial^\beta A^\alpha] + [\partial_\alpha A^\alpha, A^\beta] + [A_\alpha, [A^\alpha, A^\beta]] \\
&+ A^\sigma(\Gamma^\alpha_{\alpha\gamma}\Gamma^{\gamma\beta}_\sigma - \Gamma^\alpha_{\alpha\gamma}\Gamma^{\beta\gamma}_\sigma + \partial_\alpha \Gamma^{\alpha\beta}_\sigma - \partial_\alpha \Gamma^{\beta\alpha}_\sigma) + \Gamma^\alpha_{\alpha\gamma}(\partial^\gamma A^\beta - \partial^\beta A^\gamma) + (\Gamma^{\alpha\beta}_\gamma - \Gamma^{\beta\alpha}_\gamma)\partial_\alpha A^\gamma \\
&+ (\Gamma^{\alpha\beta}_\sigma - \Gamma^{\beta\alpha}_\sigma)[A_\alpha, A^\sigma] + \Gamma^\alpha_{\alpha\gamma}[A^\gamma, A^\beta] = 0
\end{aligned}$$

Notice that if the metric g is the Minkowski metric then the latter equation reduces to equation (2.1) because Christoffel symbols vanish. Moreover, if we come back to the commutative setting, hence if the Lie bracket vanishes, we recover Maxwell's equations on a curved manifold. Finally observe that the first line of the equation is exactly the Yang-Mills equation on Minkowski spacetime where the d'Alembertian $\Box$ has been replaced by the curved wave operator $\Box_g$. Loosely speaking, when we study Yang-Mills equation on curved background we have to add extra lower-order terms which arise from the Christoffel symbols in the covariant derivative. We can restate Yang-Mills equation on curved background as

$$\Box_g A - \partial(\partial_\alpha A^\alpha) + \mathfrak{M}(A, \partial A) + \mathfrak{M}(A, A, A) + \mathscr{E}(A) + \mathscr{E}(\partial A) + \mathscr{E}(A, A) = 0 \tag{2.9}$$

We now turn to the proof of Proposition 51.

Proof. Consider the manifolds M and $\widetilde{M}$ to be of dimensions $1+n$. Denote by $(x^\mu)_{\mu=0,\dots,n}$ the local coordinates of $U \subset M$, and by $(\tilde{x}^\mu)_{\mu=0,\dots,n}$ the local coordinates of $\widetilde{U} = \psi(U) \subset \widetilde{M}$. Denote the transformation ψ as $\tilde{x}^\mu = \tilde{x}^\mu(x)$ and its inverse by $x^\mu = x^\mu(\tilde{x})$. Recall that $\psi : (M, g) \to (\widetilde{M}, \tilde{g})$ is a confomorphism hence in local coordinates we have $\Omega^2 g_{ab} = \Pi^\alpha_a \Pi^\beta_b \tilde{g}_{\alpha\beta}$ where we use the shorthand notation $\Pi^\alpha_a = \frac{\partial \tilde{x}^\alpha}{\partial x^a}$. We define $F_{M,\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$ the Maxwell component of the Yang-Mills curvature tensor, then clearly $F_{\alpha\beta} = F_{M,\alpha\beta} + [A_\alpha, A_\beta]$, and the Yang-Mills equation could be stated as

$$D_\alpha F^{\alpha\beta} = \nabla_\alpha F^{\alpha\beta}_M + \nabla_\alpha [A^\alpha, A^\beta] + [A_\alpha, F^{\alpha\beta}]$$

Since $F_{M,ab}$ is a rank-2 covariant tensor it should transform accordingly, thus define $\widetilde{F}_{M,\alpha\beta} = Ш^a_\alpha Ш^b_\beta F_{M,ab}$, where $Ш^a_\alpha$ is defined as the inverse of Π^α_a: $Ш^a_\alpha = \frac{\partial x^a}{\partial \tilde{x}^\alpha}$. We will also use $\Pi^\alpha_{ab} = \partial_a \Pi^\alpha_b$

and $\text{Ш}^b_{\alpha\beta} = \widetilde{\partial}_\alpha \text{Ш}^b_\beta$. Then when raising indices we obtain $\widetilde{F}^{\alpha\beta}_M = \Omega^{-4}\Pi^\alpha_a \Pi^\beta_b F^{ab}_M$. We compute

$$\begin{aligned}
\widetilde{\partial}_\alpha \widetilde{F}^{\alpha\beta}_M &= \text{Ш}^c_\alpha \partial_c (\Omega^{-4}\Pi^\alpha_a \Pi^\beta_b F^{ab}_M) \\
&= -4\Omega^{-5}\partial_a \Omega \Pi^\beta_b F^{ab}_M + \Omega^{-4}\Pi^\beta_b \partial_a F^{ab}_M + \Omega^{-4}\,\text{Ш}^\beta_{ab}\, F^{ab}_M + \Omega^{-4}\Pi^\beta_b\, \text{Ш}^c_\alpha\, \Pi^\alpha_{ac} F^{ab}_M \\
&= -4\Omega^{-5}\partial_a \Omega \Pi^\beta_b F^{ab}_M + \Omega^{-4}\Pi^\beta_b \partial_a F^{ab}_M + \Omega^{-4}\Pi^\beta_b\, \text{Ш}^c_\alpha\, \Pi^\alpha_{ac} F^{ab}_M
\end{aligned}$$

For general conformal transformations the Christoffel symbols are computed to be linked by the relations

$$\Gamma^c_{ab} = \text{Ш}^\alpha_a \Pi^\beta_b \text{Ш}^c_\gamma \widetilde{\Gamma}^\gamma_{\alpha\beta} + \text{Ш}^c_\nu \Pi^\nu_{ab} - \Omega^{-1}\widetilde{\partial}_\alpha \Omega (\Pi^\alpha_a \delta^c_b + \Pi^\alpha_b \delta^c_a - \text{Ш}^c_\gamma \Pi^\mu_a \Pi^\nu_b \widetilde{g}^{\gamma\alpha} \widetilde{g}_{\mu\nu})$$

In particular we will need that

$$\Pi^\gamma_a \widetilde{\Gamma}^\alpha_{\alpha\gamma} = \Gamma^c_{ca} - \text{Ш}^c_\alpha \Pi^\alpha_{ac} + (n+1)\Omega^{-1}\partial_a \Omega \tag{2.10}$$

From the transformation of Christoffel symbols formula (2.10) we obtain

$$\begin{aligned}
\widetilde{\Gamma}^\alpha_{\alpha\gamma} \widetilde{F}^{\gamma\beta}_M &= \Omega^{-4}\widetilde{\Gamma}^\alpha_{\alpha\gamma} \Pi^\gamma_a \Pi^\beta_b F^{ab}_M \\
&= \Omega^{-4}\Pi^\beta_b \Gamma^c_{ca} F^{ab}_M - \Omega^{-4}\Pi^\beta_b\, \text{Ш}^c_\alpha\, \Pi^\alpha_{ac} F^{ab}_M + (n+1)\Omega^{-5}\partial_a \Omega \Pi^\beta_b F^{ab}_M
\end{aligned}$$

Therefore if $n=3$ we obtain $\widetilde{\nabla}_\alpha \widetilde{F}^{\alpha\beta}_M = \Omega^{-4}\Pi^\beta_b \nabla_a F^{ab}_M$. In fact we have

$$\begin{aligned}
\widetilde{\nabla}_\alpha \widetilde{F}^{\alpha\beta}_M &= \widetilde{\partial}_\alpha \widetilde{F}^{\alpha\beta}_M + \widetilde{\Gamma}^\alpha_{\alpha\gamma} \widetilde{F}^{\gamma\beta}_M \\
&= \Omega^{-4}\Pi^\beta_b (\partial_a F^{ab}_M + \Gamma^c_{ca} F^{ab}) + (n+1-4)\Omega^{-5}\partial_a \Omega \Pi^\beta_b F^{ab}_M \\
&= \Omega^{-4}\Pi^\beta_b \nabla_a F^{ab}_M + (n+1-4)\Omega^{-5}\partial_a \Omega \Pi^\beta_b F^{ab}_M
\end{aligned}$$

Notice that the second term vanishes if $n+1=4$, thus we infer that Maxwell's equations are conformally invariant iff $n=3$.

Next we examine the non-commutative part of the Yang-Mills equation. Notice that the tensor $[A^\alpha, A^\beta]$ has the same antisymmetry property and transform like $F^{\alpha\beta}$, that is $\widetilde{F}^{\alpha\beta} = \Omega^{-4}\Pi^\alpha_a \Pi^\beta_b F^{ab}$. Thus we obtain the identity $\widetilde{\nabla}_\alpha [\widetilde{A}^\alpha, \widetilde{A}^\beta] = \Omega^{-4}\Pi^\beta_b \nabla_a [A^a, A^b]$. Furthermore

$$[\widetilde{A}_\alpha, \widetilde{F}^{\alpha\beta}] = [\Pi^a_\alpha A_a, \Omega^{-4}\Pi^\alpha_c \Pi^\beta_d F^{cd}] = \Omega^{-4}\Pi^\beta_b [A_a, F^{ab}]$$

Therefore we can conclude

$$\widetilde{D}_\alpha \widetilde{F}^{\alpha\beta} = \widetilde{\nabla}_\alpha \widetilde{F}^{\alpha\beta}_M + \widetilde{\nabla}_\alpha [\widetilde{A}^\alpha, \widetilde{A}^\beta] + [\widetilde{A}_\alpha, \widetilde{F}^{\alpha\beta}] = \Omega^{-4}\Pi^\beta_b D_a F^{ab}$$

Thus, exactly as Maxwell's equations, the Yang-Mills equation are conformally invariant in any $1+3$ dimensional space-time. □

We now apply the procedure outlined in §2.2 to the Yang-Mills equation. Consider the Penrose compactification composed with the stereographic protection: $\psi_\pm : (\mathbb{R}^{1+3}, \eta) \to ((\pi,\pi)\times\mathbb{R}^3, \widetilde{g})$, where

$$\widetilde{g}_{\alpha\beta} = \mathrm{diag}\Big(-1, \Big(\frac{2}{1+|Y|^2}\Big)^2, \dots, \Big(\frac{2}{1+|Y|^2}\Big)^2\Big) \tag{2.11}$$

Recall that if we use Cartesian coordinates $x^\mu = (t, x_1, \dots, x_n)$ on the domain $\mathbb{R}^{1+n}$ and $\widetilde{y}^\mu = (T, Y_1, \dots, Y_n)$ on the codomain $\mathbb{R}^{1+n}$, we have that $\psi_\pm$ are Weyl conformal rescalings since $\psi^*_\pm \widetilde{g} = \Omega^2 \eta$ where $\Omega = \cos T \mp \frac{1-|Y|^2}{1+|Y|^2}$. From Proposition 51 we know that Yangs-Mills equations are conformally invariant thus $D_\alpha F^{\alpha\beta} = 0$ iff $\widetilde{D}_\alpha \widetilde{F}^{\alpha\beta} = 0$, where $F_{\alpha\beta} : (\mathbb{R}^{1+3}, \eta) \to \mathfrak{g}$ and $\widetilde{F}_{\alpha\beta} = F_{\alpha\beta} \circ \psi^{-1} : ((\pi,\pi)\times\mathbb{R}^3, \widetilde{g}) \to \mathfrak{g}$. Therefore Yangs-Mills equations in term of the connection $\widetilde{A}_\alpha = A_\alpha \circ \psi^{-1} : ((\pi,\pi)\times\mathbb{R}^3, \widetilde{g}) \to \mathfrak{g}$ in stereographic coordinates are

$$\begin{aligned}
&\widetilde{\Box}_{\widetilde{g}} \widetilde{A}^\beta - \widetilde{\partial}_\alpha \widetilde{\partial}^\beta \widetilde{A}^\alpha + 2[\widetilde{A}_\alpha, \widetilde{\partial}^\alpha \widetilde{A}^\beta] - [\widetilde{A}_\alpha, \widetilde{\partial}^\beta \widetilde{A}^\alpha] + [\widetilde{\partial}_\alpha \widetilde{A}^\alpha, \widetilde{A}^\beta] + [\widetilde{A}_\alpha, [\widetilde{A}^\alpha, \widetilde{A}^\beta]] \\
&+ \widetilde{A}^\sigma (\widetilde{\Gamma}^\alpha_{\alpha\gamma} \widetilde{\Gamma}^{\gamma\beta}_\sigma - \widetilde{\Gamma}^\alpha_{\alpha\gamma} \widetilde{\Gamma}^{\beta\gamma}_\sigma + \widetilde{\partial}_\alpha \widetilde{\Gamma}^{\alpha\beta}_\sigma - \widetilde{\partial}_\alpha \widetilde{\Gamma}^{\beta\alpha}_\sigma) \\
&+ \widetilde{\Gamma}^\alpha_{\alpha\gamma} (\widetilde{\partial}^\gamma \widetilde{A}^\beta - \widetilde{\partial}^\beta \widetilde{A}^\gamma) + (\widetilde{\Gamma}^{\alpha\beta}_\gamma - \widetilde{\Gamma}^{\beta\alpha}_\gamma) \widetilde{\partial}_\alpha \widetilde{A}^\gamma \\
&+ (\widetilde{\Gamma}^{\alpha\beta}_\sigma - \widetilde{\Gamma}^{\beta\alpha}_\sigma)[\widetilde{A}_\alpha, \widetilde{A}^\sigma] + \widetilde{\Gamma}^\alpha_{\alpha\gamma} [\widetilde{A}^\gamma, \widetilde{A}^\beta] = 0
\end{aligned}$$

From the definition (2.11) of the stereographic metric $\widetilde{g}$, we can compute explicitly every term of the previous equation. Let us compute the Christoffel symbols of the stereographic metric

$$\widetilde{\Gamma}^\mu_{\alpha\beta} = \begin{cases} -\frac{2Y_j}{1+|Y|^2} & \text{if } \mu = \alpha = i \text{ and } \beta = j, \\ \frac{2Y^j}{1+|Y|^2} & \text{if } \beta = \alpha = i \text{ and } \mu = j \neq i, \\ 0 & \text{if } \mu = 0, \text{ or } \alpha = 0 \text{ or } \beta = 0, \end{cases}$$

Therefore when $\beta = 0$ the temporal component of the connection satisfies the equation

$$\begin{aligned}
&\widetilde{\Box}_{\widetilde{g}} \widetilde{A}_0 - \widetilde{\partial}_0 (\widetilde{\partial}_\alpha \widetilde{A}^\alpha) + 2[\widetilde{A}^i, \widetilde{\partial}_i \widetilde{A}_0] - [\widetilde{A}^i, \widetilde{\partial}_0 \widetilde{A}_i] + [\widetilde{\partial}_i \widetilde{A}^i, \widetilde{A}_0] + [\widetilde{A}^i, [\widetilde{A}_i, \widetilde{A}_0]] \\
&- \widetilde{a}^i (\widetilde{\partial}_i \widetilde{A}_0 - \widetilde{\partial}_0 \widetilde{A}_i + [\widetilde{A}_i, \widetilde{A}_0]) = 0
\end{aligned} \tag{2.12}$$

where the coefficients are $\widetilde{a}^k = \frac{2Y^k}{1+|Y|^2}$,and when $\beta = j$ spatial component of the connection satisfies the equation

$$\begin{aligned}
&\widetilde{\Box}_{\widetilde{g}} \widetilde{A}_j - \widetilde{\partial}_j (\widetilde{\partial}_\alpha \widetilde{A}^\alpha) + 2[\widetilde{A}_i, \widetilde{\partial}^i \widetilde{A}_j] - [\widetilde{A}_i, \widetilde{\partial}_j \widetilde{A}^i] + [\widetilde{\partial}_i \widetilde{A}^i, \widetilde{A}_j] + [\widetilde{A}_i, [\widetilde{A}^i, \widetilde{A}_j]] \\
&+ \widetilde{a}^k \widetilde{\partial}_j \widetilde{A}_k - \widetilde{b}^k \widetilde{\partial}_k \widetilde{A}_j + \widetilde{c}_j \widetilde{\partial}_k \widetilde{A}^k - \widetilde{d}_k [\widetilde{A}^k, \widetilde{A}_j] - e \widetilde{A}_j = 0
\end{aligned} \tag{2.13}$$

where the coefficients are $\widetilde{b}^k = \frac{|Y|^4+2|Y|^2+3}{1+|Y|^2} Y^k$, $\widetilde{c}_j = (1+|Y|^2) Y_j$, $\widetilde{d}_k = \frac{|Y|^4+2|Y|^2-1}{1+|Y|^2} Y_k$, and $e = 2(1+|Y|^2)$.

We then reduce Yang-Mills equation on curved background in schematic form by imposing the temporal gauge, thus $A_0 = 0$. Let us divide the connection in its temporal and spatial components $A_\alpha = (A_0, A)$ where $A = (A_1, A_2, A_3)$, then the Yang-Mills equation (2.12) simplifies to

$$\widetilde{\partial}_0(\widetilde{\mathrm{div}}\widetilde{A}) + [\widetilde{A}^i, \widetilde{\partial}_0\widetilde{A}_i] + \widetilde{a}^i\widetilde{\partial}_0\widetilde{A}_i = 0$$

Then we shall write the Yang-Mills equation in stereographic coordinates and in the temporal gauge as

$$\begin{cases} \widetilde{\partial}_0(\widetilde{\mathrm{div}}\widetilde{A}) + [\widetilde{A}^i, \widetilde{\partial}_0\widetilde{A}_i] + \mathscr{F}(\widetilde{\partial}\widetilde{A}) = 0 \\ \widetilde{\Box}_{\widetilde{g}}\widetilde{A}_j - \widetilde{\partial}_j(\widetilde{\mathrm{div}}\widetilde{A}) + \mathfrak{M}_j(\widetilde{A}, \widetilde{\partial}\widetilde{A}) + \mathfrak{C}_j(\widetilde{A}, \widetilde{A}, \widetilde{A}) + \mathscr{E}_j(\widetilde{\partial}\widetilde{A}) + \mathscr{E}_j(\widetilde{A}, \widetilde{A}) + \mathscr{E}_j(\widetilde{A}) = 0 \end{cases} \tag{2.14}$$

where

$$\begin{aligned} \mathfrak{M}_j(\widetilde{A}, \widetilde{\partial}\widetilde{A}) &= 2[\widetilde{A}_i, \widetilde{\partial}^i\widetilde{A}_j] - [\widetilde{A}_i, \widetilde{\partial}_j\widetilde{A}^i] + [\widetilde{\mathrm{div}}\widetilde{A}, \widetilde{A}_j], \\ \mathfrak{C}_j(\widetilde{A}, \widetilde{A}, \widetilde{A}) &= [\widetilde{A}_i, [\widetilde{A}^i, \widetilde{A}_j]], \\ \mathscr{E}_j(\widetilde{\partial}\widetilde{A}) &= \widetilde{a}^k\widetilde{\partial}_j\widetilde{A}_k - \widetilde{b}^k\widetilde{\partial}_k\widetilde{A}_j + \widetilde{c}_j\widetilde{\mathrm{div}}\widetilde{A}, \\ \mathscr{E}_j(\widetilde{A}, \widetilde{A}) &= -\widetilde{d}_k[\widetilde{A}^k, \widetilde{A}_j], \\ \mathscr{E}_j(\widetilde{A}) &= -e\widetilde{A}_j, \\ \mathscr{F}(\widetilde{\partial}\widetilde{A}) &= \widetilde{a}^i\widetilde{\partial}_0\widetilde{A}_i. \end{aligned}$$

In the next section we shall show how to simplify the second order term $\widetilde{\partial}_0(\widetilde{\mathrm{div}}\widetilde{A})$ and how to extract from the quadratic term $\mathfrak{M}_j(\widetilde{A}, \widetilde{\partial}\widetilde{A})$ a corresponding null structure.

2.4 Preliminary reductions and reformulation of the problem

We shall abuse notation a bit and not write the tilde sign to denote quantities which depends on stereographic coordinates. In order to obtain, from the second equation in (2.14), a hyperbolic equation and to highlight its subjacent null structure, we apply the Helmholtz decomposition and we separate A into its divergence-free and curl-free part: $A = A^{df} + A^{cf}$. This procedure was already employed by Tao in [106] in the case of a flat metric. Let us define the Leray divergence-free and curl-free projections:

$$A^{df} = P(A) = (-\triangle)^{-1}(\mathrm{curl}\,\mathrm{curl}A),$$

$$A^{cf} = (I - P)(A) = -(-\triangle)^{-1}(\mathrm{grad}\,\mathrm{div}A),$$

that is $(A^{df})_i = \epsilon_{ijk}\epsilon^{klm}R^jR_lA_m = R^k(R_iA_k - R_kA_i)$ and $(A^{cf})_i = -R_iR^jA_j$, where ϵ_{ijk} is the Levi-Civita symbol and $R^j = |\nabla|^{-1}\partial^j$ is the Riesz transform. We shall see that the interesting dynamic is concentrated in the divergence-free component, while the curl-free component contains the elliptic portion of the gauge and can be treated easily. In fact, when we apply the operator $I - P$ to the first equation in system (2.14) we obtain a nonlinear elliptic equation for the time derivative of the curl-free component

$$\partial_t A^{cf} = (-\triangle)^{-1}\nabla([A^i, \partial_t A_i] + \mathscr{F}(\partial A)).$$

Furthermore, when we apply the operator P to the second equation in system (2.14) we find the following nonlinear hyperbolic equation for the divergence-free component

$$\begin{aligned}&\Box_g A^{df} + 2P[A^i, \partial_i A] - P[A^i, \nabla A_i] + P[\mathrm{div}A, A] + P[A^i, [A_i, A]]\\ &+P(a^k \nabla A_k) - P(b^k \partial_k A) + P(c\,\mathrm{div}A) - P(d_k[A^k, A]) - P(eA) = 0.\end{aligned}$$

To exploit some cancellation in the critical quadratic terms of the latter equation we split $A = A^{df} + A^{cf}$ in the quadratic nonlinear terms and we isolate the factors where A^{df} interacts with ∇A^{df}, then we obtain

$$\begin{aligned}\Box_g A^{df} + 2P[A^{df,i}, \partial_i A^{df}] - P[A^{df,i}, \nabla A_i^{df}] + \mathfrak{M}_{dc}(A^{df}, \partial A^{cf}) + \mathfrak{M}_{cd}(A^{cf}, \partial A^{df}) + \mathfrak{M}_{cc}(A^{cf}, \partial A^{cf})\\ + P\mathfrak{C}(A, A, A) + P\mathscr{E}(\partial A) + P\mathscr{E}(A, A) + P\mathscr{E}(A) = 0\end{aligned}$$

where

$$\begin{aligned}\mathfrak{M}_{dc}(A^{df}, \partial A^{cf}) &= 2P[A^{df,i}, \partial_i A^{cf}] - P[A^{df,i}, \nabla A_i^{cf}] + P[\mathrm{div}A^{cf}, A^{df}],\\ \mathfrak{M}_{cd}(A^{cf}, \partial A^{df}) &= 2P[A^{cf,i}, \partial_i A^{df}] - P[A^{cf,i}, \nabla A_i^{df}],\\ \mathfrak{M}_{cc}(A^{cf}, \partial A^{cf}) &= P[\mathrm{div}A^{cf}, A^{cf}].\end{aligned}$$

We have isolated two critical components of the nonlinearity, specifically the two nonlinear terms which involve the self-interaction of A^{df}. We shall now show that these terms are linear combination of null-forms. Let us recall that the N_{ij}-null-form for $\mathfrak{g}$-valued functions $f, g : \mathbb{R}^3 \to \mathfrak{g}$ is defined as

$$N_{ij}(f, g) = [\partial_i f, \partial_j g] - [\partial_j f, \partial_i g].$$

We have the following key property, see also equation (9) in [85].

Proposition 52. *For every scalar $\mathfrak{g}$-valued functions $f, g : \mathbb{R}^3 \to \mathfrak{g}$, we have*

$$(P[f, \nabla g])_i = |\nabla|^{-1} R^j N_{ij}(f, g).$$

Proof. From the definition of the divergence-free operator and the Riesz transform all we need prove is that

$$\epsilon_{ijk}\epsilon^{klm}\partial^j \partial_l [f, \partial_m g] = \partial^j N_{ij}(f, g).$$

From the classical identity $(\mathrm{curl}(A \times B))_i = [A_i, \mathrm{div}B] - [\mathrm{div}A, B_i] - [A^j, \partial_j B_i] + [\partial_j A_i, B^j]$ we infer

$$\begin{aligned}
\epsilon_{ijk}\epsilon^{klm}\partial^j\partial_l[f,\partial_m g] &= \epsilon_{ijk}\epsilon^{klm}\partial^j[\partial_l f,\partial_m g] = (\mathrm{curl}(\nabla f\times\nabla g))_i \\
&= [\partial_i f,\triangle g]-[\triangle f,\partial_i g]-[\partial^j f,\partial_{ij}g]+[\partial_{ij}f,\partial^j g]= \\
&= \partial^j([\partial_i f,\partial_j g]-[\partial_j f,\partial_i g]) \\
&= \partial^j N_{ij}(f,g).
\end{aligned}$$

□

It follows directly form the previous proposition that $P[A^{df,i},\nabla A^{df}{}_i]=|\nabla|^{-1}N(A^{df},A^{df})$, where $N=(N_1,N_2,N_3)$ is a vector such that each component is a linear combination with constant coefficients of N_{ij}-null-forms:

$$N_j(A,A)=R^k N_{jk}(A^i,A_i).$$

To handle the subsequent quadratic term $P[A^{df,i},\partial_i A^{df}]$ we need the following

Proposition 53. *For every vector* $\mathfrak{g}$*-valued function* $A:\mathbb{R}^3\to\mathfrak{g}^3$ *and scalar* $\mathfrak{g}$*-valued functions* $f:\mathbb{R}^3\to\mathfrak{g}$ *such that* A *is divergence free, we have*

$$[A^j,\partial_j f]=N_{jk}\big(|\nabla|^{-1}R^jA^k,f\big).$$

Proof. Define $\triangle T_{jk}=\partial_j A_k-\partial_k A_j$, then $\partial^j\triangle T_{jk}=\triangle A_k-\partial_k(\mathrm{div}A)$, thus from the divergence-free hypothesis we infer $\partial^j T_{jk}=A_k$. Moreover notice that T_{jk} is antisymmetric and by straightforward commutation we obtain

$$\begin{aligned}
[A^j,\partial_j f] &= [\partial_k T^{kj},\partial_j f]=\frac{1}{2}\Big([\partial_k T^{kj}-\partial_k T^{jk},\partial_j f]\Big) \\
&= \frac{1}{2}\Big([\partial_k T^{kj},\partial_j f]-[\partial_j T^{kj},\partial_k f]\Big) \\
&= \frac{1}{2}N_{jk}\big(T^{jk},f\big)=\frac{1}{2}N_{jk}\big(|\nabla|^{-1}(R^jA^k-R^kA^j),f\big) \\
&= N_{jk}\big(|\nabla|^{-1}R^jA^k,f\big).
\end{aligned}$$

□

Hence it follows that $P[A^{df,i},\partial_i A^{df}]=N(|\nabla|^{-1}A^{df},A^{df})$, where $N=(N_1,N_2,N_3)$ is a vector such that each component is a linear combination with constant coefficients of N_{ij}-null-forms:

$$N_i(|\nabla|^{-1}A,A)=\epsilon_{ijk}\epsilon^{klm}R^jR_lN_{ab}(|\nabla|^{-1}R^aA^b,A_m)$$

In light of these results the Yang-Mills system (2.14) then becomes essentially an hyperbolic system:

$$\begin{cases} \partial_t A^{cf} = (-\triangle)^{-1}\nabla([A^i,\partial_t A_i] + \mathscr{F}(\partial A)) \\ \Box_g A^{df} = |\nabla|^{-1} N(A^{df}, A^{df}) + N(|\nabla|^{-1} A^{df}, A^{df}) + \mathfrak{M}_{dc}(A^{df},\partial A^{cf}) + \mathfrak{M}_{cd}(A^{cf},\partial A^{df}) \\ \qquad +\mathfrak{M}_{cc}(A^{cf},\partial A^{cf}) + P\mathfrak{C}(A,A,A) + P\mathscr{E}(\partial A) + P\mathscr{E}(A,A) + P\mathscr{E}(A) = 0 \end{cases} \tag{2.15}$$

Notice that the second equation for the div-free component is hyperbolic and the nonlinearities on the right-hand side are written in decreasing order of harshness: the first two contains the interactions between div-free components and they possess a null structure, then the next two terms, $\mathfrak{M}_{dc}$ and $\mathfrak{M}_{cd}$, contains the interactions between the div-free and one derivative of the curl-free components. The subsequent term $\mathfrak{M}_{cc}$ enclose the self-interaction of curl-free components and it is of the same type as the two previous ones. Note that the three bilinear terms $\mathfrak{M}_{dc}$, $\mathfrak{M}_{cd}$ and $\mathfrak{M}_{cc}$ do not possess any sort of null structure. Finally, we have four lower-order terms for which the splitting in curl-free and div-free is not emphasized. In order we encounter a trilinear term, a simple term involving one derivative, a bilinear term involving no derivatives, and a simple linear term without derivatives.

The proof of Conjecture 48 is then reduced to the proof of the following Conjecture:

Conjecture 54. *Let* $s > 3/4$, *then the initial value problem for the Yang-Mills equation* (2.15) *is globally well-posed on* $(-\pi,\pi)\times\mathbb{R}^3$ *for initial data* $(A_0, A_1) \in C_0^\infty(\mathbb{R}^3)\times C_0^\infty(\mathbb{R}^3)$ *satisfying the compatibility condition* (2.3) *and the smallness condition*

$$\|A_0\|_{H^s(\mathbb{R}^3)} + \|A_1\|_{H^{s-1}(\mathbb{R}^3)} < \epsilon.$$

To prove the global well-posedness of system (2.15) we shall employ a contraction argument, thus it is sufficient to find two Banach spaces X and Y endowed with the norms

$$\|A\|_X = \|A^{df}\|_{X^{df}} + \|A^{cf}\|_{X^{cf}}, \quad \|A\|_Y = \|A^{df}\|_{Y^{df}} + \|A^{cf}\|_{Y^{cf}}$$

for which the following mapping properties hold:

- *Linear estimate for curl-free component*

$$\|\partial_t^{-1} A^{cf}\|_{X^{cf}} \lesssim \|(A_0^{cf}, A_1^{cf})\|_{H^s\times H^{s-1}} + \|F\|_{Y^{cf}}. \tag{2.16}$$

- *Linear estimate for div-free component*

$$\|\mathscr{H}_g(A_0^{df}, A_1^{df}) + \Box_g^{-1} F\|_{X^{df}} \lesssim \|(A_0^{df}, A_1^{df})\|_{H^s\times H^{s-1}} + \|F\|_{Y^{df}}. \tag{2.17}$$

- *Nonlinearity for curl-free component*

$$\|(-\triangle)^{-1}\nabla([A^i,\partial_0 A_i]+a^i\partial_0 A_i)\|_{Y^{cf}}\lesssim\|A\|_X^2. \tag{2.18}$$

$$\|\mathscr{F}(\partial A)\|_{Y^{cf}}\lesssim\|A\|_X. \tag{2.19}$$

- *Null-form estimate for div-free component*

$$\||\nabla|^{-1}N(A_1,A_2)+N(|\nabla|^{-1}A_1,A_2)\|_{Y^{df}}\lesssim\|A_1\|_{X^{df}}\|A_2\|_{X^{df}}. \tag{2.20}$$

- *Multiplicative estimate for div-free and curl-free interactions*

$$\|\mathfrak{M}_{dc}(A^{df},\partial A^{cf})+\mathfrak{M}_{cd}(A^{cf},\partial A^{df})\|_{Y^{df}}\lesssim\|A^{df}\|_{X^{df}}\|A^{cf}\|_{X^{cf}}. \tag{2.21}$$

- *Multiplicative estimate for curl-free interactions*

$$\|\mathfrak{M}_{cc}(A^{cf},\partial A^{cf})\|_{Y^{df}}\lesssim\|A^{cf}\|_{X^{cf}}^2. \tag{2.22}$$

- *Trilinear estimate*

$$\|P\mathfrak{C}(A,A,A)\|_{Y^{df}}\lesssim\|A\|_X^3. \tag{2.23}$$

- *Estimates for lower-order terms*

$$\begin{aligned}\|P\mathscr{E}(\partial A)\|_{Y^{df}} &\lesssim \|A\|_X,\\ \|P\mathscr{E}(A,A)\|_{Y^{df}} &\lesssim \|A\|_X^2,\\ \|P\mathscr{E}(A)\|_{Y^{df}} &\lesssim \|A\|_X.\end{aligned} \tag{2.24}$$

The Banach spaces used in the contraction argument for the curl-free part are

$$X^{cf}=H_x^{s+\frac14}H_t^{\theta-\frac14}\qquad\text{and}\qquad Y^{cf}=H_x^{s+\frac14}H_t^{\theta-\frac54}$$

where $\frac34<\theta<1$. On the other hand, due to the hyperbolic nature of the equation for the div-free component, the Banach spaces employed are more involved and will be introduced in the next section, they are the curved-analogue of hyperbolic Sobolev spaces:

$$X^{df}=X^{s,\theta}\qquad\text{and}\qquad Y^{df}=X^{s-1,\theta-1}.$$

Furthermore, since we are interested in finding a local solution, we shall consider such spaces for function in the time slice S_T by applying a cutoff in time and then use the scaling property of such spaces to recover a function of T which will be needed to assure smallness in the contraction argument.

We shall outline how to reduce the proof of Conjecture 54 to the proof of the null-form estimate for the div-free component (2.20). Let us start by analyzing the linear estimates. The linear estimate for the curl-free component (2.16) is trivial. Let A^{cf} satisfies

$$\begin{cases} \partial_t A^{cf} = F \\ A^{cf}(t=0) = A_0^{cf}, \quad \partial_t A^{cf}(t=0) = A_1^{cf} \end{cases}$$

then

$$\|\partial_t^{-1} A^{cf}\|_{H_x^{s+\frac{1}{4}} H_t^{\theta-\frac{1}{4}}} \lesssim \|(A_0^{cf}, A_1^{cf})\|_{H^s \times H^{s-1}} + \|F\|_{H_x^{s+\frac{1}{4}} H_t^{\theta-\frac{5}{4}}}$$

Let us consider the linear estimate for the div-free component (2.17): suppose that A^{df} is the solution to the Cauchy problem

$$\begin{cases} \Box_g A^{df} = F \\ (A^{df}, \partial_t A^{df})(0) = (A_0^{df}, A_1^{df}) \end{cases}$$

then Corollary 64 and Proposition 66 implies that A^{df} verifies

$$\|A^{df}\|_{X^{s,\theta}} \lesssim \|(A_0^{df}, A_1^{df})\|_{H^s \times H^{s-1}} + \|F\|_{X^{s-1,\theta-1}}$$

Notice that the hypothesis on the metric g given in (2.11) are satisfied.

Now let us turn to the more complicated nonlinear estimates. The bounds (2.18), (2.21), (2.22) and (2.23) in the flat setting are proved in [106], see also [85], and should be easily adapted to the presence of a curved metric. The proof of estimates (2.19) and (2.24) will not represent a mayor issue since they can be treated by Strichartz estimates in the same line as in [106]. Thus, the main obstacle to perform a fixed point argument is the proof of the estimate involving null-forms (2.20). As previously mentioned, we shall not provide a complete proof of such a bound here, nevertheless we prove a somewhat easier null-form bound for the pure N_{ij} null-form. This intermediate step provides guidance on the proof of the more difficult estimate (2.20) which will be addressed in a subsequent work.

2.5 $X^{s,\theta}$ spaces for curved metrics

Following [30], [28], and [27], we define wave-Sobolev type-spaces adapted to prove local well-posedness of Cauchy problems for nonlinear waves equation on curved backgrounds, i.e. with non-flat metrics. Observe that other variable-coefficient extensions of hyperbolic Sobolev spaces to treat waves equations on smooth compact non-flat manifolds have been introduced in [6] via special theory. In what follows we assume that the matrices $(g^{\alpha\beta}(t,x))_{\alpha,\beta}$, $(g_{\alpha\beta}(t,x))_{\alpha,\beta}$ are uniformly bounded in t, x and of signature $(1, n)$. Furthermore, we also assume, without loss of generality, that $g^{00} = 1$, thus the surfaces $x_0 = const$ are space-like uniformly in x. The following definition is based on Definition 2.1 of [30].

Definition. Let $n \geq 3$ the space dimension, and let g be a metric such that $\partial_t^h \partial_x^\alpha g \in L^2 L^\infty$. Moreover let $s \in \mathbb{R}$ and $\theta \in (0,1)$, we define the $X^{s,\theta}$ norm by

$$\|u\|^2_{X^{s,\theta}} = \inf\Big\{ \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}} : u = \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} P_\lambda u_{\lambda,d} \Big\} \tag{2.25}$$

and

$$\|u\|^2_{X^{s,\theta}_{\lambda,d}} = \lambda^{2s} d^{2\theta} \|u\|_2^2 + \lambda^{2s-2} d^{2\theta-2} \|\Box_{g<\lambda^{1/2}} u\|_2^2$$

where $\Box_{g<\lambda^{1/2}} = (P_{<\lambda^{1/2}}(D_t, D_x) g^{\alpha\beta}) \partial_\alpha \partial_\beta$ and we consider Lebesgue norms to be over $\mathbb{R}^{1+n}$. Furthermore for negative value of θ we define the norm

$$\|f\|^2_{X^{s-1,\theta-1}} = \inf\Big\{ \|f_0\|^2_{L^2 H^{s-1}} + \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \|f_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}} : f = f_0 + \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \Box_{g<\lambda^{1/2}} P_\lambda f_{\lambda,d} \Big\} \tag{2.26}$$

In the definition above we are using inhomogeneous Littlewood-Paley decomposition in the space variable $P_\lambda u = P_\lambda(D_x) u = \mathscr{F}_\xi^{-1}(\varphi_\lambda(\xi)\widehat{u}(t,\xi))$, where φ_λ is a smooth function supported in the set $\{\lambda/2 \leq |\xi| \leq 2\lambda\}$ and $\sum_{\lambda=1}^{\infty} \varphi_\lambda(\xi) = 1$, here P_1 incorporates all the low-frequencies contributions. Notice that only the coefficients of the metric are truncated using space-time Littlewood-Paley cutoffs: $P_\lambda(D_t, D_x) g^{\alpha\beta} = \mathscr{F}^{-1}_{\tau,\xi}(\varphi_\lambda(\tau,\xi)\widetilde{u}(\tau,\xi))$. Define the spatial multiplier $\widetilde{P}_\lambda$ with slightly bigger support, so that $\widetilde{P}_\lambda P_\lambda = P_\lambda$ and its Fourier transform is supported in the annulus $\{\lambda/8 \leq |\xi| \leq 8\lambda\}$. Observe that, since $\widetilde{P}_\lambda P_\lambda = P_\lambda$, one can take $u_{\lambda,d} = \widetilde{P}_\lambda u_{\lambda,d}$, therefore we have the equivalent definition

$$\|u\|^2_{X^{s,\theta}} = \inf\Big\{ \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \|\widetilde{P}_\lambda u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}} : u = \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} P_\lambda \widetilde{P}_\lambda u_{\lambda,d} \Big\}$$

and an analogous formula holds for $X^{s-1,\theta-1}$. This means that we can consider the functions $u_{\lambda,d}$ and $f_{\lambda,d}$ to be localized around frequency λ.

The first properties that we prove is that variable coefficients wave-Sobolev spaces defined above are indeed an extension of classical wave-Sobolev spaces. Thus in the case of flat metrics $g = \eta$, the infimum is reach by the classical modulations cutoffs with respect to distance form the light cone.

Proposition 55. *Let $g = \eta$ the Minkowski metric in the definition of the norm* (2.25)*. Then*

$$\|u\|_{X^{s,\theta}} \approx \|u\|_{H^{s,\theta}} + \|\Box u\|_{L^2 H^{s+\theta-2}}$$

where

$$\|u\|^2_{H^{s,\theta}} = \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\infty} \lambda^{2s} d^{2\theta} \|P_\lambda Q_d u\|_2^2$$

Recall that we denote by Q_d the Littlewood-Paley type operator defined by $Q_d u = \mathscr{F}^{-1}_{\tau,\xi}(\psi_d(\tau,\xi)\widetilde{u}(\tau,\xi))$, where $\psi_d(\tau,\xi) = \varphi_d(|\tau| - |\xi|)$. Here ψ_d cuts at distance $\approx d$ from the light cone.

Proof. Observe that if $g = \eta$, then the $X^{s,\theta}_{\lambda,d}$ norm reduced to

$$\|u\|^2_{X^{s,\theta}_{\lambda,d}} = \lambda^{2s} d^{2\theta} \|u\|_2^2 + \lambda^{2s-2} d^{2\theta-2} \|\Box u\|_2^2$$

since the Fourier transform of a constant tempered distribution is the delta function. Let us begin by showing the easier part: the $X^{s,\theta}$ norm is smaller than $\|u\|_{H^{s,\theta}} + \|\Box u\|_{L^2 H^{s+\theta-2}}$. Set

$$u_{\lambda,d} = \begin{cases} \widetilde{P}_\lambda Q_d u & \text{if } d < \lambda \\ \sum_{d \geq \lambda} \widetilde{P}_\lambda Q_d u & \text{if } d = \lambda \end{cases}$$

then we have the decomposition $\sum_{\lambda=1}^\infty \sum_{d=1}^\lambda P_\lambda u_{\lambda,d} = \sum_{\lambda=1}^\infty \sum_{d=1}^\infty P_\lambda Q_d u = u$ and therefore

$$\|u\|^2_{X^{s,\theta}} = \sum_{\lambda=1}^\infty \sum_{d<\lambda} \|\widetilde{P}_\lambda Q_d u\|^2_{X^{s,\theta}_{\lambda,d}} + \|\sum_{d\geq\lambda} \widetilde{P}_\lambda Q_d u\|^2_{X^{s,\theta}_{\lambda,\lambda}}$$

For the low modulation term we have the bound

$$\begin{aligned} \sum_{\lambda=1}^\infty \sum_{d<\lambda} \|\widetilde{P}_\lambda Q_d u\|^2_{X^{s,\theta}_{\lambda,d}} &\leq \sum_{\lambda=1}^\infty \sum_{d<\lambda} \lambda^{2s} d^{2\theta} \|\widetilde{P}_\lambda Q_d u\|_2^2 + \lambda^{2s-2} d^{2\theta-2} \|\widetilde{P}_\lambda Q_d \Box u\|_2^2 \\ &\lesssim \sum_{\lambda=1}^\infty \sum_{d=1}^\infty \lambda^{2s} d^{2\theta} \|\widetilde{P}_\lambda Q_d u\|_2^2 \\ &\lesssim \|u\|^2_{H^{s,\theta}} \end{aligned}$$

On the other hand the high modulation term is controlled by

$$\begin{aligned} \sum_{\lambda=1}^\infty \sum_{d\geq\lambda} \|\widetilde{P}_\lambda Q_d u\|^2_{X^{s,\theta}_{\lambda,\lambda}} &\lesssim \sum_{\lambda=1}^\infty \sum_{d\geq\lambda} \lambda^{2s+2\theta} \|\widetilde{P}_\lambda Q_d u\|_2^2 + \lambda^{2s+2\theta-4} \|\widetilde{P}_\lambda Q_d \Box u\|_2^2 \\ &\lesssim \sum_{\lambda=1}^\infty \sum_{d=1}^\infty \lambda^{2s} d^{2\theta} \|\widetilde{P}_\lambda Q_d u\|_2^2 + \lambda^{2s+2\theta-4} \|\widetilde{P}_\lambda Q_d \Box u\|_2^2 \\ &\lesssim \|u\|^2_{H^{s,\theta}} + \|\Box u\|^2_{L^2 H^{s+\theta-2}} \end{aligned}$$

To show the reverse inequality, i.e. $\|u\|_{H^{s,\theta}} + \|\Box u\|_{L^2 H^{s+\theta-2}} \lesssim \|u\|_{X^{s,\theta}}$, we argue as follows. Suppose that the function u decomposes into $u = \sum_{\lambda=1}^\infty \sum_{d=1}^\lambda P_\lambda u_{\lambda,d}$, then we must show that

$$\|u\|^2_{H^{s,\theta}} + \|\Box u\|^2_{L^2 H^{s+\theta-2}} \lesssim \sum_{\lambda=1}^\infty \sum_{d=1}^\lambda \lambda^{2s} d^{2\theta} \|u_{\lambda,d}\|_2^2 + \lambda^{2s-2} d^{2\theta-2} \|\Box u_{\lambda,d}\|_2^2 \tag{2.27}$$

Let us begin to bound the second term on the left-hand-side. Notice that $P_\lambda u = \sum_{d'=1}^\lambda P_\lambda u_{\lambda,d'}$, then by Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\|\Box u\|^2_{L^2H^{s+\theta-2}} &\approx \sum_{\lambda=1}^{\infty} \lambda^{2s+2\theta-4}\|P_\lambda \Box u\|_2^2 \\
&\lesssim \sum_{\lambda=1}^{\infty} \lambda^{2s+2\theta-4}\Big(\sum_{d=1}^{\lambda} d^{2\theta-2}\|\Box u_{\lambda,d}\|_2^2\Big)\Big(\sum_{d=1}^{\lambda} d^{2-2\theta}\Big) \\
&\lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \lambda^{2s-2} d^{2\theta-2}\|\Box u_{\lambda,d}\|_2^2
\end{aligned}$$

To close we need to bound the first term on the left-hand-side in (2.27). As above Cauchy-Schwarz inequality yield to

$$\begin{aligned}
\|u\|^2_{H^{s,\theta}} &\lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\infty} \lambda^{2s} d^{2\theta}\Big(\sum_{d'=1}^{\lambda} \|P_\lambda Q_d u_{\lambda,d'}\|_2\Big)^2 \\
&\lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\infty}\Big(\sum_{d'=1}^{\lambda} \lambda^{2s}(d'^{2\theta}+d'^{2\theta-2}d^2)\|Q_d u_{\lambda,d'}\|_2^2\Big)\Big(\sum_{d'=1}^{\lambda}\frac{d^{2\theta}}{d'^{2\theta}+d'^{2\theta-2}d^2}\Big) \\
&\lesssim \sum_{\lambda=1}^{\infty}\sum_{d'=1}^{\lambda} \lambda^{2s} d'^{2\theta}\|u_{\lambda,d'}\|_2^2 + \lambda^{2s-2}d'^{2\theta-2}\|\Box u_{\lambda,d'}\|_2^2
\end{aligned}$$

The sum $\sum_{d'=1}^{\lambda}\frac{d^{2\theta}}{d'^{2\theta}+d'^{2\theta-2}d^2}$ is bounded for $0<\theta<1$ uniformly with respect to λ and d. In fact, let $\lambda = 2^k$, $d' = 2^j$, and $d = 2^h$, then

$$2^{2\theta h}\sum_{j=1}^{k}\frac{1}{2^{2\theta j}+2^{(2\theta-2)j}2^{2h}} < \sum_{j=1}^{\infty}\frac{2^{(2-2\theta)(j-h)}}{2^{2(j-h)}+1}$$

and since

$$\lim_{j\to\infty} 2^{2-2\theta}\frac{2^{2(j-h)}+1}{2^{2(j+1-h)}+1} = 2^{-2\theta} < 1$$

by the ratio test we conclude that the series is convergent for every $h\in\mathbb{N}$. □

Next, we verify via an analogous argument used in the proof of the previous proposition, that a similar property holds for the variable-coefficient extension of hyperbolic Sobolev spaces with negative θ, that is for the norm introduced in (2.26).

Proposition 56. *Let $g=\eta$ the Minkowski metric in the definition of the norm* (2.26). *Then for $s\in\mathbb{R}$, and $\theta\in(0,1)$ we have*

$$\|f\|_{X^{s-1,\theta-1}} \approx \|f\|_{H^{s-1,\theta-1}} + \|f\|_{L^2H^{s+\theta-2}}$$

Proof. Notice that in flat space-time we have

$$\|f\|^2_{X^{s-1,\theta-1}} = \inf\Big\{\|f_0\|^2_{L^2H^{s-1}} + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|f_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}} : f = f_0 + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\Box P_\lambda f_{\lambda,d}\Big\}$$

We begin by proving the following: $\|f\|_{X^{s-1,\theta-1}} \lesssim \|f\|_{H^{s-1,\theta-1}} + \|f\|_{L^2H^{s+\theta-2}}$. Set $f_0 = 0$ and define

$$f_{\lambda,d} = \begin{cases} \Box^{-1}\widetilde{P}_\lambda Q_d f & \text{if } d < \lambda \\ \sum_{d\geq\lambda}\Box^{-1}\widetilde{P}_\lambda Q_d f & \text{if } d = \lambda \end{cases}$$

Then clearly $f = \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\infty} P_\lambda Q_d f = f_0 + \sum_{\lambda=1}^{\infty}\sum_{d\leq\lambda}\Box P_\lambda f_{\lambda,d}$, and

$$\begin{aligned}
\|f\|^2_{X^{s-1,\theta-1}} &\lesssim \sum_{\lambda=1}^{\infty}\sum_{d<\lambda}\lambda^{2s}d^{2\theta}\|\Box^{-1}\widetilde{P}_\lambda Q_d f\|_2^2 + \lambda^{2s-2}d^{2\theta-2}\|\widetilde{P}_\lambda Q_d f\|_2^2 \\
&+ \sum_{\lambda=1}^{\infty}\sum_{d\geq\lambda}\lambda^{2s+2\theta}\|\Box^{-1}\widetilde{P}_\lambda Q_d f\|_2^2 + \lambda^{2s+2\theta-4}\|\widetilde{P}_\lambda Q_d f\|_2^2 \\
&\lesssim \sum_{\lambda=1}^{\infty}\sum_{d<\lambda}\lambda^{2s-2}d^{2\theta-2}\|\widetilde{P}_\lambda Q_d f\|_2^2 + \sum_{\lambda=1}^{\infty}\sum_{d\geq\lambda}\lambda^{2s+2\theta-4}\|\widetilde{P}_\lambda Q_d f\|_2^2 \\
&\lesssim \|f\|^2_{H^{s-1,\theta-1}} + \|f\|^2_{L^2H^{s+\theta-2}}
\end{aligned}$$

To show the reverse inequality, suppose that the function f decomposes into $f = f_0 + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\Box P_\lambda f_{\lambda,d}$, it suffices to prove the bound

$$\|f\|^2_{H^{s-1,\theta-1}} + \|f\|^2_{L^2H^{s+\theta-2}} \lesssim \|f_0\|^2_{L^2H^{s-1}} + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|f_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}}$$

Notice that $P_\lambda f \approx P_\lambda f_0 + \sum_{d=1}^{\lambda}\Box P_\lambda f_{\lambda,d}$, then by Cauchy-Schwarz inequality and the fact that $\theta < 1$ we obtain

$$\begin{aligned}
\|f\|^2_{L^2H^{s+\theta-2}} &\lesssim \sum_{\lambda=1}^{\infty}\lambda^{2s+2\theta-4}\Big(\|P_\lambda f_0\|_2 + \sum_{d\leq\lambda}\|\Box P_\lambda f_{\lambda,d}\|_2\Big)^2 \\
&\lesssim \sum_{\lambda=1}^{\infty}\lambda^{2s-2}\|P_\lambda f_0\|_2^2 + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\lambda^{2s-2}d^{2\theta-2}\|\Box f_{\lambda,d}\|_2^2 \\
&\lesssim \|f_0\|^2_{L^2H^{s-1}} + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|f_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}}
\end{aligned}$$

Next we prove the corresponding bound for $H^{s-1,\theta-1}$. Recall that $P_\lambda Q_d f \approx P_\lambda Q_d f_0 + \sum_{d'=1}^{\lambda}\Box P_\lambda Q_d f_{\lambda,d'}$, Cauchy-Schwarz inequality yield to

$$\begin{aligned}\|f\|^2_{H^{s-1,\theta-1}} &\lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\infty}\lambda^{2s-2}d^{2\theta-2}\Big(\|P_\lambda Q_d f_0\|_2+\sum_{d'=1}^{\lambda}\|\Box P_\lambda Q_d f_{\lambda,d'}\|_2\Big)^2\\ &\lesssim \sum_{\lambda=1}^{\infty}\lambda^{2s-2}\|P_\lambda f_0\|_2^2+\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\infty}\Big(\sum_{d'=1}^{\lambda}\lambda^s d^\theta\|P_\lambda Q_d f_{\lambda,d'}\|_2\Big)^2\\ &\lesssim \|f_0\|^2_{L^2H^{s-1}}+\sum_{\lambda=1}^{\infty}\sum_{d\le\lambda}\Big(\sum_{d'=1}^{\lambda}\lambda^s d^\theta\|P_\lambda Q_d f_{\lambda,d'}\|_2\Big)^2+\sum_{\lambda=1}^{\infty}\sum_{d>\lambda}\Big(\sum_{d'=1}^{\lambda}\lambda^{s-1}d^{\theta-1}\|\Box P_\lambda Q_d f_{\lambda,d'}\|_2\Big)^2\end{aligned}$$

To control the second term we apply Cauchy-Schwarz inequality

$$\sum_{\lambda=1}^{\infty}\sum_{d\le\lambda}\Big(\sum_{d'=1}^{\lambda}\lambda^s d^\theta\|P_\lambda Q_d f_{\lambda,d'}\|_2\Big)^2 \lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\infty}\sum_{d'=1}^{\lambda}\|P_\lambda Q_d f_{\lambda,d'}\|^2_{X^{s,\theta}_{\lambda,d'}}\Big(\sum_{d'=1}^{\lambda}\frac{d^{2\theta}}{d'^{2\theta}+d'^{2\theta-2}d^2}\Big)$$

and analogously we can control the third term by

$$\sum_{\lambda=1}^{\infty}\sum_{d>\lambda}\Big(\sum_{d'=1}^{\lambda}\lambda^{s-1}d^{\theta-1}\|\Box P_\lambda Q_d f_{\lambda,d'}\|_2\Big)^2 \lesssim \sum_{\lambda=1}^{\infty}\sum_{d>\lambda}\sum_{d'=1}^{\lambda}\|P_\lambda Q_d f_{\lambda,d'}\|^2_{X^{s,\theta}_{\lambda,d'}}\Big(\sum_{d'=1}^{\lambda}\frac{d^{2\theta}}{\lambda^2 d'^{2\theta}d^{-2}+d'^{2\theta-2}d^2}\Big)$$

Notice that the reminder sum $\sum_{d'=1}^{\lambda}\frac{d^{2\theta}}{\lambda^2 d'^{2\theta}d^{-2}+d'^{2\theta-2}d^2}$ is bounded for $0<\theta<1$ uniformly with respect to λ and d by a similar argument used in the previous proposition. The proof is then completed. □

2.6 Basic properties of $X^{s,\theta}$ spaces

In this section we review some of the properties of the variable coefficients extension of hyperbolic Sobolev spaces introduced in the previous section. Our presentation is based on the works [30], [28], and [27]. Let $s\in\mathbb{R}$, $0<\theta<1$ and I a finite time interval around the origin, we shall define the norms $X^{s,\theta}[I]$ and $X^{s-1,\theta-1}[I]$ respectively as in (2.25) and (2.26) where we replace the Lebesgue norms $L^2(\mathbb{R}^{1+n})$ by $L^2(I\times\mathbb{R}^n)$. Hence we define the associate norm

$$\|u\|^2_{X^{s,\theta}_{\lambda,d}[I]} := \lambda^{2s}d^{2\theta}\|u\|^2_{L^2(I\times\mathbb{R}^n)}+\lambda^{2s-2}d^{2\theta-2}\|\Box_{g<\lambda^{1/2}}u\|^2_{L^2(I\times\mathbb{R}^n)}.$$

The first goal of this section is to prove the following two embedding properties:

i. Let $-1<\theta<0$ then $X^{s,\theta}\subset L^2H^{s+\theta}$.

ii. Let $1/2<\theta<1$ then $X^{s,\theta}[I]\subset C^0(I,H^s)\cap C^1(I,H^{s-1})$.

Let us decompose the function u into $u=\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}P_\lambda u_{\lambda,d}$, then the previous two embeddings properties follows from the two estimates *i.* $\|u\|^2_{L^2H^{s+\theta}} \lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}}$

ii. $\|\nabla_{t,x}u\|^2_{L^\infty H^{s-1}(S_I)} \lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}[I]}$

where $S_I = I\times\mathbb{R}^n$. The first estimate is easier to establish since for $-1<\theta<0$ we have

$$\|u\|^2_{L^2H^{s+\theta}} \lesssim \sum_{\lambda=1}^{\infty}\lambda^{2s+2\theta}\Big(\sum_{d=1}^{\lambda}\|P_\lambda u_{\lambda,d}\|_2\Big)^2 \lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\lambda^{2s}d^{2\theta}\|P_\lambda u_{\lambda,d}\|_2^2 \lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}}$$

In order to establish *ii.* we use as well Cauchy-Schwarz inequality to obtain

$$\begin{aligned}\|\nabla_{t,x}u\|^2_{L^\infty H^{s-1}(S_I)} &\lesssim \sum_{\lambda=1}^{\infty}\lambda^{2s-2}\Big(\sum_{d=1}^{\lambda}\|\nabla_{t,x}P_\lambda u_{\lambda,d}\|_{L^\infty L^2(S_I)}\Big)^2\\ &\lesssim \sum_{\lambda=1}^{\infty}\Big(\sum_{d=1}^{\lambda}\lambda^{2s-2}d^{2\theta-1}\|\nabla_{t,x}P_\lambda u_{\lambda,d}\|^2_{L^\infty L^2(S_I)}\Big)\Big(\sum_{d=1}^{\lambda}d^{1-2\theta}\Big)\end{aligned}$$

The second sum is controlled by the series $\sum_{d=1}^{\infty}d^{1-2\theta}$ which converges due to the fact $\theta>1/2$. Therefore to prove *ii.* it suffices to establish the following proposition that is taken from [30].

Proposition 57 ([30])**.** *Let $\partial_t^h\partial_x^\alpha g\in L^2L^\infty$ for every $h+|\alpha|\le 2$. Suppose $s\in\mathbb{R}$ and $\theta\in(0,1)$ then for any dyadic numbers λ and $d\gtrsim|I|^{-1}$ we have*

$$\lambda^{2s-2}d^{2\theta-1}\|\nabla_{t,x}P_\lambda v\|^2_{L^\infty L^2(S_I)} \lesssim \|v\|^2_{X^{s,\theta}_{\lambda,d}[I]}$$

Thence in view of the previous discussion from Proposition 57 it easily follows:

Corollary 58 (Embedding into solutions space)**.** *Let $\partial_t^h\partial_x^\alpha g\in L^2L^\infty$ for every $h+|\alpha|\le 2$. Suppose $s\in\mathbb{R}$ and $\theta\in(1/2,1)$, then $X^{s,\theta}[I]\subset C^0(I,H^s)\cap C^1(I,H^{s-1})$*

We split the proof of the Proposition 57 into two parts. First we define a new norm which is easier to handle in the context of wave equations:

$$\|v\|_{\overline{X}^{s,\theta}_{\lambda,d}} = \lambda^{s-1}d^\theta\|\nabla_{t,x}v\|_2 + \lambda^{s-1}d^{\theta-1}\|\Box_{g<\lambda^{1/2}}v\|_2$$

In the lemma below we compare $\overline{X}^{s,\theta}_{\lambda,d}$ with $X^{s,\theta}_{\lambda,d}$ and we shall prove that the two norms are equivalent. Thus to prove Proposition 57 it suffices to replace the norm $X^{s,\theta}_{\lambda,d}$ with $\overline{X}^{s,\theta}_{\lambda,d}$.

Lemma 59 (Norms equivalence, [30])**.** *Let $\partial_t^h\partial_x^\alpha g\in L^2L^\infty$ for every $h+|\alpha|\le 2$, then the following estimate holds*

$$\|P_\lambda v\|_{\overline{X}^{s,\theta}_{\lambda,d}} \lesssim \|v\|_{X^{s,\theta}_{\lambda,d}} \tag{2.28}$$

Moreover the norms $X^{s,\theta}_{\lambda,d}$ and $\overline{X}^{s,\theta}_{\lambda,d}$ are equivalent over functions which are localized in frequencies $\approx \lambda$.

The proof of the norm equivalence Lemma requires the fixed-time commutator estimate below, that tell us that the commutator between the low-frequencies component of the d'Alembert operator $\Box_{g<\lambda^{1/2}}$ and the Littlewood-Paley cutoff P_λ is bounded by the L^2_x norm of $\nabla_{t,x}P_\lambda u$.

Lemma 60. *(Commutator estimate) Let $\nabla^k_x g(t,\cdot) \in L^\infty(\mathbb{R}^n)$ then*

$$\|[\Box_{g<\lambda^{1/2}}, P_\lambda]v(t)\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{1-k}\|\nabla^k_x P_{<\lambda^{1/2}} g(t)\|_{L^\infty(\mathbb{R}^n)}\|\nabla_{t,x}\widetilde{P}_\lambda v\|_{L^2(\mathbb{R}^n)}$$

Proof. Let us assume without losing generality that $g^{00} = 1$, thus

$$[\Box_{g<\lambda^{1/2}}, P_\lambda] = 2[P_{<\lambda^{1/2}}g^{0i}, P_\lambda]\partial_0\partial_i + [P_{<\lambda^{1/2}}g^{ij}, P_\lambda]\partial_i\partial_j$$

Therefore for any function f such that $\nabla^k_x f(t,\cdot) \in L^\infty(\mathbb{R}^n)$ we have the fixed-time estimate

$$\begin{aligned}
\|[P_{<\lambda^{1/2}}f, P_\lambda]v(t)\|_{L^2(\mathbb{R}^n)} &\lesssim \|(P_{<\lambda^{1/2}}f)(P_\lambda v)(t)\|_{L^2(\mathbb{R}^n)} + \|P_\lambda((P_{<\lambda^{1/2}}f)v)(t)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|P_{<\lambda^{1/2}}f(t)\|_{L^\infty(\mathbb{R}^n)}\|v(t)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \lambda^{-k}\|\nabla^k_x P_{<\lambda^{1/2}}f(t)\|_{L^\infty(\mathbb{R}^n)}\|v(t)\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

Now we apply the previous bound where f now represents the different components of the metric. Notice that the commutator is localized around frequency $\approx \lambda$, hence we have

$$\begin{aligned}
\|[\Box_{g<\lambda^{1/2}}, P_\lambda]v(t)\|_{L^2(\mathbb{R}^n)} &\approx \|[\Box_{g<\lambda^{1/2}}, P_\lambda]\widetilde{P}_\lambda v(t)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|[P_{<\lambda^{1/2}}g^{0i}, P_\lambda]\partial_0\partial_i\widetilde{P}_\lambda v(t)\|_{L^2(\mathbb{R}^n)} + \|[P_{<\lambda^{1/2}}g^{ij}, P_\lambda]\partial_i\partial_j\widetilde{P}_\lambda v(t)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|\nabla^k_x g(t)\|_{L^\infty(\mathbb{R}^n)}(\lambda^{-k}\|\partial_t\nabla_x\widetilde{P}_\lambda v(t)\|_{L^2(\mathbb{R}^n)} + \lambda^{-k/2}\|\nabla^2_x\widetilde{P}_\lambda v(t)\|_{L^2(\mathbb{R}^n)}) \\
&\lesssim \lambda^{1-k}\|\nabla^k_x g(t)\|_{L^\infty(\mathbb{R}^n)}(\|\partial_t\widetilde{P}_\lambda v(t)\|_{L^2(\mathbb{R}^n)} + \|\nabla_x\widetilde{P}_\lambda v(t)\|_{L^2(\mathbb{R}^n)})
\end{aligned}$$

□

Next we prove the norm equivalence lemma.

Proof of Lemma 59. First we prove that $\|P_\lambda v\|_{X^{s,\theta}} \lesssim \|P_\lambda v\|_{\overline{X}^{s,\theta}}$. By Littlewood-Paley theory we have $\lambda^s d^\theta\|P_\lambda v\|_2 \approx \lambda^{s-1}d^\theta\|\nabla_x P_\lambda v\|_2 \lesssim \lambda^{s-1}d^\theta\|\nabla_{t,x}P_\lambda v\|_2$, thus $\|P_\lambda v\|_{X^{s,\theta}_{\lambda,d}} \lesssim \|P_\lambda v\|_{\overline{X}^{s,\theta}_{\lambda,d}}$. Now let us turn to the proof of (2.28), notice that, in view of Lemma 60, it suffices to show that

$$\lambda^{s-1}d^\theta\|\partial_t P_\lambda v\|_2 \lesssim \lambda^s d^\theta\|v\|_2 + \lambda^{s-1}d^{\theta-1}\|\Box_{g<\lambda^{1/2}}v\|_2 \tag{2.29}$$

In fact, we bound the second term in the $\|P_\lambda v\|_{\overline{X}^{s,\theta}_{\lambda,d}}$ via the following commutator inequality

$$\|\Box_{g<\lambda^{1/2}} P_\lambda v\|_2 \leq \|[\Box_{g<\lambda^{1/2}}, P_\lambda]v\|_2 + \|P_\lambda \Box_{g<\lambda^{1/2}} v\|_2$$

We employ Lemma 60 to control the first term on the right-hand side, we place the metric in $L^\infty_t L^\infty_x$ and we apply Bernstein inequality to recover the $L^2_t L^\infty_x$ norm since for the metric we are using space-time Littlewood-Paley operator, hence we obtain

$$\|[\Box_{g<\lambda^{1/2}}, P_\lambda]v\|_2 \lesssim \|\nabla^2_x P_{<\lambda^{1/2}} g(t)\|_{L^1L^\infty} \|\nabla_{t,x} P_\lambda v\|_{L^2L^2}$$

Hence to control this term $\nabla^2_x g \in L^2 L^\infty$ suffices. We now turn to the proof of (2.29). We actually prove the stronger inequality:

$$\lambda^s d^\theta (\|\partial^2_t v\|_{L^2(H^{-2}+\lambda^2 L^2)} + \|\partial_t v\|_{L^2(H^{-1}+\lambda L^2)}) \lesssim \|v\|_{X^{s,\theta}_{\lambda,d}} \tag{2.30}$$

Estimate (2.30) follow form the interpolation inequality

$$\|\partial_t v\|^2_{L^2(H^{-1}+\lambda L^2)} \lesssim (\|\partial^2_t v\|_{L^2(H^{-2}+\lambda^2 L^2)} + \|v\|_2)\|v\|_2$$

and the bound

$$\|\partial^2_t P_\lambda v\|_{L^2(H^{-2}+\lambda^2 L^2)} \lesssim \lambda^{-2} \|\Box_{g<\sqrt{\lambda}} P_\lambda v\|_2 + \|\partial_t P_\lambda v\|_{L^2(H^{-1}+\lambda L^2)} + \|P_\lambda v\|_2 \tag{2.31}$$

Notice that (2.30) implies (2.29) since $\lambda^s d^\theta (\|\partial^2_t P_\lambda v\|_{L^2(H^{-2}+\lambda^2 L^2)} + \|\partial_t P_\lambda v\|_{L^2(H^{-1}+\lambda L^2)}) = \lambda^{s-2} d^\theta \|\partial^2_t P_\lambda v\|_2 + \lambda^{s-1} d^\theta \|\partial_t P_\lambda v\|_2$. The bound (2.31) follows form the fixed-time estimates

$(i.)$ $\|g_{<\sqrt{\lambda}} \partial^2_{tx} P_\lambda v(t)\|_{H^{-2}+\lambda^2 L^2} \lesssim \|\partial_t P_\lambda v(t)\|_{H^{-1}+\lambda L^2}$

$(ii.)$ $\|g_{<\sqrt{\lambda}} \partial^2_x P_\lambda v(t)\|_{H^{-2}+\lambda^2 L^2} \lesssim \|P_\lambda v(t)\|_{L^2}$

Indeed by $(i.)$ and $(ii.)$ and the fact that $g^{00} = 1$, we obtain

$$\begin{aligned}\|\partial^2_t P_\lambda v\|_{L^2(H^{-2}+\lambda^2 L^2)} &\lesssim \|\Box_{g<\sqrt{\lambda}} P_\lambda v\|_{L^2(H^{-2}+\lambda^2 L^2)} + \|g_{<\sqrt{\lambda}} \partial^2_{tx} v\|_{L^2(H^{-2}+\lambda^2 L^2)} + \|g_{<\sqrt{\lambda}} \partial^2_x P_\lambda v\|_{L^2(H^{-2}+\lambda^2 L^2)} \\ &\lesssim \lambda^{-2} \|\Box_{g<\sqrt{\lambda}} P_\lambda v\|_2 + \|\partial_t P_\lambda v\|_{L^2(H^{-1}+\lambda L^2)} + \|P_\lambda v\|_2\end{aligned}$$

To conclude we prove $(i.)$ and $(ii.)$ by replacing each space derivative with its frequency. We have

$$g_{<\sqrt{\lambda}} \partial^2_x P_\lambda v = \partial^2_x (g_{<\sqrt{\lambda}} P_\lambda v) - 2\partial_x(\partial_x g_{<\sqrt{\lambda}} P_\lambda v) + (\partial^2_x g_{<\sqrt{\lambda}}) P_\lambda v$$

And the bounds $|g_{<\sqrt{\lambda}}(t)| \lesssim 1$, $|\partial_x g_{<\sqrt{\lambda}}(t)| \lesssim \lambda$, and $|\partial^2_x g_{<\sqrt{\lambda}}(t)| \lesssim \lambda^2$ yield to $\|g_{<\sqrt{\lambda}} \partial^2_{tx} P_\lambda v(t)\|_{H^{-2}} \lesssim \|\partial_t P_\lambda v(t)\|_{H^{-1}}$ and $\|g_{<\sqrt{\lambda}} \partial^2_{tx} P_\lambda v\|_{L^2} \lesssim \lambda \|\partial_t P_\lambda v\|_{L^2}$. Thus $(i.)$ holds, the proof of $(ii.)$ is similar. □

In order to prove Proposition 57 we will need the following version of the energy estimate for the wave equation on curved backgrounds which required more relaxed assumptions on the metric.

Lemma 61 (Classical energy estimate, [30]). *Let* $\nabla_{t,x} g \in L^1 L^\infty(S_I)$ *then*

$$\|\nabla_{t,x} v\|^2_{L^\infty L^2(S_I)} \lesssim |I|^{-1} \|\nabla_{t,x} v\|^2_{L^2 L^2(S_I)} + \|\nabla_{t,x} v\|_{L^2 L^2(S_I)} \|\Box_{g<\lambda^{1/2}} v\|_{L^2 L^2(S_I)}.$$

Notice that this estimate hold only for functions supported in the time slice $I \times \mathbb{R}^n$ and in general it fails of unbounded time intervals.

Proof. The standard energy estimate in curved background with metric which satisfies $\nabla_{t,x} g \in L^1 L^\infty$ reads

$$\|\nabla_{t,x} v(t)\|_{L^2(\mathbb{R}^n)} \lesssim \|\nabla_{t,x} v(0)\|_{L^2(\mathbb{R}^n)} + \|\Box_{g<\lambda^{1/2}} v\|_{L^1 L^2([0,t]\times\mathbb{R}^n)}.$$

Hence the function $h(t) = \|\nabla_{t,x} v(t)\|_{L^2(\mathbb{R}^n)}$ is increasing. Let t varying in the interval $[0,\epsilon^2]$, then by Cauchy-Schwarz we obtain the bound

$$h(0) = \frac{1}{\epsilon^2}\epsilon^2 h(0) \le \frac{1}{\epsilon^2}\int_0^{\epsilon^2} h(s)\,ds \le \frac{1}{\epsilon}\Big(\int_0^{\epsilon^2} |h(s)|^2 ds\Big)^{1/2}.$$

Hence $\|\nabla_{t,x} v(0)\|_{L^2(\mathbb{R}^n)} \le \frac{1}{\epsilon}\|\nabla_{t,x} v\|_{L^2 L^2([0,\epsilon^2]\times\mathbb{R}^n)}$. Moreover, Cauchy-Schwarz yield to the following bound for the inhomogeneous term

$$\|\Box_{g<\lambda^{1/2}} v\|_{L^1 L^2([0,\epsilon^2]\times\mathbb{R}^n)} \lesssim \epsilon \|\Box_{g<\lambda^{1/2}} v\|_{L^2 L^2([0,\epsilon^2]\times\mathbb{R}^n)}.$$

Therefore we obtain

$$\|\nabla_{t,x} v\|_{L^\infty L^2([0,\epsilon^2]\times\mathbb{R}^n)} \lesssim \frac{1}{\epsilon}\|\nabla_{t,x} v\|_{L^2 L^2([0,\epsilon^2]\times\mathbb{R}^n)} + \epsilon \|\Box_{g<\lambda^{1/2}} v\|_{L^2 L^2([0,\epsilon^2]\times\mathbb{R}^n)}.$$

Covering the time interval I with a finite number of intervals of length ϵ^2 and summing up over such small intervals gives the desired estimate. □

We are now ready to prove the $L^2 L^\infty$ embedding in the context of $X^{s,\theta}$ spaces.

Proof of Proposition 57. Let us apply the energy estimate of Lemma 61, we obtain

$$\begin{aligned} \lambda^{2s-2} d^{2\theta-1} \|\nabla_{t,x} P_\lambda v\|^2_{L^\infty L^2(S_I)} &\lesssim \lambda^{2s-2} d^{2\theta-1} |I|^{-1} \|\nabla_{t,x} P_\lambda v\|^2_{L^2 L^2(S_I)} \\ &+ \lambda^{s-2} d^{2\theta-1} \|\nabla_{t,x} P_\lambda v\|_{L^2 L^2(S_I)} \|\Box_{g<\lambda^{1/2}} P_\lambda v\|_{L^2 L^2(S_I)} \\ &\lesssim \|P_\lambda v\|^2_{\overline{X}^{s,\theta}_{\lambda,d}[I]} \end{aligned}$$

Therefore Proposition 57 follows from the equivalence of the $X^{s,\theta}_{\lambda,d}$ and $\overline{X}^{s,\theta}_{\lambda,d}$ norms, precisely from (2.28). □

Next, consider the Cauchy problem for the linear wave equation on a curved background metric g:

$$\begin{cases} \Box_g u = f \\ (u, u_t)(0) = (u_0, u_1) \end{cases} \tag{2.32}$$

In the following, we study properties of the homogeneous and inhomogeneous solution operators defined as

$$\Box_g \mathscr{H}(u_0, u_1) = 0, \quad \mathscr{H}(u_0, u_1)|_{t=0} = u_0, \quad \partial_t \mathscr{H}(u_0, u_1)|_{t=0} = u_1$$

$$\Box_g (\Box_g^{-1} f) = f, \quad \Box_g^{-1} f|_{t=0} = 0, \quad \partial_t \Box_g^{-1} f|_{t=0} = 0$$

The next lemma, which we will need below, allow us to handle the high frequency component of the curved d'Alembert operator. Based on Remark 2.14 in [27], we extend Lemma 2.9 in [30] to a wider range of exponents s by imposing more regularity on the metric coefficients.

Lemma 62 (High-frequencies product estimate)**.** *Let $k \geq 2$, $2-k \leq s \leq k+1$, and $\nabla_x^k g \in L^2 L^\infty(S_I)$ then*

$$\|\Box_{g \geq \lambda^{1/2}} v\|_{L^2 H^{s-1}(S_I)} \lesssim \|\nabla_x^k g\|_{L^2 L^\infty(S_I)} \|\nabla_x v\|_{L^\infty H^{s-1}(S_I)}$$

Proof. It suffices to prove the following

$$\sum_{\lambda=1}^{\infty} \lambda^{2s-2} \|P_\lambda(\Box_{g\geq \lambda^{1/2}} v)\|^2_{L^2 L^2(S_I)} \lesssim \|\nabla_x^k g\|^2_{L^2 L^\infty(S_I)} \sum_{\lambda=1}^{\infty} \lambda^{2s-2} \|\nabla_x P_\lambda v\|^2_{L^\infty L^2(S_I)}$$

which follows, by Hölder inequality, form the following fixed-time estimate

$$\sum_{\lambda=1}^{\infty} \lambda^{2s-2} \|P_\lambda(P_{\geq \lambda^{1/2}} g\ v)(t)\|^2_{L^2(\mathbb{R}^n)} \lesssim \|\nabla_x^k g(t)\|^2_{L^\infty(\mathbb{R}^n)} \sum_{\lambda=1}^{\infty} \lambda^{2s-4} \|P_\lambda v(t)\|^2_{L^2(\mathbb{R}^n)}$$

Indeed notice that $P_\lambda(\Box_{g\geq\lambda^{1/2}} v) = P_\lambda(P_{\geq \lambda^{1/2}} g^{\alpha\beta} \partial_\alpha \partial_\beta v)$ and $P_\lambda \partial_\alpha \partial_\beta v \approx \lambda \nabla P_\lambda v$, since $g^{00} = 1$, and thus both derivatives can not be two time derivatives. Let us take the Littlewood-Paley decomposition of both factors of the left-hand-side:

$$\begin{aligned} \sum_{\lambda=1}^{\infty} \lambda^{2s-2} \|P_\lambda(P_{\geq \lambda^{1/2}} g\ v)(t)\|^2_{L^2(\mathbb{R}^n)} &\approx \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{\infty} \sum_{\nu=\lambda^{1/2}}^{\infty} \lambda^{2s-2} \|P_\lambda(P_\nu g P_\mu v)(t)\|^2_{L^2(\mathbb{R}^n)} \\ &\lesssim \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{\infty} \sum_{\nu=\lambda^{1/2}}^{\infty} \lambda^{2s-2} \nu^{-2k} \|P_\nu \nabla_x^k g(t)\|^2_{L^\infty(\mathbb{R}^n)} \|P_\mu v(t)\|^2_{L^2(\mathbb{R}^n)} \end{aligned}$$

However, notice that the sum is nonzero only in the following three regimes:

i. $\nu \ll \lambda \approx \mu$. Then since $\nu^{-2k} \le \lambda^{-k}$ because of $\nu \ge \lambda^{1/2}$ we obtain

$$\sum_{\lambda=1}^{\infty}\sum_{\nu=\lambda^{1/2}}^{\lambda/8} \lambda^{2s-2-k}\|P_\nu \nabla_x^k g(t)\|_\infty^2 \|P_\lambda v(t)\|_2^2 \lesssim \|\nabla_x^k g(t)\|_\infty^2 \sum_{\lambda=1}^{\infty} \lambda^{2s-4}\|P_\lambda v(t)\|_2^2$$

Because the ν sum is finite.

ii. $\nu \approx \lambda \gg \mu$. Since $s \le k+1$ we obtain

$$\begin{aligned}\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{8\lambda} \lambda^{2s-2-2k}\|P_\lambda \nabla_x^k g(t)\|_\infty^2 \|P_\mu v(t)\|_2^2 &\lesssim \|\nabla_x^k g(t)\|_\infty^2 \sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{8\lambda} \mu^{2s-2-2k}\|P_\mu v(t)\|_2^2 \\ &\lesssim \|\nabla_x^k g(t)\|_\infty^2 \sum_{\mu=1}^{\infty} \mu^{2s-4}\|P_\mu v(t)\|_2^2\end{aligned}$$

iii. $\nu \approx \mu \gg \lambda$. Since $s \ge 2-k$ we obtain

$$\sum_{\lambda=1}^{\infty}\sum_{\nu=8\lambda}^{\infty} \lambda^{2s-2}\nu^{-2k}\|\nabla_x^k P_\nu g(t)\|_\infty^2 \|P_\nu v(t)\|_2^2 \lesssim \|\nabla_x^k g(t)\|_\infty^2 \Big(\sum_{\nu=1}^{\infty} \nu^{2s-4}\|P_\nu v(t)\|_2^2\Big)\Big(\sum_{\lambda=1}^{\infty}\lambda^{-2k+2}\Big)$$

□

We prove a corresponding energy-type inequality for the $X^{s,\theta}$ space, which is built on Lemma 2.11 in [30].

Proposition 63 (Linear estimate)**.** *Let* $k \ge 2$, $2-k \le s \le k+1$, $\theta \in (0,1)$, $\nabla_x^k g \in L^2 L^\infty(S_I)$, $\nabla_{t,x} g \in L^1 L^\infty(S_I)$. *Suppose that* u *is the solution to the Cauchy problem* (2.32)*, then* u *verifies*

$$\|u\|_{X^{s,\theta}[I]} \lesssim \|(u_0,u_1)\|_{H^s\times H^{s-1}} + \|\Box_g u\|_{L^2 H^{s-1}(S_I)}$$

Proof. We exploit the fact that in the definition of the $X^{s,\theta}$ norm one has to take the infimum over all possible decompositions, hence we can concentrate all the modulations on the $d=1$ term. Precisely set

$$u_{\lambda,d} = \begin{cases} \widetilde{P}_\lambda u & \text{if } d=1 \\ 0 & \text{if } d \neq 1 \end{cases}$$

Then we obtain

$$\begin{aligned}
&\|u\|^2_{X^{s,\theta}[I]} \lesssim \sum_{\lambda=1}^{\infty} \|\widetilde{P}_\lambda u\|^2_{X^{s,\theta}_{\lambda,1}[I]} \\
&\lesssim \sum_{\lambda=1}^{\infty} \lambda^{2s}\|\widetilde{P}_\lambda u\|^2_{L^2L^2(S_I)} + \lambda^{2s-2}\|\Box_{g<\lambda^{1/2}}\widetilde{P}_\lambda u\|^2_{L^2L^2(S_I)} \\
&\lesssim \|u\|^2_{L^2H^s(S_I)} + \sum_{\lambda=1}^{\infty} \lambda^{2s-2}(\|[\Box_{g<\lambda^{1/2}}, \widetilde{P}_\lambda]u\|^2_{L^2L^2(S_I)} + \|\widetilde{P}_\lambda \Box_{g<\lambda^{1/2}} u\|^2_{L^2L^2(S_I)}) \\
&\lesssim \|\nabla_{t,x} u\|^2_{L^2H^{s-1}(S_I)} + \sum_{\lambda=1}^{\infty} \lambda^{2s-2}(\|[\Box_{g<\lambda^{1/2}}, \widetilde{P}_\lambda]u\|^2_{L^2L^2(S_I)} + \|\Box_{g\geq\lambda^{1/2}} u\|^2_{L^2H^{s-1}(S_I)} + \|\Box_g u\|^2_{L^2H^{s-1}(S_I)})
\end{aligned}$$

The third term is controlled by the high-frequency product estimate (Lemma 62), moreover let us apply the commutator estimate from Lemma 60 to control the second term, observe that

$$\sum_{\lambda=1}^{\infty} \lambda^{2s-2}\|[\Box_{g<\lambda^{1/2}}, \widetilde{P}_\lambda]u\|^2_{L^2L^2(S_I)} \lesssim \sum_{\lambda=1}^{\infty} \lambda^{2s-2}\|\nabla_x^k g\|^2_{L^2L^\infty(S_I)}\|\nabla_{t,x}\widetilde{P}_\lambda u\|^2_{L^\infty L^2(S_I)} \lesssim \|\nabla_{t,x} u\|^2_{L^\infty H^{s-1}(S_I)}$$

To close we control the remaining terms via the classical energy estimate

$$\|\nabla_{t,x} u\|_{L^2H^{s-1}(S_I)} + \|\nabla_{t,x} u\|_{L^\infty H^{s-1}(S_I)} \lesssim \|(u_0,u_1)\|_{H^s\times H^{s-1}} + \|\Box_g u\|_{L^2H^{s-1}(S_I)}$$

which holds on a finite in time interval and the constant depends on the length of the time interval I considered. □

Corollary 64. *Let $\partial_t^h\partial_x^\alpha g \in L^2L^\infty$ for every $h+|\alpha| \leq 2$ and for $h=0$ and $|\alpha| = k \geq 2$. Suppose $2-k \leq s \leq k+1$ and $1/2 < \theta < 1$, then the homogeneous operator satisfies*

$$\|\mathscr{H}(u_0,u_1)\|_{X^{s,\theta}[I]} \lesssim \|(u_0,u_1)\|_{H^s\times H^{s-1}}$$

Moreover $\|\mathscr{H}\|_{\mathscr{L}} \approx |I|$.

We conclude this section by analysing the properties of the curved d'Alembert operator between $X^{s,\theta}$ spaces. The following proposition is based on Proposition 3.1 in [28].

