Comment. Math. Helvetici 58 (1983) 086–095 0010-2571/83/001086-10$01.50 + 0.20/0

Lifting idempotents and Clifford theory

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Let N be a normal subgroup of a finite group G and let R be a noetherian complete local commutative ring. Clifford theory deals with the relationship between RG-modules and RN-modules, using induction from N to G or restriction from G to N. Since Clifford's 1937 paper [1], the theory is well understood for irreducible representations (see also [2, §11C]). For an indecomposable RN-module W, several authors have proved a going-up theorem describing how $\mathrm{Ind}_N^G W$ decomposes (see [2, §19C]).

One purpose of this paper is to prove (in Section 2) a going-down theorem for indecomposable modules (analogous to Clifford's theorem), based on a refinement of the lifting idempotents theorem, presented in Section 1. The going-up and going-down theorems are actually equivalent in the sense that each can be derived as a corollary to the other one. One main assumption is necessary for the going-down theorem: the RG-module we start from must be projective relative to H. The whole procedure is presented in the more general context of Clifford systems. The paper concludes in Section 3 with another application of the lifting idempotents theorem, concerning the behaviour of indecomposable modules under ground ring extensions.

1. Lifting idempotents

THEOREM 1. *Let A be a ring and J a two-sided ideal contained in* Rad A. *Assume that A is complete in the J-adic topology (that is the natural map $A \to \varprojlim A/J^n$ is an isomorphism). Let Π be a finite group acting on A by automorphisms leaving J globally invariant. Let $\{\bar{e}_1, \ldots, \bar{e}_n\}$ be a set of orthogonal idempotents of $\bar{A} = A/J$ satisfying $\sum_{i=1}^n \bar{e}_i = 1$. Assume the following three conditions:*

a) *The induced action of Π on $\bar{A}$ permutes the idempotents $\bar{e}_i$ transitively.*

b) *There exists $u \in A$ such that $Tr_\Omega(u) = 1$ where Ω is the stabilizer of $\bar{e}_1$ and $Tr_\Omega(u) = \sum_{\omega\in\Omega} \omega u$.*

c) *$\bar{u}$ commutes with each $\bar{e}_i$.*

Then $\{\bar{e}_1, \ldots, \bar{e}_n\}$ lifts to a set $\{e_1, \ldots, e_n\}$ of orthogonal idempotents of A which are permuted by Π transitively and such that $\sum_{i=1}^n e_i = 1$.

Remarks. 1) If A is the ring of endomorphisms of a representation V, we shall see that the condition b) corresponds to a condition of relative projectivity for V.

2) There are two situations where c) is always satisfied: either the idempotents $\bar{e}_i$ are central or the order $|\Omega|$ of Ω is invertible in A in which case one can choose u to be the central element $|\Omega|^{-1}$.

3) When Π acts regularly on the idempotents $\bar{e}_i$, that is when Ω is trivial, one can take $u=1$ so that b) and c) are trivially satisfied. This special case appears already in [3].

Proof. It suffices to prove the theorem when J is nilpotent because, since $A \cong \varprojlim A/J^n$, the lifted idempotents are constructed as limits of idempotents of A/J^n for $n \to \infty$.

For $\sigma \in \Pi$, write $\bar{e}_\sigma = \sigma\bar{e}_1$ so that $\bar{e}_\sigma = \bar{e}_\tau$ if and only if $\sigma\Omega = \tau\Omega$. Since Π acts transitively, every idempotent $\bar{e}_i$ can be written in that form.

We proceed by induction on the nilpotent index n of J. There is nothing to prove if $n=1$. If $n \geqslant 2$, let $I = J^{n-1}$ and write $\tilde{a}$ for the image of $a \in A$ modulo I. By induction, there exist idempotents $\tilde{e}_\sigma$ of A/I such that $\sigma\tilde{e}_\tau = \tilde{e}_{\sigma\tau}$ and $\sum_{\sigma\in\Pi/\Omega} \tilde{e}_\sigma = 1$. First lift arbitrarily the idempotents $\tilde{e}_\sigma$ to get orthogonal idempotents e_σ of A satisfying $\sum_{\sigma\in\Pi/\Omega} e_\sigma = 1$. This is well known to be possible (see [2, §6A]). Of course the notation implies that we keep the convention:

$$e_\sigma = e_\tau \quad \text{if and only if} \quad \sigma\Omega = \tau\Omega.$$

Since $\sigma\tilde{e}_\tau = \tilde{e}_{\sigma\tau}$, we have:

$$\sigma e_\tau = e_{\sigma\tau} + r_{\sigma,\tau} \quad \text{for some} \quad r_{\sigma,\tau} \in I.$$

We list several properties of the elements $r_{\sigma,\tau}$:

(1) If $\omega \in \Omega$, $r_{\sigma,\tau\omega} = r_{\sigma,\tau}$.

