Towards an (∞,2)-category of homotopy coherent monads in an ∞-cosmos

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I dedicate this thesis to my family, who has always been supportive and encouraging, and in particular to my wife Cynthia, which provides me with so much love and happiness.

Abstract

This thesis is part of a program initiated by Riehl and Verity to study the category theory of $(\infty, 1)$-categories in a model-independent way. They showed that most models of $(\infty, 1)$-categories form an ∞-cosmos $\mathcal{K}$, which is essentially a category enriched in quasi-categories with some additional structure reminiscent of a category of fibrant objects. Riehl and Verity showed that it is possible to formulate the category theory of $(\infty, 1)$-categories directly with ∞-cosmos axioms. This should also help organize the category theory of $(\infty, 1)$-*categories with structure.*

Given an ∞-cosmos $\mathcal{K}$, we build via a nerve construction a stratified simplicial set $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ whose objects are homotopy coherent monads in $\mathcal{K}$. If two ∞-cosmoi are weakly equivalent, their respective stratified simplicial sets of homotopy coherent monads are also equivalent. This generalizes a construction of Street for 2-categories. We also provide an $(\infty, 2)$-category $\mathbf{Adj}_r(\mathcal{K})$ whose objects are homotopy coherent adjunctions in $\mathcal{K}$, that we use to classify the 1-simplices of $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ up to homotopy.

Key words: higher category, ∞-cosmos, $(\infty, 2)$-category, $(\infty, 1)$-category, homotopy coherent monad, model category

Résumé

Cette thèse s'inscrit dans un programme initié par Riehl et Verity pour étudier la théorie des $(\infty, 1)$-catégories d'une façon qui ne dépend pas du modèle choisi. Ils ont montré que la plupart des modèles de $(\infty, 1)$-catégories forme un ∞-cosmos, c'est-à-dire essentiellement une catégorie enrichie sur les quasi-catégories, munie de plus d'une structure rappelant celle d'une catégorie d'objets fibrants. Riehl et Verity ont montré qu'il est possible de formuler la théorie des catégories satisfaite par les $(\infty, 1)$-catégories directement à partir des axiomes d'∞-cosmos. Ceci devrait également aider à organiser la théorie des $(\infty, 1)$-*catégories munies d'une structure.*

Étant donné un ∞-cosmos $\mathcal{K}$, nous construisons, grâce à une construction de nerf, un ensemble simplicial stratifié $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ dont les objets sont les monades homotopiquement cohérentes dans $\mathcal{K}$. Si deux ∞-cosmoi sont faiblement équivalents, leurs ensembles simpliciaux stratifiés des monades homotopiquement cohérentes respectifs sont également équivalents. Ceci généralise une construction de Street pour les 2-catégories. Nous fournissons également une $(\infty, 2)$-catégorie $\mathbf{Adj}_r(\mathcal{K})$ dont les objets sont les adjonctions homotopiquement cohérentes dans $\mathcal{K}$ et que nous utilisons pour classifier les 1-simplexes de $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ à homotopie près.

Mots clefs: catégorie d'ordre supérieur, ∞-cosmos, $(\infty, 2)$-catégorie, $(\infty, 1)$-catégorie, monades homotopiquement cohérentes, catégorie de modèles

Contents

Chapter 1

Introduction

1.1 Historical motivations

1.1.1 Monads and adjunctions in classical category theory

A category consists of a collection of objects, for instance sets, groups or topological spaces, together with a collection of morphisms between objects, for instance functions, group homomorphisms or continuous functions, and a composition law. Category theory began as a field in 1945 with the first definitions by Eilenberg and Mac Lane in [17]. Since then, it has become an organizational principle and a tool used in many fields of modern mathematics, such as logic, computer science, universal algebra, algebraic topology, algebraic geometry and mathematical physics. This development was quite fast, as was already noted by Mac Lane in 1989 in [36].

The concept of adjunction is one of the most important in category theory. It was studied first by Kan in 1958 in [26] and describes a relationship between two functors $\begin{tikzcd}\mathscr{B} \arrow[r, shift left, "L"] & \mathscr{A} \arrow[l, shift left, "R"]\end{tikzcd}$. The functor L is left adjoint to R if and only if there is a natural isomorphism $\mathscr{B}(-, R(-)) \cong \mathscr{A}(L(-), -)$, which we denote by $L \dashv R$. This generalizes the classical relations between tensor and hom functors as well as between free and forgetful functors.

Monads where first introduced by Godement in 1958 in [19] under the name of “standard constructions”, in order to obtain resolutions to compute sheaf cohomology. A monad on a category $\mathscr{C}$ consists of an endofunctor $T : \mathscr{C} \to \mathscr{C}$ with a multiplication $\mu : T \circ T \to T$ and unit $\eta : 1_{\mathscr{C}} \to T$ satisfying associativity and unit axioms. This can be summarized by saying that η and μ endow T with the structure of a monoid in the category of endofunctors of $\mathscr{C}$ with respect to composition. Huber, in 1961 in [24], proved that every adjunction generates a monad on the domain of the left adjoint. In 1965, Kleisli in [29] and Eilenberg-Moore in [18] proved independently that every monad is generated by an adjunction, with two different constructions. The Kleisli construction is initial among adjunctions generating a given monad T, whereas the Eilenberg-Moore construction is terminal. The Kleisli construction has applications to type theory and computer science, and is in use in functional programming languages such as Haskell or Scala. The Eilenberg-Moore construction provides a category of algebras for each monad T, denoted $\mathrm{Alg}(T)$, and a free-forgetful adjunction $\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F^T"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(T) \arrow[l, shift left, "U^T"] \end{tikzcd}$. Given another adjunction $\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, shift left, "R"] \end{tikzcd}$ generating T, there is a unique comparison functor $\mathscr{D} \to \mathrm{Alg}(T)$ such that the two triangles involving left or right adjoints commute. The Barr-Beck criterion, also called the “monadicity theorem” (see for instance [7, Theorem 4.4.4]) provides a necessary and sufficient condition to determine when this comparison functor is an equivalence of categories. It turns out that this monadicity theorem implies that categories of objects of an algebraic nature, such as monoids, groups, rings, modules are equivalent to the categories of algebras for the corresponding monads associated to the free-forgetful adjunctions. As a consequence, monads over the category of sets were extensively studied in order to understand the properties shared by varieties of algebras (in the sense of universal algebra). More surprisingly, a-priori non-algebraic categories can be obtained as algebras for a monad on sets, such as compact Hausdorff spaces, using the ultrafilter monad.

There are many applications of monads in modern mathematics. To give just two examples, they are used by May in [38] to study iterated loop spaces, and by Janelidze and Borceux in [8] in order to study categorical Galois theory.

In this thesis, we consider the generalizations of monads and adjunctions defined by Riehl and Verity in [45]. These generalizations take place in the context of higher categories, which we informally introduce now.

1.1.2 Higher category theory

As category theory expanded, it became clear that a theory able to deal with *categories with structure* rather than *sets with structure* was needed. This was provided by Eilenberg and Kelly, as Street explains in [54].

> The long Eilenberg–Kelly paper [16] in the 1965 La Jolla Conference Proceedings was important for higher category theory in many ways; I shall mention only two. One of these ways was the realization that 2-categories could be used to organize category theory just as category

> theory organizes the theory of sets with structure. [...] The other way worth mentioning here is their efficient definition of (strict) n-category and (strict) n-functor using enrichment. If $\mathcal{V}$ is symmetric monoidal then $\mathcal{V}$-**Cat** is too and so the enrichment process can be iterated. In particular, starting with $\mathcal{V}_0 = \mathbf{Set}$ using cartesian product, we obtain cartesian monoidal categories $\mathcal{V}_n$ defined by $\mathcal{V}_{n+1} = \mathcal{V}_n$-**Cat**. This $\mathcal{V}_n$ is the category n-**Cat** of n-categories and n-functors. In my opinion, processes like $\mathcal{V} \mapsto \mathcal{V}$-**Cat** are fundamental in dimension raising.

A 2-category is a category which also has a collections of morphisms *between morphisms*. The prototypical example is the category **Cat** of (small) categories, functors and natural transformations. The definitions of adjunctions and monads can be internalized in any 2-category $\mathscr{C}$. Adjunctions and monads in the classical sense are then obtained by setting $\mathscr{C} = \mathbf{Cat}$. In 1972 in [50], Street constructed a 2-category $\mathbf{Mnd}(\mathscr{C})$ of monads in any 2-category $\mathscr{C}$. This approach unified the treatment of monads living in different categorical settings, in other words, in different 2-categories of categories with structure. For instance, one can pick $\mathscr{C}$ to be a 2-category of enriched categories or of internal categories. Also, duality can be exploited, for instance comonads in $\mathscr{C}$ are monads in $\mathscr{C}^{\mathrm{co}}$, and the Eilenberg-Moore construction in $\mathscr{C}^{\mathrm{op}}$ is precisely the Kleisli construction in $\mathscr{C}$.

A distributive law between two monads T and S on the same object is the necessary and sufficient data to equip the composite $T \circ S$ with a monad structure appropriately compatible with T and S. The classical example of this phenomenon relates the monad for monoids to the monad for abelian groups to produce the ring monad. Street's construction shed a new light on distributive laws. Indeed, a distributive law can be understood as a monad in $\mathbf{Mnd}(\mathscr{C})$, and one can take advantage of the different dualities to produce different types of mixed distributive laws, relating a monad and a comonad.

The definition of an n-category given in the previous excerpt is a very rigid notion of higher category, since all compositions are supposed to be strictly associative and unital. In practice, many examples are naturally weak n-categories, which are morally n-categories where associativity and unitality of the composition holds only up to coherent equivalence. The difficulty to determine a precise meaning of *coherence* grows drastically when n grows. One of the approaches to coherence is algebraic. It uses operadic machinery, with contributions by Grothendieck, Maltsiniotis, Batanin and Trimble, among others. Instead, it can be convenient to adopt a more geometric approach. The first step is to characterize the (stratified) simplicial sets arising as nerves of n-categories, usually as having the *unique* right lifting property against a set of special maps. Then, one defines weak n-categories as those (stratified) simplicial sets satisfying the right lifting property, without uniqueness, with respect to the same set of maps. This kind of consideration immediately summons homotopy theory, and the next step is often to provide a Quillen model structure whose fibrant objects are weak n-categories, with generally all objects being cofibrant. Remark that in this approach a weak n-category has cells of all dimensions, in the same way the nerve of a category can have non-degenerate m-simplicies for any m. However, it turns out that those m-cells are invertible for $m > n$. In the literature, weak n-categories are often referred to as (∞, n)-categories.

Quasi-categories

In [5], Boardman and Vogt introduced *weak Kan complexes* as the simplicial sets satisfying the right lifting property for *inner* horns. These can be thought of as "weak 1-categories" or $(\infty, 1)$-categories, since the definition is a relaxation of the well known characterization of the simplicial sets arising as nerves of categories. Joyal built a model structure, showed that much of basic category theory extends from categories to weak Kan complexes, and thus renamed those special simplicial sets *quasi-categories*. Lurie, in [35], developed even further the theory of these objects, which he renamed ∞-category. We prefer to stick to Joyal's terminology, in order to avoid confusion with other notions of higher categories.

$(\infty, 1)$-Categories have been the object of a particular interest because of their deep connection with homotopy theory. In [15], [13], and [14] Dwyer and Kan built a simplicial localization associated to any category with a subcategory of weak equivalences. They showed that given a model category $\mathscr{M}$, its simplicial localization contains the homotopical information of $\mathscr{M}$, for instance the homotopy type of the derived mapping spaces. It follows that the category of simplicial categories can be understood as the category of homotopy theories. In [4], Bergner showed the existence of a model structure with *Dwyer-Kan equivalences* as weak equivalences. Roughly, a Dwyer-Kan equivalence is a simplicial functor that is essentially surjective up to homotopy and induces weak equivalences on hom-spaces. This encodes the *homotopy theory of homotopy theories*. The homotopy coherent nerve, which we review in 2.5, provides a Quillen equivalence between the model categories of simplicial categories and of quasi-categories. As a consequence, one can equally well consider that quasi-categories represent homotopy theories.

In [35] and [34], Lurie developed the analogs of adjunction and monads for quasi-categories, along with a corresponding monadicity theorem. These should be thought of as homotopically meaningful versions of adjunctions and monads, and thus should be of particular interest to the homotopy theorist. Note that an analog of the monadicity theorem for monads over model categories does not exist in the literature. This suggests that the quasi-categorical technology developed by Lurie is better behaved than anything known for model categories in this particular context. However, the technicality and difficulty of his approach are quite high.

As a rough indication of this fact, adjunctions are defined in [35, Section 5.2, page 337] whereas monads and the monadicity theorem are discussed in [34, Section 4.7, page 464-507].