Proposition 65 (□ estimate)**.** *Let $\partial_t^h\partial_x^\alpha g \in L^2L^\infty$ for every $h+|\alpha| \leq 2$ and for $h=0$ and $|\alpha| = k \geq 2$. Suppose $2-k \leq s \leq k+1$ and $1/2 < \theta < 1$, then we have*

$$\|\Box_g u\|_{X^{s-1,\theta-1}[I]} \lesssim \|u\|_{X^{s,\theta}[I]}$$

Proof. Let us decompose $u = \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} P_\lambda u_{\lambda,d}$, and split the curved d'Alembert operator $\Box_g$ into its low and high frequencies components. We obtain

$$\Box_g u = \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \Box_{g\geq\lambda^{1/2}} P_\lambda u_{\lambda,d} + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \Box_{g<\lambda^{1/2}} P_\lambda u_{\lambda,d}$$

The first term will play the role of f_0 in the definition of $X^{s-1,\theta-1}$ space, and $f_{\lambda,d} = u_{\lambda,d}$. Therefore we obtain

$$\|\Box_g u\|^2_{X^{s-1,\theta-1}[I]} \leq \Big\| \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \Box_{g\geq\lambda^{1/2}} P_\lambda u_{\lambda,d}\Big\|^2_{L^2H^{s-1}(S_I)} + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}[I]}$$

To control the first term we apply Lemma 62: observe that the function $\Box_{g\geq\lambda^{1/2}} P_\lambda u_{\lambda,d}$ has support localized around frequency λ, thus we have

$$\begin{aligned}
\Big\| \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \Box_{g\geq\lambda^{1/2}} P_\lambda u_{\lambda,d}\Big\|^2_{L^2H^{s-1}(S_I)} &\lesssim \sum_{\mu=1}^{\infty} \mu^{2s-2}\Big(\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \|P_\mu(\Box_{g\geq\lambda^{1/2}} P_\lambda u_{\lambda,d})\|_{L^2L^2(S_I)}\Big)^2 \\
&\lesssim \sum_{\lambda=1}^{\infty} \lambda^{2s-2}\|\sum_{d=1}^{\lambda} \Box_{g\geq\lambda^{1/2}} P_\lambda u_{\lambda,d}\|^2_{L^2H^{s-1}(S_I)} \\
&\lesssim \sum_{\lambda=1}^{\infty} \lambda^{2s-2}\|\sum_{d=1}^{\lambda} \nabla_x P_\lambda u_{\lambda,d}\|^2_{L^\infty H^{s-1}(S_I)} \\
&\lesssim \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \lambda^{2s-2} d^{2\theta-1}\|\nabla_{t,x} P_\lambda u_{\lambda,d}\|^2_{L^\infty L^2(S_I)}
\end{aligned}$$

Notice that we have used Cauchy-Schwarz inequality in the modulation sum over d which requires $\theta > 1/2$. Thus by Proposition 57 the proof is completed. □

As next Proposition shows, one can prove that the inhomogeneous map is bounded form $X^{s-1,\theta-1}$ to $X^{s,\theta}$. The proof is build on the proof of Lemma 2.12 of [30].

Proposition 66 (Inhomogeneous linear estimate)**.** *Let $\partial_t^h\partial_x^\alpha g \in L^2L^\infty$ for every $h+|\alpha| \leq 2$ and for $h = 0$ and $|\alpha| = k \geq 2$. Suppose $2-k \leq s \leq k+1$ and $1/2 < \theta < 1$, then the inhomogeneous operator satisfies*

$$\|\Box_g^{-1} f\|_{X^{s,\theta}[I]} \lesssim \|f\|_{X^{s-1,\theta-1}[I]}$$

Proof. Let $f \in X^{s-1,\theta-1}[I]$, we shall show that the solution to the inhomogeneous wave equation with forcing term f and zero initial data belongs to $X^{s,\theta}$ for a small interval of time. Since $f \in X^{s-1,\theta-1}[I]$ there exists $f_0 \in L^2H^{s-1}(S_I)$ and $f_{\lambda,d} \in X^{s,\theta}_{\lambda,d}[I]$ such that $f = f_0 + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \Box_{g<\lambda^{1/2}} P_\lambda f_{\lambda,d}$. Let us define the function $u = \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} P_\lambda f_{\lambda,d}$ which clearly is in $X^{s,\theta}[I]$. Furthermore, define $v = u - \Box_g^{-1} f$, if we prove that $v \in X^{s,\theta}[I]$, then also $\Box_g^{-1} f$ will be in $X^{s,\theta}[I]$ and the proposition will follows. Observe that v satisfies the Cauchy problem

$$\begin{cases} \Box_g v = \Box_g u - f \\ (v, v_t)(0) = (u(0), u_t(0)) \end{cases}$$

Therefore by the linear estimate Proposition 63 we have

$$\begin{aligned}\|v\|_{X^{s,\theta}[I]} &\lesssim \|(u,u_t)(0)\|_{H^s\times H^{s-1}} + \|\Box_g u - f\|_{L^2H^{s-1}(S_I)}\\ &\lesssim \|\nabla_{t,x}u\|_{L^\infty H^{s-1}(S_I)} + \|f_0\|_{L^2H^{s-1}(S_I)} + \|\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\Box_{g\geq\lambda^{1/2}}P_\lambda f_{\lambda,d}\|_{L^2H^{s-1}(S_I)}\end{aligned}$$

Notice that form the embedding $X^{s,\theta}\subset C^0(I,H^s)\cap C^1(I,H^{s-1})$, which holds for $1/2<\theta<1$, we have $\|\nabla u\|_{L^\infty H^{s-1}(S_I)}\lesssim\|u\|_{X^{s,\theta}}$. Let us define $f_\lambda=\sum_{d=1}^{\lambda}f_{\lambda,d}$, then we apply Proposition 57 to handle the last term:

$$\|\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\Box_{g\geq\lambda^{1/2}}P_\lambda f_{\lambda,d}\|^2_{L^2H^{s-1}(S_I)}\lesssim\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\lambda^{2s-2}d^{2\theta-1}\|\nabla_{t,x}P_\lambda u_{\lambda,d}\|^2_{L^\infty L^2(S_I)}\lesssim\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|f_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}}$$

Observe that we have applied Cauchy-Schwarz inequality in the modulation sum over d, this requires $\theta>1/2$. Therefore the $X^{s,\theta}[I]$ norm of v is bounded. □

2.7 Strichartz estimates for $X^{s,\theta}$ spaces

In this section we collect Strichartz estimates for $X^{s,\theta}$ spaces. Hereafter we suppose $\partial_t^h\partial_x^\alpha g\in L^2L^\infty(S_I)$ for every $h+|\alpha|\leq 2$. Recall that (σ,p,q) is a *Strichartz triplet* if the following conditions are fulfilled:

$$2\leq p\leq\infty,\qquad\text{and}\qquad 2\leq q\leq\infty$$
$$\frac{2}{p}+\frac{n-1}{q}\leq\frac{n-1}{2}$$
$$\sigma=\frac{n}{2}-\frac{1}{p}-\frac{n}{q}$$

and when $n=3$ then $(p,q,s)\neq(2,\infty,1)$. The strichartz estimates for the variable coefficient wave equation have the form

$$\||D|^{-\sigma}\nabla_{t,x}u\|_{L^pL^q(S_I)}\lesssim\|(u,u_t)(0)\|_{H^1\times L^2}+\|\Box_{g<\lambda^{1/2}}u\|_{L^1L^2(S_I)}$$

where I is a bounded time interval containing the origin and $S_I=I\times\mathbb{R}^n$. The preceding estimate follows form Corollary 1.5 in the previous work by Tataru [115]. The reason why we include the gradient is to have also a bound for u_t. In a flat spacetime this version of Strichartz estimate is equivalent to the classical one:

$$\|u\|_{L^pL^q(S_I)}\lesssim\|(u,u_t)(0)\|_{H^\sigma\times H^{\sigma-1}}+\|\Box u\|_{L^{\tilde{q}'}L^{\tilde{r}'}(S_I)}$$

where $(1-\sigma,\tilde{p},\tilde{q})$ is another Strichartz triplet. By the same argument used in the proof of the energy estimate, Lemma 61, we obtain the following version of the Strichartz estimates:

$$\||D|^{-\sigma}\nabla_{t,x}u\|^2_{L^pL^q(S_I)}\lesssim|I|^{-1}\|\nabla_{t,x}u\|^2_{L^2L^2(S_I)}+\|\nabla_{t,x}u\|_{L^2L^2(S_I)}\|\Box_{g<\lambda^{1/2}}u\|_{L^2L^2(S_I)}\tag{2.33}$$

We are now ready to prove the embedding property of the wave-Sobolev spaces into the Strichartz spaces.

Proposition 67 (Strichartz estimate for $X^{s,\theta}$ space)**.** *Let (σ, p, q) a Strichartz triplet, $s \geq \sigma$, and $1/2 < \theta < 1$, then $X^{s,\theta}[I] \subset L^p L^q(S_I)$. In fact we have the estimate:*

$$\|u\|_{L^p L^q(S_I)} \lesssim \|u\|_{X^{s,\theta}}$$

Proof. Let $u = \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} P_\lambda u_{\lambda,d}$ then it suffices to prove the bound

$$\|u\|^2_{L^p L^q(S_I)} \lesssim \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}}$$

Notice that by the decomposition of u and Lemma 68 below we obtain

$$\begin{aligned} \|u\|^2_{L^p L^q(S_I)} &\lesssim \sum_{\lambda=1}^{\infty} \Big(\sum_{d=1}^{\lambda} \lambda^{2s-2-2\sigma} d^{2\theta-1} \|P_\lambda \nabla_{t,x} u_{\lambda,d}\|^2_{L^p L^q(S_I)} \Big) \Big(\sum_{d=1}^{\lambda} \lambda^{-2s+2\sigma} d^{1-2\theta} \Big) \\ &\lesssim \sum_{\lambda=1}^{\infty} \Big(\sum_{d=1}^{\lambda} \|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}} \Big) \Big(\sum_{d=1}^{\lambda} \lambda^{-2s+2\sigma} d^{1-2\theta} \Big) \end{aligned}$$

Now use the fact that $s \geq \sigma$ and $\theta > 1/2$ to obtain the desired estimate. □

All it remains to prove is the following lemma, which is reminiscent of Proposition 57.

Lemma 68 (Strichartz estimate for $X^{s,\theta}_{\lambda,d}$ space)**.** *Let $s \in \mathbb{R}$ and $\theta \in (0,1)$, and (σ, p, q) a Strichartz triplet, then for any dyadic number λ and $d \gtrsim |I|^{-1}$ we have*

$$\lambda^{2s-2-2\sigma} d^{2\theta-1} \|\nabla_{t,x} P_\lambda v\|^2_{L^p L^q(S_I)} \lesssim \|v\|^2_{X^{s,\theta}_{\lambda,d}[I]}$$

Proof. It follows easily form the Strichartz estimate (2.33) that

$$\begin{aligned} \lambda^{2s-2-2\sigma} d^{2\theta-1} \|\nabla_{t,x} P_\lambda v\|^2_{L^p L^q(S_I)} &\lesssim \lambda^{2s-2} d^{2\theta-1} \||D|^{-\sigma} \nabla_{t,x} P_\lambda v\|^2_{L^p L^q(S_I)} \\ &\lesssim \lambda^{2s-2} d^{2\theta-1} |I|^{-1} \|\nabla_{t,x} P_\lambda u\|^2_{L^2 L^2(S_I)} \\ &+ \lambda^{2s-2} d^{2\theta-2} \|\nabla_{t,x} P_\lambda u\|_{L^2 L^2(S_I)} \|\Box_{g<\lambda^{1/2}} P_\lambda u\|_{L^2 L^2(S_I)} \end{aligned}$$

From the equivalence of the $X^{s,\theta}_{\lambda,d}$ and $\overline{X}^{s,\theta}_{\lambda,d}$ norms (Lemma 59) allow us to reach the thesis. □

Next we discuss Strichartz estimate for non-admissible triple. Let us define a *non-admissible triple* to be (σ, p, q) such that

$$2 \leq p \leq \infty, \qquad \text{and} \qquad 2 \leq q \leq \infty$$

$$\frac{2}{p}+\frac{n-1}{q}\geq\frac{n-1}{2}$$

$$\sigma=\frac{n}{2}-\frac{1}{p}-\frac{n}{q}$$

Notice that we only have reversed the inequality.

Lemma 69 (Strichartz estimate for non-admissible triple)**.** *Let $s\in\mathbb{R}$ and $\theta\in(0,1)$, and (σ,p,q) a non-admissible triple, then for any dyadic number λ and $d\gtrsim|I|^{-1}$ we have*

$$\lambda^{2s-2-2\sigma-(\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2})}d^{2\theta-1+(\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2})}\|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)}\lesssim\|v\|^2_{X^{s,\theta}_{\lambda,d}[I]}$$

Proof. We interpolate between an admissible pair and the L^2L^2 bound

$$\|\nabla_{t,x}P_\lambda v\|_{L^2L^2(S_I)}\lesssim\lambda^{1-s}d^{-\theta}\|v\|_{X^{s,\theta}_{\lambda,d}[I]}$$

which follows from Lemma 59. To find the right admissible pair simply trace the line between $(1/2,1/2)$ and $(1/p,1/q)$ and take the first point at the edge of the admissibility region. In Figures 2.2, 2.3, 2.4 below we explain this concept. Define $\alpha=\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2}$ the distance from the non admissible pair to the sharp admissible line. Notice that $0\leq\alpha\leq1$. Define the sharp admissible pair $(\tilde{p},\tilde{q})$ such that

$$\frac{1}{p}=\frac{\alpha}{2}+\frac{1-\alpha}{\tilde{p}},\qquad\frac{1}{q}=\frac{\alpha}{2}+\frac{1-\alpha}{\tilde{q}}$$

Since $(\tilde{p},\tilde{q})$ is sharp admissible the Strichartz estimates for the admissible pair tell us that

$$\|\nabla_{t,x}P_\lambda v\|_{L^{\tilde{p}}L^{\tilde{q}}(S_I)}\lesssim\lambda^{1-s+\tilde{\sigma}}d^{\frac{1}{2}-\theta}\|v\|_{X^{s,\theta}_{\lambda,d}[I]}$$

where $\tilde{\sigma}=\frac{n}{2}-\frac{1}{\tilde{p}}-\frac{n}{\tilde{q}}$. By interpolation we obtain

$$\begin{aligned}\|\nabla_{t,x}P_\lambda v\|_{L^pL^q(S_I)}&\lesssim(\lambda^{1-s}d^{-\theta})^{\alpha}(\lambda^{1-s+\tilde{\sigma}}d^{\frac{1}{2}-\theta})^{1-\alpha}\|v\|_{X^{s,\theta}_{\lambda,d}[I]}\\&\lesssim\lambda^{1-s+(1-\alpha)\tilde{\sigma}}d^{\frac{1}{2}-\theta-\frac{\alpha}{2}}\|v\|_{X^{s,\theta}_{\lambda,d}[I]}\\&\lesssim\lambda^{1-s+\sigma+\frac{\alpha}{2}}d^{\frac{1}{2}-\theta-\frac{\alpha}{2}}\|v\|_{X^{s,\theta}_{\lambda,d}[I]}\end{aligned}$$

since $(1-\alpha)\tilde{\sigma}=\sigma+\frac{\alpha}{2}$.

□

Let us define additional norms that allow us to work with functions concentrated on a smaller modulation range. Define

$$\|u\|^2_{X^{s,\theta}_{\lambda,<d}}=\inf\Big\{\sum_{d_1\leq d}\|u_{d_1}\|^2_{X^{s,\theta}_{\lambda,d_1}}:u=\sum_{d_1\leq d}u_{d_1}\Big\}$$

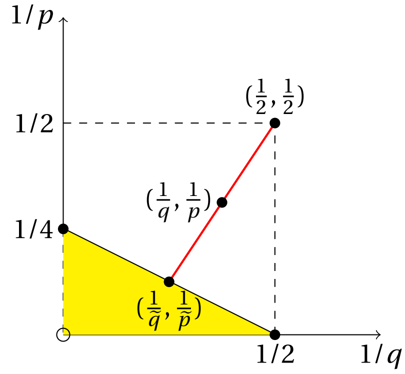


Figure 2.2: $n = 2$

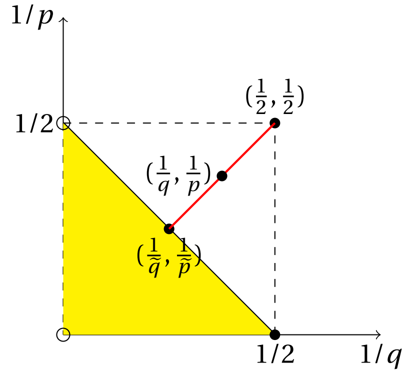


Figure 2.3: $n = 3$

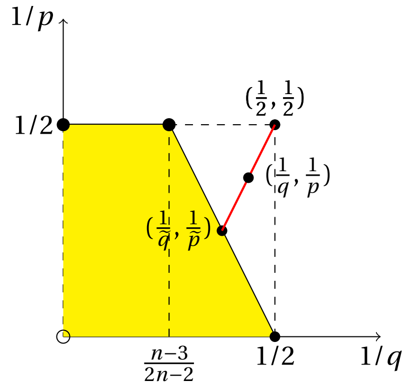


Figure 2.4: $n \geq 4$

and

$$\|u\|^2_{X^{s,\theta}_\lambda} = \inf\Big\{\sum_{d\leq\lambda}\|u_d\|^2_{X^{s,\theta}_{\lambda,d}} : u = \sum_{d\leq\lambda} u_d\Big\}$$

Notice that $\|u\|_{X^{s,\theta}_\lambda} \leq \|u\|_{X^{s,\theta}_{\lambda,\lambda}}$; it suffices to take $u_d = u$ if $d = \lambda$ and $u_d = 0$ if $d \neq \lambda$. In general if $\lambda_1 \leq \lambda_2$ then $\|u\|_{X^{s,\theta}_{\lambda_2}} \leq \|u\|_{X^{s,\theta}_{\lambda_2,\lambda_1}}$; just take $u_d = u$ if $d = \lambda_1$ and $u_d = 0$ if $d \neq \lambda_1$. The corresponding version for $X^{s,\theta}_{\lambda,<\lambda_1}$ is $\|u\|_{X^{s,\theta}_{\lambda_2,<\lambda_1}} \leq \|u\|_{X^{s,\theta}_{\lambda_2,d}}$ if $d \leq \lambda_1$. Another useful property of such norms is that they can be used to compute the $X^{s,\theta}$ norm:

$$\|u\|^2_{X^{s,\theta}} = \inf\Big\{\sum_\lambda \|u_\lambda\|^2_{X^{s,\theta}_\lambda} : u = \sum_\lambda P_\lambda u_\lambda\Big\}$$

In fact, we have

$$\begin{aligned}\|u\|^2_{X^{s,\theta}} &= \inf_{u=\sum_\lambda\sum_{d\leq\lambda}P_\lambda u_{\lambda,d}} \sum_\lambda\sum_{d\leq\lambda}\|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}} = \inf_{u=\sum_\lambda P_\lambda u_\lambda}\sum_\lambda \inf_{u_\lambda=\sum_{d\leq\lambda}u_{\lambda,d}}\sum_{d\leq\lambda}\|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}}\\ &= \inf_{u=\sum_\lambda P_\lambda u_\lambda}\sum_\lambda\|u_\lambda\|^2_{X^{s,\theta}_\lambda}\end{aligned}$$

The $X^{s,\theta}_\lambda$ space are useful to study multiplicative properties of $X^{s,\theta}$ spaces. In oder to transfer estimates form $X^{s,\theta}_{\lambda,d}$ to $X^{s,\theta}_{\lambda,<d}$ or $X^{s,\theta}_\lambda$ we need the following lemma.

Lemma 70 (Transfer principle)**.** *Suppose that* $\|u\|_N \lesssim d^{1/2-\theta}\|u\|_{X^{s,\theta}_{\lambda,d}}$ *for some norm* $\|\cdot\|_N$*,*

- *if* $\theta > 1/2$ *then* $\|u\|_N \lesssim \|u\|_{X^{s,\theta}_{\lambda,<d_1}}$ *; moreover if* $d_1 = \lambda$*, then* $\|u\|_N \lesssim \|u\|_{X^{s,\theta}_\lambda}$*,*
- *if* $\theta < 1/2$ *then* $\|u\|_N \lesssim d_1^{1/2-\theta}\|u\|_{X^{s,\theta}_{\lambda,<d_1}}$ *; moreover if* $d_1 = \lambda$*, then* $\|u\|_N \lesssim \lambda^{1/2-\theta}\|u\|_{X^{s,\theta}_\lambda}$*.*

Proof. It is an application of Cauchy-Schwarz: decompose $u = \sum_{d\leq d_1} u_d$, then

$$\begin{aligned}\|u\|^2_N &\lesssim \Big(\sum_{d\leq d_1} d^{1/2-\theta}\|u_d\|_{X^{s,\theta}_{\lambda,d}}\Big)^2\\ &\lesssim \Big(\sum_{d\leq d_1}\|u_d\|^2_{X^{s,\theta}_{\lambda,d_1}}\Big)\Big(\sum_{d\leq d_1} d^{1-2\theta}\Big)\end{aligned}$$

The second serie is bounded if $\theta > 1/2$, whereas if $\theta < 1/2$ we can control it by $d_1^{1-2\theta}$. □

As an application of the previous lemma we can extend Strichartz estimates of Lemma 68 to $X^{s,\theta}_{\lambda,<d}$ and $X^{s,\theta}_\lambda$ spaces.

Corollary 71 (Strichartz estimates for $X^{s,\theta}_{\lambda,<d}$ space)**.** *Suppose* $s \in \mathbb{R}$ *and* (σ, p, q) *a Strichartz triple.*

i. Assume that $1/2 < \theta < 1$ *then*

$$\|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)} \lesssim \lambda^{2-2s+2\sigma}\|v\|^2_{X^{s,\theta}_\lambda[I]}, \qquad \|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)} \lesssim \lambda^{2-2s+2\sigma}\|v\|^2_{X^{s,\theta}_{\lambda,<d}[I]}$$

ii. Assume that $0 < \theta < 1/2$ *then*

$$\|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)} \lesssim \lambda^{3-2s-2\theta+2\sigma}\|v\|^2_{X^{s,\theta}_\lambda[I]}, \qquad \|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)} \lesssim \lambda^{2-2s+2\sigma}d^{1-2\theta}\|v\|^2_{X^{s,\theta}_{\lambda,<d}[I]}$$

Furthermore as a consequence of the transfer principle and Strichartz estimates for non admissible pairs we have the following.

Corollary 72 (Strichartz estimates for $X^{s,\theta}_{\lambda,<d}$, non-admissible triple)**.** *Suppose* $s \in \mathbb{R}$ *and* (σ, p, q) *a non-admissible Strichartz triple.*

i. Assume that $\frac{1}{2} - \frac{1}{2}\left(\frac{2}{p} + \frac{n-1}{q} - \frac{n-1}{2}\right) < \theta < 1$ *then*

$$\|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)} \lesssim \lambda^{2-2s+2\sigma+(\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2})}\|v\|^2_{X^{s,\theta}_\lambda[I]}$$

$$\|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)} \lesssim \lambda^{2-2s+2\sigma+(\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2})}\|v\|^2_{X^{s,\theta}_{\lambda,<d}[I]}$$

ii. Assume that $0 < \theta < \frac{1}{2} - \frac{1}{2}\left(\frac{2}{p} + \frac{n-1}{q} - \frac{n-1}{2}\right)$ *then*

$$\|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)} \lesssim \lambda^{3-2s-2\theta+2\sigma}\|v\|^2_{X^{s,\theta}_\lambda[I]}$$

$$\|\nabla_{t,x}P_\lambda v\|^2_{L^pL^q(S_I)} \lesssim \lambda^{2-2s+2\sigma+(\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2})}d^{1-2\theta-(\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2})}\|v\|^2_{X^{s,\theta}_{\lambda,<d}[I]}$$

2.8 Bilinear estimates and wave maps on curved space-times

The aim of this section is to prove some multiplicative properties of $X^{s\theta}$ spaces, namely the algebra and the asymmetric estimates. As a byproduct of such bilinear bounds we obtain a sharp local well-posedness result for subcritical wave maps on curved background:

Theorem 73. *Let* $n \geq 3$, $n/2 < s \leq n/2+1$, *and suppose that the metric coefficients satisfies* $\partial_t^h\partial_x^\alpha g \in L^2L^\infty(S_I)$ *for every* $h+|\alpha| \leq 2$ *and for* $h = 0$ *and* $|\alpha| = n/2$, *where* I *is a bounded time interval containing the origin and* $S_I = I \times \mathbb{R}^n$. *Then there exist an unique local solution* $u \in C([0,T],H^s) \cap C^1([0,T],H^{s-1})$ *to the Cauchy problem for wave maps with curved background metric* g*:*

$$\begin{cases} \Box_g u^i = \Gamma^i_{jk}(u)\partial_\alpha u^j\partial^\alpha u^k \\ (u,\partial_t u)|_{t=0} = (u_0,u_1) \in H^s(\mathbb{R}^n,\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^n,\mathbb{R}^N) \end{cases}$$

where the Christoffel symbols Γ^i_{jk} *are smooth functions.*

This theorem extends the previous work by Geba [28] to dimensions $n \geq 6$. Our theorem, together with the result by Gavrus, Jao and Tataru [27], where the more difficult $n = 2$ case is treated, settle the local well-posedness theory for scaling subcritical wave maps equation on curved backgrounds. Observe that the upper bound $s \leq n/2+1$ is not restrictive since if $s > n/2+1$ then one can run an energy type argument to easily obtain local well-posedness. On the other hand, the lower bound $n/2 < s$ is dictated by scaling considerations. The proof of Theorem 73 hinges on a fixed point argument based on the following estimates:

i. Linear estimate: the homogeneous and inhomogeneous solution operator satisfies

$$\|\mathscr{H}(u_0,u_1) + \Box_g^{-1} f\|_{X^{s,\theta}[I]} \lesssim \|(u_0,u_1)\|_{H^s \times H^{s-1}} + \|f\|_{X^{s-1,\theta-1}[I]}$$

ii. Embedding into solution space:

$$\|\nabla_{t,x} u\|_{L^\infty H^s(S_I)} \lesssim \|u\|_{X^{s,\theta}}$$

iii. Curved d'Alembert operator estimate:

$$\|\Box_g u\|_{X^{s-1,\theta-1}[I]} \lesssim \|u\|_{X^{s,\theta}[I]}$$

iv. Algebra property:

$$\|uv\|_{X^{s,\theta}[I]} \lesssim \|u\|_{X^{s,\theta}[I]} \|v\|_{X^{s,\theta}[I]}$$

v. Asymmetric bilinear estimate:

$$\|uv\|_{X^{s-1,\theta-1}[I]} \lesssim \|u\|_{X^{s,\theta}[I]} \|v\|_{X^{s-1,\theta-1}[I]}$$

vi. Moser estimate: let f, g positive increasing continuous functions, then

$$\|\Gamma(u)\|_{X^{s,\theta}[I]} \lesssim f(\|u\|_{L^\infty(S_I)}) g(\|u\|_{X^{s,\theta}[I]})$$

We already discuss the first three estimates in §2.6. Notice that on top of the conditions $\partial_t^h \partial_x^\alpha g \in L^2 L^\infty(S_I)$ for every $h + |\alpha| \leq 2$, Theorem 73 requires that up to $n/2$ spatial derivatives of the metrics lies in $L^2 L^\infty(S_I)$, that is $\nabla_x^{n/2} g \in L^2 L^\infty(S_I)$. These latter regularity conditions are imposed by the linear theory. However, as we shall see for the nonlinear theory the condition $\partial_t^h \partial_x^\alpha g \in L^2 L^\infty(S_I)$ for every $h + |\alpha| \leq 2$ suffices. Let us assume the previous estimates $iv. - vi.$ for now, the proof of Theorem 73 follows closely the one for flat space-times.

Proof. Recall the relationship between the null-form $N_0(u,v) = \partial_\alpha u \partial^\alpha v$, here we are raising and lowering the indices with respect to the metric g, and the curved d'Alembert operator: $N_0(u,v) = \frac{1}{2}[\Box_g(uv) - v\Box_g u - u\Box_g v]$. Therefore by estimates $i. - vi.$ we obtain the bound for the nonlinearity:

$$\begin{aligned}
&\|\Gamma(u)N_0(u,v)\|_{X^{s-1,\theta-1}[I]}\\
&\lesssim \|\Gamma(u)\|_{X^{s,\theta}[I]}\big(\|\Box_g(uv)\|_{X^{s-1,\theta-1}[I]}+\|v\Box_g u\|_{X^{s-1,\theta-1}[I]}+\|u\Box_g v\|_{X^{s-1,\theta-1}[I]}\big)\\
&\lesssim h(\|u\|_{X^{s,\theta}[I]})\big(\|uv\|_{X^{s,\theta}[I]}+\|v\|_{X^{s,\theta}[I]}\|\Box_g u\|_{X^{s-1,\theta-1}[I]}+\|u\|_{X^{s,\theta}[I]}\|\Box_g v\|_{X^{s-1,\theta-1}[I]}\big)\\
&\lesssim h(\|u\|_{X^{s,\theta}[I]})\|u\|_{X^{s,\theta}}\|v\|_{X^{s,\theta}[I]}
\end{aligned}$$

Notice that we have applied Moser inequality and Sobolev embedding to obtain the bound $\|\Gamma(u)\|_{X^{s,\theta}[I]} \lesssim f(\|u\|_{L^\infty(S_I)})g(\|u\|_{X^{s,\theta}[I]}) \lesssim h(\|u\|_{X^{s,\theta}[I]})$. □

The goal of this section is to prove the last three bounds: $iv.-vi$. Throughout this section we suppose $\partial_t^h\partial_x^\alpha g \in L^2L^\infty$ for every $h+|\alpha|\le 2$. The proof of the algebra property for the $X^{s,\theta}$ spaces relies strongly on Strichartz estimates. We shall need Strichartz estimates which holds in any dimension $n\ge 3$, thus we will rely on the following Strichartz triplets

- $(\sigma,p,q)=(0,\infty,2)$
- $(\sigma,p,q)=(\frac{n}{2}-\frac{1}{2}+\epsilon,2+,\infty)$
- $(\sigma,p,q)=(\frac{n}{2},\infty,\infty)$
- $(\sigma,p,q)=(\frac{n-1}{4},4,4)$

where we denote $1/(2+)=1/2-\epsilon$ and $0<\epsilon\ll 1$ is a small constant. We obtain the following Strichartz estimates

$$\lambda^{2s-2}d^{2\theta-1}\|\nabla P_\lambda v\|^2_{L^\infty L^2(S_I)}+\lambda^{2s-n-1+2\epsilon}d^{2\theta-1}\|\nabla P_\lambda v\|^2_{L^{2+}L^\infty(S_I)}+\lesssim \|v\|^2_{X^{s,\theta}_{\lambda,d}}$$

$$\lambda^{2s-n-2}d^{2\theta-1}\|\nabla P_\lambda v\|^2_{L^\infty L^\infty(S_I)}+\lambda^{2s-\frac{n+3}{2}}d^{2\theta-1}\|\nabla P_\lambda v\|^2_{L^4L^4(S_I)}\lesssim \|v\|^2_{X^{s,\theta}_{\lambda,d}}$$

Notice that when $n\ge 4$ we can take $\epsilon=0$. The following property extends to $n=3$ and $n\ge 5$ spatial dimensions the corresponding algebra algebra of Proposition 3.7 in [30].

Proposition 74. *Suppose that $n\ge 3$, $1/2<\theta<1$, and $s-\theta>(n-1)/2$, then $X^{s,\theta}[I]$ is an algebra.*

Proof. Let $u,v\in X^{s,\theta}[I]$, we have the representations $u=\sum_{\lambda=1}^\infty P_\lambda u_\lambda$ and $v=\sum_{\lambda=1}^\infty P_\lambda v_\lambda$, where $u_\lambda=\sum_{d\le\lambda}u_{\lambda,d}$ and $v_\lambda=\sum_{d\le\lambda}v_{\lambda,d}$. We also have

$$uv=\sum_{\mu=1}^\infty\sum_{\lambda_1=1}^\infty\sum_{\lambda_2=1}^\infty P_\mu\widetilde{P}_\mu(P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2})$$

Therefore using the property of the $X^{s,\theta}_\mu$ space we have

$$\|uv\|^2_{X^{s,\theta}[I]}\le\sum_{\mu=1}^\infty\Big(\sum_{\lambda_1=1}^\infty\sum_{\lambda_2=1}^\infty\|\widetilde{P}_\mu(P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2})\|_{X^{s,\theta}_\mu[I]}\Big)^2$$

By Littlewood-Paley trichotomy we infer that the sum in nonzero only in the following three cases: $\lambda_1 \approx \lambda_2 \gg \mu$, $\lambda_1 \ll \lambda_2$ and $\lambda_2 \approx \mu$, $\lambda_1 \gg \lambda_2$ and $\lambda_1 \approx \mu$. Thus by symmetry we split the proof into two separate parts:

(i) *High-high-low interaction*: $\lambda_1 \approx \lambda_2 \gg \mu$. For simplicity let us denote $\lambda \approx \lambda_1 \approx \lambda_2$, we must estimate the following term:

$$\sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\|P_\lambda u_\lambda P_\lambda v_\lambda\|_{X^{s,\theta}_{\mu}[I]}\Big)^2$$

We take a precise decomposition of the product $P_\lambda u_\lambda P_\lambda v_\lambda$ where all the modulations are concentrated on a single "frequency": define the decomposition $P_\lambda u_\lambda P_\lambda v_\lambda = \sum_{d\le\mu}(P_\lambda u_\lambda P_\lambda v_\lambda)_d$ where we take

$$(P_\lambda u_\lambda P_\lambda v_\lambda)_d = \begin{cases} P_\lambda u_\lambda P_\lambda v_\lambda & \text{if } d=\mu \\ 0 & \text{if } d\neq\mu \end{cases}$$

Then we obtain

$$\begin{aligned}
&\sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\|P_\lambda u_\lambda P_\lambda v_\lambda\|_{X^{s,\theta}_{\mu[I]}}\Big)^2 \qquad (2.34)\\
&\lesssim \sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\sum_{d_1\le\lambda}\sum_{d_2\le\lambda}\|P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2}\|_{X^{s,\theta}_{\mu,\mu}[I]}\Big)^2\\
&\lesssim \sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\sum_{d_1\le\lambda}\sum_{d_2\le\lambda}\mu^{s+\theta}\|P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2}\|_2\Big)^2\\
&+\sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\sum_{d_1\le\lambda}\sum_{d_2\le\lambda}\mu^{s+\theta-2}\|\Box_{g<\mu^{1/2}}(P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2})\|_2\Big)^2
\end{aligned}$$

Let us begin by estimating the first term on the right-hand side of (2.34). We apply Hölder inequality, Cauchy-Schwarz inequality, and the L^4L^4 Strichartz estimate to get

$$\begin{aligned}
&\sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\sum_{d_1\le\lambda}\sum_{d_2\le\lambda}\mu^{s+\theta}\|P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2}\|_2\Big)^2\\
&\lesssim \sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\sum_{d_1\le\lambda}\sum_{d_2\le\lambda}\mu^{s+\theta}\lambda^{-2}\|\nabla P_\lambda u_{\lambda,d_1}\|_4\|\nabla P_\lambda v_{\lambda,d_2}\|_4\Big)^2\\
&\lesssim \sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_2}\mu^{-s+\theta+\frac{n-1}{2}}\lambda^{2s-\frac{n+3}{2}}\|\nabla P_\lambda u_{\lambda,d_1}\|_4\|\nabla P_\lambda v_{\lambda,d_2}\|_4\Big)^2\\
&\lesssim \Big(\sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-\frac{n+3}{2}}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_4^2\Big)\Big(\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{2s-\frac{n+3}{2}}d_2^{2\theta-1}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_4^2\Big)\\
&\lesssim \sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\|u_{\lambda_1,d_1}\|^2_{X^{s,\theta}_{\lambda_1,d_1}[I]}\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\|v_{\lambda_2,d_2}\|^2_{X^{s,\theta}_{\lambda_2,d_2}[I]}
\end{aligned}$$

Notice that to treat the μ-sum we have used the fact that $s-\theta > (n-1)/2$, since $-2s-2+\frac{n+3}{2} < 0$ we have transferred this extra high frequency term into low frequency μ. Moreover notice that we have used the fact that $\theta > 1/2$ and Cauchy-Schwarz once more in the sums over modulations d_1 and d_2. Next, let us turn to the second term on the right-hand side of (2.34). Here we have to distinguish further into two cases: when both derivatives of the $\Box_{g<\mu^{1/2}}$ operator act on a single term and when the derivatives are spread and act on each term once:

$$\|\Box_{g<\mu^{1/2}}(P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2})\|_2 \lesssim \|(\Box_{g<\mu^{1/2}} P_\lambda u_{\lambda,d_1}) P_\lambda v_{\lambda,d_2}\|_2 + \|P_\lambda u_{\lambda,d_1}(\Box_{g<\mu^{1/2}} P_\lambda v_{\lambda,d_2})\|_2 + \|P_{<\mu^{1/2}} g \nabla P_\lambda u_{\lambda,d_1} \nabla P_\lambda v_{\lambda,d_2}\|_2$$

In order to control the last term in the pevious inequality we apply Hölder inequality to obtain the bound

$$\begin{aligned} \|P_{<\mu^{1/2}} g \nabla P_\lambda u_{\lambda,d_1} \nabla P_\lambda v_{\lambda,d_2}\|_2 &\lesssim \|P_{<\mu^{1/2}} g\|_\infty \|\nabla P_\lambda u_{\lambda,d_1}\|_4 \|\nabla P_\lambda v_{\lambda,d_2}\|_4 \\ &\lesssim \mu^{-3/4} \|\nabla^2 P_{<\mu^{1/2}} g\|_{L^2 L^\infty} \|\nabla P_\lambda u_{\lambda,d_1}\|_4 \|\nabla P_\lambda v_{\lambda,d_2}\|_4 \end{aligned}$$

Notice that the Littlewood-Paley cutoff of the metric g is with respect to both space and time variables, hence we can use Bernstein inequality in the time variable: $\|P_{<\mu^{1/2}} g\|_\infty \lesssim \mu^{1/4} \|P_{<\mu^{1/2}} g\|_{L^2 L^\infty} \lesssim \mu^{-3/4} \|\nabla^2 P_{<\mu^{1/2}} g\|_{L^2 L^\infty}$. Therefore one can run the same argument as above with different powers of μ and λ:

$$\mu^{s+\theta-2-3/4} = \mu^{s+\theta-2-3/4} \lambda^{-2s+\frac{n+3}{2}} \lambda^{s-\frac{n+3}{4}} \lambda^{s-\frac{n+3}{4}} \leq \mu^{-s+\theta+\frac{n}{2}-\frac{5}{4}} \lambda^{s-\frac{n+3}{4}} \lambda^{s-\frac{n+3}{4}}$$

Notice that $-2s + (n+3)/2 < 0$ since $n \geq 3$ and $s > n/2$. All it remains to prove, as far as the HHL interaction is concern, is the bound for the two terms containing two derivatives. We vary a little the above argument: we use the $L^\infty L^\infty$ Strichartz estimate to get

$$\begin{aligned} &\sum_{\mu=1}^\infty \Big(\sum_{\lambda \gg \mu} \sum_{d_1 \leq \lambda} \sum_{d_2 \leq \lambda} \mu^{s+\theta-2} \|(\Box_{g<\mu^{2/n}} P_\lambda u_{\lambda,d_1}) P_\lambda v_{\lambda,d_2}\|_2 \Big)^2 \\ &\lesssim \sum_{\mu=1}^\infty \sum_{d=1}^\mu \Big(\sum_{\lambda \gg \mu} \sum_{d_1 \leq \lambda} \sum_{d_2 \leq \lambda} \mu^{s+\theta-2} \lambda^{-1} \|\Box_{g<\lambda^{1/2}} P_\lambda u_{\lambda,d_1}\|_2 \|\nabla P_\lambda v_{\lambda,d_2}\|_\infty \Big)^2 \\ &\lesssim \sum_{\lambda_1=1}^\infty \lambda_1^{2s-3} \Big(\sum_{d_1 \leq \lambda_1} \|\Box_{g<\lambda^{1/2}} P_{\lambda_1} u_{\lambda_1,d_1}\|_2 \Big)^2 \sum_{\lambda_2=1}^\infty \lambda_2^{2s-n-2} \Big(\sum_{d_2 \leq \lambda_2} \|\nabla P_{\lambda_2} v_{\lambda_2,d_2}\|_\infty \Big)^2 \\ &\lesssim \sum_{\lambda_1=1}^\infty \sum_{d_1 \leq \lambda_1} \lambda_1^{2s-2} d_1^{2\theta-2} \|\Box_{g<\lambda^{1/2}} P_{\lambda_1} u_{\lambda_1,d_1}\|_2^2 \sum_{\lambda_2=1}^\infty \sum_{d_2 \leq \lambda_2} \lambda_2^{2s-n-2} d_2^{2\theta-1} \|\nabla P_{\lambda_2} v_{\lambda_2,d_2}\|_\infty^2 \\ &\lesssim \sum_{\lambda_1=1}^\infty \sum_{d_1 \leq \lambda_1} \|u_{\lambda_1,d_1}\|^2_{X^{s,\theta}_{\lambda_1,d_1}[I]} \sum_{\lambda_2=1}^\infty \sum_{d_2 \leq \lambda_2} \|v_{\lambda_2,d_2}\|^2_{X^{s,\theta}_{\lambda_2,d_2}[I]} \end{aligned}$$

To control the μ-sum we have used a similar argument as above:

$$\mu^{s+\theta-2} \lambda^{-1} = \mu^{s+\theta-2} \lambda^{-2s+\frac{n}{2}+\frac{3}{2}} \lambda^{s-\frac{3}{2}} \lambda^{s-\frac{n}{2}-1} \leq \mu^{-s+\theta+\frac{n-1}{2}} \lambda^{s-\frac{3}{2}} \lambda^{s-\frac{n}{2}-1}$$

since $\lambda \gg \mu$, $\lambda \geq d_1$, and $s > n/2$. Finally in the d_1-sum we have used Cauchy-Schwarz where $d_1^{\theta-1} d_1^{1-\theta} \leq d_1^{\theta-1} d_1^{1/2-\theta} \lambda_1^{1/2}$. To estimate the term $\|P_\lambda u_{\lambda,d_1} \Box_{g<\mu^{2/n}}(P_\lambda v_{\lambda,d_2})\|_2$ we run the same argument as above. This conclude the proof of the high-high-low interaction case.

(ii) *Low-high-high interaction*: $\lambda_1 \ll \lambda_2$ and $\lambda_2 \approx \mu$. We must estimate the following term:

$$\sum_{\lambda_2=1}^{\infty}\Big(\sum_{\lambda_1\ll\lambda_2}\|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2}\|_{X^{s,\theta}_{\lambda_2}[I]}\Big)^2$$

Here we take the precise decomposition of the product $P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2}$ as follows: let $P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2} = \sum_{d\le\lambda_2}(P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2})_d$ where we define

$$(P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2})_d = \begin{cases} 0 & \text{if } d<\lambda_1 \\ \sum_{d\le\lambda_1}P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2,d} & \text{if } d=\lambda_1 \\ P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2,d} & \text{if } \lambda_1<d\le\lambda_2 \end{cases}$$

Then

$$\|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2}\|^2_{X^{s,\theta}_{\lambda_2}[I]} \le \Big(\sum_{d_2\le\lambda_1}\|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X^{s,\theta}_{\lambda_2,\lambda_1}[I]}\Big)^2 + \sum_{\lambda_1<d_2\le\lambda_2}\|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{X^{s,\theta}_{\lambda_2,d_2}[I]}$$

Moreover to get the square inside the λ_1 sum we use Cauchy-Schwarz inequality: we are lead to the bound

$$\begin{aligned}\sum_{\lambda_2=1}^{\infty}\Big(\sum_{\lambda_1\ll\lambda_2}\|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2}\|_{X^{s,\theta}_{\lambda_2}[I]}\Big)^2 \;\lesssim\;& \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X^{s,\theta}_{\lambda_2,\lambda_1}[I]}\Big)^2 \\ &+ \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\le\lambda_2}\Big(\sum_{d_1\le\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X^{s,\theta}_{\lambda_2,d_2}[I]}\Big)^2\end{aligned}$$

Let us split the argument in two parts based on the d_2 modulation sum: we consider the two terms separately.

(ii) (a) *Low modulation:* $d_2\le\lambda_1$. We need to control

$$\begin{aligned}&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X^{s,\theta}_{\lambda_2,\lambda_1}[I]}\Big)^2 \\ &\lesssim \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{s}\lambda_1^{s-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2 \\ &+ \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{s-1}\lambda_1^{s-\frac{n+1}{2}}\|\Box_{g<\lambda_2^{1/2}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2\end{aligned} \tag{2.35}$$

Let us start by estimate the first term on the right-hand side of (2.35). We apply Hölder and

Cauchy-Schwarz inequalities to obtain the bound

$$\begin{aligned}
&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\sum_{d_2\leq\lambda_1}\lambda_2^{s}\lambda_1^{s-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\
&\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\sum_{d_2\leq\lambda_1}\lambda_1^{s-\frac{n+1}{2}}\lambda_2^{s-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^{\infty}}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^{\infty}L^2}\Big)^2\\
&\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_{L^{2+}L^{\infty}}\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{2s-2}d_2^{2\theta-1}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{L^{\infty}L^2}
\end{aligned}$$

Thus the $L^{\infty}L^2$ and the $L^{2+}L^{\infty}$ Strichartz estimates yield to the desired result. Observe that we have used the fact that we are working on a finite time interval thus the L^2_t norm is controlled by the L^{2+}_t norm. In order to estimate the second term on the right-hand side of (2.35) we have to split the argument further and to consider the three possible scenarios:

$$\begin{aligned}
\|\Box_{g<\lambda_2^{1/2}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2 \;&\lesssim\; \|(\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\\
&+\;\|P_{\lambda_1}u_{\lambda_1,d_1}(\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\\
&+\;\|P_{<\lambda_2^{1/2}}g\nabla P_{\lambda_1}u_{\lambda_1,d_1}g\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2
\end{aligned}\tag{2.36}$$

Here we have split the $\Box_{g<\lambda_2^{1/2}}$ operator into a term involving both time derivatives ∂_{tt} and a term involving al least one space derivative, and for the latter we have

$$(P_{<\lambda_2^{1/2}}g)(\nabla\nabla_x P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\approx\lambda_1(P_{<\lambda_2^{1/2}}g)\nabla P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\lesssim(P_{<\lambda_2^{1/2}}g)\nabla P_{\lambda_1}u_{\lambda_1,d_1}\nabla P_{\lambda_2}v_{\lambda_2,d_2}$$

since $\lambda_1\ll\lambda_2$. Consider the first term on the right-hand side of (2.36), we used Bernstein inequality to obtain:

$$\begin{aligned}
\|\partial_{tt}(P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2 \;&\lesssim\; \lambda_2^{-1}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^2L^{\infty}}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^{\infty}L^2}\\
&\lesssim\; \lambda_1^{n/2-1}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^2L^2}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^{\infty}L^2}
\end{aligned}$$

Next we compute

$$\begin{aligned}
&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\sum_{d_2\leq\lambda_1}\lambda_2^{s-1}\lambda_1^{s-\frac{n+1}{2}}\|(\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\
&\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\sum_{d_2\leq\lambda_1}\lambda_2^{s-1}\lambda_1^{s-3/2}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_2\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^{\infty}L^2}\Big)^2\\
&\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-2}d_1^{2\theta-2}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_2^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{2s-2}d_2^{2\theta-1}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{L^{\infty}L^2}
\end{aligned}$$

In the previous estimate we have used the following trick in the modulation sum for d_1: from the beginning we have an extra factor of $\lambda_1^{-1/2}$ and we are missing the factor $d_1^{1/2}$. Therefore in the d_1-sum we use Cauchy-Schwarz to obtain

$$(\sum_{d_1\leq\lambda_1}\lambda_1^{-1/2}d_1^{\theta-1}d_1^{1-\theta})^2\leq(\sum_{d_1\leq\lambda_1}d_1^{2\theta-2})(\sum_{d_1\leq\lambda_1}d_1^{1-2\theta})\lesssim\sum_{d_1\leq\lambda_1}d_1^{2\theta-2}$$

Thus the $L^\infty L^2$ Strichartz estimate allow us obtain the correct bound. The estimate for the second term on the right-hand side of (2.36) is more delicate: here we need to use the fact the the modulation sum over d_2 is restricted below λ_1. We have

$$\begin{aligned}&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\sum_{d_2\leq\lambda_1}\lambda_2^{s-1}\lambda_1^{s-\frac{n+1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}(\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2\\&\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\sum_{d_2\leq\lambda_1}\lambda_1^{s-\frac{n+3}{2}}\lambda_2^{s-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\&\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-n-2}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{2s-2}d_2^{2\theta-2}\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2^2\end{aligned}$$

Here we have used the fact that we can squeeze a factor of $\lambda_1^{-1/2}$ out of the λ_1 sum which can be control by a much needed $d_2^{-1/2}$ term since we are working on the low modulation regime. Finally to estimate the third term on the right-hand side of (2.36) we apply Cauchy-Schwarz inequality in d_1 and d_2 sums to obtain

$$\begin{aligned}&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\sum_{d_2\leq\lambda_1}\lambda_2^{s-1}\lambda_1^{s-\frac{n+1}{2}}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\&\lesssim\sum_{\lambda_1}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^\infty}^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{2s-2}d_2^{2\theta-1}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}^2\end{aligned}$$

The $L^\infty L^2$ and the $L^{2+}L^\infty$ Strichartz estimates allow us to conclude. This prove the desired estimate for the low modulation case.

(ii) (b) *High modulation:* $\lambda_1<d_2\leq\lambda_2$. For this interaction we have to estimate the two terms below:

$$\begin{aligned}&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X^{s,\theta}_{\lambda_2,d_2}[I]}\Big)^2\\&\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{s}d_2^{\theta}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\&+\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{s-1}d_2^{\theta-1}\|\Box_{g<\lambda_2^{2/n}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2\end{aligned}\tag{2.37}$$

Let us start by estimate the first term on the right-hand side of (2.37). Hölder and Cauchy-

Schwarz inequalities yield to

$$\begin{aligned}
&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{s}d_2^{\theta}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\
&\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\frac{n+2}{2}}\lambda_2^{s}d_2^{\theta}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty\|P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\
&\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-n-2}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{2s}d_2^{2\theta}\|P_{\lambda_2}v_{\lambda_2,d_2}\|_2^2
\end{aligned}$$

We use the $L^\infty L^\infty$ Strichartz estimates to conclude. Next we analyse the second term on the right-hand side of (2.37); we shall split the argument into three parts based on the following estimate

$$\begin{aligned}
\|\Box_{g<\lambda_2^{1/2}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2 \;&\lesssim\; \|(\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\\
&+\;\|P_{\lambda_1}u_{\lambda_1,d_1}(\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\\
&+\;\|P_{<\lambda_2^{1/2}}g\nabla P_{\lambda_1}u_{\lambda_1,d_1}g\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2
\end{aligned}\tag{2.38}$$

Consider the first term on the right-hand side of (2.38). Bernstein inequality yield to

$$\begin{aligned}
&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{s-1}d_2^{\theta-1}\|(\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\
&\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\frac{3}{2}}\lambda_2^{s-1}d_2^{\theta-1/2}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_2\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2\\
&\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-2}d_1^{2\theta-2}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_2^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{2s-2}d_2^{2\theta-1}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}^2
\end{aligned}$$

On the first step we have transfer a $d_2^{-1/2}$ factor into $\lambda_1^{-1/2}$ by taking advantage of the high modulation regime. Next the estimate for the second term on the right-hand side of (2.38) resembles the one in the low modulation case. In fact we have

$$\begin{aligned}
&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{s-1}d_2^{\theta-1}\|P_{\lambda_1}u_{\lambda_1,d_1}(\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2\\
&\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\frac{n+2}{2}}\lambda_2^{s-1}d_2^{\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\
&\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-n-2}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{2s-2}d_2^{2\theta-2}\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2^2
\end{aligned}$$

The estimate the third term on the right-hand side of (2.38) hinges on the $L^\infty L^\infty$ Strichartz estimate. As above we transfer a $d_2^{-1/2}$ factor into $\lambda_1^{-1/2}$ and we proceed as in the low modulation case. We obtain

$$\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\le\lambda_2}\Big(\sum_{d_1\le\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{s-1}d_2^{\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\le\lambda_2}\Big(\sum_{d_1\le\lambda_1}\lambda_1^{s-\frac{n+1}{2}}\lambda_2^{s-1}d_2^{\theta-1/2}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^{\infty}}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^{\infty}L^2}\Big)^2$$
$$\lesssim\sum_{\lambda_1}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_{L^{2+}L^{\infty}}\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{2s-2}d_2^{2\theta-1}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{L^{\infty}L^2}$$

This conclude the proof of the low-high-high interaction case and thus the proof of the algebra property is completed. □

We summarise in the table below all the different interaction term that we have bounded in the previous proof and we highlight the Strichartz estimates used for each case.

Type of interaction		Strichartz estimates
HHL	1st term	L^4L^4
	$\Box\,\cdot$	$L^\infty L^\infty$
	$\nabla\nabla$	L^4L^4
	$\cdot\,\Box$	$L^\infty L^\infty$
LHHa	1st term	$L^{2+}L^\infty$, $L^\infty L^2$
	$\Box\,\cdot$	$L^\infty L^2$
	$\cdot\,\Box$	$L^\infty L^\infty$
	$\nabla\nabla$	$L^{2+}L^\infty$, $L^\infty L^2$
LHHb	1st term	$L^\infty L^\infty$
	$\Box\,\cdot$	$L^\infty L^2$
	$\cdot\,\Box$	$L^\infty L^\infty$
	$\nabla\nabla$	$L^{2+}L^\infty$, $L^\infty L^2$

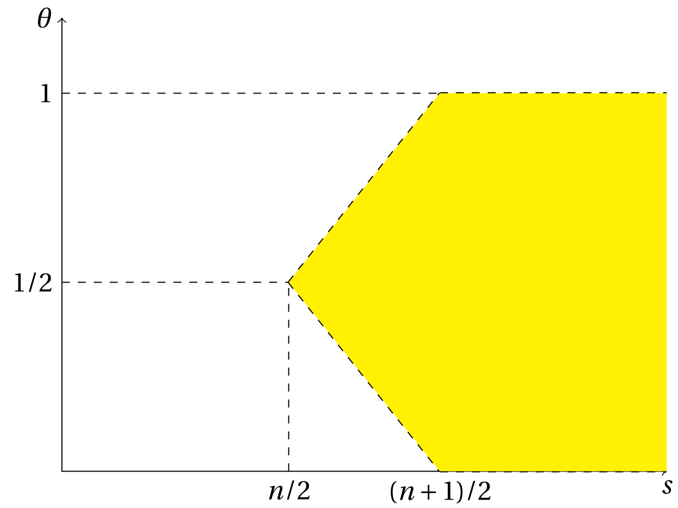
A slightly modification of the previous proof allow one to obtain an analogous algebra property for $0<\theta<1/2$ and for an appropriate modification of the s and θ relationship.

Corollary 75. *Suppose that $n\ge3$, $0<\theta<1/2$, and $s+\theta>(n+1)/2$ then $X^{s,\theta}$ is an algebra for functions with compact support in time.*

The major difference in working with θ restricted to the range $0<\theta<1/2$ is that when using Cauchy-Schwarz to control the modulation sums we end up with a term of the form

$$\sum_{d=1}^{\lambda}d^{1-2\theta}$$

when $1/2<\theta<1$ we can bound it by a constant independent from λ. On the other hand, if $0<\theta<1/2$ then we can only control it by $\lambda^{1-2\theta}$. This is the reason why we have to require that $s+\theta > (n+1)/2$. Therefore notice that $X^{s,\theta}[I]$ is an algebra when the indices s,θ are in the range: $1/2 < \theta < 1$ and $s-\theta > (n-1)/2$, or $0 < \theta < 1/2$ and $s+\theta > (n+1)/2$.



The next proposition that we need, following the analogy with the constant coefficients case, is a $X^{s-1,\theta-1}$ multiplicative estimate. As for the algebra property of Proposition 74, we extend to $n=3$ and $n \geq 5$ spatial dimensions the corresponding $X^{s-1,\theta-1}$ multiplicative estimate of Proposition 3.8 in [30].

Proposition 76. *Let $n \geq 3$, $1/2 < \theta < 1$, and $s-\theta > (n-1)/2$, then the following estimate holds*

$$\|uf\|_{X^{s-1,\theta-1}[I]} \lesssim \|u\|_{X^{s,\theta}[I]} \|f\|_{X^{s-1,\theta-1}[I]}$$

Working with the $X^{s-1,\theta-1}[I]$ norm is tedious, since for negative modulation exponents we have to rely on the decomposition $f = f_0 + \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \Box_{g<\lambda^{1/2}} P_\lambda f_{\lambda,d}$. Therefore we shall prove the $X^{s-1,\theta-1}$ multiplicative estimate by a duality argument that allow us to recover the $X^{1-s,1-\theta}[I]$ norm. In the proof of Proposition 76 we shall need the following duality relationship.

Lemma 77 ([30])**.** *Let $s \in \mathbb{R}$ and $1/2 < \theta < 1$, then*

$$X^{s-1,\theta-1} = (X^{1-s,1-\theta} + L^2 H^{2-s-\theta})'$$

Proof. See Lemma 2.13 in [30]. □

Now we are ready to prove the asymmetric multiplicative estimate.

Proof of Proposition 76. It suffices to show the multiplicative bound

$$X^{s,\theta}[I] \cdot \big(X^{1-s,1-\theta}[I] + L^2 H^{2-s-\theta}(S_I)\big) \subset X^{1-s,1-\theta}[I] + L^2 H^{2-s-\theta}(S_I)$$

Indeed, the duality relationship of Lemma 77 yield to

$$\|uf\|_{X^{s-1,\theta-1}[I]} = \|uf\|_{(X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I))'} = \sup_w \frac{|\int ufw\, dxdt|}{\|w\|_{X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I)}}$$

Moreover the previous multiplication inequality implies

$$|\int ufw\, dxdt| \leq \|uw\|_{X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I)}\|f\|_{X^{s-1,\theta-1}[I]} \leq \|u\|_{X^{s,\theta}[I]}\|w\|_{X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I)}\|f\|_{X^{s-1,\theta-1}[I]}$$

Therefore all it remains to prove are the two estimates

(1) $X^{s,\theta}[I]\cdot L^2H^{2-s-\theta}(S_I) \subset X^{1-s,1-\theta}[I] + L^2H^{2-s-\theta}(S_I)$

(2) $X^{s,\theta}[I]\cdot X^{1-s,1-\theta}[I] \subset L^2H^{2-s-\theta}(S_I) + X^{1-s,1-\theta}[I]$

In fact, assuming (1), (2) by definition we have

$$\begin{aligned}
\|uv\|_{X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I)} &\leq \min_{v=v_1+v_2} \|uv_1\|_{X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I)} + \|uv_2\|_{X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I)} \\
&\leq \|u\|_{X^{s,\theta}[I]} \min_{v=v_1+v_2}(\|v_1\|_{X^{1-s,1-\theta}[I]} + \|v_2\|_{L^2H^{2-s-\theta}(S_I)}) \\
&= \|u\|_{X^{s,\theta}[I]}\|v\|_{X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I)}
\end{aligned}$$

To prove the estimate (1) we notice that since $s > n/2$ we have the Sobolev embedding

$$H^s(\mathbb{R}^n)\cdot H^{2-s-\theta}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)\cdot H^{2-s-\theta}(\mathbb{R}^n) \subset H^{2-s-\theta}(\mathbb{R}^n)$$

Therefore Hölder inequality gives the space-time estimate

$$L^\infty H^s(S_I)\cdot L^2H^{2-s-\theta}(S_I) \subset L^2H^{2-s-\theta}(S_I) \subset X^{1-s,1-\theta}[I] + L^2H^{2-s-\theta}(S_I)$$

Finally the energy inequality of Corollary 58 implies that $X^{s,\theta} \subset L^\infty H^s$. Let us prove (2), suppose $u \in X^{s,\theta}[I]$ and $v \in X^{s-1,\theta-1}[I]$, then we have to show that $\|uv\|_{X^{1-s,1-\theta}[I]+L^2H^{2-s-\theta}(S_I)} \lesssim \|u\|_{X^{s,\theta}[I]}\|v\|_{X^{1-s,1-\theta}[I]}$. By decomposing the functions u and v into frequencies and modulations we claim that the previous estimate follows form

$$\sum_{\mu=1}^{\infty}\Big(\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_2=1}^{\infty}\|\widetilde{P}_\mu(P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2})\|_{X_\mu^{1-s,1-\theta}[I]+\mu^{-2+s+\theta}L^2L^2(S_I)}\Big)^2$$
$$\leq \Big(\sum_{\lambda_1=1}^{\infty}\sum_{d_1=1}^{\lambda_1}\|u_{\lambda_1,d_1}\|^2_{X^{s,\theta}_{\lambda_1,d_1}}\Big)\Big(\sum_{\lambda_2=1}^{\infty}\sum_{d_2=1}^{\lambda_2}\|v_{\lambda_2,d_2}\|^2_{X^{1-s,1-\theta}_{\lambda_2,d_2}}\Big)$$

Because of lack of symmetry we must consider all three types of interactions.

(i) *High-high-low interaction*: $\lambda := \lambda_1 \approx \lambda_2 \gg \mu$. We employ the well known facts from Littlewood-Paley theory: $\|\widetilde{P}_\mu u\|_{H^s} \approx \mu^{\widetilde{s}}\|P_\mu u\|_{H^{s-\widetilde{s}}}$ together with the classical multiplication estimate $\|uv\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^s}$ to obtain

$$\begin{aligned}
&\|\widetilde{P}_\mu(P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2})\|_{\mu^{-2+s+\theta}L^2L^2(S_I)} \\
&\approx \mu^{-s+\theta+\frac{n-1}{2}} \|\widetilde{P}_\mu(P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2})\|_{L^2H^{2-2\theta-\frac{n-1}{2}}(S_I)} \\
&\lesssim \mu^{-s+\theta+\frac{n-1}{2}} \lambda^{-2\theta-\frac{n-1}{2}} \left(\|\nabla P_\lambda u_{\lambda,d_1}\|_{L^\infty L^2(S_I)} \|\nabla P_\lambda v_{\lambda,d_2}\|_{L^{2+}L^\infty(S_I)} + \|\nabla P_\lambda u_{\lambda,d_1}\|_{L^{2+}L^\infty(S_I)} \|\nabla P_\lambda v_{\lambda,d_2}\|_{L^\infty L^2(S_I)}\right)
\end{aligned}$$

Therefore Strichartz estimates yield to

$$\begin{aligned}
&\|\nabla P_\lambda u_{\lambda,d_1}\|_{L^\infty L^2(S_I)} \|\nabla P_\lambda v_{\lambda,d_2}\|_{L^{2+}L^\infty(S_I)} + \|\nabla P_\lambda u_{\lambda,d_1}\|_{L^{2+}L^\infty(S_I)} \|\nabla P_\lambda v_{\lambda,d_2}\|_{L^\infty L^2(S_I)} \\
&\lesssim \lambda^{1/2+n/2-} d_1^{1/2-\theta} d_2^{-1/2+\theta} \|u_{\lambda,d_1}\|_{X^{s,\theta}_{\lambda,d_1}} \|v_{\lambda,d_2}\|_{X^{1-s,1-\theta}_{\lambda,d_2}}
\end{aligned}$$

Thus for this term Cauchy-Schwarz inequality and the bound $s-\theta > (n-1)/2$ implies

$$\begin{aligned}
&\sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\sum_{d_1\le\lambda}\sum_{d_2\le\lambda} \|\widetilde{P}_\mu(P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2})\|_{\mu^{-2+s+\theta}L^2L^2(S_I)}\Big)^2 \\
&\lesssim \sum_{\mu=1}^{\infty}\Big(\sum_{\lambda\gg\mu}\sum_{d_1\le\lambda}\sum_{d_2\le\lambda} \mu^{-s+\theta+\frac{n-1}{2}} \lambda^{-2\theta+1-} d_1^{1/2-\theta} d_2^{-1/2+\theta} \|u_{\lambda,d_1}\|_{X^{s,\theta}_{\lambda,d_1}} \|v_{\lambda,d_2}\|_{X^{1-s,1-\theta}_{\lambda,d_2}}\Big)^2 \\
&\lesssim \Big(\sum_{\lambda=1}^{\infty}\sum_{d_1\le\lambda}\sum_{d_2\le\lambda} d_1^{1/2-\theta} d_2^{1/2-\theta} \|u_{\lambda,d_1}\|_{X^{s,\theta}_{\lambda,d_1}} \|v_{\lambda,d_2}\|_{X^{1-s,1-\theta}_{\lambda,d_2}}\Big)^2 \\
&\lesssim \sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1} \|u_{\lambda_1,d_1}\|^2_{X^{s,\theta}_{\lambda_1,d_1}} \sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2} \|v_{\lambda_2,d_2}\|^2_{X^{1-s,1-\theta}_{\lambda_2,d_2}}
\end{aligned}$$

This conclude the high-high-low case.

(ii) *Low-high-high interaction*: $\lambda_1 \ll \lambda_2$ and $\lambda_2 \approx \mu$. Here the proof follow closely the one of Property 74. In oder to estimate

$$\sum_{\lambda_2=1}^{\infty}\Big(\sum_{\lambda_1\ll\lambda_2} \|P_{\lambda_1}u_{\lambda_1} P_{\lambda_2}v_{\lambda_2}\|_{X^{1-s,1-\theta}_{\lambda_2}}\Big)^2$$

we take the precise decomposition of the product $P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2}$ as follows: let $P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2} = \sum_{d\le\lambda_2}(P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2})_d$ where

$$(P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2})_d = \begin{cases} 0 & \text{if } d<\lambda_1 \\ \sum_{d\le\lambda_1} P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2,d} & \text{if } d=\lambda_1 \\ P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2,d} & \text{if } \lambda_1<d\le\lambda_2 \end{cases}$$

Then

$$\|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2}\|^2_{X^{1-s,1-\theta}_{\lambda_2}} \le \Big(\sum_{d_2\le\lambda_1} \|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X^{1-s,1-\theta}_{\lambda_2,\lambda_1}}\Big)^2 + \sum_{\lambda_1<d_2\le\lambda_2} \|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{X^{1-s,1-\theta}_{\lambda_2,d_2}}$$

Moreover to get the square inside the λ_1 sum we apply Cauchy-Schwarz inequality and take advantage of the bound $s-\theta > (n-1)/2$, we obtain

$$\begin{aligned}
&\sum_{\lambda_2=1}^{\infty}\Big(\sum_{\lambda_1\ll\lambda_2}\|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2}\|_{X^{1-s,1-\theta}_{\lambda_2}}\Big)^2\\
&\lesssim \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X^{1-s,1-\theta}_{\lambda_2,\lambda_1}}\Big)^2\\
&+\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1<d_2\le\lambda_2}\Big(\sum_{d_1\le\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X^{1-s,1-\theta}_{\lambda_2,d_2}}\Big)^2
\end{aligned}$$

We split the argument in two parts based on the d_2 modulation sum: $d_2\le\lambda_1$ and $\lambda_1<d_2\le\lambda_2$.

(ii) (a) Low modulation: $d_2\le\lambda_1$. For this interaction we estimate the following two terms

$$\begin{aligned}
&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{1-s}\lambda_1^{s-2\theta-\frac{n-3}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\
&+\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{-s}\lambda_1^{s-2\theta-\frac{n-1}{2}}\|\Box_{g<\lambda_2^{1/2}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2
\end{aligned}\tag{2.39}$$

Let us start by estimate the first term of (2.39). We apply Hölder and Cauchy-Schwarz inequalities to obtain

$$\begin{aligned}
&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{1-s}\lambda_1^{s-2\theta-\frac{n-3}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\
&\lesssim \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{-s}\lambda_1^{s-\theta-\frac{n}{2}}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^\infty}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2\\
&\lesssim \sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_{L^{2+}L^\infty}\sum_{\lambda_2=1}^{\infty}\Big(\sum_{d_2\le\lambda_1}\lambda_2^{-s}\lambda_1^{1/2-\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2\\
&\lesssim \sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_{L^{2+}L^\infty}\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s}d_2^{1-2\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{L^\infty L^2}
\end{aligned}$$

Here we have used the fact that the modulation is restricted to the low range thus $\lambda_1^{1-2\theta}(\sum_{d_2\le\lambda_1}d_2^{1/2-\theta}d_2^{\theta-1/2})^2\lesssim\sum_{d_2\le\lambda_2}d_2^{1-2\theta}$. The $L^\infty L^2$ and the $L^{2+}L^\infty$ Strichartz estimates implies the correct bound. In order to estimate the second term of (2.39) we have to further split the argument to consider three possible scenarios:

$$\begin{aligned}
\|\Box_{g<\lambda_2^{1/2}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2 &\lesssim \|(\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\\
&+\|P_{\lambda_1}u_{\lambda_1,d_1}(\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\\
&+\|P_{<\lambda_2^{1/2}}g\nabla P_{\lambda_1}u_{\lambda_1,d_1}g\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2
\end{aligned}\tag{2.40}$$

Consider the first term on the right-hand side of (2.40), we used Bernstein inequality to obtain:

$$\begin{aligned}\|\partial_{tt}(P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2 &\lesssim \lambda_2^{-1}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^2L^\infty}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\\ &\lesssim \lambda_1^{n/2-1}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^2L^2}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\end{aligned}$$

Thus to bound the first term, Cauchy-Schwarz inequality yield to

$$\begin{aligned}&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{-s}\lambda_1^{s-2\theta-\frac{n-1}{2}}\|(\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\ &\lesssim \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{-s}\lambda_1^{s-3/2}\lambda_1^{1/2-\theta}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_2\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2\\ &\lesssim \sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-2}d_1^{2\theta-2}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_2^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s}d_2^{1-2\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}^2\end{aligned}$$

The estimate for the second term on the right-hand side of (2.40) is more delicate since we are force to put the high frequency term into L^2L^2. We have

$$\begin{aligned}&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{-s}\lambda_1^{s-2\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}(\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2\\ &\lesssim \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_1^{s-n/2-1}\lambda_1^{-2\theta+1/2}\lambda_2^{-s}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\ &\lesssim \sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-2}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s}d_2^{-2\theta}\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2^2\end{aligned}$$

Here we have used the fact that we can squeeze out a factor of $\lambda_1^{-2\theta+1/2}$ from the λ_1 sum and we have used the following trivial bound

$$\lambda_1^{-4\theta+1}(\sum_{d_2\le\lambda_1}d_2^{-\theta}d_2^{\theta})^2\lesssim\sum_{d_2\le\lambda_1}d_2^{-2\theta}$$

Finally to estimate the third term on the right-hand side of (2.40) we place the high frequency term into $L^\infty L^2$ and the low frequency term into $L^{2+}L^\infty$, thus

$$\begin{aligned}&\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_2^{-s}\lambda_1^{s-2\theta-\frac{n-1}{2}}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2\\ &\lesssim \sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\Big(\sum_{d_1\le\lambda_1}\sum_{d_2\le\lambda_1}\lambda_1^{s-n/2-1/2+\epsilon}\lambda_2^{-s}\lambda_1^{1/2-\theta}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^\infty}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2\\ &\lesssim \sum_{\lambda_1}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^\infty}^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s}d_2^{1-2\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}^2\end{aligned}$$

Hence the proof for the low-high-high and low-modulation interaction is concluded.

(ii) (b) High modulation: $\lambda_1 < d_2 \leq \lambda_2$. For this case we must estimate the two terms:

$$\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\leq d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{1-s}d_2^{1-\theta}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2 \tag{2.41}$$
$$+\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\leq d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{-s}d_2^{-\theta}\|\Box_{g<\lambda_2^{1/2}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2$$

To estimate the first term of (2.41) we apply Hölder and Cauchy-Schwarz inequalities:

$$\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\leq d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{1-s}d_2^{1-\theta}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\leq d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-n/2-1}\lambda_2^{1-s}d_2^{1-\theta}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty\|P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-n-2}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{2-2s}d_2^{2-2\theta}\|P_{\lambda_2}v_{\lambda_2,d_2}\|_2^2$$

As usual to estimate the second term of (2.41) we split the argument into three parts based on the following estimate

$$\begin{aligned}\|\Box_{g<\lambda_2^{1/2}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2 &\lesssim \|\partial_{tt}(P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\\ &+ \|P_{\lambda_1}u_{\lambda_1,d_1}\Box_{g<\lambda_2^{1/2}}(P_{\lambda_2}v_{\lambda_2,d_2})\|_2\\ &+ \|P_{<\lambda_2^{1/2}}g\nabla P_{\lambda_1}u_{\lambda_1,d_1}g\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2\end{aligned} \tag{2.42}$$

Consider the first term on the right-hand side of (2.42). Bernstein inequality applied to the low frequency term yield to

$$\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\leq d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{-s}d_2^{-\theta}\|\partial_{tt}(P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\leq d_2\leq\lambda_2}\Big(\sum_{d_1\leq\lambda_1}\lambda_1^{s-3/2}\lambda_2^{-s}d_2^{1/2-\theta}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_2\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-2}d_1^{2\theta-2}\|\partial_{tt}P_{\lambda_1}u_{\lambda_1,d_1}\|_2^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{-2s}d_2^{1-2\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}^2$$

In the first inequality we have transferred a $d_2^{-1/2}$ factor into $\lambda_1^{-1/2}$ by taking advantage of the high modulation regime. Next, the estimate for the second term on the right-hand side of

(2.42) resembles the one in the low modulation case. In fact we have

$$\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\le d_2\le\lambda_2}\Big(\sum_{d_1\le\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{-s}d_2^{-\theta}\|P_{\lambda_1}u_{\lambda_1,d_1}\Box_{g<\lambda_2^{1/2}}(P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\le d_2\le\lambda_2}\Big(\sum_{d_1\le\lambda_1}\lambda_1^{s-n/2-1}\lambda_2^{-s}d_2^{-\theta}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-2}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s}d_2^{-2\theta}\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2^2$$

The estimate the third term on the right-hand side of (2.42) hinges on the $L^\infty L^\infty$ Strichartz estimate:

$$\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\le d_2\le\lambda_2}\Big(\sum_{d_1\le\lambda_1}\lambda_1^{s-\theta-\frac{n-1}{2}}\lambda_2^{-s}d_2^{-\theta}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_2=1}^{\infty}\sum_{\lambda_1\ll\lambda_2}\sum_{\lambda_1\le d_2\le\lambda_2}\Big(\sum_{d_1\le\lambda_1}\lambda_1^{s-n/2-1}\lambda_2^{1-s}d_2^{1-\theta}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty\|P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-2}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{2-2s}d_2^{2-2\theta}\|P_{\lambda_2}v_{\lambda_2,d_2}\|_2^2$$

here we moved a factor d_2^{-1} into λ_1^{-1}. Therefore the estimate for the low-high-high interaction holds.

(iii) *High-low-high interaction*: $\lambda_1\gg\lambda_2$ and $\lambda_1\approx\mu$. We proceed as in the low-high-high interaction case swapping the role of the high and low frequencies: we split the argument in two parts based on the d_1 modulation sum: $d_1\le\lambda_2$ and $\lambda_2<d_1\le\lambda_1$. Moreover to get the square inside the λ_2 sum we use Cauchy-Schwarz inequality. After this initial reductions, it suffices to estimate

$$\sum_{\lambda_1=1}^{\infty}\Big(\sum_{\lambda_1\gg\lambda_2}\|P_{\lambda_1}u_{\lambda_1}P_{\lambda_2}v_{\lambda_2}\|_{X_{\lambda_1}^{1-s,1-\theta}}\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_2^{s-\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X_{\lambda_1,\lambda_2}^{1-s,1-\theta}}\Big)^2$$
$$+\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\sum_{\lambda_2<d_1\le\lambda_1}\Big(\sum_{d_2\le\lambda_2}\lambda_2^{s-\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_{X_{\lambda_1,d_1}^{1-s,1-\theta}}\Big)^2$$

(iii) (a) Low modulation: $d_1\le\lambda_2$. Let us first consider when d_1 is restricted to the low modulation, here we need to bound the following two terms:

$$\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{1-s}\lambda_2^{s-2\theta-\frac{n-3}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2 \tag{2.43}$$
$$+\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{-s}\lambda_2^{s-2\theta-\frac{n-1}{2}}\|\Box_{g<\lambda_1^{1/2}}(P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2$$

For the first term of (2.43) we have

$$\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{1-s}\lambda_2^{s-2\theta-\frac{n-3}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{-2s+\frac{n}{2}+\frac{1}{2}}\lambda_1^{s-\frac{n+1}{2}}\lambda_2^{2s-\theta-\frac{n}{2}}\lambda_2^{-s-\theta+\frac{1}{2}}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^\infty}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_{L^{2+}L^\infty}\sum_{\lambda_2=1}^{\infty}\Big(\sum_{d_2\le\lambda_2}\lambda_2^{-s-\theta+1/2}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_{L^{2+}L^\infty}\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s}d_2^{1-2\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{L^\infty L^2}$$

since

$$\lambda_1^{-2s+\frac{n}{2}+1/2}\lambda_2^{2s-\theta-\frac{n}{2}}\le(\lambda_2/\lambda_1)^{2s-\theta-\frac{n}{2}}$$

and $2s-\theta-\frac{n}{2}>s-1/2\ge0$. To estimate the second term of (2.43) we proceed as in (ii), we split the argument into three parts based on which term is being hit by the d'Alembert operator. The estimate of the first term goes as follows

$$\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{-s}\lambda_2^{s-2\theta-\frac{n-1}{2}}\|(\Box_{g<\lambda_1^{1/2}}P_{\lambda_1}u_{\lambda_1,d_1})P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{s-1}\lambda_2^{-1/2}\lambda_2^{-s-\theta-\frac{n-1}{2}}(\lambda_2/\lambda_1)^{2s-1}\|\Box_{g<\lambda_1^{1/2}}P_{\lambda_1}u_{\lambda_1,d_1}\|_2\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_\infty\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-2}d_1^{2\theta-2}\|\Box_{g<\lambda_1^{1/2}}P_{\lambda_1}u_{\lambda_1,d_1}\|_2^2\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s-n}d_2^{1-2\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|^2_\infty$$

The estimate for the second term is given below.

$$\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{-s}\lambda_2^{s-2\theta-\frac{n-1}{2}}\|P_{\lambda_1}u_{\lambda_1,d_1}\ (\partial_{tt}P_{\lambda_2}v_{\lambda_2,d_2})\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{s-\frac{n}{2}-1}\lambda_2^{-s}\lambda_2^{-\theta}(\lambda_2/\lambda_1)^{2s-n/2}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_\infty\|\partial_{tt}P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-2}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_\infty\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s}d_2^{-2\theta}\|\Box_{g<\lambda_2^{1/2}}P_{\lambda_2}v_{\lambda_2,d_2}\|_2^2$$

Finally for the third term we obtain:

$$\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{-s}\lambda_2^{s-2\theta-\frac{n-1}{2}}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\Big(\sum_{d_1\le\lambda_2}\sum_{d_2\le\lambda_2}\lambda_1^{s-n/2-1/2+\epsilon}\lambda_2^{-s-\theta+1/2}(\lambda_2/\lambda_1)^{2s-\theta-n/2}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^\infty}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{d_1\le\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_{L^{2+}L^\infty}\sum_{\lambda_2=1}^{\infty}\sum_{d_2\le\lambda_2}\lambda_2^{-2s}d_2^{1-2\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{L^\infty L^2}$$

(iii) (b) High modulation: $\lambda_2 < d_1 \leq \lambda_1$. Now we estimate the terms where the sum over d_1 is restricted to the high terms. We need to control

$$\sum_{\lambda_1=1}^{\infty} \sum_{\lambda_1 \gg \lambda_2} \sum_{\lambda_2 < d_1 \leq \lambda_1} \Big(\sum_{d_2 \leq \lambda_2} \lambda_1^{1-s} \lambda_2^{s-\theta-\frac{n-1}{2}} d_1^{1-\theta} \| P_{\lambda_1} u_{\lambda_1,d_1} P_{\lambda_2} v_{\lambda_2,d_2}) \|_2 \Big)^2$$

$$\sum_{\lambda_1=1}^{\infty} \sum_{\lambda_1 \gg \lambda_2} \sum_{\lambda_2 < d_1 \leq \lambda_1} \Big(\sum_{d_2 \leq \lambda_2} \lambda_1^{-s} \lambda_2^{s-\theta-\frac{n-1}{2}} d_1^{-\theta} \| \Box_{g<\lambda_1^{1/2}} (P_{\lambda_1} u_{\lambda_1,d_1} P_{\lambda_2} v_{\lambda_2,d_2}) \|_2 \Big)^2 \tag{2.44}$$

For the first term of (2.44) which does not involve the d'Alembert operator we have

$$\begin{aligned}
&\sum_{\lambda_1=1}^{\infty} \sum_{\lambda_1 \gg \lambda_2} \sum_{\lambda_2 < d_1 \leq \lambda_1} \Big(\sum_{d_2 \leq \lambda_2} \lambda_1^{1-s} \lambda_2^{s-\theta-\frac{n-1}{2}} d_1^{1-\theta} \| P_{\lambda_1} u_{\lambda_1,d_1} P_{\lambda_2} v_{\lambda_2,d_2}) \|_2 \Big)^2 \\
&\lesssim \sum_{\lambda_1=1}^{\infty} \sum_{\lambda_1 \gg \lambda_2} \sum_{\lambda_2 < d_1 \leq \lambda_1} \Big(\sum_{d_2 \leq \lambda_2} \lambda_1^{s-\frac{n}{2}-\frac{1}{2}} d_1^{\theta-1/2} \lambda_2^{-s-\theta+\frac{1}{2}} (\lambda_2/\lambda_1)^{2s-n/2-1} \| \nabla P_{\lambda_1} u_{\lambda_1,d_1} \|_{L^{2+}L^\infty} \| \nabla P_{\lambda_2} v_{\lambda_2,d_2} \|_{L^\infty L^2} \Big)^2 \\
&\lesssim \sum_{\lambda_1=1}^{\infty} \sum_{d_1 \leq \lambda_1} \lambda_1^{2s-n-1+2\epsilon} d_1^{2\theta-1} \| \nabla P_{\lambda_1} u_{\lambda_1,d_1} \|_{L^{2+}L^\infty}^2 \sum_{\lambda_2=1}^{\infty} \sum_{d_2 \leq \lambda_2} \lambda_2^{-2s} d_2^{1-2\theta} \| \nabla P_{\lambda_2} v_{\lambda_2,d_2} \|_{L^\infty L^2}^2
\end{aligned}$$

In the first line we have used the fact that $d_1^{1-\theta} = d_1^{\theta-1/2} d_1^{3/2-2\theta} \leq d_1^{\theta-1/2} \lambda_1^{1/2}$. To estimate the second term of (2.44) we split the argument as usual into three parts. The estimate of the term where the d'Alembert operator hits the high frequency term goes as follows

$$\begin{aligned}
&\sum_{\lambda_1=1}^{\infty} \sum_{\lambda_1 \gg \lambda_2} \sum_{\lambda_2 < d_1 \leq \lambda_1} \Big(\sum_{d_2 \leq \lambda_2} \lambda_1^{-s} \lambda_2^{s-\theta-\frac{n-1}{2}} d_1^{-\theta} \| (\Box_{g<\lambda_1^{1/2}} P_{\lambda_1} u_{\lambda_1,d_1}) P_{\lambda_2} v_{\lambda_2,d_2} \|_2 \Big)^2 \\
&\lesssim \sum_{\lambda_1=1}^{\infty} \sum_{\lambda_1 \gg \lambda_2} \sum_{\lambda_2 < d_1 \leq \lambda_1} \Big(\sum_{d_2 \leq \lambda_2} \lambda_1^{s-1} d_1^{\theta-1} \lambda_2^{-s-\theta-\frac{n-1}{2}} (\lambda_2/\lambda_1)^{2s-1} \| \Box_{g<\lambda_1^{1/2}} P_{\lambda_1} u_{\lambda_1,d_1} \|_2 \| \nabla P_{\lambda_2} v_{\lambda_2,d_2} \|_\infty \Big)^2 \\
&\lesssim \sum_{\lambda_1=1}^{\infty} \sum_{d_1 \leq \lambda_1} \lambda_1^{2s-2} d_1^{2\theta-2} \| \Box_{g<\lambda_1^{1/2}} P_{\lambda_1} u_{\lambda_1,d_1} \|_2^2 \sum_{\lambda_2=1}^{\infty} \sum_{d_2 \leq \lambda_2} \lambda_2^{-2s-n} d_2^{1-2\theta} \| \nabla P_{\lambda_2} v_{\lambda_2,d_2} \|_\infty^2
\end{aligned}$$

The estimate of the term where the d'Alembert operator hits the low frequency term is given below:

$$\begin{aligned}
&\sum_{\lambda_1=1}^{\infty} \sum_{\lambda_1 \gg \lambda_2} \sum_{\lambda_2 < d_1 \leq \lambda_1} \Big(\sum_{d_2 \leq \lambda_2} \lambda_1^{-s} \lambda_2^{s-\theta-\frac{n-1}{2}} d_1^{-\theta} \| P_{\lambda_1} u_{\lambda_1,d_1} \ (\partial_{tt} P_{\lambda_2} v_{\lambda_2,d_2}) \|_2 \Big)^2 \\
&\lesssim \sum_{\lambda_1=1}^{\infty} \sum_{\lambda_1 \gg \lambda_2} \sum_{\lambda_2 < d_1 \leq \lambda_1} \Big(\sum_{d_2 \leq \lambda_2} \lambda_1^{s-\frac{n}{2}-1} d_1^{\theta-1/2} \lambda_2^{-s-\theta} (\lambda_2/\lambda_1)^{2s-n/2} \| \nabla P_{\lambda_1} u_{\lambda_1,d_1} \|_\infty \| \partial_{tt} P_{\lambda_2} v_{\lambda_2,d_2} \|_2 \Big)^2 \\
&\lesssim \sum_{\lambda_1=1}^{\infty} \sum_{d_1 \leq \lambda_1} \lambda_1^{2s-n-2} d_1^{2\theta-1} \| \nabla P_{\lambda_1} u_{\lambda_1,d_1} \|_\infty^2 \sum_{\lambda_2=1}^{\infty} \sum_{d_2 \leq \lambda_2} \lambda_2^{-2s} d_2^{-2\theta} \| \Box_{g<\lambda_2^{1/2}} P_{\lambda_2} v_{\lambda_2,d_2} \|_2^2
\end{aligned}$$

Finally when the d'Alembert operator distributes one derivative on each term we obtain:

$$\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\sum_{\lambda_2<d_1\leq\lambda_1}\Big(\sum_{d_2\leq\lambda_2}\lambda_1^{-s}\lambda_2^{s-\theta-\frac{n-1}{2}}d_1^{-\theta}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_2\Big)^2$$
$$\lesssim\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_1\gg\lambda_2}\sum_{\lambda_2<d_1\leq\lambda_1}\Big(\sum_{d_2\leq\lambda_2}\lambda_1^{s-\frac{n}{2}-\frac{1}{2}+\epsilon}d_1^{\theta-\frac{1}{2}}\lambda_2^{-s-\theta+\frac{1}{2}}(\lambda_2/\lambda_1)^{2s-\theta-\frac{n}{2}}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|_{L^{2+}L^\infty}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|_{L^\infty L^2}\Big)^2$$
$$\lesssim\sum_{\lambda_1}\sum_{d_1\leq\lambda_1}\lambda_1^{2s-n-1+2\epsilon}d_1^{2\theta-1}\|\nabla P_{\lambda_1}u_{\lambda_1,d_1}\|^2_{L^{2+}L^\infty}\sum_{\lambda_2=1}^{\infty}\sum_{d_2\leq\lambda_2}\lambda_2^{-2s}d_2^{1-2\theta}\|\nabla P_{\lambda_2}v_{\lambda_2,d_2}\|^2_{L^\infty L^2}$$

This conclude the proof of the high-low-high interaction case. □

We end this section by stating the following Moser type estimate corresponding to the estimate of Proposition 3.9 in [30], see also Proposition 8.1 in [27].