This follows from $e_{\eta\omega} = e_\eta$ for all $\eta \in \Pi$.

(2) $\sum_{\tau\in\Pi/\Omega} r_{\sigma,\tau} = 0$.

This follows when σ is applied to $1 = \sum_{\tau\in\Pi/\Omega} e_\tau$.

(3) $\eta r_{\sigma,\tau} = r_{\eta\sigma,\tau} - r_{\eta,\sigma\tau}$.

This is a consequence of $(\eta\sigma)e_\tau = \eta(\sigma e_\tau)$.

(4) $r_{\sigma,\tau} = e_{\sigma\tau} r_{\sigma,\tau} + r_{\sigma,\tau} e_{\sigma\tau}$.

This follows from the equality $\sigma e_\tau = (\sigma e_\tau)^2$ using also $I^2 = 0$. Multiplying (4) by e_η on the right or e_λ on the left (or both in the first case below), we get:

$$(5)\quad e_\lambda r_{\sigma,\tau} e_\eta = \begin{cases} 0 & \text{if } \lambda\Omega \neq \sigma\tau\Omega \neq \eta\Omega \\ r_{\sigma,\tau} e_\eta & \text{if } \lambda\Omega = \sigma\tau\Omega \neq \eta\Omega \\ e_\lambda r_{\sigma,\tau} & \text{if } \lambda\Omega \neq \sigma\tau\Omega = \eta\Omega \\ 0 & \text{if } \lambda\Omega = \sigma\tau\Omega = \eta\Omega \end{cases}$$

(6) If $\lambda\Omega \neq \eta\Omega$, $e_{\mu\lambda} r_{\mu,\eta} + r_{\mu,\lambda} e_{\mu\eta} = 0$.
This is a consequence of $(\mu e_\lambda)\cdot(\mu e_\eta) = 0$ using again $I^2 = 0$.

Now define: $f_\sigma = e_\sigma + \sum_{\lambda\in\Pi} r_{\lambda,\lambda^{-1}\sigma}\cdot e_\lambda \cdot \lambda u$ where $u \in A$ satisfies hypotheses b) and c). By (1), we have:

(7) $f_{\sigma\omega} = f_\sigma$ if $\omega \in \Omega$.

(8) $\sum_{\sigma\in\Pi/\Omega} f_\sigma = 1$.

For

$$\sum_{\sigma\in\Pi/\Omega} f_\sigma = \sum_{\sigma\in\Pi/\Omega} e_\sigma + \sum_{\sigma\in\Pi/\Omega}\sum_{\lambda\in\Pi} r_{\lambda,\lambda^{-1}\sigma}\cdot e_\lambda\cdot\lambda u$$

$$= 1 + \sum_{\lambda\in\Pi}\Big(\sum_{\sigma\in\Pi/\Omega} r_{\lambda,\lambda^{-1}\sigma}\Big) e_\lambda\cdot\lambda u = 1 \quad \text{by (2)}.$$

(9) $f_\sigma f_\tau = 0$ if $\sigma\Omega \neq \tau\Omega$.

$$f_\sigma f_\tau = \sum_{\lambda\in\Pi} e_\sigma r_{\lambda,\lambda^{-1}\tau} e_\lambda \cdot \lambda u + \sum_{\lambda\in\Pi} r_{\lambda,\lambda^{-1}\sigma} e_\lambda \cdot \lambda u \cdot e_\tau.$$

By hypothesis c), $\lambda\bar{u}\cdot\bar{e}_\tau = \lambda(\bar{u}\cdot\bar{e}_{\lambda^{-1}\tau}) = \lambda(\bar{e}_{\lambda^{-1}\tau}\cdot\bar{u}) = \bar{e}_\tau\cdot\lambda\bar{u}$. Hence λu commutes with e_τ modulo J. Since $I\cdot J = J^{n-1}\cdot J = 0$, we have $r\cdot\lambda u\cdot e_\tau = r\cdot e_\tau\cdot\lambda u$ for all $r\in I$ and so we can permute λu and e_τ in the second sum. Therefore, the only non-zero terms appear for $\lambda\in\tau\Omega$. By (5), the same holds for the first sum. Consequently:

$$f_\sigma f_\tau = \sum_{\omega\in\Omega} (e_\sigma r_{\tau\omega,\omega^{-1}} + r_{\tau\omega,\omega^{-1}\tau^{-1}\sigma} e_{\tau\omega}) e_{\tau\omega}\cdot\tau\omega u.$$