1.1.3 ∞-Cosmoi

The proliferation of models of $(\infty, 1)$-categories made apparent the need for a unifying treatment of some of their basic category theory. Riehl and Verity initiated a new approach, where most of the definitions and proofs are of a 2-categorical nature. Following these authors, a good context to develop the category theory of $(\infty, 1)$-categories is an ∞-cosmos, which is essentially a category enriched over quasi-categories with some additional structure reminiscent of a category of fibrant objects (see 2.4). Most of the category theory can then be understood in its homotopy 2-category and is thus independent of the chosen model.

These ideas were developed in a series of articles, [43, 45, 44, 46, 47]. These articles were summarized by Riehl in a series of lectures at EPFL, [41]. Many of the usual models for $(\infty, 1)$-categories do form an ∞-cosmos, such as the simplicially enriched categories of quasi-categories, Segal categories, complete Segal spaces and marked simplicial sets. The 2-category of small categories is also an ∞-cosmos, and many models for (∞, n)-categories as well (see 2.4). As the theory of 2-categories organizes the study of categories with structures, ∞-cosmoi should organize the study of $(\infty, 1)$-*categories with structure.*

Homotopy coherent monads and adjunctions

One of the interesting achievements of Riehl and Verity's work is to provide another approach to adjunctions and monads in $(\infty, 1)$-categories, and more generally in any ∞-cosmos. Since we are working with weak categories, homotopy coherence is unavoidable. One of the key ideas is to encode homotopy coherence in a simplicial category $\mathcal{K}$ as simplicial functors $\mathscr{C} \to \mathcal{K}$, where $\mathscr{C}$ is a well chosen simplicial category. This idea goes back at least to Cordier and Porter [11] and originated in earlier work of Vogt [58] on homotopy coherent diagrams. For instance, the homotopy coherent nerve is constructed in this way. In Riehl and Verity's paper [45], $\mathscr{C}$ is the universal 2-category containing the object of study, either a monad or an adjunction (see 2.4.1). The notion of homotopy coherent monad is well behaved: Riehl and Verity show that in the case of a homotopy coherent monad $\mathbb{T}$ on a quasi-category, there is an Eilenberg-Moore quasi-category of (homotopy coherent) $\mathbb{T}$-algebras. Moreover, they show that the classical monadicity theorem has a generalization to this context. These definitions and results are recalled in 2.4.1. Strikingly, homotopy coherence can be constructed from minimal data for adjunctions, as shown by Riehl and Verity in [45, Theorem 4.3.8]. However, a similar statement does not hold for homotopy coherent monads.

Some of Riehl and Verity's arguments are combinatorial in nature and thus rely heavily on the use of a very concrete description of the universal 2-categories containing a monad or an adjunction, using a *graphical calculus of squiggles* that we review in Section 2.3.2. The universal adjunction was previously described partially by Auderset in [3] and by Schanuel and Street in [49]. We review this description in 2.3.1 and the correspondence between the two models in 2.3.3.

1.2 Main contributions and organization of the thesis

The main contribution of this thesis is the construction of stratified simplicial set of homotopy coherent monads and of homotopy coherent adjunctions in an ∞-cosmos $\mathcal{K}$. To any category $\mathcal{K}$ enriched in quasi-categories, we associate a stratified simplicial set $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$, that we call its *homotopy coherent monadic nerve*. Its vertices are homotopy coherent monads in $\mathcal{K}$, and is analogous to Street's construction of a 2-category of monads in a 2-category $\mathscr{C}$. Our motivation is the following conjecture.

Conjecture B. *Let $\mathcal{K}$ be an ∞-cosmos. The stratified simplicial set $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ is a 2-trivial and saturated weak complicial set.*

Here, the term "2-trivial and saturated weak complicial set" refers to our preferred model of $(\infty, 2)$-categories, which is reviewed in 2.2.2. If $\mathcal{K}$ and $\mathcal{L}$ are ∞-cosmoi that are appropriately equivalent, the homotopy coherent monadic nerves of $\mathcal{K}$ and of $\mathcal{L}$ are also equivalent. This model-independence is established in Theorem 6.15. When $\mathcal{C}$ is a 2-category, $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{C})$ is isomorphic to the homotopy coherent nerve of $\mathbf{Mnd}(\mathcal{C})$, which confirms that our construction generalizes Street's construction.

For any ∞-cosmos $\mathcal{K}$, we also build a stratified simplicial set $\mathbf{Adj}_r(\mathcal{K})$ related to $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ and whose vertices are homotopy coherent adjunctions in $\mathcal{K}$. We are able to prove that $\mathbf{Adj}_r(\mathcal{K})$ is indeed an $(\infty, 2)$-category which is moreover model-independent, as stated in the following theorem.

Theorem C. *Let $\mathcal{K}$ be an ∞-cosmos. The stratified simplicial set $\mathbf{Adj}_r(\mathcal{K})$ is a 2-trivial and saturated weak complicial set. Moreover, if $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ is a weak equivalence of ∞-cosmoi, $\mathbf{Adj}_r(\mathbb{P}) : \mathbf{Adj}_r(\mathcal{K}) \to \mathbf{Adj}_r(\mathcal{L})$ is a weak equivalence of weak complicial sets.*

From the homotopy theory standpoint, an $(\infty, 2)$-category can be thought of as a homotopy theory (weakly) enriched in homotopy theories. More precisely, there is an underlying $(\infty, 1)$-category obtained by forgetting non-invertible cells of dimension 2. For a fixed quasi-category B, Lurie provides a quasi-category of monads on B. If Conjecture B holds, the underlying $(\infty, 1)$-category of $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ provides instead a global homotopy theory for homotopy coherent monads in $\mathcal{K}$. This is obtained in Corollary 6.23. Moreover, given two homotopy coherent monads $\mathbb{T}$ and $\mathbb{S}$, it would also be possible to provide a homotopy theory of homotopy coherent monad morphisms $\mathbb{T} \to \mathbb{S}$, even though this aspect has not been explored in this thesis.

To define the homotopy coherent monadic nerve, we construct a cosimplicial 2-category $\mathbf{Mnd}_{\mathrm{hc}}[-] : \Delta \to \mathbf{sCat}$. The 2-category $\mathbf{Mnd}_{\mathrm{hc}}[n]$ satisfies the following 2-universal property

$$2\text{-}\mathbf{Cat}(\mathbf{Mnd}_{\mathrm{hc}}[n], \mathscr{C}) \cong 2\text{-}\mathbf{Cat}(\mathcal{C}\Delta[n]^{\mathrm{co}}, \mathbf{Mnd}(\mathscr{C})),$$

where $\mathcal{C}\Delta[-]^{\mathrm{co}} : \Delta \to \mathbf{sCat}$ is the cosimplicial 2-category defining the homotopy coherent nerve. In order to attack Conjecture B, it is essential to have a combinatorially amenable description of $\mathbf{Mnd}_{\mathrm{hc}}[n]$. In Chapter 3, we construct first a 2-category $\mathbf{Adj}_{\mathrm{hc}}[n]$. Chapter 4 is devoted to the proof of Theorem A, which is a lifting result that is the technical core of the thesis, generalizing [45, Theorem 4.3.8]. The original idea of the proof by Riehl and Verity is unchanged, but adapted to work in our context.

Corollary 5.1 proves that $\mathbf{Adj}_{\mathrm{hc}}[n]$ has a 2-universal property analogous to that of $\mathbf{Mnd}_{\mathrm{hc}}[n]$, obtained by replacing $\mathbf{Mnd}(\mathscr{C})$ by a suitable 2-category of adjunctions. It formally follows that $\mathbf{Mnd}_{\mathrm{hc}}[n]$ is a full 2-subcategory of $\mathbf{Adj}_{\mathrm{hc}}[n]$. Theorem A also implies that a simplicial functor $\mathbf{Adj}_{\mathrm{hc}}[n] \to \mathcal{K}$ is determined up to a contractible space of choices by the underlying simplicial functor

$$\mathcal{C}\Delta[n]^{\mathrm{co}} \times (\bullet \to \bullet) \to \mathcal{K},$$

as stated in Corollary 5.13, and thus up to a zig-zag of natural weak equivalences by Corollary 5.14. These results are collected in Chapter 5, which ends with the study of right Kan extensions along the inclusion $j_n : \mathbf{Mnd}_{\mathrm{hc}}[n] \to \mathbf{Adj}_{\mathrm{hc}}[n]$. It is shown that, up to equivalence, simplicial functors $\mathbf{Mnd}_{\mathrm{hc}}[n] \to \mathcal{K}$ encode homotopy coherent diagrams between their respective objects of algebras.

In particular, homotopy coherent monad morphisms induces morphisms on the level of the objects of algebras. In Chapter 6 we classify these homotopy coherent monad morphisms up to homotopy. Indeed, in Theorem 6.30, we prove that the homotopy category of our candidate for the global homotopy theory for homotopy coherent monads in $\mathcal{K}$ is a reflective subcategory of the homotopy category of homotopy coherent adjunctions in $\mathcal{K}$. This also permits to classify equivalences of homotopy coherent monads in $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$, which is done in Corollary 6.33.

We review rapidly some of the relevant background material discussed in 1.1 in Chapter 2. However, this thesis is not self-contained. The reader is supposed to be familiar with basic notions of category theory (see [6] or [37]), model category theory (see [39] or [23]), and simplicial sets (see [20]). Note that size issues have been ignored in this thesis, in order to keep the exposition as simple as possible. We leave it to the careful reader to use Grothendieck universes to avoid any size problem.

Notations and Terminology

- For a natural number n, we will write $\mathbf{n}$ when we consider it as an ordinal. That is, we set $\mathbf{0} = \emptyset$ and $\mathbf{n} = \{\mathbf{0}, \dots, \mathbf{n-1}\}$. The set of all natural numbers is denoted $\mathbb{N}$.

- We usually use curly calligraphic letters $\mathscr{A}, \mathscr{B}, \mathscr{C}$, to denote categories. For a category $\mathscr{C}$, $|\mathscr{C}|$ denotes its class of objects. If $A, B \in |\mathscr{C}|$, $\mathscr{C}(A, B)$ denotes the set of morphisms from A to B in $\mathscr{C}$. Functors are usually denoted by capital letters F, G. We usually denote natural transformations by Greek letters, η, ϵ, μ.

- For 2-categories we usually use the same typographical conventions as for categories, with the exception that $\mathscr{C}(A, B)$ now denotes the category of morphisms from A to B in a 2-category $\mathscr{C}$.

- We usually use calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}$, to denote simplicial categories, and double-struck letters $\mathbb{F}, \mathbb{G}$, for simplicial functors.

- The category Δ_+ is the category of finite ordinals and order-preserving maps. Among those maps, it is well known that the cofaces $d^i : \mathbf{n-1} \to \mathbf{n}$ and codegeneracies $s^j : \mathbf{n+1} \to \mathbf{n}$ for $0 \le i, j < n$ generate this category. The coface d^i is the only order-preserving injective map not containing i in its image, while $s^j : \mathbf{n+1} \to \mathbf{n}$ is the only order-preserving surjective map such that j is the image of two elements of $\mathbf{n+1}$. More precisely, for $k \in \mathbf{n-1}$ and $l \in \mathbf{n+1}$,

$$d^i(k) = \left\{ \begin{array}{ccc} k & : & k < i \\ k+1 & : & k \ge i \end{array} \right. , \quad s^j(l) = \left\{ \begin{array}{ccc} l & : & l \le i \\ l-1 & : & l > i \end{array} \right. .$$

  Remark that the more common notation $[n] = \{0, \dots, n\}$ for the objects of Δ is such that $\mathbf{n} = [n-1]$. This change of notation is more convenient when considering the ordinal sum, because $[n] + [m] = [n+m-1]$, whereas the ordinal sum $\mathbf{n} + \mathbf{m}$ is exactly the ordinal with $n+m$ elements and is also denoted $\mathbf{n} + \mathbf{m}$. While working with simplicial sets we usually keep the standard notation $[n]$.

- If X is a simplicial set and $x \in X_n$ is a simplex, we say that x is *degenerated at* k if there exists $y \in X_{n-1}$ such that $s_k(y) = x$.
- If $(X, \leq)$ is a poset and E is a set, the set of functions $E \to X$ is naturally ordered. Indeed, for $f, g : E \to X$, we define $f \leq g$ if and only if $f(e) \leq g(e)$ for all $e \in E$. As a notational convenience, if $U, V \subseteq X$, we write $U \leq V$ to mean that for all $u \in U$ and all $v \in V$, $u \leq v$. Similarly, by $U < V$ we mean that for all $u \in U$ and all $v \in V$, $u < v$.