Proposition 78 ([30],[27])**.** *Let $n\geq 3$, $1/2<\theta<1$, $s-\theta>(n-1)/2$, and Γ a smooth function vanishing at the origin, then the following estimate holds*

$$\|\Gamma(u)\|_{X^{s,\theta}[I]}\lesssim f(\|u\|_{L^\infty(S_I)})g(\|u\|_{X^{s,\theta}[I]})$$

where f, g are positive increasing continuous functions.

2.9 Half-waves and angular localization operators

Let us introduce an equivalent definition of $X^{s,\theta}_{\lambda,d}$ norms in terms of half-waves norms. For each dyadic number $\alpha\leq 1$, consider the symbol for the principal part of the space-time mollified curved d'Alembert operator $\Box_{g<\alpha^{-1}}=(P_{<\alpha^{-1}}(D_x,D_t)g^{\mu\nu})(t,x)\partial_\mu\partial_\nu=g^{\mu\nu}_{<\alpha^{-1}}(t,x)\partial_\mu\partial_\nu$:

$$p_{<\alpha^{-1}}(t,x,\tau,\xi)=\tau^2-2g^{0i}_{<\alpha^{-1}}(t,x)\tau\xi_i-g^{ij}_{<\alpha^{-1}}(t,x)\xi_i\xi_j.$$

Since $\Box_{g<\alpha^{-1}}$ is hyperbolic we know that we can decompose its symbol into

$$p_{<\alpha^{-1}}(t,x,\tau,\xi)=(\tau+a^+_{<\alpha^{-1}}(t,x,\xi))(\tau+a^-_{<\alpha^{-1}}(t,x,\xi)),$$

where

$$a^\pm_{<\alpha^{-1}}(t,x,\xi)=-g^{0j}_{<\alpha^{-1}}(t,x)\xi_j\mp\sqrt{(g^{0j}_{<\alpha^{-1}}(t,x)\xi_j)^2+g^{ij}_{<\alpha^{-1}}(t,x)\xi_i\xi_j}. \tag{2.45}$$

Notice that $a^\pm_{<\alpha^{-1}}$ is an homogeneous function or order one with respect to ξ. Let us introduce the operators $A^\pm_{<\alpha^{-1}}(t,x,D)$ defined as the pseudo-differential operators with symbols $a^\pm_{<\alpha^{-1}}(t,x,\xi)$, that is

$$\big(A^\pm_{<\alpha^{-1}}(t,x,D)u\big)(t,x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{ix\xi}\,a^\pm_{<\alpha^{-1}}(t,x,\xi)\widehat{u}(t,\xi)\,d\xi.$$

Moreover we define the *half-waves operators* associated to the mollified curved d'Alembert operator $\Box_{g<\alpha^{-1}}$ as:

$$\big(D_t + A^{\pm}_{<\alpha^{-1}}(t,x,D)\big)u(t,x) = (2\pi)^{-n-1}\int_{\mathbb{R}^{n+1}} e^{i(x\xi+t\tau)}\,(\tau + a^{\pm}_{<\alpha^{-1}}(t,x,\xi))\,\widetilde{u}(\tau,\xi)\,d\xi\,d\tau.$$

The terminology make sense since we can decompose $\Box_{g<\alpha^{-1}} = (D_t + A^{+}_{<\alpha^{-1}}(t,x,D))(D_t + A^{-}_{<\alpha^{-1}}(t,x,D))$.

Remark. Notice that if g is replaced by the Minkowski metric, then the symbol of the flat d'Alembert operator is

$$p_{<\alpha^{-1}}(t,x,\tau,\xi) = (\tau^2 - |\xi|^2) = (\tau - |\xi|)(\tau + |\xi|)$$

and it does not depend on t, x, nor on α moreover $A^{\pm}(t,x,D) = \pm|D|$ since $a^{\pm}_{<\alpha^{-1}}(t,x,\xi) = \pm|\xi|$.

Definition. (Classical Symbol class) Let $m \in \mathbb{R}$, denote S^m as the set of functions $a \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ such that for every multi-indices $\alpha, \beta \in \mathbb{N}^n$

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta} a(x,\xi)| \lesssim \langle\xi\rangle^{m-|\beta|}.$$

We have the following results from [27].

Lemma 79 ([27] Lemma 2.17)**.** *In the definition of $X^{s,\theta}_{\lambda,d}$ norm, we may replace the term $\|\Box_{g<\sqrt{\lambda}} P_\lambda u\|_2$ with $\|(D_t + A^{-}_{<\sqrt{\lambda}})(D_t + A^{+}_{<\sqrt{\lambda}})P_\lambda u\|_2$ or $\|(D_t + A^{+}_{<\sqrt{\lambda}})(D_t + A^{-}_{<\sqrt{\lambda}})P_\lambda u\|_2$. In other words the following norms are equivalents:*

i. $\lambda^{2s} d^{2\theta}\|u_{\lambda,d}\|_2 + \lambda^{s-1} d^{\theta-1}\|\Box_{g<\sqrt{\lambda}} u_{\lambda,d}\|_2,$

ii. $\lambda^{2s} d^{2\theta}\|u_{\lambda,d}\|_2 + \lambda^{s-1} d^{\theta-1}\|(D_t + A^{\pm}_{<\sqrt{\lambda}})(D_t + A^{\mp}_{<\sqrt{\lambda}})u_{\lambda,d}\|_2.$

Proposition 80 ([27] Lemma 2.17)**.** *Suppose u satisfies $u = P_\lambda(D_x)u$, and write*

$$u = P_{>-\lambda/64}(D_t)u_\lambda + P_{<-\lambda/64}(D_t)u_\lambda := u^+ + u^-,$$

then

$$\|\nabla_{t,x}u^{\pm}\|_2 + \lambda\|(D_t \pm A^{\pm}_{<\sqrt{\lambda}})u^{\pm}\|_2 + \|\Box_{g<\sqrt{\lambda}}u^{\pm}\|_2 \lesssim \|\nabla_{t,x}u\|_2 + \|\Box_{g<\sqrt{\lambda}}u\|_2.$$

In view of this proposition let us introduce the norm

$$\|u\|_{X_{\pm,\lambda}} = \|u\|_{L^2} + \|(D_t + A^{\pm}_{<\sqrt{\lambda}}(t,x,D))u\|_{L^2}.$$

Observe that from Proposition 80 it follows immediately that:

Corollary 81. *Suppose u satisfies $u = P_\lambda(D_x)u$, then we have*

$$\|(D_t \pm A^{\pm}_{<\sqrt{\lambda}})u\|_2 \lesssim d\|u\|_{X^{0,0}_{\lambda,d}}.$$

Angular localization operators

Consider the Hamilton equations for the operators $A^{\pm}_{<\alpha^{-1}}(t,x,D)$, defined as

$$\begin{cases} \dot{x}^{\pm}_{<\alpha^{-1}}(t) = \nabla_{\xi} a^{\pm}_{<\alpha^{-1}}(t, x^{\pm}_{<\alpha^{-1}}(t), \xi^{\pm}_{<\alpha^{-1}}(t)), \\ \dot{\xi}^{\pm}_{<\alpha^{-1}}(t) = -\nabla_{x} a^{\pm}_{<\alpha^{-1}}(t, x^{\pm}_{<\alpha^{-1}}(t), \xi^{\pm}_{<\alpha^{-1}}(t)), \\ x^{\pm}_{<\alpha^{-1}}(0) = x, \qquad \xi^{\pm}_{<\alpha^{-1}}(0) = \xi. \end{cases} \tag{2.46}$$

The two curves $x^{\pm}_{<\alpha^{-1}}, \xi^{\pm}_{<\alpha^{-1}} : \mathbb{R}^{+} \to \mathbb{R}^{n}$ are called *bicharacteristics*, here $x, \xi \in \mathbb{R}^{n}$ are initial data. Let us introduce the Hamilton flow as the following map on the phase space:

$$\begin{aligned} \Phi^{\alpha,\pm}_{t} : \mathbb{R}^{n} \times \mathbb{R}^{n} &\to \mathbb{R}^{n} \times \mathbb{R}^{n}, \\ (x,\xi) &\mapsto (x^{\pm}_{<\alpha^{-1}}(t), \xi^{\pm}_{<\alpha^{-1}}(t)). \end{aligned}$$

From the theory of Hamilton-Jacobi equations we know that for small time the Hamilton equations admit a classical solution, thus the Hamilton flow is well defined (again for small time). Moreover observe that the Hamilton flow is 1-homogeneous in the second variable.

Remark (Flat Metric). Notice that if g is replaced by the Minkowski metric, then Hamilton equations read

$$\begin{cases} \dot{x}^{\pm}(t) = \pm \frac{\xi^{\pm}(t)}{|\xi^{\pm}(t)|}, \\ \dot{\xi}^{\pm}(t) = 0, \\ x^{\pm}(0) = x, \qquad \xi^{\pm}(0) = \xi. \end{cases}$$

Hence $\Phi^{\alpha,\pm}_{t}(x,\xi) = (\pm t \frac{\xi}{|\xi|} + x, \xi)$ the Hamilton flow is constant in the frequency variable.

Remark (A toy-model). Consider the following equation on $\mathbb{R}^{1+1}$: $\partial^2_{tt} u + c^2(x) \partial^2_{xx} u = 0$. The metric is simply given by the matrix

$$g = \begin{pmatrix} -1 & 0 \\ 0 & c^2(x) \end{pmatrix}$$

and the half-waves multipliers are constant in time and $a^{\pm}(t,x,\xi) = \pm\sqrt{c^2(x)\xi^2} = \pm|c(x)||\xi|$. Take the space-dependent speed of light to be $c^2(x) = 1 + x^2$, then the Hamilton flow is described by the system

$$\begin{cases} \dot{x}^{\pm}(t) = \pm(1 + (x^{\pm})^2) \frac{\xi^{\pm}(t)}{|\xi^{\pm}(t)|}, \\ \dot{\xi}^{\pm}(t) = \mp 2 x^{\pm}(t) |\xi^{\pm}(t)|, \\ x(0) = x, \quad \xi(0) = \xi. \end{cases}$$

We solve the first ODE by the separation of variable technique and obtain

$$x^{\pm}(t) = \tan\big(\pm \int_0^t \frac{\xi^{\pm}(s)}{|\xi^{\pm}(s)|} ds + \arctan x \big).$$

The second ODE has a solution given by $\xi^{\pm}(t) = \xi \exp\left(\mp 2\mathrm{sgn}(\xi)\int_0^t x^{\pm}(s)ds\right)$, hence the explicit solution for the curve in the physical space is

$$x^{\pm}(t) = \tan(\pm\frac{\xi}{|\xi|}t + \arctan x) = \frac{\sin(\frac{\xi}{|\xi|}t) + x\cos(\frac{\xi}{|\xi|}t)}{\cos(\frac{\xi}{|\xi|}t) - x\sin(\frac{\xi}{|\xi|}t)}.$$

Plugging it back to the formula we had for $\xi(t)$ we obtain the explicit solution for the curve on the frequencies side:

$$\xi^{\pm}(t) = \xi\left(\cos(\frac{\xi}{|\xi|}t) - x\sin(\frac{\xi}{|\xi|}t)\right)^{\pm 2}.$$

Extending these explicit solutions to the high dimensional case is problematic due to the present of the absolute value in the second ODE.

Following [30] we define the metric g_α on the phase space by

$$g_{\alpha,(x,\xi)}(y,\eta) = \frac{|y\cdot\xi|^2}{\alpha^4|\xi|^2} + \frac{|y\wedge\xi|^2}{\alpha^2|\xi|^2} + \frac{|\eta\cdot\xi|^2}{|\xi|^4} + \frac{|\eta\wedge\xi|^2}{\alpha^2|\xi|^4}.$$

Here we have used the following decomposition. Let $\xi,\eta\in\mathbb{R}^n$ two vectors and let us denote $\xi\wedge\eta = \xi\otimes\eta - \eta\otimes\xi$, that is $\xi\wedge\eta$ is a $\mathbb{R}^n\times\mathbb{R}^n$ matrix with components $(\xi\wedge\eta)_{ij} = \xi_i\eta_j - \xi_j\eta_i$. Then we can decompose the vector η into two components, one parallel and the respectively perpendicular to ξ by the formula

$$\eta = \frac{\xi\cdot\eta}{\xi\cdot\xi}\xi + \frac{1}{\xi\cdot\xi}(\eta\wedge\xi)\xi.$$

In fact $[(\eta\wedge\xi)\xi]\cdot\xi = (\eta\wedge\xi)_{ij}\xi_i\xi_j = 0$.

We shall need the following:

Lemma 82 ([27] Lemma 5.1)**.** *The components of the flow $\Phi_t^{\alpha,\pm}$ are Lipschitz and g_α-smooth.*

Let $\theta\in\mathbb{S}^{n-1}$ a given direction at time $t=0$, we introduce the angular opening $\beta\in 2^{-\mathbb{N}}$ (a dyadic number less then 1), the cone of angular opening β centered at θ as

$$C_\beta(\theta) = \{\xi\in\mathbb{R}^n : \measuredangle(\xi,\theta) < \beta\},$$

and the intersection of two of such cones as

$$\widetilde{C}_\beta(\theta) = \{\xi\in\mathbb{R}^n : C\beta < \measuredangle(\xi,\theta) < 2C\beta\}$$

for some constant $C>0$. Denote by $\kappa_\beta = C_\beta(\theta)\cap\mathbb{S}^{n-1}$ the spherical cap which generates the angular sector $C_\beta(\theta)$. The image of the sets $\mathbb{R}^n\times C_\beta(\theta)$ and $\mathbb{R}^n\times\widetilde{C}_\beta(\theta)$ along the Hamilton flow $\Phi_t^{\alpha,\pm}$ are denoted $H_\alpha^{\pm}C_\beta(\theta)$ and $H_\alpha^{\pm}\widetilde{C}_\beta(\theta)$, that is

$$H_\alpha^{\pm}C_\beta(\theta) = \{(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n : (y,\eta) = \Phi_t^{\alpha,\pm}(x,\xi),\ \text{where } \xi\in C_\beta(\theta)\}.$$

We define $\xi_\theta^{\alpha,\pm}(t,x)$ as the second component of the Hamilton flow defined via (2.46) with initial data

$$(x^{\pm}_{\alpha^{-1}}(0), \xi^{\pm}_{\alpha^{-1}}(0)) = (x,\theta)$$

at time t, hence it can be seen as the second component of the Hamilton flow $\Phi_t^{\alpha,\pm}(x,\theta)$ and it represent the evolution along the flow of the initial direction θ at x. Notice the the initial spherical cone $C_\beta(\theta)$ at initial time $t=0$ is transformed by the Hamilton flow into the cone $C_\beta(\xi_\theta^{\alpha,\pm}(t,x))$ centered $\xi_\theta^{\alpha,\pm}(t,x)$ and with angular opening β.

Observe that in the flat case, since the flow is constant in the second variable we have $H^{\pm}_\alpha C_\beta(\theta) = \mathbb{R}^n \times C_\beta(\theta)$. However, for a general metric, the initial localization of the frequency in the cone $C_\beta(\theta)$ will not be preserved by the Hamilton flow.

Define Ω_β as the finite collection of spherical caps on $\mathbb{S}^{n-1}$ of size $\beta \in 2^{-\mathbb{N}}$ with finite overlapping property and such that the union of all such caps cover all the sphere $\mathbb{S}^{n-1}$: $\Omega_\beta = \{(\kappa^j_\beta)\}_{j\in J}$. Consider a partition of unity at initial time $t=0$, given by

$$\sum_{\kappa_\beta \in \Omega_\beta} \chi_{\kappa_\beta}(x,\xi) = 1,$$

where χ_{κ_β} are 0-homogeneous symbols with support contained in $\mathbb{R}^n \times C_\beta(\theta)$, here $\kappa_\beta = C_\beta(\theta)\cap\mathbb{S}^{n-1}$. Recall that in general a m-homogeneous symbol is a function $a \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$ which satisfies the following condition: for every $p,q\in\mathbb{N}^n$ there exists constant $C_{p,q}>0$ such that

$$|\partial_x^p \partial_\xi^q a(x,\xi)| \le C_{p,q}|\xi|^{m-|q|+|p|}.$$

To define an appropriate time-dependent symbols we transport χ_{κ_β} along the Hamilton flow, thus we set

$$\chi^{\alpha,\pm}_{\kappa_\beta}(t,x,\xi) = \chi_{\kappa_\beta}(\Phi_t^{\alpha,\pm}(x,\xi)).$$

Clearly we have a time-dependent partition of unity

$$\sum_{\kappa_\beta \in \Omega_\beta} \chi^{\alpha,\pm}_{\kappa_\beta}(t,x,\xi) = 1.$$

Moreover, for each $\lambda \in 2^{\mathbb{Z}}$ we localize the symbols $\chi^{\alpha,\pm}_{\kappa_\beta}$ to frequencies less than $\lambda/8$. Define the symbols

$$\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi) = P_{<\lambda/8}(D_x)\chi^{\alpha,\pm}_{\kappa_\beta}(t,x,\xi)\widetilde{p}_\lambda(\xi).$$

Furthermore if $\alpha = \sqrt{\lambda}$ in the definition of $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}$, we shall simply write $\chi^{\pm}_{\kappa_\beta,\lambda}(t,x,\xi)$. We use $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi)$ to split a frequency localized wave into directionally localized pieces:

$$P_\lambda u = \sum_{\kappa_\beta\in\Omega_\beta} \chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)P_\lambda u.$$

Remark. Observe that the final cutoff $P_{<\lambda/8}(D_x)$ it is necessary to obtain the correct frequency output. Precisely let us consider the Fourier support of the function $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)P_\lambda u$: we have

$$\begin{aligned}\mathrm{supp}\mathscr{F}[\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)P_\lambda u] &= \mathrm{supp}\{\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi)*\widehat{P_\lambda u}\}\\ &\subset \mathrm{supp}\{\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi)\}+\mathrm{supp}\{\widehat{P_\lambda u}\}\\ &= \{|\xi|\le\lambda/8\}+\{|\xi|\approx\lambda\}\\ &\subset \{|\xi|\approx\lambda\}.\end{aligned}$$

The drawback of applying the cutoff $P_{<\lambda/8}(D_x)$ is that the symbol $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi)$ is no longer sharply localized to the sector $C_\beta(\xi^{\alpha,\pm}_{\theta_{\kappa_\beta}}(t,x))$, where θ_{κ_β} is the center direction of the spherical cap κ_β at the initial time. In fact, we can write $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}$ as the sum of a symbol which sharply localizes into the set $\{|\xi|\approx\lambda,\ \measuredangle(\xi,\xi^{\alpha,\pm}_{\theta_{\kappa_\beta}})\lesssim\lambda\beta\}$ plus an error which has better regularity properties:

$$\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi)=\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi)\chi_{\lesssim\lambda\beta}(|\xi-\xi^{\alpha,\pm}_{\theta_{\kappa_\beta}}(t,x)|)+r^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi).$$

The first symbol on the right-hand side has the same regularity properties as the original one $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}$ and the second symbol is much more regular $r^{\alpha,\pm}_{\kappa_\beta,\lambda}=O(\lambda^{-\infty})$. In what follows, we shall ignore the error $r^{\alpha,\pm}_{\kappa_\beta,\lambda}$ and make the harmless assumption that $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}$ sharply localize into the corresponding angular sector.

Remark. Observe that $\chi^{\alpha,\pm}_{\kappa_\beta}(t,x,\xi)$ depends on the initial spherical cap κ_β with center direction θ and cap size β, the truncation cutoff on the half-wave operators α. Whereas $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,\xi)$ depends on all the above parameters plus the frequency cutoff λ.

Remark. Let us analyze here the phase space localization of the symbol $\chi^{\alpha,\pm}_{\kappa_\beta}$ and $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}$. Consider the Hamilton flow with initial data given by (x_0,θ), the solution after t_0-time is given by the vector $\Phi^{\alpha,\pm}_{t_0}(x_0,\theta)$, denote its second component by $\xi^{\alpha,\pm}_\theta(t_0,x_0)$. Hence, for each spacetime point (t_0,x_0), and any initial direction θ, we obtain a corresponding *center direction* $\xi^{\alpha,\pm}_\theta(t_0,x_0)\in\mathbb{R}^n$. As for the symbols we drop the subscript α if $\alpha=\sqrt{\lambda}$ and we simply write $\xi^{\pm}_\theta(t_0,x_0)$. Observe that in the flat case $\xi^{\alpha,\pm}_\theta(t_0,x_0)=\theta$ is constant for every spacetime point (t_0,x_0). The symbol $\chi^{\alpha,\pm}_{\kappa_\beta}(t_0,x_0,\cdot)$ localize frequencies in a cone sector of angle β centered at $\widehat{\xi}^{\alpha,\pm}_\theta(t_0,x_0)$ and $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t_0,x_0,\cdot)$ localize frequencies in a cone sector of angle β centered at $\widehat{\xi}^{\alpha,\pm}_\theta(t_0,x_0)$ that intersects with an annulus around $|\xi|\sim\lambda$. The hat here denotes the renormalized vector of norm one. Let us denote this second set by $A_{\beta,\lambda}(\xi^{\alpha,\pm}_\theta)\subset\mathbb{R}^n$, we have $|A_{\beta,\lambda}(\xi^{\alpha,\pm}_\theta)|\approx\lambda^n\beta^{n-1}$. What is crucial here is that the size of $A_{\beta,\lambda}(\xi^{\alpha,\pm}_\theta)$ does not depend on the center direction $\widehat{\xi}^{\alpha,\pm}_\theta(x_0,t_0)$, thus nor on the space-time location (x_0,t_0) nor on the initial direction θ and the truncation α. However the region where $\chi^{\alpha,\pm}_{\kappa_\beta}$ and $\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}$ localize in frequency depends on the center direction $\widehat{\xi}^{\alpha,\pm}_\theta$, thus where we are on space-time.

In view of the previous Remark we have the corresponding Bernstein type inequality:

Lemma 83. *Let* $1 \le r \le q \le \infty$*, then*

$$\|\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)u\|_{L^p_t L^q_x} \lesssim (\lambda^n \beta^{n-1})^{\frac{1}{r}-\frac{1}{q}} \|u\|_{L^p_t L^r_x}.$$

Therefore one can combine Bernstein inequality with Strichartz estimate to obtain better bounds. In fact in dimension $n > 3$ the Pecher pair $(2, \frac{2n-2}{n-3})$ is admissible hence we have the Strichartz estimate

$$\|\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)u\|_{L^2_t L_x^{\frac{2n-2}{n-3}}} \lesssim \lambda^{\frac{n}{2}-\frac{1}{2}-\frac{n^2-3n}{2n-2}} d^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,d}} = \lambda^{\frac{n+1}{2n-2}} d^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,d}}.$$

Combine this with Bernstein to obtain

$$\begin{aligned}\|\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)u\|_{L^2_t L^\infty_x} &\lesssim (\lambda^n \beta^{n-1})^{\frac{n-3}{2n-2}} \|\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)u\|_{L^2_t L_x^{\frac{2n-2}{n-3}}} \\ &\lesssim \lambda^{\frac{n-1}{2}} \beta^{\frac{n-3}{2}} d^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,d}}.\end{aligned}$$

If you compare this with the pure Strichartz bound

$$\|\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)u\|_{L^2_t L^\infty_x} \lesssim \lambda^{\frac{n-1}{2}} d^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,d}}.$$

We see that we have gain a factor of $\beta^{\frac{n-3}{2}}$ (recall $\beta < 1$). Moreover in dimension $n = 3$ when the Pecher pair is not available one directly apply Bernstein inequality to obtain

$$\|\chi^{\alpha,\pm}_{\kappa_\beta,\lambda}(t,x,D)u\|_{L^2_t L^\infty_x} \lesssim \lambda^{\frac{n}{2}} \beta^{\frac{n-1}{2}} \|u\|_{X^{0,0}_{\lambda,d}}. \tag{2.47}$$

We end this section with the following Proposition that give us an almost orthogonal decomposition with respect to the $X_{\pm,\lambda}$ and $X^{0,0}_{\lambda,d}$ norms:

Proposition 84 ([30] Proposition 4.5)**.** *Fix a frequency* λ *and let* $\alpha > \lambda^{-1/2}$ *then we have the* l^2*-decomposition*

$$\sum_{\kappa_\alpha \in \Omega_\alpha} \|\chi^{\pm}_{\kappa_\alpha,\lambda}(t,x,D)u\|^2_{X_{\pm,\lambda}} = \|u\|^2_{X_{\pm,\lambda}}.$$

Corollary 85. *Fix a frequency* λ *and a modulation* d *and let* $\alpha > \lambda^{-1/2}$ *then we have the* l^2*-decomposition*

$$\sum_{\kappa_\alpha \in \Omega_\alpha} \|\chi^{\pm}_{\kappa_\alpha,\lambda}(t,x,D)u\|^2_{X^{0,0}_{\lambda,d}} = \|u\|^2_{X^{0,0}_{\lambda,d}}.$$

Angular bilinear and trilinear decompositions

In the last part of this section we shall show how to perform an angular decomposition in terms of the angular localization operators defined previously. Precisely, a Whitney-type decomposition with respect to the smallest localization threshold α gives:

$$\begin{aligned}\widetilde{P}_\lambda u(t,x)\widetilde{P}_\lambda w(t,x) &= \sum_{\kappa_\alpha^1,\kappa_\alpha^2\in\Omega_\alpha: dist(\pm\kappa_\alpha^1,\kappa_\alpha^2)\lesssim\alpha} \chi^\pm_{\kappa_\alpha^1,\lambda}(t,x,D)P_\lambda u\,\chi^\pm_{\kappa_\alpha^2,\lambda}(t,x,D)P_\lambda w \\ &+ \sum_{\beta=\alpha}^{1}\sum_{\kappa_\beta^1,\kappa_\beta^2\in\Omega_\beta: dist(\pm\kappa_\beta^1,\kappa_\beta^2)\approx\beta} \chi^\pm_{\kappa_\beta^1,\lambda}(t,x,D)P_\lambda u\,\chi^\pm_{\kappa_\beta^2,\lambda}(t,x,D)P_\lambda w.\end{aligned}$$

In the first term is denoted parallel interaction since the two spherical caps κ_α^1 and κ_α^2 have opening α and are separated by an angle *up* to α, that is the angular separation is zero. On the other hand, the second term is denoted perpendicular interaction since here the caps κ_β^1 and κ_β^2 have opening β and are separated by an angle proportional to β. To simplify notation we shall write $\kappa_\alpha^1 \approx_\gamma \kappa_\alpha^2$ if two spherical caps $\kappa_\alpha^1,\kappa_\alpha^2\in\Omega_\alpha$ are such that $dist(\pm\kappa_\alpha^1,\kappa_\alpha^2)\lesssim\gamma$ and we shall denote $\kappa_\beta^1\perp_\gamma\kappa_\beta^2$ if the two spherical caps $\kappa_\beta^1,\kappa_\beta^2\in\Omega_\beta$ are such that $dist(\pm\kappa_\beta^1,\kappa_\beta^2)\approx\gamma$. Here $dist$ denote the angular distance between two subset of $\mathbb{S}^{n-1}$. Therefore we obtain the handy formula

$$\begin{aligned}\widetilde{P}_\lambda u(t,x)\widetilde{P}_\lambda w(t,x) &= \sum_{\kappa_\alpha^1\approx_\alpha\kappa_\alpha^2} \chi^\pm_{\kappa_\alpha^1,\lambda}(t,x,D)P_\lambda u\,\chi^\pm_{\kappa_\alpha^2,\lambda}(t,x,D)P_\lambda w \\ &+ \sum_{\beta=\alpha}^{1}\sum_{\kappa_\beta^1\perp_\beta\kappa_\beta^2} \chi^\pm_{\kappa_\beta^1,\lambda}(t,x,D)P_\lambda u\,\chi^\pm_{\kappa_\beta^2,\lambda}(t,x,D)P_\lambda w.\end{aligned}$$

Notice that in the first line the two high frequency terms have supports on a spherical cap of angular size $\alpha\lambda$ around the sphere $\{|\xi|\approx\lambda\}$ and of length λ in the perpendicular direction, that are separated by an angle of at most α. On the other hand in the term on the second line the supports have sizes $(\beta\lambda)^{n-1}\lambda$ and have an angular separation proportional to β.

We shall use the short hand notation $u^\pm_{\kappa,\lambda} = \chi^\pm_{\kappa,\lambda}(t,x,D)P_\lambda u$. The full trilinear decomposition with respect to the smallest localization angle α has the form (recall that $\mu\ll\lambda$):

$$\begin{aligned}&\widetilde{P}_\lambda u\,\widetilde{P}_\lambda w\,\widetilde{P}_\mu v \\ &= \sum_{\kappa_\alpha^1\approx_\alpha\kappa_\alpha^2\approx_\alpha\kappa_\alpha^3} u^\pm_{\kappa_\alpha^1,\lambda}\,w^\pm_{\kappa_\alpha^2,\lambda}\,v^\pm_{\kappa_\alpha^3,\mu} + \sum_{\beta=\alpha}^{1}\sum_{\kappa_\alpha^1\approx_\alpha\kappa_\alpha^2\perp_\beta\kappa_\beta^3} u^\pm_{\kappa_\alpha^1,\lambda}\,w^\pm_{\kappa_\alpha^2,\lambda}\,v^\pm_{\kappa_\beta^3,\mu} \\ &+ \sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa_\beta^1\perp_\beta\kappa_\beta^2} u^\pm_{\kappa_\beta^1,\lambda}\,w^\pm_{\kappa_\beta^2,\lambda}\,v^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu} + \sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\approx_{\frac{\beta\lambda}{\mu}}\kappa_\beta^1\perp_\beta\kappa_\beta^2} u^\pm_{\kappa_\beta^1,\lambda}\,w^\pm_{\kappa_\beta^2,\lambda}\,v^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu} \\ &+ \sum_{\beta=\alpha}^{1}\sum_{\gamma=\frac{\beta\lambda}{\mu}}^{1}\sum_{\kappa_\beta^1\perp_\beta\kappa_\beta^2\perp_\gamma\kappa_\gamma^3} u^\pm_{\kappa_\beta^1,\lambda}\,w^\pm_{\kappa_\beta^2,\lambda}\,v^\pm_{\kappa_\gamma^3,\mu} \qquad (2.48)\end{aligned}$$

The previous trilinear decomposition will be needed in the following section to exploit the null structure of the non linearity.

2.10 null-form estimate

In this section we prove a multiplicative estimate involving pure N_{ij} null-forms. The proposition below is the extension of Proposition 43 to curved metrics for $n = 3$ spatial dimensions.

Proposition 86. *Assume $n = 3$ and $3/4 < \theta < s - 1$, then we have the multiplicative estimate*

$$\|N(u, v)\|_{X^{s-1,\theta-1}} \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}},$$

where N be a linear combination with constant coefficients of N_{ij} null-forms.

In the previous proposition and throughout this section we shall made the implicit assumption that $s < 2$ since we shall need that $3/4 < \theta < 1$. By duality this estimate is equivalent to

$$\left|\int N(u, v) w \; dxdt\right| \lesssim \|u\|_{X^{s,\theta}} \|v\|_{X^{s,\theta}} \|w\|_{X^{1-s,1-\theta}+L^2H^{2-s-\theta}}.$$

We apply the Littlewood-Paley trilinear decomposition, due to symmetry we just need to consider two interaction cases: high-low-high and high-high-low.

High-low-high interaction

Suppose $\mu \ll \lambda$, we have to control the quantity $|\int N(P_\lambda u_\lambda, P_\mu v_\mu) P_\lambda w_\lambda \; dxdt|$. We start to dispense the cases that can be treated by Strichartz estimate and do not require the null-form structure nor the $n = 3$ hypothesis, thus in this first part relax the assumption to $n \geq 3$. Suppose that $w_\lambda \in L^2 H^{2-s-\theta}$ then we have

$$\begin{aligned}\left|\int N(P_\lambda u_\lambda, P_\mu v_\mu) P_\lambda w_\lambda \; dxdt\right| &\lesssim \|\nabla P_\lambda u_\lambda\|_{L^\infty L^2} \|\nabla P_\mu v_\mu\|_{L^{2+}L^\infty} \|P_\lambda w_\lambda\|_{L^2} \\ &\lesssim \lambda^{\theta-1} \mu^{1-s+n/2-1/2+} \|u_\lambda\|_{X^{s,\theta}_\lambda} \|v_\mu\|_{X^{s,\theta}_\mu} \|P_\lambda w_\lambda\|_{L^2 H^{2-s-\theta}}.\end{aligned}$$

Since the exponent of the high frequency is negative, we can transfer all the high frequencies to low frequency and close since $-s+\theta+n/2-1/2 < 0$. Next, let us consider the case $w_\lambda \in X^{1-s,1-\theta}_\lambda$, we split into high and low modulations

$$u_\lambda = \sum_{d=1}^{\lambda} u_{\lambda,d} = \sum_{d=1}^{\mu} u_{\lambda,d} + \sum_{d=2\mu}^{\lambda} u_{\lambda,d} =: u_{\lambda,\leq\mu} + u_{\lambda,>\mu}$$

and we observe that for the high modulation term we have the improved bound:

$$\|\nabla P_\lambda u_{\lambda,>\mu}\|_{L^2} \lesssim \lambda^{1-s} \sum_{d=2\mu}^{\lambda} d^{-\theta} \lambda^{s-1} d^{\theta} \|\nabla P_\lambda u_{\lambda,d}\|_{L^2} \lesssim \lambda^{1-s} \mu^{-\theta} \|u_\lambda\|_{X^{s,\theta}_\lambda}.$$

Therefore we can control the high modulation part of u_λ via Strichartz inequality as follows:

$$\begin{aligned}\Big|\int N(P_\lambda u_{\lambda,>\mu} P_\mu v_\mu) P_\lambda w_\lambda \, dxdt\Big| &\lesssim \|\nabla P_\lambda u_{\lambda,>\mu}\|_{L^2} \|\nabla P_\mu v_\mu\|_{L^\infty} \|P_\lambda w_\lambda\|_{L^2} \\ &\lesssim \mu^{-s-\theta+n/2+1} \|u_\lambda\|_{X^{s,\theta}_\lambda} \|v_\mu\|_{X^{s,\theta}_\mu} \|P_\lambda w_\lambda\|_{X^{1-s,1-\theta}_\lambda}.\end{aligned}$$

Recall that by Cauchy-Schwarz we have $\|P_\lambda w_\lambda\|_{L^2} \lesssim \lambda^{s-1} \|P_\lambda w_\lambda\|_{X^{1-s,1-\theta}_\lambda}$. Thus the high frequency exponent vanishes. Notice that the μ exponent is negative since $s > n/2 + 1/4$ and $\theta > 3/4$. Analogously we split w_λ into low modulations $w_{\lambda,\le\mu}$ and high modulations $w_{\lambda,>\mu}$, then we bound the high modulation term using Strichartz inequality:

$$\|\nabla P_\lambda w_{\lambda,>\mu}\|_{L^2} \lesssim \lambda^{s-1} \mu^{\theta-1} \|w_\lambda\|_{X^{1-s,1-\theta}_\lambda}.$$

Thus, even without the null-form gain we obtain:

$$\begin{aligned}\Big|\int N(P_\lambda u_\lambda P_\mu v_\mu) P_\lambda w_{\lambda,>\mu} \, dxdx\Big| &\lesssim \|\nabla P_\lambda u_\lambda\|_{L^\infty L^2} \|\nabla P_\mu v_\mu\|_{L^{2+} L^\infty} \|P_\lambda w_{\lambda,>\mu}\|_{L^2} \\ &\lesssim \mu^{-s+\theta+n/2-1/2+} \|u_\lambda\|_{X^{s,\theta}_\lambda} \|v_\mu\|_{X^{s,\theta}_\mu} \|w_\lambda\|_{X^{1-s,1-\theta}_\lambda}.\end{aligned}$$

Hence, we have reduced the proof of the high-low-high interaction to the boundedness of the low modulations term:

$$\Big|\int N(P_\lambda u_{\lambda,\le\mu} P_\mu v_\mu) P_\lambda w_{\lambda,\le\mu} \, dxdx\Big|.$$

This term can not be controlled by Strichartz estimate, thus to bound it we need to decompose it further into angular sectors. However, before performing such decomposition we shall simplify further to 1-modulations. First observe that, in view of the previous discussion, the proof of the high-low-high interaction follows from the following:

Proposition 87. *Let $n \ge 3$ and $3/4 < \theta < s - (n-1)/2$. Assume that we have the following bounds:*

i. if $d_2 < d_{max}$, where $d_{max} = \max\{d_1, d_2, d_3\}$, then

$$\|N(P_\lambda u_{\lambda,d_1} P_\mu v_{\mu,d_2}) P_\lambda w_{\lambda,d_3}\|_{L^1L^1} \lesssim \mu^{\frac{n}{2}+} \lambda d_1^{\frac12} d_2^{\frac12} d_3^{\frac12} \|u_{\lambda,d_1}\|_{X^{0,0}_{\lambda,d_1}} \|v_{\mu,d_2}\|_{X^{0,0}_{\mu,d_2}} \|w_{\lambda,d_3}\|_{X^{0,0}_{\lambda,d_3}};$$

ii. if $d_2 = d_{max}$ then

$$\|N(P_\lambda u_{\lambda,d_1} P_\mu v_{\mu,d_2}) P_\lambda w_{\lambda,d_3}\|_{L^1L^1} \lesssim \mu^{\frac{n+3}{4}} \lambda d_2^{\frac{n+1}{4}} d_{min}^{\frac12} \|u_{\lambda,d_1}\|_{X^{0,0}_{\lambda,d_1}} \|v_{\mu,d_2}\|_{X^{0,0}_{\mu,d_2}} \|w_{\lambda,d_3}\|_{X^{0,0}_{\lambda,d_3}};$$

where $d_{min} = \min\{d_1, d_2, d_3\}$.

Then we have

$$\left|\int N(P_\lambda u_{\lambda,\le\mu} P_\mu v_\mu) P_\lambda w_{\lambda,\le\mu}\, dxdt\right| \lesssim \mu^\alpha \|u_\lambda\|_{X^{s,\theta}_\lambda} \|v_\mu\|_{X^{s,\theta}_\mu} \|w_\lambda\|_{X^{1-s,1-\theta}_\lambda},$$

where $\alpha < 0$, thus the high-low-high interaction term is controlled.

Proof. It is an easy application of Cauchy-Schwarz inequality. Suppose $i.$ holds, then

$$\begin{aligned}
&\|N(P_\lambda u_{\lambda,\le\mu} P_\mu v_\mu) P_\lambda w_{\lambda,<\mu}\|_{L^1L^1} \lesssim \sum_{d_1,d_2,d_3=1}^{\mu} \|N(P_\lambda u_{\lambda,d_1} P_\mu v_{\mu,d_2}) P_\lambda w_{\lambda,d_3}\|_{L^1L^1}\\
&\lesssim \sum_{d_1,d_2,d_3=1}^{\mu} \mu^{-s+\frac{n}{2}+} d_1^{\frac12-\theta} d_2^{\frac12-\theta} d_3^{\theta-\frac12} \|u_{\lambda,d_1}\|_{X^{s,\theta}_{\lambda,d_1}} \|v_{\mu,d_2}\|_{X^{s,\theta}_{\mu,d_2}} \|w_{\lambda,d_3}\|_{X^{1-s,1-\theta}_{\lambda,d_3}}\\
&\lesssim \mu^{-s+\theta+n/2-1/2+} \|u_{\lambda,}\|_{X^{s,\theta}_\lambda} \|v_\mu\|_{X^{s,\theta}_\mu} \|w_\lambda\|_{X^{1-s,1-\theta}_\lambda}.
\end{aligned}$$

On the other hand if $ii.$ holds, suppose without loosing generality that $d_{min} = d_1$, then

$$\begin{aligned}
&\|N(P_\lambda u_{\lambda,\le\mu} P_\mu v_\mu) P_\lambda w_{\lambda,\le\mu}\|_{L^1L^1} \lesssim \sum_{d_1,d_2,d_3=1}^{\mu} \|N(P_\lambda u_{\lambda,d_1} P_\mu v_{\mu,d_2}) P_\lambda w_{\lambda,d_3}\|_{L^1L^1}\\
&\lesssim \sum_{d_1,d_2,d_3=1}^{\mu} \mu^{-s+\frac{n+3}{4}} d_1^{\frac12-\theta} d_2^{\frac{n+1}{4}-\theta} d_3^{\theta-1} \|u_{\lambda,d_1}\|_{X^{s,\theta}_{\lambda,d_1}} \|v_{\mu,d_2}\|_{X^{s,\theta}_{\mu,d_2}} \|w_{\lambda,d_3}\|_{X^{1-s,1-\theta}_{\lambda,d_3}}\\
&\lesssim \mu^{-s-\theta+n/2+1} \|u_{\lambda,}\|_{X^{s,\theta}_\lambda} \|v_\mu\|_{X^{s,\theta}_\mu} \|w_\lambda\|_{X^{1-s,1-\theta}_\lambda}.
\end{aligned}$$

Here we have use the fact that the d_1 and d_3 exponents are negative, while for d_2 we obtain, after having applied Cauchy-Schwarz inequality a positive exponent that is controlled by $\mu^{\frac{n+1}{4}-\theta}$. □

The next technical lemma allow us to reduce the proof of Proposition 87 further to one modulation.

Lemma 88. *Let $n \ge 3$ and suppose that the following bounds hold:*

$$\|N(u,v)w\|_{L^1L^1} \lesssim \begin{cases} \mu^{\frac{n}{2}+}\lambda d_1^{\frac12} \|u\|_{X^{0,0}_{\lambda,d_1}} \|v\|_{X^{0,0}_{\mu,1}} \|w\|_{X^{0,0}_{\lambda,1}} & \text{if } d_1 = d_{\max},\\ \mu^{\frac{n+3}{4}}\lambda d_2^{\frac{n+1}{4}} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\mu,d_2}} \|w\|_{X^{0,0}_{\lambda,1}} & \text{if } d_2 = d_{\max},\\ \mu^{\frac{n}{2}+}\lambda d_3^{\frac12} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\mu,1}} \|w\|_{X^{0,0}_{\lambda,d_3}} & \text{if } d_3 = d_{\max}, \end{cases} \tag{2.49}$$

then the estimates $i.$ and $ii.$ in Proposition 87 hold.

Proof. Let (χ_j) a smooth partition of unity of the time interval $[0,1]$ so that $I_j = \operatorname{supp}\chi_j$ and $|I_j| = \delta$. Notice that for $1 \le \delta^{-1} \le d$ we have the l^2-summability property

$$\|u\|^2_{X^{s,\theta}_{\lambda,d}} \approx \sum_{j\in\mathbb{N}} \|\chi_j(t)u\|^2_{X^{s,\theta}_{\lambda,d}}.$$

Moreover define $u^\delta(t,x) = u(\delta t, \delta x)$ then if $\delta d \geq 1$ we have the scaling law

$$\|u^\delta\|_{X^{s,\theta}_{\delta\lambda,\delta d}[0,1]} \approx \delta^{s+\theta-\frac{n+1}{2}} \|u\|_{X^{s,\theta}_{\lambda,d}[0,\delta]}.$$

See Proposition 2.6 on [27] for a proof of these properties. Moreover let $(\widetilde{\chi}_j)$ a similar family such that $\widetilde{\chi}_j = 1$ in I_j.

Now suppose $d_2 = d_{\max}$, and without loosing generality $d_1 = d_{min}$. We carry out a two steps argument: first we reduce the estimate $ii.$ in Proposition 87 to the case $d_{min} = 1$, that is suppose that such a estimate holds. Notice that by a change of variable we obtain

$$\begin{aligned} \|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} &\lesssim \sum_{j\in\mathbb{N}} \|\chi_j(t)N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \\ &\lesssim \sum_{j\in\mathbb{N}} \|N(\widetilde{\chi}_j u, \widetilde{\chi}_j v)\widetilde{\chi}_j w\|_{L^1_{t,x}(I_j\times\mathbb{R}^n)} \\ &\lesssim \delta^{n-1} \sum_{j\in\mathbb{N}} \|(N(\widetilde{\chi}_j u, \widetilde{\chi}_j v)\widetilde{\chi}_j w)^\delta\|_{L^1_{t,x}(I\times\mathbb{R}^n)}. \end{aligned}$$

We now apply our hypothesis, estimate $ii.$ in Proposition 87 where $d_1 = 1$, and use the scaling law to obtain

$$\begin{aligned} &\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \\ &\lesssim \sum_{j\in\mathbb{N}} \delta^{n-1}(\delta\mu)^{\frac{n+3}{4}}\delta\lambda(\delta d_2)^{\frac{n+1}{4}} \|(\widetilde{\chi}_j u)^\delta\|_{X^{0,0}_{\delta\lambda,1}[I]} \|(\widetilde{\chi}_j v)^\delta\|_{X^{0,0}_{\delta\mu,\delta d_2}[I]} \|(\widetilde{\chi}_j w)^\delta\|_{X^{0,0}_{\delta\lambda,\delta d_3}[I]} \\ &\lesssim \sum_{j\in\mathbb{N}} \delta^{-\frac{1}{2}+}\mu^{\frac{n+3}{4}}\lambda d_2^{\frac{n+1}{4}} \|\widetilde{\chi}_j u\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]} \|\widetilde{\chi}_j v\|_{X^{0,0}_{\mu,d_2}[I]} \|\widetilde{\chi}_j w\|_{X^{0,0}_{\lambda,d_3}[I]}. \end{aligned}$$

To close we use the l^2-summability property and the fact that $l^2 \subset l^3$, then we obtain

$$\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \lesssim \delta^{-\frac{1}{2}+}\mu^{\frac{n+3}{4}}\lambda d_2^{\frac{n+1}{4}} \|u\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]} \|v\|_{X^{0,0}_{\mu,d_2}[I]} \|w\|_{X^{0,0}_{\lambda,d_3}[I]}.$$

Choosing $\delta^{-1} = d_{\min}$ conclude the reduction of estimate $ii.$ in Proposition 87 to $d_1 = 1$.

Next, in the second reduction step, we reduce estimate $ii.$ with $d_1 = 1$ further to estimate $ii.$ with $d_1 = d_3 = 1$, yielding to the second line in (2.49), i.e. suppose that

$$\|N(u,v)w\|_{L^1_{t,x}} \lesssim \mu^{(n+3)/4}\lambda d_2^{(n+1)/4} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\mu,d_2}} \|w\|_{X^{0,0}_{\lambda,1}},$$

then

$$\|N(u,v)w\|_{L^1_{t,x}} \lesssim \mu^{(n+3)/4}\lambda d_2^{(n+1)/4} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\mu,d_2}} \|w\|_{X^{0,0}_{\lambda,d_3}}.$$

We proceed by following a similar argument as in the previous step but at the end we set a different value for δ. We have

$$\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \lesssim \sum_{j\in\mathbb{N}} \delta^{-\frac{1}{2}+}\mu^{\frac{n+3}{4}}\lambda d_2^{\frac{n+1}{4}}\|\widetilde{\chi}_j u\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]}\|\widetilde{\chi}_j v\|_{X^{0,0}_{\mu,d_2}[I]}\|\widetilde{\chi}_j w\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]}.$$

Set $\delta^{-1}=d_3$ and notice that the term involving u we apply the following simple bound (see Proposition 2.6 on [27]) $\|\widetilde{\chi}_j u\|_{X^{0,0}_{\lambda,\delta^{-1}}} \lesssim \delta^{\frac{1}{2}}\|u\|_{X^{0,0}_{\lambda,1}}$ for $\delta<1$, to recover the $d_1=1$ exponent. Then by Hölder inequality and the square summability property of the terms involving v and w we obtain the desired bound

$$\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \lesssim \mu^{\frac{n+3}{4}}\lambda d_2^{\frac{n+1}{4}}\|u\|_{X^{0,0}_{\lambda,1}[I]}\|v\|_{X^{0,0}_{\mu,d_2}[I]}\|w\|_{X^{0,0}_{\lambda,d_3}[I]}.$$

This conclude the proof for the $d_2=d_{max}$ since a similar argument holds in the case $d_3=d_{\min}$, where the role of d_1 and d_3 are swapped.

Now suppose that $d_2<d_{max}$, then we apply a similar two steps reduction algorithm. Suppose without loosing generality that $d_1=d_{max}$ and $d_2=d_{min}$, and assume that

$$\|N(u,v)w\|_{L^1_{t,x}} \lesssim \mu^{n/2+}\lambda d_1^{1/2}d_3^{1/2}\|u\|_{X^{0,0}_{\lambda,d_1}}\|v\|_{X^{0,0}_{\mu,1}}\|w\|_{X^{0,0}_{\lambda,d_3}} \tag{2.50}$$

holds. Then we obtain

$$\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \lesssim \delta^{-\frac{1}{2}+}\mu^{\frac{n}{2}+}\lambda d_1^{1/2}d_3^{1/2}\|u\|_{X^{0,0}_{\lambda,d_1}[I]}\|v\|_{X^{0,0}_{\mu,\delta^{-1}}[I]}\|w\|_{X^{0,0}_{\lambda,d_3}[I]}.$$

Setting $\delta^{-1}=d_2$ we obtain estimate i. in Proposition 87. Next to reduce (2.50) further to the first line in (2.49) we carry out a similar argument: suppose that

$$\|N(u,v)w\|_{L^1L^1} \lesssim \mu^{\frac{n}{2}+}\lambda d_1^{\frac{1}{2}}\|u\|_{X^{0,0}_{\lambda,d_1}}\|v\|_{X^{0,0}_{\mu,1}}\|w\|_{X^{0,0}_{\lambda,1}}$$

holds. Then we have

$$\begin{aligned}\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} &\lesssim \sum_{j\in\mathbb{N}}\delta^{-1+}\mu^{\frac{n}{2}+}\lambda d_1^{1/2}\|\widetilde{\chi}_j u\|_{X^{0,0}_{\lambda,d_1}[I]}\|\widetilde{\chi}_j v\|_{X^{0,0}_{\mu,\delta^{-1}}[I]}\|\widetilde{\chi}_j w\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]}\\ &\lesssim \delta^{-1/2+}\mu^{\frac{n}{2}+}\lambda d_1^{1/2}\|u\|_{X^{0,0}_{\lambda,d_1}[I]}\|v\|_{X^{0,0}_{\mu,1}[I]}\|w\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]}.\end{aligned}$$

Set $\delta^{-1}=d_3$ to obtain (2.50). The other cases, $d_1=d_{max}$ and $d_3=d_{min}$, or $d_3=d_{max}$, are treated in a similar way. □

The previous Lemma allow us to reduce the proof of Proposition 87 to the proof of the following key Proposition. Here we have to impose the condition $n=3$ on the space dimensions.

Proposition 89 (High-low-high interaction). *Let $n=3$, $3/4<\theta<s-1$, and $1\le d_1,d_2,d_3\le\mu\lesssim$*

λ, then following bounds hold:

$$\|N(P_\lambda u, P_\mu v)P_\lambda w\|_{L^1L^1} \lesssim \begin{cases} \lambda\mu^{\frac{3}{2}+} d_1^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,d_1}} \|v\|_{X^{0,0}_{\mu,1}} \|w\|_{X^{0,0}_{\lambda,1}} & \text{if } d_1 = d_{\max}, \\ \lambda\mu^{\frac{3}{2}} d_2 \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\mu,d_2}} \|w\|_{X^{0,0}_{\lambda,1}} & \text{if } d_2 = d_{\max}, \\ \lambda\mu^{\frac{3}{2}+} d_3^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\mu,1}} \|w\|_{X^{0,0}_{\lambda,d_3}} & \text{if } d_3 = d_{\max}. \end{cases}$$

In the proof of the previous proposition we shall need the following lemma:

Lemma 90 (Angular gain)**.** *Let $\chi^{\pm}_{\kappa^1_\alpha,\lambda}(t,x,D)P_\lambda u$ and $\chi^{\pm}_{\kappa^2_\beta,\mu}(t,x,D)P_\mu v$ two inputs of the null-form N such that the angular caps at time $t=0$ are separated by a constant γ : dist$(\kappa^1_\alpha,\kappa^2_\beta)\approx\gamma$. Then for $0<t<1$ we have*

$$\mathit{dist}\big(\Phi^{\sqrt{\lambda},\pm}_{t,2}(\kappa^1_\alpha), \Phi^{\sqrt{\mu},\pm}_{t,2}(\kappa^2_\beta)\big) \approx \gamma$$

where $\Phi^{\sqrt{\lambda},\pm}_{t,2}$ is the second component of the Hamilton flow defined via (2.46)*.*

The previous lemma highlights the reason why the null-form N_{ij} in this is particularly well suited with respect to angular localization. Recall that

$$N_{ij}(u,v) = \iint e^{ix\cdot(\xi+\eta)}[-\xi_i\eta_j + \eta_j\xi_i]\,\widehat{u}(t,\xi)\widehat{v}(t,\eta)\, d\eta d\xi$$

and $|-\xi_i\eta_j+\eta_j\xi_i| \lesssim |\xi||\eta||\angle(\xi,\eta)|$. Therefore the previous lemma allow us to conclude that dist$(\kappa^1_\alpha,\kappa^2_\beta)\approx\gamma$, then for small time we have

$$|N\big(\chi^{\pm}_{\kappa^1_\alpha,\lambda}(t,x,D)P_\lambda u, \chi^{\pm}_{\kappa^2_\beta,\mu}(t,x,D)P_\mu v\big)| \le (\alpha+\beta+\gamma)\lambda\mu|\chi^{\pm}_{\kappa^1_\alpha,\lambda}(t,x,D)P_\lambda u||\chi^{\pm}_{\kappa^2_\beta,\mu}(t,x,D)P_\mu v|$$

Proof of Lemma 90. Let θ_α and θ_β be respectively the two centers of the caps κ^1_α and κ^2_β. Define the corresponding centers at time t, denoted $\xi^{\pm}_{\theta_\alpha}(t,x)$ and $\xi^{\pm}_{\theta_\beta}(t,x)$, as the second component of the Hamilton flow with initial data (x,θ_α) and (x,θ_β) respectively. That is

$$\begin{cases} \dot{x}^{\pm} = \nabla_\xi a^{\pm}_{<\sqrt{\lambda}}(t,x^{\pm},\xi^{\pm}_{\theta_\alpha}), \\ \dot{\xi}^{\pm}_{\theta_\alpha} = -\nabla_x a^{\pm}_{<\sqrt{\lambda}}(t,x^{\pm},\xi^{\pm}_{\theta_\alpha}), \\ (x^{\pm},\xi^{\pm}_{\theta_\alpha})|_{t=0} = (x,\theta_\alpha), \end{cases} \qquad \begin{cases} \dot{x}^{\pm} = \nabla_\xi a^{\pm}_{<\sqrt{\mu}}(t,x^{\pm},\xi^{\pm}_{\theta_\beta}), \\ \dot{\xi}^{\pm}_{\theta_\beta} = -\nabla_x a^{\pm}_{<\sqrt{\mu}}(t,x^{\pm},\xi^{\pm}_{\theta_\beta}), \\ (x^{\pm},\xi^{\pm}_{\theta_\beta})|_{t=0} = (x,\theta_\beta). \end{cases}$$

Notice that the lemma is proved if we show that dist$(\xi^{\pm}_{\theta_\alpha}(t,x),\xi^{\pm}_{\theta_\beta}(t,x))\approx\gamma$. Furthermore, by localizing the coefficients of the metric to a fixed smaller space-time scale and rescaling back to the unit scale, we can ensure that the coefficients satisfies

$$\|\nabla g^{\alpha\beta}_{<\sqrt{\lambda}}\|_{L^1_t(0,1)L^\infty_x(B(0,1))} \le \epsilon$$

thus the don't vary too much inside a ball of radius 1. Then, thanks to (2.45) the half-wave symbols a inherit the bounds on the metric, that is for bounded ξ we have

$$\|\nabla a(t,x,\xi)\|_{L^1_t(0,1)L^\infty_x(B(0,1))} \leq \epsilon.$$

Hence we obtain that the second component of the Hamilton flow is a $O(\epsilon)$-perturbation of the constant map: $\dot{\xi}^\pm_{\theta_\alpha} = \xi^\pm_{\theta_\alpha} + O(\epsilon t)$. □

We now turn to the proof of Proposition 89.

Proof. Hereafter we shall denote $u^\pm_{\kappa,\lambda} = \chi^\pm_{\kappa,\lambda}(t,x,D)P_\lambda u$. In order to prove Proposition 89 we apply the trilinear angular decomposition (2.48) from §2.9 which yield us to the following five terms:

$$\begin{aligned} & N(\widetilde{P}_\lambda u, \widetilde{P}_\mu v)\widetilde{P}_\lambda w \\ &\quad = \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \approx_\alpha \kappa^3_\alpha} N(u^\pm_{\kappa^1_\alpha,\lambda}, v^\pm_{\kappa^3_\alpha,\mu}) w^\pm_{\kappa^2_\alpha,\lambda} + \sum_{\beta=\alpha}^{1} \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \perp_\beta \kappa^3_\beta} N(u^\pm_{\kappa^1_\alpha,\lambda}, v_{\kappa^3_\beta,\mu}) w^\pm_{\kappa^2_\alpha,\lambda} \\ &\quad + \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \perp_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} N(u^\pm_{\kappa^1_\beta,\lambda}, v^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}) w^\pm_{\kappa^2_\beta,\lambda} + \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \approx_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} N(u^\pm_{\kappa^1_\beta,\lambda}, v^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}) w^\pm_{\kappa^2_\beta,\lambda} \\ &\quad + \sum_{\beta=\alpha}^{1} \sum_{\gamma=\frac{\beta\lambda}{\mu}}^{1} \sum_{\kappa^1_\beta \perp_\beta \kappa^2_\beta \perp_\gamma \kappa^3_\gamma} N(u^\pm_{\kappa^1_\beta,\lambda}, v^\pm_{\kappa^3_\gamma,\mu}) w^\pm_{\kappa^2_\beta,\lambda} =: I + II + III + IV + V. \end{aligned}$$

Motivated by the analysis in the constant coefficients case we set the angular separation threshold to be $\alpha = \mu^{-1/2} d^{1/2}_{max}$. In what follows we shall carefully estimate each of the five terms.

Estimate for I - Small angle interactions

First, suppose that $d_{max} = d_2$, thus $\alpha = \mu^{-\frac{1}{2}} d_2^{\frac{1}{2}}$. Then we use Strichartz for high frequency term and Bernstein for the low-frequency one. Since the sum over spherical caps is diagonal we obtain:

$$\begin{aligned} \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \approx_\alpha \kappa^3_\alpha} \|N(u^\pm_{\kappa^1_\alpha,\lambda}, v^\pm_{\kappa^3_\alpha,\mu})\, w^\pm_{\kappa^2_\alpha,\lambda}\|_{L^1_{t,x}} &\lesssim \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \approx_\alpha \kappa^3_\alpha} \lambda\mu\alpha \|u^\pm_{\kappa^1_\alpha,\lambda}\|_{L^\infty L^2} \|w^\pm_{\kappa^2_\alpha,\lambda}\|_{L^2L^2} \|v^\pm_{\kappa^3_\alpha,\mu}\|_{L^2L^\infty} \\ &\lesssim \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \approx_\alpha \kappa^3_\alpha} \lambda\mu\alpha\mu^{\frac{3}{2}}\alpha \|u^\pm_{\kappa^1_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}} \|w^\pm_{\kappa^2_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}} \|v^\pm_{\kappa^3_\alpha,\mu}\|_{X^{0,0}_{\mu,d_2}} \\ &\lesssim \mu^{\frac{3}{2}} \lambda d_2 \|u_\lambda\|_{X^{0,0}_{\lambda,1}} \|v_\mu\|_{X^{0,0}_{\mu,d_2}} \|w_\lambda\|_{X^{0,0}_{\lambda,1}}. \end{aligned}$$

Notice that the L^2L^∞ norm of the low-frequency term is estimated via the Bernstein type inequality (2.47). Moreover the summation with respect to κ^i_α is achieved via the l^2 decomposition property from Corollary 85: let $\alpha \geq \lambda^{-1/2}$, then

$$\sum_{\kappa\in\Omega_\alpha} \|u^{\pm}_{\kappa,\lambda}\|^2_{X^{0,0}_{\lambda,d}} \approx \|u_\lambda\|^2_{X^{0,0}_{\lambda,d}}.$$

Clearly a similar l^2 summation property holds as well for the low frequency term v_μ. Next, suppose without loosing generality that $d_{max} = d_1$, thus $\alpha = \mu^{-\frac{1}{2}} d_1^{\frac{1}{2}}$. Strichartz estimates yields to

$$\begin{aligned}
\sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \approx_\alpha \kappa^3_\alpha} & \|N(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\alpha,\mu})\, w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{L^1_{t,x}} \\
&\lesssim \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \approx_\alpha \kappa^3_\alpha} \lambda\mu\alpha \|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^2L^2} \|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{L^\infty L^2} \|v^{\pm}_{\kappa^3_\alpha,\mu}\|_{L^{2+}L^\infty} \\
&\lesssim \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \approx_\alpha \kappa^3_\alpha} \lambda\mu\alpha\mu^{1+} \|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{X^{0,0}_{\lambda,d_1}} \|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}} \|v^{\pm}_{\kappa^3_\alpha,\mu}\|_{X^{0,0}_{\mu,1}} \\
&\lesssim \mu^{\frac{3}{2}+} \lambda d_1^{\frac{1}{2}} \|u_\lambda\|_{X^{0,0}_{\lambda,d_1}} \|v_\mu\|_{X^{0,0}_{\mu,2}} \|w_\lambda\|_{X^{0,0}_{\lambda,1}}.
\end{aligned}$$

Notice that if the maximum modulation is coupled with the other high frequency term, i.e $d_{max} = d_3$, then we simply permute the L^2L^2 and the $L^\infty L^2$ norms in the second line of the previous estimate.

Estimate for II - Non resonant interactions

This is the most difficult term to estimate. Given a spherical cap $\kappa_\beta \in \Omega_\beta$, define its center direction by θ_{κ_β}, and its evolution along the Hamilton flows by $\xi^{\pm}_{\theta_{\kappa_\beta}}(t,x)$, where $(x, \theta_{\kappa_\beta})$ are taken as initial data. Let us denote $\widetilde{a}_{<\mu^{1/2}}(t,x,\xi)$ the linearization of $a_{<\mu^{1/2}}(t,x,\xi)$ with respect to ξ around the vector $\xi^{\pm}_{\theta_{\kappa_\beta}}(t,x)$. Recall that, by definition, the symbol $a_{<\mu^{1/2}}(t,x,\xi)$ is homogeneous of order 1, hence

$$\widetilde{a}_{<\mu^{1/2}}(t,x,\xi) = \xi\cdot\nabla_\xi a_{<\mu^{1/2}}(t,x,\theta_{\xi^{\pm}_{\kappa_\beta}}(t,x)).$$

We define the symbols $e(t,x,\xi) = \widetilde{a}_{<\mu^{1/2}}(t,x,\xi) - a_{<\mu^{1/2}}(t,x,\xi)$, we can estimate

$$|e(t,x,\xi)| \approx |\xi| |\angle\big(\xi, \xi^{\pm}_{\theta_{\kappa_\beta}}(t,x)\big)|^2$$

which, in the support of the symbol $\chi^{\pm}_{\theta_{\kappa_\beta},\mu}$, has angular size $|\angle\big(\xi,\xi^{\pm}_{\theta_{\kappa_\beta}}(t,x)\big)| \approx \beta$ and frequency has size $|\xi| \approx \mu$. Let us define its local inverse $l_{\kappa_\beta,\mu}(t,x,\xi) = \widetilde{\chi}^{\pm}_{\kappa_\beta,\mu}(t,x,\xi)e^{-1}(t,x,\xi)$. Furthermore, we localize the local inverse back to frequency μ by introducing the operator:

$$\widetilde{L}_{\kappa_\beta,\mu}(t,x) = P_\mu(D_x)\, l_{\kappa_\beta,\mu}(t,x,D).$$

Remark. Recall that in the flat case we have $a_{<\mu^{1/2}}(t,x,\xi) = |\xi|$ and $\theta_{\kappa_\beta}(t,x) = \theta_\beta$ is constant, hence $\widetilde{a}_{<\mu^{1/2}}(t,x,\xi) =$

The properties of the $\widetilde{L}_{\kappa_\beta,\mu}$ operator are analyzed in [30]. We restated them here for completeness.

Lemma 91 ([30] Lemma 5.2)**.** *The operator* $\widetilde{L}$ *satisfies the following estimates:*

(a) fixed-time L^p *mapping properties:*

$$\|\widetilde{L}_{\kappa_\beta,\mu} v\|_{L^p_x} \lesssim \beta^{-2}\mu^{-1}\|v\|_{L^p_x}, \qquad 1 \leq p \leq \infty;$$

(b) fixed-time approximate inverse of e:

$$\|((A_{<\mu^{1/2}} - \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_\beta,\mu} - 1)v\|_{L^p_x} \lesssim (\mu^{-\frac{1}{2}} + \beta^{-2}\mu^{-1})\|v\|_{L^p_x}, \qquad 1 \leq p \leq \infty.$$

In order to take advantage of the previous lemma we split the proof of the II term into five parts based on the following decomposition:

$$\begin{aligned} N(u,v)w &= N\big(u, -((A_{<\mu^{1/2}} - \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_\beta,\mu} - 1)v\big)w + N\big(u, (A_{<\mu^{1/2}} - \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_\beta,\mu} v\big)w \\ &= N\big(u, -((A_{<\mu^{1/2}} - \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_\beta,\mu} - 1)v\big)w + N\big((D_t + A_{<\mu^{1/2}})u, \widetilde{L}_{\kappa_\beta,\mu} v\big)w \\ &+ N\big(u, \widetilde{L}_{\kappa_\beta,\mu} v\big)(D_t + A_{<\mu^{1/2}})w + N\big(u, (D_t + A_{<\mu^{1/2}})\widetilde{L}_{\kappa_\beta,\mu} v\big)w - E(u,v,w) \\ &= II.a + II.b + II.c + II.d + II.e \end{aligned}$$

where

$$E(u,v,w) = N\big((D_t + A_{<\mu^{1/2}})u, \widetilde{L}_{\kappa_\beta,\mu} v\big)w + N\big(u, \widetilde{L}_{\kappa_\beta,\mu} v\big)(D_t + A_{<\mu^{1/2}})w + N\big(u, (D_t + \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_\beta,\mu} v\big)w.$$

Below we shall carefully estimate each of the following five terms.

Estimate for II.a

Let us consider the case $d_{max} = d_2$ first. We apply Lemma 91 and Strichartz inequality for the low frequency term to obtain:

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \| N\big(u^{\pm}_{\kappa_\alpha^1,\lambda}, -((A_{<\sqrt{\mu}} - \widetilde{A}_{<\sqrt{\mu}})\widetilde{L}_{\kappa_\beta,\mu} - 1)\, v^{\pm}_{\kappa_\beta^3,\mu}\big) w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{L^1_{t,x}} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu\beta(\mu^{-\frac{1}{2}} + \beta^{-2}\mu^{-1}) \| u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{L^\infty L^2} \| w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{L^2 L^2} \| v^{\pm}_{\kappa_\beta^3,\mu} \|_{L^{2+} L^\infty} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu\beta(\mu^{-\frac{1}{2}} + \beta^{-2}\mu^{-1})\mu^{1+} d_2^{\frac{1}{2}} \| u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{X^{0,0}_{\lambda,1}} \| w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{X^{0,0}_{\lambda,1}} \| v^{\pm}_{\kappa_\beta^3,\mu} \|_{X^{0,0}_{\mu,d_2}} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} (\lambda\mu^{\frac{3}{2}+} d_2^{\frac{1}{2}} + \lambda\mu^{1+} d_2^{\frac{1}{2}} (\beta\alpha^{-1})^{-1}\alpha^{-1}) \| u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{X^{0,0}_{\lambda,1}} \| w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{X^{0,0}_{\lambda,1}} \| v^{\pm}_{\kappa_\beta^3,\mu} \|_{X^{0,0}_{\mu,d_2}} \\
&\lesssim (\lambda\mu^{\frac{3}{2}+} d_2^{\frac{1}{2}} + \lambda\mu^{\frac{3}{2}+}) \| u_\lambda \|_{X^{0,0}_{\lambda,1}} \| v_\mu \|_{X^{0,0}_{\mu,d_2}} \| w_\lambda \|_{X^{0,0}_{\lambda,1}}.
\end{aligned}$$

Recall that $\beta > \alpha = \mu^{-1/2} d_2^{1/2}$ and in the β sum we have applied Cauchy-Schwarz to obtain an l^2-series. Now consider the case $d_{max} \neq d_2$, then we apply the same argument and we make sure to place the high modulation term into L^2L^2, to avoid loosing powers of the modulation. This procedure will lead to a better constant due to the better bound for Strichartz inequality for the low frequency term, i.e. μ^{1+}.

Estimate for II.b

For the case $d_{max} = d_2$ we proceed as the corresponding case in II.a. by using Bernstein inequality in the low frequency term. Therefore by Corollary 81, which allow us to control the term $\|(D_t \pm A^{\pm}_{<\sqrt{\lambda}})u\|_2$ with $\|u\|_{X^{0,0}_{\lambda,1}}$, we obtain

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \| N\big((D_t + A_{\sqrt{\lambda}})u^{\pm}_{\kappa_\alpha^1,\lambda}, \widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa_\beta^3,\mu}\big) w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{L^1_{t,x}} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu\beta(\beta^2\mu)^{-1} \| (D_t + A_{\sqrt{\lambda}})u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{L^2L^2} \| w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{L^\infty L^2} \| v^{\pm}_{\kappa_\beta^3,\mu} \|_{L^2 L^\infty} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu^{\frac{3}{2}} \| u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{X^{0,0}_{\lambda,1}} \| w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{X^{0,0}_{\lambda,1}} \| v^{\pm}_{\kappa_\beta^3,\mu} \|_{X^{0,0}_{\mu,d_2}} \\
&\lesssim \lambda\mu^{\frac{3}{2}} \| u_\lambda \|_{X^{0,0}_{\lambda,1}} \| v_\mu \|_{X^{0,0}_{\mu,1}} \| w_\lambda \|_{X^{0,0}_{\lambda,d_2}}.
\end{aligned}$$

For this term as well we have an extra room for a d_2 factor. Next consider $d_{max} = d_1$, we replace Bernstein inequality by Strichartz estimate in the previous bound to obtain

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \| N\big((D_t + A_{\sqrt{\lambda}}) u^{\pm}_{\kappa^1_\alpha,\lambda}, \widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\big) w^{\pm}_{\kappa^2_\alpha,\lambda} \|_{L^1_{t,x}} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu\beta(\beta^2\mu)^{-1} \| (D_t + A_{\sqrt{\lambda}}) u^{\pm}_{\kappa^1_\alpha,\lambda} \|_{L^2L^2} \| w^{\pm}_{\kappa^2_\alpha,\lambda} \|_{L^\infty L^2} \| v^{\pm}_{\kappa^3_\beta,\mu} \|_{L^{2+}L^\infty} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} (\beta\alpha^{-1})^{-1} \lambda \mu^{1+} \alpha^{-1} d_1 \| u^{\pm}_{\kappa^1_\alpha,\lambda} \|_{X^{0,0}_{\lambda,d_1}} \| w^{\pm}_{\kappa^2_\alpha,\lambda} \|_{X^{0,0}_{\lambda,1}} \| v^{\pm}_{\kappa^3_\beta,\mu} \|_{X^{0,0}_{\mu,1}} \\
&\lesssim \lambda \mu^{\frac{3}{2}+} d_1^{\frac{1}{2}} \| u_\lambda \|_{X^{0,0}_{\lambda,d_1}} \| v_\mu \|_{X^{0,0}_{\mu,1}} \| w_\lambda \|_{X^{0,0}_{\lambda,1}}.
\end{aligned}$$

Recall that Corollary 81 tell us that $\|(D_t \pm A^{\pm}_{<\sqrt{\lambda}}) u\|_2 \lesssim d_1 \|u\|_{X^{0,0}_{\lambda,d_1}}$. Finally if $d_{max} = d_3$ we apply the same procedure as above except that the d_1 factor is replaced by the weaker $d_3^{1/2}$ coming from Strichartz estimate for the w term.

Estimate for II.c

This is similar to II.b since the half-wave operator $D_t + A_{\sqrt{\lambda}}$ hits the high frequency term. As for the previous cases we start estimating the $d_{max} = d_2$ case. We use Bernstein and Lemma 91 the estimate the low frequency term, this yields to

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \| N\big(u^{\pm}_{\kappa^1_\alpha,\lambda}, \widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\big)(D_t + A_{\sqrt{\lambda}}) w^{\pm}_{\kappa^2_\alpha,\lambda} \|_{L^1_{t,x}} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu\beta(\beta^2\mu)^{-1} \| u^{\pm}_{\kappa^1_\alpha,\lambda} \|_{L^\infty L^2} \| (D_t + A_{\sqrt{\lambda}}) w^{\pm}_{\kappa^2_\alpha,\lambda} \|_{L^2L^2} \| v^{\pm}_{\kappa^3_\beta,\mu} \|_{L^2L^\infty} \\
&\lesssim \lambda \mu^{\frac{3}{2}} \| u_\lambda \|_{X^{0,0}_{\lambda,1}} \| v_\mu \|_{X^{0,0}_{\mu,1}} \| w_\lambda \|_{X^{0,0}_{\lambda,d}}.
\end{aligned}$$

The case $d_{max} = d_3$ is similar to the corresponding one for II.b, we write it below for completeness. Here we place the high frequency term involving the half-wave operator $D_t + A_{\sqrt{\lambda}}$ into L^2L^2 in order to avoid loosing extra powers the modulation. Recall that for such interaction we have $\alpha = \mu^{-1/2} d_3^{1/2}$, hence

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \| N\big(u^{\pm}_{\kappa_\alpha^1,\lambda}, \widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa_\beta^3,\mu}\big)(D_t + A_{\sqrt{\lambda}}) w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{L^1_{t,x}} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu\beta(\beta^2\mu)^{-1} \| u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{L^\infty L^2} \| (D_t + A_{\sqrt{\lambda}}) w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{L^2 L^2} \| v^{\pm}_{\kappa_\beta^3,\mu} \|_{L^{2+} L^\infty} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} (\beta\alpha^{-1})^{-1} \lambda\mu^{1+} \alpha^{-1} d_3 \| u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{X^{0,0}_{\lambda,1}} \| w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{X^{0,0}_{\lambda,d_3}} \| v^{\pm}_{\kappa_\beta^3,\mu} \|_{X^{0,0}_{\mu,1}} \\
&\lesssim \lambda\mu^{\frac{3}{2}+} d_3^{\frac{1}{2}} \| u_\lambda \|_{X^{0,0}_{\lambda,1}} \| v_\mu \|_{X^{0,0}_{\mu,1}} \| w_\lambda \|_{X^{0,0}_{\lambda,d_3}}.
\end{aligned}$$

To estimate the case $d_{max} = d_1$ we use the same Hölder pairs and we obtain $d_1^{1/2}$ instead of d_3, thus yielding to a better estimate then needed.

Estimate for II.d

Consider $d_{max} = d_2$. We place the low frequency term into $L^2 L^\infty$ and use Bernstein, then we apply Corollary 81 which yield to a factor of d_2. Therefore we obtain the sharp estimate

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \| N(u^{\pm}_{\kappa_\alpha^1,\lambda}, (D_t + A_{\sqrt{\mu}}) \widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa_\beta^3,\mu}) w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{L^1_{t,x}} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu\beta(\beta^2\mu)^{-1} \| u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{L^\infty L^2} \| w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{L^2 L^2} \| (D_t + A_{\sqrt{\mu}}) v^{\pm}_{\kappa_\beta^3,\mu} \|_{L^2 L^\infty} \\
&\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \perp_\beta \kappa_\beta^3} \lambda\mu^{\frac{3}{2}} d_2 \| u^{\pm}_{\kappa_\alpha^1,\lambda} \|_{X^{0,0}_{\lambda,1}} \| w^{\pm}_{\kappa_\alpha^2,\lambda} \|_{X^{0,0}_{\lambda,1}} \| v^{\pm}_{\kappa_\beta^3,\mu} \|_{X^{0,0}_{\mu,d_2}} \\
&\lesssim \lambda\mu^{\frac{3}{2}} d_2 \| u_\lambda \|_{X^{0,0}_{\lambda,1}} \| v_\mu \|_{X^{0,0}_{\mu,d_2}} \| w_\lambda \|_{X^{0,0}_{\lambda,1}}.
\end{aligned}$$

Notice that when $d_{max} \neq d_2$ then we place the maximum modulation term into $L^2 L^2$ and use Bernstein and Corollary 81 for the low frequency term. This yields to a better factor since we do not loose a power of the high modulation when we apply Corollary 81:

$$\lambda\mu\beta(\beta^2\mu)^{-1} \mu^{\frac{3}{2}} \beta \leq \lambda\mu^{\frac{3}{2}}.$$

Estimate for II.e

To estimate the E term we first dispense the time derivatives. Observe that $E(u, v, w) = E_0(u, v, w) + E_1(u, w, v)$, where

$$E_0(u, v, w) = N\big(A_{\mu^{1/2}} u, \widetilde{L}_{\kappa_\beta,\mu} v\big) w + N\big(u, \widetilde{L}_{\kappa_\beta,\mu} v\big) A_{\mu^{1/2}} w + N\big(u, \widetilde{A}_{\mu^{1/2}} \widetilde{L}_{\kappa_\beta,\mu} v\big) w,$$

and

$$E_1(u,v,w) = N\big(D_t u, \widetilde{L}_{\kappa_\beta,\mu} v\big) w + N\big(u, \widetilde{L}_{\kappa_\beta,\mu} v\big) D_t w + N\big(u, D_t \widetilde{L}_{\kappa_\beta,\mu} v\big) w.$$

We first treat the easier E_1 term, suppose $d_{max} = d_2$ then when computing the $L^1_{t,x}$ norm of the E_1 term we can apply the fundamental Theorem of calculus to obtain

$$\begin{aligned}\|E_1(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\beta,\mu}, w^{\pm}_{\kappa^2_\beta,\lambda})\|_{L^1_{t,x}} &\lesssim \|N\big(u^{\pm}_{\kappa^1_\alpha,\lambda}, \widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\big) w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^1_x}\Big|_{t=0}^{t=1} \\ &\lesssim \lambda\mu\beta \|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^\infty_t L^2_x} \|\widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^\infty_t L^\infty_x} \|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^\infty_t L^2_x}.\end{aligned}$$

By Lemma 91 the $\widetilde{L}_{\kappa_\beta,\mu}$ operator produces $(\beta^2\mu)^{-1}$ and we apply Bernstein inequality for the low modulation term to reduce it to $L^\infty_t L^2_x$ yielding to factor of $\mu^{3/2}\beta$ and Strichartz inequality to reach the $X^{0,0}_{\mu,d_2}$ space at the price of an extra $d_2^{1/2}$ factor. Therefore we obtain

$$\|E_1(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\beta,\mu}, w^{\pm}_{\kappa^2_\beta,\lambda})\|_{L^1_{t,x}} \lesssim \lambda\mu^{\frac{3}{2}} d_2^{\frac{1}{2}} \|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}} \|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}} \|v^{\pm}_{\kappa^3_\beta,\mu}\|_{X^{0,0}_{\mu,d_2}}.$$

The same argument can be applied to the case $d_{\max} \neq d_2$ case, yielding to a factor of $\lambda\mu^{\frac{3}{2}} d_{min}^{\frac{1}{2}}$, which is acceptable.

To estimate the E_0 term, let us define

$$\widetilde{E}_0(u,v,w) = (A_{<\mu^{1/2}} u)(\widetilde{L}_{\kappa_\beta,\mu} v) w + u(\widetilde{L}_{\kappa_\beta,\mu} v)(A_{<\mu^{1/2}} w) + u(\widetilde{A}_{<\mu^{1/2}} \widetilde{L}_{\kappa_\beta,\mu} v) w.$$

We shall refer to the bound for $\widetilde{E}_0$ provided in [30].