Now apply (6) with $\eta = 1$, $\mu = \tau\omega$ and $\lambda = \omega^{-1}\tau^{-1}\sigma$, using also (1). The condition $\lambda\Omega \neq \eta\Omega$ is equivalent to $\sigma\Omega \neq \tau\Omega$. We get $f_\sigma f_\tau = 0$, as required.

Clearly (8) and (9) imply that f_σ is idempotent. There remains to prove the additional property we are looking for:

(10) $\tau f_\sigma = f_{\tau\sigma}$.

By (3), we have:

$$\tau f_\sigma = e_{\tau\sigma} + r_{\tau,\sigma} + \sum_{\lambda\in\Pi} (r_{\tau\lambda,\lambda^{-1}\sigma} - r_{\tau,\sigma}) \cdot (e_{\tau\lambda} + r_{\tau,\lambda}) \cdot \tau\lambda u.$$

Since $I^2 = 0$, we get:

$$\begin{aligned}\tau f_\sigma &= e_{\tau\sigma} + \sum_{\lambda\in\Pi} r_{\tau\lambda,\lambda^{-1}\sigma} \cdot e_{\tau\lambda} \cdot \tau\lambda u + r_{\tau,\sigma}\Big(1 - \sum_{\lambda\in\Pi} e_{\tau\lambda} \cdot \tau\lambda u\Big)\\ &= e_{\tau\sigma} + \sum_{\mu\in\Pi} r_{\mu,\mu^{-1}\tau\sigma} \cdot e_\mu \cdot \mu u + r_{\tau,\sigma}\Big(1 - \sum_{\mu\in\Pi/\Omega} e_\mu \cdot \sum_{\omega\in\Omega} \mu\omega u\Big)\\ &= f_{\tau\sigma} + r_{\tau,\sigma}\Big(1 - \sum_{\mu\in\Pi/\Omega} e_\mu \cdot \mu Tr_\Omega(u)\Big) = f_{\tau\sigma},\end{aligned}$$

using $Tr_\Omega(u) = 1$ and $\sum_{\mu\in\Pi/\Omega} e_\mu = 1$. ■

2. Clifford theory

Let N be a normal subgroup of a finite group G and $S = G/N$. Throughout this section, R denotes a noetherian local commutative ring which is complete in its natural topology of local ring. These assumptions are made in order to have the following properties:

(i) Every finitely generated RG-module is a direct sum of indecomposable submodules.

(ii) If M is an indecomposable RG-module, then $\mathrm{End}_{RG} M$ is a local ring. Hence Krull-Schmidt theorem holds for RG-modules.

In order to study the restriction to N of an indecomposable RG-module, we consider the more general case of an S-graded Clifford system $A = \oplus_{s\in S} A_s$ over R, in the sense of [2, §11C]. The case of group algebras corresponds to $A = RG$ and $A_1 = RN$. Recall that there exist units $a_s \in A_s$ such that $A_s = a_s A_1 = A_1 a_s$. Also $a_s a_t a_{st}^{-1} \in A_1$ because $A_s A_t = A_{st}$.

For the rest of this paper, all modules will be finitely generated left modules. For an A_1-module W, denote by W^A the induced module $\mathrm{Ind}_{A_1}^A W = A \otimes_{A_1} W$, while for an A-module V, we denote by V_{A_1} the restriction $\mathrm{Res}_{A_1}^A V$. If V is an A-module, then S acts on $\mathrm{End}_{A_1} V$ by $sf = a_s f a_s^{-1}$ and the set of fixed points is exactly $\mathrm{End}_A V$.

DEFINITIONS. 1) An A-module V is said to be *projective relative* to A_1 if V is a direct summand of a module induced from A_1 which actually can be chosen to be $(V_{A_1})^A$. This is equivalent to the existence of an endomorphism $u \in \mathrm{End}_{A_1} V$ such that $Tr_S(u) = 1$ where $Tr_S(u) = \sum_{s \in S} su$. The equivalence of these definitions is well known in the case of group algebras [2, §19A], but the proof can be carried over without change to the case of Clifford systems.