Chapter 2

Background Material

The goal of this chapter is to give an overview of the background material needed to read the core of this thesis. In order to make it easier to skim through, we do not include proofs as long as we can find a similar statement in the literature. We do not claim originality for results presented with proof in this section, as they are probably well known to the specialists. If a proof is mostly technical and would be prejudicial to the exposition, we defer it to Appendix A. Some material is necessary only to read very specific parts of the thesis, and in this case we included it in Appendix B. We refer the reader to the corresponding paragraphs of the introduction for motivation and historical comments.

2.1 Enriched categories

In this thesis, we will mainly use categories enriched over (stratified) simplicial sets or small categories. We do not want to suppose that the reader is familiar with $\mathcal{V}$-category theory for a general symmetric closed monoidal category $\mathcal{V}$. Instead, we review both with very explicit descriptions of these structures in Sections 2.1.1 and 2.1.2. However, we use the language of $\mathcal{V}$-category theory to make available to us the concepts of weighted limits and enriched right Kan extensions for both contexts in Section 2.1.3. Sections 2.1.1 and 2.1.2 can be used as dictionaries to translate these concepts into elementary terms.

2.1.1 Simplicial categories

Definition 2.1. A *simplicial category* $\mathcal{C}$ is a category enriched in the cartesian closed category of simplicial sets **sSet**. It can also be seen as a simplicial object $\mathcal{C} : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ which is constant on objects. More explicitly it contains

(i) a class of objects $|\mathcal{C}|$;

(ii) for all objects $A, B \in |\mathcal{C}|$, a simplicial set of morphisms $\mathcal{C}(A, B)$, often called *homspace*, of which an n-simplex $f \in \mathcal{C}(A, B)_n$ is called an *n-morphism.*

(iii) for all object $A \in |\mathcal{C}|$, an *identity* 0*-morphism* $1_A : A \to A$;

(iv) for all objects $A, B, C \in |\mathcal{C}|$, and $n \geq 0$, an associative and unital composition

$$\circ_n : \mathcal{C}(A, B)_n \times \mathcal{C}(B, C)_n \longrightarrow \mathcal{C}(A, C)_n$$

compatible with the simplicial structure (the n-th identity is the degeneracy of the 0-th identity).

We write $\mathcal{C}_m$ for the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are the m-morphisms of $\mathcal{C}$.

Definition 2.2. A *simplicial functor* $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ is an **sSet**-functor. More explicitly, it consists of

(i) a function $\mathbb{F} : |\mathcal{C}| \longrightarrow |\mathcal{D}|$;

(ii) for each $C, C' \in |\mathcal{C}|$, and $n \geq 0$, a function

$$(\mathbb{F}_{C,C'})_n : \mathcal{C}(C, C')_n \longrightarrow \mathcal{D}(\mathbb{F}(C), \mathbb{F}(C'))_n$$

preserving composition, units and the simplicial structure.

Definition 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two simplicial categories (with $\mathcal{A}$ small). The simplicial category of simplicial functors from $\mathcal{A}$ to $\mathcal{B}$ is written $[\mathcal{A}, \mathcal{B}]$ or $\mathcal{B}^{\mathcal{A}}$. It is defined, for simplicial functors $\mathbb{F}, \mathbb{G} : \mathcal{A} \to \mathcal{B}$, by letting $[\mathcal{A}, \mathcal{B}](\mathbb{F}, \mathbb{G})$ be the simplicial set defined as follows. A n-simplex $\alpha \in [\mathcal{A}, \mathcal{B}](\mathbb{F}, \mathbb{G})_n$ is a family of n-morphisms $\alpha_A : \mathbb{F}(A) \to \mathbb{G}(A)$ of $\mathcal{B}$ indexed by the objects of $\mathcal{A}$, such that for every simplicial operator $\phi : [m] \to [n]$ and m-morphism $a : A \to A'$ of $\mathcal{A}$, the following diagram commutes in $\mathcal{B}_m$,

$$\begin{tikzcd}
\mathbb{F}(A) \arrow[r, "\phi^*(\alpha_A)"] \arrow[d, "\mathbb{F}(a)"'] & \mathbb{G}(A) \arrow[d, "\mathbb{G}(a)"] \\
\mathbb{F}(A') \arrow[r, "\phi^*(\alpha_{A'})"] & \mathbb{G}(A').
\end{tikzcd}$$

The simplicial structure is inherited from the simplicial structure of the homspace of $\mathcal{B}$. The **sSet**-natural transformations $\mathbb{F} \Rightarrow \mathbb{G}$ are the 0-simplices of the simplicial set $[\mathcal{A}, \mathcal{B}](\mathbb{F}, \mathbb{G})$. They are also referred to as *simplicial natural transformations.*

Definition 2.4. A simplicial functor $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ is said to be a *local isofibration* if for all pairs of objects $C, C' \in |\mathcal{C}|$,

$$\mathbb{F}_{C,C'} : \mathcal{C}(C, C') \to \mathcal{D}(\mathbb{F}(C), \mathbb{F}(C'))$$

is an isofibration of simplicial sets (See Proposition 2.53).

Relative simplicial computads

Definition 2.5. We define a functor $\mathbb{2}[-] : \mathbf{sSet} \longrightarrow \mathbf{sCat}$. For a simplicial set X, $\mathbb{2}[X]$ is the simplicial category with two objects $0, 1$ and

- $\mathbb{2}[X](0,1) = X$;
- $\mathbb{2}[X](0,0) = \mathbb{2}[X](1,1) = *$;
- $\mathbb{2}[X](1,0) = \emptyset$;

with the unique compositions and units possible. If $f : X \longrightarrow Y$ is a simplicial map, $\mathbb{2}[f]$ is the simplicial functor which is the identity on objects and given by $f : \mathbb{2}[X](0,1) \longrightarrow \mathbb{2}[Y](0,1)$ on morphisms.

Definition 2.6. Let $I = \{\partial\Delta[n] \to \Delta[n] : n \geq 0\}$ be the set of generating cofibrations of **sSet**. A simplicial functor $\mathcal{C} \to \mathcal{D}$ is said to be a *relative simplicial computad* if it is a relative $\mathbb{2}[I] \cup \{\emptyset \to *\}$-cell complex. More precisely, we mean that that relative simplicial computads are those maps which can be obtained as a transfinite composition of pushout of maps in $\mathbb{2}[I] \cup \{\emptyset \to *\}$. A *simplicial computad* is a simplicial category $\mathcal{C}$ such that the unique simplicial functor $\emptyset \to \mathcal{C}$ is a relative simplicial computad.

Simplicial computads are exactly the cofibrant objects in Bergner's model structure on simplicial categories [42, Lemma 16.2.2], while relative simplicial computads are cofibrations which are transfinite composite of pushouts of generating cofibrations. The justification for the terminology comes from Lemmas 2.8 and 2.13. It is shown that they are the appropriate notion of (relative) *free simplicial categories* on a collection of generators.

Definition 2.7. A morphism $f : A \to B$ in a category $\mathscr{C}$ is said to be atomic if it is not an identity and whenever $f = g \cdot h$, then g or h is an identity.

We present now a slight generalization of a characterization of simplicial computads by Riehl and Verity to relative simplicial computads.

Lemma 2.8. *Let $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ be a simplicial functor which is bijective on objects. Then, the following propositions are equivalent.*

(i) $\mathbb{F}$ *is a relative simplicial computad;*

(ii)
- $\mathbb{F}$ *is faithful;*
- *for every m-morphism h of $\mathcal{D}$ there exist a unique integer $k \in \mathbb{N} \cup \{-1\}$ and m-morphisms $c_i \in \mathbb{F}(\mathcal{C})$ and $h_i \in \mathcal{D} \setminus \mathbb{F}(\mathcal{C})$ such that*

$$h = c_{k+1} \cdot h_k \cdots h_1 \cdot c_1 \cdot h_0 \cdot c_0 \tag{2.1}$$

and the h_i are atomic;

- *atomic morphisms of $\mathcal{D}$ not in the image of $\mathcal{C}$ are closed under degeneracies.*

Moreover, under these conditions, images of atomic morphisms of $\mathcal{C}$ are atomic in $\mathcal{D}$.

Proof. We defer the proof to the Appendix A. □

Corollary 2.9. *Let $\mathbb{J} : \mathcal{C} \to \mathcal{D}$ be a relative simplicial computad which is bijective on objects and g a morphism of $\mathcal{D}_m$. There exist unique $k \in \mathbb{N}$, $c_i \in \mathcal{C}_m$, non-degenerate atomic morphisms $g_i \in \mathcal{D}_{m_i} \setminus \mathbb{J}(\mathcal{C})$ and degeneracy operators $\sigma_i : [m] \to [m_i]$ such that*

$$g = \mathbb{J}(c_{k+1}) \cdot \sigma_k^*(g_k) \cdot \mathbb{J}(c_k) \cdot \sigma_{k-1}^*(g_{k-1}) \cdots \sigma_0^*(g_0) \cdot \mathbb{J}(c_0). \tag{2.2}$$

Corollary 2.10. *A simplicial category $\mathcal{D}$ is a simplicial computad if and only if*

- *for every m-morphism h of $\mathcal{D}$ there exist a unique integer k and unique morphisms $h_i \in \mathcal{D}$ such that*

  $$h = h_k \cdots h_0.$$

  and the h_i are atomic (in $\mathcal{D}_m$);

- *atomic morphisms of $\mathcal{D}$ are closed under degeneracies.*

Proof. Remark that $\mathcal{D}$ is a simplicial computad if and only if the inclusion of the subcategory containing only the objects and identities is a relative simplicial computad. Apply lemma 2.8. □

Definition 2.11. Let $\mathcal{C}$ be a simplicial subcategory of a simplicial computad $\mathcal{D}$. We say that $\mathcal{C}$ is *atom-complete* if for all $c : D \to D'$ in $\mathcal{C}$, and $c = h_n \dots h_0$ a decomposition of c in atomic arrows (in $\mathcal{D}$), then $h_i \in \mathcal{C}$ for all $i = 0, \dots, n$.

Corollary 2.12. *Let $\mathcal{D}$ be a simplicial computad and $\mathcal{C} \subseteq \mathcal{D}$ an atom-complete simplicial subcategory containing all objects. Then, $\mathcal{C}$ is a simplicial computad and $\mathcal{C} \to \mathcal{D}$ is a relative simplicial computad.*

Proof. Let $c : D \to D'$ be a morphism of $\mathcal{C}$. It decomposes in $\mathcal{D}$ in a unique way as $c = h_n \dots h_0$, with h_i atomic in $\mathcal{C}$ by assumption. Thus, $\mathcal{C}$ verifies the hypothesis of corollary 2.10, and $\mathcal{C}$ is a simplicial computad. Moreover, the inclusion $\mathcal{C} \to \mathcal{D}$ verifies the hypothesis of 2.8. □

Proposition 2.13. *Let* $\mathbb{J} : \mathcal{C} \to \mathcal{D}$ *be a relative simplicial computad which is bijective on objects and* $\mathcal{E}$ *a simplicial category. An extension of a simplicial functor* $\mathbb{F} : \mathcal{C} \to \mathcal{E}$ *to a simplicial functor* $\mathcal{D} \to \mathcal{E}$ *is uniquely specified by choosing for each non-degenerate atomic morphism* $g : x \to y$ *of* $\mathcal{D}_m \setminus \mathbb{J}(\mathcal{C})$*, a morphism* $\mathbb{F}(g) : \mathbb{F}x \to \mathbb{F}y$ *in* $\mathcal{E}_m$*; such that if* $d_i g = \mathbb{J}(c_{k+1}) \cdot \sigma_k^*(g_k) \cdots \sigma_0^*(g_0) \cdot \mathbb{J}(c_0)$ *is the decomposition of* $d_i g$ *as in* (2.2)*, then*

$$d_i \mathbb{F}(g) = \mathbb{F}(c_{k+1}) \cdot \sigma_k^* \mathbb{F}(g_k) \cdots \sigma_0^* \mathbb{F}(g_0) \cdot \mathbb{F}(c_0).$$

Proof. We defer the proof to Appendix A. ☐

Simplicial presheaves

In this thesis, simplicial presheaves are endowed with the projective model structure, which we recall now.

Definition 2.14. Let $\mathcal{D}$ be a small simplicial category and $\mathscr{M}$ a simplicial category whose underlying category is a model category. The *projective model structure* on $\mathscr{M}^{\mathcal{D}}$, if it exists, is defined as follows.

- The class of projective weak equivalences is the class of objectwise weak equivalences.
- The class of projective fibrations is the class of objectwise fibrations.
- The class of projective cofibrations is the class of maps having the left lifting property with respect to the projective acyclic fibrations.