Lemma 92 ([30] Lemma 5.3)**.** *Let $1 \leq \mu \lesssim \lambda$. Assume that $\xi^{\pm}_{\theta_{\kappa^1_\alpha}}(t,x)$ is a Lipschitz function of x with $|\xi^{\pm}_{\theta_{\kappa^1_\alpha}}(t,x) - \theta_{\kappa^1_\alpha}| \ll 1$ and suppose that the symbol is C^1 with respect to time and homogeneous or order one, that is $a \in C^1_t S^1_{hom}$. Then the trilinear form $\widetilde{E}_0$ satisfies the fixed-time estimate:*

$$\begin{aligned}\|\widetilde{E}_0(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\beta,\mu}, w^{\pm}_{\kappa^2_\beta,\lambda})\|_{L^1_x} &\lesssim \|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^{p_1}_x} \|\widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^{q_1}_x} \|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^{r_1}_x} \\ &+ \lambda^{-1} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^{p_2}_x} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) \widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^{q_2}_x} \|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^{r_2}_x} \\ &+ \lambda^{-1} \|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^{p_2}_x} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) \widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^{q_2}_x} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^{r_2}_x} \\ &+ \mu\lambda^{-2} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^{p_3}_x} \|\widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^{q_3}_x} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^{r_3}_x}\end{aligned}$$

for all indices $1/p_j + 1/q_j + 1/r_j = 1$.

We shall now show how Lemma 92 yield to the right bound for the II.e term. Consider the case

$d_{max} = d_2$, then choosing the triplet $(2,\infty,2)$ in Lemma 92 yields to

$$\begin{aligned}
\|E_0\big(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\beta,\mu}, w^{\pm}_{\kappa^2_\alpha,\lambda}\big)\|_{L^1_{t,x}} &\lesssim \lambda\mu\beta\|\widetilde{E}_0\big(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\beta,\mu}, w^{\pm}_{\kappa^2_\alpha,\lambda}\big)\|_{L^1_{t,x}} \\
&\lesssim \lambda\mu\beta\big(\|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^\infty L^2}\|\widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^2L^\infty}\|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{L^2L^2} \\
&+ \lambda^{-1}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^\infty L^2}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)\widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^{2+}L^\infty}\|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{L^2L^2} \\
&+ \lambda^{-1}\|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^\infty L^2}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)\widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^{2+}L^\infty}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{L^2L^2} \\
&+ \mu\lambda^{-2}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{L^\infty L^2}\|\widetilde{L}_{\kappa_\beta,\mu} v^{\pm}_{\kappa^3_\beta,\mu}\|_{L^2L^\infty}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{L^2L^2}\big).
\end{aligned}$$

Notice that the operator $(\xi_\theta \wedge D_x)$ when applied to $u^{\pm}_{\kappa^1_\alpha,\lambda}$ or $w^{\pm}_{\kappa^2_\alpha,\lambda}$ yield to a factor of $\lambda\alpha$, and when applied to $v^{\pm}_{\kappa^1_\beta,\mu}$ lead to a loss of $\mu\beta$. Moreover in the first and fourth term we apply Bernstein inequality while in the second we employ Strichartz estimate to avoid loosing a β factor. Thus we obtain

$$\begin{aligned}
&\|E_0\big(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\beta,\mu}, w^{\pm}_{\kappa^2_\alpha,\lambda}\big)\|_{L^1_{t,x}} \\
&\lesssim \lambda\mu\beta\frac{1}{\beta^2\mu}\big(\mu^{\frac{3}{2}}\beta + \alpha\beta\mu^{2+}d_2^{\frac{1}{2}} + \alpha\beta\mu^{2+}d_2^{\frac{1}{2}} + \mu^{\frac{3}{2}}\beta\alpha^2\mu\big)\|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^3_\beta,\mu}\|_{X^{0,0}_{\mu,d_2}}.
\end{aligned}$$

Therefore, since $\alpha = \mu^{-1/2}d_2^{1/2}$ we obtain

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1}\sum_{\kappa^1_\alpha\approx_\alpha\kappa^2_\alpha\perp_\beta\kappa^3_\beta}\|E_0\big(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\beta,\mu}, w^{\pm}_{\kappa^2_\alpha,\lambda}\big)\|_{L^1_{t,x}} \\
&\lesssim \sum_{\beta=\alpha}^{1}\sum_{\kappa^1_\alpha\approx_\alpha\kappa^2_\alpha\perp_\beta\kappa^3_\beta}\lambda\mu^{3/2}(1+\mu^+d_2+d_2)\|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^3_\beta,\mu}\|_{X^{0,0}_{\mu,d_2}} \\
&\lesssim \lambda\mu^{\frac{3}{2}+}d_2\|u_\lambda\|_{X^{0,0}_{\lambda,1}}\|v_\mu\|_{X^{0,0}_{\mu,d_2}}\|w_\lambda\|_{X^{0,0}_{\lambda,1}}.
\end{aligned}$$

Notice the the loss of μ^+ is still acceptable. On the other hand if $d_{max} = d_1$, to avoid loosing a power of d_1, we place the term $u_{\kappa^1_\alpha,\lambda}$ into L^2L^2 and the term $w_{\kappa^2_\alpha,\lambda}$ into the energy space $L^\infty L^2$. This implies that the low frequency term $v_{\kappa^3_\alpha,\mu}$ when placed in the Strichartz space $L^{2+}L^\infty$ does not looses a $d_2^{1/2}$ factor, i.e. it contributes only with μ^{1+}. Moreover to bound the fourth term we shall place $v_{\kappa^3_\alpha,\mu}$ into $L^{2+}L^\infty$ and use Strichartz, this will allow to compensate the α^2 factor. Then a similar argument as in the $d_{max} = d_2$ case yields to

$$\begin{aligned}
&\|E_0\big(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^3_\beta,\mu}, w^{\pm}_{\kappa^2_\alpha,\lambda}\big)\|_{L^1_{t,x}} \\
&\lesssim \lambda\mu\beta\frac{1}{\beta^2\mu}\big(\mu^{\frac{3}{2}}\beta + \alpha\beta\mu^{2+} + \alpha\beta\mu^{2+} + \mu^{1+}\alpha^2\mu\big)\|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{X^{0,0}_{\lambda,d_1}}\|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^3_\beta,\mu}\|_{X^{0,0}_{\mu,1}} \\
&\lesssim \lambda\mu^{3/2}(1+\mu^+d_1^{\frac{1}{2}} + (\beta\alpha^{-1})^{-1}d_1^{\frac{1}{2}})\|u^{\pm}_{\kappa^1_\alpha,\lambda}\|_{X^{0,0}_{\lambda,d_1}}\|w^{\pm}_{\kappa^2_\alpha,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^3_\beta,\mu}\|_{X^{0,0}_{\mu,1}}.
\end{aligned}$$

This concludes the estimate for the II term.

Estimate for III - Non resonant interactions

The argument to estimate the III term follows closely the one employed for the previous II term with the following differences. Since the angle between the support of the high frequency input term u and the low frequency input term v is of order $\beta\lambda/\mu$ we shall employ an operator $\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu}$ instead of $\widetilde{L}_{\kappa_{\beta,\mu}}$ used for II. Therefore from the fixed-time L_x^p estimate in Lemma 91 we obtain the extra gain $\mu/\lambda^2\beta^2$. However, the angular separation of the supports of u and v is $\beta\lambda/\mu$ which is larger then what we had in II. Moreover the size of the low frequency support is also lager then before: $(\lambda\beta)^2\mu$. To clarify things we have highlighted the differences in Table 2.1 below.

	II	III
$\widetilde{L}$	$\Vert\widetilde{L}_{\kappa_{\beta,\mu}}v\Vert_{L_x^p}\lesssim\frac{1}{\beta^2\mu}\Vert v\Vert_{L_x^p}$	$\Vert\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu}v\Vert_{L_x^p}\lesssim\frac{\mu}{\lambda^2\beta^2}\Vert v\Vert_{L_x^p}$
Nullform	$\lambda\mu\beta$	$\beta\lambda^2$
Low freq. Bernstein	$\Vert v^{\pm}_{\kappa^3_\beta,\mu}\Vert_{L^2L^\infty}\lesssim\mu^{\frac{3}{2}}\beta\Vert v^{\pm}_{\kappa^3_\beta,\mu}\Vert_{X^{0,0}_{\mu,d_2}}$	$\Vert v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\Vert_{L^2L^\infty}\lesssim\mu^{\frac{1}{2}}\lambda\beta\Vert v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\Vert_{X^{0,0}_{\mu,d_2}}$
Low freq. Strichartz	$\Vert v^{\pm}_{\kappa^3_\beta,\mu}\Vert_{L^{2+}L^\infty}\lesssim\mu^{1+}d_2^{\frac{1}{2}}\Vert v^{\pm}_{\kappa^3_\beta,\mu}\Vert_{X^{0,0}_{\mu,d_2}}$	$\Vert v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\Vert_{L^{2+}L^\infty}\lesssim\mu^{1+}d^{\frac{1}{2}}\Vert v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\Vert_{X^{0,0}_{\mu,d_2}}$
Low freq. perp.	$\Vert(\xi_{\theta_{\kappa^1_\alpha}}\wedge D_x)v^{\pm}_{\kappa^2_\beta,\mu}\Vert_{L^2L^2}\lesssim\mu\beta\Vert v^{\pm}_{\kappa^2_\beta,\mu}\Vert_{L^2L^2}$	$\Vert(\xi_{\theta_{\kappa^1_\alpha}}\wedge D_x)v^{\pm}_{\kappa^2_{\frac{\beta\lambda}{\mu}},\mu}\Vert_{L^2L^2}\lesssim\mu\beta\Vert v^{\pm}_{\kappa^2_{\frac{\beta\lambda}{\mu}},\mu}\Vert_{L^2L^2}$
High freq. perp.	$\Vert(\xi_{\theta_{\kappa^1_\alpha}}\wedge D_x)u^{\pm}_{\kappa^1_\alpha,\lambda}\Vert_{L^2L^2}\lesssim\lambda\alpha\Vert u^{\pm}_{\kappa^1_\alpha,\lambda}\Vert_{L^2L^2}$	$\Vert(\xi_{\theta_{\kappa^1_\alpha}}\wedge D_x)u^{\pm}_{\kappa^1_\beta,\lambda}\Vert_{L^2L^2}\lesssim\lambda\mu^{-1}\alpha\Vert u^{\pm}_{\kappa^1_\beta,\lambda}\Vert_{L^2L^2}$

Table 2.1: Gains/losses comparison

With the help of this table it is then easy to compute, based on the procedure employed for II, the constants for the III.a-e terms. Indeed for III.a, case $d_{max}=d_2$ we apply the null-form bound, the $\widetilde{L}$ bound and the low frequency Bernstein to obtain

$$\beta\lambda^2\,\frac{\mu}{\lambda^2\beta^2}\,\mu^{\frac{1}{2}}\lambda\beta=\lambda\mu^{\frac{3}{2}},$$

which is far than acceptable, since we still have room for d_2. In the case III.a $d_{max}\neq d_2$ we replace the low frequency Bernstein by the low frequency Strichartz, that leads us to the allowed constant

$$\beta\lambda^2\,\frac{\mu}{\lambda^2\beta^2}\,\mu^{1+}=\lambda\mu^{\frac{3}{2}+}d_{max}^{-1/2}$$

since $\beta>\alpha=\mu^{-1/2}d_{max}^{1/2}$.

The case III.b for $d_{max}=d_2$ is treated as the corresponding III.a $d_{max}=d_2$ case. Furthermore the case $d_1=d_{max}$ yield to an extra d_1 factor from the half-wave operator, thus we obtain the strict bound

$$\beta\lambda^2\,\frac{\mu}{\lambda^2\beta^2}\,\mu^{1+}d_1=\lambda\mu^{\frac{3}{2}+}d_1^{1/2}.$$

On the other hand the case $d_3=d_{max}$ is analogous to the previous III.a $d_3=d_{max}$.

In the III.c term when $d_{max} = d_2$ we obtain the same constant as in the III.a $d_{max} = d_2$ case. Moreover as for the two $d_{max} \neq d_2$ cases, they are similar to the III.a $d_{max} \neq d_2$ cases but with the cases $d_1 = d_{max}$ and $d_3 = d_{max}$ swapped since here the half-wave operator hits the term involving the d_3 modulation.

As for the bound for III.d, here the half-wave operator hits the low frequency term, thus in the $d_{max} = d_2$ case we obtain an extra d_2 factor yielding to

$$\beta\lambda^2 \frac{\mu}{\lambda^2\beta^2} \mu^{\frac{1}{2}} \lambda\beta d_2 = \lambda\mu^{\frac{3}{2}} d_2.$$

On the other hand the cases $d_{max} \neq d_2$ are analogous to the III.a $d_{max} \neq d_2$.

There is a slight difference between the estimates of II.e and III.e, in particular in the bound for E_0, since E_1 is estimated in the same way. We shall take advantage of the following bound which is a consequence of Lemma 92:

$$\begin{aligned} \|\widetilde{E}_0(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}, w^{\pm}_{\kappa^2_\beta,\lambda})\|_{L^1_{t,x}} \lesssim\ & \|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^2L^2} \|\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^2L^\infty} \|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^\infty L^2} \\ +\ & \lambda^{-1} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^2L^2} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty} \|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^\infty L^2} \\ +\ & \lambda^{-1} \|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^2L^2} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^\infty L^2} \\ +\ & \mu\lambda^{-2} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^2L^2} \|\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty} \|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}} \wedge D_x) w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^\infty L^2} \end{aligned} \tag{2.51}$$

Thus when $d_{max} = d_2$ we use Bernstein for the first term and Strichartz for the second, third and fourth, then we obtain:

$$\begin{aligned} & \|E_0\big(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}, w^{\pm}_{\kappa^2_\beta,\lambda}\big)\|_{L^1_{t,x}} \\ & \lesssim \beta\lambda^2 \frac{\mu}{\lambda^2\beta^2} \big(\mu^{\frac{1}{2}}\lambda\beta + \alpha\beta\mu^{1+} d_2^{\frac{1}{2}} + \alpha\beta\mu^{1+} d_2^{\frac{1}{2}} + \alpha^2\mu^{+} d_2^{\frac{1}{2}}\big) \|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}} \|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}} \|v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d_2}} \\ & \lesssim \big(\lambda\mu^{\frac{3}{2}} + \mu^{\frac{3}{2}+} d_2 + \beta^{-1}\mu^{+} d_2^{\frac{3}{2}}\big) \|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}} \|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}} \|v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d_2}}. \end{aligned}$$

Therefore, since $\beta > \alpha = \mu^{-1/2} d_2^{1/2}$, we have the acceptable constant for the third factor:

$$\beta^{-1}\mu^{+} d_2^{\frac{3}{2}} = (\beta\alpha^{-1})^{-1}\mu^{1/2+} d_2.$$

On the other hand if $d_{\max} \neq d_2$ we place the maximum modulation term into L^2L^2 and the second high frequency term into the energy space $L^\infty L^2$. Hence we have to place the low frequency term into L^2L^∞ or $L^{2+}L^\infty$. Then we proceed as above, but in this case we don't loose a $d_2^{1/2}$ factor when applying Strichartz, thus we obtain the bound with an acceptable constant. This proves the bound for III.

Estimate for IV

The argument employed to bound this term follows the corresponding IV term in [30]. Recall that

$$IV = \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \approx \frac{\beta\lambda}{\mu} \kappa^1_\beta \perp_\beta \kappa^2_\beta} N(u^\pm_{\kappa^1_\beta,\lambda}, v^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}) w^\pm_{\kappa^2_\beta,\lambda}.$$

This term, as the following one, is better behaved since it is supported at distance $\beta\lambda$ from the diagonal set $D = \{(\xi,\eta,\zeta) : \xi+\eta+\zeta = 0\}$. In fact, let us denote the Fourier variable of u, v, w respectively with ξ,ζ,η, that is

$$u^\pm_{\kappa^1_\beta,\lambda}(t,x) = \chi^\pm_{\kappa^1_\beta,\lambda}(t,x,D)P_\lambda u = \int e^{ix\cdot\xi} \chi^\pm_{\kappa^1_\beta,\lambda}(t,x,\xi)\widehat{u}(t,\xi)d\xi$$

and analogous formulas for $w^\pm_{\kappa^3_\beta,\lambda}$ and $v^\pm_{\kappa^2_{\frac{\beta\lambda}{\mu}},\mu}$ hold. Let us denote $\widehat{u}^\pm_{\kappa^1_\beta,\lambda}(t,x,\xi) = \chi^\pm_{\kappa^1_\beta,\lambda}(t,x,\xi)\widehat{u}(t,\xi)$. Then since

$$|\xi| \approx \lambda,\ |\xi \wedge \xi^\pm_{\theta_{\kappa^1_\beta}}| \lesssim \beta\lambda, \qquad |\eta| \approx \lambda,\ |\eta \wedge \xi^\pm_{\theta_{\kappa^1_\beta}}| \approx \beta\lambda, \qquad |\zeta| \approx \mu,\ |\zeta \wedge \xi^\pm_{\theta_{\kappa^1_\beta}}| \lesssim \beta\lambda,$$

we obtain $|(\xi+\eta+\zeta) \wedge \xi^\pm_{\theta_{\kappa^1_\beta}}| \approx \beta\lambda$. We take advantage of this fact in the following way: define the spatial elliptic operator $F(t,x,D_x) = (D_x \wedge \xi^\pm_{\theta_{\kappa^1_\beta}}(t,x))^{2N}$, where $N > 0$, that is

$$F(t,x,D_x)u = \int e^{-ix\cdot\xi}(\xi \wedge \xi^\pm_{\theta_{\kappa^1_\beta}}(t,x))^{2N}\widehat{u}(t,\xi)d\xi.$$

Then since the Fourier transform of the complex exponential is the delta function we obtain

$$F(t,x,D_x)e^{ix\cdot(\xi+\eta+\zeta)} = e^{ix\cdot(\xi+\eta+\zeta)}\Big((\xi+\eta+\zeta) \wedge \xi^\pm_{\theta_{\kappa^1_\beta}}(t,x)\Big)^{2N}$$

where we think of $e^{ix\cdot(\xi+\eta+\zeta)}$ as a function of the space variable x with parameters (ξ,η,ζ). Moreover notice that we can write

$$\begin{aligned}
&\|N(u^\pm_{\kappa^1_\beta,\lambda}, v^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}) w^\pm_{\kappa^2_\beta,\lambda}\|_{L^1_{t,x}} \\
&= \beta\lambda^2 \int e^{ix\cdot(\xi+\eta+\zeta)}\,\widehat{u}^\pm_{\kappa^1_\beta,\lambda}(t,x,\xi)\,\widehat{w}^\pm_{\kappa^2_\beta,\lambda}(t,x,\eta)\,\widehat{v}^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}(t,x,\zeta)\ d\xi d\eta d\zeta dt dx \\
&= \beta\lambda^2 \int \frac{F(t,x,D_x)(e^{ix\cdot(\xi+\eta+\zeta)})}{((\xi+\eta+\zeta)\wedge \xi^\pm_{\theta_{\kappa^1_\beta}}(t,x))^{2N}}\,\widehat{u}^\pm_{\kappa^1_\beta,\lambda}(t,x,\xi)\,\widehat{w}^\pm_{\kappa^2_\beta,\lambda}(t,x,\eta)\,\widehat{v}^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}(t,x,\zeta)\ d\xi d\eta d\zeta dt dx.
\end{aligned}$$

Therefore if we integrate by parts the above formula and introduce the adjoint operator $F^*(t,x,D_x) = (D_x \wedge \xi^{\pm}_{\theta_{\kappa^1_\beta}}(t,x))^{-2N}$, we obtain

$$\begin{aligned}
&\|N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}) w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^1_{t,x}} \\
&= \beta\lambda^2 \int e^{ix\cdot(\xi+\eta+\zeta)} F^*(t,x,D_x)\Big(\frac{\widehat{u}^{\pm}_{\kappa^1_\beta,\lambda}(t,x,\xi)\,\widehat{w}^{\pm}_{\kappa^2_\beta,\lambda}(t,x,\eta)\,\widehat{v}^{\pm}_{\kappa^3_{\beta\lambda/\mu},\mu}(t,x,\zeta)}{((\xi+\eta+\zeta)\wedge\xi^{\pm}_{\theta_{\kappa^1_\beta}}(t,x))^{2N}}\Big) d\xi d\eta d\zeta dt dx \\
&\approx \beta\lambda^2(\beta\lambda)^{-2N} \int e^{ix\cdot(\xi+\eta+\zeta)} F^*(t,x,D_x)\Big(\widehat{u}^{\pm}_{\kappa^1_\beta,\lambda}(t,x,\xi)\,\widehat{w}^{\pm}_{\kappa^2_\beta,\lambda}(t,x,\eta)\,\widehat{v}^{\pm}_{\kappa^3_{\beta\lambda/\mu},\mu}(t,x,\zeta)\Big) d\xi d\eta d\zeta dt dx
\end{aligned}$$

since on the support of $\widehat{u}^{\pm}_{\kappa^1_\beta,\lambda}(t,x,\xi)\,\widehat{w}^{\pm}_{\kappa^2_\beta,\lambda}(t,x,\eta)\,\widehat{v}^{\pm}_{\kappa^3_{\beta\lambda/\mu},\mu}(t,x,\zeta)$ we have that $|(\xi+\eta+\zeta)\wedge\xi^{\pm}_{\theta_{\kappa^1_\beta}}| \approx \beta\lambda$. Now each elliptic derivative $(D_x \wedge \xi^{\pm}_{\theta_{\kappa^1_\beta}}(t,x))^{-1}$, when hits one of the three terms $u^{\pm}_{\kappa^1_\beta,\lambda}(t,x,\xi)$, $\widehat{w}^{\pm}_{\kappa^2_\beta,\lambda}(t,x,\eta)$, or $\widehat{v}^{\pm}_{\kappa^3_{\beta\lambda/\mu},\mu}(t,x,\zeta)$, contributes with a factor of $(\beta\lambda)^{-1}$. Hence we finally obtain the arbitrary large gain:

$$\|N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}) w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^1_{t,x}} \approx \beta\lambda^2(\beta\lambda)^{-4N}\|u^{\pm}_{\kappa^1_\beta,\lambda} v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu} w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^1_{t,x}}. \tag{2.52}$$

Finally Strichartz estimates allow us to conclude

$$\begin{aligned}
\|N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}) w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^1_{t,x}} &= \beta\lambda^2(\beta\lambda)^{-4N}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty}\|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^2L^2} \\
&\lesssim \lambda^{2-4N}\beta^{1-4N}\mu^{1+}d_2^{\frac{1}{2}}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d_2}}.
\end{aligned}$$

Then choosing $N = 1/4$ yields to the constant $\lambda\mu^{1+}d_2^{1/2}$, which is acceptable.

Estimate for V

To estimate this last term we apply a similar argument as in IV. First recall that

$$V = \sum_{\beta=\alpha}^{1}\sum_{\gamma=\frac{\beta\lambda}{\mu}}^{1}\sum_{\kappa^1_\beta\perp_\beta\kappa^2_\beta\perp_\gamma\kappa^3_\gamma} N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^3_\gamma,\mu}) w^{\pm}_{\kappa^2_\beta,\lambda},$$

thus the Fourier variables satisfies:

$$|\xi|\approx\lambda,\ |\xi\wedge\xi^{\pm}_{\theta_{\kappa^1_\beta}}|\lesssim\beta\lambda, \qquad |\eta|\approx\lambda,\ |\eta\wedge\xi^{\pm}_{\theta_{\kappa^1_\beta}}|\approx\beta\lambda, \qquad |\zeta|\approx\mu,\ |\zeta\wedge\xi^{\pm}_{\theta_{\kappa^1_\beta}}|\approx\gamma\mu.$$

we then obtain $|(\xi+\eta+\zeta)\wedge\xi^{\pm}_{\theta_{\kappa^1_\beta}}| \approx \beta\lambda$. Therefore we can applied the same argument used in IV which yield to a gain of arbitrary powers of $\beta\gamma\lambda$. □

High-high-low interaction

The high-high-low interaction case consists in proving the following estimate:

$$\sum_{\lambda=1}^{\infty}\sum_{\mu\lesssim\lambda}\|N(P_\lambda u_\lambda, P_\lambda v_\lambda)P_\mu w_\mu\|_{L^1_{t,x}} \lesssim \|u\|_{X^{s,\theta}}\|v\|_{X^{s,\theta}}\|w\|_{X^{1-s,1-\theta}+L^2H^{2-s-\theta}}.$$

As for the previously studied high-low-high interaction we shall first dispense the easier cases which do not require the use of the specific null-form structure, and thus hold true for any general bilinear form. For such cases we also can relax the hypothesis on the space dimension to $n\geq 3$.

First suppose that $P_\mu w_\mu \in L^2H^{2-s-\theta}$ then we can easily close the bound by means of Hölder and Strichartz inequalities:

$$\begin{aligned}\|N(P_\lambda u_\lambda P_\lambda v_\lambda)P_\mu w_\mu\|_{L^1L^1} &\lesssim \|\nabla P_\lambda u_\lambda\|_{L^\infty L^2}\|\nabla P_\lambda v_\lambda\|_{L^{2+}L^\infty}\|P_\mu w_\mu\|_{L^2}\\ &\lesssim \lambda^{2-2s+n/2-1/2+}\mu^{s+\theta-2}\|u_\lambda\|_{X^{s,\theta}_\lambda}\|v_\lambda\|_{X^{s,\theta}_\lambda}\|P_\mu w_\mu\|_{L^2H^{2-s-\theta}}.\end{aligned}$$

Since the exponent of the high frequency is negative for $n\geq 3$ and $s>n/2$, we shall transfer all the high frequencies to low frequency to obtain the negative exponent $-s+\theta+n/2-1/2$. Hence the case $P_\mu w_\mu\in L^2H^{2-s-\theta}$ is settled. Next, let us consider the case $w_\mu\in X^{1-s,1-\theta}_\mu$. We split the high frequency term into high and low modulations as follows:

$$u_\lambda = \sum_{d=1}^{\lambda}u_{\lambda,d} = \sum_{d=1}^{\mu}u_{\lambda,d}+\sum_{d=2\mu}^{\lambda}u_{\lambda,d} =: u_{\lambda,\leq\mu}+u_{\lambda,>\mu}$$

and we observe that for the high modulation term we have the improved bound:

$$\|\nabla P_\lambda u_{\lambda,>\mu}\|_{L^2}\lesssim \lambda^{1-s}\sum_{d=2\mu}^{\lambda}d^{-\theta}\lambda^{s-1}d^{\theta}\|\nabla P_\lambda u_{\lambda,d}\|_{L^2}\lesssim \lambda^{1-s}\mu^{-\theta}\|u_\lambda\|_{X^{s,\theta}_\lambda}.$$

Therefore we can control the high modulation part of u_λ via Strichartz inequality as follows:

$$\begin{aligned}\|N(P_\lambda u_{\lambda,>\mu}P_\lambda v_\lambda)P_\mu w_\mu\|_{L^1L^1} &\lesssim \|\nabla P_\lambda u_{\lambda,>\mu}\|_{L^2}\|\nabla P_\lambda v_\lambda\|_{L^\infty L^2}\|P_\mu w_\mu\|_{L^{2+}L^\infty}\\ &\lesssim \lambda^{2-2s}\mu^{-\theta}\mu^{1/2-2+s+\theta+n/2-1/2+}\|u_\lambda\|_{X^{s,\theta}_\lambda}\|v_\lambda\|_{X^{s,\theta}_\lambda}\|P_\mu w_\mu\|_{X^{1-s,1-\theta}_\mu}.\end{aligned}$$

The high frequency exponent, being negative, can be estimated in term of the low frequency μ, we obtain $\mu^{-s+n/2+}$ which is negative. Notice that we cannot apply a similar argument for the low modulation term since we are missing the improved bound, precisely the factor $\mu^{-\theta}$, thus an analogous argument will lead to the exponent $\mu^{-s+n/2+\theta+}$ which can be positive. We shall see later that to overcome this difficulty we will need to analyse in more detail the null structure present in the nonlinearity. First notice that if we split the other high frequency term v_λ into low modulations $v_{\lambda,\le\mu}$ and high modulations $w_{\lambda,>\mu}$, then a similar argument lead to an acceptable bound for the high modulation component:

$$\begin{aligned}\|N(P_\lambda u_\lambda P_\lambda v_{\lambda,>\mu})P_\mu w_\mu\|_{L^1L^1} &\lesssim \|\nabla P_\lambda u_\lambda\|_{L^\infty L^2}\|\nabla P_\lambda v_{\lambda,>\mu}\|_{L^2L^2}\|P_\mu w_\mu\|_{L^{2+}L^\infty}\\ &\lesssim \lambda^{2-2s}\mu^{-\theta}\mu^{1/2-2+s+\theta+n/2-1/2+}\|u_\lambda\|_{X^{s,\theta}_\lambda}\|v_\lambda\|_{X^{s,\theta}_\lambda}\|P_\mu w_\mu\|_{X^{1-s,1-\theta}_\mu}\\ &\lesssim \lambda^{2-2s}\mu^{s+n/2-2}\|u_\lambda\|_{X^{s,\theta}_\lambda}\|v_\lambda\|_{X^{s,\theta}_\lambda}\|P_\mu w_\mu\|_{X^{1-s,1-\theta}_\mu}.\end{aligned}$$

Hence, we have reduced the proof of the high-high-low interaction to the boundedness of the low modulations term:

$$\|N(P_\lambda u_{\lambda,\le\mu}P_\lambda v_{\lambda,\le\mu})P_\mu w_\mu\|_{L^1L^1}.$$

This term in the current form can not be controlled by Strichartz estimates. In order to bound it we need to decompose it further into angular sectors. However, following the argument as in the high-low-high case, before performing such decomposition we shall simplify further to 1-modulations. We have the analogous of Proposition 87 in the high-low-high case:

Proposition 93. *Let $n\ge 3$ and $3/4<\theta<s-(n-1)/2$. Assume that we have the following bound:*

$$\|N(P_\lambda u_{\lambda,d_1}P_\lambda v_{\lambda,d_2})P_\mu w_{\mu,d_3}\|_{L^1L^1}\lesssim \lambda^{\frac{n}{2}+\frac{1}{2}+}\mu^{\frac12}d_1^{\frac12}d_2^{\frac12}d_3^{\frac12}\|u_{\lambda,d_1}\|_{X^{0,0}_{\lambda,d_1}}\|v_{\lambda,d_2}\|_{X^{0,0}_{\lambda,d_2}}\|w_{\mu,d_3}\|_{X^{0,0}_{\mu,d_3}}; \tag{2.53}$$

Then we have the corresponding high-high-low estimate for low modulations:

$$\|N(P_\lambda u_{\lambda,\le\mu}P_\lambda v_{\lambda,\le\mu})P_\mu w_\mu\|_{L^1L^1}\lesssim \lambda^\alpha\mu^\beta\|u_{\lambda,}\|_{X^{s,\theta}_\lambda}\|v_\lambda\|_{X^{s,\theta}_\lambda}\|w_\mu\|_{X^{1-s,1-\theta}_\mu},$$

where the exponents $\alpha<0$ and $\alpha+\beta<0$.

Proof. The proof is a straightforward application of Cauchy-Schwarz inequality. In fact we have:

$$\begin{aligned}&\|N(P_\lambda u_{\lambda,\le\mu}P_\lambda v_{\lambda,\le\mu})P_\mu w_\mu\|_{L^1L^1}\lesssim \sum_{d_1,d_2,d_3=1}^{\mu}\|N(P_\lambda u_{\lambda,d_1}P_\lambda v_{\lambda,d_2})P_\mu w_{\mu,d_3}\|_{L^1L^1}\\ &\lesssim \sum_{d_1,d_2,d_3=1}^{\mu}\lambda^{-2s+\frac{n}{2}+\frac12+}\mu^{s-\frac12}d_1^{\frac12-\theta}d_2^{\frac12-\theta}d_3^{\theta-\frac12}\|u_{\lambda,d_1}\|_{X^{s,\theta}_{\lambda,d_1}}\|v_{\lambda,d_2}\|_{X^{s,\theta}_{\lambda,d_2}}\|w_{\mu,d_3}\|_{X^{1-s,1-\theta}_{\mu,d_3}}\\ &\lesssim \lambda^{-2s+\frac{n}{2}+\frac12+}\mu^{s+\theta-1}\|u_{\lambda,}\|_{X^{s,\theta}_\lambda}\|v_\lambda\|_{X^{s,\theta}_\lambda}\|w_\mu\|_{X^{1-s,1-\theta}_\mu}.\end{aligned}$$

Then $\alpha=-2s+\frac{n}{2}+\frac12+$ is negative for $n\ge3$ and $s>n/2$. Moreover $\alpha+\beta=-s+\theta+n/2-1/2+<0$, hence the proof is completed. □

By means of a similar scaling and l^2-summation argument already employed in the high-low-high case (see Lemma 88) we shall reduce the proof of the high-high-low interaction further to 1-modulation spaces.

Lemma 94. *Let $n \geq 3$ and suppose that the following bounds hold:*

$$\|N(u,v)w\|_{L^1L^1} \lesssim \lambda^{\frac{n}{2}+\frac{1}{2}+}\mu^{\frac{1}{2}} d_{max}^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,d_1}} \|v\|_{X^{0,0}_{\lambda,d_2}} \|w\|_{X^{0,0}_{\mu,d_3}}, \quad \text{where } d_{min} = d_{med} = 1 \tag{2.54}$$

then (2.53) *holds.*

Proof. Let (χ_j) a smooth partition of unity of the time interval $[0,1]$ as in the proof of Lemma 88: $I_j = \operatorname{supp}\chi_j$ and $|I_j| = \delta$. Notice that for $1 \leq \delta^{-1} \leq d$ we have the l^2-summability property

$$\|u\|^2_{X^{s,\theta}_{\lambda,d}} \approx \sum_{j\in\mathbb{N}} \|\chi_j(t)u\|^2_{X^{s,\theta}_{\lambda,d}}.$$

Moreover define $u^\delta(t,x) = u(\delta t, \delta x)$ then if $\delta d \geq 1$ we have the scaling law

$$\|u^\delta\|_{X^{s,\theta}_{\delta\lambda,\delta d}[0,1]} \approx \delta^{s+\theta-\frac{n+1}{2}} \|u\|_{X^{s,\theta}_{\lambda,d}[0,\delta]}.$$

Moreover let $(\widetilde{\chi}_j)$ a similar family such that $\widetilde{\chi}_j = 1$ in I_j.

Now suppose without loosing generality that $d_1 = d_{min}$ and $d_3 = d_{max}$. We carry out a two step argument: first we reduce the estimate (2.53) to the case $d_1 = 1$. Notice that by a change of variable we obtain

$$\begin{aligned}
\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} &\lesssim \sum_{j\in\mathbb{N}} \|\chi_j(t)N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \\
&\lesssim \sum_{j\in\mathbb{N}} \|N\widetilde{\chi}_j u, \widetilde{\chi}_j v)\widetilde{\chi}_j w\|_{L^1_{t,x}(I_j\times\mathbb{R}^n)} \\
&\lesssim \delta^{n-1} \sum_{j\in\mathbb{N}} \|(N(\widetilde{\chi}_j u, \widetilde{\chi}_j v)\widetilde{\chi}_j w)^\delta\|_{L^1_{t,x}(I\times\mathbb{R}^n)}.
\end{aligned}$$

We now apply our hypothesis, (2.53) where $d_1 = 1$, and use the scaling law to obtain

$$\begin{aligned}
&\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \\
&\lesssim \sum_{j\in\mathbb{N}} \delta^{n-1}(\delta\lambda)^{\frac{n}{2}+\frac{1}{2}+}(\delta\mu)^{\frac{1}{2}}(\delta d_2)^{\frac{1}{2}}(\delta d_3)^{\frac{1}{2}} \|(\widetilde{\chi}_j u)^\delta\|_{X^{0,0}_{\delta\lambda,1}[I]} \|(\widetilde{\chi}_j v)^\delta\|_{X^{0,0}_{\delta\lambda,\delta d_2}[I]} \|(\widetilde{\chi}_j w)^\delta\|_{X^{0,0}_{\delta\mu,\delta d_3}[I]} \\
&\lesssim \sum_{j\in\mathbb{N}} \delta^{-\frac{1}{2}+}\lambda^{\frac{n}{2}+\frac{1}{2}+}\mu^{\frac{1}{2}} d_2^{\frac{1}{2}} d_3^{\frac{1}{2}} \|\widetilde{\chi}_j u\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]} \|\widetilde{\chi}_j v\|_{X^{0,0}_{\lambda,d_2}[I]} \|\widetilde{\chi}_j w\|_{X^{0,0}_{\mu,d_3}[I]}.
\end{aligned}$$

To close we use the l^2-summability property and the fact that $l^2 \subset l^3$, then we obtain

$$\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \lesssim \delta^{-\frac{1}{2}+}\lambda^{\frac{n}{2}+\frac{1}{2}+}\mu^{\frac{1}{2}} d_2^{\frac{1}{2}} d_3^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]} \|v\|_{X^{0,0}_{\lambda,d_2}[I]} \|w\|_{X^{0,0}_{\mu,d_3}[I]}.$$

Choose $\delta^{-1} = d_{\min}$ to conclude the reduction of estimate (2.53) to the case $d_1 = 1$.

Next, in the second reduction step, we reduce estimate (2.53) with $d_1 = 1$ further to estimate (2.53) with $d_1 = d_2 = 1$, yielding to (2.54). Therefore suppose that

$$\|N(u,v)w\|_{L^1_{t,x}} \lesssim \lambda^{\frac{n}{2}+\frac{1}{2}+}\mu^{\frac{1}{2}} d_2^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\lambda,d_2}} \|w\|_{X^{0,0}_{\mu,1}},$$

then

$$\|N(u,v)w\|_{L^1_{t,x}} \lesssim \lambda^{\frac{n}{2}+\frac{1}{2}+}\mu^{\frac{1}{2}} d_2^{\frac{1}{2}} d_3^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\lambda,d_2}} \|w\|_{X^{0,0}_{\mu,d_3}}.$$

We proceed by following a similar argument as in the previous step but at the end we set a different value for δ. We have

$$\|N(u,v)w\|_{L^1_{t,x}(I\times\mathbb{R}^n)} \lesssim \sum_{j\in\mathbb{N}} \delta^{-1+}\lambda^{\frac{n}{2}+\frac{1}{2}+}\mu^{\frac{1}{2}} d_3^{\frac{1}{2}} \|\widetilde{\chi}_j u\|_{X^{0,0}_{\lambda,\delta^{-1}}[I]} \|\widetilde{\chi}_j v\|_{X^{0,0}_{\lambda,d_2}[I]} \|\widetilde{\chi}_j w\|_{X^{0,0}_{\mu,\delta^{-1}}[I]}.$$

Set $\delta^{-1} = d_3$ and notice that the term involving u we apply the simple bound (see Proposition 2.6 on [27])

$$\|\widetilde{\chi}_j u\|_{X^{0,0}_{\lambda,\delta^{-1}}} \lesssim \delta^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,1}}, \qquad \delta < 1,$$

to recover the $d_1 = 1$ exponent. Then by Hölder inequality and the square summability property of the terms involving v and w we obtain the desired bound. □

Hereafter we shall impose the $n = 3$ condition on the space dimensions. Notice that, due the symmetry of the N_{ij} null-form we shall show how to reduce the proof of estimate (2.54) in the case $d_{max} \neq d_3$ to the previously studied high-low-high interaction case. In fact, recall that N is a linear combination with constant coefficients of pure N_{ij} null-form:

$$N(u,v) = c^{ij}(\partial_i u \partial_j v - \partial_i v \partial_j u)$$

and notice that we can isolate a derivative in the following sense:

$$N(u,v) = c^{ij}\partial_i(u\partial_j v) - c^{ij}\partial_j(u\partial_i v)$$

This formulation has the advantage of having one derivative outside hence we can perform an integration by parts to obtain the following:

$$\begin{aligned}
\|N(P_\lambda u P_\lambda v)P_\mu w\|_{L^1L^1} &= \|c^{ij}\partial_i(P_\lambda u \partial_j P_\lambda v)P_\mu w - c^{ij}\partial_j(P_\lambda u_{\lambda,\le\mu}\partial_i P_\lambda v_{\lambda,\le\mu})P_\mu w_\mu\|_{L^1L^1} \\
&= \|c^{ij}P_\lambda u \partial_i P_\lambda v \partial_j P_\mu w - c^{ij}P_\lambda u \partial_j P_\lambda v \partial_i P_\mu w\|_{L^1L^1} \\
&= \|N(P_\lambda v, P_\mu w)P_\lambda u\|_{L^1L^1}.
\end{aligned}$$

Then, when the maximum modulation is not coupled with the minimum frequency, Lemma

88 yield to the bound

$$\begin{aligned} \|N(P_\lambda u_{\lambda,d_1} P_\lambda v_{\lambda,d_2}) P_\mu w_{\mu,d_3}\|_{L^1L^1} &\lesssim \mu^{\frac{3}{2}+}\lambda d_1^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,d_1}} \|v\|_{X^{0,0}_{\mu,1}} \|w\|_{X^{0,0}_{\lambda,1}} \\ &\lesssim \lambda^{2+}\mu^{\frac{1}{2}+} d_1^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,d_1}} \|v\|_{X^{0,0}_{\mu,1}} \|w\|_{X^{0,0}_{\lambda,1}} \end{aligned}$$

Here we have suppose without loosing generality that $d_{max} = d_1$, the other case $d_{max} = d_2$ yielding to a similar bound. Notice that the previous estimate is exactly (2.54), therefore Lemma 94 and the previous discussion allow us to reduce the proof of Proposition 93 to the proof of the following key Proposition which cannot be reconnected to the high-low-high case.

Proposition 95 (High-high-low interaction)**.** *Let $n = 3$, $3/4 < \theta < s-1$, and $1 \le d \le \mu \lesssim \lambda$, then following bound holds:*

$$\|N(P_\lambda u, P_\lambda v) P_\mu w\|_{L^1L^1} \lesssim \lambda^{2+}\mu^{\frac{1}{2}} d^{\frac{1}{2}} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{0,0}_{\lambda,1}} \|w\|_{X^{0,0}_{\mu,d}},$$

The remaining part of the section we shall prove Proposition 95.

Proof. We denote $u^{\pm}_{\kappa,\lambda} = \chi^{\pm}_{\kappa,\lambda}(t,x,D) P_\lambda u$. In order to prove Proposition 95 we apply the trilinear angular decomposition (2.48) from §2.9 which yield us to the following five terms:

$$\begin{aligned} &N(\widetilde{P}_\lambda u, \widetilde{P}_\lambda v)\widetilde{P}_\mu w \\ &= \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \approx_\alpha \kappa^3_\alpha} N(u^{\pm}_{\kappa^1_\alpha,\lambda}, v^{\pm}_{\kappa^2_\alpha,\lambda}) w^{\pm}_{\kappa^3_\alpha,\mu} + \sum_{\beta=\alpha}^{1} \sum_{\kappa^1_\alpha \approx_\alpha \kappa^2_\alpha \perp_\beta \kappa^3_\beta} N(u^{\pm}_{\kappa^1_\alpha,\lambda}, v_{\kappa^2_\alpha,\lambda}) w^{\pm}_{\kappa^3_\beta,\mu} \\ &+ \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \perp_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}) w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu} + \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \approx_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}) w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu} \\ &+ \sum_{\beta=\alpha}^{1} \sum_{\gamma=\frac{\beta\lambda}{\mu}}^{1} \sum_{\kappa^1_\beta \perp_\beta \kappa^2_\beta \perp_\gamma \kappa^3_\gamma} N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}) w^{\pm}_{\kappa^3_\gamma,\mu} =: I + II + III + IV + V. \end{aligned}$$

We shall considered two different scales for the angular separation threshold α based on the following cases:

i. if $\frac{\mu^{1/2} d^{1/2}}{\lambda} \ge \mu^{-1/2}$ then motivated by the analysis in the constant coefficients case we set

$$\alpha = \mu^{1/2} d^{1/2} \lambda^{-1};$$

ii. if instead $\mu^{-1/2} > \frac{\mu^{1/2} d^{1/2}}{\lambda}$ then we simply set

$$\alpha = \mu^{-1/2}.$$

This choice is enforced by the condition in Corollary 85 ($\alpha \geq \mu^{-1/2}$) to have an l^2 decomposition property we must have the cap size bigger then the negative square root of the frequency. In what follows we shall carefully estimate each of the five terms in the above trilinear decomposition.

Estimate for I - Small angle interactions

First suppose that $i.$ holds. Then we use Strichartz for high frequency term, since the sum over spherical caps is diagonal we obtain:

$$\begin{aligned}
\sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \approx_\alpha \kappa_\alpha^3} \| N(u^\pm_{\kappa_\alpha^1,\lambda}, v^\pm_{\kappa_\alpha^2,\lambda})\, w^\pm_{\kappa_\alpha^3,\mu} \|_{L^1_{t,x}} &\lesssim \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \approx_\alpha \kappa_\alpha^3} \lambda^2 \alpha \| u^\pm_{\kappa_\alpha^1,\lambda} \|_{L^\infty L^2} \| v^\pm_{\kappa_\alpha^2,\lambda} \|_{L^{2+} L^\infty} \| w^\pm_{\kappa_\alpha^3,\mu} \|_{L^2 L^2} \\
&\lesssim \sum_{\kappa_\alpha^1 \approx_\alpha \kappa_\alpha^2 \approx_\alpha \kappa_\alpha^3} \lambda^{3+} \alpha \| u^\pm_{\kappa_\alpha^1,\lambda} \|_{X^{0,0}_{\lambda,1}} \| v^\pm_{\kappa_\alpha^2,\lambda} \|_{X^{0,0}_{\lambda,1}} \| w^\pm_{\kappa_\alpha^3,\mu} \|_{X^{0,0}_{\mu,d}} \\
&\lesssim \lambda^{3+} \alpha \| u_\lambda \|_{X^{0,0}_{\lambda,1}} \| v_\lambda \|_{X^{0,0}_{\lambda,1}} \| w_\mu \|_{X^{0,0}_{\mu,d}}.
\end{aligned}$$

Notice that the $L^{2+} L^\infty$ norm of the high-frequency term is estimated via Strichrtz type inequality. The summation with respect to κ_α^i is achieved via the l^2 decomposition property from Corollary 85: let $\alpha \geq \lambda^{-1/2}$, then

$$\sum_{\kappa \in \Omega_\alpha} \| u^\pm_{\kappa,\lambda} \|^2_{X^{0,0}_{\lambda,d}} \approx \| u_\lambda \|^2_{X^{0,0}_{\lambda,d}}.$$

Clearly a similar l^2 summation property holds for the low frequency term w_μ. Then in case $i.$ we obtain

$$\lambda^{3+} \alpha = \lambda^{2+} \mu^{1/2} d^{1/2}$$

On the other hand for the second case $ii.$ the argument needs to be modified slightly. Precisely we place the two high frequency components into $L^2 L^2$ and $L^\infty L^2$ to avoid any high frequency loss, and we place the low frequency component in $L^{2+} L^\infty$. Strichartz estimates yield to a factor of $\mu^{1+} d^{1/2}$ which is added to the factor coming from the null-form $\lambda^2 \alpha$, thus we obtain

$$\lambda^2 \alpha \mu^{1+} d^{1/2} = \lambda^2 \mu^{1/2+} d^{1/2}.$$

Therefore the estimate for the term I holds.

Estimate for II - Small angle interactions

Observe that in term II the angle between the two high frequency inputs is still bounded by α thus we can apply exactly the same argument as for the previous term. The only difference is that when performing a Cauchy-Schwarz inequality in the β sum we loose a small power of α which is harmless.

Estimate for III - Non resonant interactions

This is the most difficult term to estimate and the argument adopted below follows the one used to estimate the term III in the high-low-high interaction. Recall that the operator $\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu}(t,x) = P_\mu(D_x) l_{\kappa_{\frac{\beta\lambda}{\mu}},\mu}(t,x,D)$ has the following properties:

(a) fixed-time L^p mapping properties:

$$\|\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} v\|_{L^p_x} \lesssim \big(\mathbb{R}(\frac{\beta^2\lambda^2}{\mu^2}\mu)\big)^{-1}\|v\|_{L^p_x}, \qquad 1 \le p \le \infty;$$

(b) fixed-time approximate inverse of e:

$$\|((A_{<\mu^{1/2}} - \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} - 1)v\|_{L^p_x} \lesssim \Big(\mu^{-\frac{1}{2}} + (\frac{\beta^2\lambda^2}{\mu^2}\mu)^{-1}\Big)\|v\|_{L^p_x}, \qquad 1 \le p \le \infty.$$

In order to take advantage of such bounds we split further the proof of the II term into five parts based on the following decomposition:

$$\begin{aligned}
N(u,v)w &= N(u,v)\big(-(A_{<\mu^{1/2}} - \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} + 1\big)w + N(u,v)(A_{<\mu^{1/2}} - \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w \\
&= N(u,v)\big(-(A_{<\mu^{1/2}} - \widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} + 1\big)w + N\big((D_t + A_{<\mu^{1/2}})u, v\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w \\
&+ N\big(u,(D_t + A_{<\mu^{1/2}})v\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w + N\big(u,v\big)(D_t + A_{<\mu^{1/2}})\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w - E(u,v,w) \\
&= II.a + II.b + II.c + II.d + II.e
\end{aligned}$$

where

$$E(u,v,w) = N\big((D_t+A_{<\mu^{1/2}})u,v\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w + N\big(u,(D_t+A_{<\mu^{1/2}})v\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w + N\big(u,v\big)(D_t+\widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w.$$

Below we shall carefully estimate each of the following five terms.

Estimate for III.a

Let us consider the case $i.$ first. We apply Lemma 91 and Strichartz inequality for the low frequency term to obtain:

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa^1_\beta\perp_\beta\kappa^2_\beta}\|N(u^\pm_{\kappa^1_\beta,\lambda},v^\pm_{\kappa^2_\beta,\lambda})\big(-(A_{<\mu^{1/2}}-\widetilde{A}_{<\mu^{1/2}})\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu}+1\big)w^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^1_{t,x}}\\
&\lesssim\sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa^1_\beta\perp_\beta\kappa^2_\beta}\lambda^2\beta\Big(\mu^{-\frac{1}{2}}+(\frac{\beta^2\lambda^2}{\mu^2}\mu)^{-1}\Big)\|u^\pm_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|v^\pm_{\kappa^2_\beta,\lambda}\|_{L^2L^2}\|w^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty}\\
&\lesssim\sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa^1_\beta\perp_\beta\kappa^2_\beta}(\lambda^2\beta\mu^{\frac{1}{2}+}d^{\frac{1}{2}}+\mu^{2+}d^{\frac{1}{2}}(\beta\alpha^{-1})^{-1}\alpha^{-1})\|u^\pm_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^\pm_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}}\\
&\lesssim(\lambda^2\mu^{\frac{1}{2}+}d^{\frac{1}{2}}+\lambda\mu^{\frac{3}{2}+})\|u_\lambda\|_{X^{0,0}_{\lambda,1}}\|v_\lambda\|_{X^{0,0}_{\lambda,1}}\|w_\mu\|_{X^{0,0}_{\mu,d}}.
\end{aligned}$$

Recall that $\beta>\alpha=\mu^{1/2}d_2^{1/2}\lambda^{-1}$ and in the β sum we have applied Cauchy-Schwarz to obtain an l^2-series. Now let us consider the case $ii.$, we apply the same strategy as in the estimate of term I, thus we obtain a factor

$$\lambda^2\beta\Big(\mu^{-\frac{1}{2}}+(\frac{\beta^2\lambda^2}{\mu^2}\mu)^{-1}\Big)\mu^{1+}d^{\frac{1}{2}}\le\lambda^2\mu^{\frac{1}{2}+}d^{\frac{1}{2}}+\mu^{2+}d^{\frac{1}{2}}(\beta\alpha^{-1})^{-1}\alpha^{-1}\le\lambda^2\mu^{\frac{1}{2}+}d^{\frac{1}{2}}+\mu^{\frac{5}{2}+}d^{\frac{1}{2}}\quad(2.55)$$

which is still acceptable.

Estimate for III.b

We proceed as the corresponding case in III.a.ii. by using Strichartz inequality in the low frequency term. Therefore by Corollary 81, which allow us to control the term $\|(D_t\pm A^\pm_{<\sqrt{\lambda}})u\|_2$ with $\|u\|_{X^{0,0}_{\lambda,1}}$, we obtain

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa^1_\beta\perp_\beta\kappa^2_\beta}\|N\big((D_t+A_{<\sqrt{\lambda}})u^\pm_{\kappa^1_\beta,\lambda},v^\pm_{\kappa^2_\beta,\lambda}\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu}w^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^1_{t,x}}\\
&\lesssim\sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa^1_\beta\perp_\beta\kappa^2_\beta}\lambda^2\beta(\frac{\beta^2\lambda^2}{\mu^2}\mu)^{-1}\|(D_t+A_{\sqrt{\lambda}})u^\pm_{\kappa^1_\beta,\lambda}\|_{L^2L^2}\|v^\pm_{\kappa^2_\beta,\lambda}\|_{L^\infty L^2}\|w^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty}\\
&\lesssim\sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa^1_\beta\perp_\beta\kappa^2_\beta}\mu^{2+}d^{\frac{1}{2}}(\beta\alpha^{-1})^{-1}\alpha^{-1}\|u^\pm_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^\pm_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^\pm_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}}.
\end{aligned}$$

For the case $i.$ we then obtain the not sharp bound $\mu^{2+}d^{\frac{1}{2}}\alpha^{-1}\le\lambda\mu^{\frac{3}{2}+}$, and for case $ii.$ we obtain $\mu^{2+}d^{\frac{1}{2}}\alpha^{-1}\le\mu^{\frac{5}{2}+}d^{\frac{1}{2}}$. Thus this case is closed.

Estimate for III.c

This case is symmetric to III.b since the half-wave operator $D_t + A_{\sqrt{\lambda}}$ hits the high frequency term. As for the previous cases we use Strichartz and Lemma 91 to obtain the bound

$$\sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \perp_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} \|N\big(u^{\pm}_{\kappa^1_\beta,\lambda}, (D_t + A_{\sqrt{\lambda}}) v^{\pm}_{\kappa^2_\beta,\lambda}\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^1_{t,x}}$$
$$\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \perp_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} \lambda^2\beta(\frac{\beta^2\lambda^2}{\mu^2}\mu)^{-1}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|(D_t + A_{\sqrt{\lambda}}) v^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^2L^2}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty}$$
$$\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \perp_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} \mu^{2+}d^{\frac{1}{2}}(\beta\alpha^{-1})^{-1}\alpha^{-1}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}}.$$

Therefore $\mu^{2+}d^{\frac{1}{2}}\alpha^{-1} \le \lambda\mu^{\frac{3}{2}+}$ in case $i.$ and $\mu^{2+}d^{\frac{1}{2}}\alpha^{-1} \le \mu^{\frac{5}{2}+}d^{\frac{1}{2}}$ in case $ii.$, and both factors are aceptables.

Estimate for III.d

To estimate this term we are forced to place the low frequency term into L^2L^2 and to apply Lemma 91 and Corollary 81, thus the high frequencies terms are estimated via Strichartz inequalities. We then compute the bound :

$$\sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \perp_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} \|N\big(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu}(D_t + A_{\sqrt{\mu}}) w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^1_{t,x}}$$
$$\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \perp_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} \lambda^2\beta(\frac{\beta^2\lambda^2}{\mu^2}\mu)^{-1}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^{2+}L^\infty}\|(D_t + A_{\sqrt{\mu}}) w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^2L^2}$$
$$\lesssim \sum_{\beta=\alpha}^{1} \sum_{\kappa^3_{\frac{\beta\lambda}{\mu}} \perp_{\frac{\beta\lambda}{\mu}} \kappa^1_\beta \perp_\beta \kappa^2_\beta} \lambda^{1+}\mu d(\beta\alpha^{-1})^{-1}\alpha^{-1}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}}.$$

In case $i.$ we have $\lambda^{1+}\mu\alpha^{-1}d \le \lambda^{2+}\mu^{\frac{1}{2}}d_2^{\frac{1}{2}}$ and in case $ii.$ we obtain $\lambda^{1+}\mu d\alpha^{-1} \le \lambda^{1+}\mu^{\frac{3}{2}}d$. This latter constant seems a priori too big, however in case $ii.$ we have imposed the bound $d^{\frac{1}{2}}\mu \le \lambda$, therefore we obtain $\lambda^{1+}\mu^{\frac{3}{2}}d \le \lambda^{2+}\mu^{\frac{1}{2}}d^{\frac{1}{2}}$ which is acceptable.

Estimate for III.e

We proceed as for the case II.e in the high-low-high interaction case. Let us split the error term by isolating the factors which contains time derivatives, we write $E(u,v,w) = E_0(u,v,w) +$

$E_1(u,v,w)$, where

$$E_0(u,v,w) = N\big(A_{<\mu^{1/2}}u, v\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w + N\big(u, A_{<\mu^{1/2}}v\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w + N\big(u,v\big)\widetilde{A}_{<\mu^{1/2}}\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w,$$

and

$$E_1(u,v,w) = N\big(D_t u, v\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w + N\big(u, D_t v\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w + N\big(u,v\big) D_t \widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w.$$

We estimate the E_1 term by means of the fundamental theorem of calculus:

$$\begin{aligned}\|E_1(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}, w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu})\|_{L^1_{t,x}} &\lesssim \|N\big(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}\big)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w^{\pm}_{\kappa^3_\beta,\mu}\|_{L^1_x}\Big|_{t=0}^{t=1} \\ &\lesssim \lambda^2\beta \|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty_t L^2_x}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^\infty_t L^2_x}\|\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^\infty_t L^\infty_x}.\end{aligned}$$

By Lemma 91 the $\widetilde{L}_{\kappa_\beta,\mu}$ operator produces $(\frac{\beta^2\lambda^2}{\mu^2}\mu)^{-1}$ and we apply Bernstein inequality for the low modulation term to reduce it to $L^\infty_t L^2_x$ yielding to factor of $\lambda\mu^{1/2}\beta$ and Strichartz inequality to reach the $X^{0,0}_{\mu,d}$ space at the price of an extra $d^{1/2}$ factor. Therefore we obtain

$$\|E_1(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}, w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu})\|_{L^1_{t,x}} \lesssim \lambda\mu^{\frac{3}{2}} d^{\frac{1}{2}} \|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}}.$$

Thus the same argument holds in the two cases $i.$ and $ii.$ yielding to an acceptable constant.

To estimate the E_0 term, let us define

$$\widetilde{E}_0(u,v,w) = (A_{<\mu^{1/2}}u)v(\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w) + u(A_{<\mu^{1/2}}v)(\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w) + uv(\widetilde{A}_{<\mu^{1/2}}\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w).$$

We shall refer to Lemma 5.3 of [30] see also (2.51) and Lemma 92 to obtain a bound for $\widetilde{E}_0$. We have :

$$\begin{aligned}&\|E_0\big(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}, w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\big)\|_{L^1_{t,x}} \\ &\lesssim \lambda^2\beta\|\widetilde{E}_0\big(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}, w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\big)\|_{L^1_{t,x}} \\ &\lesssim \lambda^2\beta\big(\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^2L^2}\|\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^2L^\infty} \\ &+ \mu\lambda^{-2}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)v^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^2L^2}\|\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty} \\ &+ \lambda^{-1}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)v^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^2L^2}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty} \\ &+ \lambda^{-1}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^2L^2}\|(\xi^{\pm}_{\theta_{\kappa^1_\alpha}}\wedge D_x)\widetilde{L}_{\kappa_{\frac{\beta\lambda}{\mu}},\mu} w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^2L^\infty}\big).\end{aligned}$$

Notice that the operator $(\xi_\theta \wedge D_x)$ when applied to $u^{\pm}_{\kappa^1_\beta,\lambda}$ or $v^{\pm}_{\kappa^2_\beta,\lambda}$ yield to a factor of $\lambda\mu^{-1}\alpha$, and when applied to $w^{\pm}_{\kappa^1_{\frac{\beta\lambda}{\mu}},\mu}$ lead to a loss of $\mu\beta$. Moreover in the first and fourth term we apply Bernstein inequality while in the second we employ Strichartz estimate to avoid loosing a β factor. Thus we obtain

$$\begin{aligned}
&\|E_0(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}, w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu})\|_{L^1_{t,x}} \\
&\lesssim \lambda^2\beta\frac{\mu}{\lambda^2\beta^2}\big(\lambda\mu^{\frac12}\beta + \alpha\beta\mu^{1+}d^{\frac12} + \alpha\beta\mu^{1+}d^{\frac12} + \alpha^2\mu^{+}d^{\frac12}\big)\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}}.
\end{aligned}$$

Therefore in case $i.$, since $\alpha = \mu^{1/2}d_2^{1/2}\lambda^{-1}$, we obtain

$$\begin{aligned}
&\sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa^1_\beta\perp_\beta\kappa^2_\beta}\|E_0(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}, w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu})\|_{L^1_{t,x}} \\
&\lesssim \sum_{\beta=\alpha}^{1}\sum_{\kappa^3_{\frac{\beta\lambda}{\mu}}\perp_{\frac{\beta\lambda}{\mu}}\kappa^1_\beta\perp_\beta\kappa^2_\beta}(\lambda\mu^{3/2} + \alpha\mu^{2+}d + (\beta\alpha^{-1})^{-1}\alpha\mu^{1+}d^{\frac12})\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}} \\
&\lesssim \lambda\mu^{\frac32+}d_2\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}}
\end{aligned}$$

where the constant

$$\lambda\mu^{3/2} + \alpha\mu^{2+}d + (\beta\alpha^{-1})^{-1}\alpha\mu^{1+}d^{\frac12} \le \lambda\mu^{3/2} + \lambda^{-1}\mu^{\frac52+}d^{\frac32} + \lambda^{-1}\mu^{\frac32+}d$$

is small enough to close the argument. A similar augment can be applied in the $ii.$ case yielding to a constant

$$\lambda\mu^{3/2} + \alpha\mu^{2+}d + (\beta\alpha^{-1})^{-1}\alpha\mu^{1+}d^{\frac12} \le \lambda\mu^{3/2} + \mu^{\frac32+}d + \mu^{\frac32+}d^{\frac12}$$

which is still acceptable. This concludes the estimate for the III term.

Estimate for IV and V

These two terms are better behaved since they are supported away from the diagonal set $D = \{(\xi,\eta,\zeta) : \xi+\eta+\zeta = 0\}$. Therefore we employ the same integration by parts procedure used to estimate IV in the high-low-high interaction to gain an arbitrary power of high frequencies. To control IV we then apply the bound (2.52) to obtain

$$\|N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda})w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^1_{t,x}} \approx \beta\lambda^2(\beta\lambda)^{-4N}\|u^{\pm}_{\kappa^1_\beta,\lambda}v^{\pm}_{\kappa^2_\beta,\lambda}w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^1_{t,x}}.$$

Finally Strichartz estimates allow us to conclude

$$\begin{aligned}\|N(u^{\pm}_{\kappa^1_\beta,\lambda}, v^{\pm}_{\kappa^2_\beta,\lambda}) w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^1_{t,x}} &= \beta\lambda^2(\beta\lambda)^{-4N}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{L^\infty L^2}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{L^2L^2}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{L^{2+}L^\infty} \\ &\lesssim \lambda^{2-4N}\beta^{1-4N}\mu^{1+}d_2^{\frac{1}{2}}\|u^{\pm}_{\kappa^1_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|v^{\pm}_{\kappa^2_\beta,\lambda}\|_{X^{0,0}_{\lambda,1}}\|w^{\pm}_{\kappa^3_{\frac{\beta\lambda}{\mu}},\mu}\|_{X^{0,0}_{\mu,d}}.\end{aligned}$$

Then choosing $N = 1/4$ yields to the constant $\lambda\mu^{1+}d_2^{1/2}$, which is acceptable in both cases $i.$ and $ii.$.

To estimate the term V we apply the same argument as for the previous term. In this case the integration by parts procedure yield to a gain of arbitrary powers of $\beta\gamma\lambda$. Therefore we can run the same argument used in IV. This conclude the proof of the high-high-low interaction and thus Proposition 86 holds. □

Part II
Stability of slow blow-up solutions

3 Construction and stability of type II blow-up solutions

In this chapter we outline the recent advances on the stability issues of certain finite time type II blow-up solutions for the energy critical focusing wave equation $\Box u = -u^5$ in $\mathbb{R}^{3+1}$. Hereafter we use the convention $\Box = -\partial_t^2 + \triangle$. The objective of this article is twofold: firstly we describe the construction of singular solutions contained in [62] and [59], and secondly we undertake a detailed analysis of its stability properties enclosed in [49] and [7].

3.1 Introduction

Despite its naive appearance, the semilinear wave equation

$$\Box u = -u^5, \quad u : \mathbb{R}^{1+3}_{t,x} \to \mathbb{R} \tag{3.1}$$

is an excellent simplistic model since its main features are shared with multiple geometric and physical equations such as critical Wave-Maps and Yang-Mills equation. However, as we shall see, the price to pay to avoid many technical issues is the ubiquity of type I blow-up solutions which constitute the generic blow-up scenario.

Local well-posedness up to the optimal regularity class $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ of the Cauchy problem for equation (3.1) coupled with initial data was proved by Lindblad and Sogge [69] and it relies on the celebrated Strichartz estimates, see also [95] for a detailed description. Moreover, as a typical trademark for focusing equations, the conserved energy

$$E(u)(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla_{t,x} u|^2 - \frac{1}{6}|u|^6 dx$$

is not positive definite, making the extensions of local solutions to global one a highly non-trivial question.

In fact, several obstructions to long-time existence of solution of (3.1) have been uncovered. For instance, Levine [66] demonstrated via a convexity argument that break down in finite time occur for initial data with negative energy. Nonetheless, Levine's argument is indirect, and it does not provide much information about the exact nature of the blow-up. More primitive blow-up solutions can be explicitly constructed by the ODE technique: let $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\phi(x) = 1$ if $|x| \le 2T$, set the initial data $u_0(0,x) = (\frac{3}{4})^{1/4} T^{-1/2}\phi(x)$, and $u_t(0,x) = (\frac{3}{64})^{1/4} T^{-3/2}\phi(x)$. Then the solution of (3.1) behaves like the so called *fundamental self-similar solution*

$$u(t,x) = (\frac{3}{4})^{1/4}(T-t)^{-1/2} \tag{3.2}$$

for $0 < t < T$ and $|x| < T-t$. As this example shows, singularities can arise in finite time even for smooth compactly supported initial data. Observe that for these solutions the critical Sobolev norm diverges as time approaches the maximum time of existence:

$$\limsup_{t\to T} \|\nabla_{t,x} u(t,\cdot)\|_{L^2(\mathbb{R}^3)} \to +\infty. \tag{3.3}$$

Motivated by such blow-up mechanism it is common to define a blow-up solution u with maximum forward time of existence $T < +\infty$ of *type I* if (3.3) holds, and of *type II* otherwise, that is if $\|\nabla_{t,x} u(t,\cdot)\|_{L^2}$ remains bounded up to the break down time. The dichotomy between type I and type II blow-up solutions is well understood at this point in time.

Another explicit solution of (3.1) is the Aubin-Talenti function

$$W(x) = \Big(1 + \frac{|x|^2}{3}\Big)^{-1/2}$$

which is the unique (up to symmetries) positive solution to the associated elliptic equation $\triangle W = -W^5$ and it is the minimizer of the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, see [1] and [102]. Through a remarkable series of works Duyckaerts, Kenig, and Merle [20, 22, 21, 23] provided a complete abstract classification of all possible type II blow-up solutions in finite time in terms of a finite number of rescaled W plus a small radiation term.

Theorem 96 ([23])**.** *Let u be a radial type II solution of* (3.1) *which breaks down in finite time T. Then there exist finitely many continuous functions $\lambda_j(t)$, $j = 1,\ldots,k$, with* $\lim_{t\to T}(T-t)\lambda_j(t) = +\infty$*, and*

$$\lim_{t\to T} \Big|\log\Big(\frac{\lambda_i(t)}{\lambda_j(t)}\Big)\Big| = +\infty, \qquad \text{for } i \neq j,$$

such that

$$u(t,x) = \sum_{j=1}^{k} \pm W_{\lambda_j(t)}(x) + \eta(t,x)$$

and where $(\eta, \partial_t\eta) \in C([0,T], \dot{H}^1 \times L^2)$ and $W_\lambda(x) = \lambda^{1/2} W(\lambda x)$.

The extent of the previous result is essential in the progress of understanding type II blow-up solutions. However, due to the nature of the arguments, namely the famous concentration compactness method, the Duyckaerts, Kenig, and Merle program does not demonstrate the existence of all such possible blow-up dynamics. In fact, at the best of author's knowledge it emerges that only finite time blow-up solutions with one bulk term W are known to exist. Moreover, the precise blow-up dynamics is unknown and it does not appear to give any information on the stability of such solutions.

Complementary, an explicit finite time type II blow-up was constructed by Krieger, Schlag and Tataru [62]. The breakthrough [62] consists in establishing the existence of a family of rough blow-up solutions displaying a continuum of blow-up rates slower then the one provided by the self-similar blow-up. In addition, all previously known blow-up solutions become singular along a hypersurface, vice-versa the ones furnished in [62] exhibit a one-point blow-up. In a subsequent work [59], the first two authors extended the range of allowed blow-up speeds up to reach arbitrary close the self-similar blow-up speed.

Other concrete realizations for finite-time type II dynamics where established: Hillairet and Raphaël [31] constructed type II smooth solution for the energy critical semilinear wave equation $\Box u = -u^3$ in $\mathbb{R}^{4+1}$, with the fixed scaling law

$$\lambda(t) = t^{-1} e^{\sqrt{|\log t|}}, \qquad \text{as } t \to 0.$$

The set of initial data leading to such type II blow-up is given by a co-dimension one Lipschitz manifold. Another constructive approach was given in Krieger, Donninger, Huang, and Schlag [50], where the authors provided a finite time blow-up solution of type II with oscillating scaling law, that is of the form $u(t,x) = W_{\lambda(t)}(x) + \eta(t,x)$ where $\lambda(t) = t^{-\nu(t)}$ and $\nu(t) = \nu + \epsilon_0 \frac{sin(\log t)}{\log t}$, with $\nu > 3$ and $|\epsilon_0| \ll 1$ be arbitrary and η a small error.

A deeper study of the stability of such blow-up scenarios has been the subject of a number of recent works. Persuaded by numerical evidence provided by Bizon et al. [3], which suggested that finite time blow-up for (3.1) are generically of type I, in a sequence of pioneering works Donninger and Schörkhuber [18] and Donninger [16] settled the asymptotic stability of the ODE blow-up solution (3.2) in the energy norm. On the other hand, Krieger, Nakanishi, and Schlag [56] elucidated that type II solutions are unstable in the energy norm in the following precise sense.

Theorem 97 ([56])**.** *Let $\lambda(t) \to +\infty$ as $t \to T$, and*

$$u(t,x) = W_{\lambda(t)}(x) + \eta(t,x)$$

be a type II blow-up solution on $I \times \mathbb{R}^3$ for (3.1)*, such that*

$$\sup_{t \in I} \|\nabla_{t,x}\eta(t,\cdot)\|_{L^2_x} \le \delta \ll 1$$

for some sufficiently small $\delta > 0$, where $I = [0,T]$ denotes the maximal life span of the Shatah-Struwe solution u. Also, assume that $t_0 \in I$. Then there exists a co-dimension one Lipschitz manifold Σ in a small neighborhood of the data $\big(u(t_0,\cdot), u_t(t_0,\cdot)\big) \in \Sigma$ in the energy topology

$\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, such that initial data $(f,g) \in \Sigma$ result in a type II solution, while initial data

$$(f,g) \in B_\delta \backslash \Sigma,$$

where $B_\delta \subset \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is a sufficiently small ball centered at $(u(t_0,\cdot), u_t(t_0,\cdot))$, either lead to blow-up in finite time, or solutions scattering to zero, depending on the 'side of Σ' these data are chosen from.

In spite of the universality of type I blow-up for equation (3.1), with the purpose of study more sophisticated equations at the critical regime where only type II dynamic is present, it is fundamental to investigate further type II blow-up solutions and its stability properties. The stability of solutions constructed in [59] and [62] was analyzed by Krieger [49] where a conditional result requiring two extra co-dimensions was obtained for solutions which blow-up at a rate sufficiently close to the self-similar one. The optimal stability result was achieved by the author and Krieger in [7]. In the second part of this article we outline the proof of the latter results.

To place these results in a proper context, some more discussion on similar results for different equations is in order. As a matter of fact, the work [62] is an occurrence in a triplets of works [62, 60, 61], dedicated to the explicit construction of rough type II singular solutions respectively for semilinear wave equations, for the co-rotational critical wave maps from $\mathbb{R}^{2+1} \to S^2$, and for the critical Yangs-Mills equations in $4+1$ dimensions under the spherically symmetric ansatz. A parallel construction of a smooth finite time type II singular solution with fixed blow-up speed was carried out by Raphaël and Rodnianski [87] for the co-rotational critical wave maps in $2+1$ dimensions with S^2 target, and for the critical $SO(4)$ Yangs-Mills equations in $4+1$ dimensions. Concerning the stability issue, the method employed by Raphaël and Rodnianski implies that their solutions are stable. Furthermore, in a recent breakthrough Krieger and Miao [52] were able to show that the solutions constructed in [60] for the co-rotational critical wave maps are stable in a suitable topology. The corresponding result for the Yang-Mills problem is still open.

3.2 The construction of slow blow-up solutions

In this section we describe the construction of explicit finite time type II blow-up solutions contained in the works [59] and [62]. We shall be interested exclusively in the case of radial solutions, thus the energy critical focusing semilinear wave equation under radial symmetry can be written as:

$$-u_{tt} + u_{rr} + \frac{2}{r} u_r = -u^5. \tag{3.4}$$

The goal is to construct a solution $u \in C((0,t_0], H^{1+}) \times C^1((0,t_0], H^+)$ of (3.4) which blows-up at the space-time origin, and the blow-up is of type II, hence its space-time gradient remains bounded on the interval of existence: $\sup_{t\in[0,t_0]} \|\nabla_{t,x} u(t,\cdot)\|_{L^2(\mathbb{R}^3)} < \infty$. Notice that, due to the time reversibility of the wave equation, we start evolving the dynamics from initial data at time $t_0 > 0$ and solve the wave equation backwards in time until the blow-up time $t = 0$. Here t_0 is a small positive constant that will be defined later.

We state here the results of [62] and [59]. The main difference between them is the lower bound for ν. In [62] the restriction $\nu > 1/2$ was imposed, and in [59] the result was extended to include $\nu > 0$.

Theorem 98. *([62], [59]) Let $\nu > 0$ and $\lambda(t) = t^{-1-\nu}$ the scaling parameter. There exists a class of solutions to equation* (3.4) *of the form*

$$u_\nu(t,r) = W_{\lambda(t)}(r) + \eta(t,r) =: u_0(t,r) + \eta(t,r)$$

inside the truncated light cone $K = \{(t,r) \in (0,t_0) \times \mathbb{R}^+ : \ t > r\}$. The term u_0 is called bulk term or non-oscillatory elliptic term and it is given by the rescaling of W. The second term η is called oscillatory radiation part and it is composed by two distinct functions: $\eta = \eta^e + \varepsilon$. Here η^e is an non-oscillatory term satisfying $\eta^e \in C^\infty(K)$ and $\mathscr{E}_{loc}(\eta^e)(t) \lesssim (t\lambda(t))^{-2}|\log t|^2$ as $t \to 0$, hence its local energy vanishes as time $t \to 0$. The local energy relative to the origin is defined as

$$\mathscr{E}_{loc}(u)(t) = \int_{|x|<t} \frac{1}{2}|\nabla_{t,x} u|^2 - \frac{1}{6}|u|^6 dx.$$

On the other hand ε is rougher, that is $(\varepsilon(t,\cdot), \varepsilon_t(t,\cdot)) \in (H^{\frac{\nu}{2}+1-}(\mathbb{R}^3) \times H^{\frac{\nu}{2}-}(\mathbb{R}^3))$ and $\mathscr{E}_{loc}(\varepsilon)(t) \to 0$ as $t \to 0$. Moreover, outside the light cone we have the bound

$$\int_{|x|\geq t} \frac{1}{2}|\nabla_{t,x} u_\nu|^2 - \frac{1}{6}|u_\nu|^6 dx \leq C < \infty.$$

Notice that the bulk term $u_0 \in C^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ blows-up at the space-time origin. However $u_0(t,\cdot) \notin L^2(\mathbb{R}^3)$, since it does not decay sufficiently fast at space infinity. To obtain a solution in H^{1+} it suffices to multiply u_0 by a bump function $\chi \in C_0^\infty(\mathbb{R})$ which equals one on the ball of radius t_0. In this way, on every fixed positive time slice we clearly have $u_0(t,\cdot)\chi(t) \in H^{1+}(\mathbb{R}^3)$ and $\partial_t(u_0(t,\cdot)\chi(t)) \in H^+(\mathbb{R}^3)$. Notice that $u_0(t,\cdot)\chi(t,\cdot)$ is in $C_0^\infty(\mathbb{R}^3)$ therefore clearly is type II. The rougher part of the solution which gives the overall regularity $C((0,t_0], H^{\frac{\nu}{2}+1-}) \times C^1((0,t_0], H^{\frac{\nu}{2}-})$ is the term η^e. In fact, although it is smooth inside the light cone, namely $\eta^e \in C^\infty(K)$, it reveals a cusp singularity along the boundary ∂K of the light cone, implying that $\eta^e(t,\cdot) \in H_{loc}^{1+\frac{\nu}{2}-}(\mathbb{R}^3)$ and $\partial_t \eta^e(t,\cdot) \in H_{loc}^{\frac{\nu}{2}-}(\mathbb{R}^3)$.

Before outlining the proof of Theorem 98, a final remark on the blow-up speed of u_ν is in order. Clearly, the $L^\infty(\mathbb{R}^3)$ norm of the ODE blow-up solution (3.2) concentrates, as time approaches the break-down time, at a rate which is proportional to $t^{-1/2}$. Diversely, the blow-up of speed of u_ν solutions is proportional to $t^{-1/2-\nu/2}$. Hence type II solutions blow-up faster that type

I and, as ν approaches 0, the different blow-up speeds become comparables. Moreover, by varying the parameter $\nu > 0$, which it is not a priori fixed, we obtain a blow-up solution with prescribed blow-up speed, i.e. a *continuum of blow-up speeds*.

The proof of Theorem 98 is based on a two steps procedure mimicking the strategy of others constructions of type II dynamics contained in [31, 87, 71]. Firstly, one constructs a sequence of approximate solutions which solve (3.4) up to a small error. This approximation method will not lead to an exact solution by passing to the limit due to the divergence of the coefficients. Hence one needs to terminate the process after finitely many steps. Secondly, one completes the approximate solution to an exact solution via a fixed point argument. In regard to the second step, the argument used to prove Theorem 98 differs drastically from the strategy employed in [31, 87, 71]. In the latter pioneering works the remaining error is controlled via Morawetz and viral type identities, whereas the present proof hinges on a constructive parametrix approach.

3.2.1 The renormalization step

The aim of this step is to iteratively construct a very accurate approximate solution near the singularity depending on two parameters k and ν which has the form

$$u_k(t,r) = W_{\lambda(t)}(r) + \eta_k^e(t,r)$$

where the k-th non-oscillatory term $\eta_k^e(t,r) = \sum_{j=1}^k v_j(t,r)$ is a sum of small corrections and $\lambda(t) = t^{-1-\nu}$. The bulk term $u_0(t,r) = W_{\lambda(t)}(r)$ is very far from being an approximate solution of (3.4), indeed it produces an error $e_0 = \Box u_0 + (u_0)^5$ which blows up like t^{-2} as $t \to 0$. In [62] the authors adopt the strategy of adding successive corrections functions v_j so that the the error $e_k = \Box u_k + u_k^5$ generated by the approximate solution u_k can be made arbitrary small in a suitable sense by picking k suitable large. More precisely, the corrections v_j are chosen in order to force e_k to go to zero like t^N as $t \to 0$ in the energy norm restricted to a light cone, where N can be made arbitrarily large by taking k large.

The construction consists in a delicate bookkeeping procedure to iteratively reduce the size of the generated error by alternating between amelioration near the spatial origin and improvements near the light cone. The finite sequence of approximate solution u_k is defined recursively. Set $u_0 = W_{\lambda(t)}(r)$, then for $k \geq 1$ the k-th approximation u_k is given in terms of the previous one via the following algorithm: let u_{k-1} be the approximate solution which generates the error $e_{k-1} = \Box u_{k-1} + u_{k-1}^5$, then one updates u_{k-1} by adding a correction, i.e. $u_k = u_{k-1} + v_k = u_0 + v_1 + \cdots + v_k$, thereby the error e_k produced by the improved approximation u_k is smaller than e_{k-1} in a suitable sense. To define the appropriate correction v_k we distinguish between k even or k odd. The odd corrections are the solutions of the following inhomogeneous second order ODEs:

$$\begin{cases} \left(\partial_r^2 + \frac{2}{r}\partial_r + 5u_0^4(t,r)\right)v_{2k-1}(t,r) = e_{2k-2}(t,r) & \text{in } \mathbb{R}_r^+, \\ v_{2k-1}(t,0) = \partial_r v_{2k-1}(t,0) = 0. \end{cases} \tag{3.5}$$

The heuristic which leads to such formulation is that when $r \ll t$ we expect the term involving the time derivative in (3.4) to be negligible. On the other hand, for even corrections we improve the approximate solution near the light cone $r \approx t$, thus we can roughly estimate u_0 by zero and we are led to the $1+1$-inhomogeneous hyperbolic equation

$$\begin{cases} \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right)v_{2k}(t,r) = e_{2k-1}(t,r) & \text{in } \mathbb{R}_t^+ \times \mathbb{R}_r^+, \\ v_{2k}(t,0) = \partial_r v_{2k}(t,0) = 0. \end{cases} \tag{3.6}$$

The Cauchy problem (3.5) is a standard Sturm-Liouville problem and it is solved via the variation of parameter method. Whereas the hyperbolic character of (3.6) is controlled by using self-similar coordinate $a = r/t$ and a brillant ansatz on the form of the solution.