2) If W is an A_1-module, then $a_s \otimes W$ has a natural structure of A_1-module and is called a *conjugate* of W.

3) Let $M = \oplus_{i,j} M_{ij}$ be a decomposition of a module M into indecomposable summands such that $M_{ij} \cong M_{ik}$ for all i, j, k and $M_{ij} \not\cong M_{km}$ if $i \neq k$. Then $M_i = \oplus_j M_{ij}$ is called a *homogeneous component* of M. Contrary to the case of semi-simple modules, note that in general M_i is not uniquely determined by M.

Now we can state the going-down theorem analogous to Clifford's theorem:

THEOREM 2. *Let A be an S-graded Clifford system over R and V an indecomposable A-module. Assume that V is projective relative to A_1, that is there exists an indecomposable summand W of V_{A_1} such that V is a direct summand of W^A. Let $T = \{t \in S \mid a_t \otimes W \cong W\}$ be the inertial subgroup of W and let $\{s_1, \ldots, s_n\}$ be a set of coset representatives of T in S. Finally let $B = \oplus_{t \in T} A_t$ be the T-graded subalgebra of A. Then:*

(i) *V_{A_1} is isomorphic to a direct sum of conjugates of W.*

(ii) *$\{a_{s_i} \otimes W \mid i = 1, \ldots, n\}$ is a complete set of non-isomorphic conjugates of W and each appears with the same multiplicity in a decomposition of V_{A_1}.*

(iii) *There exists a decomposition $V_{A_1} = \oplus_{i=1}^n U_i$ into homogeneous components which are permuted transitively by $\{a_s \mid s \in S\}$ and such that $\{a_t \mid t \in T\}$ stabilizes U_1.*

(iv) *U_1 is an indecomposable B-module and V is isomorphic to U_1^A.*

Beside Theorem 1, the main ingredient for the proof of Theorem 2 is the following:

PROPOSITION 3. *Let A be an R-algebra, finitely generated as R-module, and M an A-module. Denote by a bar the reduction modulo the radical of $\mathrm{End}_A M$. Let $M = \oplus_{i=1}^n M_i$ (respectively $M = \oplus_{i=1}^n M_i'$) be any decomposition of M corresponding to idempotents $e_1, \ldots, e_n \in \mathrm{End}_A M$ (respectively $e_1', \ldots, e_n'$).*

(i) *The modules M_i are homogeneous components of M if and only if $\bar{e}_1, \ldots, \bar{e}_n$ are the primitive central idempotents of $\overline{\mathrm{End}_A M}$.*

(ii) *Assume the modules M_i and M_i' are homogeneous components of M, labelled in order to have $M_i \cong M_i'$ for all i. Then there exists $f \in \mathrm{Aut}_A M$ such that $f(M_i) = M_i'$ for all i and $\bar{f} = 1$.*

(iii) *Assume the modules M_i and M_i' are homogeneous components of M. Then $M_1 \cong M_1'$ if and only if $\bar{e}_1 = \bar{e}_1'$.*

Proof. (i) If the modules M_i are homogeneous components of M, write $M_i \cong m_i N_i$ with N_i indecomposable. Let $E_i = \mathrm{End}_A N_i$ and $D_i = \overline{\mathrm{End}_A N_i}$. By Fitting's theorem [2, §19C, lemma], there is a commutative diagram

$$\begin{array}{ccc} E = \mathrm{End}_A M & \searrow & \\ \uparrow & & \overline{\mathrm{End}_A M} \cong \prod_{i=1}^{n} M_{m_i}(D_i) \\ \prod_{i=1}^{n} M_{m_i}(E_i) \cong \prod_{i=1}^{n} \mathrm{End}_A M_i & \nearrow & \end{array}$$

Since e_i is the unit matrix of $M_{m_i}(E_i)$ (with zeros in all other components), $\bar{e}_i$ is the unit matrix of $M_{m_i}(D_i)$, i.e. $\bar{e}_i$ is a primitive central idempotent of $\overline{\mathrm{End}_A M}$.