We are mainly interested in the case $\mathscr{M} = \mathbf{sSet}_J$, the category of simplicial sets endowed with the Joyal model structure (See 2.53). In this particular case, the projective model structure exists and the set of generating cofibrations can be chosen to be the set of *projective cells*

$$\{\partial\Delta[n] \times \mathcal{D}(D, -) \to \Delta[n] \times \mathcal{D}(D, -) : n \geq 0, D \in |\mathcal{D}|\}.$$

A *relative projective cell complex* is a transfinite composite of pushouts of projective cells. By standard arguments of model category theory, every projective cofibration is a retract of a relative projective cell complex.

There is a way to relate simplicial presheaves and simplicial categories, through the following construction.

Definition 2.15. Let $\mathcal{C}$ be a simplicial category and $\mathbb{F} : \mathcal{C} \to \mathbf{sSet}$ a simplicial functor. The *collage construction of* $\mathbb{F}$ is the simplicial category $\operatorname{coll}\mathbb{F}$ containing $\mathcal{C}$ as a simplicial subcategory and containing precisely one extra object $\star$ whose endomorphism space is a point. Moreover, for an object $C \in |\mathcal{C}|$, $\operatorname{coll}\mathbb{F}(C,\star) = \emptyset$ and $\operatorname{coll}\mathbb{F}(\star, C) = \mathbb{F}(C)$. The action maps $\mathscr{C}(C,D) \times F(C) \to F(D)$ provide the missing composition laws. This construction is functorial: a simplicial natural transformation $\alpha : \mathbb{F} \to \mathbb{G}$ induces a simplicial functor $\operatorname{coll}\alpha : \operatorname{coll}\mathbb{F} \to \operatorname{coll}\mathbb{G}$ given by identity on $\mathcal{C}$ and $\operatorname{coll}\alpha_{\star,C} = \alpha_C : \mathbb{F}(C) \to \mathbb{G}(C)$.

Proposition 2.16. *Let $\mathscr{A}$ be a small simplicial category. A natural transformation $i : V \to W$ in $\mathbf{sSet}^{\mathscr{A}}$ is a relative projective cell complex if and only if its collage* $\operatorname{coll}(i) : \operatorname{coll}V \to \operatorname{coll}W$ *is a relative simplicial computad.*

Proof. See [45, Proposition 3.3.3]. □

2.1.2 2-Categories

Definition 2.17. A 2-*category* is a category enriched over the cartesian closed category **Cat** of small categories. More explicitly, a 2-category $\mathscr{C}$ contains

(i) a class of objects $|\mathscr{C}|$;

(ii) for all objects $A, B \in |\mathscr{C}|$, a small category of morphisms $\mathscr{C}(A,B)$ where

  (a) an object $f \in |\mathscr{C}(A,B)|$ is called a 1-*cell* and written $f : A \to B$;

  (b) a morphism $\alpha \in \mathscr{C}(A,B)(f,g)$ is called a 2-*cell* and written $$\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & B \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$;

  (c) composition in $\mathscr{C}(A,B)$ is called *vertical composition* and written $\circ$;

(iii) for all object $A \in |\mathscr{C}|$, an *identity* 1-*cell* $A \to A$ denoted by 1_A or id_A;

(iv) for all objects $A, B, C \in |\mathscr{C}|$, a *horizontal composition* functor

$$\cdot : \mathscr{C}(A,B) \times \mathscr{C}(B,C) \longrightarrow \mathscr{C}(A,C).$$

These data satisfy the following axioms.

(i) Horizontal and vertical compositions are associative and unital;

(ii) If $$\begin{tikzcd}[column sep=large] A \arrow[r, bend left=50, "f_1"{name=F1}] \arrow[r, "f_2"{name=F2, description}] \arrow[r, bend right=50, "f_3"'{name=F3}] & B \arrow[r, bend left=50, "g_1"{name=G1}] \arrow[r, "g_2"{name=G2, description}] \arrow[r, bend right=50, "g_3"'{name=G3}] & C \arrow[Rightarrow, from=F1, to=F2, "\alpha_1"] \arrow[Rightarrow, from=F2, to=F3, "\alpha_2"] \arrow[Rightarrow, from=G1, to=G2, "\beta_1"] \arrow[Rightarrow, from=G2, to=G3, "\beta_2"] \end{tikzcd}$$ is a diagram in $\mathscr{C}$, the *interchange law*

$(\beta_2 \circ \beta_1) \cdot (\alpha_2 \circ \alpha_1) = (\beta_2 \cdot \alpha_2) \circ (\beta_1 \cdot \alpha_1)$ holds.

Examples 2.18.

- Let $\mathcal{V}$ be a monoidal category. There is a 2-category $\mathcal{V}$-**Cat** whose objects, 1-cells and 2-cells are respectively small $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations.

- To an ∞-cosmos $\mathcal{K}$ is associated its homotopy 2-category $h_*(\mathcal{K})$ (see section 2.4).

Definition 2.19. Let $\mathscr{C}$ be a 2-category. We define the two following dual 2-categories.

- $\mathscr{C}^{\mathrm{op}}$ has the same class of objects and $\mathscr{C}^{\mathrm{op}}(A,B) = \mathscr{C}(B,A)$, only the direction of the 1-cells is reversed.

- $\mathscr{C}^{\mathrm{co}}$ has the same class of objects and $\mathscr{C}^{\mathrm{co}}(A,B) = \mathscr{C}(A,B)^{\mathrm{op}}$, only the direction of the 2-cells is reversed.

Definition 2.20. Let $\mathscr{C}$, $\mathscr{D}$ be 2-categories. A strict 2-functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a **Cat**-enriched functor. More explicitly, F consists of

(i) a function $F : |\mathscr{C}| \longrightarrow |\mathscr{D}|$;

(ii) for each $C, C' \in |\mathscr{C}|$, a functor $F_{C,C'} : \mathscr{C}(C,C') \longrightarrow \mathscr{D}(F(C),F(C'))$ preserving unit 1-cells and horizontal composition.

The experienced reader may have noticed that 2-categories embeds fully faithfully in simplicial categories. This is discussed in Remark 2.50.

Definition 2.21. Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be strict 2-functors. A 2-natural transformation $\alpha : F \Rightarrow G$ is a **Cat**-natural transformation. More explicitly, α consists of a collection $\alpha_C : FC \longrightarrow GC$ of 1-cells of $\mathscr{D}$, indexed by the objects of $\mathscr{C}$ such that

- for all 1-cells $f : C \longrightarrow C'$ of $\mathscr{C}$, $\alpha_{C'} Ff = Gf \alpha_C$;

- for all 2-cells $\sigma : f \longrightarrow g$, we have $\alpha_{C'} F\sigma = G\sigma \alpha_C$.

$$\begin{tikzcd}
F(C) \arrow[r, "\alpha_C"] \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\overset{F\sigma}{\Rightarrow}"] & G(C) \arrow[d, bend right, "Gf"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\overset{G\sigma}{\Rightarrow}"] \\
F(C') \arrow[r, "\alpha_{C'}"] & G(C')
\end{tikzcd}$$

Definition 2.22. Let
$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha \quad \Downarrow\beta"] & \mathscr{D} \end{tikzcd}$$
be a diagram in 2-**Cat**. A modification $\alpha \overset{m}{\Rrightarrow} \beta$ is a collection $m_C : \alpha_C \longrightarrow \beta_C$ of 2-cells of $\mathscr{D}$ indexed by the objects of $\mathscr{C}$, such that for all 1-cell $f : C \longrightarrow C'$ of $\mathscr{C}$, $m_{C'}F(f) = G(f)m_C$.

$$\begin{tikzcd}
F(C) \arrow[r, bend left, "\alpha_C"] \arrow[r, bend right, "\beta_C"'] \arrow[r, phantom, "\Downarrow m_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\
F(C) \arrow[r, bend left, "\alpha_{C'}"] \arrow[r, bend right, "\beta_{C'}"'] \arrow[r, phantom, "\Downarrow m_{C'}"] & G(C')
\end{tikzcd}$$

Definition 2.23. Let $\mathscr{A}$ and $\mathscr{B}$ be two 2-categories (with $\mathscr{A}$ small). The 2-category of 2-functors from $\mathscr{A}$ to $\mathscr{B}$ is written $[\mathscr{A}, \mathscr{B}]$. It is defined, for 2-functors $F, G : \mathscr{A} \to \mathscr{B}$, by letting $[\mathscr{A}, \mathscr{B}](F, G)$ be the category with 2-natural transformations as objects and modifications as morphisms. Remark that vertical composition of modifications is induced by vertical composition of 2-cells.

Remark 2.24. When $\mathcal{V}$ is a monoidal category, and $\mathscr{A}, \mathscr{B}$ are $\mathcal{V}$-categories, there is a general definition of a $\mathcal{V}$-category of $\mathcal{V}$-functors $[\mathscr{A}, \mathscr{B}]$, that one can find for instance in [27, Paragraph 2.2]. The previous definition 2.23 coincides with it when $\mathcal{V} = \mathbf{Cat}$. There are weaker notions of functors between 2-categories and natural transformations, given by replacing equalities by coherent families of 2-cells, as in John Gray's book [21].

Adjunctions and equivalences

Definition 2.25. Let $\mathscr{C}$ be a 2-category. An adjunction in $\mathscr{C}$ is a couple of 1-cells $l : B \longrightarrow A$, $r : A \longrightarrow B$, together with 2-cells $\eta : \mathrm{id}_B \Rightarrow rl$, $\epsilon : lr \Rightarrow \mathrm{id}_A$ satisfying the triangle equalities

(i) $(\epsilon 1_l) \circ (1_l \eta) = 1_l$;

(ii) $(1_r \epsilon) \circ (\eta 1_r) = 1_r$.

We say that l is *left adjoint* to r and r is *right adjoint* to l, which we denote by $l \dashv r$.

Definition 2.26. A 1-cell $f : X \longrightarrow Y$ in a 2-category $\mathscr{C}$ is a called an *equivalence* if and only if there exists a 1-cell $g : Y \longrightarrow X$ in $\mathscr{C}$ such that $fg \cong 1_Y$ and $gf \cong 1_X$.

As in the categorical context, one has the following propositions.

Proposition 2.27. *Let $\mathscr{C}$ be a 2-category and suppose that $f : X \longrightarrow Y$ is an equivalence in $\mathscr{C}$, and let $g : Y \longrightarrow X$ be such that $fg \cong \mathrm{id}_Y$ and $gf \cong \mathrm{id}_X$. Then f is left adjoint to g.*

Since equivalence is a self dual notion, one gets that also $g \dashv f$. As in the categorical context, one can show that right adjoints to a given 1-cell are unique up to isomorphism if they exists, and that adjunctions compose, that is if $f \dashv g$ and $f' \dashv g'$ and f and f' are composable, then $ff' \dashv g'g$. For a proof, the reader can see [21, Proposition I,6.3]. One has the following proposition.

Proposition 2.28. *A 2-functor sends an adjunction to an adjunction and an equivalence to an equivalence.*

Equivalences can be detected representably, as the following proposition states.

Proposition 2.29. *A 1-cell $f : X \longrightarrow Y$ in a 2-category $\mathscr{C}$ is an equivalence if and only if it induces for all $C \in |\mathscr{C}|$ an equivalence of hom-categories*

$$\mathscr{C}(C, f) : \mathscr{C}(C, X) \longrightarrow \mathscr{C}(C, Y).$$

Proof. By Yoneda's lemma. □

Formal Eilenberg-Moore 2-Adjunction and the Eilenberg-Moore objects of algebras

In this section, we review the classical correspondence between monads and adjunctions (see Mac Lane [37]), but in the general context of monads in a 2-category $\mathscr{C}$ that is sufficiently complete, as part of a 2-adjunction situation. We start by reviewing the classical 2-category of monads in $\mathscr{C}$ as defined by Street in [50].

Definition 2.30. A *monad* over an object B in a 2-category $\mathscr{C}$, is a quadruple $\mathcal{T} = (B, t, \mu, \eta)$, where $B \in |\mathscr{C}|$, $t : B \longrightarrow B$ is a 1-cell, and $\mu : t^2 \longrightarrow t$, $\eta : \mathrm{id}_B \longrightarrow t$ are 2-cells satisfying the following properties:

- $\mu \circ t\mu = \mu \circ \mu t$,
- $\mu \circ t\eta = 1_t = \mu \circ \eta t$.

Street first defined *monad functors* in [50] and later used the terminology *monad morphisms* with Lack in [31]. Some authors call them *lax monad morphisms.* In this thesis, all monad morphism will be lax unless otherwise mentioned, thus we will drop the adjective lax, and add the adjective strict or oplax instead when necessary.