3.2.2 Completion to an exact solution

The main point of this second part of the argument is to perturb around the approximate solution constructed in the previous step, and thus to look for an exact solution of (3.4) of the form $u_\nu = u_{2k-1} + \varepsilon$. Notice that we stop the approximation algorithm after an odd number of cycles. By imposing that u_ν to be an exact solution we force an equation for ε:

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\varepsilon - 5\lambda^2(t)W^4\big(\lambda(t)r\big)\varepsilon = e_{2k-1} + \widetilde{N}(u_{2k-1},\varepsilon) \tag{3.7}$$

where $\widetilde{N}(u,\varepsilon) = (\varepsilon+u)^5 - u^5 - 5\varepsilon u_0^4$. To avoid treating a nonlinear hyperbolic equation with time-dependent potential one removes the time dependency of the potential by introducing new coordinates $(t,r) \to (\tau, R)$, where

$$\tau(t) = \int_t^{t_0} \lambda(s)ds + \nu^{-1}t_0^{-\nu} = \nu^{-1}t^{-\nu}, \qquad R(t,r) = \lambda(t)r.$$

The price to pay is that the time derivative ∂_t is transformed into the operator $\lambda(\tau)\partial_\tau + \frac{\lambda'(\tau)}{\lambda(\tau)}R\partial_R$. Let us set $v(\tau,R) = \varepsilon(t(\tau), \lambda^{-1}(\tau)R)$ and $\beta(\tau) = \lambda'(\tau)/\lambda(\tau)$, then equation (3.7) is transformed into

$$\begin{aligned} \left[\Big(\partial_\tau + \beta(\tau)R\partial_R\Big)^2 - \beta(\tau)\Big(\partial_\tau + \beta(\tau)R\partial_R\Big) - \partial_R^2 - \frac{2}{R}\partial_R\right] v(\tau,R) - 5W^4(R)v(\tau,R) = \\ \lambda^{-2}(\tau)e_{2k-1}(\tau,R) + \lambda^{-2}(\tau)\widetilde{N}(u_{2k-1},v)(\tau,R). \end{aligned}$$

Subsequently, in order to get rid of the first derivative in the R variable, we consider the function $\widetilde{\varepsilon}(\tau,R) = Rv(\tau,R)$, this new function satisfies the equation

$$(\mathscr{D}^2 + \beta(\tau)\mathscr{D} + \mathscr{L})\widetilde{\varepsilon}(\tau,R) = f[\widetilde{\varepsilon}](\tau,R), \qquad \text{in } \mathbb{R}^+_\tau \times \mathbb{R}^+_R \tag{3.8}$$

where $\mathscr{D} = \partial_\tau + \beta(\tau)(R\partial_R - 1)$, $\mathscr{L} = -\partial_R^2 - 5W^4(R)$, and

$$f[\widetilde{\varepsilon}](\tau,R) = \lambda^{-2}(\tau)\Big(Re_{2k-1} + N(u_{2k-1},\widetilde{\varepsilon})\Big)$$

and

$$N(u_{2k-1},\widetilde{\varepsilon})(\tau,R) = 5\widetilde{\varepsilon}(u_{2k-1}^4 - u_0^4) + R\Big(\frac{\widetilde{\varepsilon}}{R} + u_{2k-1}\Big)^5 - Ru_{2k-1}^5 - 5u_{2k-1}^4\widetilde{\varepsilon}.$$

To look for a solution of equation (3.8) a prototypical Fourier transform, namely the *distorted Fourier transform associated to the operator* $\mathscr{L}$, is applied imitating the procedure to convert to the frequencies sides the free wave equation. The spectral properties of the operator $\mathscr{L}$ play a pivotal role and are analyzed in details in [62]. This operator, when restricted to functions on $[0,\infty)$ with Dirichlet condition at $R = 0$, has a simple negative eigenvalue $\xi_d < 0$ (the subscript d referring to *discrete spectrum*), and a corresponding L^2-normalized positive ground state $\phi_d \in L^2(0,\infty) \cap C^\infty([0,\infty))$ decaying exponentially and vanishing at the origin $R = 0$. This mode will cause exponential growth for the linearized evolution $e^{it\sqrt{\mathscr{L}}}$. However, in [62] and [59] the authors avoid this problem by imposing vanishing initial data at $t = 0$ for the function ε, which is equivalent to impose zero data at $\tau = \infty$ for the function $\widetilde{\varepsilon}$. In the subsequent works [49] and [7], where no such freedom of imposing zero initial data is acceptable, only a co-dimension one condition will ensure that the forward flow will remains bounded.

Let us present below the pivotal result which summarize the main properties of the distorted Fourier transform.

Proposition 99. *([62]) There exists a generalized Fourier basis $\phi(R,\xi)$, $\xi \geq 0$, a eigenstate $\phi_d(R)$, and a spectral measure $\rho(\xi) \in C^\infty\big((0,\infty)\big)$ with the asymptotic behaviors*

$$\rho(\xi) \sim \begin{cases} \xi^{-\frac{1}{2}}, & \text{if } 0 < \xi \ll 1, \\ \xi^{\frac{1}{2}}, & \text{if } \xi \gg 1, \end{cases}$$

as well as symbol behaviour with respect to differentiation, and such that by defining

$$\mathscr{F}(f)(\xi) := \widehat{f}(\xi) := \lim_{b\to+\infty}\int_0^b \phi(R,\xi)f(R)\,dR,$$
$$\widehat{f}(\xi_d) = \int_0^\infty \phi_d(R)f(R)\,dR,$$

the map $f \longrightarrow \widehat{f}$ is an isometry from L^2_{dR} to $L^2(\{\xi_d\} \cup \mathbb{R}^+, \rho)$, and we have

$$f(R) = \widehat{f}(\xi_d)\phi_d(R) + \lim_{\mu\to\infty} \int_0^\mu \phi(R,\xi)\widehat{f}(\xi)\rho(\xi)\,d\xi,$$

the limits being in the suitable L^2-sense.

The mayor issue in applying the distorted Fourier transform to equation (3.8) is the term involving $R\partial_R$ contained in the $\mathscr{D}$ operator since $\mathscr{F}(R\partial_R) \neq \xi\partial_\xi\mathscr{F}$. Therefore one defines the *error operator* $\mathscr{K}$ via the equation

$$\mathscr{F}[(R\partial_R - 1)u(\tau,R)](\xi) = \mathscr{A}\,\widehat{u}(\tau,\xi) + \mathscr{K}\,\widehat{u}(\tau,\xi)$$

where

$$\mathscr{A} = \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{A}_c \end{pmatrix}, \qquad \mathscr{K} = \begin{pmatrix} \mathscr{K}_{dd} & \mathscr{K}_{dc} \\ \mathscr{K}_{cd} & \mathscr{K}_{cc} \end{pmatrix}$$

and $\mathscr{A}_c = -2\xi\partial_\xi - \left(\frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}\right)$. We add the second term in $\mathscr{A}_c$ because later on we shall need the relation $(SM)^{-1}\partial_\tau SM = \mathscr{D}_\tau$ where $\mathscr{D}_\tau = \partial_\tau + \beta(\tau)\mathscr{A}_c$. In other words $\mathscr{K}$ is defined as the solution to the system

$$\begin{cases} \langle (R\partial_R - 1)u(\tau,R), \phi_d\rangle_{L^2_{dR}} = \mathscr{K}_{dd}\langle u(\tau,R),\phi_d\rangle_{L^2_{dR}} + \mathscr{K}_{dc}\widehat{u} \\ \mathscr{F}[(R\partial_R - 1)u(\tau,R)] = \mathscr{A}_c\widehat{u} + \mathscr{K}_{cd}\langle u(\tau,R),\phi_d\rangle_{L^2_{dR}} + \mathscr{K}_{cc}\widehat{u} \end{cases}$$

and we have

$$\mathscr{K}_{dd} = -\frac{1}{2}, \qquad \mathscr{K}_{cd}(\xi) = k_d(\xi),$$

$$\mathscr{K}_{dc}(f) = -\int_0^\infty f(\xi)k_d(\xi)\rho(\xi)d\xi, \qquad \mathscr{K}_{cc}[f](\xi) = \int_0^\infty \frac{F(\xi,\eta)}{\xi-\eta} f(\eta)\rho(\eta)d\eta.$$

where $k_d(\xi)$ is a smooth and rapidly decaying function at $\xi = +\infty$ and the function F is of regularity at least C^2 on $(0,\infty)\times(0,\infty)$, and satisfies further smoothness and decay properties listed in Theorem 5.1 in [62].

We now proceed to transpose equation (3.8) to the Fourier side. Notice that the time variable remain invariant since we are dealing with Fourier transform in space only. Let us denote the distorted Fourier transform of the unknown function in (3.8) by $\big(x_d(\tau), x(\tau,\xi)\big) = \mathscr{F}(\widetilde{\varepsilon})(\tau,\xi)$, that is:

$$x(\tau,\xi) = \int_0^\infty \phi(R,\xi)\widetilde{\varepsilon}(\tau,R)\,dR, \qquad x_d(\tau) = \int_0^\infty \phi_d(R)\widetilde{\varepsilon}(\tau,R)\,dR.$$

Notice that once the Fourier representation $\big(x_d(\tau), x(\tau,\xi)\big)$ is known one can easily recover the original function $\widetilde{\varepsilon}$ via

$$\widetilde{\varepsilon}(\tau,R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)d\xi. \tag{3.9}$$

Applying the distorted Fourier transform to equation (3.8) yields to the following system involving one equation for the discrete spectral part and a second equation for the continuous spectral part:

$$(\vec{\mathscr{D}}_\tau^2 + \beta(\tau)\vec{\mathscr{D}}_\tau + \vec{\xi})\vec{x}(\tau,\xi) = \vec{\mathscr{R}}\vec{x}(\tau,\xi) + \vec{f}(\tau,\xi) \tag{3.10}$$

where $(\tau,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+$, $\vec{x}(\tau,\xi) = \big(x_d(\tau), x(\tau,\xi)\big)^T$, and

$$(\vec{\mathscr{D}}_\tau^2 + \beta(\tau)\vec{\mathscr{D}}_\tau + \vec{\xi}) = \begin{pmatrix} \partial_\tau^2 + \beta(\tau)\partial_\tau + \xi_d & 0 \\ 0 & \mathscr{D}_\tau^2 + \beta(\tau)\mathscr{D}_\tau + \xi \end{pmatrix}.$$

The inhomogeneous terms on the right-hand-side of (3.10) are composed of a linear source term

$$\vec{\mathscr{R}} = \begin{pmatrix} \mathscr{R}_{dd} & \mathscr{R}_{dc} \\ \mathscr{R}_{cd} & \mathscr{R}_{cc} \end{pmatrix}$$

where

$$\begin{aligned}
\mathscr{R}_{dd} &= -2\beta(\tau)\mathscr{K}_{dd}\partial_\tau - \beta^2(\tau)\Big(\mathscr{K}_{dd}^2 + \mathscr{K}_{dc}\mathscr{K}_{cd} + \mathscr{K}_{dd} + \frac{\beta'(\tau)}{\beta^2(\tau)}\mathscr{K}_{dd}\Big),\\
\mathscr{R}_{dc} &= -2\beta(\tau)\mathscr{K}_{dc}\mathscr{D}_\tau - \beta^2(\tau)\Big(\mathscr{K}_{dd}K_{dc} + \mathscr{K}_{dc}\mathscr{K}_{cc} - \mathscr{K}_{dc}\mathscr{A}_c + \mathscr{K}_{dc} + \frac{\beta'(\tau)}{\beta^2(\tau)}\mathscr{K}_{dc}\Big),\\
\mathscr{R}_{cd} &= -2\beta(\tau)\mathscr{K}_{cd}\partial_\tau - \beta^2(\tau)\Big(\mathscr{K}_{cd}\mathscr{K}_{dd} + \mathscr{K}_{dc}\mathscr{K}_{cd} + \mathscr{A}_c\mathscr{K}_{cd} + \mathscr{K}_{cd} + \frac{\beta'(\tau)}{\beta^2(\tau)}\mathscr{K}_{cd}\Big),\\
\mathscr{R}_{cc} &= -2\beta(\tau)\mathscr{K}_{cc}\mathscr{D}_\tau - \beta^2(\tau)\Big(\mathscr{K}_{cd}\mathscr{K}_{dc} + \mathscr{K}_{cc}^2 + [\mathscr{A}_c,\mathscr{K}_{cc}] + \mathscr{K}_{cc} + \frac{\beta'(\tau)}{\beta^2(\tau)}\mathscr{K}_{cc}\Big),
\end{aligned} \tag{3.11}$$

plus the nonlinear term (observe that $\widetilde{\varepsilon}$ depends on the unknown functions $x_d(\tau), x(\tau,\xi)$ via (3.9)):

$$\vec{f}(\tau,\xi) = \begin{pmatrix} f_d(\tau) \\ f(\tau,\xi) \end{pmatrix} = \begin{pmatrix} \lambda^{-2}(\tau)\langle\phi_d, Re_{2k-1} + N(u_{2k-1},\widetilde{\varepsilon})\rangle_{L^2_{dR}} \\ \lambda^{-2}(\tau)\mathscr{F}\Big(Re_{2k-1} + N(u_{2k-1},\widetilde{\varepsilon})\Big)(\tau,\xi) \end{pmatrix}.$$

We coupled system (3.10) with initial conditions $\lim_{\tau\to\infty} x_d(\tau) = \partial_\tau x_d(\tau) = 0$, and $\lim_{\tau\to\infty} x(\tau,\xi) = \mathscr{D}_\tau x(\tau,\xi) = 0$.

The advantage of system (3.10) is the crucial observation that it can be solved completely explicitly. In fact, define $(Sf)(\tau,\xi) = f(\tau,\lambda^{-2}(\tau)\xi)$ and $(Mf)(\tau,\xi) = \lambda^{-5/2}(\tau)\rho^{1/2}(\xi)f(\tau,\xi)$, then we have the essential identity

$$(\mathscr{D}_\tau^2 + \beta(\tau)\mathscr{D}_\tau + \xi) = (SM)^{-1}[\big(\partial_\tau^2 + \beta(\tau)\partial_\tau + \lambda^{-2}(\tau)\xi\big)]SM$$

which provides the following parametrix

$$x_d(\tau) = \int_{\tau_0}^{\infty} -\frac{1}{2|\xi_d|^{1/2}} e^{-|\xi_d|^{1/2}|\tau-\sigma|}\big(g_d(\sigma) - \beta(\sigma)\partial_\sigma x_d(\sigma)\big)d\sigma,$$
$$x(\tau,\xi) = \int_{\tau}^{\infty} \frac{\lambda^{3/2}(\tau)}{\lambda^{3/2}(\sigma)} \frac{\rho^{1/2}\big(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\big)}{\rho^{1/2}(\xi)} \frac{\sin\left[\lambda(\tau)\xi^{1/2}\int_\tau^\sigma \lambda^{-1}(u)du\right]}{\xi^{1/2}} g\Big(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\Big)d\sigma, \tag{3.12}$$

where $\vec{g} = (g_d, g)^T$ represent respectively the right-hand-side of (3.10).

A contraction argument allow us to conclude the proof by finding an appropriate solution to (3.10). The fix point iteration is carried out in a weighted Sobolev type spaces defined by means of the following norms. Let $\alpha \in \mathbb{R}^+$, and a function $\vec{u}(\xi) = \big(u_d, u(\xi)\big)^T$, then define the norm

$$\|\vec{u}\|^2_{L^{2,\alpha}_{d\rho}} = |u_d|^2 + \|u\|^2_{L^{2,\alpha}_{d\rho}} := |u_d|^2 + \int_0^\infty |u(\xi)|^2 \langle\xi\rangle^{2\alpha}\rho(\xi)d\xi.$$

Notice that $\mathscr{F}$ is an isometry from $H^{2\alpha}_{dR}(\mathbb{R}^+)$ to $L^{2,\alpha}_{d\rho}(\mathbb{R}^+)$. Moreover for every τ-dependent function $\vec{f}(\tau,\xi) = \big(f_d(\tau), f(\tau,\xi)\big)^T$ let us define the norm $\|\vec{f}\|_{L^{2,\alpha,N}_\rho} = \sup_{\tau>\tau_0}\tau^N\|\vec{f}(\tau,\cdot)\|_{L^{2,\alpha}_{d\rho}}$. Defining $\vec{x}$ via the explicit formulas (3.12), we obtain the linear estimate

$$\|(\vec{x}, \vec{\mathscr{D}}_\tau\vec{x})\|_{L^{2,\alpha+\frac{1}{2},N-2}_{d\rho}\times L^{2,\alpha,N-1}_{d\rho}} \lesssim \frac{1}{N}\|\vec{g}\|_{L^{2,\alpha,N}_{d\rho}}.$$

The small factor N^{-1} is crucial for the fixed point argument to work. A similar estimate holds for the inhomogeneous terms on the right-hand-side of (3.10). More precisely the map $\vec{g}$ satisfies the bound

$$\|\vec{g}\|_{L^{2,\alpha,N}_{d\rho}} \lesssim \|(\vec{x}, \vec{\mathscr{D}}_\tau\vec{x})\|_{L^{2,\alpha+\frac{1}{2},N-2}_{d\rho}\times L^{2,\alpha,N-1}_{d\rho}} \tag{3.13}$$

and it is locally Lipschitz as a map from $L^{2,\alpha+1/2,N-2}_{d\rho}$ to $L^{2,\alpha,N}_{d\rho}$. Here the smallness of the constant is a consequences of the smallness of the error generated by the approximate solution built in the first part of the argument and the smallness of the time interval $(0, t_0]$ where the construction holds. The lack of smoothness of the approximate solution limits the decay in frequencies, hence the nonlinear estimate (3.13) holds only for $\nu/4 > \alpha$.

In [62], to control the nonlinear factors enclosed in the $f(\tau,\xi)$ term, precisely to obtain the quintilinear bound

$$H^{2\alpha+1}(\mathbb{R}^3)\cdot H^{2\alpha+1}(\mathbb{R}^3)\cdot H^{2\alpha+1}(\mathbb{R}^3)\cdot H^{2\alpha+1}(\mathbb{R}^3)\cdot H^{2\alpha+1}(\mathbb{R}^3) \subset H^{2\alpha}(\mathbb{R}^3),$$

the authors relies on a standard application of the Leibniz rule and Sobolev embedding, which holds for $\alpha \geq 1/8$, leading to the lower bound on the blow-up speed: $\nu > 1/2$. The latter restriction was removed in [59] by a more detailed analysis of the first iterate of the exact solution, thus yielding to the full expected range $\nu > 0$.

3.3 The stability of slow blow-up solutions

In what follows we outline the stability results of type II blow-up solutions u_ν constructed in [62] and [59]. The continuum of blow-up rates proper to u_ν and their limited regularity seem to indicate that these solutions are less stable than their smooth analogs built in [31]. Moreover, taking into consideration the parallel results in the parabolic setting [88], [117] it was commonly assumed that imposing a stability condition will single out a quantized set of allowed blow-up speeds. Although these observations had solid foundations, they were disproved in [49] and [7], making the stability of a family of rough solutions with varying concentration rates a unique feature of hyperbolic equations. In fact, the results [49] and [7] demonstrate that rough solutions u_ν are stable along a co-dimension one Lipschitz manifold of data perturbations in a suitable topology, provided that the blow-up speed is sufficiently close to the self-similar ones, i. e. $\nu > 0$ is sufficiently small. The result is optimal in view of [56], since any type II solution with data close enough to the ground state W can be at best stable for perturbations of the data along a co-dimension one hypersurface in energy space.

The main improvement of [7] over [49] is essentially in the number of co-dimensions imposed on the perturbations. In [49] Krieger showed that type II solutions u_ν are stable under an appropriate co-dimension three condition. Precisely, there exists a co-dimension three Lipschitz hypersurface $\Sigma_0 \subset H^{3/2+}_{rad,loc}(\mathbb{R}^3) \times H^{1/2+}_{rad,loc}(\mathbb{R}^3)$ such that if we take the perturbation of the initial data $(\varepsilon_0, \varepsilon_1) \in \Sigma_0$ small enough, then the solution of the perturbed problem

$$\begin{cases} \Box u = -u^5 \quad \text{in } (0, t_0] \times \mathbb{R}^3 \\ u[t_0] = u_\nu[t_0] + (\varepsilon_0, \varepsilon_1) \end{cases} \tag{3.14}$$

is a type II blow-up solution of exactly of same type as u_ν. In the subsequent work [7] the extra co-dimensions two condition was removed yielding to the optimal result. The precise statement is given below.

To properly enunciate the co-dimension conditions imposed in [49] we have to closely analyse the initial value problem on the Fourier side. We shall seek to construct a solution of (3.14) by perturbing around the exact solution u_ν, thus we make the following ansatz:

$$u(t, r) = u_\nu(t, r) + \varepsilon(t, r)$$

where $(\varepsilon, \partial_t \varepsilon)$ matches the initial data at time $t = t_0$: $(\varepsilon, \partial_t \varepsilon)|_{t=t_0} = (\varepsilon_0, \varepsilon_1)$. In analogy with the argument of the previous section we introduce the renormalized coordinates $(\tau, R) = (\nu^{-1} t^{-\nu}, \lambda(t) r)$, we set $\widetilde{\varepsilon} = R\varepsilon$, and we apply the distorted Fourier transform to the equation satisfied by $\widetilde{\varepsilon}$. Thus we obtain the following equation in terms of the Fourier variable $\vec{x}(\tau, \xi) = \mathscr{F}(\widetilde{\varepsilon})(\tau, \xi)$:

$$(\vec{\mathscr{D}}_\tau^2 + \beta(\tau)\vec{\mathscr{D}}_\tau + \vec{\xi})\vec{x}(\tau, \xi) = \vec{\mathscr{R}}\vec{x}(\tau, \xi) + \vec{f}[\widetilde{\varepsilon}](\tau, \xi) \tag{3.15}$$

where $(\tau, R) \in [\tau_0, \infty) \times \mathbb{R}^+$ and the linear source terms $\vec{\mathscr{R}}$ are as in (3.11) and the nonlinear terms are defined by

$$\vec{f}[\widetilde{\varepsilon}](\tau,\xi) = \begin{pmatrix} f_d[\widetilde{\varepsilon}](\tau) \\ f[\widetilde{\varepsilon}](\tau,\xi) \end{pmatrix} = \begin{pmatrix} \lambda^{-2}(\tau)\langle\phi_d, N(u_\nu,\widetilde{\varepsilon})\rangle_{L^2_{dR}} \\ \lambda^{-2}(\tau)\mathscr{F}\Big(N(u_\nu,\widetilde{\varepsilon})\Big)(\tau,\xi) \end{pmatrix}.$$

Instead of coupling the system (3.15) with vanishing initial data at $\tau = +\infty$ we shall impose initial data at the corresponding initial time $\tau = \tau_0$:

$$\begin{aligned} x_d(\tau_0) &= x_{0d}, & \partial_\tau x_d(\tau_0) &= x_{1d}, \\ x(\tau_0,\xi) &= x_0(\xi), & \mathscr{D}_\tau x(\tau_0,\xi) &= x_1(\xi). \end{aligned} \tag{3.16}$$

One can compute the initial data on the physical side $(\widetilde{\varepsilon}_0,\widetilde{\varepsilon}_1)$ in terms of the initial data on the Fourier side $(\vec{x}_0,\vec{x}_1)$, and vice-versa, via the formulas:

$$\begin{aligned} \mathscr{F}(\widetilde{\varepsilon}_0) &= x_0, & -\mathscr{F}\big(\frac{\widetilde{\varepsilon}_1}{\lambda}\big) &= x_1 + \beta_\nu(\tau_0)\mathscr{K}_{cc}x_0 + \beta_\nu(\tau_0)\mathscr{K}_{cd}x_{0d}, \\ \langle\phi_d,\widetilde{\varepsilon}_0\rangle_{L^2_{dR}} &= x_{0d}, & -\langle\phi_d,\frac{\widetilde{\varepsilon}_1}{\lambda}\rangle_{L^2_{dR}} &= x_{1d} + \beta_\nu(\tau_0)\mathscr{K}_{dd}x_{0d} + \beta_\nu(\tau_0)\mathscr{K}_{dc}x_0. \end{aligned} \tag{3.17}$$

We now present the Theorem contained in [7] which states that the blow-up phenomenon described in Theorem 98 is stable under a suitable co-dimension one class of data perturbations.

Theorem 100. *([7]) Assume* $0 < \nu \ll 1$, *and assume* $t_0 = t_0(\nu) > 0$ *is sufficiently small, so that the solutions* u_ν *constructed in [59] and [62] exist on* $(0,t_0]\times\mathbb{R}^3$. *Let* $\delta_1 = \delta_1(\nu) > 0$ *be small enough, and let* $\mathscr{B}_{\delta_1} \subset \widetilde{S}\times\mathbb{R}$ *be the* δ_1*-vicinity of* $\big((0,0),0\big)\in\widetilde{S}\times\mathbb{R}$, *where* $\widetilde{S}$ *is the Banach space defined as the completion of* $C_0^\infty(0,\infty)\times C_0^\infty(0,\infty)$ *with respect to the norm*

$$\|(x_0,x_1)\|_{\widetilde{S}} = \|\langle\xi\rangle^{\frac{1}{2}+2\delta_0}\min\{\tau_{0,0}\xi^{\frac{1}{2}},1\}^{-1}\xi^{\frac{1}{2}-\delta_0}x_0\|_{L^2_{d\xi}} + \|\langle\xi\rangle^{\frac{1}{2}+2\delta_0}\xi^{-\delta_0}x_1\|_{L^2_{d\xi}}.$$

Then there is a Lipschitz function $\gamma_1 : \mathscr{B}_{\delta_1}\to\mathbb{R}$, *such that for any triple* $\big((x_0,x_1),x_{0d}\big)\in\mathscr{B}_{\delta_1}$, *the quadruple*

$$\big((x_0,x_1),(x_{0d},x_{1d})\big),\ x_{1d} = \gamma_1(x_0,x_1,x_{0d})$$

determines a data perturbation pair $(\varepsilon_0,\varepsilon_1)\in H^{\frac{3}{2}+2\delta_0}_{rad,loc}(\mathbb{R}^3)\times H^{\frac{1}{2}+2\delta_0}_{rad,loc}(\mathbb{R}^3)$ *via* (3.17), *and such that the perturbed initial data*

$$u_\nu[t_0] + (\varepsilon_0,\varepsilon_1) \tag{3.18}$$

lead to a solution $\widetilde{u}(t,x)$ *on* $(0,t_0]\times\mathbb{R}^3$ *admitting the description*

$$\widetilde{u}(t,x) = W_{\widetilde{\lambda}(t)}(x) + \epsilon(t,x),\ \big(\varepsilon(t,\cdot),\varepsilon_t(t,\cdot)\big)\in H^{1+\frac{\nu}{2}-}_{rad,loc}\times H^{\frac{\nu}{2}-}_{rad,loc}$$

where the parameter $\widetilde{\lambda}(t)$ *equals* $\lambda(t)$ *asymptotically*

$$\lim_{t\to 0}\frac{\widetilde{\lambda}(t)}{\lambda(t)} = 1.$$

The proof of Theorem 108 builds on the previous work [49] thence let us describe below the main ingredients contained in the latter breakthrough.

3.3.1 Conditional stability result

The strategy of [49] consists in solving system (3.15) coupled with (3.16) iteratively: define the following sequence $\vec{x}^{(j)}(\tau,\xi) = \vec{x}^{(0)}(\tau,\xi) + \sum_{k=1}^{j}\triangle\vec{x}^{(k)}(\tau,\xi)$, where the *zero-th iterate* solves the homogeneous system:

$$\begin{cases}\big(\vec{\mathscr{D}}_\tau^2 + \beta(\tau)\vec{\mathscr{D}}_\tau + \vec{\xi}\big)\vec{x}^{(0)}(\tau,\xi) = 0\\ (\vec{x}^{(0)},\vec{\mathscr{D}}_\tau\vec{x}^{(0)})|_{\tau=\tau_0} = (\vec{x}_0,\vec{x}_1)\end{cases}$$

and the the *k-th increment* $\triangle\vec{x}^{(k)}$ satisfies the inhomogeneous equation:

$$\begin{cases}\big(\vec{\mathscr{D}}_\tau^2 + \beta(\tau)\vec{\mathscr{D}}_\tau + \vec{\xi}\big)\triangle\vec{x}^{(k)}(\tau,\xi) = \vec{\mathscr{R}}\triangle\vec{x}^{(k-1)}(\tau,\xi) + \triangle\vec{f}^{(k-1)}(\tau,\xi)\\ (\triangle\vec{x}^{(k)},\vec{\mathscr{D}}_\tau\triangle\vec{x}^{(k)})|_{\tau=\tau_0} = (\triangle\tilde{\vec{x}}_0^{(k)},\triangle\tilde{\vec{x}}_1^{(k)})\end{cases}$$

where $\triangle\vec{f}^{(0)} = \vec{f}[\widetilde{\varepsilon}^{(0)}]$ and $\triangle\vec{f}^{(k-1)} = \vec{f}[\widetilde{\varepsilon}^{(k-1)}] - \vec{f}[\widetilde{\varepsilon}^{(k-2)}]$ for $j \geq 2$.

As expected from the presence of a resonance of the operator $\mathscr{L}$, an accurate analysis of the zero-th iterate reveals that this term is fast growing toward $\tau = +\infty$. The growth of its discrete spectral part is easily controlled by imposing a vanishing condition on x_{0d} and x_{1d}. However, the growth of the continuous spectral part is more fundamental and it can be investigated via the explicit homogeneous parametrix:

$$\begin{aligned} x^{(0)}(\tau,\xi) &= S[x_0,x_1](\tau,\xi)\\ &:= \frac{\lambda^{5/2}(\tau)}{\lambda^{5/2}(\tau_0)}\frac{\rho^{1/2}\big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\big)}{\rho^{1/2}(\xi)}\cos\Big[\lambda(\tau)\xi^{1/2}\int_{\tau_0}^{\tau}\lambda^{-1}(u)du\Big]x_0\Big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\Big)\\ &+ \frac{\lambda^{3/2}(\tau)}{\lambda^{3/2}(\tau_0)}\frac{\rho^{1/2}\big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\big)}{\rho^{1/2}(\xi)}\frac{\sin\Big[\lambda(\tau)\xi^{1/2}\int_{\tau_0}^{\tau}\lambda^{-1}(u)du\Big]}{\xi^{1/2}}x_1\Big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\Big). \end{aligned}$$

Since $\lambda(\tau) \approx \tau^{1+\nu^{-1}}$, hence $x^{(0)}$ grows polynomially in τ. To control such a growth, the following natural co-dimensions two condition on the initial data (x_0, x_1) is imposed:

$$\begin{aligned}
&\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(s)\,ds]\,d\xi = 0,\\
&\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_1(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(s)\,ds]\,d\xi = 0.
\end{aligned} \tag{3.19}$$

Albeit such *vanishing relations* do not eliminate completely the growth of $x^{(0)}$ at infinity but only reduces it to linear growth, it is sufficient to run the iteration scheme. In fact, by choosing $\nu \leq 1/3$ and thanks to the decaying factor $\lambda^{-2}(\tau)$ appearing in the nonlinear terms $\triangle\vec{f}^{(k-1)}$ one can control them in a relatively straightforward way.

Let us briefly discuss the role of the corrections $(\triangle\widetilde{\widetilde{x}}_0^{(k)}, \triangle\widetilde{\widetilde{x}}_1^{(k)})$, which a priori should be both set to zero. At each iterative step, the continuous spectral part of the k-th increments are computed via the two explicit parametrices:

$$\triangle x^{(k)} = I[\mathscr{R}\triangle x^{(k-1)} + \triangle f^{(k-1)}] + S[\triangle\widetilde{\widetilde{x}}_0^{(k)}, \triangle\widetilde{\widetilde{x}}_1^{(k)}]$$

where $I[g]$ is the Duhamel parametrix for the inhomogeneous problem with source g and vanishing initial data at $\tau = \tau_0$:

$$I[g] = \int_{\tau_0}^\tau \frac{\lambda^{3/2}(\tau)}{\lambda^{3/2}(\sigma)} \frac{\rho^{1/2}\big(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\big)}{\rho^{1/2}(\xi)} \frac{\sin\Big[\lambda(\tau)\xi^{1/2}\int_\tau^\sigma \lambda^{-1}(u)du\Big]}{\xi^{1/2}} g\Big(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\Big)d\sigma.$$

To control the $\widetilde{S}$ norm of the low-frequencies component of the k-th increment $(\triangle x^{(k)}, \mathscr{D}_\tau\triangle x^{(k)})$ one splits $I[\mathscr{R}\triangle x^{(k-1)} + \triangle f^{(k-1)}]$, the inhomogeneous parametrix with vanishing initial data at $\tau = \tau_0$, into $I_{>\tau}[\mathscr{R}\triangle x^{(k-1)} + \triangle f^{(k-1)}]$, an inhomogeneous parametrix with vanishing initial data at $\tau = +\infty$, plus $S[\widetilde{\triangle}\widetilde{x}_0^{(k)}, \widetilde{\triangle}\widetilde{x}_1^{(k)}]$, a homogeneous solutions with non-vanishing initial data at $\tau = \tau_0$. Therefore we obtain

$$\triangle x^{(k)} = I_{>\tau}[\mathscr{R}\triangle x^{(k-1)} + \triangle f^{(k-1)}] + S[\widetilde{\triangle}\widetilde{x}_0^{(k)} + \triangle\widetilde{\widetilde{x}}_0^{(k)}, \widetilde{\triangle}\widetilde{x}_1^{(k)} + \triangle\widetilde{\widetilde{x}}_1^{(k)}].$$

The corrections $\triangle\widetilde{\widetilde{x}}_{0,1}^{(k)}$ ensure that the small error introduced in the initial data will preserve the vanishing conditions (3.19), leading to an approximation $\varepsilon^{(j)}$ on the physical side with controlled growth. Therefore to guarantee that the vanishing conditions holds throughout each step one needs to adjust the initial data by adding a small correction.

The final portion of [49] consists in proving that such iteration scheme converges by picking τ_0 sufficiently large. This is achieved via a re-iteration argument of the inhomogeneous parametrix which allows to gain enough smallness and to obtain a convergent series. A similar procedure was employed in [61] and [50]. Once the convergence is established, we obtain a solution of system (3.15) that fulfills the initial data requirements where $(\vec{x}_0, \vec{x}_1)$ have been replaced by $(\vec{x}_0 + \triangle\vec{x}_0, \vec{x}_1 + \triangle\vec{x}_1)$. The corrections $\triangle\vec{x}_{0,1}$ are obtained by summing up all the k-

th step corrections $\triangle \tilde{\tilde{\vec{x}}}_{0,1}^{(k)}$. Moreover, they are small with respect to the $\widetilde{S}$ norm when compared to the original initial data $(\vec{x}_0, \vec{x}_1)$ and they depend in Lipschitz continuous fashion on $\vec{x}_{0,1}$.

3.3.2 Optimal stability result

The elimination of the extra-vanishing conditions (3.19) imposed on the perturbation accomplished in [7] is attained in a four steps argument. Firstly, notice that one cannot time translate the solution u_ν without introducing an error of regularity $H^{1+\nu/2-}_{rad,loc}(\mathbb{R}^3)$ on each time-slice, that is too weak since the tollerate regularity of the perturbations is $H^{3/2+}_{rad,loc}(\mathbb{R}^3)$. Therefore a subtle modulation of the scaling law $\lambda(t) = t^{-1-\nu}$ is required. Precisely, one needs to work with a more flexible scaling law depending on two additional parameters γ_1 and γ_2. We stipulate the following ansatz:

$$\lambda^{(\gamma)}(t) = \Big(1 + \gamma_1 \frac{t^N}{\langle t^N \rangle} + \gamma_2 \log t \frac{t^N}{\langle t^N \rangle}\Big) t^{-1-\nu}, \tag{3.20}$$

here $N \gg 1$ is sufficiently large. Notice that $\lambda^{(\gamma)}$ asymptotically equals to λ as $t \to 0$, and such alteration implies a corresponding adjustment of renormalized coordinates (τ, R): let us introduce

$$\tau^{(\gamma)}(t) = \int_t^{t_0} \lambda^{(\gamma)}(s) ds + \nu^{-1} t_0^{-\nu}, \qquad R^{(\gamma)}(t,r) = \lambda^{(\gamma)}(t) r.$$

A similar iterative procedure that gave rise to the approximate solutions in [62] and [59] can be applied for the more general scaling law (3.20) to build approximate solutions of the form

$$u^{(\gamma)}_{app}(t,r) = W_{\lambda^{\gamma}(t)}(r) + \sum_{l=1}^{2k-1} v_l(t,r) + \sum_{a=1,2} v_{smooth,a}(t,r) + v(t,r)$$

which solves $\Box u^{(\gamma)}_{app} + (u^{(\gamma)}_{app})^5 = e^{(\gamma)}_{app}$ and where the error satisfies

$$\begin{aligned} e^{(\gamma)}_{app} = (|\gamma_1| + |\gamma_2|)\Big[\mathcal{O}\Big(\log t \frac{\lambda^{1/2} R}{(t\lambda)^{k_0+4}}(1 + (1-a)^{1/2+\nu/2})\Big) \\ + \mathcal{O}\Big(\log t \frac{\lambda^{1/2} R^{-1}}{(t\lambda)^{k_0+2}}(1 + (1-a)^{1/2+\nu/2})\Big)\Big]. \end{aligned}$$

The main novelty is that we perturb around $W_{\lambda^{(\gamma)}}$ as opposed to W_λ, which when inserted into the equation (3.4), generates additional error terms. We isolate the terms of the error which depend on $\gamma_{1,2}$ from the part which do not depend on $\gamma_{1,2}$. The former error terms are treated by adding a finite number of corrections v_l following the iterative scheme in [62] and [59]. On the other hand, the latter error terms are decimated by the two corrections $v_{smooth,a}$ which have better regularity property than the previous corrections. The final correction v is introduced to further improve the overall regularity to the error term.

Next, in the modulation step, one shows how to tune the parameters γ_1 and γ_2 such that a comparable procedure from [49] can be applied. Precisely, our point of departure is a singular type II solution constructed in the previous papers [62] and [59] which has the form $u_\nu = u_{2k-1}+\varepsilon$. Denote the associated initial data on the $t = t_0$ time slice by $(\varepsilon_1, \varepsilon_2)$ and consider $(\vec{x}_0, \vec{x}_1)$ the corresponding initial data at $\tau = \tau_0$ on the distorted Fourier side (with respect to R) computed via the relations (3.17). The point is that the initial data $(\vec{x}_0, \vec{x}_1)$ do not satisfy the vanishing conditions (3.19) with respect to scaling law λ anymore, thence we can not directly apply the argument of [49] as outlined in the previous section. To circumvent this impasse, we shall seek to complete the approximation $u_{app}^{(\gamma)}$ to an exact solution to the critical focusing wave equation (3.4) by introducing the function $\overline{\varepsilon}$:

$$u = u_{app}^{(\gamma)} + \overline{\varepsilon}. \tag{3.21}$$

Denote $(\overline{\varepsilon}_1, \overline{\varepsilon}_2) = \overline{\varepsilon}[t_0]$ the associated initial data of the new perturbation on the $t = t_0$ time slice and consider $(\vec{x}_0^{(\gamma)}, \vec{x}_1^{(\gamma)})$ the corresponding initial data at $\tau = \tau_0$ on the distorted Fourier side with respect to $R^{(\gamma)}$. We impose the following relations on the $t = t_0$ time slice

$$\overline{\varepsilon}_0 = \chi_{r \lesssim t_0}\big[W_\lambda(r) - W_{\lambda^{(\gamma)}}(r) - v_{smooth,1,2} - v\big] + \varepsilon_0,$$

as well as

$$\overline{\varepsilon}_1 = \chi_{r \lesssim t_0}\big[\partial_t\big[W_\lambda(r) - W_{\lambda^{(\gamma)}}(r)\big] - \partial_t v_{smooth,1,2} - \partial_t v\big] + \varepsilon_1.$$

Then one proves that there exists a unique choice of the parameters $\gamma_{1,2}$ such that the corresponding vanishing conditions (3.19) for $(x_0^{(\gamma)}, x_1^{(\gamma)})$ with respect to the scaling law (3.20) are satisfied.

Subsequently, we plug the ansatz (3.21) into (3.4) to find a corresponding equation for the perturbation $\overline{\varepsilon}$. Proceeding as in the previous section we solve such equation by passing to the distorted Fourier side with respect to $R^{(\gamma)}$. Let us denote the distorted Fourier transform of $\overline{\varepsilon}$ by $\vec{x}^{(\gamma)}$, then we obtain the corresponding transport equation on the distorted Fourier side:

$$(\vec{\mathscr{D}}_\tau^2 + \beta(\tau)\vec{\mathscr{D}}_\tau + \vec{\xi})\vec{x}^{(\gamma)}(\tau,\xi) = \vec{\mathscr{R}}\vec{x}^{(\gamma)}(\tau,\xi) + \vec{f}[\widetilde{\varepsilon}^{(\gamma)}](\tau,\xi) \tag{3.22}$$

where the linear source terms $\vec{\mathscr{R}}$ are as in (3.11) and the nonlinear terms are defined by

$$\vec{f}[\widetilde{\varepsilon}^{(\gamma)}](\tau,\xi) = \begin{pmatrix} f_d[\widetilde{\varepsilon}^{(\gamma)}](\tau) \\ f[\widetilde{\varepsilon}^{(\gamma)}](\tau,\xi) \end{pmatrix} = \begin{pmatrix} \lambda^{-2}(\tau)\langle \phi_d, R^{(\gamma)} e_{app}^{(\gamma)} + N(u_{app}^{(\gamma)}, \widetilde{\varepsilon}^{(\gamma)})\rangle_{L^2_{dR}} \\ \lambda^{-2}(\tau)\mathscr{F}\Big(R^{(\gamma)} e_{app}^{(\gamma)} + N(u_{app}^{(\gamma)}, \widetilde{\varepsilon}^{(\gamma)})\Big)(\tau,\xi) \end{pmatrix}.$$

The system (3.22) is coupled with initial data $(\vec{x}_0^{(\gamma)}, \vec{x}_1^{(\gamma)})$ which satisfy the vanishing relations (3.19) with respect to the scaling law (3.20). Thus we can apply a similar iterative scheme as in [49] to show that there exist corrections $\triangle\vec{x}_{0,1}^{(\gamma)}$ such that a solution $\vec{x}^{(\gamma)}$ to (3.22) with perturbed initial data $(\vec{x}_0^{(\gamma)} + \triangle\vec{x}_0^{(\gamma)}, \vec{x}_1^{(\gamma)} + \triangle\vec{x}_1^{(\gamma)})$ exists.

The last step consists to estimate the error induced by the small correction terms $\triangle\vec{x}_{0,1}^{(\gamma)}$ which have been introduced in the iterative scheme in terms of the original variable R. Hence we analyse $\triangle\varepsilon_{0,1}^{(\gamma)}$ the inverse distorted Fourier transform of $\triangle\vec{x}_{0,1}^{(\gamma)}$, with respect to the variable $R^{(\gamma)}$, and we prove that such errors are small when compared to the initial data perturbation. To show smallness one needs to compute the Fourier transform of $\triangle\varepsilon_{0,1}^{(\gamma)}$ with respect to the original variable R yielding to corrections denoted $\triangle\vec{x}_{0,1}$, and prove that the latter corrections are small in the $\widetilde{S}$ norm when compared to the original initial data $\vec{x}_{0,1}$.

Finally, the investigation of the Lipschitz dependence of the corrections $\triangle\vec{x}_{0,1}^{(\gamma)}$ with respect to the original data $\vec{x}_{0,1}^{(\gamma)}$ is carried out in details in [7] by carefully analyzing the dependence of the error $e_{app}^{(\gamma)}$ from the parameters $\gamma_{1,2}$.

4 Type II blow-up solutions with optimal stability properties

In this chapter we show that the finite time type II blow-up solutions for the energy critical nonlinear wave equation

$$\Box u = -u^5$$

on $\mathbb{R}^{3+1}$ constructed in [62], [61] are stable along a co-dimension one Lipschitz manifold of data perturbations in a suitable topology, provided the scaling parameter $\lambda(t) = t^{-1-\nu}$ is sufficiently close to the self-similar rate, i. e. $\nu > 0$ is sufficiently small. This result is qualitatively optimal in light of the result of [56]. The paper builds on the analysis of [49] and it is joint work with my thesis advisor Prof. J. Krieger.

4.1 Introduction

The critical focussing nonlinear wave equation on $\mathbb{R}^{3+1}$ given by

$$\Box u = -u^5, \Box = -\partial_t^2 + \triangle, \tag{4.1}$$

has received a lot of attention recently as a key model for a critical nonlinear wave equation displaying interesting type II dynamics, the latter referring to energy class Shatah-Struwe type solutions $u(t,x)$ (see [94]) which have a priori bounded $\dot{H}^1$ norm on their life-span I, i. e. with the property

$$\sup_{t\in I} \|\nabla_{t,x} u(t,\cdot)\|_{L^2_x} < \infty. \tag{4.2}$$

Throughout the paper, we shall be interested exclusively in the case of radial solutions. In that case, a rather complete abstract classification theory for type II dynamics in terms of the ground state[1]

$$W(x) = \frac{1}{\left(1+\frac{|x|^2}{3}\right)^{\frac{1}{2}}}$$

has been developed in [23], see the discussion in [49]. On the other hand, the first 'non-trivial' type II dynamics, were constructed explicitly in [57], [62], [59], [17], [50]. As far as finite time type II blow-up solutions are concerned, the issue of their *stability properties* has been shrouded in some mystery. The fact that there is a *continuum of blow-up rates* in the works [62], [59], seemed to suggest that these solutions, and maybe also their analogues for critical Wave Maps and other models, such as in [61], [60], [26], [8], are intrinsically less stable than 'generic type II blow-ups', and that the *requirement of optimal stability* of some sort may in fact single out a more or less *unique blow-up dynamics* for type II solutions, for example in the parabolic context see the deep work [88]. Two instances of 'optimally stable' type II blow-up were exhibited in the context of the $4+1$-dimensional critical NLW in the work [31], and in the context of critical co-rotational wave maps and equivariant Yang-Mills in [87], see also the brief historical comments in [49]. Note that the linearisation of (4.1) around the ground state W has a unique unstable eigenmode ϕ_d, and in accordance with this, [31] exhibits a co-dimensional one manifold of data perturbations of W (in the $4+1$-dimensional context) resulting in the stable blow-up.

In this article we show that the solutions constructed in [62], [59], corresponding to $\lambda(t) = t^{-1-\nu}$ and *with* $\nu > 0$ *small enough* are also optimally stable in a suitable sense. However, due to the fact that these solutions are only of finite regularity, and in effect experience a shock along the light cone centered at the singularity, an appreciation of our result requires carefully reviewing the nature of them.

4.1.1 The type II blow-up solutions of [62], [59]

Solutions of (4.1) are divided into those of type II, satisfying (4.2), as well as solutions of type I which violate this condition. The celebrated result in [23], (see also [20], [21]) provides a general criterion characterising abstract radial type II solutions in terms of the ground states $\pm W(x)$. In particular, assuming that $u(t,x)$ is a type II solution of (4.1) which is radial and develops a singularity at time $t = T$, then near T, we can write

$$u(t,x) = \sum_{j=1}^{N} \kappa_j W_{\lambda_j(t)}(x) + \epsilon(t,x),\ W_\lambda(x) = \lambda^{\frac{1}{2}} W(\lambda x), \tag{4.3}$$

where $\epsilon(t,x)$ can be extended continuously as an energy class solutions past the singularity $t = T$, $\kappa_j = \pm 1$, $\lim_{t\to T}(T-t)\lambda_j(t) = +\infty$, and $\lim_{t\to T}|\log(\frac{\lambda_j(t)}{\lambda_k(t)})| = +\infty$, provided $j \neq k$.

We note here that this appears the only result for a non-integrable PDE where this kind of a continuous in time soliton resolution has been proved. We also observe that the solutions in

[1] Also known as Aubin-Talenti solution (see [1], [102]) from its geometric origins.

[62], [59], [50], appear to be the only known finite time type II blow-up solutions for (4.1), all with $N = 1$, and that in fact solutions of the form (4.3) with $N \geq 2$ are not known at this time (and might not exist).

We now detail briefly these specific blow-up solutions. Let $\nu > 0$, but otherwise arbitrary, and denote $\lambda(t) = t^{-1-\nu}$.

Theorem 101. *([62], [59]) There is $t_0 > 0$ and a radial solution $u_\nu(t,x)$ of the form*

$$u_\nu(t,x) = W_{\lambda(t)}(x) + \eta(t,x),$$

where the term[2] $\eta(t,\cdot) \in H^{1+\frac{\nu}{2}-}, \eta_t(t,\cdot) \in H^{\frac{\nu}{2}-}$ for any $t \in (0, t_0]$, and we have the asymptotic vanishing relation

$$\lim_{t\to 0}\int_{|x|<t} [|\nabla_{t,x}\eta|^2 + \frac{1}{6}\eta^6]\, dx = 0.$$

The correction term η satisfies $\eta|_{|x|<t} \in C^\infty$, while it is only of regularity $H^{1+\frac{\nu}{2}-}$ across the light cone. In fact, there is a splitting $\eta(t,x) = \eta_e(t,x) + \epsilon(t,x)$, with (here N may be picked arbitrarily large, depending on the number of steps used to construct η_e)

$$\|\epsilon(t,\cdot)\|_{H^{1+\frac{\nu}{2}-}} + \|\epsilon_t(t,\cdot)\|_{H^{\frac{\nu}{2}-}} < t^N,$$

and such that using the new variables $R = \lambda(t)r$, $r = |x|$, $a = \frac{r}{t}$, there is an expansion near $a = 1$ of the form

$$\eta_e(t,x) = \frac{\lambda^{\frac{1}{2}}}{(\lambda t)^2}\cdot \sum_{l\geq 0}\sum_{0\leq k\leq k(l)} c_l(a,t)(\log R)^k R^{1-l}, \tag{4.4}$$

and such that

$$c_l(a,t) = q_0^{(l)}(t,a) + \sum_{i=1}^{\infty}(1-a)^{\beta(i)+1}\sum_{j=0}^{j(i)} q_{ij}^{(l)}(t,a)\big(\log(1-a)\big)^j. \tag{4.5}$$

Here the coefficients $q_0^{(l)}(t,a), q_{ij}^{(l)}(t,a)$ are of class C^∞, while the exponents $\beta(i)$ are of the form

$$\beta(i) = \sum_{k\in K_i}\big((2k-\frac{3}{2})\nu - \frac{1}{2})\big) + \sum_{k\in K_i'}\big((2k-\frac{1}{2})\nu - \frac{1}{2})\big)$$

for suitable finite sets of positive integers K_i, K_i'. In particular, $\beta(i) \geq \frac{\nu-1}{2}$. The sums in (4.4), (4.5) *are absolutely convergent, and the most singular terms in* (4.5) *are of the form*

$$(1-a)^{\frac{\nu+1}{2}}\log(1-a).$$

We observe that a similar asymptotic expansion as in (4.4), (4.5) near $a = 1$ may also be inferred

[2]The notation H^{s-} means in any $H^{s'}$, $s' < s$.

for the error $\epsilon(t,x)$, and thus the singularity of $\eta(t,x)$ is indeed confined exactly to the forward light cone $|x| = t$ centered in the singularity, see [59]. However, the methods for determining $\eta_e(t,x)$ and $\epsilon(t,x)$ differ importantly. The first is in fact obtained by approximating the wave equation by a finite number of elliptic equations approximating the wave equation in a suitable sense, while the second quantity is obtained as solution of a wave equation via a suitable parametrix method. Both of these techniques will play an important role in this paper. We shall see next that the limited regularity and more precisely the shock across the light cone (see Figure 4.1) entails a certain rigidity for such solutions, which will be reflected in terms of the stability properties of this kind of blow-up.

4.1.2 The effect of symmetries on the solutions of Theorem 101

In the sequel, we shall assume $0 < \nu \ll 1$. Restricting to the radial setting, the symmetry group acting on solutions of (4.1) is restricted to time translations $u(t,x) \longrightarrow u(t-T,x)$, as well as scaling transformations $u(t,x) \longrightarrow \lambda^{\frac{1}{2}} u(\lambda t, \lambda x)$, and it is then natural to subject the special solutions $u_\nu(t,x)$ to such transformations. Let us consider the effect on the principal singular term, which is of the schematic form

$$\frac{\lambda^{\frac{1}{2}}(t)}{(\lambda(t)\cdot t)^2} \cdot R\log(1+R^2)\cdot(1-a)^{\frac{1}{2}+\frac{\nu}{2}}\log(1-a),\ a = \frac{r}{t},\ R = \lambda(t) r,\ \lambda(t) = t^{-1-\nu}.$$

Calling this term $\eta_p(t,x)$, we find by simple inspection that for $T \neq 0$

$$\eta_p(t,\cdot) - \eta_p(t-T,\cdot) \in H_{loc}^{1+\frac{\nu}{2}-}$$

and this is in effect optimal, i. e. the preceding difference is in no H^s_{loc} for any $s \geq 1 + \frac{\nu}{2}$. This is of course simply due to the fact that time translating u_ν will shift the forward light cone on which the solution experiences a shock, and so the difference will be no smoother than u_ν. The same phenomenon occurs for the difference

$$\eta_p(t,x) - \lambda^{\frac{1}{2}}\eta_p(\lambda t, \lambda x),\ \lambda \neq 1.$$

What we shall intend in this article is to consider *smooth perturbations* of the solutions $u_\nu(t,x)$, i. e. consider the evolution corresponding to the initial data

$$u_\nu[t_0] + \big(\epsilon_0, \epsilon_1\big),\ u_\nu[t_0] = \big(u_\nu(t_0,\cdot), \partial_t u_\nu(t_0,\cdot)\big),$$

where $\big(\epsilon_0,\epsilon_1\big) \in H^{\frac{3}{2}+}(\mathbb{R}^3) \times H^{\frac{1}{2}+}(\mathbb{R}^3)$, in a way made more precise in the sequel. In particular, we see that the differences

$$u_\nu[t_0 - T] - u_\nu[t_0],\ u_{\nu,\lambda}[t_0], -u_\nu[t_0],\ T \neq 0,\ \lambda \neq 1,$$

are not of this form, since $1 + \frac{\nu}{2} < \frac{3}{2}+$ for $\nu \ll 1$. In Figure 4.1 below we have plotted the leading behavior of the function $u_\nu(t,x)$ for $t>0$, the shock along the forward light cone generated from the origin is evident. Moreover, in Figure 4.2 one can indubitably see that the difference $u_\nu[t_0-T]-u_\nu[t_0]$ manifest cusp type singularities at $|x|=t_0$ and $|x|=t_0-T$.

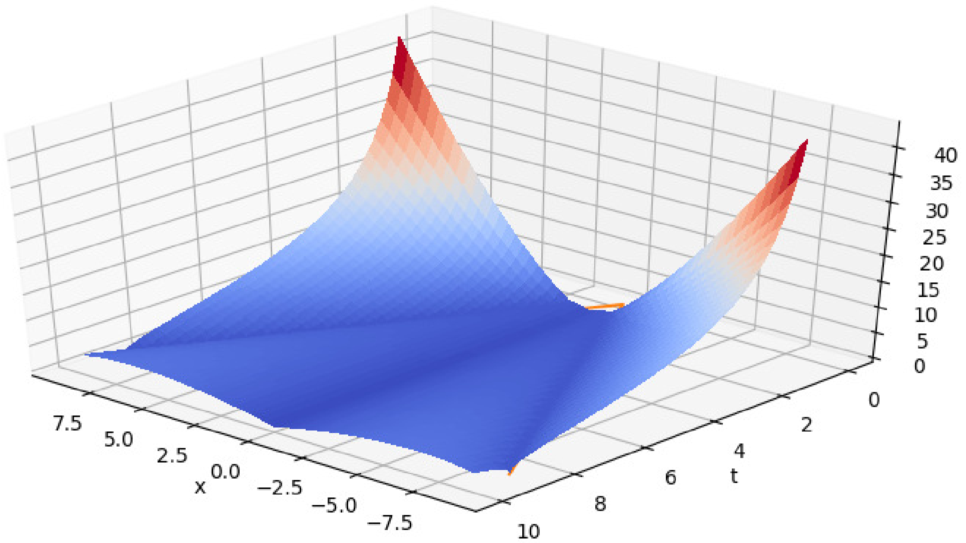


Figure 4.1: Graph of $u_\nu(t,x)$ for $t>0$ and $\nu=0.1$.

This reveals that the role of the symmetries in describing the evolutions of the initial data

$$u_\nu[t_0]+\big(\epsilon_0,\epsilon_1\big),\ \big(\epsilon_0,\epsilon_1\big)\in H^{\frac{3}{2}+}(\mathbb{R}^3)\times H^{\frac{1}{2}+}(\mathbb{R}^3), \tag{4.6}$$

is not a priori clear, and in fact, we shall show that the blow-up corresponding to (a certain subclass of) such initial data perturbations takes place in the same space-time location and with the same scaling law, which may sound paradoxical at first, but is explained by the role of the topology of the data.

In fact, what our main result shall reveal, and what is also borne out by the result [57], while the abstract general classification theory by Duyckaerts-Kenig-Merle ([20], [22], [21], [23]) takes place in the largest possible space H^1 in which the problem (4.1) is well-posed, an understanding of the precise possible dynamics (involving blow-up speeds and stability properties) rely crucially on finer topological properties of the data in spaces more restrictive than H^1. It is conceivable that such considerations have much broader applicability for certain nonlinear *hyperbolic* problems.

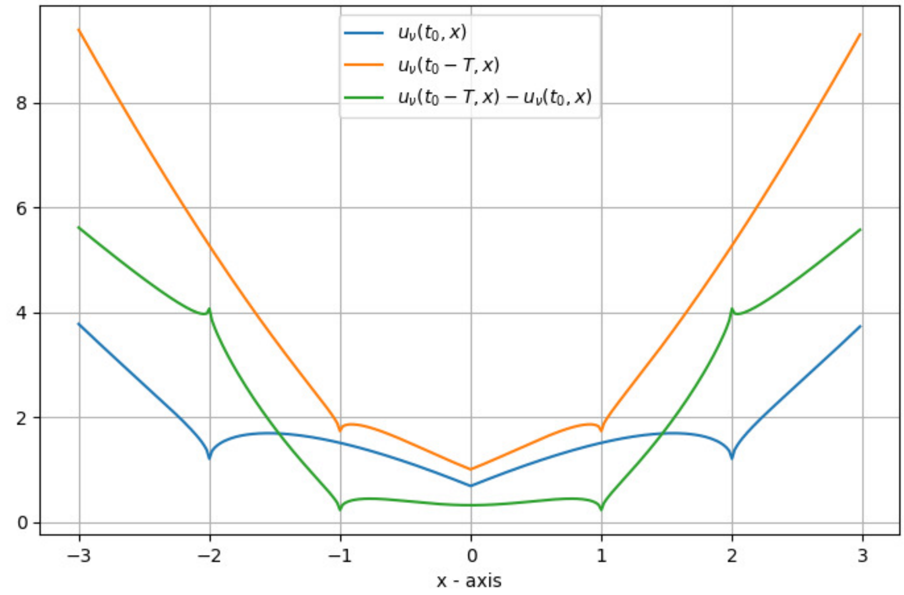


Figure 4.2: Graphs of initial data with $\nu = 0.1$, $t_0 = 2$, and $T = 1$.

4.1.3 Conditional stability of type II solutions

Before stating the main theorem of this paper about stability properties of the solutions in Theorem 101 with $\nu \ll 1$, we place it briefly into a broader context. It is intuitively clear that when analysing the stability of any of the type II solutions in (4.3) with $N = 1$, say, the linearisation of the equation (4.1) around W, and thence the operator[3]

$$\mathscr{L} = -\partial_R^2 - 5W^4(R), \; W(R) := \frac{1}{(1 + \frac{R^2}{3})^{\frac{1}{2}}}, \tag{4.7}$$

will play a pivotal role. This operator, when restricted to functions on $[0,\infty)$ with Dirichlet condition at $R = 0$, has a simple negative eigenvalue $\xi_d < 0$ (the subscript d referring to 'discrete spectrum'), and a corresponding L^2-normalized positive ground state ϕ_d with

$$\mathscr{L}\phi_d = \xi_d \phi_d,$$

see [57], [62]. Thus, $\phi_d \in L^2(0,\infty) \cap C^\infty([0,\infty))$, decaying exponentially and with $\phi_d(R) > 0$ for $R > 0$, but $\phi_d(0) = 0$. This mode will cause exponential growth for the linearised flow $e^{it\sqrt{\mathscr{L}}}$, and only a co-dimension one condition will ensure that the forward flow will remain bounded. That a corresponding center-stable manifold may be constructed for perturbations of type II solutions for the nonlinear problem (4.1) was first shown in the context of the special solution $u(t,x) = W(x)$ and perturbations in a topology which is significantly stronger than H^1 in [57],

[3] It arises by passing from radial $u(x) = v(R)$ to $Rv(R)$, $R = |x|$

and later in vastly larger generality (for perturbations of only regularity $\dot{H}^1$) in [56]. Here we let $u(t,x)$ be a general solution of regularity $\dot{H}^1$, which may be obtained as limit of a sequence of smooth solutions. We shall refer to such solutions as 'Shatah-Struwe solutions' (see [94]) .

Theorem 102. *([56]) Let*

$$u(t,x) = W_{\lambda(t)}(x) + v(t,x)$$

be a type II blow-up solution on $I \times \mathbb{R}^3$ *for* (4.1), *such that*

$$\sup_{t\in I} \|\nabla_{t,x} v(t,\cdot)\|_{L^2_x} \leq \delta \ll 1$$

for some sufficiently small $\delta > 0$, *where as usual* I *denotes the maximal life span of the Shatah-Struwe solution* u. *Also, assume that* $t_0 \in I$. *Then there exists a co-dimension one Lipschitz manifold* Σ *in a small neighborhood of the data* $\big(u(t_0,\cdot), u_t(t_0,\cdot)\big) \in \Sigma$ *in the energy topology* $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *and such that initial data* $\big(u_0, u_1\big) \in \Sigma$ *result in a type II solution, while initial data*

$$\big(u_0, u_1\big) \in B_\delta \backslash \Sigma,$$

where $B_\delta \subset \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *is a sufficiently small ball centered at* $\big(u(t_0,\cdot), u_t(t_0,\cdot)\big)$, *either lead to blow-up in finite time, or solutions scattering to zero, depending on the 'side of* Σ*' these data are chosen from.*

Note that by contrast to the result in [57] which precisely describes the dynamics of the perturbed solutions but at the expense of a much more restrictive class of perturbations, there is no description of the perturbed solutions in the preceding theorem other than the assertion that the solutions are of type II.

The question we shall now address is whether the *specific dynamics* of the solutions in Theorem 101 are preserved for a suitable class of perturbations, essentially as in (4.6). Note that such perturbations only constitute a very small subset of the surface Σ in the preceding theorem, as evidenced by the fact that if $\mathscr{S}_{T,\lambda}$ denotes re-scaling by λ and time-tranlsation by T, then if $(T,\lambda) \neq (0,1)$, any two data pairs

$$u_\nu[t_0] + \big(\epsilon_0, \epsilon_1\big), \ \big(\mathscr{S}_{T,\lambda} u_\nu\big)[t_0] + \big(\epsilon_0', \epsilon_1'\big)$$

with $\big(\epsilon_0, \epsilon_1\big) \in H^{\frac{3}{2}+} \times H^{\frac{1}{2}+}, \big(\epsilon_0', \epsilon_1'\big) \in H^{\frac{3}{2}+} \times H^{\frac{1}{2}+}$ will be distinct. We aim now to understand the evolution of a certain class of data $u_\nu[t_0] + \big(\epsilon_0, \epsilon_1\big)$, with t_0 as in Theorem 101, backward in time. Precisely stating the conditions on the perturbation $\big(\epsilon_0, \epsilon_1\big)$ requires certain technical preliminaries involving the spectral theory and representation associated to $\mathscr{L}$, mostly developed in [62].

4.1.4 Spectral theory associated with the linearisation $\mathscr{L}$

Here we quote from [62], specifically Lemma 4.2 as well as Proposition 4.3 in loc. cit. Let $\mathscr{L}$ be given by (4.7), restricted to $L^2((0,\infty)$, with domain

$$\mathrm{Dom}(\mathscr{L}) = \{f \in L^2((0,\infty) : f, f' \in AC([0,R])\forall R>0, f(0)=0, f'' \in L^2((0,\infty)\}$$

Then $\mathscr{L}$ is self-adjoint with this domain, and its spectrum consists of

$$\mathrm{spec}(\mathscr{L}) = \{\xi_d\} \cup [0,\infty),$$

with $\xi_d < 0$ the unique negative eigenvalue of $\mathscr{L}$ and associated L^2-normalized and positive ground state $\phi_d(R)$. There is a resonance at zero given by the function

$$\phi_0(R) = R(1-\frac{R^2}{3})(1+\frac{R^2}{3})^{-\frac{3}{2}},\ \mathscr{L}\phi_0 = 0.$$

The latter is simply a reflection of the scaling invariance of the problem.

Importantly, the operator $\mathscr{L}$ induces a '*distorted Fourier transform*' $\mathscr{F}(f) = \widehat{f}$, which allows for a nice Fourier representation in terms of generalised eigenfunctions $\phi(R,\xi)$. For a qualitative behavior of the functions $\phi(R,\xi)$ in the small and high frequencies regime see respectively Figure 4.5 and 4.6 at the end of the chapter. We have

Proposition 103. *([62]) For each $z \in \mathbb{C}$, one can define a basis of generalised eigenfunctions*

$$\phi(R,z) = \phi_0(R) + R^{-1}\sum_{j=1}^{\infty}(R^2 z)^j \phi_j(R^2)$$

given by an absolutely convergent sum, with $\phi_j(u)$ holomorphic on the complex numbers with $\mathrm{Re}\, u > -1/2$*, and satisfying bounds*

$$|\phi_j(u)| \le \frac{C^j}{(j-1)!}|u|\langle u\rangle^{-\frac{1}{2}}.$$

Denoting the Jost solutions $f_\pm(R,\xi)$ which satisfy $\mathscr{L}f_\pm = \xi f_\pm$ as well as $f_\pm(R,\xi) \sim e^{\pm iR\xi^{\frac{1}{2}}}$ as $R \to +\infty$, there is a representation

$$\phi(R,\xi) = \sum_{\pm} a_\pm(\xi) f_\pm(R,\xi),$$

with $a_\pm(\xi) \to 1$ as $\xi \to 0$ as well as $|a_\pm(\xi)| \lesssim \xi^{-\frac{1}{2}}$ as $\xi \to \infty$. Further, there is a function $\rho(\xi) \in C^\infty\big((0,\infty)\big)$ with the asymptotic behaviour

$$\rho(\xi) \sim \xi^{-\frac{1}{2}},\ 0<\xi \ll 1,\ \rho(\xi) \sim \xi^{\frac{1}{2}},\ \xi \gg 1,$$

as well as symbol behaviour with respect to differentiation, and such that defining

$$\mathscr{F}(f)(\xi) := \widehat{f}(\xi) := \lim_{b\to+\infty}\int_0^b \phi(R,\xi)f(R)\,dR,\ \xi \geq 0,$$
$$\widehat{f}(\xi_d) = \int_0^\infty \phi_d(R)f(R)\,dR,$$

the map $f \longrightarrow \widehat{f}$ *is an isometry from* L^2_{dR} *to* $L^2(\{\xi_d\}\cup\mathbb{R}^+,\rho)$*, and we have*

$$f(R) = \widehat{f}(\xi_d)\phi_d(R) + \lim_{\mu\to\infty}\int_0^\mu \phi(R,\xi)\widehat{f}(\xi)\rho(\xi)\,d\xi,$$

the limits being in the suitable L^2*-sense.*

The precise structure of the Jost solutions shall sometimes be important, and the following result, which we cite verbatim from [62], gives a precise asymptotic expansion:

Proposition 104. *For any* $\xi > 0$*, the Jost solution* $f_+(\cdot,\xi)$ *satisfying*

$$\mathscr{L}f_+(\cdot,\xi) = \xi f_+(\cdot,\xi),\ f_+(R,\xi) \sim e^{i\sqrt{\xi}R} \text{ as } R\to\infty,$$

is of the form

$$f_+(R,\xi) = e^{iR\xi^{\frac{1}{2}}}\sigma(R\xi^{\frac{1}{2}},R),$$

where σ *admits the asymptotic series approximation*

$$\sigma(q,R) \sim \sum_{j=0}^\infty q^{-j}\psi_j^+(R),$$

in the sense that for all integers $j_0 \geq 0$*, and all indices* α,β*, we have*

$$\sup_{R>0}\langle R\rangle^2|(R\partial_R)^\alpha(q\partial_q)^\beta\big[\sigma(q,R) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(R)\big]| \leq c_{\alpha,\beta,j_0}q^{-j_0-1}$$

for all $q > 1$*. Here*

$$\psi_0^+ = 1,\ \psi_1^+(R) = \left\{\begin{array}{c} ic_1R^{-2} + iO(R^{-4} \text{ as } R\to\infty \\ ic_2R + iO(R^2) \text{ as } R\to 0\end{array}\right\}$$

with some real constants c_1, c_2*. More generally,* $\psi_j^+(R)$ *are smooth symbols of order* -2 *for* $j\geq 1$*, i. e. for all* $k \geq 0$

$$\sup_{R>0}\langle R\rangle^2|\big(\langle R\rangle\partial_R\big)^k\psi_j^+(R)| < \infty.$$

Finally, $\psi_j^+(R) = O(R^j)$ *as* $R\to 0$.

We also observe the following estimate describing classical H^s_{dR}-norms in terms of the distorted Fourier transform, and which follows by a simple interpolation argument:

Lemma 105. *Assume $s \geq 0$. Then we have*

$$\|f(R)\|_{H^s(\mathbb{R}_+)} \lesssim \|\langle\xi\rangle^{\frac{s}{2}}\hat{f}(\xi)\|_{L^2_{d\rho}} + |\xi_d|,$$

When passing from the standard coordinates $r = |x|$, t to the new ones $R = \lambda(t)r, \tau = \int_t^\infty \lambda(s)\,ds$, the time derivative will be replaced by a dilation type operator of essentially the form $\partial_\tau + \frac{\dot{\lambda}}{\lambda}R\partial_R$, and translation to the Fourier variables will require expressing the operator $R\partial_R$ in terms of the distorted Fourier transform. Specifically, we need to understand how $R\partial_R$ acts on $x_d, x(\xi)$ for a function

$$\tilde{\epsilon}(R) = x_d\phi_d(R) + \int_0^\infty x(\xi)\phi(R,\xi)\rho(\xi)\,d\xi.$$

The precise result here comes also from [62]:

Theorem 106. *([62]) We have the identity*

$$\begin{pmatrix} \widehat{(R\partial_R)f}(\xi_d) \\ \widehat{(R\partial_R)f}(\xi) \end{pmatrix} = (\widetilde{\mathscr{A}} + \mathscr{K})\begin{pmatrix} \hat{f}(\xi_d) \\ \hat{f}(\xi) \end{pmatrix},$$

where the matrix operators $\mathscr{A}, \mathscr{K}$ on the right are given by

$$\widetilde{\mathscr{A}} = \begin{pmatrix} 0 & 0 \\ 0 & -2\xi\partial_\xi - \frac{3}{2} - \frac{\rho'(\xi)\xi}{\rho(\xi)} \end{pmatrix}, \mathscr{K} = \begin{pmatrix} \mathscr{K}_{dd} & \mathscr{K}_{dc} \\ \mathscr{K}_{cd} & \mathscr{K}_{cc} \end{pmatrix},$$

and the individual components of $\mathscr{K}$ are given by $\mathscr{K}_{dd} = -\frac{1}{2}$, $\mathscr{K}_{cd}(\xi) = K_d(\xi)$ a smooth function rapidly decaying toward $\xi = +\infty$,

$$\mathscr{K}_{dc}f = -\int_0^\infty K_d(\xi)f(\xi)\rho(\xi)\,d\xi,$$

and finally, $\mathscr{K}_{cc}$ is a Calderon-Zygmund type operator given by a kernel

$$K_0(\xi,\eta) = \frac{\rho(\eta)}{\xi - \eta}F(\xi,\eta),$$

where the function $F(\cdot,\cdot)$ is of regularity at least C^2 on $(0,\infty)\times(0,\infty)$, and satisfies the bounds

$$|F(\xi,\eta)| \lesssim \begin{cases} \xi + \eta, & if\, \xi + \eta \leq 1, \\ (\xi+\eta)^{-1}(1 + |\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}|)^{-N}, & if\, \xi + \eta \geq 1 \end{cases}$$

$$|\partial_\xi F(\xi,\eta)| + |\partial_\eta F(\xi,\eta)| \lesssim \begin{cases} 1, & if\, \xi + \eta \leq 1, \\ (\xi+\eta)^{-\frac{3}{2}}(1 + |\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}|)^{-N}, & if\, \xi + \eta \geq 1 \end{cases}$$

$$\sum_{j+k=2}|\partial_\xi^j\partial_\eta^k F(\xi,\eta)|+|\partial_\eta F(\xi,\eta)|\lesssim\begin{cases}(\xi+\eta)^{-\frac{1}{2}}, & if\,\xi+\eta\leq 1,\\ (\xi+\eta)^{-2}(1+|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|)^{-N}, & if\,\xi+\eta\geq 1\end{cases}$$

Here N can be chosen arbitrarily (with the implicit constant depending on N).

4.1.5 Description of the data perturbation in terms of the distorted Fourier transform

In the sequel, we shall mainly describe functions $f(R)$ in terms of their distorted Fourier transform $\widehat{f}(\xi)$, $\widehat{f}(\xi_d)$. In particular, we shall describe the precise class of data perturbations (ϵ_0,ϵ_1) via properties of their distorted Fourier transforms: for a pair of functions $\big(x_0(\xi),x_1(\xi)\big)$, which will represent the continuous spectral part[4] of $R\epsilon_0$, and in a more roundabout way the continuous spectral part of $R\epsilon_1(R)$, we introduce the following norm:

$$\begin{aligned}\|(x_0,x_1)\|_{\widetilde{S}}&:=\|x_0\|_{\widetilde{S}_1}+\|x_1\|_{\widetilde{S}_2}\\ &:=\|\langle\xi\rangle^{\frac{1}{2}+2\delta_0}\min\{\tau_{0,0}\xi^{\frac{1}{2}},1\}^{-1}\xi^{\frac{1}{2}-\delta_0}x_0\|_{L^2_{d\xi}}\\ &+\|\langle\xi\rangle^{\frac{1}{2}+2\delta_0}\xi^{-\delta_0}x_1\|_{L^2_{d\xi}}.\end{aligned}\tag{4.8}$$

Here $1\gg\delta_0>0$ is a small constant held fixed throughout, and the constant $\tau_{0,0}$ is defined via

$$\tau_{0,0}:=\int_{t_0}^\infty s^{-1-\nu}\,ds.$$

This norm is in fact exactly the same as the one used in [49]. We easily observe that

$$\|\langle\xi\rangle^{1+\delta_0}x_0\|_{L^2_{d\xi}}+\|\langle\xi\rangle^{\frac{1}{2}+\delta_0}x_1\|_{L^2_{d\xi}}\lesssim_{\tau_{0,0}}\|(x_0,x_1)\|_{\widetilde{S}},$$

as well as

$$\|\xi^{-\delta_0}x_0\|_{L^2_{d\xi}(0<\xi<1)}\lesssim_{\tau_{0,0}}\|(x_0,x_1)\|_{\widetilde{S}},$$

and so we find, setting[5] $(P_c\widetilde{\epsilon}_0)(R)=\int_0^\infty\phi(R,\xi)x_0(\xi)\rho(\xi)\,d\xi$, we have

$$\begin{aligned}&\|\chi_{R\leq C\tau_{0,0}}(P_c\widetilde{\epsilon}_0)(R)\|_{H^{\frac{3}{2}+2\delta_0}_{dR}}\\ &\lesssim\|\chi_{R\leq C\tau_{0,0}}\int_0^1\phi(R,\xi)x_0(\xi)\rho(\xi)\,d\xi\|_{H^{\frac{3}{2}+2\delta_0}_{dR}}\\ &+\|\chi_{R\leq C\tau_{0,0}}\int_1^\infty\phi(R,\xi)x_0(\xi)\rho(\xi)\,d\xi\|_{H^{\frac{3}{2}+2\delta_0}_{dR}}\\ &\lesssim(C\tau_{0,0})^{\frac{1}{2}}\|\langle\xi\rangle^{\frac{3}{4}}x_0(\xi)\rho(\xi)\|_{L^1_{d\xi}(\xi<1)}+\|\langle\xi\rangle^{\frac{3}{4}+\delta_0}x_0(\xi)\|_{L^2_{d\rho}(\xi>1)}.\end{aligned}$$

[4] Recall that $R=\lambda(t)r$, where for now $\lambda(t)=t^{-1-\nu}$.

[5] The notation P_c means projection onto continuous spectral part, i. e. projecting away the discrete spectral part which is the multiple of ϕ_d.

We have used here that $\phi(R,\xi)$ us uniformly bounded, which follows from the preceding proposition. Then we have

$$\|\langle\xi\rangle^{\frac{3}{4}}x_0(\xi)\rho(\xi)\|_{L^1_{d\xi}(\xi<1)} \lesssim \|\xi^{-\delta_0}x_0(\xi)\|_{L^2_{d\xi}(\xi<1)},$$
$$\|\langle\xi\rangle^{\frac{3}{4}+\delta_0}x_0(\xi)\|_{L^2_{d\rho}(\xi>1)} \lesssim \|\langle\xi\rangle^{1+\delta_0}x_0\|_{L^2_{d\xi}(\xi>1)},$$

where we have used the asymptotic for ρ from Proposition 103, and so we in fact have

$$\|\chi_{R\le C\tau_{0,0}}(P_c\widetilde{\epsilon}_0)(R)\|_{H^{\frac{3}{2}+2\delta_0}_{dR}} \lesssim_{\tau_{0,0}} \|x_0\|_{\widetilde{S}_1}. \tag{4.9}$$

Thus the 'physical data' corresponding to the distorted Fourier variable in $\widetilde{S}_1$ is actually of regularity $H^{\frac{3}{2}+}_{loc}$. To reconstruct the full perturbation ϵ_0, we also need to prescribe the discrete spectral part $x_{0,d}$, and then set

$$\epsilon_0(r) = R^{-1}\widetilde{\epsilon}_0(R) = R^{-1}[x_{0,d}\phi_d(R) + \int_0^\infty x_0(\xi)\phi(R,\xi)\rho(\xi)\,d\xi],\ R = \lambda(t_0)r, \tag{4.10}$$

where $\lambda(t) = t^{-1-\nu}$.

The relation of the second Fourier variable $x_1(\xi)$ and ϵ_1 is a bit more complicated, see [49], due to the fact that here all of $x_0, x_{0,d}, x_1, x_{1,d}$ are involved. This is due to the fact that the description of the perturbed solution $u(t,x)$ shall actually be in terms of the new variables[6] $R = \lambda(t)r$, $\tau = \int_t^\infty \lambda(s)\,ds$, which mix time and space. Specifically, consider a function

$$\widetilde{\epsilon}(\tau,R) = x_d(\tau) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi,\ \widetilde{\epsilon} = R\epsilon.$$

Then we obtain the relation

$$-\frac{R\epsilon_t}{\lambda} = \big(\partial_\tau + \frac{\dot{\lambda}}{\lambda}(R\partial_R - 1)\big)\widetilde{\epsilon},\ \dot{\lambda} = \lambda_\tau.$$

Introducing the notation $\underline{x} := \begin{pmatrix} x_d \\ x \end{pmatrix}$ and passing to the Fourier variables by using Theorem 106, we find

$$-\begin{pmatrix} \mathscr{F}\big(\frac{R}{\lambda}\epsilon_t\big) \\ \langle\phi_d, \frac{R}{\lambda}\epsilon_t\rangle \end{pmatrix} = \mathscr{D}_\tau\underline{x}(\tau,\cdot) + \beta_\nu(\tau)\mathscr{K}\,\underline{x}(\tau,\cdot),\ \beta_\nu(\tau) = \frac{\dot{\lambda}}{\lambda}(\tau),$$

where we have introduced the important dilation type operator

$$\mathscr{D}_\tau := \partial_\tau + \frac{\dot{\lambda}}{\lambda}\mathscr{A},\ \mathscr{A} = \begin{pmatrix} 0 & 0 \\ 0 & -2\xi\partial_\xi - \frac{5}{2} - \frac{\rho'(\xi)\xi}{\rho(\xi)} \end{pmatrix}$$

[6]Actually, we shall be more specific later, and in fact introduce slightly perturbed $\lambda_{\gamma_1,\gamma_2}$, τ_{γ_1,γ_2} to get the right description.

More explicitly, we have

$$-\mathscr{F}\big(\frac{R}{\lambda}\epsilon_t\big)|_{t=t_0} = x_1 + \beta_\nu(\tau_{0,0})\mathscr{K}_{cc}x_0 + \beta_\nu(\tau_{0,0})\mathscr{K}_{cd}x_{0d} \tag{4.11}$$

$$-\langle\phi_d, \frac{R}{\lambda}\epsilon_t\rangle|_{t=t_0} = x_{1d} + \beta_\nu(\tau_{0,0})\mathscr{K}_{dd}x_{0d} + \beta_\nu(\tau_{0,0})\mathscr{K}_{dc}x_0 \tag{4.12}$$

where we have set $x_1 = \big(\partial_\tau - \frac{\dot\lambda}{\lambda}(2\xi\partial_\xi + \frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)})\big)x(\tau,\xi)|_{\tau=\tau_{0,0}}$, as well as $x_{1d} = \partial_\tau x_d(\tau)|_{\tau=\tau_{0,0}}$, and as before we use

$$\tau_{0,0} = \int_{t_0}^\infty s^{-1-\nu}\,ds,$$

which thus corresponds to the new time variable with respect to the scaling law $\lambda(t) = t^{-1-\nu}$ evaluated at initial time $t = t_0$.

For future reference, we note that we shall sometimes use the notation $\mathscr{D}_\tau = \partial_\tau - \frac{\dot\lambda}{\lambda}(2\xi\partial_\xi + \frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)})$ when this operator acts on scalar functions $x(\tau,\xi)$, while it acts on vector valued functions $\underline{x}$ via the above formula.

Finally, the relations (4.10) in conjunction with (4.11), (4.12) give the translation from the data quadruple $(x_0, x_1) \in \widetilde{S}$, $(x_{0d}, x_{1d}) \in \mathbb{R}^2$ to a data pair $(\epsilon, \epsilon_t)|_{t=t_0} = (\epsilon_0, \epsilon_1)$. An argument analogous to the one used to establish (4.9) implies then that $\epsilon_1 \in H_{loc}^{\frac{1}{2}+2\delta_0}$.

4.1.6 Outline of the main result from [49]

Now that the technical preliminaries involving the representation associated to the operator $\mathscr{L}$ have been introduced, we give a clear description of the conditional stability result contained in [49].

Theorem 107 ([49])**.** *There is a $\nu_0 > 0$ sufficiently small, such that the following holds: let u_ν, $0 < \nu < \nu_0$ be one of the solutions constructed in [62], [59], on a time slice $(0, t_0] \times \mathbb{R}^3$, with $0 < t_0 \ll 1$ sufficiently small. Then there exists a co-dimension two Lipschitz hyper surface Σ_0 in a Hilbert space $\widetilde{S} \times \mathbb{R}$ where $\widetilde{S}$ is essentially $(H^{\frac{3}{2}+}_{rad,loc}(\mathbb{R}^3) \cap \{\phi_d\}^\perp) \times (H^{\frac{1}{2}+}_{rad,loc}(\mathbb{R}^3) \cap \{\phi_d\}^\perp)$, and a positive $\delta_1 \ll 1$, such that for any $(u_0, u_1, \gamma) \in \Sigma_0 \cap \big(B_{\delta_1,\widetilde{S}}(0) \times (-\delta_1, \delta_1)\big)$ and a suitable Lipschitz functions*

$$\gamma_{1,2} : \Sigma_0 \cap (B_{\delta_1,\widetilde{S}}(0) \times (-\delta_1, \delta_1)) \longrightarrow \mathbb{R},$$

the solution of (4.1) *with data*

$$\begin{aligned} u[t_0] &:= u_\nu[t_0] + (u_0, u_1) + (\gamma\phi_d + \gamma_1(u_0, u_1, \gamma)\phi_d, \gamma_2(u_0, u_1, \gamma)\phi_d) \\ &\in (H^{1+}_{rad}(\mathbb{R}^3) \times H^{0+}_{rad}(\mathbb{R}^3)) \cap \Sigma \end{aligned}$$

exists on $I = (0, t_0]$ *and can be written in the form*

$$u(t,x) = W_{\lambda(t)}(x) + v_1(t,x),\ \lambda(t) = t^{-1-\nu}$$

with $(v_1, v_{1,t}) \in H^{1+\frac{\nu}{2}-} \times H^{\frac{\nu}{2}-}$ *on each time slice* $t = t_1 \in I$*, and furthermore*

$$\big(E_{loc}(v)\big)(t) := \int_{|x|\le t} \frac{1}{2}|\nabla_{t,x} v_1|^2 dx \longrightarrow 0$$

as $t \to 0$.

The preceding theorem reveals that for small enough $\nu > 0$, the solutions constructed in [62], [59] are stable under perturbations along a co-dimension three manifold in a suitable topology, and a clear description of the dynamics of the perturbed solutions is provided. However, as already noticed in Section 4.1.4, since the operator $\mathscr{L}$ has a unique negative eigenvalue and a corresponding positive ground state ϕ_d causing exponential growth for the linearised flow $e^{it\sqrt{\mathscr{L}}}$, a co-dimension one condition suffices to ensure that the forward flow will remain bounded. Moreover, in light of Theorem 102, the optimal stability result should require only a co-dimension one condition on the space of initial perturbations. In order to highlight the enforced extra two conditions in Theorem 107 consider the perturbed initial data

$$u_\nu[t_0] + (\epsilon_0, \epsilon_1),$$

where the perturbation

$$(\epsilon_0, \epsilon_1) = (u_0, u_1) + (\gamma\phi_d + \gamma_1(u_0, u_1, \gamma)\phi_d, \gamma_2(u_0, u_1, \gamma)\phi_d)$$

is associated with a data quadruple $(\underline{x}_0, \underline{x}_1)$ as in (4.10), (4.11), (4.12). In [49], to prevent the growth of the linear approximation oft the perturbed solution, the following vanishing relations producing Σ_0 are imposed:

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi) x_0(\xi)}{\xi^{\frac{1}{4}}} \cos[\nu\tau_0\xi^{\frac{1}{2}}]\, d\xi = 0, \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi) x_1(\xi)}{\xi^{\frac{3}{4}}} \sin[\nu\tau_0\xi^{\frac{1}{2}}]\, d\xi = 0.$$

The goal of this paper is to remove these extra two conditions responsible for the loss of two co-dimensions, and thus to obtain a qualitatively optimal result. To gain two co-dimensions, we work with a more flexible blow-up scaling law $\lambda(t)$, depending on two additional parameters.