If conversely $\bar{e}_i$ is primitive central, decompose it into primitive idempotents $\bar{e}_i = \bar{e}_{i1} + \cdots + \bar{e}_{im_i}$ and lift them to get $e_i = e_{i1} + \cdots + e_{im_i}$. Now $e_{ij}E \cong e_{ik}E$ because $\bar{e}_{ij}\bar{E} \cong \bar{e}_{ik}\bar{E}$. Therefore:

$$e_{ij}M \cong e_{ij}E \otimes_E M \cong e_{ik}E \otimes_E M \cong e_{ik}M.$$

So $e_iM = \bigoplus_{i=1}^{m_i} e_{ij}M$ is a homogeneous decomposition of e_iM into indecomposable summands. If some indecomposable summand of e_iM was isomorphic to a summand of e_kM for $k \neq i$, there would be less than n homogeneous components in M and so, by the first part of the proof, less than n primitive central idempotents in $\bar{E}$.

(ii) Consider again the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_A M & \overset{q}{\searrow} & \\ {\scriptstyle j}\uparrow & & \overline{\mathrm{End}_A M} \\ \prod_{i=1}^{n} \mathrm{End}_A M_i & \underset{p}{\nearrow} & \end{array}$$

We emphasize that not only q but also p is surjective. Choose an isomorphism $g_i : M_i \to M_i'$ for each i and define an automorphism g of M by $g|_{M_i} = g_i$. Since g is invertible, so is $q(g)$ and since p is onto, there exists $h \in \prod_{i=1}^{n} \mathrm{End}_A M_i$ such that $p(h) = q(g)^{-1}$. Clearly $f = g \cdot j(h)$ satisfies $f(M_i) = M_i'$ and $\bar{f} = 1$.

(iii) By (ii), if $M_1 \cong M_1'$, there exists $f \in \mathrm{Aut}_A M$ such that $f(M_1) = M_1'$, $f(\bigoplus_{i=2}^{n} M_i) = \bigoplus_{i=2}^{n} M_i'$ and $\bar{f} = 1$. It follows easily that $e_1' = fe_1f^{-1}$ and therefore $\bar{e}_1' = \bar{e}_1$.

Conversely suppose $\bar{e}_1' = \bar{e}_1$. By Krull–Schmidt theorem, $M_1' \cong M_i$ for some i. By the first part of this proof, $\bar{e}_1' = \bar{e}_i$. Hence $\bar{e}_i = \bar{e}_1$ and so $i = 1$. ∎

Proof of Theorem 2. (i) Write $V_{A_1} = \bigoplus_{i=1}^{r} W_i$ with the W_i indecomposable. Since V is a direct summand of W^A, V_{A_1} is a summand of $(W^A)_{A_1} \cong \bigoplus_{s \in S} a_s \otimes W$. By Krull–Schmidt theorem, each W_i is isomorphic to some $a_s \otimes W$.

(ii) Changing notations write $V_{A_1} = \bigoplus_{i=1}^n m_i W_i$ where $m_i W_i$ denotes the direct sum of m_i copies of W_i and $W_i \not\cong W_j$ if $i \neq j$. By (i), $W_i \cong a_s \otimes W$ for some s. Applying a_s to V, we get:

$$\bigoplus_{i=1}^{n} m_i W_i \cong V_{A_1} = (a_s V)_{A_1} \cong \bigoplus_{i=1}^{n} m_i (a_s \otimes W_i).$$

Comparing the multiplicities of W_i in both decompositions, we get $m_i = m_1$. The same argument applied with an arbitrary a_s shows that $a_s \otimes W$ must be isomorphic to some W_i. Therefore, by definition of T, $\{a_{s_i} \otimes W \mid i = 1, \ldots, n\}$ is a complete set of non-isomorphic conjugates of W.

(iii) Let $E = \mathrm{End}_{A_1} V$ and $\bar{E} = E/\mathrm{rad}\,(E)$. The group S acts on E via $sf = a_s f a_s^{-1}$ and induces an action on $\bar{E}$ which necessarily permutes the primitive central idempotents of $\bar{E}$.

Let $V_{A_1} = \bigoplus_{i=1}^n U_i$ be a decomposition of V_{A_1} into homogeneous components, corresponding to idempotents $e_1, \ldots, e_n$. Assume W is a summand of U_1. For $s \in S$, $V_{A_1} = \bigoplus_{i=1}^n a_s U_i$ is also a decomposition of V_{A_1} into homogeneous components, corresponding to idempotents $a_s e_i a_s^{-1} = se_i$. By Proposition 3(i), $\{\bar{e}_1, \ldots, \bar{e}_n\}$ are the primitive central idempotents of $\bar{E}$. Since $a_s U_1 \cong a_s \otimes U_1 \cong U_i$ for some i, we have $s\bar{e}_1 = \bar{e}_i$ by Proposition 3(iii). Moreover each U_i is isomorphic to some $a_s U_1$ by part (i) and (ii). This implies that S acts transitively on the set $\{\bar{e}_1, \ldots, \bar{e}_n\}$. Since W is a summand of U_1, T is the stabilizer of $\bar{e}_1$ (again by Proposition 3(iii)).