Definition 2.31. Let $\mathcal{T} = (B, t, \mu, \eta)$ and $\mathcal{S} = (C, s, \rho, \iota)$ be two monads in a 2-category $\mathscr{C}$. A *monad morphism* $\mathcal{T} \longrightarrow \mathcal{S}$ is a couple (f, ϕ), where

- $f : B \longrightarrow C$ is a 1-cell of $\mathscr{C}$;
- $\phi : sf \longrightarrow ft$ is a 2-cell of $\mathscr{C}$;

such that the following diagrams are commutative.

$$\begin{tikzcd} s^2 f \arrow[r, "s\phi"] \arrow[d, "\rho f"] & sft \arrow[r, "\phi t"] & ft^2 \arrow[d, "f\mu"] \\ sf \arrow[rr, "\phi"] & & ft \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, "\iota f"] \arrow[dr, "f\eta"'] & sf \arrow[d, "\phi"] \\ & ft \end{tikzcd}$$

A monad morphism (f, ϕ) is said to be *strict* when its 2-cell ϕ is an identity.

Definition 2.32. Let $\begin{tikzcd} (B, t, \mu, \eta) \arrow[r, bend left, "(f,\phi)"] \arrow[r, bend right, "(g,\gamma)"'] & (C, s, \rho, \iota) \end{tikzcd}$ be a diagram of monads in $\mathscr{C}$. A *monad transformation* $(f, \phi) \longrightarrow (g, \gamma)$ is a 2-cell $\alpha : f \to g$ such that the following diagram is commutative.

$$\begin{tikzcd} sf \arrow[r, "s\alpha"] \arrow[d, "\phi"] & sg \arrow[d, "\gamma"] \\ ft \arrow[r, "\alpha t"] & gt \end{tikzcd}$$

In the 2-category $\mathbf{Mnd}(\mathscr{C})$ constructed by Street in [50], the objects, 1-cells and 2-cells are respectively monads, monad morphisms, and monad transformations in $\mathscr{C}$.

Let $\mathcal{T} = (B, t, \mu, \eta)$ be a monad in a 2-category $\mathscr{C}$. We define now the Eilenberg-Moore object of algebras of the monad $\mathcal{T}$, a construction similar to that in category theory (See [7, Chapter 4]). For any $X \in |\mathscr{C}|$, since $\mathscr{C}(X, -)$ is a 2-functor,

$$\mathcal{T}_*(X) = (\mathscr{C}(X, B), \mathscr{C}(X, t), \mathscr{C}(X, \mu), \mathscr{C}(X, \eta))$$

is a monad in **Cat**. Moreover, any 1-cell $f : X' \longrightarrow X$ induces a strict monad morphism $\mathscr{C}(f, B) : \mathcal{T}_*(X) \longrightarrow \mathcal{T}_*(X')$ and thus a functor

$$\mathscr{C}(X, B)^{\mathcal{T}_*(X)} \longrightarrow \mathscr{C}(X', B)^{\mathcal{T}_*(X')}$$

between the categories of algebras. One can also check that any 2-cell induces a natural transformation between these functors. This defines a 2-functor

$$\mathscr{C}(-, B)^{T_*(-)} : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}. \tag{2.3}$$

Definition 2.33. Let $\mathcal{T} = (B, T, \mu, \eta)$ be a monad in a 2-category $\mathscr{C}$. The *Eilenberg-Moore object of algebras* of $\mathcal{T}$, if it exists, is a representing object $\mathrm{Alg}(\mathcal{T})$ of the 2-functor (2.3). A 2-category $\mathscr{C}$ *admits the construction of algebras* if the Eilenberg-Moore object of algebras of any monad in $\mathscr{C}$ exists.

In part 2.3.4, it becomes obvious that a 2-category $\mathscr{C}$ admits the construction of algebras as long as $\mathscr{C}$ is sufficiently complete. In particular, 2-**Cat** admits the construction of algebras, and the Eilenberg-Moore object of algebras is of course the usual Eilenberg-Moore category of algebras.

We follow now [9] and [32], and define a 2-category of adjunctions in $\mathscr{C}$. The definitions are motivated by Proposition 2.36.

Definition 2.34. An *adjunction morphism* from an adjunction $\begin{tikzcd} B \arrow[r, shift left, "l"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "r"] \end{tikzcd}$ to an adjunction $\begin{tikzcd} B' \arrow[r, shift left, "l'"] \arrow[r, phantom, "\perp"] & A' \arrow[l, shift left, "r'"] \end{tikzcd}$ in a 2-category $\mathscr{C}$ is a pair of 1-cells $b : B \longrightarrow B'$ and $a : A \longrightarrow A'$ such that $br = r'a$.

Definition 2.35. Let $\begin{tikzcd} B \arrow[r, shift left, "l"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "r"] \end{tikzcd}$ and $\begin{tikzcd} B' \arrow[r, shift left, "l'"] \arrow[r, phantom, "\perp"] & A' \arrow[l, shift left, "r'"] \end{tikzcd}$ be two adjunctions in a 2-category $\mathscr{C}$ and $(b, a), (b', a') : l \dashv r \to l' \dashv r'$ two adjunction morphisms. An *adjunction transformation* $(b, a) \Rightarrow (b', a')$ is a pair (β, α) where $\beta : b \Rightarrow b'$ and $\alpha : b \Rightarrow b'$ are two-cells of $\mathscr{C}$ such that $\beta \cdot r = r' \cdot \alpha$.

Let $\mathbf{Adj}_r(\mathscr{C})$ be the 2-category whose objects are adjunctions in $\mathscr{C}$, whose 1-cells are adjunction morphisms and whose 2-cells are adjunction transformations. As in category theory, adjunctions, adjunction morphisms and adjunction transformations in $\mathscr{C}$ induce respectively monads, monad morphisms and monad transformations. This constitutes a 2-functor $M : \mathbf{Adj}_r(\mathscr{C}) \to \mathbf{Mnd}(\mathscr{C})$ which is natural in $\mathscr{C}$. When $\mathscr{C}$ admits the construction of algebras, this 2-functor has a right adjoint, as the next proposition states. The proof of the next proposition can be found in [32, Section 3] or in [9].

Proposition 2.36. *Suppose $\mathscr{C}$ admits the construction of algebras. There is an adjunction in* 2-**Cat**

$$\begin{tikzcd} \mathbf{Adj}_r(\mathscr{C}) \arrow[r, shift left, "M"] \arrow[r, phantom, "\perp"] & \mathbf{Mnd}(\mathscr{C}) \arrow[l, shift left, "\mathrm{Alg}(-)"] \end{tikzcd}$$

whose counit is an identity.

2.1.3 Weighted limits and enriched right Kan extensions

In this section, we fix a symmetric closed monoidal category $(\mathcal{V}, \otimes, I)$. We recall the concepts of weighted limit and enriched right Kan extension in this general context. We are interested mainly in the case where $\mathcal{V}$ is the cartesian closed category **sSet** or **Cat**. The underlying category functor $(-)_0 : \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Cat}$, which sends a $\mathcal{V}$ category $\mathscr{C}$ to the category $\mathscr{C}_0$ with same objects and morphisms given by $\mathscr{C}_0(A, B) = \mathcal{V}(I, \mathscr{C}(A, B))$, is not conservative in either case. The reader can find proofs of results stated in this section in [27].

Weighted limits

Definition 2.37. Let $\Gamma : \mathscr{D} \longrightarrow \mathcal{V}$ and $F : \mathscr{D} \longrightarrow \mathscr{C}$ be two $\mathcal{V}$-functors, with $\mathscr{D}$ a small $\mathcal{V}$-category. The *weighted limit of F indexed by* Γ is a contravariant representation $(\{\Gamma, F\}, \alpha)$ of the $\mathcal{V}$-functor $[\mathscr{D}, \mathcal{V}](\Gamma, \mathscr{C}(-, F-)) : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathcal{V}$. This means that $\{\Gamma, F\} \in |\mathscr{C}|$ and α is a $\mathcal{V}$-natural isomorphism

$$\mathscr{C}(-, \{\Gamma, F\}) \cong [\mathscr{D}, \mathcal{V}](\Gamma, \mathscr{C}(-, F-)). \tag{2.4}$$

The functor Γ is said to be the *weight*.

Remark 2.38. (i) By the weak Yoneda lemma [27, Paragraph 1.9], $\{\Gamma, F\}$ is unique, up to an isomorphism in the underlying category. Moreover, α corresponds to a unique morphism

$$I \longrightarrow [\mathscr{D}, \mathcal{V}](\Gamma, \mathscr{C}(\{\Gamma, F\}, F-)),$$

which means a $\mathcal{V}$-natural transformation, called the counit of the weighted limit, of the form

$$e : \Gamma \longrightarrow \mathscr{C}(\{\Gamma, F\}, F-).$$

The $\mathcal{V}$-natural isomorphism (2.4) induces a bijection (by applying the underlying category functor)

$$\mathscr{C}_0(-, \{\Gamma, F\}) \cong \mathcal{V}\text{-Nat}(\Gamma, \mathscr{C}(-, F-)) \tag{2.5}$$

This bijection (2.5) is equivalent to the fact that $\{\Gamma, F\}$ satisfies a universal property (which already determines $\{\Gamma, F\}$ up to an isomorphism in the underlying category).

The universal property tells that for every $X \in |\mathscr{C}|$ and every $\mathcal{V}$-natural transformation $\phi : \Gamma \longrightarrow \mathscr{C}(X, F-)$, there exists a unique map $\bar{\phi} : X \longrightarrow$

$\{\Gamma, F\}$ such that the following diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "e"] \arrow[dr, "\phi"'] & \mathscr{C}(\{\Gamma, F\}, F-) \arrow[d, "\mathscr{C}(\bar{\phi}{,}F-)"] \\
& \mathscr{C}(X, F-)
\end{tikzcd}$$

is commutative.

(ii) One should note that the universal property is enough to recognize the weighted limit, when it is known to exist, but since the underlying category functor need not be conservative, there is in general no reason for (2.4) to be satisfied when one only has (2.5). We give a counter-example when $\mathcal{V} = \mathbf{Cat}$. If $\mathscr{D}$ is the empty 2-category, then T verifies (2.5) if it is terminal in the underlying category, while T is the weighted limit (or the weighted terminal object) if (2.4) holds. This is the case if $\mathscr{C}(X, T)$ is terminal in **Cat** for all $X \in |\mathscr{C}|$. So let $\mathscr{C}$ be the 2-category with one object A, one 1-cell $\mathrm{id}_A : A \longrightarrow A$ and $\mathbb{N}$ as monoid of 2-cells over id_A, with vertical and horizontal composition given by sum. Then A verifies (2.5) but not (2.4).

(iii) When $\mathcal{V} = \mathbf{Set}$ and Γ is the terminal functor, then natural transformations $\phi : \Gamma \longrightarrow \mathscr{C}(X, F-)$ are in bijective correspondence with cones $\Delta_X \Rightarrow F$, and the universal property of the previous point reduces to the universal property of the limit. Moreover, since the underlying category functor is the identity in the case $\mathcal{V} = \mathbf{Set}$, this universal property is enough to characterize $\{\Gamma, F\}$. When $\mathcal{V} = \mathbf{Set}$, every weighted limit can be expressed in terms of usual conical limits, see [27, Paragraph 3.4].