4.1.7 Figures

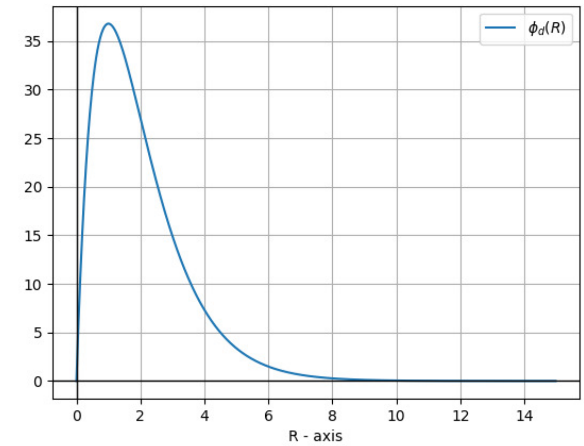


Figure 4.3: The ground state $\phi_d(R)$.

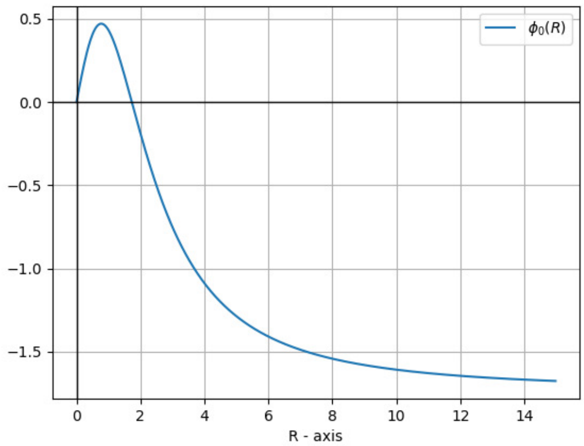


Figure 4.4: The resonance $\phi_0(R)$.

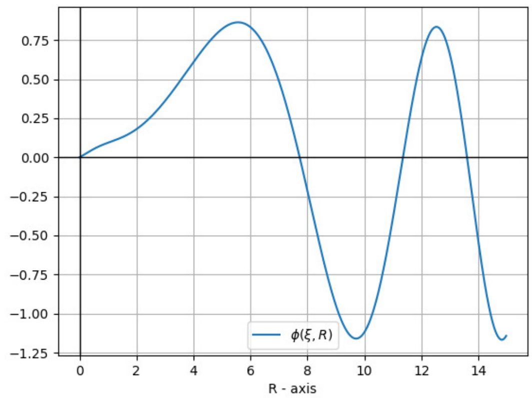


Figure 4.5: The generalized Fourier basis $\phi(\xi, R)$ for ξ small

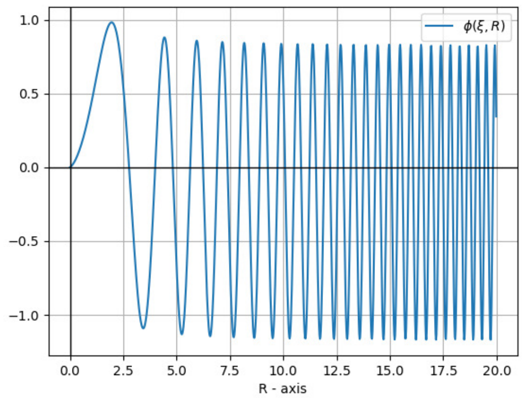


Figure 4.6: The generalized Fourier basis $\phi(\xi, R)$ for ξ large

4.2 The main theorem and outline of the proof

4.2.1 The main theorem

We shall now consider what happens to the evolution of the perturbed initial data $u_\nu[t_0]$, with u_ν as in Theorem 101. In light of Theorem 102, we only expect such perturbations to yield a type II dynamics (backwards in time, i. e. for $t < t_0$), provided we impose a suitable co-dimension one condition on the perturbation imposed. That this is indeed all that is required follows from:

Theorem 108. *Assume* $0 < \nu \ll 1$, *and assume* $t_0 = t_0(\nu) > 0$ *is sufficiently small, so that the solutions* $u_\nu(t,x)$ *in Theorem 101 exist on* $(0, t_0] \times \mathbb{R}^3$. *Let* $\delta_1 = \delta_1(\nu) > 0$ *be small enough, and let* $\mathscr{B}_{\delta_1} \subset \widetilde{S} \times \mathbb{R}$ *be the* δ_1*-vicinity of* $\big((0,0),0\big) \in \widetilde{S} \times \mathbb{R}$, *where* $\widetilde{S}$ *is the Banach space defined as the completion of* $C_0^\infty(0,\infty) \times C_0^\infty(0,\infty)$ *with respect to the norm* (4.8). *Then there is a Lipschitz* *function* $\gamma_1 : \mathscr{B}_{\delta_1} \to \mathbb{R}$*, such that for any triple* $\big((x_0, x_1), x_{0d}\big) \in \mathscr{B}_{\delta_1}$*, the quadruple*

$$\big((x_0, x_1), (x_{0d}, x_{1d})\big),\; x_{1d} = \gamma_1(x_0, x_1, x_{0d})$$

determines a data perturbation pair $(\epsilon_0, \epsilon_1) \in H^{\frac{3}{2}+2\delta_0}_{rad,loc}(\mathbb{R}^3) \times H^{\frac{1}{2}+2\delta_0}_{rad,loc}(\mathbb{R}^3)$ *via* (4.10), (4.11), (4.12)*, and such that the perturbed initial data*

$$u_\nu[t_0] + (\epsilon_0, \epsilon_1) \tag{4.13}$$

lead to a solution $\widetilde{u}(t,x)$ *on* $(0, t_0] \times \mathbb{R}^3$ *admitting the description*

$$\widetilde{u}(t,x) = W_{\widetilde{\lambda}(t)}(x) + \epsilon(t,x),\; \big(\epsilon(t,\cdot), \epsilon_t(t,\cdot)\big) \in H^{1+\frac{\nu}{2}-}_{loc} \times H^{\frac{\nu}{2}-}_{loc}$$

where the parameter $\widetilde{\lambda}(t)$ *equals* $\lambda(t)$ *asymptotically*

$$\lim_{t\to 0} \frac{\widetilde{\lambda}(t)}{\lambda(t)} = 1,\; \lambda(t) = t^{-1-\nu}.$$

In particular, the blow-up phenomenon described in Theorem 101 is stable under a suitable co-dimension one class of data perturbations.

Remark. Notice that this result is qualitatively optimal in light of the result of [56], for the construction of a center-stable manifold see also [53], [54], [55].

Remark. We could have replaced $\widetilde{\lambda}$ by λ in the preceding theorem and included the arising modification in the error term $\epsilon(t,x)$. The formulation of the theorem emphasises part of the proof strategy, which shall indeed consist in a (very slight) modification of the scaling law $\lambda(t) = t^{-1-\nu}$ to force two important vanishing conditions. It is this part which is indeed analogous to the usual 'modulation method'.

4.2.2 Outline of the proof

The proof will consist of two stages, the first replacing the blow-up solution $u_\nu(t,x)$ by a two parameter family $u^{(\gamma_1,\gamma_2)}_{approx}$ of approximate blow-up solutions, where the parameters γ_1, γ_2 will depend on the perturbation (ϵ_0, ϵ_1) and thus on the original data set (x_0, x_1, x_{0d}), and the second stage will involve completing the approximate solution $u^{(\gamma_1,\gamma_2)}_{approx}$ to an exact one of the form

$$u^{(\gamma_1,\gamma_2)}_{approx} + \epsilon(t,x),$$

whose data at time $t = t_0$ will coincide with $u_\nu[t_0] + (\epsilon_0, \epsilon_1)$ at time $t = t_0$, *provided we restrict the data to a suitable dilate of the light cone* $r \le Ct_0$. In fact, we do not care about what happens outside of the light cone, as our solutions will remain regular there for simple a priori non-concentration of energy reasons, exactly as in [62].

Explaining the reason for introducing $u^{(\gamma_1,\gamma_2)}_{approx}$ necessitates outlining the strategy for controlling the error term $\epsilon(t,x)$, which will be done via Fourier methods, exactly as was done in

[49].

The method of [49].

Assume we intend to construct a solution of the form $u(t,x) = u_\nu(t,x) + \epsilon(t,x)$, with u_ν as in Theorem 101. Recall that u_ν consists of a bulk part $W_{\lambda(t)}(x)$ with $\lambda(t) = t^{-1-\nu}$ and an error part. Passing to the new variables $R = \lambda(t)r, \tau = \int_t^\infty \lambda(s)\,ds$, one derives the following equation for the variable $\widetilde{\epsilon}(\tau,R) := R\epsilon(t,r)$, see also [62], [59]:

$$\begin{aligned}&(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)^2\widetilde{\varepsilon} - \beta_\nu(\tau)(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)\widetilde{\varepsilon} + \mathscr{L}\widetilde{\varepsilon}\\ &= \lambda^{-2}(\tau)RN(\varepsilon) + \partial_\tau(\dot{\lambda}\lambda^{-1})\widetilde{\varepsilon};\ \beta_\nu(\tau) = \dot{\lambda}(\tau)\lambda^{-1}(\tau),\end{aligned} \tag{4.14}$$

where the operator $\mathscr{L}$ is given by

$$\mathscr{L} = -\partial_R^2 - 5W^4(R)$$

and we have

$$RN(\varepsilon) = 5(u_\nu^4 - u_0^4)\widetilde{\varepsilon} + RN(u_\nu,\widetilde{\varepsilon}),\ u_0 = W_{\lambda(t)}(x) = \frac{\lambda^{\frac{1}{2}}(t)}{\left(1 + \frac{(\lambda(t)r)^2}{3}\right)^{\frac{1}{2}}},$$

$$RN(u_\nu,\widetilde{\varepsilon}) = R(u_\nu + \frac{\widetilde{\varepsilon}}{R})^5 - Ru_\nu^5 - 5u_\nu^4\widetilde{\varepsilon}$$

To solve this equation inside the forward light cone centered at the origin, one translates it to the Fourier side, i. e. one writes

$$\widetilde{\varepsilon}(\tau,R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi, \tag{4.15}$$

see Proposition 103. Taking advantage of Theorem 106, and using simple algebraic manipulations, see (2.3) in [49], one derives the following equation system in terms of the Fourier coefficients

$$\underline{x}(\tau,\xi) = \begin{pmatrix} x_d(\tau)\\ x(\tau,\xi)\end{pmatrix}:$$

$$\left(\mathscr{D}_\tau^2 + \beta_\nu(\tau)\mathscr{D}_\tau + \underline{\xi}\right)\underline{x}(\tau,\xi) = \mathscr{R}(\tau,\underline{x}) + \underline{f}(\tau,\underline{\xi}), \tag{4.16}$$

where we have

$$\mathscr{R}(\tau,\underline{x})(\xi) = \Big(-4\beta_\nu(\tau)\mathscr{K}\mathscr{D}_\tau\underline{x} - \beta_\nu^2(\tau)(\mathscr{K}^2 + [\mathscr{A},\mathscr{K}] + \mathscr{K} + \beta_\nu'\beta_\nu^{-2}\mathscr{K})\underline{x}\Big)(\xi) \tag{4.17}$$

with $\beta_\nu(\tau) = \frac{\dot\lambda(\tau)}{\lambda(\tau)}$, and we set $\underline{f} = \begin{pmatrix} f_d \\ f \end{pmatrix}$ where

$$\begin{aligned} f(\tau,\xi) &= \mathscr{F}\big(\lambda^{-2}(\tau)\big[5(u_\nu^4 - u_0^4)\widetilde{\varepsilon} + RN(u_\nu,\widetilde{\varepsilon})\big]\big)(\xi) \\ f_d(\tau) &= \langle\lambda^{-2}(\tau)\big[5(u_\nu^4 - u_0^4)\widetilde{\varepsilon} + RN(u_\nu,\widetilde{\varepsilon})\big],\phi_d(R)\rangle. \end{aligned} \tag{4.18}$$

Also the key operator

$$\mathscr{D}_\tau = \partial_\tau + \beta_\nu(\tau)\mathscr{A}, \quad \mathscr{A} = \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{A}_c \end{pmatrix}$$

and we have

$$\mathscr{A}_c = -2\xi\partial_\xi - \Big(\frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}\Big),$$

while $\mathscr{K}$ is as in Theorem 106.

As *initial data* for the problem (4.16), we shall of course use

$$\underline{x}(\tau_0) = \begin{pmatrix} x_{0d} \\ x_0(\xi) \end{pmatrix}, \mathscr{D}_\tau\underline{x}(\tau_0) = \begin{pmatrix} x_{1d} \\ x_1(\xi) \end{pmatrix}$$

where the components $x_{0,1}(\xi), x_{0d}$ shall be freely described (within the constraints of Theorem 108), while the last component x_{1d} shall be determined via a suitable Lipschitz function in terms of the first three components. This is again due to the exponential growth of the component $x_d(\tau)$ due to the unstable mode. The method of solution of (4.16) uses an iterative scheme, beginning with the zeroth iterate solving

$$\big(\mathscr{D}_\tau^2 + \beta_\nu(\tau)\mathscr{D}_\tau + \underline{\xi}\big)\underline{x}(\tau,\xi) = \underline{0},\ \underline{x}(\tau_0) = \begin{pmatrix} x_{0d} \\ x_0(\xi) \end{pmatrix}, \mathscr{D}_\tau\underline{x}(\tau_0) = \begin{pmatrix} x_{1d} \\ x_1(\xi) \end{pmatrix}. \tag{4.19}$$

This can be solved explicitly as in Lemma 2.1 in [49], which we quote here:

Lemma 109. *The equation* (4.19) *is solved for the continuous spectral part* $x(\tau,\xi)$ *via the following parametrix:*

$$\begin{aligned} x(\tau,\xi) =& \frac{\lambda^{\frac{5}{2}}(\tau)}{\lambda^{\frac{5}{2}}(\tau_0)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi)}{\rho^{\frac{1}{2}}(\xi)}\cos\Big[\lambda(\tau)\xi^{\frac{1}{2}}\int_{\tau_0}^{\tau}\lambda^{-1}(u)\,du\Big]x_0\Big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\Big) \\ &+ \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\tau_0)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi)}{\rho^{\frac{1}{2}}(\xi)}\frac{\sin\Big[\lambda(\tau)\xi^{\frac{1}{2}}\int_{\tau_0}^{\tau}\lambda^{-1}(u)\,du\Big]}{\xi^{\frac{1}{2}}}x_1\Big(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\Big) \end{aligned} \tag{4.20}$$

Moreover, writing $\underline{x}_0 = \begin{pmatrix} x_{0d} \\ x_0(\xi) \end{pmatrix}, \underline{x}_1 = \begin{pmatrix} x_{1d} \\ x_1(\xi) \end{pmatrix}$ *and picking* $\tau_0 \gg 1$ *sufficiently large, there is* $c_d = 1 + O(\tau_0^{-1})$ *as well as* $\gamma_d = -|\xi_d|^{\frac{1}{2}} + O(\tau_0^{-1})$ *such that if we impose the co-dimension one condition*

$$x_{1d} = \gamma_d x_{0d}, \tag{4.21}$$

then the discrete spectral part of $\underline{x}(\tau,\xi)$ *admits for any* $\kappa > 0$ *the representation*

$$x_d(\tau) = \big(1 + O_\kappa(\tau^{-1}e^{\kappa(\tau-\tau_0)})\big)e^{-|\xi_d|^{\frac{1}{2}}(\tau-\tau_0)}c_d x_{0d}$$

One also has for $i \geq 1$

$$(-\partial_\tau)^i x_d(\tau) = \big(1 + O_\kappa(\tau^{-1}e^{\kappa(\tau-\tau_0)})\big)|\xi_d|^{\frac{i}{2}}e^{-|\xi_d|^{\frac{1}{2}}(\tau-\tau_0)}c_d x_{0d}$$

This co-dimension one condition will have to be slightly modified in nonlinear ways for the higher iterates, but to leading order remains the same throughout and is responsible for the co-dimension one condition of Theorem 108.

Extra vanishing conditions.

There are, however, two additional vanishing conditions in the work [49], imposed on the *continuous spectral parts* $x_{0,1}(\xi)$, and which arise due to the need to bound the nonlinear terms in $N(u_\nu,\tilde{\varepsilon})$ in (4.64). These conditions arise when bounding $\frac{\tilde{\varepsilon}(\tau,R)}{R}$ upon expressing $\tilde{\varepsilon}(\tau,R)$ as in (4.15) and inserting the parametrix (4.20) for the continuous spectral parts. This is the content of Propostion 3.1 in [49]:

Proposition 110. *Assume the initial data*

$$(x_0,x_1) \in \langle\xi\rangle^{-1-\delta_0}\xi^{0+\delta_0}L^2_{d\xi} \times \langle\xi\rangle^{-\frac{1}{2}-\delta_0}\xi^{0+\delta_0}L^2_{d\xi}.$$

Furthermore, assume that we have the vanishing relations

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi = 0, \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_1(\xi)}{\xi^{\frac{3}{4}}}\sin[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi = 0. \tag{4.22}$$

at time $\tau = \tau_0$. *Assume that* $x(\tau,\xi)$ *is given by* (4.20). *Then the function*[7] $P_c\tilde{\varepsilon}(\tau,R)$ *represented by the Fourier coefficients* $x(\tau,\xi)$ *via*

$$P_c\tilde{\varepsilon}(\tau,R) = \int_0^\infty \phi(R,\xi)x(\tau,\xi)\rho(\xi)\,d\xi$$

satisfies

$$P_c\tilde{\varepsilon}(\tau,R) = \tilde{\epsilon}_1(\tau,R) + \tilde{\epsilon}_2(\tau,R),$$

where we have

$$\|\frac{\tilde{\epsilon}_1(\tau,R)}{R}\|_{L^\infty_{dR}} \lesssim \|(\langle\xi\rangle^{\frac{1}{2}+2\delta_0}\xi^{\frac{1}{2}-\delta_0}x_0, \langle\xi\rangle^{\frac{1}{2}+2\delta_0}\xi^{-\delta_0}x_1)\|_{L^2_{d\xi}}$$

$$\|\tilde{\epsilon}_2(\tau,R)\|_{L^\infty_{dR}} \lesssim \tau\|(\langle\xi\rangle^{\frac{1}{2}+2\delta_0}\xi^{\frac{1}{2}-\delta_0}x_0, \langle\xi\rangle^{\frac{1}{2}+2\delta_0}\xi^{-\delta_0}x_1)\|_{L^2_{d\xi}}$$

Here $\delta_0 > 0$ *is the small constant used to define* $\tilde{S}$ *in* (4.8).

[7] Here P_c denotes the projection onto the continuous spectral part

We note here that the growth of $\widetilde{\epsilon}_2(\tau,R)$ is precisely due to the growth of the 'resonant part' of $\widetilde{\epsilon}$, i. e. a multiple of the resonance $\phi_0(R)$, see the discussion preceding Prop. 103. We also observe that the expressions $\cos[\nu\tau_0\xi^{\frac{1}{2}}]$, $\sin[\nu\tau_0\xi^{\frac{1}{2}}]$ can alternatively be written as

$$\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(s)\,ds],\ \sin[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(s)\,ds],$$

upon noting that in terms of the variable $\tau\in[\tau_0,\infty)$, we have (abuse of notation) $\lambda(\tau)=c(\nu)\tau^{-1-\nu^{-1}}$. As the parametrix in (4.20) is valid for arbitrary λ, we see that the generalisation of the vanishing conditions (4.22) to more general $\lambda(t)\sim t^{-1-\nu}$ as $t\to\infty$ becomes (again upon passing to the new time variable $\tau=\int_t^{\infty}\lambda(s)\,ds$)

$$\int_0^{\infty}\frac{\rho^{\frac{1}{2}}(\xi)x_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(s)\,ds]\,d\xi=0,$$
$$\int_0^{\infty}\frac{\rho^{\frac{1}{2}}(\xi)x_1(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(s)\,ds]\,d\xi=0.$$

The road map

The key for proving Theorem 108 shall be to get rid of these two conditions on the continuous spectral part of the data, and thence reduce things to the unique condition involving the discrete spectral part. To achieve this, we shall pass from the splitting $u(t,x)=u_\nu(t,x)+\epsilon(t,x)$ to a slightly modified one

$$u(t,x)=u_{approx}^{(\gamma_1,\gamma_2)}(t,x)+\overline{\epsilon}(t,x), \tag{4.23}$$

where

$$u_{approx}^{(\gamma_1,\gamma_2)}(t,x)=W_{\lambda_{\gamma_1,\gamma_2}(t)}(x)+\eta(t,x)$$

will be an approximate solution built analogously to $u_\nu(t,x)$ (as in Theorem 101), but where the bulk part $W_{\lambda_{\gamma_1,\gamma_2}(t)}(x)$ is now scaled according to

$$\lambda_{\gamma_1,\gamma_2}(t):=\Big(1+\gamma_1\cdot\frac{t^{k_0\nu}}{\langle t^{k_0\nu}\rangle}+\gamma_2\log t\cdot\frac{t^{k_0\nu}}{\langle t^{k_0\nu}\rangle}\Big)t^{-1-\nu},\ k_0=[N\nu^{-1}], \tag{4.24}$$

for some $N\gg 1$, and this is clearly asymptotically equal to $\lambda(t)$: $\lim_{t\to 0}\frac{\lambda_{\gamma_1,\gamma_2}(t)}{\lambda(t)}=1$. As we shall want to match the data (4.13) at time $t=t_0$, at least in the forward light cone, we impose for some $C>1$ the condition

$$\chi_{r\le Ct_0}u_\nu[t_0]+(\epsilon_0,\epsilon_1)=\chi_{r\le Ct_0}u_{approx}^{(\gamma_1,\gamma_2)}[t_0]+(\overline{\epsilon}_0,\overline{\epsilon}_1) \tag{4.25}$$

on the data $(\overline{\epsilon}_0,\overline{\epsilon}_1)=\overline{\epsilon}[t_0]$ of the 'new perturbation' $\overline{\epsilon}$ at $t=t_0$.

We note that the proper re-scaled variables to describe $\overline{\epsilon}$ are now given by

$$\tau_{\gamma_1,\gamma_2} := \int_t^\infty \lambda_{\gamma_1,\gamma_2}(s)\,ds,\ R_{\gamma_1,\gamma_2} = \lambda_{\gamma_1,\gamma_2}(t)r, \tag{4.26}$$

In analogy to (4.10), (4.11), (4.12), we can then determine $(x_0^{(\gamma_1,\gamma_2)}, x_1^{(\gamma_1,\gamma_2)})$ as well as $(x_{0d}^{(\gamma_1,\gamma_2)}, x_{1d}^{(\gamma_1,\gamma_2)})$, such that

$$\overline{\epsilon}_0(r(R_{\gamma_1,\gamma_2})) = R_{\gamma_1,\gamma_2}^{-1}[x_{0d}^{(\gamma_1,\gamma_2)}\phi_d(R_{\gamma_1,\gamma_2}) + \int_0^\infty x_0^{(\gamma_1,\gamma_2)}(\xi)\phi(R_{\gamma_1,\gamma_2},\xi)\rho(\xi)\,d\xi] \tag{4.27}$$

$$\begin{aligned}-\mathscr{F}\big(\frac{R_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}\overline{\epsilon}_1\big)|_{t=t_0} = x_1^{(\gamma_1,\gamma_2)} + \beta_\nu^{(\gamma_1,\gamma_2)}(\tau_{\gamma_1,\gamma_2})|_{t=t_0}\mathscr{K}_{cc}x_0^{(\gamma_1,\gamma_2)}\\ + \beta_\nu^{(\gamma_1,\gamma_2)}(\tau_{\gamma_1,\gamma_2})|_{t=t_0}\mathscr{K}_{cd}x_{0d}\end{aligned} \tag{4.28}$$

$$\begin{aligned}-\langle\phi_d, \frac{R_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}\overline{\epsilon}_1\rangle|_{t=t_0} = x_{1d}^{(\gamma_1,\gamma_2)} + \beta_\nu^{(\gamma_1,\gamma_2)}(\tau_{\gamma_1,\gamma_2})|_{t=t_0}\mathscr{K}_{dd}x_{0d}^{(\gamma_1,\gamma_2)}\\ + \beta_\nu^{(\gamma_1,\gamma_2)}(\tau_{\gamma_1,\gamma_2})|_{t=t_0}\mathscr{K}_{dc}x_0^{(\gamma_1,\gamma_2)},\end{aligned} \tag{4.29}$$

and we use the notation $\beta_\nu^{(\gamma_1,\gamma_2)} = \frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}$, where the $\dot{\lambda}_{\gamma_1,\gamma_2}$ indicates differentiation with respect to the new time variable τ_{γ_1,γ_2}.

At this stage, we can succinctly formulate the key technical steps required to complete the proof of Theorem 108.

- *Show that given x_0, x_1, x_{0d} as in the statement of Theorem 108, there are unique choices of $\gamma_1, \gamma_2, x_{1d}$ such that the Fourier variables*

  $$x_0^{(\gamma_1,\gamma_2)}, x_1^{(\gamma_1,\gamma_2)}, x_{0d}^{(\gamma_1,\gamma_2)}, x_{1d}^{(\gamma_1,\gamma_2)}$$

  satisfy the vanishing relations

  $$\begin{aligned}\int_0^\infty \frac{\rho^{\frac12}(\xi)x_0^{(\gamma_1,\gamma_2)}(\xi)}{\xi^{\frac14}}\cos[\lambda_{(\gamma_1,\gamma_2)}(\tau_0)\xi^{\frac12}\int_{\tau_0}^\infty \lambda_{(\gamma_1,\gamma_2)}^{-1}(s)\,ds]\,d\xi = 0,\\ \int_0^\infty \frac{\rho^{\frac12}(\xi)x_1^{(\gamma_1,\gamma_2)}(\xi)}{\xi^{\frac34}}\sin[\lambda_{(\gamma_1,\gamma_2)}(\tau_0)\xi^{\frac12}\int_{\tau_0}^\infty \lambda_{(\gamma_1,\gamma_2)}^{-1}(s)\,ds]\,d\xi = 0.\end{aligned} \tag{4.30}$$

  as well as condition (4.21).

- *Using the splitting*

  $$u(t,x) = u_{approx}^{(\gamma_1,\gamma_2)}(t,x) + \overline{\epsilon}(t,x),$$

pass to the Fourier representation

$$R_{\gamma_1,\gamma_2}\overline{\epsilon} = x_d^{(\gamma_1,\gamma_2)}\phi_d(R_{\gamma_1,\gamma_2}) + \int_0^\infty x^{(\gamma_1,\gamma_2)}(\tau_{\gamma_1,\gamma_2},\xi)\phi(R_{\gamma_1,\gamma_2},\xi)\rho(\xi)\,d\xi,$$

and use the analog of (4.16) *to construct a solution*

$$\underline{x}^{(\gamma_1,\gamma_2)}(\tau,\xi) = \begin{pmatrix} x_d^{(\gamma_1,\gamma_2)}(\tau) \\ x^{(\gamma_1,\gamma_2)}(\tau,\xi) \end{pmatrix}$$

'closely matching' the initial conditions $x_{0,1}^{(\gamma_1,\gamma_2)}, x_{0,1d}^{(\gamma_1,\gamma_2)}$. In fact, the method from [49] furnishes such a solution with data

$$\underline{x}^{(\gamma_1,\gamma_2)}(\tau_{\gamma_1,\gamma_2},\xi)|_{t=t_0} = \begin{pmatrix} x_{0d}^{(\gamma_1,\gamma_2)} + \triangle x_{0d}^{(\gamma_1,\gamma_2)} \\ x_0^{(\gamma_1,\gamma_2)}(\xi) + \triangle x_0^{(\gamma_1,\gamma_2)}(\xi) \end{pmatrix}$$

$$\mathscr{D}_\tau \underline{x}^{(\gamma_1,\gamma_2)}(\tau_{\gamma_1,\gamma_2},\xi)|_{t=t_0} = \begin{pmatrix} x_{1d}^{(\gamma_1,\gamma_2)} + \triangle x_{1d}^{(\gamma_1,\gamma_2)} \\ x_1^{(\gamma_1,\gamma_2)}(\xi) + \triangle x_1^{(\gamma_1,\gamma_2)}(\xi) \end{pmatrix}$$

- *Translating things back to the original perturbation in terms of the old Fourier variables $x_0(\xi), x_1(\xi), x_{0d}, x_{1d}$, show that we have found an initial data pair corresponding to Fourier variables*

$$x_0(\xi) + \triangle x_0(\xi), x_1(\xi) + \triangle x_1(\xi), x_{0d} + \triangle x_{0d}, x_{1d} + \triangle x_{1d},$$

where the corrections $\triangle x_0(\xi)$ etc are small and depend in Lipschitz continuous fashion on the original data x_0 etc, with small Lipschitz constant.

4.3 Construction of a two parameter family of approximate blow-up solutions

Here we construct the approximate blow-up solutions $u_{approx}^{(\gamma_1,\gamma_2)}$ which replace the previous $u_\nu(t,x)$, see the decomposition (4.23). The idea behind the construction is to closely mimic the steps in section 2 of [59], which in turn follows closely the steps in section 2 of [62]. In particular, to describe the successive corrections in the construction, we shall rely on the same algebras of functions as in [59].

In the sequel, we shall work mostly with respect to the scaling parameter $\lambda_{\gamma_1,\gamma_2}(t)$ given by (4.24). To simplify the notation, we shall henceforth set

$$\lambda(t) := \lambda_{\gamma_1,\gamma_2}(t),\ R := \lambda_{\gamma_1,\gamma_2}(t)r,\ R_{0,0} := \lambda_{0,0}(t)r \tag{4.31}$$

Theorem 111. *Let $\nu > 0$, $t_0 = t_0(\nu) > 0$ sufficiently small, and $\gamma_{1,2} \ll 1$. Also, let $N \gg 1$, $k_0 =$*

$[N\nu^{-1}]$, $k_* = [\frac{1}{2}N\nu^{-1}]$. *Then there exists an approximate solution* $u_{approx} = u_{approx}^{(\gamma_{1,2})}$ *for* $\Box u = -u^5$ *of the form (putting* $\lambda(t) := \lambda_{\gamma_{1,2}}(t)$ *for simplicity)*

$$u_{approx}^{(\gamma_{1,2})} = \lambda^{\frac{1}{2}}(t)\big[W(R) + \frac{c}{(\lambda t)^2}R^2(1+R^2)^{-\frac{1}{2}} + O((\lambda t)^{-2}\log R R^2(1+R^2)^{-\frac{3}{2}})\big],$$

such that the corresponding error

$$e_{approx} = \Box u_{approx} + u_{approx}^5$$

is of the form

$$\begin{aligned} &t^2 e_{approx} \\ &= [|\gamma_1| + |\gamma_2|]\big[O\big(\log t \frac{\lambda^{\frac{1}{2}}R}{(\lambda t)^{k_0+4}}(1+(1-a)^{\frac{1}{2}+\frac{\nu}{2}})\big) \\ &\qquad + O\big(\log t \frac{\lambda^{\frac{1}{2}}}{(\lambda t)^{k_0+2}}R^{-1}(1+(1-a)^{\frac{1}{2}+\frac{\nu}{2}})\big)\big] \end{aligned}$$

and such that the above expansions may be formally differentiated, where we use the notation $a = \frac{r}{t}$. *Furthermore, writing* $u_{approx}^{(\gamma_{1,2})} = u_{approx}^{(\gamma_{1,2})}(t, r, \gamma_{1,2}, \nu)$ *we have the* γ*-dependence*

$$\partial_{\gamma_1} u_{approx}^{(\gamma_{1,2})} = O(t^{k_0\nu}\lambda^{\frac{1}{2}}\frac{R}{(\lambda t)^2}),$$

with symbol type behaviour with respect to the $\partial_{t,r}$ *derivatives up to order two, and similarly for*

$$\partial_{\gamma_2} u_{approx}^{(\gamma_{1,2})} = O(t^{k_0\nu}\log t\lambda^{\frac{1}{2}}\frac{R}{(\lambda t)^2}),$$

Remark. The key point here is the last part, which ensures that the γ-dependent part of the solutions $u_{\text{approx}}^{(\gamma_{1,2})}$ is smoother than the solutions themselves (they are only of class $H^{1+\frac{\nu}{2}-}$ regularity).

Remark. Observe from the preceding construction that $e_{approx} = 0$ provided $\gamma_1 = \gamma_2 = 0$. Thus in that case the function $u_{approx}^{(0,0)}(t,x) = u_\nu(t,x)$ reproduces an exact solution as in Theorem 101.

The rest of the chapter is devoted to the proof of Theorem 111. We shall obtain the functions $u_{\text{approx}}^{(\gamma_{1,2})}$ by adding corrections v_j to the bulk part $u_0 := W_{\lambda(t)}(r)$, the latter as in the paragraph following (4.14). The precise description of these corrections is a bit cumbersome, but in principle elementary, as they arise by solving certain explicit ordinary differential equations. The following definitions come directly from [59]:

Definition. We define $\mathcal{Q}$ to be the algebra of continuous functions $q : [0,1] \to \mathbb{R}$ with the following properties:

i. q is analytic in $[0,1)$ with an even expansion at 0 and with $q(0) = 0$.

ii. Near $a = 1$ we have an expansion of the form

$$q(a) = q_0(a) + \sum_{i=1}^{\infty} (1-a)^{\beta(i)+1} \sum_{j=0}^{\infty} q_{ij}(a)(\log(1-a))^j$$

with analytic coefficients q_0, q_{ij}; if ν is irrational, then $q_{ij} = 0$ if $j > 0$. The $\beta(i)$ are of the form

$$\sum_{k\in K} \big((2k-3/2)\nu - 1/2\big) + \sum_{k\in K'} \big((2k-1/2)\nu - 1/2\big)$$

where K, K' are finite sets of positive integers. Moreover, only finitely many of the q_{ij} are nonzero.

We remark that the exponents of $1-a$ in the above series all exceed $\frac{1}{2}$ because of $\nu > 0$. For the errors e_k we introduce

Definition. $\mathcal{Q}'$ is the space of continuous functions $q : [0,1) \to \mathbb{R}$ with the following properties:

i. q is analytic in $[0,1)$ with an even expansion at 0.

ii. Near $a = 1$ we have an expansion of the form

$$q(a) = q_0(a) + \sum_{i=1}^{\infty} (1-a)^{\beta(i)} \sum_{j=0}^{\infty} q_{ij}(a)(\log(1-a))^j$$

with analytic coefficients q_0, q_{ij}, of which only finitely many are nonzero. The $\beta(i)$ are as above.

By construction, $\mathcal{Q} \subset \mathcal{Q}'$. The family $\mathcal{Q}'$ is obtained by applying $a^{-1}\partial_a$ to the algebra $\mathcal{Q}$. The exact number of $\log(1-a)$ factors can of course be determined, but is irrelevant for our purposes. We will also need the following definition:

Definition. Denote by $\mathcal{Q}_{smooth}$ the algebra of continuous functions $q : [0,1] \longrightarrow \mathbb{R}$ with the following properties:

(i) q is analytic in $[0,1)$ with an even expansion at 0 and with $q(0) = 0$.

(ii) Near $a = 1$ we have an expansion of the form

$$q(a) = q_0(a) + \sum_{i=1}^{\infty} (1-a)^{\beta(i)+1} \sum_{j=0}^{\infty} q_{ij}(a)\big(\log(1-a)\big)^j$$

with analytic coefficients q_0, q_{ij}. The $\beta(i)$ are of the form

$$\sum_{k\in K,\, k\geq [N\nu^{-1}]} a_k\big((k-\frac{1}{2})\nu - \frac{1}{2}\big)$$

where K consist of finite sets of natural numbers and $a_k \in \mathbb{N}$. Only finitely many of the $q_{ij}(a)$ are non-zero.

The next definition is also taken from [59], except that we formulate it in terms of the variable $R_{0,0}$, which is independent of γ_1, γ_2. This shall be important in clarifying which of the corrections are independent of $\gamma_{1,2}$, and which indeed depend on these variables. Introduce the variables $b(t) = \mu_{0,0}(t)^{-1}$, $\mu_{0,0}(t) = \lambda_{0,0}(t) \cdot t$, as well as b_1, which will represent $\frac{\log t}{\mu_{0,0}(t)} = \frac{\log t}{t\lambda_{0,0}(t)}$. Then

Definition. Let us define the following class of analytic functions:

(a) $S^m(R_{0,0}^k(\log R_{0,0})^l, \mathcal{Q})$ is the class of analytic functions

$$v : [0,\infty) \times [0,1] \times [0,b_0] \times [0,b_0] \longrightarrow \mathbb{R}$$

such that

(i) v is analytic as a function of $R_{0,0}, b, b_1$ and $v : [0,\infty) \times [0,b_0] \times [0,b_0] \longrightarrow \mathcal{Q}$.

(ii) v vanishes of order m relative to R, and $R^{-m}v$ has an even Taylor expansion at $R_{0,0} = 0$.

(iii) v has a convergent expansion at $R_{0,0} = +\infty$.

$$v(R_{0,0}, a, b, b_1) = \sum_{i=0}^{\infty} \sum_{j=0}^{l+i} c_{ij}(a, b, b_1) R_{0,0}^{k-i} (\log R_{0,0})^j$$

where the coefficients $c_{ij}(\cdot, b) \in \mathcal{Q}$ and $c_{ij}(a, b, b_1)$ are analytic in $b, b_1 \in [0, b_0]$ for all $0 \le a \le 1$.

(b) $IS^m(R_{0,0}^k(\log R_{0,0})^l, \mathcal{Q})$ is the class of analytic functions w on the cone C_0 which can be represented as

$$w(r,t) = v(R_{0,0}, a, b, b_1)$$

where $v \in S^m(R_{0,0}^k(\log R_{0,0})^l, \mathcal{Q})$, $b = \frac{1}{\mu_{0,0}(t)}$, $b_1 = \frac{\log t}{\mu_{0,0}(t)}$, $\mu_{0,0}(t) = t \cdot \lambda_{0,0}(t)$.

(c) Define $S^m(R_{0,0}^k(\log R_{0,0})^l, \mathcal{Q}_{smooth}), IS^m(R_{0,0}^k(\log R_{0,0})^l, \mathcal{Q}_{smooth})$ as in (a), (b) above. We shall also use the notation $IS^m(R_{0,0}^k(\log R_{0,0})^l)$ to denote functions analytic in $b, b_1, R_{0,0}$ with the indicated vanishing and decay properties.

Observe that functions in $\mathcal{Q}_{smooth}$ are at least of regularity $C^{N+\frac{1}{2}-\frac{\nu}{2}-}$ at $a = 1$, and we can extend them past the light cone $a = 1$ by replacing $(1-a)$ by $|1-a|$ in the logarithmic terms.

The proof now proceeds by first building a solution u_{prelim} by solving suitable elliptic problems approximating the wave equation (4.1), and finally adding a further correction to produce the u_{approx}, by solving a suitable wave equation via the parametrix method of [62], [59]. The method here in particular makes it clear that when $\gamma_1 = \gamma_2 = 0$ we simply reproduce the solutions if [62], [59]. To construct the preliminary approximate solution, we use

Lemma 112. *For any $k_* := [\frac{1}{2}N\nu^{-1}] \geq k \geq 1$ there exist corrections v_{2k}, v_{2k-1} such that the approximations $u_{2k-1} = u_0 + \sum_{j=1}^{2k-1} v_j$, $u_{2k} = u_0 + \sum_{j=1}^{2k} v_j$ generate errors e_{2k-1}, e_{2k} as below:*

$$v_{2k-1} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{2k}} IS^2(R_{0,0}(\log R_{0,0})^{m_k}, \mathcal{Q}) \tag{4.32}$$

$$t^2 e_{2k-1} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{2k}} IS^0(R_{0,0}(\log R_{0,0})^{p_k}, \mathcal{Q}') \tag{4.33}$$

$$v_{2k} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{2k+2}} IS^2(R_{0,0}^3(\log R_{0,0})^{p_k}, \mathcal{Q}) \tag{4.34}$$

$$t^2 e_{2k} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{2k}} \left[IS^0(R_{0,0}^{-1}(\log R_{0,0})^{q_k}, \mathcal{Q}) + b^2 IS^0(R_{0,0}(\log R_{0,0})^{q_k}, \mathcal{Q}')\right] \tag{4.35}$$

Here the functions v_{2k-1}, v_{2k} are independent of $\gamma_{1,2}$, but not the errors e_{2k-1}, e_{2k}. Furthermore, we may pick two more corrections $v_{smooth,1}, v_{smooth,2}$, such that

$$\partial_{\gamma_1} v_{smooth,1} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} IS^2(R_{0,0}, \mathcal{Q}_{smooth}),$$

$$\partial_{\gamma_2} v_{smooth,1} \in \log t \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} IS^2(R_{0,0}, \mathcal{Q}_{smooth}),$$

$$\partial_{\gamma_1} v_{smooth,2} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+4}} IS^2(R_{0,0}^3, \mathcal{Q}_{smooth}),$$

$$\partial_{\gamma_2} v_{smooth,2} \in \log t \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+4}} IS^2(R_{0,0}^3, \mathcal{Q}_{smooth}),$$

such that the final error generated by $u_{prelim} := u_0 + \sum_{j=1}^{2k_*-1} v_j + \sum_{a=1,2} v_{smooth,a}$ satisfies

$$\begin{aligned} t^2 e_{prelim} &:= t^2(\Box u_{prelim} + u_{prelim}^5) \\ &\in \gamma_1 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} \left[IS^0(R_{0,0}^{-1}, \mathcal{Q}) + b^2 IS^0(R_{0,0}, \mathcal{Q})\right] \\ &+ \gamma_2 \log t \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} \left[IS^0(R_{0,0}^{-1}, \mathcal{Q}) + b^2 IS^0(R_{0,0}, \mathcal{Q})\right] + t^2 \tilde{e}_{prelim}, \end{aligned}$$

where the remaining error $t^2\tilde{e}_{prelim}$ does not depend on $\gamma_{1,2}$ and resides in

$$t^2 \tilde{e}_{prelim} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{2k_*}} IS^0(R_{0,0}(\log R_{0,0})^{p_{k_*}}, \mathcal{Q}')$$

Proof. We follow closely the procedure in [59], section 2. The only novelty is that we perturb around $u_0 = \lambda^{\frac{1}{2}}(t)W(\lambda(t)r)$ as opposed to $\lambda_{0,0}^{\frac{1}{2}}(t)W(\lambda_{0,0}(t)r)$, which will generate additional error terms during the construction of the v_j, $1 \le j \le 2k_* - 1$. We relegate these to the end of the procedure, and use the final two corrections $v_{smooth,a}$ to decimate this remaining error, leaving only e_{prelim}.

Step 0: the bulk term

We put $u_0(t,r) = \lambda^{\frac{1}{2}}(t)W(R)$, $R = \lambda(t)r$, $\lambda(t) = \lambda_{\gamma_1,\gamma_2}(t)$. Then (with $\mathscr{D} = \frac{1}{2} + R\partial_R$)

$$
\begin{aligned}
e_0 &:= \partial_t^2 u_0 = \lambda^{\frac{1}{2}}(t)\Big[\Big(\frac{\lambda'}{\lambda}\Big)^2(t)(\mathscr{D}^2 W)(R) + \Big(\frac{\lambda'}{\lambda}\Big)'(t)(\mathscr{D}W)(R)\Big] \\
t^2 e_0 &=: \lambda_{0,0}^{\frac{1}{2}}(t)\Big[\omega_1 \frac{1 - R_{0,0}^2/3}{(1 + R_{0,0}^2/3)^{\frac{3}{2}}} + \omega_2 \frac{9 - 30R_{0,0}^2 + R_{0,0}^4}{(1 + R_{0,0}^2/3)^{\frac{5}{2}}}\Big] + \epsilon_0 \\
&=: t^2 e_0^0 + \epsilon_0
\end{aligned}
\tag{4.36}
$$

where we have

$$
\epsilon_0 \in \gamma_1 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}} IS^0(R_{0,0}^{-1}) + \gamma_2 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}} \log t IS^0(R_{0,0}^{-1})
$$

Further, importantly the constants $\omega_{1,2}$ do not depend on $\gamma_{1,2}$. We shall then treat ϵ_0 as a lower-order error which can be neglected in the first k_0 stages of the iteration process.

Step 1: choice of the first correction v_1

Introduce the operator

$$
L_0 := \partial_{R_{0,0}}^2 + \frac{2}{R_{0,0}}\partial_{R_{0,0}} + 5W^4(R_{0,0})
$$

Then we solve the ordinary differential equation

$$
\mu_{0,0}^2(t) L_0 v_1 = t^2 e_0^0, \quad v_1(0) = v_1'(0) = 0
$$

Introducing the conjugated operator $\widetilde{L}_0 := R_{0,0} L_0 R_{0,0}^{-1}$, which has fundamental system

$$
\begin{aligned}
\widetilde{\varphi}_1(R_{0,0}) &:= \frac{R_{0,0}(1 - R_{0,0}^2/3)}{(1 + R_{0,0}^2/3)^{\frac{3}{2}}} \\
\widetilde{\varphi}_2(R_{0,0}) &:= \frac{1 - 2R_{0,0}^2 + R_{0,0}^4/9}{(1 + R_{0,0}^2/3)^{\frac{3}{2}}},
\end{aligned}
$$

we find the following expression for v_1:

$$\mu_{0,0}^2(t)\,v_1(t,R_{0,0}) = R_{0,0}^{-1}\Big(\widetilde{\varphi}_1(R_{0,0})\int_0^{R_{0,0}} \widetilde{\varphi}_2(R'_{0,0})R'_{0,0}\,t^2 e_0^0(R'_{0,0})\,dR'_{0,0} - \widetilde{\varphi}_2(R_{0,0})\int_0^{R_{0,0}} \widetilde{\varphi}_1(R'_{0,0})R'\,t^2 e_0^0(R'_{0,0})\,dR'_{0,0}\Big)$$

Then using (4.36), we infer

$$v_1(t,r) = \lambda_{0,0}^{\frac{1}{2}}(t)\mu_{0,0}^{-2}(t)(\omega_1 f_1(R_{0,0}) + \omega_2 f_2(R_{0,0})) =: \lambda_{0,0}^{\frac{1}{2}}(t)\mu_{0,0}^{-2}(t) f(R_{0,0}) \tag{4.37}$$

where further

$$\begin{aligned} f_j(R_{0,0}) &= R_{0,0}(b_{1j} + b_{2j}R_{0,0}^{-1} + R_{0,0}^{-2}\log R_{0,0}\,\varphi_{1j}(R_{0,0}^{-2}) + R_{0,0}^{-2}\varphi_{2j}(R_{0,0}^{-1})) \\ &=: R_{0,0}(F_j(\rho) + \rho^2 G_j(\rho^2)\log\rho) \end{aligned}$$

where $\varphi_{1j},\varphi_{2j}$ and F_j, G_j are analytic around zero, with $\rho := R_{0,0}^{-1}$. Moreover, the coefficients of these analytic functions do not depend on $\gamma_{1,2}$.

Step 2: the e_1 error

Here we analyse the error e_1 generated by the approximate solution $u_1 = u_0 + v_1$, which equals

$$\begin{aligned} e_1 &= \partial_t^2 v_1 - 10u_0^3 v_1^2 - 10u_0^2 v_1^3 - 5u_0 v_1^4 - v_1^5 \\ &+ 5\lambda_{0,0}^2(t)[\frac{\lambda^2(t)}{\lambda_{0,0}^2(t)} W^4(R) - W^4(R_{0,0})]\,v_1 + \epsilon_0. \end{aligned}$$

Using the (4.37), we can write $t^2 e_1$ as a sum as follows

$$t^2 e_1 = \sum_{j=1}^{3} A_j + \epsilon_0,$$

where up to sign, the terms are given by

$$A_1 = \lambda_{0,0}^{\frac{1}{2}}(t)\mu_{0,0}^{-2}(t)\sum_{k=2}^{5}\begin{pmatrix} 5 \\ k \end{pmatrix}\mu_{0,0}^{4-2k}(t)W^{5-k}(R_{0,0})f^k((R_{0,0}),$$

$$A_2 = \lambda_{0,0}^{\frac{1}{2}}(t)\Big(\big(t\partial_t + t\lambda'_{0,0}(t)\lambda_{0,0}^{-1}(t)\mathscr{D}\big)^2 - \big(t\partial_t + t\lambda'_{0,0}(t)\lambda_{0,0}^{-1}(t)\mathscr{D}\big)\Big)\big(\mu_{0,0}^{-2}(t)f(R_{0,0})\big),$$

$$A_3 = \lambda_{0,0}^{\frac{1}{2}}(t)\sum_{k=1}^{4}\begin{pmatrix} 5 \\ k \end{pmatrix}\mu_{0,0}^{2-2k}(t)f^k(R_{0,0})[W^{5-k}(R)\frac{\lambda^{\frac{5-k}{2}}(t)}{\lambda_{0,0}^{\frac{5-k}{2}}(t)} - W^{5-k}(R_{0,0})],$$

with $\mathscr{D} = \frac{1}{2} + R_{0,0}\partial_{R_{0,0}}$. Also, ϵ_0 is as in Step 0. Then we can write

$$\begin{aligned} A_2 &= \lambda_{0,0}^{\frac{1}{2}}(t)\mu_{0,0}^{-2}(t)\big[(2\nu-(1+\nu)\mathscr{D})^2-(2\nu-(1+\nu)\mathscr{D})\big]f(R_{0,0}) \\ &=: \lambda_{0,0}^{\frac{1}{2}}(t)\mu_{0,0}^{-2}(t)g(R_{0,0}), \end{aligned}$$

where the last term on the right admits an expansion like for v_1 in (4.37), with coefficients that are independent of $\gamma_{1,2}$. On the other hand, the term A_3 is dependent on $\gamma_{1,2}$, and can in fact be placed in the space

$$\gamma_1\frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}}IS^0(R_{0,0}^{-1})+\gamma_2\frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}}\log tIS^0(R_{0,0}^{-1})$$

We shall deal with it when we define $v_{smooth,a}$. At any rate, the error e_1 satisfies (4.33) for $k=1$.

Step 3: choice of second correction v_2

Choice of second correction v_2. It is in this step where the shock along the light cone, as evidenced by the expansion (4.5), as well as the definition of $\mathscr{Q}$, is introduced into $u_{\text{approx}}^{(\gamma_{1,2})}$ (whence also into $u_{\text{approx}}^{(0,0)} = u_\nu$, the solutions being described in Theorem 101). The key in this step shall be to ensure that the singular part of v_2 will be independent of $\gamma_{1,2}$. This we can achieve since by our preceding construction the principal part of the error e_1 is independent of $\gamma_{1,2}$. Write

$$e_1 = e_1^0 + t^{-2}\epsilon_1, \epsilon_1 := A_3+\epsilon_0.$$

Then as in [59], equation (2.32), we infer the leading behaviour of the term e_1^0 (where we change the notation with respect to [59]), as follows:

$$t^2e_1^{00}(t,r) := \lambda_{0,0}^{\frac{1}{2}}(t)\mu_{0,0}^{-1}(t)(c_1a+c_2b)$$

where we have $a=\frac{r}{t}$, $b=b(t)=\frac{1}{\mu_{0,0}(t)}$, and as remarked before the coefficients c_j do not depend on $\gamma_{1,2}$. Also, recall

$$\mu_{0,0}(t) = (\lambda_{0,0}(t)\cdot t).$$

The second correction will then be obtained by neglecting the effect of the potential term $5W^4(R)$, and setting

$$t^2\big(v_{2,tt}-v_{2,rr}-\frac{2}{r}v_{2,r}\big) = -t^2e_1^{00}$$

To solve this we make the ansatz

$$v_2(t,r) = \lambda_{0,0}(t)^{\frac{1}{2}}\big(\mu_{0,0}^{-1}(t)q_1(a)+\mu_{0,0}^{-2}(t)q_2(a)\big)$$

In fact, proceeding exactly as in [59], section 2.5, we then infer the equations

$$L_{\frac{\nu-1}{2}}\, q_1 = c_1 a, \quad L_{\frac{3\nu-1}{2}}\, q_2 = c_2,$$

where we set

$$L_\beta := (1-a^2)\partial_a^2 + (2(\beta-1)a + 2a^{-1})\partial_a - \beta^2 + \beta.$$

In fact, our $\lambda_{0,0}, \mu_{0,0}$ are exactly the λ, μ in [59]. To uniquely determine $q_{1,2}$, we impose the vanishing conditions

$$q_j(0) = q_j'(0) = 0,\ j = 1,2.$$

As in [59], equation (2.45), one can then write (using $a = \frac{R_{0,0}}{\mu_{0,0}(t)}$ where $R_{0,0} := r\lambda_{0,0}(t)$)

$$v_2 = \frac{\lambda_{0,0}(t)^{\frac{1}{2}}}{\mu_{0,0}^2(t)}(R_{0,0}\widetilde{q}_1(a) + q_2(a)),$$

where now $\widetilde{q}_1, q_2$ both have even power expansions around $a = 0$. In order to ensure the necessary parity of exponents in the power series expansions around $R_{0,0} = 0$ imposed by the definition of $\mathcal{Q}$, we sacrifice some accuracy in the approximation, relabel the preceding expression $v_2^0(t,r)$ (as in [59]), and then use for the true correction v_2 the formula

$$v_2 = \frac{\lambda_{0,0}(t)^{\frac{1}{2}}}{\mu_{0,0}^2(t)}(R_{0,0}^2\langle R_{0,0}\rangle^{-1}\widetilde{q}_1(a) + q_2(a)),\ \langle R_{0,0}\rangle = \sqrt{R_{0,0}^2 + 1}.$$

Again by construction $\widetilde{q}_1, q_2$ and thence v_2 do not depend on $\gamma_{1,2}$.

Step 4: the e_2 error

Here we analyse the error generated by the approximate solution $u_2 = u_0 + v_1 + v_2$, which is given by the expression

$$\begin{aligned} e_2 &= e_1 - e_1^{00} - 5u_1^4 v_2 - 10u_1^3 v_2^2 - 10u_1^2 v_2^3 - 5u_1 v_2^4 - v_2^5 \\ &\quad + (\partial_{tt} - \partial_{rr} - \frac{2}{r}\partial_r)(v_2 - v_2^0) \end{aligned}$$

Then according to the preceding we have

$$\begin{aligned} &t^2(e_1 - e_1^{00}) - \epsilon_0 \\ &\in O(R_{0,0}^{-1}\lambda_{0,0}(t)^{\frac{1}{2}}\mu_{0,0}^{-2}(t)) + \gamma_1\frac{\lambda^{\frac{1}{2}}(t)}{\mu^{k_0+2}(t)}IS^0(R_{0,0}) + \gamma_2\frac{\lambda^{\frac{1}{2}}}{\mu(t)^{k_0+2}}\log t IS^0(R_{0,0}), \end{aligned}$$

where the first term $O(R_{0,0}^{-1}\lambda_{0,0}(t)^{\frac{1}{2}}\mu_{0,0}^{-2}(t))$ is independent of $\gamma_{1,2}$. The sum of the last two terms on the right will then be deferred until the last stage, when we define $v_{smooth,a}$. Next, consider the term

$$t^2\big[-5u_1^4v_2 - 10u_1^3v_2^2 - 10u_1^2v_2^3 - 5u_1v_2^4 - v_2^5 + (\partial_{tt} - \partial_{rr} - \frac{2}{r}\partial_r)(v_2 - v_2^0)\big]$$

Here the interaction terms $u_1^{5-j}v_2^j$, $j \leq 4$, are only of the smoothness implied by $\mathcal{Q}$, but do depend on $\gamma_{1,2}$ on account of $u_1 = u_0 + v_1$ and the γ-dependence of u_0. However, writing

$$u_1 = [u_0 - \lambda_{0,0}^{\frac{1}{2}}W(\lambda_{0,0}r)] + [v_1 + \lambda_{0,0}^{\frac{1}{2}}W(\lambda_{0,0}r)],$$

and expanding out u_1^{5-j}, we can place any term of the form

$$t^2[u_0 - \lambda_{0,0}^{\frac{1}{2}}W(\lambda_{0,0}r)]^{l_1}[v_1 + \lambda_{0,0}^{\frac{1}{2}}W(\lambda_{0,0}r)]^{l_2}v_2^{l_3}, \sum l_j = 5,$$

and with $l_1 \geq 1, l_3 \geq 1$ into

$$\begin{aligned}&\gamma_1\frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}}\big[IS^0(R_{0,0}^{-1},\mathcal{Q}) + b^2IS^0(R_{0,0},\mathcal{Q})\big]\\ &+\gamma_2\log t\frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}}\big[IS^0(R_{0,0}^{-1},\mathcal{Q}) + b^2IS^0(R_{0,0},\mathcal{Q})\big],\end{aligned}$$

and so this can be placed into t^2e_{prelim}. Finally, the preceding also implies (4.35) for $k = 1$.

Step 5: inductive step

The inductive step. Here we again follow [59], section 2.7, closely, but need to carefully keep track of various parts of e_k First consider the case of even indices, i. e. assume e_{2k-2}, $2 \leq k \leq k_*$, satisfies (4.35) with k replaced by $k-1$, and more precisely, that we can decompose

$$e_{2k-2} = e_{2k-2}^1 + e_{2k-2}^2 + e_{2k-2}^3, \tag{4.38}$$

where we have

$$\begin{aligned}&t^2e_{2k-2}^1 \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{2k-2}}\big[IS^0(R_{0,0}^{-1}(\log R_{0,0})^{q_{k-1}},\mathcal{Q}) + b^2IS^0(R_{0,0}(\log R_{0,0})^{q_{k-1}},\mathcal{Q}')\big],\\ &t^2e_{2k-2}^2 \in \gamma_1\frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}}IS^0(R_{0,0}^{-1}) + \gamma_2\frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}}\log tIS^0(R_{0,0}^{-1}),\end{aligned}$$

the term e^1_{2k-2} being independent of $\gamma_{1,2}$, while for the third term we have

$$t^2 e^3_{2k-2} \in \gamma_1 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} \left[IS^0(R^{-1}_{0,0}, \mathcal{Q}) + b^2 IS^0(R_{0,0}, \mathcal{Q}) \right] + \gamma_2 \log t \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} \left[IS^0(R^{-1}_{0,0}, \mathcal{Q}) + b^2 IS^0(R_{0,0}, \mathcal{Q}) \right].$$

We have verified such a structure for the case $k = 2$ in the preceding step. Then we introduce the correction v_{2k-1} in order to improve the error e^1_{2k-1}, exactly mirroring Step 1 in section 2.7 of [59]. We completely forget about e^3_{2k-2} as it can be moved into the final error e_{prelim}, while we shall deal with the intermediate term e^2_{2k-2} when introducing $v_{smooth,a}$. Returning to v_{2k-1}, and proceeding just as in Step 1, we see that v_{2k-1} will satisfy (4.32), and moreover be independent of $\gamma_{1,2}$. The error e_{2k-1} generated by the approximation $u_0 + \sum_{j=1}^{2k-1} v_j$ will be mostly independent of $\gamma_{1,2}$, and satisfy (4.33), except for the cross interaction terms of v_{2k-1} and u_0, of the form $u_0^{5-j} v^j_{2k-1}$, $1 \leq j \leq 4$. However, splitting

$$u_0 = [u_0 - \lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0}(t) r)] + [\lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0}(t) r)],$$

we may replace u_0 by $u_0 - \lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0}(t) r)$, and then the corresponding cross interactions, multiplied by t^2, can again be seen to be in

$$\gamma_1 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} \left[IS^0(R^{-1}_{0,0}, \mathcal{Q}) + b^2 IS^0(R_{0,0}, \mathcal{Q}) \right] + \gamma_2 \log t \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} \left[IS^0(R^{-1}_{0,0}, \mathcal{Q}) + b^2 IS^0(R_{0,0}, \mathcal{Q}) \right],$$

whence these error terms may be placed into e_{prelim} and discarded.

The case of odd indices, i. e. departing from e_{2k-1}, $k \leq k_*$, is handled just the same.

Repeating this procedure leads to the v_j, $1 \leq j \leq 2k_* - 1$. Moreover, each of the errors generated satisfies a decomposition analogous to (4.38), replacing (4.35) by (4.33) for odd indices.

Step 6: choice of $v_{smooth,a}$, $a = 1, 2$

Here we depart from the approximation $u_{2k_*-1} = u_0 + \sum_{j=1}^{2k_*-1} v_j$, which generates an error e_{2k_*-1} satisfying (4.33) for $k = k_*$, as well as a decomposition

$$e_{2k_*-1} = \sum_{j=1}^{3} e^j_{2k_*-1} \tag{4.39}$$

analogous to (4.38). Importantly, the first error

$$t^2 e^1_{2k_*-1} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{2k_*}} IS^0(R_{0,0}(\log R_{0,0})^{p_{k_*}}, \mathcal{Q}')$$

is independent of $\gamma_{1,2}$, and the last error $e^3_{2k_*-1}$ may be placed into e_{prelim}, and so it remains to deal with the middle error which for technical reasons is still too large. Recall that the middle error satisfies

$$t^2 e^2_{2k_*-1} \in \gamma_1 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}} IS^0(R_{0,0}^{-1}) + \gamma_2 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0}} \log t IS^0(R_{0,0}^{-1}),$$

and in particular is C^∞-smooth. Then set

$$\mu^2_{0,0}(t) L_0 v_{smooth,1} = t^2 e^2_{2k_*-1},$$

leading to

$$v_{smooth,1} \in \gamma_1 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} IS^2(R_{0,0}) + \gamma_2 \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} \log t IS^2(R_{0,0})$$

Then all errors generated by $v_{smooth,1}$ by interaction with the bulk part u_{2k_*-1} can be placed into e_{prelim}. On the other hand, the error $t^2\partial_t^2 v_{smooth,1}$ is of the same form as $v_{smooth,1}$. We next construct $v_{smooth,2}$, proceeding in analogy to Step 3, to improve the error generated by $\partial_t^2 v_{smooth,1}$. The key here is that on the account of the rapid temporal decay of this term, the method of [59] applied to it results in a term of sufficient smoothness, to be acceptable for a correction depending on $\gamma_{1,2}$. Specifically, we write the leading order term of $t^2\partial_t^2 v_{smooth,1}$ in the form

$$(c_1 + c_3 \log t) \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} R_{0,0} + (c_2 + c_4 \log t) \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}},$$

and then set (where the coefficients $c_{1,2}$ depend on $\gamma_{1,2}$)

$$\begin{aligned} &t^2\big(\partial_t^2 v_{smooth,2} - \partial_r^2 v_{smooth,2} - \frac{2}{r}\partial_r v_{smooth,2}\big) \\ &= (c_1 + c_3 \log t) \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}} R_{0,0} + (c_2 + c_4 \log t) \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+2}}. \end{aligned}$$

Making the correct ansatz as in [59] this is solved by

$$v_{smooth,2} \in \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+4}} IS^2(R^3_{0,0}, \mathcal{Q}_{smooth}) + \log t \frac{\lambda_{0,0}^{\frac{1}{2}}}{\mu_{0,0}(t)^{k_0+4}} IS^2(R^3_{0,0}, \mathcal{Q}_{smooth}).$$

The effect of this correction is that we replace the middle term in (4.39) by one in e_{prelim}, i. e. our final approximate solution

$$u_{\text{prelim}} := u_0 + \sum_{j=1}^{2k_*-1} v_j + \sum_{a=1,2} v_{smooth,a}$$

generates an error e_{prelim} as claimed in the lemma.

□

In order to complete the proof of the Theorem 111, we need to improve the approximate solution obtained in the preceding lemma a bit in order to replace the generated error e_{prelim} by one which is smoother. More precisely, we need to get rid of the rough part of the error $\widetilde{e}_{\text{prelim}}$. For this, we replace u_{prelim} by

$$u_{approx} := u_{\text{prelim}} + v,$$

where v solves the equation

$$\Box v + 5\widetilde{u}_{\text{prelim}}^4 v + \sum_{2\leq j\leq 5} \binom{5}{j} v^j \widetilde{u}_{\text{prelim}}^{5-j} = -\widetilde{e}_{\text{prelim}},$$

where

$$\widetilde{u}_{\text{prelim}} = u_{\text{prelim}} - v_{smooth} + \lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0}(t)r) - \lambda^{\frac{1}{2}} W(\lambda(t)r),\ v_{smooth} = \sum_{a=1}^{2} v_{smooth,a}$$

is the γ-independent part of u_{prelim}. Also, we shall impose vanishing of v at $t = 0$. Then it is clear that v will not depend on $\gamma_{1,2}$. The fact that such a v can be computed with the required smoothness and bounds, provided N is chosen large enough, follows exactly as in [62], see the discussion there after equation (3.1). Also, we have for any $t \in (0, t_0]$

$$\|\nabla_{t,x} v(t)\|_{H^{\frac{\nu}{2}-}} \lesssim t^{N-3}$$

Then we arrive at the error

$$\begin{aligned}
&\Box u_{approx} + u_{approx}^5 \\
&= \Box u_{\text{prelim}} + u_{\text{prelim}}^5 + \sum_{2\leq j\leq 5} \binom{5}{j} v^j u_{\text{prelim}}^{5-j} \\
&+ \Box v + 5\widetilde{u}_{\text{prelim}}^4 v \\
&+ 5(-\widetilde{u}_{\text{prelim}}^4 + u_{\text{prelim}}^4) v
\end{aligned}$$

It follows that

$$e_{approx} = e_{\text{prelim}} - \widetilde{e}_{\text{prelim}} + \sum_{2\le j\le 5} \binom{5}{j} v^j [u_{\text{prelim}}^{5-j} - \widetilde{u}_{\text{prelim}}^{5-j}] \\ + 5(-\widetilde{u}_{\text{prelim}}^4 + u_{\text{prelim}}^4)\, v \tag{4.40}$$

This remaining error is easily seen to satisfy the claimed properties of Theorem 111.

4.4 Modulation theory; determination of the parameters $\gamma_{1,2}$.

4.4.1 Re-scalings and the distorted Fourier transform

The discussion following (4.26) shows that we intend to pass to a slightly altered coordinate system, depending on the parameters $\gamma_{1,2}$, and given by

$$\tau_{\gamma_1,\gamma_2}, R_{\gamma_1,\gamma_2}$$

differing from the old one which corresponded to $\tau_{0,0}, R_{0,0}$ (and which served as the basis for the discussion following (4.14)). We then have to reinterpret functions given in terms of $R_{0,0}$ as functions in terms of R_{γ_1,γ_2}, and understand the effect of such a change of scale on the distorted Fourier transform. Infinitesimally, this is explained in terms of Theorem 106, and we state here a simple variation on this theme:

Lemma 113. *Assume $\widetilde{\epsilon}$ has the Fourier representation given above. Then we have the formula*

$$\mathscr{F}\big(\widetilde{\epsilon}(e^{-\kappa}R)\big)(\xi) = x(e^{2\kappa}\xi) + \kappa \cdot \widetilde{\mathscr{K}}_\kappa x + O_{\|\cdot\|_{\widetilde{S}_1}}(\kappa|x_d|)$$

where $\widetilde{\mathscr{K}}_\kappa$ can be written as

$$\big(\widetilde{\mathscr{K}}_\kappa x\big)(\xi) = h_\kappa(\xi)x(e^{2\kappa}\xi) + \int_0^\infty \frac{F_\kappa(\xi,\eta)\rho(\eta)}{\xi-\eta} x(\eta)\, d\eta$$

where F_κ satisfies the same bounds as the function $F(\cdot,\cdot)$ in Theorem 106, and the function h_κ is of class $C^\infty(0,\infty)$ and uniformly bounded (both in ξ as well as κ), and satisfies symbol type bounds with respect to ξ.

In particular, we have

$$\|\mathscr{F}\big(\widetilde{\epsilon}(e^{\kappa}R)\big)\|_{\widetilde{S}_1} \lesssim_{\tau_0,\kappa} (\|x\|_{\widetilde{S}_1} + |x_d|).$$

and more precisely, we have

$$\|\mathscr{F}\big(\widetilde{\epsilon}(e^{\kappa}R)\big) - \big(\mathscr{F}(\widetilde{\epsilon})\big)(e^{2\kappa}\xi)\|_{\widetilde{S}_1} \lesssim_{\tau_0} \kappa(\|x\|_{\widetilde{S}_1} + |x_d|).$$

as well as

$$\|\mathscr{F}\big(\widetilde{\epsilon}(e^{\kappa}R)\big)\|_{\widetilde{S}_1} \lesssim (1+\tau_0\kappa)\|\mathscr{F}\big(\widetilde{\epsilon}(R)\big)\|_{\widetilde{S}_1} + \kappa|x_d|.$$

Proof. This is entirely analogous to the proof of Theorem 5.1 in [62]; in effect the latter deals with the 'infinitesimal version' of the current situation. Consider the expression

$$(\Xi_\kappa x)(\eta) := \langle \int_0^\infty x(\xi)\phi(e^{-\kappa}R, e^{2\kappa}\xi)\rho(\xi)\,d\xi,\, \phi(R,\eta)\rangle,$$

where $x \in C_0^\infty(0,\infty)$. Under the latter restriction the integral converges absolutely. Then proceeding as in [62], see in particular Lemma 4.6 and the proof of Theorem 5.1, we get

$$(\Xi_\kappa x)(\xi) = \mathrm{Re}\,[\frac{a_+(e^{2\kappa}\xi)}{a_+(\xi)}]\,x(\xi) + \int_0^\infty f_\kappa(\xi,\eta)x(\eta)\,d\eta,$$

where a_+ is the function occurring in Prop. 103. Here in order to determine the kernel f_κ of the 'off-diagonal' operator at the end, we use

$$\begin{aligned}&(\eta-\xi)f_\kappa(\xi,\eta)\\ &= \langle \int_0^\infty x(\xi)5[e^{-2\kappa}W^4(e^{-\kappa}R) - W^4(R)]\phi(e^{-\kappa}R, e^{2\kappa}\xi)\rho(\xi)\,d\xi,\, \phi(R,\eta)\rangle\end{aligned}$$

Then by following the argument of [62], proof of Theorem 5.1, one infers that

$$f_\kappa(\xi,\eta) = \kappa\cdot\frac{\rho(\eta)F_\kappa(\xi,\eta)}{\xi-\eta},$$

with F_κ having the same asymptotic and vanishing properties as the kernel $F(\xi,\eta)$ in Theorem 106, uniformly in $\kappa\in[0,1]$, say. It remains to translate the properties of Ξ_κ to those of the re-scaling operator. Let Ψ be the operator which satisfies

$$\mathscr{F}\big(\Psi(\widetilde{\epsilon})\big)(\xi) = e^{-2\kappa}\frac{\rho(\frac{\xi}{e^{2\kappa}})}{\rho(\xi)}x(\frac{\xi}{e^{2\kappa}})$$

and leaves the discrete spectral part invariant, while $S_{e^{-\kappa}}(\widetilde{\epsilon})(R) = \widetilde{\epsilon}(\frac{R}{e^\kappa})$ is the scaling operator. Then we have

$$(\Xi_\kappa x)(\xi) = \mathscr{F}\big(S_{e^{-\kappa}}\Psi(\widetilde{\epsilon})\big)(\xi). + O_{\|\cdot\|_{\widetilde{S}_1}}(\kappa|x_d|).$$

We conclude that

$$\mathscr{F}\big(S_{e^{-\kappa}}\widetilde{\epsilon}\big)(\xi) = \Xi_\kappa\big(\mathscr{F}\big(\Psi^{-1}(\widetilde{\epsilon})\big)\big) + O_{\|\cdot\|_{\widetilde{S}_1}}(\kappa|x_d|).$$

It follows that we can write

$$\begin{aligned}\mathscr{F}\big(S_{e^{-\kappa}}\widetilde{\epsilon}\big)(\xi) =& x(e^{2\kappa}\xi) + \big[e^{2\kappa}\,\mathrm{Re}\,[\frac{a_+(e^{2\kappa}\xi)}{a_+(\xi)}]\cdot\frac{\rho(e^{2\kappa}\xi)}{\rho(\xi)} - 1\big]x(e^{2\kappa}\xi)\\ &+ \int_0^\infty \widetilde{f}_\kappa(\xi,\eta)x(\eta)\,d\eta + O_{\|\cdot\|_{\widetilde{S}_1}}(\kappa|x_d|),\end{aligned}$$

where we put

$$\widetilde{f}_\kappa(\xi,\eta) := f_\kappa(\xi,\frac{\eta}{e^{2\kappa}})\cdot\frac{\rho(\eta)}{\rho(\frac{\eta}{e^{2\kappa}})}.$$

This implies the claims of the lemma.

□

4.4.2 The effect of scaling the bulk part

Here we investigate how changing the bulk part from $\lambda_{0,0}^{\frac{1}{2}}W(\lambda_{0,0}r)$ to $\lambda^{\frac{1}{2}}W(\lambda r)$ affects the distorted Fourier transform of the new perturbation term. Specifically, recall (4.25), which defines a new data pair $(\overline{\epsilon}_0, \overline{\epsilon}_1)$, which in turn uniquely define a new quadruple of Fourier components $(x_0^{(\gamma_1,\gamma_2)}, x_{0d}^{(\gamma_1,\gamma_2)}, x_1^{(\gamma_1,\gamma_2)}, x_{1d}^{(\gamma_1,\gamma_2)})$, via (4.27), (4.28), (4.29). We can then derive the analogue of (4.16), and try to repeat the iterative process in [49], but for this we shall have to ensure the two key vanishing conditions (4.30), as well as the condition (4.21). That this is indeed possible is the content of the following

Proposition 114. *Given a fixed* $\nu \in (0, \nu_0]$, $t_0 \in (0,1]$, *there is a* $\delta_1 = \delta_1(\nu, t_0) > 0$ *small enough such that the following holds. Given a triple of data*

$$(x_0(\xi), x_1(\xi), x_{0d}) \in \widetilde{S} \times \mathbb{R}$$

and with

$$\|(x_0, x_1)\|_{\widetilde{S}} + |x_{0d}| < \delta_1,$$

there is a unique pair $\gamma_{1,2}$ *with* $|\gamma_1| + |\gamma_2| \lesssim_{\nu, t_0} \|(x_0, x_1)\|_{\widetilde{S}}$ *and a unique parameter* x_{1d} *satisfying* $|x_{1d}| \lesssim_\nu |x_{0d}|$ *such that determining* (ϵ_0, ϵ_1) *via* (4.10), (4.11), (4.12), *and from there* $(\overline{\epsilon}_0, \overline{\epsilon}_1)$ *via* (4.25) *which in turn defines the quadruple of Fourier data* $(x_0^{(\gamma_1,\gamma_2)}, x_{0d}^{(\gamma_1,\gamma_2)}, x_1^{(\gamma_1,\gamma_2)}, x_{1d}^{(\gamma_1,\gamma_2)})$, *we have*

$$A(\gamma_1, \gamma_2) := \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi) x_1^{(\gamma_1,\gamma_2)}(\xi)}{\xi^{\frac{3}{4}}} \sin[\lambda_{\gamma_1,\gamma_2}(\tau_0)\xi^{\frac{1}{2}} \int_{\tau_0}^\infty \lambda_{\gamma_1,\gamma_2}^{-1}(s)\,ds]\,d\xi = 0$$

$$B(\gamma_1, \gamma_2) := \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi) x_0^{(\gamma_1,\gamma_2)}(\xi)}{\xi^{\frac{1}{4}}} \cos[\lambda_{\gamma_1,\gamma_2}(\tau_0)\xi^{\frac{1}{2}} \int_{\tau_0}^\infty \lambda_{\gamma_1,\gamma_2}^{-1}(s)\,ds]\,d\xi = 0,$$

and the discrete spectral part $(x_{0d}^{(\gamma_1,\gamma_2)}, x_{1d}^{(\gamma_1,\gamma_2)})$ *satisfies the vanishing property of Lemma 109,* (4.21), *with respect to the scaling law* $\lambda = \lambda_{\gamma_1,\gamma_2}$. *We have the precise bound*

$$|\gamma_1 \lambda_{0,0}^{\frac{1}{2}} t_0^{k_0\nu}| + |\gamma_2 \lambda_{0,0}^{\frac{1}{2}} \log t_0\, t_0^{k_0\nu}| \lesssim \tau_0 \log \tau_0 (\|(x_0, x_1)\|_{\widetilde{S}} + |x_{0d}|). \tag{4.41}$$

Finally, we have the bound

$$\|x_0^{(\gamma_1,\gamma_2)} - \frac{\lambda_{0,0}}{\lambda} S_{\frac{\lambda_{0,0}^2}{\lambda^2}} x_0\|_{\widetilde{S}_1} + \|x_1^{(\gamma_1,\gamma_2)} - \frac{\lambda_{0,0}}{\lambda} S_{\frac{\lambda_{0,0}^2}{\lambda^2}} x_1\|_{\widetilde{S}_2} \lesssim \log \tau_0 \cdot \tau_0^{0+} \cdot (\|(x_0, x_1)\|_{\widetilde{S}} + |x_{0d}|).$$

where $S_{\frac{\lambda_{0,0}^2}{\lambda^2}} x_i(\xi) = x_i(\frac{\lambda_{0,0}^2}{\lambda^2}\xi)$ is the scaling operator.

Proof. The strategy shall be to first fix the discrete spectral part to (x_{0d}, x_{1d}) while choosing $\gamma_{1,2}$, and at the end finalising the choice of x_{1d} to satisfy the required co-dimension one condition.

Observe that from our definition and the structure of $u_{approx}^{(\gamma_1,\gamma_2)}$, in particular Lemma 112, and the end of the proof of Theorem 111, we can write

$$\overline{\epsilon}_0 = \chi_{r\leq Ct_0}\big[\lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0} r) - \lambda_{\gamma_1,\gamma_2}^{\frac{1}{2}} W(\lambda_{\gamma_1,\gamma_2} r) - v_{smooth}\big] + \epsilon_0, \tag{4.42}$$

as well as

$$\overline{\epsilon}_1 = \chi_{r\leq Ct_0}\big[\partial_t\big[\lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0} r) - \lambda_{\gamma_1,\gamma_2}^{\frac{1}{2}} W(\lambda_{\gamma_1,\gamma_2} r)\big] - \partial_t v_{smooth}\big] + \epsilon_1, \tag{4.43}$$

where we have introduced the notation $v_{smooth} = \sum_{a=1,2} v_{smooth,a}$, the latter as in the statement of Lemma 112. Also, it is implied that the expressions gets evaluated at $t = t_0$.

We shall think of $\overline{\epsilon}_0, \overline{\epsilon}_1$ as functions of $R = \lambda_{\gamma_1,\gamma_2}(t_0) r$, and we shall keep the latter definition of R for the rest of the paper, as this is the correct variable to use for the sequel.