Now since V is projective relative to A_1, there exists $v \in \mathrm{End}_{A_1} V$ such that $Tr_S(v) = 1$. Let $u = \sum_{i=1}^n r_i v$ where $r_1, \ldots, r_n$ are representatives of the cosets Tr. Then $Tr_T(u) = \sum_{t \in T} tu = Tr_S(v) = 1$. Moreover $\bar{u}$ commutes with $\bar{e}_i$ for $\bar{e}_i$ is central. Therefore the hypotheses of Theorem 1 are satisfied. It follows that there exist orthogonal idempotents $f_1, \ldots, f_n$ of E (lifting $\bar{e}_1, \ldots, \bar{e}_n$) which are permuted transitively by S and such that T stabilizes f_1.

By Proposition 3(i), the modules $f_i V_{A_1}$ are homogeneous components of V_{A_1}. The equation $f_i = sf_1 = a_s f_1 a_s^{-1}$ means exactly that $a_s(f_1 V_{A_1}) = f_i V_{A_1}$. This completes the proof of part (iii).

(iv) Since $\{a_t \mid t \in T\}$ stabilizes $U_1 = f_1 V_{A_1}$, U_1 is a B-module. Now $V = \bigoplus_{i=1}^n a_{s_i} U_1$ which is the definition of an induced module. Finally U_1 is indecomposable otherwise V would be decomposable. ■

Counter-example. Without the assumption of relative projectivity for V, Theorem 2 does not hold any more. Take K a field of characteristic 2, $G = C_4$, $N = C_2$ and $V = K[X]/(X-1)^3$ (the generator of C_4 acting by multiplication by X). Then: $\mathrm{Res}_N V = S_1 \oplus S_2$ where $S_i = K[Y]/(Y-1)^i$ (the generator of C_2

acting by multiplication by Y). Since S_1 and S_2 do not have the same dimension, they cannot be conjugate. In fact, the two primitive central idempotents of $\overline{\text{End}_{KN} V}$ are fixed under the action of $S = G/N$, and each of them can be lifted in four ways in $\text{End}_{KN} V$. But no idempotent of $\text{End}_{KN} V$ is fixed by S.

Now we can recall the going-up theorem, which we shall prove to be equivalent to Theorem 2.

THEOREM 4 (Conlon, Tucker, Ward [2, §19C]). *Let A be an S-graded Clifford system over R, W an indecomposable A_1-module, T the inertial subgroup of W and $B = \bigoplus_{t\in T} A_t$. If $W^B = \bigoplus_{i=1}^m Z_i$ is a decomposition of W^B into indecomposable B-modules, then each Z_i^A is an indecomposable A-module, that is $W^A = \bigoplus_{i=1}^m Z_i^A$ gives a decomposition of W^A into indecomposable A-modules.*

Proof. The notation $X \mid Y$ will mean: X is a direct summand of Y. Let Z be an indecomposable summand of W^B. Since T is the inertial subgroup of W, $(W^B)_{A_1} = |T| \cdot W$ and so Z_{A_1} is a multiple of W. Since $Z \mid (Z^A)_B$, there exists an indecomposable summand V of Z^A such that $Z \mid V_B$. Then $V \mid W^A$ and $W \mid V_{A_1}$. By Theorem 2, there exists an indecomposable B-module U such that $V \cong U^A$ and U_{A_1} is a multiple of W. Now $U \mid (Z^A)_B$ because $V \mid Z^A$ and $U \mid (U^A)_B = V_B$. But Z is the only indecomposable summand of $(Z^A)_B$ whose restriction to A_1 is a multiple of W, for $(Z^A)_{A_1} = \bigoplus_{i=1}^n a_{s_i} \otimes Z_{A_1}$ (where $\{s_1, \ldots, s_n\}$ is a set of coset representatives of T in S) and $a_{s_i} \otimes Z_{A_1}$ is a proper conjugate of Z_{A_1} (a multiple of a proper conjugate of W). It follows that $U \cong Z$ and so $Z^A \cong U^A \cong V$ is indecomposable. ∎