(iv) Remark also that the strong Yoneda lemma [27, Paragraph 2.4] implies that the weighted limit $\{\mathscr{D}(D, -), F\}$ is given by $F(D)$ with counit given by

$$F : \mathscr{D}(D, -) \longrightarrow \mathscr{C}(F(D), F-).$$

Cotensor products

Let $\mathscr{C}$ be a $\mathcal{V}$-category, $X \in \mathcal{V}$ and $C \in |\mathscr{C}|$. Denote by $\mathscr{I}$ the $\mathcal{V}$-category with one object which has the unit I as endomorphism object. The *cotensor product* C^X, if it exists, is defined to be the weighted limit of the $\mathcal{V}$-functor $C : \mathscr{I} \longrightarrow \mathscr{C}$ which picks C, weighted by the $\mathcal{V}$-functor $X : \mathscr{I} \longrightarrow \mathcal{V}$ which picks X. More explicitly, C^X is defined by the existence of a $\mathcal{V}$-natural isomorphism

$$\mathscr{C}\left(-, C^X\right) \cong \mathcal{V}(X, \mathscr{C}(-, C)).$$

If this weighted limit exists for all X and for all C, we get a $\mathcal{V}$-functor

$$(-)^{-} : \mathcal{V}^{\text{op}} \times \mathscr{C} \longrightarrow \mathscr{C},$$

and we say that $\mathscr{C}$ is *cotensored.*

Proposition 2.39. *Let $\mathscr{D}$ a small $\mathcal{V}$-category and $\Gamma : \mathscr{D} \longrightarrow \mathcal{V}$ a weight. If $\mathscr{C}$ is a cotensored $\mathcal{V}$-category which admits all (weighted) conical limits and $F : \mathscr{D} \longrightarrow \mathscr{C}$ is any $\mathcal{V}$-functor, $\{\Gamma, F\}$ exists.*

Proof. See [27, Theorem 3.73]. □

Proposition 2.40. *Let $\mathscr{D}$ be a small $\mathcal{V}$-category, $\Gamma : \mathscr{D} \longrightarrow \mathcal{V}$ and $F : \mathscr{D} \longrightarrow \mathcal{V}$ be two $\mathcal{V}$-functors with $\mathscr{D}$ small. Then $\{\Gamma, F\} = [\mathscr{D}, \mathcal{V}](\Gamma, F)$.*

Enriched right Kan extensions

In this subsection, we present enriched right Kan extensions. We follow the terminology of Kelly [27]. Some authors call them *pointwise* Kan extensions. Let $\Gamma : \mathscr{A} \to \mathcal{V}$ be a weight. If $\beta : F \longrightarrow F'$ is a $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors from $\mathscr{A}$ to $\mathscr{C}$, then it induces a $\mathcal{V}$-natural transformation

$$\bar{\beta} : [\mathscr{A}, \mathcal{V}](\Gamma, \mathscr{C}(-, F-)) \longrightarrow [\mathscr{A}, \mathcal{V}](\Gamma, \mathscr{C}(-, F'-))$$

and by consequence, if $\{\Gamma, F\}$ and $\{\Gamma, F'\}$ exist, there is a $\mathcal{V}$-natural transformation

$$\mathscr{C}(-, \{\Gamma, F\}) \longrightarrow \mathscr{C}(-, \{\Gamma, F'\}).$$

which, by the (contravariant) weak Yoneda lemma, corresponds to a unique morphism in the underlying category of $\mathscr{C}$

$$\{1, \beta\} : \{\Gamma, F\} \longrightarrow \{\Gamma, F'\}.$$

Similarly, given a (not necessarily commutative) diagram of $\mathcal{V}$-functors

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "j"'] & \mathscr{C} \\
\mathscr{B} \arrow[ur, "T"'] &
\end{tikzcd}$$

together with a $\mathcal{V}$-natural transformation $\alpha : Tj \longrightarrow F$, one gets for any weight $\Gamma : \mathscr{B} \longrightarrow \mathcal{V}$, an induced $\mathcal{V}$-natural transformation

$$[\mathscr{B}, \mathcal{V}](\Gamma, \mathscr{C}(-, T-)) \xrightarrow{j^*} [\mathscr{A}, \mathcal{V}](\Gamma j, \mathscr{C}(-, Tj-)) \xrightarrow{\bar{\alpha}} [\mathscr{A}, \mathcal{V}](\Gamma j, \mathscr{C}(-, F-)) \,.$$

By the Yoneda lemma, this corresponds to a unique morphism in the underlying category of $\mathscr{C}$ $(j, \alpha)_* : \{\Gamma, T\} \longrightarrow \{\Gamma j, F\}$.

Definition 2.41. Let $j : \mathscr{A} \longrightarrow \mathscr{B}$ be a $\mathcal{V}$-functor and $F : \mathscr{A} \longrightarrow \mathscr{C}$ be another $\mathcal{V}$-functor. The *right Kan extension of F along j* is a $\mathcal{V}$-functor $\mathrm{Ran}_j F : \mathscr{B} \longrightarrow \mathscr{C}$ together with a $\mathcal{V}$-natural transformation $\phi : (\mathrm{Ran}_j F)j \longrightarrow F$ such that for every weight $\Gamma : \mathscr{B} \longrightarrow \mathcal{V}$, the morphism $(j, \phi)_* : \{\Gamma, \mathrm{Ran}_j F\} \longrightarrow \{\Gamma j, F\}$ is an isomorphism, either limit existing when the other does.

Proposition 2.42. *Let*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "j"'] & \mathscr{C} \\
\mathscr{B} \arrow[ur, "T"'] &
\end{tikzcd}$$

be a diagram of $\mathcal{V}$-functors and a $\mathcal{V}$-natural transformation $\phi : Tj \longrightarrow F$. Then, then following are equivalent:

(i) *(T, ϕ) is the enriched right Kan extension of F along j;*

(ii) *For all $B \in \mathscr{B}$, the weighted limit $\{\mathscr{B}(B, j(-)), F\}$ is given by $T(B)$ with counit given by*

$$\mathscr{B}(B, j-) \xrightarrow{T} \mathscr{C}(TB, Tj-) \xrightarrow{\phi_*} \mathscr{C}(TB, F-)\ .$$

Proof. See [27, Theorem 4.6]. □

Proposition 2.43. *Let*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "j"'] & \mathscr{C} \\
\mathscr{B} \arrow[ur, "T"'] &
\end{tikzcd}$$

be a diagram of $\mathcal{V}$-functors and $\phi : Tj \longrightarrow F$ be a $\mathcal{V}$-natural transformation. If j is full and faithful and (T, ϕ) is the right Kan extension of F along j, then ϕ is an isomorphism.

Proof. See [27, Proposition 4.23]. □

It is not clear from this definition that this is a generalization of the concept of right Kan extension. However, Proposition 2.45 below shows that an enriched right Kan extension satisfies a universal property similar to that of a classical right Kan extension, although this universal property is weaker than the definition.

Proposition 2.44. *Let* $(\mathrm{Ran}_j F, \phi)$ *be the enriched right Kan extension of* F *along* j *in the diagram*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "j"'] & \mathscr{C} \\
\mathscr{B} \arrow[ur, "\mathrm{Ran}_j F"'] & .
\end{tikzcd}$$

The $\mathcal{V}$*-functor* $[\mathscr{B},\mathscr{C}]^{\mathrm{op}} \xrightarrow{j^*} [\mathscr{A},\mathscr{C}]^{\mathrm{op}} \xrightarrow{[\mathscr{A},\mathscr{C}](-,F)} \mathcal{V}$ *is representable. More precisely, for* $S : \mathscr{B} \longrightarrow \mathscr{C}$*, there is a* $\mathcal{V}$*-natural isomorphism*

$$[\mathscr{A},\mathscr{C}](Sj,F) \cong [\mathscr{B},\mathscr{C}](S,\mathrm{Ran}_j F),$$

the unit of this representation being $\phi : \mathrm{Ran}_j(Fj) \longrightarrow F$.

Proposition 2.45. *Let* $(\mathrm{Ran}_j F, \phi)$ *be the enriched right Kan extension of* F *along* j *in the diagram*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, "j"'] & \mathscr{C} \\
\mathscr{B} \arrow[ur, "\mathrm{Ran}_j F"'] &
\end{tikzcd}$$

and $S : \mathscr{B} \longrightarrow \mathscr{C}$ *a* $\mathcal{V}$*-functor together with a* $\mathcal{V}$*-natural transformation* $\alpha : Sj \longrightarrow F$*. Then there exists a unique* $\bar{\alpha} : S \longrightarrow \mathrm{Ran}_j F$ *such that* $\alpha = \phi \circ (\bar{\alpha} j)$.

Moreover, since this implies $[\mathscr{B},\mathscr{C}]_0(S,\mathrm{Ran}_j F) \cong [\mathscr{A},\mathscr{C}]_0(Sj,F)$, this property determines $\mathrm{Ran}_j F$ up to a natural isomorphism. However, this property is weaker than the one of 2.44 and thus of the defining one. This means that in some cases there is no enriched right Kan extension in the strong sense (pointwise) and, there is a functor and a $\mathcal{V}$-natural transformation verifying the previous universal properties (unenriched or enriched). But if one knows that the enriched right Kan extension exists, then it can be recognized as the unique one fulfilling this universal property.

2.2 Higher categories

2.2.1 Quasi-categories

In this thesis, we recall only the constructions and concepts which are strictly needed for our purposes. The reader who wants to read a more complete introduction to quasi-categories can consult [25], [43] or [35].

We denote by $\Delta[n]$ the standard n-simplex, by $\partial\Delta[n]$ its boundary and by $\Lambda^k[n]$ its k-th horn.

Definition 2.46. A *quasi-category* is a simplicial set which satisfies the right lifting property with respect to the set of inner horns $\{\Lambda^k[n] \to \Delta[n] : 0 < k < n\}$. The full simplicial category of simplicial sets spanned by quasi-categories is denoted $\mathbf{qCat}_\infty$.

Proposition 2.47. *The nerve functor* $N : \mathbf{Cat} \to \mathbf{sSet}$ *has a left adjoint* $h : \mathbf{sSet} \to \mathbf{Cat}$*, called the* homotopy category functor*. Moreover, it exhibits* $\mathbf{Cat}$ *as a reflexive full subcategory of* $\mathbf{sSet}$*.*

As a consequence, we often write $\mathscr{C}$ for both a category and its nerve, since categories are just special simplicial sets, namely those satisfying the lifting property of Definition 2.46 with uniqueness. When X is a quasi-category, $h(X)$ has a simple description.

Proposition 2.48. *Let* X *be a quasi-category. Its homotopy category* $h(X)$ *is naturally isomorphic to the category* $\tilde{h}(X)$ *given by* $|\tilde{h}(X)| = X_0$ *and*

$$\tilde{h}(X)(x, y) = \{f \in X_1 : d_0 f = y, d_1 f = x\}/_\sim$$

where $\sim$ *is given by* $f \sim g$ *if and only if there exists* $H \in X_2$ *with* $d_0 H = f$*,* $d_1 H = g$ *and* $d_2 H = s_0 d_1 f$*.*

The units are $s_0 x$*,* $x \in X_0$ *while composition is given by* $[f_2][f_1] = [f_3]$ *if there exists* $H \in X_2$ *such that* $d_0 H = f_2, d_1 H = f_3$ *and* $d_2 H = f_1$*.*

Proof. See [25, Proposition 1.11]. ☐

Lemma 2.49. *The homotopy category functor* $h : \mathbf{sSet} \longrightarrow \mathbf{Cat}$ *preserve products.*

Proof. See [25, Proposition 1.3]. ☐

Remark 2.50. This lemma implies that there is an induced adjunction

$$\begin{tikzcd} \mathbf{sCat} \arrow[r, shift left, "h_*"] \arrow[r, phantom, "\perp"] & 2\text{-}\mathbf{Cat} \arrow[l, shift left, "N_*"] \end{tikzcd}$$

where $h_*(\mathcal{K})$ and $N_*(\mathscr{C})$ are given by applying h and N locally, that is objects are unchanged and

$$h_*(\mathcal{K})(X, Y) = h(\mathcal{K}(X, Y))$$
$$N_*(\mathscr{C})(C, D) = N(\mathscr{C}(C, D)).$$

Moreover, it exhibits 2-$\mathbf{Cat}$ as a reflexive full subcategory of $\mathbf{sCat}$. We thus often write $\mathscr{C}$ to denote a 2-category but also to denote $N_*(\mathscr{C})$, according to the context. Indeed, 2-categories are just special simplicial categories, namely those whose homspaces satisfy the lifting property of Definition 2.46 with uniqueness.

Definition 2.51. Let A be a quasi-category and $f \in A_1$. We say that f is an *isomorphism* if $[f]$ is an isomorphism in $h(A)$.

We let $\mathbb{I}$ be the category with two objects, $-, +$ and two non-identity morphisms $e_1^- : - \to +$ and $e_1^+ : + \to -$, with non-trivial compositions given by $e_1^+ e_1^- = \mathrm{id}_-$ and $e_1^- e_1^+ = \mathrm{id}_+$. The following proposition is due to Joyal.

Proposition 2.52. *Let A be a quasi-category. The following propositions are equivalent.*

- *The 1-simplex $f \in A_1$ is an isomorphism;*
- *There is a simplicial map $j : \mathbb{I} \to A$ such that $j(e_1^-) = f$.*

Joyal also introduced a model category whose fibrant objects are the quasi-categories. We recall it in the next proposition.

Proposition 2.53. *The Joyal model category* $\mathbf{sSet}_J$ *is the model category on simplicial sets uniquely determined by the following:*

- *The weak equivalences are the simplicial set maps $w : X \longrightarrow Y$ such that $h(A^w) : h(A^Y) \longrightarrow h(A^X)$ are equivalences of categories for all quasi-categories A;*
- *The cofibrations are the injective simplicial maps;*
- *The fibrant objects are the quasi-categories;*
- *The fibrations between fibrant objects, called* isofibrations*, are the morphisms $X \longrightarrow Y$ with the right lifting property relative to the inner horns inclusions $\Lambda^k[n] \longrightarrow \Delta[n] (n = 2, \dots, 0 < k < n)$, and to the monomorphisms $\Delta^0 \to \mathbb{I}$.*

Moreover, this model structure is combinatorial and with the cartesian closed structure of $\mathbf{sSet}$*, is a monoidal model category. The fact that this model category is monoidal means that in the situation where $i : X \to Y$ is a cofibration and $p : E \to B$ is a fibration, the map $\widehat{p^i}$ displayed in the following diagram*

$$\begin{tikzcd}
E^Y \arrow[dr, dashed, "\widehat{p^i}"] \arrow[drr, bend left, "p^Y"] \arrow[ddr, bend right, "E^i"'] & & \\
& E^X \times_{B^X} B^Y \arrow[r] \arrow[d] & B^Y \arrow[d, "B^i"] \\
& E^X \arrow[r, "p^X"] & B^X
\end{tikzcd}$$

is a fibration which is trivial if either i or p is. We write $\mathbf{qCat}_\infty$ *to denote the full simplicial category generated by quasi-categories.*

As a consequence of being a monoidal model category, we get the following result.