Observe that setting

$$\widetilde{x}_0^{(\gamma_1,\gamma_2)}(\xi) = \int_0^\infty \phi(R,\xi) R\epsilon_0(R_{0,0}(R))\, dR,\ \widetilde{x}_{0d}^{(\gamma_1,\gamma_2)} = \int_0^\infty \phi_d(R) R\epsilon_0(R_{0,0}(R))\, dR$$

$$\begin{aligned}\widetilde{x}_1^{(\gamma_1,\gamma_2)}(\xi) = &-\lambda_{\gamma_1,\gamma_2}^{-1}|_{t=t_0} \int_0^\infty \phi(R,\xi) R\epsilon_1(R_{0,0}(R))\, dR - \frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}|_{t=t_0}(\mathcal{K}_{cc}\widetilde{x}_0^{(\gamma_1,\gamma_2)})(\xi)\\ &- \frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}|_{t=t_0}(\mathcal{K}_{cd}\widetilde{x}_{0d}^{(\gamma_1,\gamma_2)})(\xi),\end{aligned}$$

and finally (recall (4.29))

$$\begin{aligned}\widetilde{x}_{1d}^{(\gamma_1,\gamma_2)} = &-\lambda_{\gamma_1,\gamma_2}^{-1}|_{t=t_0} \int_0^\infty \phi_d(R) R\epsilon_1(R_{0,0}(R))\, dR - \frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}|_{t=t_0}\mathcal{K}_{dd}\widetilde{x}_{0d}^{(\gamma_1,\gamma_2)}\\ &- \frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}|_{t=t_0}\mathcal{K}_{dc}\widetilde{x}_0^{(\gamma_1,\gamma_2)},\end{aligned}$$

then using Lemma 113, we have

$$\|\widetilde{x}_0^{(\gamma_1,\gamma_2)}(\xi) - \frac{\lambda}{\lambda_{0,0}} x_0(\frac{\lambda^2}{\lambda_{0,0}^2}\xi)\|_{\widetilde{S}_1} \lesssim_{\tau_0} |\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|[\|x_0\|_{\widetilde{S}_1} + |x_{0d}|],$$

while we directly infer the bound

$$|\widetilde{x}_{0d}^{(\gamma_1,\gamma_2)} - x_{0d}| \lesssim |\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|(\tau_0 \|x_0\|_{\widetilde{S}_1} + |x_{0d}|).$$

Similarly, we obtain

$$\begin{aligned}
&\|\widetilde{x}_1^{(\gamma_1,\gamma_2)}(\xi) - \frac{\lambda}{\lambda_{0,0}} x_1(\frac{\lambda^2}{\lambda_{0,0}^2}\xi)\|_{\widetilde{S}_2} \\
&\lesssim |\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|[\|x_1\|_{\widetilde{S}_2} + \|x_0\|_{\widetilde{S}_1} + |x_{1d}| + \tau_0^{-1}|x_{0d}|],
\end{aligned}$$

as well as

$$|\widetilde{x}_{1d}^{(\gamma_1,\gamma_2)} - x_{1d}| \lesssim |\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|(\tau_0 \|x_0\|_{\widetilde{S}_1} + |x_{0d}|).$$

taking advantage of the structure of $\mathscr{K}_{cc}, \mathscr{K}_{cd}$ as detailed in Theorem 106. Recall the quantities in (4.22)

$$B := \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_0(\xi)}{\xi^{\frac{1}{4}}} \cos[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi,\ A := \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_1(\xi)}{\xi^{\frac{3}{4}}} \sin[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi,$$

and thus formulated in terms of the original data $x_{0,1}(\xi)$, and independent of $\gamma_{1,2}$. Then denoting by $\widetilde{A}(\gamma_1,\gamma_2)$, resp. $\widetilde{B}(\gamma_1,\gamma_2)$ the quantity defined like $A(\gamma_1,\gamma_2)$, $B(\gamma_1,\gamma_2)$ in the statement of the proposition, but with $x_j^{(\gamma_1,\gamma_2)}$ replaced by $\widetilde{x}_j^{(\gamma_1,\gamma_2)}$, $j = 1,0$, we infer after a change of variables that

$$\widetilde{A}(\gamma_1,\gamma_2) = A + O\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\tau_0[\|x_1\|_{\widetilde{S}_2} + \tau_0^{-1}\|x_0\|_{\widetilde{S}_1} + |x_{1d}| + \tau_0^{-1}|x_{0d}|]\big), \tag{4.44}$$

$$\widetilde{B}(\gamma_1,\gamma_2) = B + O\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\tau_0[\|x_0\|_{\widetilde{S}_1} + \tau_0^{-1}|x_{0d}|]\big), \tag{4.45}$$

Finally, in light of (4.42), (4.43), introduce the Fourier transforms of the 'bulk part differences'

$$\widetilde{\widetilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi) = \int_0^\infty \phi(R,\xi)R\chi_{r\le Ct_0}\big[\lambda_{0,0}^{\frac{1}{2}}W(\lambda_{0,0}r) - \lambda_{\gamma_1,\gamma_2}^{\frac{1}{2}}W(\lambda_{\gamma_1,\gamma_2}r) - v_{smooth}\big]\,dR,$$

$$\begin{aligned}
\widetilde{\widetilde{x}}_1^{(\gamma_1,\gamma_2)}(\xi) = -\lambda_{\gamma_1,\gamma_2}^{-1}\int_0^\infty \phi(R,\xi)R\chi_{r\le Ct_0}\big[\partial_t\big[\lambda_{0,0}^{\frac{1}{2}}W(\lambda_{0,0}r) \\
- \lambda_{\gamma_1,\gamma_2}^{\frac{1}{2}}W(\lambda_{\gamma_1,\gamma_2}r)\big] - \partial_t v_{smooth}\big]\,dR,
\end{aligned}$$

and label their contributions to the expressions $A(\gamma_1,\gamma_2)$, $B(\gamma_1,\gamma_2)$, by

$$\tilde{\tilde{A}}(\gamma_1,\gamma_2) := \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\tilde{\tilde{x}}_1^{(\gamma_1,\gamma_2)}(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda_{\gamma_1,\gamma_2}(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda_{\gamma_1,\gamma_2}^{-1}(s)\,ds]\,d\xi = 0$$
$$\tilde{\tilde{B}}(\gamma_1,\gamma_2) := \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\tilde{\tilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda_{\gamma_1,\gamma_2}(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda_{\gamma_1,\gamma_2}^{-1}(s)\,ds]\,d\xi = 0.$$

Then the first two vanishing conditions of the proposition can be formulated as

$$0 = A(\gamma_1,\gamma_2) = \tilde{\tilde{A}}(\gamma_1,\gamma_2) + \tilde{A}(\gamma_1,\gamma_2),\ 0 = B(\gamma_1,\gamma_2) = \tilde{\tilde{B}}(\gamma_1,\gamma_2) + \tilde{B}(\gamma_1,\gamma_2),$$

and so, in light of (4.44), (4.45), we find

$$\tilde{\tilde{A}}(\gamma_1,\gamma_2) = -A + O\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\tau_0[\|x_1\|_{\tilde{S}_2} + \tau_0^{-1}\|x_0\|_{\tilde{S}_1} + \tau_0^{-1}|x_{0d}| + |x_{1d}|]\big), \tag{4.46}$$

$$\tilde{\tilde{B}}(\gamma_1,\gamma_2) = -B + O\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\tau_0[\|x_0\|_{\tilde{S}_1} + \tau_0^{-1}|x_{0d}|]\big). \tag{4.47}$$

It remains to compute $\tilde{\tilde{A}}(\gamma_1,\gamma_2)$, $\tilde{\tilde{B}}(\gamma_1,\gamma_2)$ in terms of $\gamma_{1,2}$, which we now do: we can write

$$\begin{aligned}\lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0} r) - \lambda_{\gamma_1,\gamma_2}^{\frac{1}{2}} W(\lambda_{\gamma_1,\gamma_2} r) &= O(|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|)\lambda_{0,0}^{\frac{1}{2}}\phi(R,0)\\ &\quad + O(|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|^2),\end{aligned} \tag{4.48}$$

Further, we find after writing $\xi\phi(R,\xi) = \mathscr{L}\phi(R,\xi)$ and performing integration by parts

$$|\int_0^\infty \phi(R,\xi)\chi_{r\le C t_0}\lambda_{0,0}^{\frac{1}{2}}\phi(R,0)\,dR| \lesssim_N \lambda_{0,0}^{\frac{1}{2}}\frac{C\tau_0}{\langle C\tau_0\xi^{\frac{1}{2}}\rangle^N}, \tag{4.49}$$

whence we infer

$$\begin{aligned}&|\int_0^\infty \phi(R,\xi)\chi_{r\le C t_0} R\big[\lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0} r) - \lambda_{\gamma_1,\gamma_2}^{\frac{1}{2}} W(\lambda_{\gamma_1,\gamma_2} r)\big]\,dR|\\ &\lesssim_N \lambda_{0,0}^{\frac{1}{2}}\frac{C\tau_0}{\langle C\tau_0\xi^{\frac{1}{2}}\rangle^N}|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}| + |\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|^2.\end{aligned} \tag{4.50}$$

We also have the important *non-degeneracy property*

$$\begin{aligned}&\lim_{R\to 0} R^{-1}\chi_{R\le C\tau_0} R\big[\lambda_{0,0}^{\frac{1}{2}} W(\lambda_{0,0} r) - \lambda_{\gamma_1,\gamma_2}^{\frac{1}{2}} W(\lambda_{\gamma_1,\gamma_2} r)\big]\\ &= \frac{1}{2}\lambda_{0,0}^{\frac{1}{2}}[\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}] + O\big(\lambda_{0,0}^{\frac{1}{2}}[\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}]^2\big),\end{aligned} \tag{4.51}$$

in the sense that the principal term on the depends linearly on $\gamma_{1,2}$ with non-vanishing factor.

This is to be contrasted with the *vanishing property*

$$\lim_{R\to 0} R^{-1}\chi_{R\leq C\tau_0} R v_{smooth}(R) = 0, \tag{4.52}$$

which follows from Lemma 112, and finally, by another integration by parts argument similar to the one for the bulk term to get (4.50), and exploiting the fine structure of v_{smooth} from Lemma 112, we get

$$\int_0^\infty \phi(R,\xi)R\chi_{R\leq C\tau_0} v_{smooth}(R)|_{t=t_0}\,dR = |\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0\, t_0^{k_0\nu}| O_N\big(\lambda_{0,0}^{\frac{1}{2}} \frac{C\tau_0}{\langle C\tau_0\xi^{\frac{1}{2}}\rangle^N}\big) \tag{4.53}$$

Finally, using the precise asymptotic relation $\lim_{\xi\to 0}\xi^{\frac{1}{2}}\rho(\xi) = c$ where in fact one has $c = \frac{1}{3\pi}$, see Lemma 3.4 in [17], we infer that

$$\begin{aligned}
\tilde{\tilde{B}}(\gamma_1,\gamma_2) &= \int_0^\infty \frac{\tilde{\tilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi)\rho^{\frac{1}{2}}(\xi)}{\xi^{\frac{1}{4}}}\cos[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi + O_{t_0}\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\cdot t_0^{k_0\nu}|^2\big)\\
&= \lim_{R\to 0} cR^{-1}\int_0^\infty \phi(R,\xi)\tilde{\tilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi)\rho(\xi)\,d\xi\\
&+ \int_0^\infty \tilde{\tilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi)[\frac{\rho^{\frac{1}{2}}(\xi)}{\xi^{\frac{1}{4}}} - c\rho(\xi)]\cos[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi\\
&+ c\int_0^\infty \tilde{\tilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi)\rho(\xi)(\cos[\nu\tau_0\xi^{\frac{1}{2}}]-1)\,d\xi\\
&+ O_{t_0}\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\cdot t_0^{k_0\nu}|^2\big).
\end{aligned} \tag{4.54}$$

Observe that the extra term $O_{t_0}\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\cdot t_0^{k_0\nu}|^2\big)$ arises from replacing

$$\lambda_{\gamma_1,\gamma_2}(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda_{\gamma_1,\gamma_2}^{-1}(s)\,ds$$

by $\nu\tau_0$. The last term on the right in the above identity is essentially quadratic and negligible in the sequel. The second and third terms are also negligible on account of the bounds (4.50), (4.53) from before for the Fourier transform of the bulk part as well as v_{smooth}: for the second term, we get (for suitable $c > 0$)

$$\Big|\int_0^\infty \tilde{\tilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi)[\frac{\rho^{\frac{1}{2}}(\xi)}{\xi^{\frac{1}{4}}} - c\rho(\xi)]\cos[\nu\tau_0\xi^{\frac{1}{2}}]\,d\xi\Big| \lesssim \lambda_{0,0}^{\frac{1}{2}}\tau_0^{-1}|\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\, t_0^{k_0\nu}|,$$

while the third term becomes small upon choosing C sufficiently large:

$$\begin{aligned}
&\Big|\int_0^\infty \tilde{\tilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi)\rho(\xi)(\cos[\nu\tau_0\xi^{\frac{1}{2}}]-1)\,d\xi\Big|\\
&\lesssim C^{-1}\lambda_{0,0}^{\frac{1}{2}}|\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\, t_0^{k_0\nu}|
\end{aligned}$$

Finally, for the first term above, we have according to the earlier limiting relations (4.51), (4.52) the key relation

$$\lim_{R\to 0} R^{-1}\int_0^\infty \phi(R,\xi)\tilde{\tilde{x}}_0^{(\gamma_1,\gamma_2)}(\xi)\rho(\xi)\,d\xi$$
$$= \frac{1}{2}\lambda_{0,0}^{\frac{1}{2}}[\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}] + O\big(\lambda_{0,0}^{\frac{1}{2}}[\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}]^2\big).$$

Combining the preceding bounds and identities for the various terms in the above identity for $\tilde{\tilde{B}}(\gamma_1,\gamma_2)$ and also recalling (4.47), we have obtained the first relation determining $\gamma_{1,2}$, given by

$$\begin{aligned} B = &-\frac{c}{2}\lambda_{0,0}^{\frac{1}{2}}[\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}] + O\big(C^{-1}\lambda_{0,0}^{\frac{1}{2}}|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 t_0^{k_0\nu}|\big)\\ &+ O\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\tau_0[\|x_0\|_{\tilde{S}_1} + \tau_0^{-1}|x_{0d}|]\big)\\ &+ O_{t_0}\big(\lambda_{0,0}^{\frac{1}{2}}[\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}]^2\big), \end{aligned} \tag{4.55}$$

where the first term on the right dominates all the remaining error terms, provided the data are chosen small enough.

To derive the second equation determining $\gamma_{1,2}$, we recall the formula (4.28) for $x_1^{(\gamma_1,\gamma_2)}$, which hinges on $\overline{\epsilon}_1$. Then by combining (4.43) with (4.48), we have (using the notation $\Lambda := \frac{1}{2} + R\partial_R$)

$$\begin{aligned} R\overline{\epsilon}_1 = &\, R\partial_t\big[(t^{k_0\nu}\gamma_1 + \log t \cdot t^{k_0\nu}\gamma_2)\lambda_{0,0}^{\frac{1}{2}}R^{-1}\phi(R,0) - v_{smooth}\big]_{t=t_0} + R\epsilon_1\\ &+ O\big(\lambda_{0,0}^{\frac{1}{2}}t_0^{k_0\nu-1}\log t_0(\textstyle\sum|\gamma_j|)|t_0^{k_0\nu}\gamma_1 + \log t_0 \cdot t_0^{k_0\nu}\gamma_2|\big)\\ = &\, c_1 t_0^{-1}(t_0^{k_0\nu}\gamma_1 + \log t_0 \cdot t_0^{k_0\nu}\gamma_2)\lambda_{0,0}^{\frac{1}{2}}\phi(R,0)\\ &+ c_2 t_0^{-1}(t_0^{k_0\nu}\gamma_1 + \log t_0 \cdot t_0^{k_0\nu}\gamma_2)\lambda_{0,0}^{\frac{1}{2}}(\Lambda^2 W)(R)\\ &+ \gamma_2 t_0^{k_0\nu-1}\lambda_{0,0}^{\frac{1}{2}}\phi(R,0) + O\big(\lambda_{0,0}^{\frac{1}{2}}t_0^{k_0\nu-1}\log t_0(\textstyle\sum|\gamma_j|)|t_0^{k_0\nu}\gamma_1 + \log t_0 \cdot t_0^{k_0\nu}\gamma_2|\big)\\ &- R\partial_t v_{smooth} + R\epsilon_1. \end{aligned}$$

Using (4.28) which gives

$$\begin{aligned} x_1^{(\gamma_1,\gamma_2)}(\xi) = &-\lambda_{\gamma_1,\gamma_2}^{-1}|_{t=t_0}\int_0^\infty \phi(R,\xi)R\overline{\epsilon}_1(R)\,dR - \frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}|_{t=t_0}(\mathcal{K}_{cc}x_0^{(\gamma_1,\gamma_2)})(\xi)\\ &-\frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}|_{t=t_0}(\mathcal{K}_{cd}x_{0d}^{(\gamma_1,\gamma_2)})(\xi), \end{aligned}$$

and also keeping in mind the corresponding relations (4.11), (4.12), we deduce in light of Lemma 113 the identity

$$x_1^{(\gamma_1,\gamma_2)}(\xi) = \sum_{j=1}^4 A_j + \sum_{j=1}^3 B_j + C$$

where the A_j are the terms coming from the 'bulk term' and are given by

$$A_1 = -c_1(t_0\lambda_{\gamma_1,\gamma_2})^{-1}\lambda_{0,0}^{\frac{1}{2}}(t_0^{k_0\nu}\gamma_1 + \log t_0 \cdot t_0^{k_0\nu}\gamma_2)\int_0^\infty \phi(R,\xi)\chi_{R\leq C\tau_0}\lambda_{0,0}^{\frac{1}{2}}\phi(R,0)\,dR$$

$$A_2 = -c_2(t_0\lambda_{\gamma_1,\gamma_2})^{-1}\lambda_{0,0}^{\frac{1}{2}}(t_0^{k_0\nu}\gamma_1 + \log t_0 \cdot t_0^{k_0\nu}\gamma_2)\int_0^\infty \phi(R,\xi)\chi_{R\leq C\tau_0}\lambda_{0,0}^{\frac{1}{2}}(\Lambda^2 W)(R,0)\,dR,$$

$$A_3 = -\lambda_{0,0}^{\frac{1}{2}}\gamma_2 t_0^{k_0\nu-1}\lambda_{\gamma_1,\gamma_2}^{-1}\int_0^\infty \phi(R,\xi)\chi_{R\leq C\tau_0}\lambda_{0,0}^{\frac{1}{2}}\phi(R,0)\,dR$$

$$A_4 = -\lambda_{\gamma_1,\gamma_2}^{-1}\int_0^\infty \phi(R,\xi)\chi_{R\leq C\tau_0}R\partial_t v_{smooth}\,dR,$$

while the terms B_j arising from the perturbation are given by

$$B_1 = \frac{\lambda}{\lambda_{0,0}}x_1(\frac{\lambda^2}{\lambda_{0,0}^2}\xi),\ B_2 = -\frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}|_{t=t_0}(\mathcal{K}_{cc}x_0^{(\gamma_1,\gamma_2)})(\xi),$$
$$B_3 = -\frac{\dot{\lambda}_{\gamma_1,\gamma_2}}{\lambda_{\gamma_1,\gamma_2}}|_{t=t_0}(\mathcal{K}_{cd}x_{0d}^{(\gamma_1,\gamma_2)})(\xi)$$

Finally, C is the error, which is given crudely by

$$C = O\big(\lambda_{0,0}^{\frac{1}{2}}|\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\cdot t_0^{k_0\nu}|\tau_0|[\|x_1\|_{\tilde{S}_1} + \tau_0^{-1}\|x_0\|_{\tilde{S}_2} + |x_{1d}| + \tau_0^{-1}|x_{0d}|]\big)$$
$$+ O_{t_0}\big((\sum_j|\gamma_j|)|\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\cdot t_0^{k_0\nu}|\big).$$

Inserting the preceding into the expression for $A(\gamma_1,\gamma_2)$ (as in the statement of the proposition) and proceeding as for the derivation for (4.55), as well as observing that

$$\tau_0^{-1}\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\mathcal{K}_{cc}x_0^{(\gamma_1,\gamma_2)})(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda_{\gamma_1,\gamma_2}(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda_{\gamma_1,\gamma_2}^{-1}(s)\,ds]\,d\xi$$
$$= D(x_0,x_{0d}) + D_1\lambda_{0,0}^{\frac{1}{2}}(\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\cdot t_0^{k_0\nu}) + O_{t_0}\big(|\gamma_1 t_0^{k_0\nu} + \gamma_2\log t_0\cdot t_0^{k_0\nu}|^2\big)$$

where $D(x_0,x_{0d})$ is independent of $\gamma_{1,2}$ and is bounded by $|D(x_0,x_{0d})| \lesssim \|x_0\|_{\tilde{S}_1} + |x_{0d}|$, while $D_1 = D_1(\nu)$ is a suitable absolute constant, we infer the following identity:

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)x_1^{(\gamma_1,\gamma_2)}(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda_{\gamma_1,\gamma_2}(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda_{\gamma_1,\gamma_2}^{-1}(s)\,ds]\,d\xi$$
$$= A + \sum_{j=1,2}E_j + D(x_0,x_{0d}) + F. \tag{4.56}$$

Here the first term on the right arises on account of (4.44), the two terms $E_{1,2}$ arise via the contribution of the bulk terms $A_1 - A_4$ (taking advantage of estimates like (4.49)), and are of the form

$$E_1 = e_1\lambda_{0,0}^{\frac{1}{2}}(\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}),\ E_2 = e_2\lambda_{0,0}^{\frac{1}{2}}\gamma_2 t_0^{k_0\nu},\ e_2 \neq 0,$$

and finally, the error term F is of the form

$$\begin{aligned} F &= O_{t_0}\big((\sum_j |\gamma_j|)|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\big) \\ &+ O\big(\lambda_{0,0}^{\frac{1}{2}}|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\tau_0|[\|x_1\|_{\tilde{S}_1} + \tau_0^{-1}\|x_0\|_{\tilde{S}_2} + |x_{1d}| + \tau_0^{-1}|x_{0d}|]\big) \end{aligned}$$

Equating the expression on the left of (4.56) with 0, we infer the second equation, analogous to (4.55):

$$\begin{aligned} A &= -e_1\lambda_{0,0}^{\frac{1}{2}}(\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}) - e_2\lambda_{0,0}^{\frac{1}{2}}\gamma_2 t_0^{k_0\nu} - D(x_0, x_{0d}) \\ &+ O_{t_0}\big((\sum_j |\gamma_j|)|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\big) \\ &+ O\big(\lambda_{0,0}^{\frac{1}{2}}|\gamma_1 t_0^{k_0\nu} + \gamma_2 \log t_0 \cdot t_0^{k_0\nu}|\tau_0|[\|x_1\|_{\tilde{S}_1} + \tau_0^{-1}\|x_0\|_{\tilde{S}_2} + |x_{1d}| + \tau_0^{-1}|x_{0d}|]\big) \end{aligned}$$

On account of the easily verified bounds

$$|A| \lesssim \tau_0\|x_1\|_{\tilde{S}_2},\ |B| \lesssim \tau_0\|x_0\|_{\tilde{S}_1},$$

we then infer

$$|\gamma_1\lambda_{0,0}^{\frac{1}{2}} t_0^{k_0\nu}| + |\gamma_2\lambda_{0,0}^{\frac{1}{2}} \log t_0 t_0^{k_0\nu}| \lesssim (\log\tau_0)\cdot\tau_0(\|(x_0, x_1)\|_{\tilde{S}} + |x_{0d}|).$$

Recall that throughout the preceding discussion we kept the discrete spectral parts (x_{0d}, x_{1d}) of the initial perturbation (ϵ_1, ϵ_2) fixed. If instead we allow x_{1d} to vary, we can think of $\gamma_{1,2}$ as functions of x_{1d}, and moreover one easily checks that

$$\tilde{x}_{1d}^{(\gamma_1,\gamma_2)} = x_{1d} + O([\|x_0\|_{\tilde{S}_1} + \|x_1\|_{\tilde{S}_2} + |x_{0d}| + |x_{1d}|]^2).$$

with a corresponding Lipschitz bound. It follows that there is a unique choice of x_{1d} such that (for given x_0, x_1, x_{0d}) the pair $(\tilde{x}_{0d}^{(\gamma_1,\gamma_2)}, \tilde{x}_{1d}^{(\gamma_1,\gamma_2)})$ satisfies the linear compatibility relation (4.21) with respect to the scaling parameter $\lambda = \lambda_{\gamma_1,\gamma_2}$.

The last bound of the proposition follows from the preceding formulas for $x_1^{(\gamma_1,\gamma_2)}$, as well as $x_0^{(\gamma_1,\gamma_2)}$ in terms of x_1, x_0. Specifically, one uses the fact that for the Fourier transform of the bulk term[8] in (4.42), we have the asymptotics (4.48), (4.49) as well as (4.53), and we get

$$\|\frac{C\tau_0}{\langle C\tau_0\xi^{\frac{1}{2}}\rangle^N}\|_{\tilde{S}_1} \lesssim \tau_0^{-(1-\delta_0)}. \tag{4.57}$$

[8]This means the sum of the first three terms on the right.

□

For later purposes, we also mention the following important Lipschitz continuity properties, which follow easily from the preceding proof:

Lemma 115. *Let* $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ *the parameters associated with a data quadruple* $(\underline{\widetilde{x}}_0, \underline{\widetilde{x}}_1) \in \widetilde{S}$. *Then using the notation from before and putting*

$$\widetilde{\lambda} = \lambda_{(\widetilde{\gamma}_1, \widetilde{\gamma}_2)},$$

we have

$$\begin{aligned} |(\gamma_1 - \widetilde{\gamma}_1)\lambda_{0,0}^{\frac{1}{2}} t_0^{k_0\nu}| + |(\gamma_2 - \widetilde{\gamma}_2)\lambda_{0,0}^{\frac{1}{2}} \log t_0 \, t_0^{k_0\nu}| &\lesssim \tau_0 \log \tau_0 [\|(x_0 - \widetilde{x}_0, x_1 - \widetilde{x}_1)\|_{\widetilde{S}} \\ &\quad + \|(x_0, x_1)\|_{\widetilde{S}} |x_{0d} - \widetilde{x}_{0d}|]. \end{aligned}$$

$$\begin{aligned} &\|x_0^{(\gamma_1,\gamma_2)} - \widetilde{x}_0^{(\widetilde{\gamma}_1,\widetilde{\gamma}_2)} - (\frac{\lambda_{0,0}}{\lambda} S_{\frac{\lambda_{0,0}^2}{\lambda^2}} x_0 - \frac{\lambda_{0,0}}{\widetilde{\lambda}} S_{\frac{\lambda_{0,0}^2}{\widetilde{\lambda}^2}} \widetilde{x}_0)\|_{\widetilde{S}_1} \\ &\qquad + \|x_1^{(\gamma_1,\gamma_2)} - \widetilde{x}_1^{(\widetilde{\gamma}_1,\widetilde{\gamma}_2)} - (\frac{\lambda_{0,0}}{\lambda} S_{\frac{\lambda_{0,0}^2}{\lambda^2}} x_1 - \frac{\lambda_{0,0}}{\widetilde{\lambda}} S_{\frac{\lambda_{0,0}^2}{\widetilde{\lambda}^2}} \widetilde{x}_1)\|_{\widetilde{S}_2} \\ &\lesssim \log \tau_0 \cdot \tau_0^{0+} \cdot [\|(x_0 - \widetilde{x}_0, x_1 - \widetilde{x}_1)\|_{\widetilde{S}} + \|(x_0, x_1)\|_{\widetilde{S}} |x_{0d} - \widetilde{x}_{0d}|]. \end{aligned}$$

Finally, we have the bound

$$\begin{aligned} &|(x_{1d}^{(\gamma_1,\gamma_2)} - x_{1d}) - (\widetilde{x}_{1d}^{(\gamma_1,\gamma_2)} - \widetilde{x}_{1d})| \\ &\lesssim [\|(x_0 - \widetilde{x}_0, x_1 - \widetilde{x}_1)\|_{\widetilde{S}} + |x_{0d} - \widetilde{x}_{0d}|] \cdot [\|(x_0, x_1)\|_{\widetilde{S}} + |x_{0d}|]. \end{aligned}$$

4.5 Iterative construction of blow-up solution almost matching the perturbed initial data

Here we carry out the actual construction of the solution, as explained in the paragraph following (4.30). Thus departing from perturbed initial data

$$u_\nu[t_0] + (\epsilon_0, \epsilon_1),$$

where the perturbation (ϵ_0, ϵ_1) is associated with a data quadruple $(\underline{x}_0, \underline{x}_1)$ as in (4.10), (4.11), (4.12), where x_{1d}, as well as parameters $\gamma_{1,2}$ have been computed according to Proposition 114, in terms of the Fourier data $(x_0(\xi), x_1(\xi), x_{0d})$, we then pass to a different representation of the data which coincides with the preceding data in a dilate of the light cone at time $t = t_0$, i. e. we have

$$\chi_{r \le Ct_0} u_{approx}^{(0,0)}[t_0] + (\epsilon_1, \epsilon_2) = \chi_{r \le Ct_0} u_{approx}^{(\gamma_1,\gamma_2)}[t_0] + (\overline{\epsilon}_1, \overline{\epsilon}_2).$$

Then, according to Proposition 114, the Fourier data associated to $(\overline{\epsilon}_1, \overline{\epsilon}_2)$ in reference to the coordinate $R := \lambda_{\gamma_1,\gamma_2}(t_0)r$, satisfy the key vanishing relations

$$A(\gamma_1,\gamma_2) = B(\gamma_1,\gamma_2) = 0,$$

these quantities being defined as in Proposition 114. We shall now strive to evolve the data

$$u_{approx}^{(\gamma_1,\gamma_2)}[t_0] + (\overline{\epsilon}_1, \overline{\epsilon}_2)$$

backwards in time from $t = t_0$, and thereby build another blow-up solution with bulk part $u_{approx}^{(\gamma_1,\gamma_2)}(t,x)$ on the time slice $(0,t_0] \times \mathbb{R}^3$.

4.5.1 Formulation of the perturbation problem on Fourier side

Re-iterating that we shall work with the coordinates

$$\tau := \int_t^\infty \lambda_{\gamma_1,\gamma_2}(s)\,ds,\ R = \lambda_{\gamma_1,\gamma_2}(t)r, \tag{4.58}$$

we shall write the desired solution in the form

$$u(t,x) = u_{approx}^{(\gamma_1,\gamma_2)}(t,x) + \epsilon(t,x),\ \epsilon[t_0] = (\overline{\epsilon}_1, \overline{\epsilon}_2), \tag{4.59}$$

and passing to the variable $\widetilde{\varepsilon}(\tau,R) := R\epsilon$, we derive the following equation completely analogous to (4.14): using from now on $\lambda(\tau) = \lambda_{\gamma_1,\gamma_2}(\tau)$,

$$\begin{aligned}&(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)^2\widetilde{\varepsilon} - \beta(\tau)(\partial_\tau + \dot{\lambda}\lambda^{-1}R\partial_R)\widetilde{\varepsilon} + \mathscr{L}\widetilde{\varepsilon}\\ &= \lambda^{-2}(\tau)R[N_{approx}(\varepsilon) + e_{approx}] + \partial_\tau(\dot{\lambda}\lambda^{-1})\widetilde{\varepsilon};\ \beta(\tau) = \dot{\lambda}(\tau)\lambda^{-1}(\tau),\end{aligned} \tag{4.60}$$

where we use the notation

$$RN_{approx}(\varepsilon) = 5(u_{approx}^4 - u_0^4)\widetilde{\varepsilon} + RN(u_{approx}, \widetilde{\varepsilon}),$$

$$RN(u_{approx}, \widetilde{\varepsilon}) = R(u_{approx} + \frac{\widetilde{\varepsilon}}{R})^5 - Ru_{approx}^5 - 5u_{approx}^4\widetilde{\varepsilon},$$

and $u_{approx} = u_{approx}^{(\gamma_{1,2})}$. The source term e_{approx} is precisely the one in Theorem 111. Also, observe that we may and shall include cutoffs to the right-hand source terms of the form $\chi_{R\le C\tau}$, since we are only interested in the behavior of the solution inside the forward light cone emanating from the origin. Ideally we will want to match

$$\epsilon[t_0] = (\overline{\epsilon}_1, \overline{\epsilon}_2),$$

but we shall have to deviate from this by a small error. In order to solve (4.60), we pass to the distorted Fourier transform of $\widetilde{\varepsilon}$, by using the representation

$$\widetilde{\varepsilon}(\tau,R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi.$$

Writing

$$\underline{x}(\tau,\xi) := \begin{pmatrix} x_d(\tau) \\ x(\tau,\xi) \end{pmatrix}, \qquad \underline{\xi} = \begin{pmatrix} \xi_d \\ \xi \end{pmatrix},$$

we infer

$$\big(\mathscr{D}_\tau^2 + \beta(\tau)\mathscr{D}_\tau + \underline{\xi}\big)\underline{x}(\tau,\xi) = \mathscr{R}(\tau,\underline{x}) + \underline{f}(\tau,\underline{\xi}),\ \underline{f} = \begin{pmatrix} f_d \\ f \end{pmatrix}, \tag{4.61}$$

combined with the initial data (which in turn obey (4.27), (4.28), (4.29))

$$\underline{x}(\tau_0,\xi) = \begin{pmatrix} x_{0d}^{(\gamma_1,\gamma_2)} \\ x_0^{(\gamma_1,\gamma_2)} \end{pmatrix}, \mathscr{D}_\tau\underline{x}(\tau_0,\xi) = \begin{pmatrix} x_{1d}^{(\gamma_1,\gamma_2)} \\ x_1^{(\gamma_1,\gamma_2)} \end{pmatrix}, \tau_0 = \tau(t_0). \tag{4.62}$$

where we have

$$\mathscr{R}(\tau,\underline{x})(\xi) = \Big(-4\beta(\tau)\mathscr{K}\mathscr{D}_\tau\underline{x} - \beta^2(\tau)(\mathscr{K}^2 + [\mathscr{A},\mathscr{K}] + \mathscr{K} + \beta'\beta^{-2}\mathscr{K})\underline{x}\Big)(\xi) \tag{4.63}$$

with $\beta(\tau) = \frac{\dot\lambda(\tau)}{\lambda(\tau)}$, and

$$\begin{aligned} f(\tau,\xi) &= \mathscr{F}\big(\lambda^{-2}(\tau)\chi_{R\lesssim\tau}\big[5(u_{approx}^4 - u_0^4)\widetilde{\varepsilon} + RN(u_{approx},\widetilde{\varepsilon}) + Re_{approx}\big]\big)(\xi) \\ f_d(\tau) &= \langle\lambda^{-2}(\tau)\chi_{R\lesssim\tau}\big[5(u_{approx}^4 - u_0^4)\widetilde{\varepsilon} + RN(u_{approx},\widetilde{\varepsilon}) + Re_{approx}\big], \phi_d(R)\rangle. \end{aligned} \tag{4.64}$$

Also the key operator

$$\mathscr{D}_\tau = \partial_\tau + \beta(\tau)\mathscr{A}, \quad \mathscr{A} = \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{A}_c \end{pmatrix}$$

and we have

$$\mathscr{A}_c = -2\xi\partial_\xi - \Big(\frac{5}{2} + \frac{\rho'(\xi)\xi}{\rho(\xi)}\Big).$$

The operator $\mathscr{K}$ is described in Theorem 106.

The main technical result of this article then furnishes a solution of (4.61), (4.62) as follows:

Theorem 116. *Let $(x_0^{(\gamma_1,\gamma_2)}, x_1^{(\gamma_1,\gamma_2)}) \in \widetilde{S}, x_{ld}^{(\gamma_1,\gamma_2)}$, $l = 0,1$, be as in Proposition 114, and assume t_0 is sufficiently small, or analogously, τ_0 is sufficiently large. Then there exist corrections*

$$(\triangle x_0^{(\gamma_1,\gamma_2)}, \triangle x_1^{(\gamma_1,\gamma_2)}),\ (\triangle x_{0d}^{(\gamma_1,\gamma_2)}, \triangle x_{1d}^{(\gamma_1,\gamma_2)})$$

satisfying

$$\|(\triangle x_0^{(\gamma_1,\gamma_2)}, \triangle x_1^{(\gamma_1,\gamma_2)})\|_{\widetilde{S}} \ll \|(x_0,x_1)\|_{\widetilde{S}} + |x_{0d}|,$$

$$|\triangle x_{0d}^{(\gamma_1,\gamma_2)}| + |\triangle x_{1d}^{(\gamma_1,\gamma_2)}| \ll \|(x_0, x_1)\|_{\widetilde{S}} + |x_{0d}|,$$

and such that the $(\triangle x_0^{(\gamma_1,\gamma_2)}, \triangle x_1^{(\gamma_1,\gamma_2)})$, $(\triangle x_{0d}^{(\gamma_1,\gamma_2)}, \triangle x_{1d}^{(\gamma_1,\gamma_2)})$ *depend in Lipschitz continuous fashion on* (x_0, x_1, x_{0d}) *with respect to* $\|\cdot\|_{\widetilde{S}} + |\cdot|$ *with Lipschitz constant* $\ll 1$, *and such that the equation* (4.61) *with data*

$$\big(x(\tau_0,\xi), (D_\tau x)(\tau_0,\xi)\big) = \big(x_0^{(\gamma_1,\gamma_2)} + \triangle x_0^{(\gamma_1,\gamma_2)}, x_1^{(\gamma_1,\gamma_2)} + \triangle x_1^{(\gamma_1,\gamma_2)}\big)$$

$$\big(x_d(\tau_0), \partial_\tau x_d(\tau_0)\big) = \big(x_{0d}^{(\gamma_1,\gamma_2)} + \triangle x_{0d}^{(\gamma_1,\gamma_2)}, x_{1d}^{(\gamma_1,\gamma_2)} + \triangle x_{1d}^{(\gamma_1,\gamma_2)}\big)$$

admits a solution $\underline{x}(\tau,\xi)$ *for* $\tau \geq \tau_0$ *satisfying*

$$\|(x(\tau,\cdot), \mathcal{D}_\tau x(\tau,\cdot)\|_{\widetilde{S}} + |x_d(\tau)| + |\partial_\tau x_d(\tau)| \lesssim_\tau \|(x_0, x_1)\|_{\widetilde{S}} + |x_{0d}|,$$

corresponding to $\widetilde{\epsilon}(\tau,R) \in H_{loc}^{\frac{3}{2}+}$ *where*

$$\widetilde{\epsilon}(\tau,R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\,d\xi.$$

Finally, we have energy decay within the light cone:

$$\lim_{t\to 0}\int_{|x|\leq t} \frac{1}{2}|\nabla_{t,x}\epsilon|^2\,dx = 0$$

where we recall $\epsilon = R^{-1}\widetilde{\epsilon}$.

Remark. In fact, the Fourier coefficients $(\triangle x_0^{(\gamma_1,\gamma_2)}, \triangle x_1^{(\gamma_1,\gamma_2)})$ will have a very specific form, which makes them well-behaved with respect to re-scalings (which hence don't entail smoothness loss when passing to differences). This shall be important when reverting to the original coordinates $R_{0,0}$ at time $t = t_0$, which were used to specify the perturbation (x_0, x_1) to begin with.

4.5.2 The proof of Theorem 116

It is divided into two parts: the existence part for the solution, which follows essentially verbatim the scheme in [49], and the more delicate verification of Lipschitz dependence of the solution on the data (x_0, x_1, x_{0d}). Here the issue is the fact that there are re-scalings involved, and the very parametrix used to solve (4.61), as well as the source terms there, depend implicitly on $\gamma_{1,2}$, which in turn depend on (x_0, x_1, x_{0d}).

Setup of the iteration scheme; the zeroth iterate

Proceeding in close analogy to [49], we shall obtain the final solution $\underline{x}(\tau,\xi)$ of (4.61) as the limit of a sequence of iterates $\underline{x}^{(j)}(\tau,\xi)$. To begin with, we introduce the zeroth iterate in the following proposition. The only difference compared to [49] is the presence of the error term e_{approx}, whose dependence on $\gamma_{1,2}$ needs to be taken carefully into account.

To formulate the bounds on the successive iterates, we introduce a number of notations. First, we recall (4.8), which is used to control data sets, and we also introduce the slightly stronger norm

$$\|(x_0, x_1)\|_S := \|x_0\|_{S_1} + \|x_1\|_{\widetilde{S}_2} := \|\langle\xi\rangle^{1+2\delta_0}\xi^{-\delta_0}x_0\|_{L^2_{d\xi}} + \|\langle\xi\rangle^{\frac{1}{2}+2\delta_0}\xi^{-\delta_0}x_1\|_{L^2_{d\xi}}. \tag{4.65}$$

Denote the homogeneous propagator (4.20) by $S(\tau)(x_0, x_1)$, and further introduce the inhomogeneous propagator solving the problem with source (this only involves the continuous spectral part)

$$\big(\mathscr{D}_\tau^2 + \beta(\tau)\mathscr{D}_\tau + \xi\big)x(\tau,\xi) = h(\tau,\xi),\ (x(\tau_0,\xi) = 0, \mathscr{D}_\tau x(\tau_0,\xi) = 0$$

by

$$\begin{aligned} x(\tau,\xi) &= \int_{\tau_0}^{\tau} U(\tau,\sigma)h(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma,\\ U(\tau,\sigma) &= \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}\frac{\sin\Big[\lambda(\tau)\xi^{\frac{1}{2}}\int_\sigma^\tau\lambda^{-1}(u)\,du\Big]}{\xi^{\frac{1}{2}}} \end{aligned} \tag{4.66}$$

Further, denote the evolution of the spectral part with inhomogeneous data

$$\big(\partial_\tau^2 + \beta(\tau)\partial_\tau + \xi_d\big)x_d(\tau) = h_d(\tau),\ x_d(\tau_0) = 0, \partial_\tau x_d(\tau_0) = 0$$

and without exponential decay at infinity (for bounded h_d for example) by

$$x_d(\tau) = \int_{\tau_0}^{\tau} H(\tau,\sigma)h_d(\sigma)\,d\sigma, \tag{4.67}$$

where we have (see Lemma 109) the bound $|H(\tau,\sigma)| \lesssim e^{-c|\tau-\sigma|}$ for some $c > 0$. Following [49], we also introduce the somewhat complicated square-sum norms over dyadic time intervals and given by

$$\|y(\tau,\xi)\|_{Sqr} := \Big(\sum_{\substack{N\gtrsim\tau_0\\ N\,\text{dyadic}}}\sup_{\tau\sim N}\big(\frac{\lambda(\tau)}{\lambda(\tau_0)}\big)^{4\delta_0}\|y(\tau,\cdot)\|^2_{L^2_{d\xi}}\Big)^{\frac{1}{2}}, \tag{4.68}$$

and we shall also use $\|y(\tau,\xi)\|_{Sqr(\xi<1)}, \|y(\tau,\xi)\|_{Sqr(\xi>1)}$ where $L^2_{d\xi}$ will be refined to $L^2_{d\xi}(\xi < 1), L^2_{d\xi}(\xi > 1)$. To define the zeroth iterate $\underline{x}^{(0)}$, we replace the source functions in (4.64) by

$$\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\tau}Re_{approx}\big),\ \langle\lambda^{-2}\chi_{R\lesssim\tau}Re_{approx}, \phi_d(R)\rangle.$$

Proposition 117. *Assume the same setup as in Theorem 116. In particular, as before, everything depends on a basic data triple $(x_0(\xi), x_1(\xi), x_{0d})$ from which the fourth component x_{1d} and further the new Fourier components $x_{0,1}^{(\gamma_1,\gamma_2)}$ are derived. There is a pair $(\triangle\widetilde{\widetilde{x}}_0^{(0)}, \triangle\widetilde{\widetilde{x}}_1^{(0)}) \in \widetilde{S}$,* *satisfying the bounds*

$$\|(\triangle\tilde{\tilde{x}}_0^{(0)}, \triangle\tilde{\tilde{x}}_1^{(0)})\|_{\tilde{S}} \lesssim \tau_0^{-(2-)}[\|(x_0, x_1)\|_{\tilde{S}} + |x_{0d}|],$$

and such that if we set for the continuous spectral part

$$x^{(0)}(\tau,\xi) := \widetilde{x^{(0)}}(\tau,\xi) + S(\tau)\big(x_0^{(\gamma_1,\gamma_2)} + \triangle\tilde{\tilde{x}}_0^{(0)}, x_1^{(\gamma_1,\gamma_2)} + \triangle\tilde{\tilde{x}}_1^{(0)}\big),$$
$$\widetilde{x^{(0)}}(\tau,\xi) := \int_{\tau_0}^{\tau} U(\tau,\sigma)\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\tau}Re_{approx}\big)(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\, d\sigma$$

then the following conclusions obtain: for high frequencies $\xi > 1$, we have

$$\sup_{\tau\geq\tau_0}(\frac{\tau}{\tau_0})^{-\kappa}\|\chi_{\xi>1}\widetilde{x^{(0)}}(\tau,\xi)\|_{S_1} + \sup_{\tau\geq\tau_0}(\frac{\tau}{\tau_0})^{\kappa}\|\chi_{\xi>1}\mathscr{D}_\tau\widetilde{x^{(0)}}(\tau,\xi)\|_{S_2}$$
$$+ \|\xi^{\frac{1}{2}+\delta_0}\mathscr{D}_\tau\widetilde{x^{(0)}}(\tau,\xi)\|_{Sqr(\xi>1)} \lesssim \tau_0^{-1}[\|(x_0, x_1)\|_{\tilde{S}} + |x_{0d}|].$$

For low frequencies $\xi < 1$, there is a decomposition

$$\widetilde{x^{(0)}}(\tau,\xi) + S(\tau)\big(\triangle\tilde{\tilde{x}}_0^{(0)}, \triangle\tilde{\tilde{x}}_1^{(0)}\big) = \triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi) + S(\tau)\big(\triangle\widetilde{x^{(0)}}_0(\xi), \triangle\widetilde{x^{(0)}}_1(\xi)\big)$$

where the data $\big(\triangle\widetilde{x^{(0)}}_0(\xi), \triangle\widetilde{x^{(0)}}_1(\xi)\big)$ satisfy the vanishing conditions

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\triangle\widetilde{x^{(0)}}_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(u)\,du]\,d\xi = 0, \tag{4.69}$$

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\triangle\widetilde{x^{(0)}}_1(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(u)\,du]\,d\xi = 0, \tag{4.70}$$

and such that we have the bound

$$\|\big(\triangle\widetilde{x^{(0)}}_0(\xi), \triangle\widetilde{x^{(0)}}_1(\xi)\big)\|_{\tilde{S}} + \sup_{\tau\geq\tau_0}(\frac{\tau}{\tau_0})^{-\kappa}\|\chi_{\xi<1}\triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi)\|_{S_1}$$
$$+ \|\xi^{-\delta_0}\mathscr{D}_\tau\triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi)\|_{Sqr(\xi<1)} \lesssim \tau_0^{-1}[\|(x_0, x_1)\|_{\tilde{S}} + |x_{0d}|].$$

Furthermore, letting $\triangle\tilde{\tilde{x}}_j^{(0)}, \triangle\tilde{\tilde{\bar{x}}}_j^{(0)}$, $j = 1, 2$, be the corrections corresponding to two initial perturbation quadruples (where the component x_{1d} is determined in terms of the other three ones via Proposition 114)

$$(\underline{x}_0, \underline{x}_1),\ (\overline{x}_0, \overline{x}_1),$$

we have we have

$$\|(\triangle\tilde{\tilde{x}}_0^{(0)} - \triangle\tilde{\tilde{\bar{x}}}_0^{(0)}, \triangle\tilde{\tilde{x}}_1^{(0)} - \triangle\tilde{\tilde{\bar{x}}}_1^{(0)})\|_{\tilde{S}} \lesssim \tau_0^{-(1-)}[\|(x_0 - \overline{x}_0, x_1 - \overline{x}_1)\|_{\tilde{S}} + |x_{0d} - \overline{x}_{0d}|].$$

For the discrete spectral part, setting

$$\triangle x_d^{(0)}(\tau) := \int_{\tau_0}^{\infty} H_d(\tau,\sigma)\langle \lambda^{-2}(\sigma) Re_{approx}, \phi_d(R)\rangle \, d\sigma,$$

we have the bound

$$\tau^2 [|\triangle x_d^{(0)}(\tau)| + |\partial_\tau \triangle x_d^{(0)}(\tau)|] \lesssim \|(x_0, x_1)\|_{\tilde{S}} + |x_{0d}|.$$

We also have the difference bound

$$\tau^2 [|\triangle x_d^{(0)}(\tau) - \triangle \overline{x}_d^{(0)}(\tau)| + |\partial_\tau \triangle x_d^{(0)}(\tau) - \partial_\tau \triangle \overline{x}_d^{(0)}(\tau)|]$$
$$\lesssim \|(x_0 - \overline{x}_0, x_1 - \overline{x}_1)\|_{\tilde{S}} + |x_{0d} - \overline{x}_{0d}|.$$

We shall then set

$$x_d^{(0)}(\tau) := x_d^{(\gamma_1,\gamma_2)}(\tau) + \triangle x_d^{(0)}(\tau),$$

where $x_d^{(\gamma_1,\gamma_2)}(\tau)$ is the 'free evolution' of the discrete spectral part constructed as in Lemma 109 with data $(x_{0d}^{(\gamma_1,\gamma_2)}, x_{1d}^{(\gamma_1,\gamma_2)})$.

Remark. Observe that the formula for the continuous spectral part $x^{(0)}(\tau,\xi)$ arises by adding the term $S(\tau)\big(\triangle\tilde{\tilde{x}}_0^{(0)}, \triangle\tilde{\tilde{x}}_1^{(0)}\big)$ to the Duhamel type parametrix coming from Lemma 109. The reason for such a correction term, which is already present in [49], comes from poor low frequency bounds for the term

$$\int_{\tau_0}^{\tau} U(\tau,\sigma)\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\tau} Re_{approx}\big)(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\, d\sigma,$$

and more generally, for any such term occurring in the iterative scheme. The idea then is to write this bad term (for small ξ) in the form

$$\triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi) + S(\tau)\big(\triangle\widetilde{x^{(0)}}_0(\xi), \triangle\widetilde{x^{(0)}}_1(\xi)\big),$$

by replacing the integral $\int_{\tau_0}^{\tau}$ by one over $\int_{\tau}^{\infty}$. Since term components

$$\triangle\widetilde{x^{(0)}}_0(\xi), \triangle\widetilde{x^{(0)}}_1(\xi)$$

don't necessarily satisfy the vanishing conditions (4.87), (4.70), we need to add the corrections $\triangle\tilde{\tilde{x}}_0^{(0)}, \triangle\tilde{\tilde{x}}_1^{(0)}$. Importantly, these can be chosen to be much smaller than the initial data $x_{0,1}, x_{0d}$. This procedure is explained in greater detail in [49].

Proof. We follow the same outline of steps as for example in the proof of Proposition 8.1 in [49].

Step 1: Proof of the high frequency bound

Recall (4.40). Correspondingly, we shall write $\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\tau}Re_{approx}\big)$ as the sum of several terms. We shall prove the somewhat more delicate square-sum type bound, the remaining bounds being more of the same.

The contribution of $e_{\text{prelim}} - \tilde{e}_{\text{prelim}}$

Write

$$\Xi_1(\tau,\xi) := \int_{\tau_0}^{\tau} U(\tau,\sigma)\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\tau}R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma$$

We need to bound $\|\xi^{\frac{1}{2}+\delta_0}\mathscr{D}_\tau\Xi_1(\tau,\xi)\|_{Sqr(\xi>1)}$. Observe that

$$\mathscr{D}_\tau\Big[\int_{\tau_0}^{\tau} U(\tau,\sigma)g(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma\Big] = \int_{\tau_0}^{\tau} V(\tau,\sigma)g(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma, \tag{4.71}$$

where we set

$$V(\tau,\sigma) := \frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}\cos\Big[\lambda(\tau)\xi^{\frac{1}{2}}\int_\sigma^\tau \lambda^{-1}(u)\,du\Big].$$

In light of Prop. 103, we infer the inequality

$$\frac{\lambda^{\frac{3}{2}}(\tau)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)} \lesssim \frac{\lambda^2(\tau)}{\lambda^2(\sigma)},\ \xi > 1,$$

and this implies

$$\begin{aligned}&\|\xi^{\frac{1}{2}+}\mathscr{D}_\tau\Xi_1(\tau,\xi)\|_{L^2_{d\xi}(\xi>1)}\\ &\lesssim \int_{\tau_0}^{\tau}\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\|\xi^{\frac{1}{2}+\delta_0}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\|_{L^2_{d\xi}(\xi>1)}\,d\sigma\end{aligned}$$

Referring to the same proposition for the isometry properties of the distorted Fourier transform, as well as Lemma 105, we obtain

$$\begin{aligned}&\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\|\xi^{\frac{1}{2}+\delta_0}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\|_{L^2_{d\xi}(\xi>1)}\\ &\lesssim (\frac{\lambda^2(\tau)}{\lambda^2(\sigma)})^{-\delta_0-\frac{1}{4}}\|\xi^{\frac{1}{2}+\delta_0}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}})\big)(\sigma,\cdot)\|_{L^2_{d\rho}}\\ &\lesssim (\frac{\lambda^2(\tau)}{\lambda^2(\sigma)})^{-\delta_0-\frac{1}{4}}\|\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}})\|_{H^{1+2\delta_0}_{dR}}.\end{aligned}$$

Referring to the end of Lemma 112, as well as Definition 4.3 preceding that lemma, for the structure of $e_{\text{prelim}} - \tilde{e}_{\text{prelim}}$, and finally also using the key bound (4.41), we infer the estimate

$$\begin{aligned}&\|\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}} - \tilde{e}_{\text{prelim}})\|_{H^{1+2\delta_0}_{dR}}\\ &\lesssim \sigma^{-2}\cdot\sigma^{\frac{1}{2}(1+\nu^{-1})-k_0-2+}\cdot\log\tau_0\tau_0^{k_0+1-\frac{1}{2}(1+\nu^{-1})-}\cdot\sigma^{\frac{1}{2}}\cdot[\|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}|].\end{aligned} \tag{4.72}$$

Finally integrating this over $\sigma \in [\tau_0, \tau]$, we get

$$\|\xi^{\frac{1}{2}+\delta_0}\mathscr{D}_\tau\Xi_1(\tau,\xi)\|_{L^2_{d\xi}(\xi>1)} \lesssim \tau_0^{-\frac{3}{2}-}\cdot\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\right)^{-\delta_0-\frac{1}{4}}\cdot[\|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}|].$$

In turn inserting this bound into the definition (4.68), we find

$$\|\xi^{\frac{1}{2}+}\mathscr{D}_\tau\Xi_1(\tau,\xi)\|_{Sqr(\xi>1)} \lesssim \tau_0^{-\frac{3}{2}-}\cdot[\|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}|],$$

which is indeed better than what we need.

The contribution of the expression:

$$G(\tau,R) := \sum_{2\le j\le 5}\binom{5}{j}v^j[u^{5-j}_{\text{prelim}} - \tilde{u}^{5-j}_{\text{prelim}}] + 5(-\tilde{u}^4_{\text{prelim}} + u^4_{\text{prelim}})v,$$

where we recall $v, u_{\text{prelim}}, \tilde{u}_{\text{prelim}}$, is described in the last part of the proof of Theorem 111. In particular, we have the bound

$$\|v(\tau,R)\|_{H^{1+2\delta_0}_{dR}} \lesssim \tau^{\frac{1}{2}(2+\nu^{-1})-2k_*},$$

with k_* defined as in Theorem 111. Then setting

$$\Xi_2(\tau,\xi) := \int_{\tau_0}^{\tau} U(\tau,\sigma)g(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma,$$

where we set

$$g(\tau,\xi) = \mathscr{F}\big(\lambda^{-2}(\tau)\chi_{R\lesssim\tau}RG(\tau,R)\big)(\xi),$$

we infer by a similar argument as for the preceding contribution the bound

$$\begin{aligned}\|\xi^{\frac{1}{2}+\delta_0}\mathscr{D}_\tau\Xi_2(\tau,\xi)\|_{L^2_{d\xi}(\xi>1)} &\lesssim \int_{\tau_0}^{\tau}\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\|\xi^{\frac{1}{2}+\delta_0}g(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\|_{L^2_{d\xi}(\xi>1)}\,d\sigma\\ &\lesssim \int_{\tau_0}^{\tau}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)^{-\delta_0-\frac{1}{4}}\|\lambda^{-2}(\sigma)RG(\sigma,R)\|_{H^{1+2\delta_0}_{dR}(R\lesssim\sigma)}\,d\sigma.\end{aligned}$$

On the other hand, the proof of Theorem 111 easily implies the crude bound

$$\|\lambda^{-2}(\sigma)RG(\sigma,R)\|_{H^{1+2\delta_0}_{dR}(R\lesssim\sigma)} \lesssim \sigma^{-N}[\|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}|],\ N \gg 1,$$

and so we obtain

$$\|\xi^{\frac{1}{2}+\delta_0}\mathscr{D}_\tau\Xi_2(\tau,\xi)\|_{L^2_{d\xi}(\xi>1)} \lesssim \tau_0^{-N}\big(\frac{\lambda^2(\tau_0)}{\lambda^2(\tau)}\big)^{\frac{1}{4}+\delta_0}[\|(x_0,x_1)\|_{\tilde{S}}+|x_{0d}|].$$

This in turn furnishes the bound

$$\|\xi^{\frac{1}{2}+}\mathscr{D}_\tau\Xi_2(\tau,\xi)\|_{Sqr(\xi>1)} \lesssim \tau_0^{-N}\cdot[\|(x_0,x_1)\|_{\tilde{S}}+|x_{0d}|],$$

which is much better than what we need.

Step 2: Choice of the corrections $(\triangle\tilde{\tilde{x}}_0^{(0)},\triangle\tilde{\tilde{x}}_1^{(0)})$

In analogy to [49], we shall pick these corrections in the specific form

$$\triangle\tilde{\tilde{x}}_0^{(0)}(\xi) = \alpha\mathscr{F}\big(\chi_{R\leq C\tau}\phi(R,0)\big)(\xi),\ \triangle\tilde{\tilde{x}}_1^{(0)}(\xi) = \beta\mathscr{F}\big(\chi_{R\leq C\tau}\phi(R,0)\big)(\xi), \tag{4.73}$$

and we need to determine the parameters α,β in order to force the required vanishing conditions (4.87), (4.70) for $\triangle\widetilde{x^{(0)}}_0(\xi),\triangle\widetilde{x^{(0)}}_1(\xi)$. The latter quantities are given by

$$\triangle\widetilde{x^{(0)}}_0(\xi) = \int_{\tau_0}^\infty U(\tau_0,\sigma)\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\tau}Re_{approx}\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\,d\sigma + \triangle\tilde{\tilde{x}}_0^{(0)}(\xi),$$
$$\triangle\widetilde{x^{(0)}}_1(\xi) = \int_{\tau_0}^\infty V(\tau_0,\sigma)\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\tau}Re_{approx}\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\,d\sigma + \triangle\tilde{\tilde{x}}_1^{(0)}(\xi),$$

where we recall (4.71). Thus writing $\widetilde{\triangle}\widetilde{x^{(0)}}_j(\xi) := \triangle\widetilde{x^{(0)}}_j(\xi) - \triangle\tilde{\tilde{x}}_j^{(0)}(\xi),\ j=0,1$, we need the following simple

Lemma 118. *We have the bounds*

$$\Big|\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\widetilde{\triangle}\widetilde{x^{(0)}}_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty\lambda^{-1}(u)\,du]\,d\xi\Big| \lesssim \tau_0^{-(1-)}[\|(x_0,x_1)\|_{\tilde{S}}+|x_{0d}|],$$

$$\Big|\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\widetilde{\triangle}\widetilde{x^{(0)}}_1(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty\lambda^{-1}(u)\,du]\,d\xi\Big| \lesssim \tau_0^{-(1-)}[\|(x_0,x_1)\|_{\tilde{S}}+|x_{0d}|],$$

Proof. We again refer to (4.40) to split this into a number of bounds. We consider here the contribution of

$$e_{\text{prelim}} - \tilde{e}_{\text{prelim}},$$

the remaining terms being treated similarly. We distinguish between three frequency regimes:

(i): $\xi < 1$. Here we get

$$
\begin{aligned}
&|\widetilde{\triangle}\widetilde{x^{(0)}}_0(\xi)| \\
&\lesssim \xi^{-\frac{1}{2}+}\tau_0^{0+}\int_{\tau_0}^{\infty}\frac{\lambda(\tau_0)}{\lambda(\sigma)}|\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)|\,d\sigma
\end{aligned}
$$

Referring to Lemma 112, we have (using the point wise bounds on $\phi(R,\xi)$ in Proposition 103)

$$
\begin{aligned}
&|\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)| \\
&\lesssim \|\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)\|_{L^1_{dR}} \qquad (4.74)\\
&\lesssim \sigma^{-2}\cdot\sigma^{\frac{1}{2}(1+\nu^{-1})-k_0-2+}\cdot\log\tau_0\tau_0^{k_0+1-\frac{1}{2}(1+\nu^{-1})-}\cdot\sigma\cdot[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|].
\end{aligned}
$$

Inserting this in the preceding σ-integral for $\widetilde{\triangle}\widetilde{x^{(0)}}_0(\xi)$, we find

$$
|\widetilde{\triangle}\widetilde{x^{(0)}}_0(\xi)| \lesssim \xi^{-\frac{1}{2}+}\tau_0^{-1+}\cdot[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|].
$$

In turn recalling the asymptotics for the spectral density $\rho(\xi)$ from Proposition 103, we obtain

$$
\begin{aligned}
&|\int_0^1\frac{\rho^{\frac{1}{2}}(\xi)\widetilde{\triangle}\widetilde{x^{(0)}}_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(u)\,du]\,d\xi| \\
&\lesssim \tau_0^{-(1-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|]\cdot\int_0^1\xi^{-(1-)}\,d\xi \\
&\lesssim \tau_0^{-(1-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|].
\end{aligned}
$$

(ii): $1\le\xi<\frac{\lambda^2(\sigma)}{\lambda^2(\tau_0)}$. Call the contribution to $\widetilde{\triangle}\widetilde{x^{(0)}}_0$ under this restriction $\widetilde{\triangle}\widetilde{x^{(0)}}_{01}$. Again referring to the ρ-asymptotics from Proposition 103 an recalling (4.66), we infer

$$
\begin{aligned}
&|\widetilde{\triangle}\widetilde{x^{(0)}}_{01}| \\
&\lesssim \xi^{-1}\int_{\tau_0}^{\infty}\chi_{\xi<\frac{\lambda^2(\sigma)}{\lambda^2(\tau_0)}}\frac{\lambda(\tau_0)}{\lambda(\sigma)}|\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)|\,d\sigma \\
&\lesssim \xi^{-\frac{3}{2}}\tau_0^{-(1-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|],
\end{aligned}
$$

where we have used the same asymptotics for $|\mathscr{F}\big(\ldots\big)|$ as in *(i)*. In turn, this implies

$$
|\int_1^{\infty}\frac{\rho^{\frac{1}{2}}(\xi)\widetilde{\triangle}\widetilde{x^{(0)}}_{01}(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(u)\,du]\,d\xi| \lesssim \tau_0^{-(1-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|].
$$

(iii): $\xi>\frac{\lambda^2(\sigma)}{\lambda^2(\tau_0)}$. Here we use that for the corresponding contribution to $\widetilde{\triangle}\widetilde{x^{(0)}}_0$, which we call

$\widetilde{\triangle}\widetilde{x^{(0)}}_{02}$, we have

$$
\begin{aligned}
&\|\xi\widetilde{\triangle}\widetilde{x^{(0)}}_{02}(\xi)\|_{L^2_{d\xi}}\\
&\lesssim \int_{\tau_0}^{\infty}\|\xi^{\frac{1}{2}}\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\|_{L^2_{d\xi}(\xi>\frac{\lambda^2(\sigma)}{\lambda^2(\tau_0)})}\,d\sigma\\
&\lesssim \int_{\tau_0}^{\infty}\|\xi^{\frac{1}{2}}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\cdot)\|_{L^2_{d\rho}}\,d\sigma\\
&\lesssim \int_{\tau_0}^{\infty}\|\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\tau}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\cdot)\|_{H^1_{dR}}\,d\sigma\\
&\lesssim \tau_0^{-\frac{3}{2}}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|],
\end{aligned}
$$

where we have used (4.72). We conclude by Cauchy-Schwarz that

$$
\begin{aligned}
&\Big|\int_1^{\infty}\frac{\rho^{\frac{1}{2}}(\xi)\widetilde{\triangle}\widetilde{x^{(0)}}_{02}(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(u)\,du]\,d\xi\Big|\\
&\lesssim \|\xi\widetilde{\triangle}\widetilde{x^{(0)}}_{02}(\xi)\|_{L^2_{d\xi}}\lesssim \tau_0^{-\frac{3}{2}}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|].
\end{aligned}
$$

The contributions of the remaining terms forming e_{approx} are handled similarly, as is the second estimate of the lemma involving $\widetilde{\triangle}\widetilde{x^{(0)}}_1$. □

We next use the same argument as in (4.54) to infer the asymptotic relations (for $C\gg 1,\tau_0\gg 1$)

$$
\Big|\int_0^{\infty}\frac{\rho^{\frac{1}{2}}(\xi)\mathscr{F}\big(\chi_{R\leq C\tau}\phi(R,0)\big)(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(u)\,du]\,d\xi\Big|\sim 1,
$$

$$
\Big|\int_0^{\infty}\frac{\rho^{\frac{1}{2}}(\xi)\mathscr{F}\big(\chi_{R\leq C\tau}\phi(R,0)\big)(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\infty}\lambda^{-1}(u)\,du]\,d\xi\Big|\sim \tau_0,
$$

The preceding lemma in conjunction with these asymptotics implies that the vanishing relations (4.87), (4.70) will be satisfied for α,β in (4.73) satisfying

$$
|\alpha|\lesssim \tau_0^{-(1-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|],\ |\beta|\lesssim \tau_0^{-(2-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|].
$$

Then **Step 2** is concluded by observing the bounds (4.49), (4.57), as well as the analogous bound (recalling (4.65))

$$
\|\frac{C\tau_0}{\langle C\tau_0\xi^{\frac{1}{2}}\rangle^N}\|_{\widetilde{S}_1}\lesssim \tau_0^{\delta_0},
$$

whence

$$
\|\triangle\widetilde{\widetilde{x}}_0^{(0)}\|_{\widetilde{S_1}}+\|\triangle\widetilde{\widetilde{x}}_1^{(0)}\|_{\widetilde{S_2}}\lesssim \tau_0^{-(2-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|].
$$

Step 3: Proof of the low frequency bounds

Here we control $\triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi)$ in the low frequency regime $\xi<1$. The choice of $\triangle\widetilde{x^{(0)}}_0, \triangle\widetilde{x^{(0)}}_1$ at the beginning of **Step 2** imply that

$$\triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi) = -\int_\tau^\infty U(\tau,\sigma)\cdot\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{approx})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma,$$

and in light of the asserted bounds of the proposition, we need to control

$$(\frac{\tau}{\tau_0})^{-\kappa}\|\chi_{\xi<1}\triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi)\|_{S_1},\ \|\xi^{-\delta_0}\mathscr{D}_\tau\triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi)\|_{Sqr(\xi<1)}.$$

We show here how to bound the first quantity, the second being more of the same. We use that

$$\|\xi^{-\delta_0}U(\tau,\sigma)\|_{L^2_{d\xi}(\xi<1)}\lesssim\tau^{\delta_0}\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},$$

which then implies

$$\begin{aligned}&\|\xi^{-\delta_0}\triangle_{>\tau}\widetilde{x^{(0)}}(\tau,\xi)\|_{L^2_{d\xi}(\xi<1)}\\&\lesssim\tau^{\delta_0}\int_\tau^\infty\frac{\lambda(\tau)}{\lambda(\sigma)}\|\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{approx})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\|_{L^\infty_{d\xi}}\,d\sigma\end{aligned}$$

Then as usual we distinguish between the different parts of e_{approx}. For example, for the contribution of the principal part $e_{\text{prelim}}-\widetilde{e}_{\text{prelim}}$, we get by arguing as in *(i)* of the proof of the preceding lemma

$$\begin{aligned}&\tau^{0+}\int_\tau^\infty\frac{\lambda(\tau)}{\lambda(\sigma)}\|\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\|_{L^\infty_{d\xi}}\,d\sigma\\&\lesssim\tau^{0+}\cdot\frac{\tau_0^{k_0+1}\log\tau_0}{\lambda_{0,0}^{\frac12}(\tau_0)}\cdot\int_\tau^\infty\frac{\lambda(\tau)}{\lambda(\sigma)}\sigma^{-k_0-3}\lambda_{0,0}^{\frac12}(\sigma)\,d\sigma\cdot[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|]\\&\lesssim\tau_0^{-(1-)}\cdot[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|].\end{aligned}$$

This is even better than what we need, since we have omitted the weight $(\frac{\tau}{\tau_0})^{-\kappa}$. The remaining terms in e_{approx} lead to similar contributions.

Step 4: Control over the data $(\triangle\widetilde{x^{(0)}}_0(\xi),\triangle\widetilde{x^{(0)}}_1(\xi))$ for the free part in the low frequency regime

In light of the low frequency bound established in the preceding step, it suffices to establish the high-frequency bound, i. e. restrict to $\xi>1$. Thus in light of (4.8) we need to bound

$$\|\xi^{1+\delta_0}\triangle\widetilde{x^{(0)}}_0(\xi)\|_{L^2_{d\xi}(\xi>1)}+\|\xi^{\frac12+\delta_0}\triangle\widetilde{x^{(0)}}_1(\xi)\|_{L^2_{d\xi}(\xi>1)}.$$

where $(\triangle\widetilde{x^{(0)}}_0(\xi), \triangle\widetilde{x^{(0)}}_1(\xi))$ are defined as at the beginning of **Step 2**. We shall establish the desired estimate for $\triangle\widetilde{x^{(0)}}_0$ and the contribution of $e_{\text{prelim}} - \widetilde{e}_{\text{prelim}}$, the remaining contributions as well as the term $\triangle\widetilde{x^{(0)}}_1$ being more of the same. Note that on account of the final bound of **Step 2**, the correction terms $\triangle\widetilde{\widetilde{x}}_0^{(0)}, \triangle\widetilde{\widetilde{x}}_1^{(0)}$ satisfy the required bounds. The norm $\|\xi^{1+}\triangle\widetilde{x^{(0)}}_0(\xi)\|_{L^2_{d\xi}(\xi>1)}$ can be bounded by

$$\begin{aligned}
&\|\xi^{1+\delta_0}\int_{\tau_0}^{\infty} U(\tau,\sigma)\cdot\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\,d\sigma\|_{L^2_{d\xi}(\xi>1)}\\
&\lesssim\int_{\tau_0}^{\infty}\frac{\lambda(\tau_0)}{\lambda(\sigma)}\|\xi^{\delta_0}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\|_{L^2_{d\xi}(1<\xi<\frac{\lambda^2(\sigma)}{\lambda^2(\tau_0)})}\,d\sigma \qquad (4.75)\\
&+\int_{\tau_0}^{\infty}\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\|\xi^{\frac12+\delta_0}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\|_{L^2_{d\xi}(\xi>\frac{\lambda^2(\sigma)}{\lambda^2(\tau_0)})}\,d\sigma.
\end{aligned}$$

Then recalling parameter $\kappa = 2(1+\nu^{-1})\delta_0$, the first term on the right (intermediate frequencies) is bounded by

$$\begin{aligned}
&\int_{\tau_0}^{\infty}\frac{\lambda(\tau_0)}{\lambda(\sigma)}\|\xi^{\delta_0}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\|_{L^2_{d\xi}(1<\xi<\frac{\lambda^2(\sigma)}{\lambda^2(\tau_0)})}\,d\sigma\\
&\lesssim\int_{\tau_0}^{\infty}(\frac{\sigma}{\tau_0})^{\kappa}\|\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\|_{L^\infty_{d\xi}}\,d\sigma
\end{aligned}$$

and further recalling (4.74), this is bounded by

$$\begin{aligned}
&\lesssim\log\tau_0\cdot\tau_0^{k_0+1-}\cdot[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|]\cdot\int_{\tau_0}^{\infty}(\frac{\sigma}{\tau_0})^{\kappa}\cdot\frac{\lambda_{0,0}^{\frac12}(\sigma)}{\lambda_{0,0}^{\frac12}(\tau_0)}\cdot\sigma^{-k_0-3+}\,d\sigma\\
&\lesssim\tau_0^{-(1-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|]
\end{aligned}$$

The second term on the right of (4.75) (large frequencies) is bounded by

$$\begin{aligned}
&\int_{\tau_0}^{\infty}\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\|\xi^{\frac12+\delta_0}\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\|_{L^2_{d\xi}(\xi>\frac{\lambda^2(\sigma)}{\lambda^2(\tau_0)})}\,d\sigma\\
&\lesssim\int_{\tau_0}^{\infty}(\frac{\sigma}{\tau_0})^{\kappa}\|\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R(e_{\text{prelim}}-\widetilde{e}_{\text{prelim}})(\sigma,\cdot)\|_{H^{1+2\delta_0}_{dR}}\,d\sigma\\
&\lesssim\tau_0^{-(\frac32-)}[\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|],
\end{aligned}$$

where we have taken advantage of (4.72).

Step 5: Lipschitz continuity of the corrections $(\triangle\widetilde{\widetilde{x}}_0^{(0)}(\xi), \triangle\widetilde{\widetilde{x}}_1^{(0)}(\xi))$ **with respect to the original perturbations** (x_0, x_1, x_{0d})

Here we prove the final assertion of the proposition. We note that on account of our construction of $(\triangle\widetilde{\widetilde{x}}_0^{(0)}(\xi), \triangle\widetilde{\widetilde{x}}_1^{(0)}(\xi))$ in **Step 2**, their dependence on (x_0, x_1) comes solely through the coefficients α, β. We consider the first of these, the second being treated similarly. Then recall that we have

$$\alpha = \frac{c_1(\gamma_1,\gamma_2)}{c_2(\gamma_1,\gamma_2)},$$

where we have introduced the functions

$$c_1(\gamma_1,\gamma_2) = -\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\widetilde{\triangle}\widetilde{x^{(0)}}_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(u)\,du]\,d\xi,$$

$$c_2(\gamma_1,\gamma_2) = \int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\widetilde{\triangle}\mathscr{F}\big(\chi_{R\leq C\tau}\phi(R,0)\big)(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(u)\,du]\,d\xi,$$

and we also recall the notation, introduced shortly before Lemma 118

$$\widetilde{\triangle}\widetilde{x^{(0)}}_0(\xi) = \int_{\tau_0}^\infty U(\tau_0,\sigma)\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\sigma}Re_{approx}\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\,d\sigma. \tag{4.76}$$

Observe that there is dependence on $\gamma_{1,2}$ via $\lambda(\tau) = \lambda_{\gamma_1,\gamma_2}(\tau)$, $\tau_0 = \int_{t_0}^\infty \lambda(s)\,ds$, as well as

$$e_{approx} = e_{approx}(\tau_{0,0}, R_{0,0}, \gamma_{1,2}),$$

with $R_{0,0}$ defined as in (4.31), and $\tau_{0,0} = \int_t^\infty s^{-1-\nu}\,ds$, and we interpret $R_{0,0}$ as a function of $\tau, R, \gamma_{1,2}$, and $\tau_{0,0}$ as a function of $\tau, \gamma_{1,2}$. Then writing

$$\widetilde{e}_{approx}(\tau,R,\gamma_{1,2}) := e_{approx}(\tau_{0,0}(\tau,\gamma_{1,2}), R_{0,0}(R,\tau,\gamma_{1,2}),\gamma_{1,2}),$$

one derives after some algebraic manipulations a relation of the form

$$\partial_{\gamma_j}\widetilde{e}_{approx} = A_j(\tau,\gamma_{1,2})R\partial_R\widetilde{e}_{approx} + B_j(\tau,\gamma_{1,2})\tau\partial_\tau\widetilde{e}_{approx} + \partial_{\gamma_j}e_{approx}, \tag{4.77}$$

where the coefficients are given in terms of

$$\begin{aligned} A_j(\tau,\gamma_{1,2}) &= \frac{\lambda}{\lambda_{0,0}}\partial_{\gamma_j}\big(\frac{\lambda_{0,0}}{\lambda}\big) - \partial_{\gamma_j}\tau_{0,0}(\partial_\tau\tau_{0,0})^{-1}\cdot\frac{\lambda}{\lambda_{0,0}}\partial_\tau\big(\frac{\lambda_{0,0}}{\lambda}\big), \\ B_j(\tau,\gamma_{1,2}) &= \tau^{-1}\cdot\partial_{\gamma_j}\tau_{0,0}(\partial_\tau\tau_{0,0})^{-1}. \end{aligned} \tag{4.78}$$

In light of the definition (4.24) as well as (4.41), we infer the bounds

$$\begin{aligned} |A_j(\tau,\gamma_{1,2})| &\lesssim \tau^{-k_0}\log\tau + \tau^{-k_0}\log\tau\cdot O_{\tau_0}\big(\|(x_0,x_1)\|_{\widetilde{S}} + |x_{0d}|\big), \\ |B_j(\tau,\gamma_{1,2})| &\lesssim \tau^{-k_0}\log\tau. \end{aligned}$$

As for the integration kernel $U(\tau_0,\sigma)$, recalling (4.66), we find

$$\begin{aligned}&|\partial_{\gamma_j}U(\tau_0,\sigma)|\\&\lesssim \log\tau_0\tau_0^{-k_0}\frac{\lambda^{\frac{3}{2}}(\tau_0)}{\lambda^{\frac{3}{2}}(\sigma)}\frac{\rho^{\frac{1}{2}}(\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)}{\rho^{\frac{1}{2}}(\xi)}\big(\tau_0+|\frac{\sin[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^{\sigma}\lambda^{-1}(u)\,du]}{\xi^{\frac{1}{2}}}|\big)\end{aligned}$$

Finally, we can bound $\partial_{\gamma_j}c_1(\gamma_1,\gamma_2)$. Observe the crude bounds

$$|\widetilde{\triangle x^{(0)}}_0(\xi)|\lesssim\begin{cases}\xi^{-(\frac{1}{2}-)}\tau_0^{0+}\cdot\tau_0^{-2}\big(\|(x_0,x_1)\|_{\tilde{S}}+|x_{0d}|\big), & \text{if}\quad \xi<1,\\ \xi^{-(\frac{3}{2}+)}\cdot\tau_0^{-2}\big(\|(x_0,x_1)\|_{\tilde{S}}+|x_{0d}|\big), & \text{if}\quad \xi>1,\end{cases}$$

which follow from (4.76), (4.66), as well as Theorem 111 and the bound (4.41). Again taking advantage of the ρ-asymptotics from Proposition 103, we infer

$$\begin{aligned}&|\int_0^\infty\frac{\rho^{\frac{1}{2}}(\xi)\widetilde{\triangle x^{(0)}}_0(\xi)}{\xi^{\frac{1}{4}}}\partial_{\gamma_j}\big(\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty\lambda^{-1}(u)\,du]\big)\,d\xi|\\&\lesssim\tau_0^{-(1-)}\big(\|(x_0,x_1)\|_{\tilde{S}}+|x_{0d}|\big).\end{aligned}\tag{4.79}$$

The preceding point wise bound for $\partial_{\gamma_j}U(\tau_0,\sigma)$ easily reveals that a similar bound is obtained when $\widetilde{\triangle x^{(0)}}_0(\xi)$ in the preceding is replaced by

$$\int_{\tau_0}^\infty\partial_{\gamma_j}U(\tau_0,\sigma)\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\sigma}Re_{approx}\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\,d\sigma.$$

It then remains to consider the case when the operator ∂_{γ_j} falls on the Fourier transform

$$\mathscr{F}\big(\lambda^{-2}\chi_{R\lesssim\sigma}Re_{approx}\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)$$

in (4.76), which we handle schematically as follows. Note that when ∂_{γ_j} falls on $\lambda(\tau)$, we obtain a function bounded by a $\lesssim\tau^{-k_0}\log\tau\cdot\lambda(\tau)$, in light of (4.24). Further, recall (4.77) as well as (4.78) and the bounds following it, as well as Theorem 106 which gives a translation of $R\partial_R$ to the Fourier side. In all, we infer a schematic relation of the form

$$\partial_{\gamma_1}\big[\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}Re_{approx}\big)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\big]=\sum_{j=1}^5A_j,$$

with the following terms on the right: writing $G(\sigma,R)=\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}Re_{approx}$,

$$\begin{aligned}&A_1=\sigma^{-k_0}\lambda(\sigma)\mathscr{F}(G)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi),\ A_2=\sigma^{-k_0}\lambda(\sigma)(\xi\partial_\xi)[\mathscr{F}(G)(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)]\\&A_3=\sigma^{-k_0}\lambda(\sigma)[\mathscr{K}\mathscr{F}(G)](\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi),\ A_4=\sigma^{-k_0}\lambda(\sigma)[(\sigma\partial_\sigma)\mathscr{F}(G)](\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\\&A_5=\mathscr{F}\big(\lambda^{-2}(\sigma)\chi_{R\lesssim\sigma}R\partial_{\gamma_1}e_{approx}\big),\end{aligned}$$

with a similar relation for ∂_{γ_2} but with σ^{-k_0} replaced by $\sigma^{-k_0}\log\sigma$.
But then performing integration by parts with respect to ξ or σ as needed, and recalling the point wise bounds on $\widetilde{\triangle}\widetilde{x^{(0)}}_0(\xi)$, we infer

$$|\int_{\tau_0}^{\infty} U(\tau_0,\sigma)\big(\sum_{j=1}^{4} A_j\big)\,d\sigma| \lesssim_{\tau_0} \|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}| \tag{4.80}$$

Finally also recalling the structure of e_{approx} from Theorem 111, we get the bound

$$|\int_{\tau_0}^{\infty} U(\tau_0,\sigma)A_5| \lesssim \tau_0^{-(k_0+2-)}\lambda_{0,0}^{\frac{1}{2}}(\tau_0) \tag{4.81}$$

It is this last term which is dominant, of course. Combining (4.79) and the remark following it with (4.80), (4.81), we finally obtain the bound (for $j=1,2$)

$$|\partial_{\gamma_j} c_1| \lesssim \tau_0^{-(k_0+2-)}\lambda_{0,0}^{\frac{1}{2}}(\tau_0) + O_{t_0}\big(\|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}|\big) \tag{4.82}$$

A simple variation of the preceding arguments also implies the much easier bound

$$|\partial_{\gamma_j} c_2| \lesssim \tau_0^{-(k_0-1)}\log\tau_0\cdot\lambda_{0,0}(\tau_0). \tag{4.83}$$

Combining (4.82), (4.83), and also recalling $c_2 \sim 1$ from the end if **Step 2**, we finally obtain the desired estimate

$$|\partial_{\gamma_j}\alpha| \lesssim \tau_0^{-(k_0+2-)}\lambda_{0,0}^{\frac{1}{2}}(\tau_0) + O_{t_0}\big(\|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}|\big). \tag{4.84}$$

Finally, comparing the corrections

$$\triangle\tilde{\tilde{x}}_0^{(0)}(\xi) = \alpha\mathscr{F}\big(\chi_{R\le C\tau}\phi(R,0)\big)(\xi),\ \triangle\overline{\tilde{\tilde{x}}}_0^{(0)}(\xi) = \overline{\alpha}\mathscr{F}\big(\chi_{R\le C\tau}\phi(R,0)\big)(\xi),$$

corresponding to different data quadruples $\underline{x}_j, \underline{\overline{x}}_j$, we find

$$\begin{aligned}\|\triangle\tilde{\tilde{x}}_0^{(0)} - \triangle\overline{\tilde{\tilde{x}}}_0^{(0)}\|_{\tilde{S}_1} &\lesssim |\alpha-\overline{\alpha}|\cdot\tau_0^{\delta_0}\\ &\lesssim \big(\sum_j |\partial_{\gamma_j}\alpha|\cdot|\gamma_j-\overline{\gamma}_j|\big)\cdot\tau_0^{\delta_0},\end{aligned}$$

where we have used a bound at the end of **Step 2** for the first inequality, and the preceding can be bounded by

$$\lesssim \big[\tau_0^{-(1-)} + O_{t_0}\big(\|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}|\big)\big]\cdot\big(\|(x_0-\overline{x}_0, x_1-\overline{x}_1)\|_{\tilde{S}} + |x_{0d}-\overline{x}_{0d}|\big),$$

in light of (4.84) as well as Lemma 115. This is the desired Lipschitz dependence of $\triangle\tilde{\tilde{x}}_0^{(0)}(\xi)$ on the data, provided the latter are chosen small enough (depending on τ_0), and that of $\triangle\tilde{\tilde{x}}_1^{(0)}(\xi)$ is similar.