Equivalence of Theorems 2 *and* 4. If Theorem 4 is proved independently (e.g. by the proof of [2, §19C]), then Theorem 2 can be derived as corollary in the following way: Let V be an indecomposable A-module which is a summand of W^A for some indecomposable summand W of V_{A_1}. Let T be the inertial subgroup of W. By Theorem 4, there exists an indecomposable summand U of W^B such that $V = U^A$. Now $U_{A_1} \cong mW$ for some m because $(W^B)_{A_1} \cong |T|\, W$. Then clearly $V \cong \bigoplus_{i=1}^n a_{s_i} \otimes U$ and $V_{A_1} \cong \bigoplus_{i=1}^n m(a_{s_i} \otimes W)$ where $s_1, \ldots, s_n$ are coset representatives of T in S. This completes the proof of Theorem 2. ∎

3. Ground field extensions

Let K be a field and A a finite dimensional K-algebra. Let F be a finite Galois extension of K, with Galois group Π, and consider the F-algebra $F \otimes A$ (note that throughout this section $\otimes$ will always mean $\otimes_K$). Every element $\sigma \in \Pi$ induces a semi-linear automorphism $\sigma : F \otimes A \to F \otimes A$. If W is an $F \otimes A$-module, one can define a new $F \otimes A$-module structure on W by scalar extension via σ (or equivalently restriction via σ^{-1}). Explicitly the new structure is given by $a \cdot w = \sigma^{-1}(a)w$, $a \in F \otimes A$, $w \in W$. This module is called a Galois conjugate of W.

Now if V is a finitely generated indecomposable A-module, then $F \otimes V$ has a natural structure of $F \otimes A$-module. Moreover, Π acts on $F \otimes V$ via $\sigma(f \otimes v) = \sigma f \otimes v$, $\sigma \in \Pi$, $f \in F$, $v \in V$. This action is semi-linear with respect to $F \otimes A$, i.e. $\sigma(aw) = \sigma(a)\sigma(w)$, $\sigma \in \Pi$, $a \in F \otimes A$, $w \in F \otimes V$. If $F \otimes V = \bigoplus_{i=1}^n W_i$ is a decomposition of $F \otimes V$ into homogeneous components, then so is $F \otimes V = \bigoplus_{i=1}^n \sigma W_i$. One can readily check that σW_i is a Galois conjugate of W_i. By Krull–Schmidt theorem, $\sigma W_i \cong W_j$ for some j. Moreover, it is easy to see that for given i and j, there exists $\sigma \in \Pi$ such that $\sigma W_i \cong W_j$. The purpose of this section is to derive from Theorem 1 a stronger result, namely that for a suitable choice of the submodules W_i, one can replace this isomorphism by an equality:

PROPOSITION 5. *In the above notations, there exists a decomposition* $F \otimes V = \bigoplus_{i=1}^n W_i$ *of* $F \otimes V$ *into homogeneous components such that the modules* W_i *are permuted transitively under the natural action of* Π *on* $F \otimes V$.

Proof. Let $E = \mathrm{End}_A V$ and $\bar{E} = E/\mathrm{Rad}\, E$. Since V is indecomposable, $\bar{E}$ is a division algebra containing K in its center. Now $F \otimes E = \mathrm{End}_{F \otimes A}(F \otimes V)$ and let $\overline{F \otimes E} = F \otimes E/\mathrm{Rad}\,(F \otimes E)$. Since F/K is separable, $\overline{F \otimes E} \cong F \otimes \bar{E}$. Let $F \otimes V = \bigoplus_{i=1}^n W_i$ be a decomposition of $F \otimes V$ into homogeneous components corresponding to idempotents $e_1, \ldots, e_n \in F \otimes E$. The decomposition $F \otimes V = \bigoplus_{i=1}^n \sigma W_i$ corresponds to the idempotents $\sigma e_1 \sigma^{-1}, \ldots, \sigma e_n \sigma^{-1}$ (where σ is viewed as a semi-linear automorphism of $F \otimes V$).