Lemma 2.54. *Let X be a quasi-category. Then,*

- *X^A is a quasi-category for any simplicial set A.*
- *If $A \longrightarrow B$ is a cofibration, the pre-composition morphism $X^A \longrightarrow X^B$ is an isofibration.*

Observe that by Proposition 2.29, the Joyal equivalences are exactly the equivalences in $\mathbf{qCat}_2 := h_*(\mathbf{qCat}_\infty)$. In particular, equivalences of categories are Joyal equivalences.

Definition 2.55. A simplicial set map $f : X \to Y$ between quasi-categories is said to be *conservative* if and only if $hf : hX \to hY$ is conservative, i.e., it reflects isomorphisms.

Proposition 2.56. *Let $f : X \to Y$ be a simplicial set map between quasi-categories. The following are equivalent:*

(i) *f is conservative;*

(ii) *$f_* : \mathbf{qCat}_\infty(W, X) \to \mathbf{qCat}_\infty(W, Y)$ is conservative for all quasi-category W.*

Proof. Observe that (ii) implies (i) by taking $W = *$. For the converse statement, it is enough to remark that by [43, Lemma 2.3.10], isomorphisms in $\mathbf{qCat}_\infty(W, X)$ are exactly pointwise isomorphisms. □

Dual of a simplicial set

Given a totally ordered set, one can always formally reverse the order. Thus, there is an involution $\mathrm{op} : \Delta_+ \to \Delta_+$, described as follows:

- for all $n \in \mathbb{N}$, $\mathbf{n}^{\mathrm{op}} = \mathbf{n}$.
- for all $f : \mathbf{n} \to \mathbf{m}$, and $x \in \mathbf{n}$ $f^{\mathrm{op}}(x) = m - 1 - f(n - 1 - x)$.

Remark that one has the following identities

- $(d^i : \mathbf{n} \to \mathbf{n+1})^{\mathrm{op}} = d^{n-i}$.
- $(s^i : \mathbf{n+1} \to \mathbf{n})^{\mathrm{op}} = s^{n-1-i}$.

As a consequence, op induces an involution $\mathrm{op} : \mathbf{sSet} \to \mathbf{sSet}$ which preserves both the Quillen and Joyal model structures on simplicial sets.

Join and quasi-categories of cones

Recall that the monoidal structure of Δ_+ given by ordinal sum induces by Day's convolution a monoidal structure $\star$ on augmented simplicial sets, such that the Yoneda embedding is strongly monoidal and that $\star : \mathbf{Set}^{\Delta_+^{\mathrm{op}}} \times \mathbf{Set}^{\Delta_+^{\mathrm{op}}} \to \mathbf{Set}^{\Delta_+^{\mathrm{op}}}$ is cocontinuous on each variable (see Day, [12]). We thus get an operation, called the *join* and described as follows.

Definition 2.57. The *join* of two augmented simplicial sets X, Y is given by

- $(X \star Y)_n = \coprod_{i_0+i_1+1=n} X_{i_0} \times Y_{i_1}$ for all $n \geq -1$,
- If $\phi : [m] \to [n] = [i_0] + [i_1]$ is a simplicial operator and $(x, y) \in X_{i_0} \times Y_{i_1}$, $\phi^*(x, y) = (\phi_0^*(x), \phi_1^*(y))$ where $\phi_\epsilon : [j_\epsilon] \to [i_\epsilon]$ is the restriction of ϕ to the preimage by ϕ of the image of $[i_\epsilon]$ in $[i_0] + [i_1]$.

Since the category of simplicial sets is isomorphic to the category of augmented simplicial sets with trivial augmentation, one can extend the operation to simplicial sets. Moreover, it is easy to see that X and Y actually sit inside $X \star Y$, since $X_{-1} = Y_{-1} = *$.

If $\mathscr{C}$ and $\mathscr{D}$ are two categories, their join is the category $\mathscr{C} \star \mathscr{D}$ given, for all $C \in |\mathscr{C}|$ and $D \in |\mathscr{D}|$ by

$$\begin{aligned} \mathscr{C} \star \mathscr{D}(C, C') &= \mathscr{C}(C, C') \\ \mathscr{C} \star \mathscr{D}(D, D') &= \mathscr{D}(C, C') \\ \mathscr{C} \star \mathscr{D}(C, D) &= * \\ \mathscr{C} \star \mathscr{D}(D, C) &= \emptyset. \end{aligned}$$

Remark 2.58. One can directly check from the formulas that the nerve functor preserves the join operation. More precisely, if $\mathscr{C}, \mathscr{D} \in |\mathbf{Cat}|$,

$$N(\mathscr{C}) \star N(\mathscr{D}) = N(\mathscr{C} \star \mathscr{D}).$$

As a consequence, $\Delta[n] \star \Delta[m] = \Delta[n + m + 1]$.

It is not true that $\star : \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$ is cocontinous in each variable because the inclusion $\mathbf{sSet} \to \mathbf{Set}^{\Delta_+^{\mathrm{op}}}$ is not. For instance, $(\Delta[0] \coprod \Delta[0]) \star \Delta[0]$ is $\Lambda^2[2]$ whereas $(\Delta[0] \star \Delta[0]) \coprod (\Delta[0] \star \Delta[0]) = \Delta[1] \coprod \Delta[1]$. Nevertheless, $- \star Y$ preserves colimits indexed over connected categories, since the inclusion $\mathbf{sSet} \to \mathbf{Set}^{\Delta_+^{\mathrm{op}}}$ does. This defect can be corrected by modifying the target category of $- \star Y$, as in Definition 2.59 below.

We introduce now the quasi-categories of cones.

Definition 2.59. For every simplicial set Y, the functor $-\star Y : \mathbf{sSet} \to Y \downarrow \mathbf{sSet}$ associating to a simplicial set X the inclusion $Y \to X \star Y$ has a right adjoint, sending a map $g : Y \to X$ to a simplicial set $X_{/g}$.

Remark that the counit ϵ^Y of this adjunction provides, for every $g : Y \to X$, a commutative diagram

$$\begin{tikzcd}
Y \arrow[d] \arrow[dr, "g"] & \\
X_{/g} \star Y \arrow[r, "\epsilon^Y_g"'] & X.
\end{tikzcd}$$

Therefore, there is a forgetful map $X_{/g} \to X$, given by precomposition of ϵ^Y_g with $X_{/g} \to X_{/g} \star Y$.

The join and slice construction are well behaved with respect to the Joyal model structure. We recall a result of Joyal, [25, Proposition 6.29].

Proposition 2.60. *The functor* $- \star Y : \mathbf{sSet}_J \to Y \downarrow \mathbf{sSet}_J$ *is a left Quillen functor.*

Definition 2.61. Given $k : Z \to X$, the simplicial set $k_{/g}$ is the pullback in the diagram

$$\begin{tikzcd}
k_{/g} \arrow[r] \arrow[d] & X_{/g} \arrow[d] . \\
Z \arrow[r, "k"] & X
\end{tikzcd}$$

2.2.2 Weak complicial sets

Complicial sets were introduced by Roberts in 1978 in [48], in order to characterize nerves of strict ω-categories, which he believed to be the right context in which to value non-abelian cohomology. In [52], Street actually constructed this ω-categorical nerve and formulated the Street-Roberts conjecture, which says that it provides an equivalence between strict ω-categories and complicial sets. This conjecture was proven by Verity in [57]. He notes that "its real importance lies in the fact that it acts as a prelude to the development of a simplicial rendition of the theory of *weak* ω-categories, known as weak complicial set theory." Weak complicial sets were first introduced by Street in [53]. Their basic homotopy theory was developed by Verity in [55]. Interestingly, one can consider n-trivial saturated weak complicial sets as a truncation in order to obtain a model of (∞, n)-categories. This path is suggested by Riehl in [40]. The cases $n = 1$ and $n = 2$ have been developed independently and similarly by Lurie in [35] and [33], respectively under the name *marked simplicial sets* and *scaled simplicial sets.* We only recall the most basic definitions and facts, and let the interested reader consult the references [57], [55] and [56].

Definition 2.62. A *stratified simplicial set* is a pair (X, tX) where $tX \subseteq \sqcup_{i \geq 1} X_i$ is a subset of its simplices of dimension at least 1, containing all degenerate simplices. The elements of tX are said to be *thin* . We usually abuse notation and write X for the stratified simplicial set (X, tX).

A stratified simplicial set map $f : X \to Y$ is a simplicial set map which preserves thin simplices, that is such that $f(tX) \subseteq tY$. The category of stratified simplicial set and maps is denoted **Strat**.

Examples 2.63. A simplicial set X can be given two natural stratifications. The minimal stratification of X declares only the degenerate simplices to be thin. The corresponding stratified simplicial set is denoted by X. The maximal stratification of X declares all simplices of dimension greater or equal to 1 to be thin. The corresponding stratified simplicial set is denoted by $X_\sharp$.

A quasi-category X can be given another natural stratification, by declaring thin all isomorphisms. The corresponding stratified simplicial set is denoted by $e(X)$.

If (X, tX) and (Y, tY) are stratified simplicial set, their product is given by

$$(X, tX) \circledast (Y, tY) = (X \times Y, \{(x, y) \in X_n \times Y_n : n \in \mathbb{N}, x \in tX, y \in tY\}).$$

This product is not denoted with × by Verity, because it should be understood as a generalization of the 2-categorical Gray tensor product (See [21, I.4]).

Definition 2.64. The *join* of two stratified simplicial set (X, tX), (Y, tY) is

$$(X, tX) \star (Y, tY) = (X \star Y, t(X \star Y))$$

where a simplex $(x, y) \in (X \star Y)_n$ is thin if and only if x is thin in X or y is thin in Y.

Definition 2.65. For $0 \leq k \leq n$,

- the k-complicial n-simplex $\Delta^k[n]$ is the stratified simplicial set

  $$(\Delta[n], \{\phi : [m] \to [n] \in \Delta[n]_m : m \geq 1 \text{ and } \{k-1, k, k+1\} \cap [n] \subseteq \operatorname{im} \phi\});$$

- $\Delta^k[n]'$ is the stratified simplicial set whose underlying simplicial set is $\Delta[n]$ and where a simplex $\phi : [m] \to [n]$ is thin if and only if it is thin in $\Delta^k[n]$ or it is the $k-1$-th face $d^{k-1} : [n-1] \to [n]$ or the $k+1$-th face $d^{k+1} : [n-1] \to [n]$.

- $\Delta^k[n]''$ is the stratified simplicial set whose underlying simplicial set is $\Delta[n]$ and where a simplex $\phi : [m] \to [n]$ is thin if and only if it is thin in $\Delta^k[n]$ or it is a codimension-one face.
- the $(n-1)$-dimensional k-complicial horn $\Lambda^k[n]$ is the stratified simplicial set whose underlying simplicial set is $\Lambda^k[n]$ with the maximal stratification making the inclusion $\Lambda^k[n] \to \Delta^k[n]$ a stratified simplicial set map.

Definition 2.66. A *weak complicial set* is a stratified simplicial set which satisfies the right lifting property with respect to the complicial horn extensions

$$\Lambda^k[n] \to \Delta^k[n]$$

for $0 \leq k \leq n, n \geq 1$ and the complicial thinness extensions

$$\Delta^k[n]' \to \Delta^k[n]''$$

for $0 \leq k \leq n, n \geq 2$. The full subcategory of **Strat** generated by weak complicial set is denoted by **wcSet**.

The n-simplices of a weak complicial set should be thought of as n-dimensional arrows. The thin simplices play two distinct roles. First, they witness weak compositions of lower dimensional simplices, which are given by complicial horn extensions. Secondly, they are "equivalences" with respect to the weak composition. Complicial thinness extensions should be understood as a guarantee that "thin simplices compose".

Definition 2.67. A weak complicial set X is said to be *n-trivial* if $\sqcup_{i>n} X_i \subseteq tX$.

Saturation and equivalences

We follow [40, Section 3] in this part.