This concluded the proof of the proposition for the continuous spectral part, and we omit the much simpler routine estimates for the discrete spectral part. □

Setup of the iteration scheme; the higher iterates

We next add a sequence of corrections $\triangle x^{(j)}(\tau,\xi)$ to the zeroth iterate in order to arrive at a solution of (4.61), but with data differing slightly from (4.62). Specifically, we set for the first iterate

$$\underline{x}^{(1)} = \underline{x}^{(0)} + \underline{\triangle x}^{(1)}$$

where

$$\left(\mathscr{D}_\tau^2 + \beta(\tau)\mathscr{D}_\tau + \underline{\xi}\right)\underline{\triangle x}^{(1)}(\tau,\xi) = \mathscr{R}(\tau,\underline{x}^{(0)}) + \underline{\triangle f}^{(0)}(\tau,\underline{\xi}),$$

and we recall (4.63) and further use the notation

$$\begin{aligned}\triangle f^{(0)}(\tau,\xi) &= \mathscr{F}\left(\lambda^{-2}(\tau)\left[5(u_{approx}^4 - u_0^4)\widetilde{\varepsilon}^{(0)} + RN(u_{approx},\widetilde{\varepsilon}^{(0)})\right]\right)(\xi),\\ \triangle f_d^{(0)}(\tau) &= \langle \lambda^{-2}(\tau)\left[5(u_{approx}^4 - u_0^4)\widetilde{\varepsilon}^{(0)} + RN(u_{approx},\widetilde{\varepsilon}^{(0)}),\phi_d(R)\right\rangle.\end{aligned}$$

and further, we naturally set

$$\widetilde{\varepsilon}^{(0)}(\tau,R) = x_d^{(0)}(\tau)\phi_d(R) + \int_0^\infty \phi(R,\xi)x^{(0)}(\tau,\xi)\rho(\xi)\,d\xi.$$

For the higher corrections $\underline{\triangle x}^{(j)}$, $j \geq 2$, defining the higher iterates, we set correspondingly

$$\left(\mathscr{D}_\tau^2 + \beta(\tau)\mathscr{D}_\tau + \underline{\xi}\right)\underline{\triangle x}^{(j)}(\tau,\xi) = \mathscr{R}(\tau,\underline{\triangle x}^{(j-1)}) + \underline{\triangle f}^{(j-1)}(\tau,\xi), \tag{4.85}$$

and we use the definitions

$$\begin{aligned}\triangle f^{(j-1)}(\tau,\xi) &= \mathscr{F}\left(\lambda^{-2}(\tau)\chi_{R\lesssim\tau}\left[5(u_{approx}^4 - u_0^4)\triangle\widetilde{\varepsilon}^{(j-1)} + RN(u_{approx},\triangle\widetilde{\varepsilon}^{(j-1)})\right]\right)(\xi),\\ \triangle f_d^{(j-1)}(\tau)&\\ &= \int_0^\infty \lambda^{-2}(\tau)\chi_{R\lesssim\tau}\left[5(u_{approx}^4 - u_0^4)\triangle\widetilde{\varepsilon}^{(j-1)} + RN(u_{approx},\triangle\widetilde{\varepsilon}^{(j-1)})\right]\phi_d(R)\,dR\end{aligned}$$

where we set

$$\triangle\widetilde{\varepsilon}^{(j-1)}(\tau,R) = \int_0^\infty \phi(R,\xi)\triangle x^{(j-1)}(\tau,\xi)\rho(\xi)\,d\xi + \triangle x_d^{(j-1)}(\tau)\phi_d(R),\ j \geq 2.$$

The fact that *upon using suitable initial conditions* these equations yield in fact iterates which rapidly converge to zero in a suitable sense follows exactly as in [49], and so we formulate the corresponding result, which is a summary of Propositions 9.1 - 9.6 and most importantly Corollary 12.2, Corollary 12.3 in [49]:

Proposition 119. *For each $j \geq 1$, there exists a pair $(\triangle\widetilde{\widetilde{x}}_0^{(j)},\triangle\widetilde{\widetilde{x}}_1^{(j)}) \in \widetilde{S}$, and such that if we set* *up the inductive scheme (recall* (4.66)*)*

$$\begin{aligned}&\triangle x^{(j)}(\tau,\xi) = \\ &\int_{\tau_0}^{\tau} U(\tau,\sigma)[\mathscr{R}(\tau,\underline{\triangle x}^{(j-1)}) + \triangle f^{(j-1)}(\tau,\underline{\xi})](\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma \\ &+ S(\tau)\big(\triangle\tilde{\tilde{x}}_0^{(j)},\triangle\tilde{\tilde{x}}_1^{(j)}\big)\end{aligned} \tag{4.86}$$

for the continuous spectral part, while we set (recall (4.67)*)*

$$\triangle_d x^{(j)}(\tau) = \int_{\tau_0}^{\infty} H_d(\tau,\sigma)\cdot[\mathscr{R}_d(\tau,\underline{\triangle x}^{(j-1)}) + \triangle_d f^{(j-1)}(\tau,\underline{\xi})](\sigma)\,d\sigma,$$

then we obtain control over the iterates in the following precise sense: there is a splitting

$$\triangle x^{(j)}(\tau,\xi) = \triangle_{>\tau} x^{(j)}(\tau,\xi) + S(\tau)\big(\triangle\tilde{x}_0^{(j)},\triangle\tilde{x}_1^{(j)}\big)$$

in which $\triangle\tilde{x}_0^{(j)},\triangle\tilde{x}_1^{(j)}$ *satisfy the vanishing conditions*

$$\int_0^{\infty}\frac{\rho^{\frac{1}{2}}(\xi)\triangle\tilde{x}_0^{(j)}(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\int_{\tau_0}^{\infty}\lambda^{-1}(u)\,du]\,d\xi = 0 \tag{4.87}$$

$$\int_0^{\infty}\frac{\rho^{\frac{1}{2}}(\xi)\triangle\tilde{x}_1^{(j)}(\xi)}{\xi^{\frac{3}{4}}}\sin[\lambda(\tau_0)\int_{\tau_0}^{\infty}\lambda^{-1}(u)\,du]\,d\xi = 0, \tag{4.88}$$

and such that if we set

$$\widetilde{\triangle x^{(j)}}(\tau,\xi) = \int_{\tau_0}^{\tau} U(\tau,\sigma)\cdot[\mathscr{R}(\tau,\underline{\triangle x}^{(j-1)}) + \triangle f^{(j-1)}(\tau,\underline{\xi})](\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi)\,d\sigma$$

and introduce the quantities (with $\kappa = 2(1+\nu^{-1})\delta_0$*)*

$$\begin{aligned}\triangle A_j :=& \sup_{\tau\geq\tau_0}(\frac{\tau_0}{\tau})^{\kappa}\|\chi_{\xi>1}\triangle x^{(j)}(\tau,\xi)\|_{S_1} + \|\xi^{\frac{1}{2}+\delta_0}\mathscr{D}_\tau\widetilde{\triangle x^{(j)}}(\tau,\xi)\|_{Sqr(\xi>1)} \\ &+ \sup_{\tau\geq\tau_0}(\frac{\tau_0}{\tau})^{\kappa}\|\chi_{\xi<1}\triangle_{>\tau}\triangle x^{(j)}(\tau,\xi)\|_{S_1} + \|\xi^{-\delta_0}\mathscr{D}_\tau\triangle_{>\tau}\triangle x^{(j)}(\tau,\xi)\|_{Sqr(\xi<1)} \\ &+ \|(\triangle\tilde{x}_0^{(j)},\triangle\tilde{x}_1^{(j)})\|_{\tilde{S}} + \|(\triangle\tilde{\tilde{x}}_0^{(j)},\triangle\tilde{\tilde{x}}_1^{(j)})\|_{\tilde{S}} + \sup_{\tau\geq\tau_0}\tau^{(1-)}(|\triangle x_d^{(j)}(\tau)| + |\partial_\tau\triangle x_d^{(j)}(\tau)|),\end{aligned}$$

where we recall (4.68) *for the definition of* $\|\cdot\|_{Sqr}$*, then we have exponential decay*

$$\triangle A_j \lesssim_\delta \delta^j[\|(x_0,x_1)\|_{\tilde{S}} + |x_{0d}|]$$

for any given $\delta > 0$*, provided* τ_0 *is sufficiently large (or equivalently,* t_0 *is sufficiently small). In particular, the series*

$$\underline{x}(\tau,\xi) = \underline{x}^{(0)}(\tau,\xi) + \sum_{j\geq 1}\underline{\triangle x}^{(j)}(\tau,\xi),$$

converges with

$$\sup_{\tau\geq\tau_0}(\frac{\tau_0}{\tau})^{\kappa}\|\xi^{1+\delta_0}x(\tau,\xi)\|_{L^2_{d\xi}(\xi>1)}+\sup_{\tau\geq\tau_0}(\frac{\tau_0}{\tau})^{-\kappa}\|\xi^{\frac{1}{2}+\delta_0}\mathscr{D}_\tau x(\tau,\xi)\|_{L^2_{d\xi}(\xi>1)}$$
$$\lesssim\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|.$$

Also, for low frequencies, i. e. $\xi<1$, there is a decomposition

$$x(\tau,\xi)=x_{>\tau}(\tau,\xi)+S(\tau)\big(\widetilde{x}_0,\widetilde{x}_1\big)$$

such that $\widetilde{x}_0,\widetilde{x}_1$ satisfy the natural analogues of (4.87), (4.88), *and we have the bounds*

$$\sup_{\tau\geq\tau_0}(\frac{\tau_0}{\tau})^{\kappa}\|\xi^{-\delta_0}x(\tau,\xi)\|_{L^2_{d\xi}(\xi<1)}+\sup_{\tau\geq\tau_0}(\frac{\tau_0}{\tau})^{-\kappa}\|\xi^{-\delta_0}\mathscr{D}_\tau x(\tau,\xi)\|_{L^2_{d\xi}(\xi<1)}$$
$$+\|(\widetilde{x}_0,\widetilde{x}_1)\|_{\widetilde{S}}\lesssim\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|.$$

Finally, we also have

$$\sup_{\tau\geq\tau_0}\tau^{1-}|x_d(\tau)-x_d^{(0)}(\tau)|\lesssim\|(x_0,x_1)\|_{\widetilde{S}}+|x_{0d}|.$$

The function

$$u(\tau,R)=u_{approx}(\tau,R)+\widetilde{\epsilon}(\tau,R)$$

with

$$\widetilde{\epsilon}(\tau,R):=x_d(\tau)\phi_d(R)+\int_0^\infty\phi(R,\xi)x(\tau,\xi)\rho(\xi)\,d\xi$$

is then the desired solution of (4.60), *satisfying the properties in terms of its Fourier transform specified in Theorem 116. In fact, we set*

$$\triangle x_\kappa^{(\gamma_1,\gamma_2)}=\sum_{j\geq1}\triangle\widetilde{\widetilde{x}}_\kappa^{(j)},\ \triangle x_{\kappa d}^{(\gamma_1,\gamma_2)}=\sum_{j\geq1}\partial_\tau^\kappa\triangle x_d^{(j)}|_{\tau=\tau_0},\ \kappa=0,1.$$

In fact, all of the assertions in the preceding long proposition follow exactly from the arguments in [49] (the only difference being the slightly different scaling law $\lambda(\tau)$), and this will easily establish almost all of Theorem 116, *except* its last statement concerning the *Lipschitz continuous dependence* of the initial data perturbation with respect to the initial perturbation (x_0,x_1). This is a somewhat delicate point which requires a special argument, analogous to the one given for the corresponding assertion in Proposition 117. We formulate this as a separate proposition at the level of the iterative corrections:

Proposition 120. *If $(\triangle\widetilde{\widetilde{x}}_0^{(j)},\triangle\widetilde{\widetilde{x}}_1^{(j)}),(\triangle\widetilde{\widetilde{\widetilde{x}}}_0^{(j)},\triangle\widetilde{\widetilde{\widetilde{x}}}_1^{(j)})$, $j\geq1$, are as in the preceding proposition and with respect to perturbations specified in terms of data quadruples $(\underline{x}_0,\underline{x}_1)$ respectively*

$(\underline{x}_0, \underline{x}_1) \in \widetilde{S}$, *then for any given* $\delta > 0$ *we have the Lipschitz bound*

$$\|(\triangle\tilde{\tilde{x}}_0^{(j)} - \triangle\tilde{\tilde{\overline{x}}}_0^{(j)}, \triangle\tilde{\tilde{x}}_1^{(j)} - \triangle\tilde{\tilde{\overline{x}}}_1^{(j)})\|_{\widetilde{S}}$$
$$\lesssim_\delta \tau_0^{-(1-)}\delta^j[\|(x_0 - \overline{x}_0, x_1 - \overline{x}_1)\|_{\widetilde{S}} + |x_{0d} - \overline{x}_{0d}|],$$

provided τ_0 *is sufficiently large compared to* δ*, and*

$$\|(x_0, x_1)\|_{\widetilde{S}} + \|(\overline{x}_0, \overline{x}_1)\|_{\widetilde{S}} + |x_{0d}| + |\overline{x}_{0d}|$$

is sufficiently small depending on τ_0*.*

To begin the sketch of the proof, we observe from the proofs of Proposition 7.1, 8.1, 9.1 in [49] that the profiles of the corrections $\triangle\tilde{\tilde{x}}_\kappa^{(j)}$, $\kappa = 0, 1$, are fixed up to a multiplication parameter, and more precisely we set

$$\triangle\tilde{\tilde{x}}_0^{(j)} = \alpha^{(j)}\mathscr{F}\big(\chi_{R\leq C\tau_0}\phi(R,0)\big),\ \triangle\tilde{\tilde{x}}_1^{(j)} = \beta^{(j)}\mathscr{F}\big(\chi_{R\leq C\tau_0}\phi(R,0)\big),$$

whence the only dependence of the corrections $\triangle\tilde{\tilde{x}}_\kappa^{(j)}$ on the data $x_{0,1}$ reside in the coefficients $\alpha^{(j)}, \beta^{(j)}$. The latter, however, depend in a complex manner on the iterative functions $\triangle x^{(j)}$,$\triangle_d x^{(j)}$, and so we cannot get around analysing the (Lipschitz)-dependence of the latter on $x_{0,1}$. This latter task is rendered somewhat cumbersome by the fact that in each iterative step we use a parametrix which re-scales the ingredients (via the factors $\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}$), which depend on $\gamma_{1,2}$ whence on $x_{0,1}$, and so differentiating with respect to γ_j will result in a loss of smoothness. What saves things here is the fact that the coefficients $\alpha^{(j)}, \beta^{(j)}$ are given by certain integrals, which are well-behaved with respect to inputs with lesser regularity, as already seen in Step 5 of the proof of Proposition 117: there differentiating the term $\mathscr{F}\big(\lambda^{-2}(\sigma)Re_{approx}\big)(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)$ with respect to γ_1 results in a term (see the term A_2 in the list of terms preceding (4.80))

$$\tau_0^{-k_0}(\xi\partial_\xi)\big[\mathscr{F}\big(\lambda^{-2}(\sigma)Re_{approx}\big)(\sigma, \frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\big]$$

which is of lesser regularity with respect to ξ, but the corresponding contribution to $\partial_{\gamma_j}\widetilde{\triangle x^{(0)}}_0(\xi)$ and thence to the integral

$$\int_0^\infty \frac{\rho^{\frac{1}{2}}(\xi)\partial_{\gamma_j}\widetilde{\triangle x^{(0)}}_0(\xi)}{\xi^{\frac{1}{4}}}\cos[\lambda(\tau_0)\xi^{\frac{1}{2}}\int_{\tau_0}^\infty \lambda^{-1}(u)\,du]\,d\xi$$

is then handled by integration by parts with respect to ξ.

The exact same type of observation applies to the higher order corrections $\triangle x^{(j)}(\tau,\xi)$ as well.

To render this intuition precise, we first need to exhibit a functional framework which will be preserved by the iterative steps and which adequately describes the γ_j differentiated corrections $\triangle x^{(j)}$. To begin with, we introduce two types of norms:

Definition. Call a pair of functions $(\triangle y(\tau,\xi),\triangle y_d(\tau))$ *strongly bounded*, provided there exist $(\triangle\tilde{\tilde{y}}_0(\xi),\triangle\tilde{\tilde{y}}_1(\xi))\in\tilde{S}$, as well as $(\triangle\tilde{y}_0(\xi),\triangle\tilde{y}_1(\xi))\in\tilde{S}$, *the latter satisfying the vanishing conditions* (4.87), (4.88), such that if we set

$$\triangle y(\tau,\xi)=\triangle_{>\tau}y(\tau,\xi)+S(\tau)\big(\triangle\tilde{y}_0(\xi),\triangle\tilde{y}_1(\xi)\big),$$
$$\triangle y(\tau,\xi)=\widetilde{\triangle y}(\tau,\xi)+S(\tau)\big(\triangle\tilde{\tilde{y}}_0(\xi),\triangle\tilde{\tilde{y}}_1(\xi)\big)$$

then we have (recall (4.68))

$$\begin{aligned}
+\infty > &\|(\triangle y(\tau,\xi),\triangle y_d(\tau))\|_{S_{strong}}\\
&:=\sup_{\tau\geq\tau_0}(\frac{\tau_0}{\tau})^{\kappa}\|\chi_{\xi>1}\triangle y(\tau,\xi)\|_{S_1}+\|\xi^{\frac12+\delta_0}\mathcal{D}_\tau\widetilde{\triangle y}(\tau,\xi)\|_{Sqr(\xi>1)}\\
&+\sup_{\tau\geq\tau_0}(\frac{\tau_0}{\tau})^{\kappa}\|\chi_{\xi<1}\triangle_{>\tau}\triangle y(\tau,\xi)\|_{S_1}+\|\xi^{-\delta_0}\mathcal{D}_\tau\triangle_{>\tau}\triangle y(\tau,\xi)\|_{Sqr(\xi<1)}\\
&+\|(\triangle\tilde{y}_0,\triangle\tilde{y}_1)\|_{\tilde{S}}+\|(\triangle\tilde{\tilde{y}}_0,\triangle\tilde{\tilde{y}}_1)\|_{\tilde{S}}+\sup_{\tau\geq\tau_0}\tau^{(1-)}(|\triangle y_d(\tau)|+|\partial_\tau\triangle y_d(\tau)|).
\end{aligned}$$

We call a pair of functions $(\triangle z(\tau,\xi),\triangle z_d(\tau))$ *weakly bounded*, provided there exist $(\triangle\tilde{\tilde{z}}_0(\xi),\triangle\tilde{\tilde{z}}_1(\xi))\in\tilde{S}$ as well as $(\triangle\tilde{z}_0(\xi),\triangle\tilde{z}_1(\xi))\in\tilde{S}$ *not necessarily satisfying any vanishing conditions*, such that if we set

$$\triangle z(\tau,\xi)=\triangle_{>\tau}z(\tau,\xi)+S(\tau)\big(\triangle\tilde{z}_0(\xi),\triangle\tilde{z}_1(\xi)\big),$$
$$\triangle z(\tau,\xi)=\widetilde{\triangle z}(\tau,\xi)+S(\tau)\big(\triangle\tilde{\tilde{z}}_0(\xi),\triangle\tilde{\tilde{z}}_1(\xi)\big)$$

then we have

$$\begin{aligned}
&+\infty>\|(\triangle z(\tau,\xi),\triangle z_d(\tau))\|_{S_{weak}}:=\\
&\tau_0^{-1}\big[\sup_{\tau\geq\tau_0}(\frac{\lambda(\tau_0)}{\lambda(\tau)})^{2\delta_0+1}\|\chi_{\xi>1}\triangle z(\tau,\xi)\|_{\langle\xi\rangle^{-\frac12}L^2_{d\xi}}+\|\xi^{\delta_0}\mathcal{D}_\tau\widetilde{\triangle z}(\tau,\xi)\|_{\widetilde{Sqr}(\xi>1)}\big]\\
&+\tau_0^{-1}\big[\sup_{\tau\geq\tau_0}((\frac{\lambda(\tau_0)}{\lambda(\tau)})^{2\delta_0+1}\|\chi_{\xi<1}\triangle_{>\tau}\triangle z(\tau,\xi)\|_{S_1}+\|\xi^{-\delta_0}\mathcal{D}_\tau\triangle_{>\tau}\triangle z(\tau,\xi)\|_{\widetilde{Sqr}(\xi<1)}\big]\\
&+\|(\langle\xi\rangle^{-\frac12}\triangle\tilde{z}_0,\langle\xi\rangle^{-\frac12}\triangle\tilde{z}_1)\|_{\tilde{S}}+\|(\triangle\tilde{\tilde{z}}_0,\triangle\tilde{\tilde{z}}_1)\|_{\tilde{S}}+\sup_{\tau\geq\tau_0}\frac{\tau\lambda(\tau_0)}{\tau_0\lambda(\tau)}|\langle\partial_\tau\rangle\triangle z_d(\tau)|.
\end{aligned}$$

Here the norm $\|\cdot\|_{\widetilde{Sqr}}$ is defined just like in (4.68), except that the power $4\delta_0$ is replaced by $-2+4\delta_0$.

Observe that by comparison to $\|\cdot\|_{S_{strong}}$, the norm $\|\cdot\|_{S_{weak}}$ loses $\xi^{-\frac12}$ in terms of decay for large ξ, and we lose a factor $\tau_0\frac{\lambda(\tau)}{\lambda(\tau_0)}$ in terms of temporal decay.
Using the preceding terminology, we can now introduce the proper norm to measure the expressions arising upon applying ∂_{γ_j} to the corrections $\triangle x^{(j)}(\tau,\xi)$. Note that the dependence on the γ_j results on the one hand from the parametrices

$$U(\tau,\sigma),S(\tau),$$

as well as from the expressions e_{approx}, u_{approx}, u_0 and $\lambda(\tau)$ in (4.64). To emphasise that we want to measure the differences of functions, we introduce the symbol $\triangle\widetilde{S}$ for the relevant space:

Definition. We define $\triangle\widetilde{S}$ as the space of pairs of functions $(\triangle x(\tau,\xi), \triangle x_d(\tau))$ which admit a decomposition

$$\triangle x(\tau,\xi) = (\xi\partial_\xi)\triangle y(\tau,\xi) + \triangle z(\tau,\xi),\ \triangle x_d(\tau) = \triangle y_d(\tau) + \triangle z_d(\tau)$$

such that $\triangle y$ is strongly bounded and $\triangle z$ is weakly bounded, and we then set

$$\|(\triangle x(\tau,\xi), \triangle x_d(\tau))\|_{\triangle\widetilde{S}} := \inf\big(\|(\triangle y(\tau,\xi), \triangle y_d(\tau))\|_{S_{strong}} + \|(\triangle z(\tau,\xi), \triangle z_d(\tau))\|_{S_{weak}}\big)$$

where the infimum is over all decompositions into differentiated strongly bounded and weakly bounded functions.

We use the norm $\|\cdot\|_{\triangle\widetilde{S}}$ to measure the pair quantities $\big(\partial_{\gamma_\kappa}\triangle x^{(j)}(\tau,\xi), \partial_{\gamma_\kappa}\triangle x_d^{(j)}(\tau)\big)$, where $\kappa = 1,2$. To achieve this for all the corrections, we need an inductive step which infers the required bound for the next iterate, as well as rapid decay of these quantities. Correspondingly we have the following two lemmas:

Lemma 121. *Provided the $(\triangle x^{(j)}, \triangle x_d^{(j)})$ are constructed as in Proposition 119, and assuming the bounds there, we have*

$$\begin{aligned}&\|\big(\partial_{\gamma_\kappa}\triangle x^{(j)}(\tau,\xi), \partial_{\gamma_\kappa}\triangle x_d^{(j)}(\tau)\big)\|_{\triangle\widetilde{S}}\\ &\lesssim \tau_0^{-k_0+}\|(\triangle x^{(j-1)}, \triangle x_d^{(j-1)})\|_{S_{strong}} + \tau_0^{0+}\|\big(\partial_{\gamma_\kappa}\triangle x^{(j-1)}(\tau,\xi), \partial_{\gamma_\kappa}\triangle x_d^{(j-1)}(\tau)\big)\|_{\triangle\widetilde{S}},\\ &\kappa = 1,2.\end{aligned}$$

Lemma 122. *For any $\delta > 0$, there is $\tau_* = \tau_*(\delta)$ large enough such that if $\tau_0 \geq \tau_*$, then we have*

$$\|\big(\partial_{\gamma_\kappa}\triangle x^{(j)}(\tau,\xi), \partial_{\gamma_\kappa}\triangle x_d^{(j)}(\tau)\big)\|_{\triangle\widetilde{S}} \lesssim_\delta \tau_0^{-(k_0+2-)}\lambda^{\frac{1}{2}}(\tau_0)\delta^j[1 + O_{\tau_0}\big(\|(x_0,x_1)\|_{\widetilde{S}} + |x_{0d}|\big)].$$

Observe that the principal contribution here arises when the operator ∂_{γ_κ} gets passed from one correction to the earlier one, until it arrives on the source term e_{approx}. All other terms arising can be bounded by

$$O_{\tau_0}\big(\|(x_0,x_1)\|_{\widetilde{S}} + |x_{0d}|\big)$$

The proofs of these lemmas follow very closely the arguments in [49], and we shall only indicate the outlines:

Outline of proof of Lemma 121. One may assume a decomposition

$$\begin{aligned}&\big(\partial_{\gamma_\kappa}\triangle x^{(j-1)}(\tau,\xi),\partial_{\gamma_\kappa}\triangle x_d^{(j-1)}(\tau)\big)\\&=\big((\xi\partial_\xi)\triangle_\kappa y^{(j-1)}(\tau,\xi)+\triangle_\kappa z^{(j-1)}(\tau,\xi),\triangle_\kappa y_d^{(j-1)}(\tau)+\triangle_\kappa z_d^{(j-1)}(\tau)\big)\end{aligned}$$

with, say,

$$\begin{aligned}&\|(\triangle_\kappa y^{(j-1)},\triangle_\kappa y_d^{(j-1)})\|_{S_{strong}}+\|(\triangle_\kappa z^{(j-1)},\triangle_\kappa z_d^{(j-1)})\|_{S_{weak}}\\&\lesssim\|\big(\partial_{\gamma_\kappa}\triangle x^{(j-1)}(\tau,\xi),\partial_{\gamma_\kappa}\triangle x_d^{(j-1)}(\tau)\big)\|_{\triangle\widetilde{S}}\end{aligned}$$

Now let the operator ∂_{γ_κ} fall on the expression for $\triangle x^{(j)}(\tau,\xi)$ in Proposition 119, given by the parametrix (4.86). Then if ∂_{γ_κ} acts on the scaling factor in

$$[\mathscr{R}(\tau,\underline{\triangle x}^{(j-1)})+\triangle f^{(j-1)}(\tau,\underline{\xi})](\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi),$$

as well as in

$$S(\tau)\big(\triangle\widetilde{\widetilde{x}}_0^{(j)},\triangle\widetilde{\widetilde{x}}_1^{(j)}\big),$$

defined as in (109), then one can incorporate the corresponding term into $(\xi\partial_\xi)\triangle_\kappa y^{(j)}(\tau,\xi)$. On the other hand, if ∂_{γ_κ} falls on the parametrix factors

$$U(\tau,\sigma),\,V(\tau,\tau_0),\,U(\tau,\tau_0),$$

where we recall (4.71), or on one of the γ_κ-dependent factors $u_{approx}-u_0$, u^l_{approx} in $N_{approx}(\epsilon^{(j-1)})-N_{approx}(\epsilon^{(j-2)})$ (recalling (4.85)), we place the corresponding contribution into $\triangle z^{(j)}$. The required bounds follow essentially directly from the proofs of Proposition 7.1, 8.1, 9.1, 9.6 in [49].

On the other hand, if ∂_{γ_κ} falls on $\underline{\triangle x}^{(j-1)}$ in $\mathscr{R}(\tau,\underline{\triangle x}^{(j-1)})$, and we assume that

$$\partial_{\gamma_\kappa}\triangle x^{(j-1)}=(\xi\partial_\xi)\triangle y^{(j-1)},\,\triangle y^{(j-1)}\in S_{strong},$$

one notices that one can 'essentially' move the operator $(\xi\partial_\xi)$ past the non-local operator $\mathscr{R}$ modulo better errors which can be placed into $\triangle z^{(j)}$, and further to the outside of the parametrix. The situation is slightly more delicate provided ∂_{γ_κ} falls on a factor $\triangle\widetilde{\varepsilon}^{(l)}$ in $\triangle f^{(j-1)}$, again recalling (4.85) and the definition of $\triangle f^{(j-1)}$. Then writing

$$\triangle\widetilde{\varepsilon}^{(l)}(\tau,R)=\triangle x_d^{(l)}(\tau)\phi_d(R)+\int_0^\infty\phi(R,\xi)\triangle x^{(l)}(\tau,\xi)\rho(\xi)\,d\xi,$$

we exploit the spatial localisation forced by the cutoff $\chi_{R\lesssim\tau}$ in order to perform an integration by parts, provided

$$\partial_{\gamma_\kappa}\triangle x^{(l)}=(\xi\partial_\xi)\triangle y^{(l)}.$$

Thus write

$$\chi_{R\leq C\tau}\int_0^\infty \phi(R,\xi)(\xi\partial_\xi)\triangle y^{(l)}(\tau,\xi)\rho(\xi)\,d\xi$$
$$=-\chi_{R\leq C\tau}\int_0^\infty (\partial_\xi\xi)[\phi(R,\xi)\rho(\xi)]\triangle y^{(l)}(\tau,\xi)\,d\xi,$$

and then use the bound

$$\sup_{\tau\geq\tau_0}\tau^{-1}\|R^{-1}\chi_{R\leq C\tau}\int_0^\infty (\partial_\xi\xi)[\phi(R,\xi)\rho(\xi)]\triangle y^{(l)}(\tau,\xi)\,d\xi\|_{L^\infty_{dR}}\lesssim\|\triangle y^{(l)}\|_{S_{strong}}.$$

Indeed, such a bound follows easily from the asymptotic expansions for $\phi(R,\xi)$ given by Prop. 103. If we assume

$$\partial_{\gamma_\kappa}\triangle x^{(l)}=\triangle z^{(l)}\in S_{weak},$$

we have the weaker estimate

$$\sup_{\tau\geq\tau_0}\frac{\lambda(\tau_0)}{\lambda(\tau)}\|R^{-1}\chi_{R\leq C\tau}\int_0^\infty \phi(R,\xi)\rho(\xi)\triangle z^{(l)}(\tau,\xi)\,d\xi\|_{L^\infty_{dR}}\lesssim\|\triangle z^{(l)}\|_{S_{weak}}.$$

Using these and arguing just as in the proof of Proposition 9.6 in [49] yields the desired bound for the corresponding contribution of $\partial_{\gamma_\kappa}\triangle f^{(j-1)}$ to $\triangle x^{(j)}(\tau,\xi)$, which is placed in S_{weak}.

Next, consider the effect of ∂_{γ_κ} on the free term, when it falls on the source term $(\triangle\tilde{\tilde{x}}_0^{(j)},\triangle\tilde{\tilde{x}}_0^{(j)})$. In light of the choice of these terms, see the paragraph after the statement of Proposition 120, we have

$$\partial_{\gamma_\kappa}\triangle\tilde{\tilde{x}}_0^{(j)}=(\partial_{\gamma_\kappa}\alpha^{(j)})\mathscr{F}(\chi_{R\leq C\tau_0}\phi(R,0)),\ \partial_{\gamma_\kappa}\triangle\tilde{\tilde{x}}_1^{(j)}=(\partial_{\gamma_\kappa}\beta^{(j)})\mathscr{F}(\chi_{R\leq C\tau_0}\phi(R,0)),$$

and we have

$$\partial_{\gamma_\kappa}\alpha^{(j)}\sim\partial_{\gamma_\kappa}\int_0^\infty\frac{\rho^{\frac12}(\xi)\widetilde{\triangle}\tilde{x}_0^{(j)}(\xi)}{\xi^{\frac14}}\cos[\lambda(\tau_0)\xi^{\frac12}\int_{\tau_0}^\infty\lambda^{-1}(u)\,du]\,d\xi,$$

where

$$\widetilde{\triangle}\tilde{x}_0^{(j)}(\xi)=\int_{\tau_0}^\infty U(\tau_0,\sigma)H(\sigma,\frac{\lambda^2(\tau_0)}{\lambda^2(\sigma)}\xi)\,d\sigma,$$

and

$$H(\sigma,\xi):=[\mathscr{R}(\tau,\underline{\triangle x}^{(j-1)})+\triangle f^{(j-1)}(\tau,\underline{\xi})](\sigma,\xi).$$

Then performing integration by parts with respect to ξ if necessary, one checks that

$$|\int_0^\infty\frac{\rho^{\frac12}(\xi)\partial_{\gamma_\kappa}\widetilde{\triangle}\tilde{x}_0^{(j)}(\xi)}{\xi^{\frac14}}\cos[\lambda(\tau_0)\xi^{\frac12}\int_{\tau_0}^\infty\lambda^{-1}(u)\,du]\,d\xi|$$
$$\lesssim\tau_0^{0+}[\tau_0^{-k}\|(\triangle x^{(j-1)},\triangle x_d^{(j-1)})\|_{S_{strong}}+\|(\partial_{\gamma_\kappa}\triangle x^{(j-1)},\partial_{\gamma_\kappa}\triangle x_d^{(j-1)})\|_{\triangle\tilde{S}}].$$

This implies the required bound for $\partial_{\gamma_\kappa}\triangle\tilde{\tilde{x}}_0^{(j)}$, and the bound for $\partial_{\gamma_\kappa}\triangle\tilde{\tilde{x}}_1^{(j)}$ is similar. One then places

$$S(\tau)\big(\partial_{\gamma_\kappa}\triangle\tilde{\tilde{x}}_0^{(j)},\partial_{\gamma_\kappa}\triangle\tilde{\tilde{x}}_1^{(j)}\big)$$

into S_{weak}. □

Outline of proof of Lemma 122. This follows in analogy to the arguments in sections 11 and 12 in [49], a key being re-iterating the iterative step leading from $\partial_{\gamma_\kappa}\triangle x_l^{(j-1)}$ to $\partial_{\gamma_\kappa}\triangle x_l^{(j-1)}$ by differentiating (4.86). □

Completion of proof of Proposition 120. Recalling Lemma 115 and also invoking Lemma 122, we find

$$\begin{aligned}&\|(\triangle\tilde{\tilde{x}}_0^{(j)}-\triangle\tilde{\tilde{\tilde{x}}}_0^{(j)},\triangle\tilde{\tilde{x}}_1^{(j)}-\triangle\tilde{\tilde{\tilde{x}}}_1^{(j)})\|_{\tilde{S}}\\&\lesssim_\delta [\|(x_0-\overline{x}_0,x_1-\overline{x}_1)\|_{\tilde{S}}+|x_{0d}-\overline{x}_{0d}|]\cdot\delta^j\,[\frac{\tau_0^{k_0+1}\log\tau_0}{\lambda_{0,0}^{\frac{1}{2}}(\tau_0)}\cdot\tau_0^{-(k_0+2-)}\lambda^{\frac{1}{2}}(\tau_0)\\&\qquad+\tau_0^{-(2-)}]\end{aligned}$$

Observe that the final $\tau_0^{-(2-)}$ arises when keeping λ fixed and varying the initial data satisfying the vanishing conditions, just as in [49], while the more complicated expression preceding $\tau_0^{-(2-)}$ reflects the effect of changing $\gamma_{1,2}$. and so we finally get

$$\|(\triangle\tilde{\tilde{x}}_0^{(j)}-\triangle\tilde{\tilde{\tilde{x}}}_0^{(j)},\triangle\tilde{\tilde{x}}_1^{(j)}-\triangle\tilde{\tilde{\tilde{x}}}_1^{(j)})\|_{\tilde{S}}\lesssim_\delta \delta^j\tau_0^{-(1-)}[\|(x_0-\overline{x}_0,x_1-\overline{x}_1)\|_{\tilde{S}}+|x_{0d}-\overline{x}_{0d}|]$$

This implies Proposition 120. □

Proof of Theorem 116

This is a consequence of Proposition 120. Recalling Proposition 117, Proposition 119, it suffices to set

$$(\triangle x_0^{(\gamma_1,\gamma_2)},\triangle x_1^{(\gamma_1,\gamma_2)})=\sum_{j=0}^{\infty}(\triangle\tilde{\tilde{x}}_0^{(j)},\triangle\tilde{\tilde{x}}_1^{(j)})$$

$$(\triangle x_{0d}^{(\gamma_1,\gamma_2)},\triangle x_{1d}^{(\gamma_1,\gamma_2)})=\sum_{j=0}^{\infty}(\triangle x_d^{(j)}(\tau_0),\partial_\tau\triangle x_d^{(j)}(\tau_0))$$

Then the correction $\tilde{\epsilon}(\tau,R)$ is given by its Fourier coefficients

$$\underline{x}(\tau,\xi)=\underline{x}^{(0)}(\tau,\xi)+\sum_{j=1}^{\infty}\underline{\triangle x}^{(j)}(\tau,\xi)$$

The decaying bounds over $\|(\triangle x^{(j)}, \triangle x_d^{(j)})\|_{S_{strong}} = \triangle A_j$ imply that (recalling (4.9))

$$\widetilde{\epsilon}(\tau, R) = x_d(\tau)\phi_d(R) + \int_0^\infty \phi(R,\xi)x(\tau,\xi)\rho(\xi)\,d\xi \in H^{\frac{3}{2}+}_{dR,loc}$$

for any $\tau \geq \tau_0$, as desired. The fact that the local energy (restricted to $|x| \leq |t|$) vanishes asymptotically follows from

$$\|r\epsilon_r\|_{L^2_{dr}(r\leq t)} \leq \lambda^{-\frac{3}{2}}\|\widetilde{\epsilon}_R\|_{L^2_{dR}(R\lesssim\tau)} + \lambda^{-\frac{3}{2}}\|\frac{\widetilde{\epsilon}}{R}\|_{L^2_{dR}(R\lesssim\tau)},$$

and invoking the Fourier representation to bound the L^2-norms on the right, resulting in

$$\|r\epsilon_r\|_{L^2_{dr}(r\leq t)} \lesssim \tau^{\frac{5}{3}-\frac{3}{2}(1+\nu^{-1})},$$

and similarly for $r\epsilon_t$.

4.5.3 Translation to original coordinate system

In the preceding sections, we have obtained a singular solution of the form (the sum of the first four terms on the right representing $u^{(\gamma_1,\gamma_2)}_{approx}$ given by Theorem 111)

$$u(\tau,R) = \lambda^{\frac{1}{2}}(\tau)W(R) + \sum_{j=1}^{2k_*-1} v_j(\tau,R) + \sum_{a=1,2} v_{smooth,a}(\tau,R) + v(\tau,R) + R^{-1}\widetilde{\epsilon}(\tau,R),$$

with the error term $\widetilde{\epsilon}(\tau,R)$ given by the Fourier expansion

$$\widetilde{\epsilon}(\tau,R) = \int_0^\infty \phi(R,\xi)[x^{(0)}(\tau,\xi) + \sum_{j=1}^\infty \triangle x^{(j)}(\tau,\xi)]\rho(\xi)\,d\xi.$$

At initial time $\tau = \tau_0$, setting $x(\tau,\xi) := x^{(0)}(\tau,\xi) + \sum_{j=1}^\infty \triangle x^{(j)}(\tau,\xi)$, we have from our construction

$$\big(x(\tau_0,\xi), \mathcal{D}_\tau x(\tau_0,\xi)\big) = (x_0^{(\gamma_1,\gamma_2)} + \triangle x_0^{(\gamma_1,\gamma_2)}, x_1^{(\gamma_1,\gamma_2)} + \triangle x_1^{(\gamma_1,\gamma_2)}),$$
$$x_d(\tau_0) = x_d^{(\gamma_1,\gamma_2)} + \triangle x_d^{(\gamma_1,\gamma_2)}$$

where we recall

$$\triangle x_l^{(\gamma_1,\gamma_2)}(\xi) = \sum_{j=1}^\infty \triangle\widetilde{\widetilde{x}}_l^{(j)}(\xi),\ l = 1,2, \triangle x_{0d}^{(\gamma_1,\gamma_2)} = \sum_{j=0}^\infty \triangle x_{0d}^{(j)}(\tau_0)$$

The fact that we have added on the correction terms $\triangle x_l^{(\gamma_1,\gamma_2)}(\xi)$ means that the data

$$\big(R^{-1}\widetilde{\epsilon}(\tau_0,R), \partial_t R^{-1}\widetilde{\epsilon}(\tau_0,R)\big)$$

will no longer match the original data $(\overline{\epsilon}_1, \overline{\epsilon}_2)$, and we need to precisely quantify this correction *at the level of the Fourier variables associated with the old radial variable* $R_{0,0}$. Doing so requires recalling (4.27) - (4.29) as well as Lemma 113. Assume that our construction has replaced the data $(\overline{\epsilon}_1, \overline{\epsilon}_2)$ in (4.59) by $(\overline{\epsilon}_1 + \triangle\epsilon_1, \overline{\epsilon}_2 + \triangle\epsilon_2)$, we have the relations

$$R\triangle\epsilon_1(R) = \int_0^\infty \phi(R,\xi)\triangle x_0^{(\gamma_1,\gamma_2)}(\xi)\rho(\xi)\,d\xi + \triangle x_d^{(\gamma_1,\gamma_2)}\phi_d(R),$$
$$\triangle x_1^{(\gamma_1,\gamma_2)}(\xi) = -\lambda^{-1}(\tau_0)\int_0^\infty \phi(R,\xi)R\triangle\epsilon_2\,dR - \frac{\dot\lambda}{\lambda}\mathcal{K}_{cc}\triangle x_0^{(\gamma_1,\gamma_2)} - \frac{\dot\lambda}{\lambda}\mathcal{K}_{cd}\triangle x_d^{(\gamma_1,\gamma_2)},$$

where we recall that $\lambda = \lambda_{\gamma_1,\gamma_2}$. Recalling the relation

$$\chi_{r\le Ct_0} u_{approx}^{(0,0)}[t_0] + (\epsilon_1,\epsilon_2) = \chi_{r\le Ct_0} u_{approx}^{(\gamma_1,\gamma_2)}[t_0] + (\overline{\epsilon}_1,\overline{\epsilon}_2)$$

for the initial data, we see that the initial data perturbation (ϵ_1,ϵ_2) in (4.13) has been replaced by

$$(\epsilon_1 + \triangle\epsilon_1, \epsilon_2 + \triangle\epsilon_2) + (1-\chi_{r\le Ct_0})\cdot(u_{approx}^{(0,0)}[t_0] - u_{approx}^{(\gamma_1,\gamma_2)}[t_0]), \tag{4.89}$$

Here we may suppress the term

$$(1-\chi_{r\le Ct_0})\cdot(u_{approx}^{(0,0)}[t_0] - u_{approx}^{(\gamma_1,\gamma_2)}[t_0])$$

since this will not affect the evolution in the backward light cone. In light of the fact that the corresponding Fourier variables (x_0, x_1) were computed from (ϵ_1,ϵ_2) via (4.27) - (4.29) with $\gamma_{1,2} = 0$, we infer that the perturbed data (4.89) with the second part suppressed correspond to Fourier variables (with respect to the physical radial variable $R_{0,0}$) given by $(x_0 + \triangle x_0, x_1 + \triangle x_1)$ for the continuous part and $x_d + \triangle x_d$ for the discrete part, where we have

$$\triangle x_0(\xi) = \int_0^\infty \phi(R_{0,0},\xi)R_{0,0}\triangle\epsilon_1(R(R_{0,0}))\,dR_{0,0},$$
$$\triangle x_{0d} = \int_0^\infty \phi_d(R_{0,0})R_{0,0}\triangle\epsilon_1(R(R_{0,0}))\,dR_{0,0},$$
$$\triangle x_1(\xi) = -\lambda_{0,0}^{-1}(\tau_0)\int_0^\infty \phi(R_{0,0},\xi)R_{0,0}\triangle\epsilon_2\,dR_{0,0} - \frac{\dot\lambda_{0,0}}{\lambda_{0,0}}\mathcal{K}_{cc}\triangle x_0 - \frac{\dot\lambda_{0,0}}{\lambda_{0,0}}\mathcal{K}_{cd}\triangle x_d,$$
$$\triangle x_{1d} = -\lambda_{0,0}^{-1}(\tau_0)\int_0^\infty \phi_d(R_{0,0})R_{0,0}\triangle\epsilon_2\,dR_{0,0} - \frac{\dot\lambda_{0,0}}{\lambda_{0,0}}\mathcal{K}_{cc}\triangle x_d - \frac{\dot\lambda_{0,0}}{\lambda_{0,0}}\mathcal{K}_{dc}\triangle x_0,$$

Then using Lemma 113 we easily infer

$$\|\triangle x_0(\xi)\|_{\tilde S_1} \lesssim \|\triangle x_0^{(\gamma_1,\gamma_2)}\|_{\tilde S_1} + |\triangle x_{0d}^{(\gamma_1,\gamma_2)}| \lesssim \tau_0^{-(1-)}[\|(x_0,x_1)\|_{\tilde S} + |x_{0d}|],$$

and similarly

$$\|\triangle x_1(\xi)\|_{\tilde S_2} \lesssim \tau_0^{-(1-)}[\|(x_0,x_1)\|_{\tilde S} + |x_{0d}|]$$

For the discrete part of the correction, we get

$$|\triangle x_{0d}| = |\int_0^\infty \phi_d(R_{0,0})R_{0,0}\triangle\epsilon_1(R(R_{0,0}))\,dR_{0,0}|$$
$$\lesssim \tau_0^{-(1-)}|x_{0d}| + [\|(x_0, x_1)\|_{\widetilde{S}} + |x_{0d}|]^2.$$

Finally, we observe that the discrete spectral part of $\epsilon_2 + \triangle\epsilon_2$ with respect to the radial variable $R_{0,0}$ is completely determined in terms of $(x_0, x_1), x_{0d}$ and in fact a Lipschitz function of these. To conclude this discussion, we note that our precise choice of $\triangle\epsilon_l$, $l = 1, 2$, as well as Theorem 116 imply that the mapping

$$\big(x_0, x_1, x_{0d}\big) \longrightarrow \big(\triangle x_0, \triangle x_1, \triangle x_{0d}\big)$$

is Lipschitz with respect to the norm $\|(\cdot,\cdot)\|_{\widetilde{S}} + |\cdot|$, with Lipschitz constant $\ll 1$.

4.6 Proof of Theorem 108

This is immediate from the preceding discussion: the implicit function theorem guarantees that the mapping

$$\big(x_0, x_1, x_{0d}\big) \longrightarrow \big(x_0 + \triangle x_0, x_1 + \triangle x_1, x_{0d} + \triangle x_{0d}\big)$$

is invertible on a sufficiently small open neighbourhood of the origin in $\widetilde{S} \times \mathbb{R}$. Moreover, the second discrete spectral component $x_{1d} + \triangle x_{1d}$ is then uniquely determined as a Lipschitz function of

$$\big(x_0 + \triangle x_0, x_1 + \triangle x_1, x_{0d} + \triangle x_{0d}\big).$$

4.7 Outlook

While Theorem 108 explains the behaviour of *radial perturbations* of the special solutions $u_\nu(t,x)$ from Theorem 101, it is just as natural to consider *non-radial perturbations.* We conjecture that for sufficiently small and smooth such perturbations, the same result obtains, and the position of the blow-up is still unperturbed. Observe that passing to the general context enlarges the symmetry group, to include spatial translations as well as Lorentz transforms. Still, in analogy to the discussion preceding (4.6), the effect of these on the rough part $\eta_p(t,x)$ in the solutions u_ν implies that the difference (with $\mathscr{S} \neq I$ a Lorentz transform or spatial translation)

$$\mathscr{S} u_\nu - u_\nu$$

is of smoothness at most $H^{1+\frac{\nu}{2}-}$, whence again incompatible with sufficiently smooth perturbations of the data. It appears that stability under non-radial perturbations for type II blow-up solutions is for the most part an open issue for any of the nonlinear Hamiltonian wave equations, including the energy critical wave equation on $\mathbb{R}^{n+1}$, the critical Wave Maps as well as the critical Yang-Mills equation. In fact, in the context of the focussing nonlinear wave equation

$\Box u = -|u|^{p-1}u$ in dimension greater than one, the only results pertaining to the stability of explicit blow-up solutions without symmetry restrictions appear to be those in the works [72], [73] in the sub-conformal context, and the works by [18], [19], [9] in a super-conformal context.

A further important issue appears the applicability of the methods developed in this paper and [49] to smoother solutions, such as those in [31]. In particular, the question arises whether there is a way to construct solutions of the latter type via an approach as in [62], with the stability analysis supplied by methods as in the present paper.

Another issue concerns more general type II solutions, including such blowing up at time $T = +\infty$. The paper [17] constructs such solutions, and specifically establishes the following:

Theorem 123. *([17]) Let $\epsilon_0 > 0$ sufficiently small and $|\mu| \le \epsilon_0$. Then for any $\delta > 0$, there is $t_0 \ge 1$, and an energy class solution of* (4.1) *of the form*

$$u(t,x) = t^{\frac{\mu}{2}} W(t^{\mu}x) + \eta(t,x)$$

on $t \ge t_0$, where

$$\|\partial_t u(t,\cdot)\|_{L^2(B_t)} + \|\nabla u(t,\cdot)\|_{L^2(B_t)} \le \delta,$$

$$\|\partial_t \eta(t,\cdot)\|_{L^2(B_t)} + \|\nabla \eta(t,\cdot)\|_{L^2(B_t)} \le \delta,$$

for all $t \ge t_0$, where $B_t = \{x \in \mathbb{R}^3 \,|\, |x| < t\}$.

The preceding theorem does not furnish any information concerning either additional smoothness or stability of these solutions, but the methods of the present paper might be applicable here as well to furnish a co-dimension one stability result of this type of dynamics (under suitably smooth perturbations). We observe here that in analogy to (4.3), the remarkable classification theorem of [23] implies that any radial solution of (4.1) which exists globally in forward time is automatically type II (forward in time) and can be written as

$$u(t,x) = \sum_{j=1}^{N} \kappa_j W_{\lambda_j(t)}(x) + \epsilon(t,x),\ W_\lambda(x) = \lambda^{\frac{1}{2}} W(\lambda x), \tag{4.90}$$

where $\lim_{t\to+\infty} t\lambda_j(t) = +\infty$, and we also have the asymptotic decoupling property

$$\lim_{t\to\infty} |\log(\frac{\lambda_j(t)}{\lambda_k(t)})| = +\infty.$$

At this point in time, the only global solutions for (4.1) with the behaviour in (4.90) have $N = 1$ and either $\lambda_1(t) \to 1 + \delta$ for some small constant δ ([57]), or else are like in Theorem 123.

We further note that similar solutions as those in the preceding theorem have been constructed in the work [84] for the critical nonlinear Schrödinger equation

$$i u_t = \triangle u + |u|^4 u$$

on $\mathbb{R}^{3+1}$, and this equation as expected to also have solutions analogous to the u_ν from Theorem 101. It is to be noted that the corresponding construction is more involved due to the infinite propagation speed, which does not allow localisation to a region bounded by characteristics.

Very recently, a result of somewhat similar flavour as Theorem 123 but using a completely different technique was obtained for the parabolic equation

$$u_t = \triangle u + u^5$$

on $\mathbb{R}^{3+1}$ in the work [15]. There a co-dimension one stability result for these solutions was also established. Remarkably, these solutions display a continuum of possible grow-up rates, analogous to Theorem 123, which may suggest that in fact the solutions in the latter theorem may be chosen of regularity C^∞.

A Dispersive and Strichartz estimates for the wave equation

The solution of the wave equation with smooth compactly supported initial data cannot remain concentrated for a long period of time since waves will spread out along the characteristic cone and decay at a rate of $t^{-(n-1)/2}$. This process is usually referred to as the dispersive phenomenon. Dispersion estimates still works when the initial data have some decay at spatial infinity, that is when they lie in some Lebesgue space L^p, with $1 \le p \le 2$. However if we take initial data in some Sobolev space H^s the best way to measure dispersion is through Strichartz estimates. Strichartz estimates originated in the seminal work by Strichartz, an harmonic analyst, in the 70s. This tradition of cross fertilization between harmonic analysis and PDEs has continued ever since. Recently there has been an explosion of work in this area.

We present below the original Strichartz estimate proved in [101]. This inequality tell us that if we take rough initial data and an integrable inhomogeneous term of a Cauchy problem, then the mixed L^4 norm in both time and space of the solution reamains bounded.

Proposition 124 (Original Strichartz inequality)**.** *Let u be a solution of*

$$\begin{cases} \Box u = F & \text{in } \mathbb{R}^{1+3} \\ (u, \partial_t u)\Big|_{t=0} = (f, g) \end{cases}$$

Suppose further that $f \in \dot{H}^{1/2}(\mathbb{R}^3)$, $g \in \dot{H}^{-1/2}(\mathbb{R}^3)$, *and* $F \in L^{4/3}(\mathbb{R}^{1+3})$, *then*

$$\|u\|_{L^4(\mathbb{R}^{3+1})} \lesssim \|(f,g)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} + \|F\|_{L^{4/3}(\mathbb{R}^{3+1})}$$

In general dimension $n \ge 2$ Strichartz's inequality generalize to the *conformal Strichartz estimate*:

$$\|u\|_{L^{2\frac{n+1}{n-1}}(\mathbb{R}^{1+n})} \lesssim \|(f,g)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} + \|F\|_{L^{2\frac{n+1}{n+3}}(\mathbb{R}^{1+n})}$$

Notice that the Lebesgue exponents $2\frac{n+1}{n-1}$ and $2\frac{n+1}{n+3}$ are conjugate. As we shall see the conformal Strichartz pairs is the only sharp admissible pair with the same Lebesgue norm in time and in space.

One can interpret Strichartz inequality in term of boundedness of solution maps. The Strichartz inequality splits natural into two parts: the homogeneous and inhomogeneous estimates

$$\mathscr{H} \in \mathscr{L}(\dot{H}^{1/2} \times \dot{H}^{-1/2}, L^4)$$

$$\Box^{-1} \in \mathscr{L}(L^{4/3}, L^4)$$

and the norm of the solution map measured in these spaces *do not* depend on the length of the time interval considered. This means that we can consider either Lebesgue spaces over $\mathbb{R}^{n+1}$ oder S_T in the conformal Strichartz inequality.

The original Strichartz estimate stated in Proposition 124 is just the beginning of a long interplay between harmonic analysis and PDEs. In 1998, almost 20 years after the publication of Strichartz, Keel and Tao [33] generalize the estimate to the $\mathbb{R}^n$ case and to a brother class of exponent including spacetime-mixed-norms. In this chapter we restrict our analysis to the wave equation, however the result of Keel and Tao includes other types of dispersive equations such as Schrödinger and KdV.

Proposition 125 ([33]). *Suppose u be a solution of the inhomogeneous wave equation.*

$$\begin{cases} \Box u = 0 \\ (u, \partial_t u)\Big|_{t=0} = (f, g) \end{cases}$$

Let $n \geq 2$, and let the triplets (q, r, s) and $(\widetilde{q}, \widetilde{r}, 1-s)$ be Strichartz wave admissible, this means that

$$2 \leq q, \widetilde{q} \leq \infty, \qquad \text{and} \qquad 2 \leq r, \widetilde{r} < \infty$$

$$\frac{2}{q} + \frac{n-1}{r} \leq \frac{n-1}{2}, \qquad \text{and} \qquad \frac{2}{\widetilde{q}} + \frac{n-1}{\widetilde{r}} \leq \frac{n-1}{2}$$

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s, \qquad \text{and} \qquad \frac{1}{\widetilde{q}'} + \frac{n}{\widetilde{r}'} - 2 = \frac{n}{2} - s$$

Moreover suppose further that when $n = 3$ we have $(q, r, s) \neq (2, \infty, 1)$. Then

$$\|u\|_{L_t^q L_x^r(X)} + \|\partial u\|_{L^\infty \dot{H}^{s-1}(X)} \lesssim \|f, g\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|\Box u\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}(X)}$$

where $X = \mathbb{R}^{1+n}$ or $X = S_T$ for any $T \in \mathbb{R}$.

The Strichartz board for the wave equation is visualized in Figures A.1, A.2, and A.3 below. We have highlighted the admissibility region for (q, r). Observe that the pair (q, ∞) is not a wave admissible one. Whereas the pair $(q, r) = (2, \frac{2n-2}{n-3})$ is called end-point. In terms of solution maps, the Strichartz estimates of Keel and Tao can be naturally decoupled into four estimates:

i. $\mathscr{H} \in \mathscr{L}\big(\dot{H}^s \times \dot{H}^{s-1}, \dot{X}_T^s\big)$,

ii. $\mathscr{H} \in \mathscr{L}\big(\dot{H}^s \times \dot{H}^{s-1}, L_t^q L_x^r\big)$,

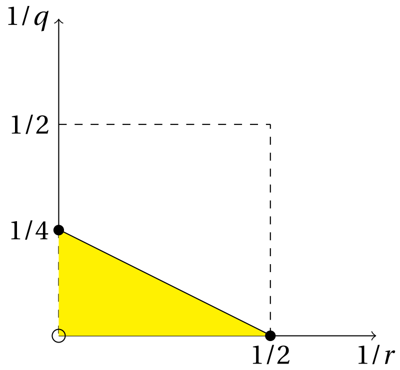


Figure A.1: $n = 2$

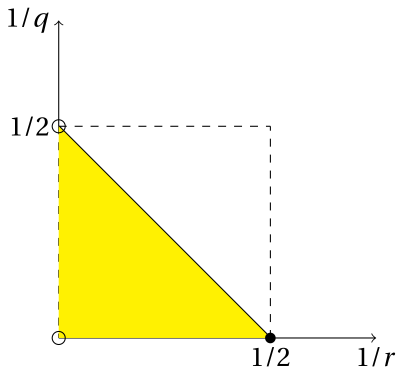


Figure A.2: $n = 3$

iii. $\Box^{-1} \in \mathscr{L}\big(L_t^{\tilde{q}'} L_x^{\tilde{r}'}, L_t^q L_x^r\big)$

iv. $\Box^{-1} \in \mathscr{L}\big(L_t^{\tilde{q}'} L_x^{\tilde{r}'}, X_T^s\big)$

Notice that we can take a finite time interval or $T = \infty$ since the constant does not depend on the length of the time interval considered. We know by the energy inequality that estimate 1. holds. We call estimate 2. the *homogeneous Strichartz estimate* and estimates 3. and 4. the *inhomogeneous Strichartz estimates.*

A.1 Dispersive estimate

In this section we show that if we assume some integrability of the initial data, then we obtain that the L^∞ norm of the solution decay polynomially in time with rate depending on the dimension; whereas the L^2 norm remains bounded. An interpolation argument between this two results yield to the so called *dispersion estimate.* In the proof of Strichartz estimates we shall need the following frequency-localized-dispersive estimates for half-wave propagator, see [75].

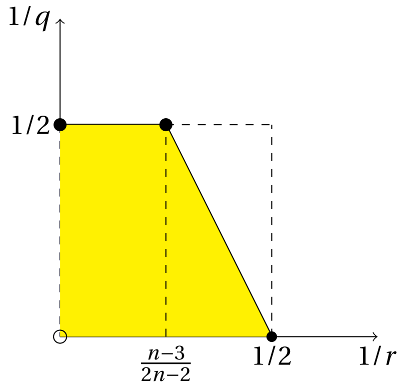


Figure A.3: $n \geq 4$

Proposition 126. *Let P_λ the Littlewood-Paley decomposition at dyadic frequency $\sim \lambda$, and $D = \sqrt{-\Delta}$, then*

(i) $\|e^{\pm itD} P_\lambda f\|_\infty \lesssim \langle \lambda t\rangle^{-\frac{n-1}{2}} \lambda^n \|P_\lambda f\|_1$

(ii) $\|e^{\pm itD} P_\lambda f\|_2 \lesssim \|P_\lambda f\|_2$

(iii) *If $2 \leq r \leq \infty$, then we have*

$$\|e^{\pm itD} P_\lambda f\|_r \lesssim \langle \lambda t\rangle^{-\frac{n-1}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} \lambda^{n\left(\frac{1}{r'}-\frac{1}{r}\right)} \|P_\lambda f\|_{r'}$$

Notice that the L^2 norm do not have to be frequency localized. Moreover, estimates (i) and (iii) implies the weaker estimates

$$\|e^{\pm itD} P_\lambda f\|_\infty \lesssim \langle t\rangle^{-\frac{n-1}{2}} \lambda^{\frac{n+1}{2}} \|P_\lambda f\|_1 \approx \langle t\rangle^{-\frac{n-1}{2}} \|D^{\frac{n+1}{2}} P_\lambda f\|_1$$

and by interpolation

$$\|e^{\pm itD} P_\lambda f\|_r \lesssim \langle t\rangle^{-\frac{n-1}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} \lambda^{\frac{n+1}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} \|P_\lambda f\|_{r'} \approx \langle t\rangle^{-\frac{n-1}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} \|D^{\frac{n+1}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)} P_\lambda f\|_{r'}$$

Proof. As every pseudo-differential operator, we can write the half-wave propagator as a convolution with a kernel, that is

$$e^{itD} P_\lambda f = e^{itD} P_{\leq \lambda+2} P_\lambda f = P_{\leq\lambda+2} K_t * P_\lambda f$$

where $P_{\leq\lambda+2} K_t = \mathscr{F}^{-1}(e^{\pm it|\xi|}\psi_{\lambda+2}(\xi))$. The dispersive estimate (i) follows from Young inequality and Lemma 127 below since

$$\|e^{\pm itD} P_\lambda f\|_\infty = \|P_{\leq\lambda+2} K_t * P_\lambda f\|_\infty \leq \|P_{\leq\lambda+2} K_t\|_\infty \|P_\lambda f\|_1 \lesssim \langle \lambda t\rangle^{-\frac{n-1}{2}} \lambda^n \|P_\lambda f\|_1$$

The energy type estimate (ii) is a straightforward application of Plancherel's theorem

$$\|e^{itD}f\|_2 = \|e^{it|\xi|}\widehat{f}(\xi)\|_2 \le \|\widehat{f}\|_2 = \|f\|_2$$

To obtain estimate (iii) just interpolate between estimates (i) and (ii) $\square$

We now compete the proof of the Proposition 126 with the following lemma, see [75].

Lemma 127 (Kernel estimate). *Let $P_{\le\lambda+2}K_t(x) = \int e^{i(x\cdot\xi\pm t|\xi|)}\psi_{\lambda+2}(\xi)d\xi$, then*

$$\|P_{\le\lambda+2}K_t\|_{L^\infty(\mathbb{R}^n)} \lesssim \langle\lambda t\rangle^{-\frac{n-1}{2}}\lambda^n$$

Proof. Recall that $\psi_\lambda(\xi) = \psi(\xi/\lambda)$. Let us make a change of variable in polar coordinates $\xi = 2^{k+2}\rho\omega$, where we write the dyadic number λ as 2^k, we then obtain

$$\begin{aligned}
P_{\le\lambda+2}K_t(x) &= \lambda^n\int_0^\infty\int_{S^{n-1}} e^{i\lambda\rho(x\cdot\omega\pm t)}\psi(\rho)\rho^{n-1}d\sigma(\omega)\,d\rho \\
&\le \lambda^n\int_0^1 e^{\pm i\lambda\rho t}\rho^{n-1}\psi(\rho)\int_{S^{n-1}} e^{i\lambda\rho x\cdot\omega}d\sigma(\omega)\,d\rho \\
&= \lambda^n\int_0^1 e^{\pm i\lambda\rho t}\rho^{n-1}\psi(\rho)\,\widehat{d\sigma}(\lambda\rho x)\,d\rho \\
&\le \langle\lambda x\rangle^{-\frac{n-1}{2}}\lambda^n\int_0^1 e^{\pm i\lambda\rho t}\rho^{\frac{n-1}{2}}\psi(\rho)\,d\rho
\end{aligned}$$

We have used the fact that the Fourier transform of the sphere surface measure has some decay, precisely $|\widehat{d\sigma}(x)| \lesssim \langle x\rangle^{-\frac{n-1}{2}}$. One can prove this with the method of stationary phase.

Let us now spit the proof into two cases. Far form the light cone, $|t| \le |x|$, and inside the light cone, $|t| \ge |x|$. In the region far from the light cone, $|t| \le |x|$, we obtain easily that

$$|P_{\le\lambda+2}K_t(x)| \lesssim \langle\lambda x\rangle^{-\frac{n-1}{2}}\lambda^n \le \langle\lambda t\rangle^{-\frac{n-1}{2}}\lambda^n$$

Since the integral in ρ is bounded. Inside the light cone, that is for $|t| \ge |x|$, we can use integration by parts m-times to obtain

$$\begin{aligned}
|P_{\le\lambda+2}K_t(x)| &\lesssim \frac{\lambda^n}{\lambda t}\Big|\int_0^1 e^{\pm i\lambda\rho t}\partial_\rho\Big(\rho^{\frac{n-1}{2}}\psi(\rho)\Big)d\rho\Big| \\
&= \frac{\lambda^n}{(\lambda t)^m}\Big|\int_0^1 e^{\pm i\lambda\rho t}\partial_\rho^m\Big(\rho^{\frac{n-1}{2}}\psi(\rho)\Big)d\rho\Big| \\
&\lesssim \langle\lambda t\rangle^{-m}\lambda^n\int_0^1 \rho^{\frac{n-1}{2}-m}d\rho
\end{aligned}$$

Observe that we have use the trivial inequality $\langle\lambda x\rangle^{-\frac{n-1}{2}} \le 1$ and the fact that the boundary terms vanish due to the cutoff function ψ. The integral is bounded only if $m \le (n-1)/2$, thus the best possible m to obtain the fastest decay in time is $m = (n-1)/2$, which gives

$$|K(t,x)| \lesssim \langle \lambda t\rangle^{-\frac{n-1}{2}} \lambda^n$$

Thus we obtain the same decay as in the region far away from the light cone. □

A.2 Proof of homogeneous Strichartz inequality

Recall that the homogeneous solution maps can be written as a linear combination of half waves

$$\mathcal{H}(f,g) = \frac{1}{2}(e^{itD} + e^{-itD})f + \frac{1}{2i}(e^{itD} - e^{-itD})D^{-1}g$$

Thus the Strichartz estimate are reduced to the question of boundedness of the half-wave propagator. We proceed in analogy with for the proof of the dispersive estimates, first we prove a frequency localized version then we sum over dyadic blocks to obtain the full estimate.

Proposition 128 (Frequency-localized Strichartz estimates for half-wave propagator)**.** *Let* (q,r,s) *a Strichartz admissible triple, then*

(i) $\|e^{\pm itD} P_\lambda f\|_{L^q L^r} \lesssim \lambda^s \|P_\lambda f\|_2$

(ii) $\left\| \int e^{\mp isD} P_\lambda F(s) ds \right\|_2 \lesssim \lambda^s \|P_\lambda F\|_{L^{q'} L^{r'}}$

(iii) $\left\| \int e^{\pm i(t-s)D} P_\lambda F(s) ds \right\|_{L^q L^r} \lesssim \lambda^{2s} \|P_\lambda F\|_{L^{q'} L^{r'}}$

In the proof of Proposition 128 we shall need the so called TT^* *principle*, see [75]. Let $T: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n+1})$ an operator form Schwartz functions in $\mathbb{R}^n$ to Schwartz functions in $\mathbb{R}^{n+1}$. We define the formal adjoint of T as the operator $T^*: \mathcal{S}(\mathbb{R}^{n+1}) \to \mathcal{S}(\mathbb{R}^n$ determined by the relation

$$\langle Tf, F\rangle_{L^2(\mathbb{R}^{n+1})} = \langle f, T^*F\rangle_{L^2(\mathbb{R}^n)}$$

for any $f \in \mathcal{S}(R^n)$ and $F \in \mathcal{S}(R^{n+1})$.

Lemma 129 (TT^*)**.** *The following statement are equivalent:*

(i) $T: L^2(\mathbb{R}^n) \to L^q(\mathbb{R}^{n+1})$ *is bounded.*

(ii) $T^*: L^{q'}(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^n)$ *is bounded.*

(ii) $TT^*: L^{q'}(\mathbb{R}^{n+1}) \to L^q(\mathbb{R}^{n+1})$ *is bounded.*

We shall not prove this elementary lemma, we rather turn to the proof of Proposition 128.

Proof. By the TT^* Lemma i, ii, and iii are equivalent. Indeed let $Tf = e^{\pm itD}f$ then

$$T^*F = \int e^{\mp isD}F(s)ds \qquad \text{and} \qquad TT^*F = \int e^{\pm i(t-s)D}F(s)ds$$

Therefore it suffices to prove estimate (iii) Observe that via the frequency-localized dispersive estimate (iii) of Proposition 126 we obtain the following bound

$$\begin{aligned} \|TT^*P_\lambda F\|_{L^qL^r} &\lesssim \Big\| \int \Big\| e^{\pm i(t-s)D}P_\lambda F(s)\Big\|_{L^r} ds\Big\|_{L^q} \\ &\lesssim \lambda^{n-\frac{2n}{r}} \Big\| \int \langle \lambda(t-s)\rangle^{-\frac{n-1}{2}+\frac{n-1}{r}} \|P_\lambda F(s)\|_{L^{r'}} ds\Big\|_{L^q} \end{aligned}$$

In the non-sharp wave admissible case we can use Young's inequality to conclude that

$$\|TT^*P_\lambda F\|_{L^qL^r} \lesssim \lambda^{n-\frac{2n}{r}} \|\langle \lambda t\rangle^{-\frac{n-1}{2}+\frac{n-1}{r}}\|_{L_t^{q/2}} \|P_\lambda F\|_{L_t^{q'}L_x^{r'}} \lesssim \lambda^{n-\frac{2n}{r}-\frac{2}{q}} \|P_\lambda F\|_{L_t^{q'}L_x^{r'}}$$

Using the fact that $\langle t\rangle^{-\alpha} \in L_t^\beta(\mathbb{R})$ iff $\alpha\beta > 1$, we obtain that the integral in time is bounded since (q,r) is a non-sharp wave admissible pair.

In the sharp wave admissible case we cannot close by Young's inequality since the integral in time will be unbounded. However we can apply its weak version, the Hardy-Littlewood inequality which state that if $1 < p < q < \infty$, $0 < \alpha < 1$, and $1 + 1/q = 1/p + \alpha$ then

$$\Big\| |\cdot|^{-\alpha} * h \Big\|_{L^q(\mathbb{R})} \lesssim \|h\|_{L^p(\mathbb{R})}$$

In our case we have $\alpha = \frac{n-1}{2} - \frac{n-1}{r}$ and to obtain the desired bound we set $p = q'$, this implies that $\alpha = 2/q$, hence we recover the sharp wave admissible condition. Notice that Hardy-Littlewood-Sobolev fractional inequality requires that the exponent of the kernel α is strictly between zero and one, hence $2 < r < (2n-2)/(n-3)$ when $n > 3$. The proof of the end-point Strichartz estimate $(q,r) = (2, \frac{2n-2}{n-3})$ for $n > 3$, require more effort and it is not presented here, we refer to [33]. Notice that when $n = 3$ the end-point Strichartz estimate fail. Finally observe that when (q,r) is a sharp wave admissible pair then $(n+1)/2 - (n+1)/r = 2s$ □

We now combine the previous frequency localized estimates to obtain a Strichartz estimates for the half-wave propagator.

Corollary 130. *Let* (q,r,s) *a Strichartz admissible triple, then*

$$\|e^{\pm itD}f\|_{L^qL^r} \lesssim \|f\|_{\dot{H}^s}$$

Proof. Notice that if we prove that

$$\|u\|_{L^qL^r} \lesssim \Big[\sum_\lambda \|P_\lambda u\|^2_{L^qL^r}\Big]^{1/2} \tag{A.1}$$

Then by the frequency-localized Strichartz estimate Proposition 128 we can conclude

$$\|u\|_{L^qL^r} \lesssim \big[\sum_{\lambda} \|P_\lambda u\|^2_{L^qL^r}\big]^{1/2} \lesssim \big[\sum_{\lambda} \lambda^{2s}\|f\|_2^2\big]^{1/2} \lesssim \|f\|_{\dot{H}^s}$$

Observe that for wave admissible pairs for which $q, r \neq \infty$ estimate (A.1) follow form Littlewood-Paley inequality. On the other hand if $q = \infty$ and $r \neq \infty$ we use Littlewood-Paley inequality for the space variable and just the triangle inequality, which holds for any value of q, for the time variable:

$$\|u\|_{L^\infty L^r} = \|\|u\|_{L^r_x}\|_{L^\infty_t} \lesssim \|\sum_{\lambda} \|P_\lambda u\|^2_{L^\infty_x}\|^{1/2}_{L^\infty_t} \lesssim \big[\sum_{\lambda} \|P_\lambda u\|^2_{L^\infty L^r}\big]^{1/2}$$

Recall that the pairs (q,∞) are not wave admissible. Therefore (A.1) holds for any wave admissible pairs. □

As a trivial consequence we obtain the Strichartz estimates for homogeneous wave equation.

Corollary 131. *Let (q, r, s) a Strichartz admissible triple, then*

$$\mathcal{H} : \dot{H}^s \times \dot{H}^{s-1} \to L^q L^r$$

Notice again that the norm of the homogeneous map measured in such spaces does not depend on the time interval, thus we can take the Lebesgue space norm over $\mathbb{R}^{n+1}$ or over S_T.

A.3 Proof of inhomogeneous Strichartz inequality

Recall that the inhomogeneous maps can be written as a linear combination of half waves

$$\Box^{-1}F = \frac{1}{2i}\int_0^t e^{i(t-s)D}D^{-1}F(s)ds - \frac{1}{2i}\int_0^t e^{-i(t-s)D}D^{-1}F(s)ds$$

Thus the Strichartz estimate are reduced to the question of boundedness of the half-wave propagator. We proceed as above, first we prove a frequency localized version then we sum over the frequencies to obtain the full estimate. Let us start by the following frequency-localized Strichartz estimates for inhomogeneous half-wave propagator.

Proposition 132. *Let (q, r, s) and $(\tilde{q}, \tilde{r}, 1-s)$ be two Strichartz admissible triple, then*

$$(i) \quad \Big\|\int e^{\pm i(t-s)D}D^{-1}P_\lambda F(s)ds\Big\|_{L^qL^r} \lesssim \|P_\lambda F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

$$(ii) \quad \Big\|\int e^{\pm i(t-s)D}D^{-1}P_\lambda F(s)ds\Big\|_{L^\infty L^2} \lesssim \lambda^{-s}\|P_\lambda F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

Proof. Using the homogeneous frequency-localized Strichartz estimates for the half-wave propagator, Proposition 128 i and ii, we obtain

$$\begin{aligned}\left\|\int e^{\pm i(t-s)D}D^{-1}P_\lambda F(s)ds\right\|_{L^qL^r} &= \left\|e^{\pm itD}\int e^{\pm isD}D^{-1}P_\lambda F(s)ds\right\|_{L^qL^r}\\ &\lesssim \lambda^{\frac{n}{2}-\frac{1}{q}-\frac{n}{r}}\left\|\int e^{\pm isD}D^{-1}P_\lambda F(s)ds\right\|_2\\ &\lesssim \lambda^{n-\frac{1}{q}-\frac{n}{r}-\frac{1}{\tilde{q}}-\frac{n}{\tilde{r}}}\|D^{-1}P_\lambda F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}\\ &\lesssim \|P_\lambda F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}\end{aligned}$$

Notice that the scaling condition we obtain that $n-\frac{1}{q}-\frac{n}{r}-\frac{1}{\tilde{q}}-\frac{n}{\tilde{r}}=1$. Therefore (i) holds. In similar fashion we prove (ii)

$$\begin{aligned}\left\|\int e^{\pm i(t-s)D}D^{-1}P_\lambda F(s)ds\right\|_{L^\infty L^2} &= \left\|e^{\pm itD}\int e^{\pm isD}D^{-1}P_\lambda F(s)ds\right\|_{L^\infty L^2}\\ &\lesssim \left\|\int e^{\pm isD}D^{-1}P_\lambda F(s)ds\right\|_2\\ &\lesssim \lambda^{\frac{n}{2}-\frac{1}{\tilde{q}}-\frac{n}{\tilde{r}}}\|D^{-1}P_\lambda F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}\\ &\lesssim \lambda^{-s}\|P_\lambda F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}\end{aligned}$$

Again by the scaling condition we obtain that $1-s=\frac{n}{2}-\frac{1}{\tilde{q}}-\frac{n}{\tilde{r}}$. □

The previous proposition and Christ-Kiselev lemma, see [95], implies the following Strichartz estimates for inhomogeneous wave equation.

Corollary 133. *Let (q,r,s) and $(\tilde{q},\tilde{r},1-s)$ be two Strichartz admissible triple, then*

(i) $\Box^{-1}: L^{\tilde{q}'}L^{\tilde{r}'} \to L^qL^r$

(ii) $\Box^{-1}: L^{\tilde{q}'}L^{\tilde{r}'} \to C^0\dot{H}^s \cap C^1\dot{H}^{s-1}$

Again observe that the constants in the latter inequalities do not depend on time.

Proof. If we sum over frequency-localized blocks, by Proposition 132 and Littlewood-Paley inequality, we obtain the estimates

$$\left\|\int e^{\pm i(t-s)D}D^{-1}F(s)ds\right\|_{L^qL^r} \lesssim \|F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

$$\left\|\int e^{\pm i(t-s)D}D^{-1}F(s)ds\right\|_{L^\infty\dot{H}^s} \lesssim \|F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

Moreover notice that when we derive with respect to time we get another D factor in front, precisely we have the following estimate

$$\left\|\partial_t\int e^{\pm i(t-s)D}D^{-1}F(s)ds\right\|_{L^\infty\dot{H}^{s-1}} = \left\|\int ie^{\pm i(t-s)D}F(s)ds\right\|_{L^\infty\dot{H}^{s-1}} \lesssim \|F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

Now we apply Christ-Kiselev lemma we get the truncated in time estimates

$$\Big\| \int_0^t e^{\pm i(t-s)D} F(s)ds \Big\|_{L^q L^r} \lesssim \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}$$

$$\Big\| \int_0^t e^{\pm i(t-s)D} D^{-1} F(s)ds \Big\|_{L^\infty \dot{H}^s} \lesssim \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}$$

and

$$\Big\| \partial_t \int_0^t e^{\pm i(t-s)D} D^{-1} F(s)ds \Big\|_{L^\infty \dot{H}^{s-1}} \lesssim \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}$$

Finally observe that the inhomogeneous map operator is a linear combination of truncated in time maps. □

A.4 Some improvements of Strichartz estimates

In the previous section we proved the classical Strichartz inequality

$$\|u\|_{L_t^q L_x^r(X)} + \|\partial u\|_{L^\infty \dot{H}^{s-1}(X)} \lesssim \|f,g\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|\Box u\|_{L_t^{\tilde{q}'} L_x^{r'}(X)}$$

where the triplets (q,r,s) and $(\tilde{q},\tilde{r},1-s)$ are Strichartz admissible. This estimate is the result of four different ones: the two homogeneous estimates

$$\|\mathcal{H}(u_0,u_1)\|_{L_t^q L_x^r(X)} \lesssim \|u_0,u_1\|_{\dot{H}^s \times \dot{H}^{s-1}}, \qquad \|\partial \mathcal{H}(u_0,u_1)\|_{L^\infty \dot{H}^{s-1}(X)} \lesssim \|u_0,u_1\|_{\dot{H}^s \times \dot{H}^{s-1}}$$

and two inhomogeneous estimates

$$\|\Box^{-1} u\|_{L_t^q L_x^r(X)} \lesssim \|u\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(X)}, \qquad \|\partial \Box^{-1} u\|_{L^\infty \dot{H}^{s-1}(X)} \lesssim \|u\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(X)}$$

notice that the second one does not follow form the energy inequality. All the four mentioned inequalities are a consequence of the following

$$\|e^{\pm itD} P_\lambda f\|_{L^q L^r} \lesssim \lambda^s \|P_\lambda f\|_2$$

If now instead we derive with respect to time or space and apply the elliptic derivative $-s$ times we obtain

$$\|\partial D^{-s} e^{\pm itD} P_\lambda f\|_{L^q L^r} \lesssim \lambda \|P_\lambda f\|_2$$

Thus form the same argument as above $\|\partial D^{-s} e^{\pm itD} f\|_{L^q L^r} \lesssim \|f\|_{\dot{H}^1}$, which yield to

$$\|\partial D^{-s} \mathcal{H}(u_0,u_1)\|_{L_t^q L_x^r(X)} \lesssim \|u_0,u_1\|_{\dot{H}^1 \times L^s}$$

We can run a similar argument for the inhomogeneous estimates. Set $(\widetilde{q},\widetilde{r}) = (\infty,2)$ then

$$\begin{aligned}
\left\|\partial D^{-s}\int e^{\pm i(t-s)D}D^{-1}P_\lambda F(s)ds\right\|_{L^qL^r} &= \left\|\partial D^{-s}e^{\pm itD}\int e^{\pm isD}D^{-1}P_\lambda F(s)ds\right\|_{L^qL^r} \\
&\lesssim \lambda\left\|\int e^{\pm isD}D^{-1}P_\lambda F(s)ds\right\|_2 \\
&\lesssim \lambda\|D^{-1}P_\lambda F\|_{L^1L^2} \\
&\lesssim \|P_\lambda F\|_{L^1L^2}
\end{aligned}$$

Therefore we obtain the improved Strichartz bound

$$\|D^{-s}\partial u\|_{L^qL^r(X)} \lesssim \|f,g\|_{\dot{H}^1\times L^2} + \|\Box u\|_{L^1L^2(X)}$$

Proposition 134 (∂ Strichartz)**.** *Let u be a solution of the inhomogeneous wave equation. Furthermore let $n \geq 2$, the triplets (q,r,s) and $(\widetilde{q},\widetilde{r},\widetilde{s})$ be Strichartz admissible, $(q,\widetilde{q}) \neq (2,2)$, and $s_1 > s$ and $s_2 > s+\widetilde{s}$. Then*

$$\|\partial u\|_{L_t^qL_x^r(X)} \lesssim \|f,g\|_{H^{s_1+1}\times H^{s_1}} + \|\langle D\rangle^{s_2}\Box u\|_{L_t^{\widetilde{q}'}L_x^{\widetilde{r}'}(X)}$$

where $X = \mathbb{R}^{1+n}$ or $X = S_T$ for any $T \in \mathbb{R}$.

Proof. Suppose $u = \mathscr{H}(f,g)$ then since ∂ and $\Box$ commute we have $\partial u = \mathscr{H}(\partial_x f,\partial_x g)$. This means $v = \partial u$ solves

$$\begin{cases} \Box v = 0 \\ (u,\partial_t v)\Big|_{t=0} = (\partial_x f,\partial_x g) \end{cases}$$

Therefore Strichartz estimates, Proposition 125 yield to

$$\|v\|_{L_t^qL_x^r(X)} \lesssim \|\partial_x f,\partial_x g\|_{H^s\times H^{s-1}} \approx \|f,g\|_{H^{s+1}\times H^s}$$

Hence the homogeneous estimate follows once we notice that $s_1 > s$ implies $\|f\|_{H^s} \leq \|f\|_{H^{s_1}}$. Furthermore the inhomogeneous estimates is slightly more involve since we have introduced $\widetilde{s}$, and $\widetilde{p}$ and $\widetilde{q}$ are not directly liked to s anymore. To prove the inhomogeneous Strichartz estimate observe that for the frequency localised version we obtain

$$\begin{aligned}
\left\|\partial\int e^{\pm i(t-s)D}D^{-1}P_\lambda F(s)ds\right\|_{L^qL^r} &\approx \left\|e^{\pm itD}\int e^{\pm isD}P_\lambda F(s)ds\right\|_{L^qL^r} \\
&\lesssim \lambda^s\left\|\int e^{\pm isD}P_\lambda F(s)ds\right\|_2 \\
&\lesssim \lambda^{s+\widetilde{s}}\|P_\lambda F\|_{L^{\widetilde{q}'}L^{\widetilde{r}'}} \\
&\lesssim \lambda^{s_2}\|P_\lambda F\|_{L^{\widetilde{q}'}L^{\widetilde{r}'}}
\end{aligned}$$

If we sum over frequency localised blocks we obtain the inhomogeneous Strichartz estimate.

□

Proposition 135 ($D^s\partial$ Strichartz). *Let u be a solution of the inhomogeneous wave equation. Suppose $n \geq 2$, the triplets (q,r,σ), and $(\tilde{q},\tilde{r},\tilde{\sigma})$ are Strichartz admissible, and $s \in \mathbb{R}$. Then*

$$\|D^s\partial u\|_{L_t^q L_x^r(X)} \lesssim \|f,g\|_{\dot{H}^{s+\sigma+1}\times\dot{H}^{s+\sigma}} + \|D^{\sigma+\tilde{\sigma}+s}\Box u\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

where $X = \mathbb{R}^{1+n}$ or $X = S_T$ for any $T \in \mathbb{R}$.

Proof. As before notice D^s and $\Box$ commute, then Strichartz estimates from Proposition 125 yield to the homogeneous estimate

$$\|D^s\mathscr{H}(f,g)\|_{L_t^q L_x^r(X)} \lesssim \|D^s f, D^s g\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \approx \|f,g\|_{\dot{H}^{s+\sigma}\times\dot{H}^{s+\sigma-1}}$$

In order to establish the frequency localised inhomogeneous estimate observe that

$$\begin{aligned}\left\|D^s\int e^{\pm i(t-s)D}D^{-1}P_\lambda F(s)ds\right\|_{L^qL^r} &\approx \left\|e^{\pm itD}\int e^{\pm isD}D^{s-1}P_\lambda F(s)ds\right\|_{L^qL^r}\\ &\lesssim \lambda^\sigma\left\|\int e^{\pm isD}D^{s-1}P_\lambda F(s)ds\right\|_2\\ &\lesssim \lambda^{\sigma+\tilde{\sigma}+s-1}\|P_\lambda F\|_{L^{\tilde{q}'}L^{\tilde{r}'}}\end{aligned}$$

□

A.5 Knapp Counterexample

The following question is in oder: are the conditions on wave admissible pair sharp? The following proposition tell us that the answer is positive: the conditions are indeed sharp.

Proposition 136 (Knapp Counterexample). *Let $n \geq 2$, $2 \leq q \leq \infty$, and $2 \leq r < \infty$ such that*

$$\frac{2}{q} + \frac{n-1}{r} > \frac{n-1}{2}$$

Then there exist at least one function f such that $\|e^{itD}f\|_{L^qL^r} \geq \|f\|_2$.

Proof. Consider a function f such that $\widehat{f}(\xi) = \chi_{B_\epsilon}(\xi)$, where B_ϵ is a block of dimensions $1\times\epsilon^{-1}\times\cdots\times\epsilon^{-1}$. For example take $-1/2 < \xi_1 < 1/2$, and $-1/(2\epsilon) < \xi_i < 1/(2\epsilon)$ for $2 \leq i \leq n$. Then clearly

$$\|f\|_2 = \|\widehat{f}\|_2 = |B_\epsilon|^{1/2} = \epsilon^{-\frac{n-1}{2}}$$

and

$$e^{itD}f = \int e^{i(t|\xi|+x\cdot\xi)}\widehat{f}(\xi)d\xi \approx \epsilon^{-(n-1)}$$

Moreover for the Heisenberg principle f has support on the physical side on a box of size $1 \times \epsilon \times \cdots \times \epsilon$. Let the t variable be restricted to a size ϵ^2 interval, then $e^{itD} f$ has support on the spacetime region S_ϵ where $-\epsilon^2/2 < t < \epsilon^2/2$, $-2 < x_1 < 2$ and $-2\epsilon < x_i < 2\epsilon$. Therefore

$$\|e^{itD} f\|_{L^q L^r} = \epsilon^{-(n-1)} \|\chi_{S_\epsilon}\|_{L^q L^r} = \epsilon^{-(n-1)+\frac{2}{q}+\frac{n-1}{r}}$$

Thus $\frac{2}{q} + \frac{n-1}{r} > \frac{n-1}{2}$ implies that $\|e^{itD} f\|_{L^q L^r} \geq \|f\|_2$. □

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Scientific Publications

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