Now Π acts on $F \otimes E$ via $\sigma \cdot (f \otimes e) = \sigma f \otimes e$, $\sigma \in \Pi$, $f \in F$, $e \in E$. We claim that $\sigma z \sigma^{-1} = \sigma \cdot z$ for all $z \in F \otimes E$. Indeed, if $z = f \otimes e$, $f \in F$, $e \in E$, and if $g \otimes v \in F \otimes V$, then:

$$(\sigma z \sigma^{-1})(g \otimes v) = \sigma(f \otimes e)(\sigma^{-1} g \otimes v) = \sigma(f \cdot \sigma^{-1} g) \otimes ev = \sigma f \cdot g \otimes ev$$
$$= (\sigma f \otimes e)(g \otimes v) = (\sigma \cdot z)(g \otimes v).$$

It follows that $\{\sigma \cdot e_1, \ldots, \sigma \cdot e_n\}$ are the idempotents corresponding to the decomposition $F \otimes V = \bigoplus_{i=1}^n \sigma W_i$. By Proposition 3(i), $\{\bar{e}_1, \ldots, \bar{e}_n\} = \{\overline{\sigma \cdot e_1}, \ldots, \overline{\sigma \cdot e_n}\}$ is the set of primitive central idempotents of $F \otimes \bar{E}$. Now Π acts transitively on $\{\bar{e}_1, \ldots, \bar{e}_n\}$ for if $\{\bar{e}_1, \ldots, \bar{e}_k\}$ is a Π-orbit, then $\bar{e} = \sum_{i=1}^k \bar{e}_k$ is an idempotent, invariant under Π, hence lies in $K \otimes \bar{E} = \bar{E}$. Since 1 is the only idempotent of $\bar{E}$, we get $\bar{e} = 1$ and so $k = n$.

Since F/K is separable, $Tr_{F/K}$ is surjective. Therefore there exists $x \in F$ such that $Tr_{F/K}(x) = \sum_{\sigma \in \Pi} \sigma x = 1$. In particular, if Ω denotes the stabilizer of $\bar{e}_1$ and $\sigma_1, \ldots, \sigma_n$ are coset representatives of Ω in Π, then $u = \sum_{i=1}^n \sigma_i x$ satisfies $\sum_{\omega \in \Omega} \omega u = 1$. Also $u \otimes \bar{1} \in F \otimes \bar{E}$ commutes with every $\bar{e}_i$. Therefore $u \otimes 1 \in F \otimes E$

satisfies the hypotheses of Theorem 1. Consequently $\{\bar{e}_1, \ldots, \bar{e}_n\}$ lifts to a set of orthogonal idempotents $f_1, \ldots, f_n$ of $F \otimes E$ which are permuted transitively by Π and such that $\sum_{i=1}^{n} f_i = 1$. By Proposition 3(i), the modules $W_i' = f_i(F \otimes V)$ are homogeneous components of $F \otimes V$. Finally, since $\sigma f_i = f_j$ for some j, we have:

$$\sigma W_i' = \sigma(f_i(F \otimes V)) = (\sigma f_i \sigma^{-1})(F \otimes V) = (\sigma \cdot f_i)(F \otimes V) = f_j(F \otimes V) = W_j'. \quad \blacksquare$$

Remarks. 1) If one replace homogeneous components of $F \otimes V$ by indecomposable summands, then one must consider sets of primitive idempotents $\{\bar{e}_1, \ldots, \bar{e}_n\}$ of $\bar{E}$ instead of primitive central idempotents of $\bar{E}$. If one can show that there exists such a set which is stable under the action of Π (this happens quite often), then the whole proof works without change, so that there exists a decomposition $F \otimes V = \bigoplus_{i=1}^{n} W_i$ into *indecomposable submodules* such that the modules W_i are permuted transitively under the natural action of Π on $F \otimes V$.

2) Proposition 5 holds more generally if one replaces the field K by a complete discrete valuation ring R and the extension F by an unramified Galois extension S (so that the Galois group of S/R is isomorphic to the Galois group of the residue field extension). Moreover, A must be an R-algebra which is finitely generated as R-module.

3) The similarity between restriction to a normal subgroup (Theorem 2) and ground field Galois extension (Proposition 5) extends a little further. If Ω denotes the stabilizer of the homogeneous component W_1 of $F \otimes V$ and if L is the fixed field of Ω, then W_1 is realizable over L, that is there exists an $L \otimes_K A$-module U such that $F \otimes_L U = W_1$. Moreover, by analogy with part (iv) of Theorem 2 (replacing group induction by scalar restriction), one can easily show that $V \cong \mathrm{Res}_K U$.

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Received May 3, 1982