Definition 2.68. Let (X, tX) be a stratified simplicial set. A 1-simplex $f : \Delta[1] \to X$ is said to be an 1-equivalence if there exist thin simplices $H, K \in X_2$ with faces as depicted in the following diagrams.

$$\begin{tikzcd}
& y \arrow[dr, "g"] \arrow[d, phantom, "H"] & & & x \arrow[dr, "f"] \arrow[d, phantom, "K"] & \\
x \arrow[ur, "f"] \arrow[rr, "s_0 x"'] & & x & y \arrow[ur, "g"] \arrow[rr, "s_0 y"'] & & y
\end{tikzcd}$$

Observe that by the right lifting property with respect to the complicial (outer) horn extensions and complicial thinness extensions, thin 1-simplices of a weak complicial set are always 1-equivalences.

Definition 2.69. Let $\Delta[3]_{\simeq}$ be the stratified simplicial sets whose underlying simplicial set is $\Delta[3]$ and where the thin simplices are given by

$$t\Delta[3]_{\simeq} = \sqcup_{i\geq 2}\Delta[3]_i \sqcup \{(0,2),(1,3)\}.$$

Definition 2.70. A weak complicial set (X, tX) is said to be *saturated* if it satisfies the right lifting property with respect to

$$\Delta[n] \star \Delta[3]_{\simeq} \star \Delta[m] \to \Delta[n] \star \Delta[3]_{\sharp} \star \Delta[m], n, m \geq -1.$$

Observe that a weak complicial set X has the right lifting property with respect to $\Delta[3]_{\simeq} \to \Delta[3]_{\sharp}$ if and only if thin 1-simplices satisfy a 2-out-of-6 property. Remark that it implies that all 1-equivalences in X are thin. Moreover, 1-equivalences satisfy a 2-out-of-6 property, because they can be characterized as isomorphisms in a homotopy category construction. More generally, the intuitive idea behind saturation is that a weak complicial set is saturated if for all n, all n-equivalences are thin.

Following Riehl and Verity, n-trivial and saturated weak complicial sets should model (∞, n)-categories. In this thesis we are mostly interested by the case $n = 0, 1, 2$. Consider $k, m \geq -1$ and a diagram

$$\begin{tikzcd}
\Delta[k] \star \Delta[3]_{\simeq} \star \Delta[m] \arrow[r, "f"] \arrow[d] & X \\
\Delta[k] \star \Delta[3]_{\sharp} \star \Delta[m] &
\end{tikzcd}$$

where X is a 2-trivial weak complicial set. The only possible lift of f is by f itself, so we have to understand when f sends thin simplices of $\Delta[k] \star \Delta[3]_{\sharp} \star \Delta[m]$ to thin simplices of X. This is automatically the case for simplices which are also thin in $\Delta[k] \star \Delta[3]_{\simeq} \star \Delta[m]$.

Let $(x, y, z) \in \Delta[k]\star\Delta[3]\star\Delta[m]$ be a simplex which is thin in $\Delta[k]\star\Delta[3]_{\sharp}\star\Delta[m]$ and not thin in $\Delta[k]\star\Delta[3]_{\simeq}\star\Delta[m]$. This happens exactly when y is of degree 1, and x and z are not thin. Observe that since X is 2-trivial, we only have to deal with simplices of degree less or equal to 2. But $\text{degree}(x, y, z) = \text{degree}\, x + \text{degree}\, z + 3$. Thus, X is saturated if and only if it has the right lifting property with respect to

$$\Delta[3]_{\simeq} \to \Delta[3]_{\sharp},$$
$$\Delta[3]_{\simeq} \star \Delta[0] \to \Delta[3]_{\sharp} \star \Delta[0],$$
$$\Delta[0] \star \Delta[3]_{\simeq} \to \Delta[0] \star \Delta[3]_{\sharp}.$$

Model structures

We close this section by advertising a theorem of Verity [55, Theorem 100], which establish the homotopy theory of weak complicial sets of several flavors.

The category **Strat** is cartesian closed. This means that there exists a functor

$$\hom(-,-) : \mathbf{Strat}^{\mathrm{op}} \times \mathbf{Strat} \to \mathbf{Strat}$$

such that there are natural isomorphisms

$$\mathbf{Strat}(X \times Y, Z) \cong \mathbf{Strat}(X, \hom(Y, Z)) \cong \mathbf{Strat}(Y, \hom(X, Z)).$$

Definition 2.71. Let J be a set of stratified inclusions which contains the complicial horn extensions and the complicial thinness extensions. We say that a stratified simplicial set is *J-fibrant* if and only if it has the right lifting property with respect to all morphisms in J. A morphism $f : X \to Y$ is said to be a *J-weak equivalence* if and only if for all J-fibrant stratified simplicial sets W, $\hom(f, W) : \hom(Y, W) \to \hom(X, W)$ is a homotopy equivalence with respect to the interval $\Delta[1]_\sharp$.

Theorem 2.72. *Let J be a set of stratified inclusions which contains the complicial horn extensions and the complicial thinness extensions, and are J-weak equivalences. There is a cofibrantly generated Quillen model structure on the category* **Strat** *of stratified sets, called the J-complicial model structure, whose:*

- *weak equivalences are the J-weak equivalences of Definition 2.71*
- *cofibrations are simply inclusions of stratified simplicial sets, and whose*
- *fibrant objects are the J-fibrants stratified simplicial sets of Definition 2.71.*

Examples 2.73. In this thesis, we are mostly interested by the following model structures, which can be obtained by Verity's theorem (See [55], [40]).

- The model structure for weak complicial sets, obtained by applying Theorem 2.72 to the set containing exactly the complicial horn extensions and the complicial thinness extensions.
- Fix $n \geq 0$. Let J_n be the set containing complicial horn extensions, complicial thinness extensions, $\Delta[k] \star \Delta[3]_{\simeq} \star \Delta[m] \to \Delta[k] \star \Delta[3]_\sharp \star \Delta[m], m, k \geq 1$ and $\Delta[r] \to \Delta[r]_t$ for $r > n$, where $\Delta[r]_t$ is obtained from $\Delta[r]$ by making thin the unique non-degenerate simplex of dimension r. The model structure

obtained by Theorem 2.72 has n-trivial and saturated weak complicial sets as fibrant objects. This is our preferred model for (∞, n)-categories.

In this thesis, we are mostly concerned by the cases $n = 0, 1, 2$. As a motivation, Verity shows in [55] that setting $n = 0$ gives the Quillen model structure on simplicial sets, whereas setting $n = 1$ gives the Joyal model structure on simplicial sets!

2.3 The universal 2-category containing an adjunction

In this section, we review several models of **Adj**, a 2-category defined as follows.

Definition 2.74. The 2-category **Adj** is the 2-category determined up to isomorphism by the existence of a natural bijection

$$2\text{-}\mathbf{Cat}(\mathbf{Adj}, \mathscr{C}) \cong |\mathbf{Adj}_r(\mathscr{C})|$$

Its existence can be derived by constructing it by generators and relations, using presentations by computads as in [51]. A short introduction to computads is provided in Appendix B. However, the consequent literature on word problems for groups indicates that understanding if two words of generators actually represents the same element is difficult in the general case, and thus this construction is not suited for studying **Adj** combinatorially. We thus prefer concrete models, provided by Auderset, Schanuel and Street on the one hand, and Riehl and Verity on the other.

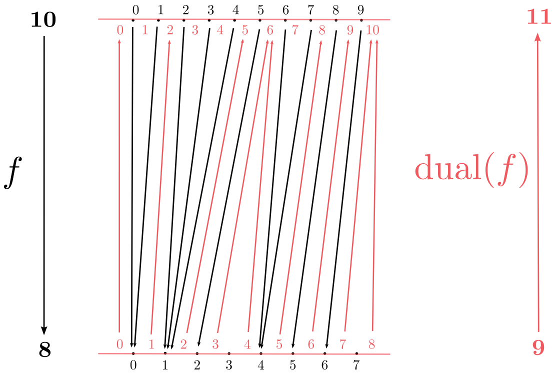
2.3.1 The 2-categorical model

In [3], Auderset defines **Adj** up to isomorphism by its universal property as above, computes half of the hom-categories, and shows that the other half are duals of the computed ones. The 2-category **Adj** has exactly two objects, A and B, and Auderset provided a quotient-free description of the hom-categories, which are

- $\mathbf{Adj}(B, B) = \Delta_+$, the category of possibly empty finite ordinals with non-decreasing maps;
- $\mathbf{Adj}(B, A) = \Delta_{-\infty}$, the category of non-empty finite ordinals with non-decreasing maps that preserve the minimal element;
- $\mathbf{Adj}(A, B) = \Delta_{-\infty}^{\mathrm{op}}$;
- $\mathbf{Adj}(A, A) = \Delta_+^{\mathrm{op}}$.

Unfortunately, the composition maps are not easy to express from this point of view, for instance $\Delta_{-\infty}^{\mathrm{op}} \times \Delta_+ \longrightarrow \Delta_{-\infty}^{\mathrm{op}}$ is not clear in Auderset's article.

There is a well-known isomorphism $\mathrm{dual} : \Delta_+^{\mathrm{op}} \longrightarrow \Delta_{-\infty,+\infty}$, (that one can call *Stone duality for intervals*, following for instance Andrade's thesis [1, page 206]) where the codomain is the category of non-empty finite ordinals with non-decreasing maps that preserve the minimal and maximal elements. Another name for this correspondence is *interval representation*, because one can have the following picture in mind.



We take the opportunity to give formulas for this correspondence. If $f : P \to Q$ is a monotone map between linearly ordered sets, its dual is given by

$$\begin{aligned} \mathrm{dual}\, f : Q \sqcup \{+\infty\} &\to P \sqcup \{+\infty\} \\ q &\mapsto \min\{p \in P : f(p) \geq q\}. \end{aligned}$$

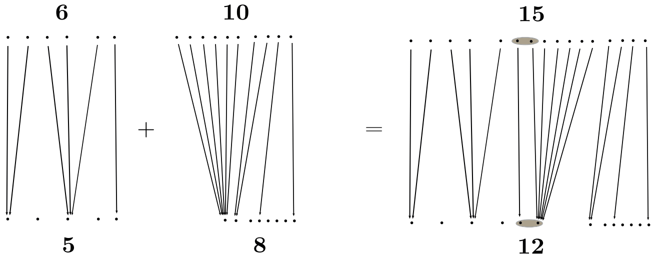
Its inverse associates to a monotone map $g : Q \to P$ preserving minimal and maximal elements the order-preserving map

$$\begin{aligned} \mathrm{dual}^{-1} g : P \setminus \max P &\to Q \setminus \max Q \\ p &\mapsto \max\{q \in Q : g(q) \leq p\}. \end{aligned}$$

In [49], Schanuel and Street use Stone duality for intervals and express **Adj** in the following way.

- $\mathbf{Adj}(B, B)$ and $\mathbf{Adj}(B, A)$ are as before;
- $\mathbf{Adj}(A, B) = \Delta_{+\infty}$, the category of non-empty finite ordinals with non-decreasing maps that preserve the maximal element;
- $\mathbf{Adj}(A, A) = \Delta_{-\infty,+\infty}$;

- the composition $\mathbf{Adj}(Y,Z) \times \mathbf{Adj}(X,Y) \longrightarrow \mathbf{Adj}(X,Z)$ is given by ordinal sum when $Y = B$. When $Y = A$, it is given by a quotient of the ordinal sum, where the minimal element of an object of $\mathbf{Adj}(X,Y)$ is identified with the maximal element of the object of $\mathbf{Adj}(Y,Z)$ in the ordinal sum. One can picture an example of such a composition in the following way,



where on the right the dark grey region is considered as a point.

The adjunction data is given as follows:

- the left adjoint is $\mathbf{1} \in \mathbf{Adj}(B,A)$ while the right adjoint is $\mathbf{1} \in \mathbf{Adj}(A,B)$;
- the unit of the adjunction is the unique map $\mathrm{id}_B = \mathbf{0} \to \mathbf{1}$;
- the counit of the adjunction is the unique map $\mathbf{2} \to \mathbf{1} = \mathrm{id}_A$.

2.3.2 The simplicial model

In [45], Riehl and Verity defined a simplicial category $\mathfrak{Adj}$ that we review now. We will use $\{A, B\}$ as object set, in order to avoid unnecessary confusion when we will generalize it in Chapter 3. Let $\mathcal{A}_n = \{A, 1, \dots, n, B\}$ be an alphabet and give it a total order by setting $A < i < B$ for all $i = 1, \dots, n$. Let W be the simplicial set given by

- $W_n = \mathcal{A}_n^*$, the set of finite words with letters in $\mathcal{A}_n$;
- Given a simplicial operator $\alpha : [n] \to [m]$, its Stone dual $\mathrm{dual}\,\alpha : [m+1] \to [n+1]$ preserves minimal and maximal elements. There is a unique order-preserving bijection $\mathcal{A}_n \cong [n+1]$ and thus $\mathrm{dual}\,\alpha$ can be interpreted as a morphism $\mathrm{dual}\,\alpha : \mathcal{A}_m \to \mathcal{A}_n$, which in turns induces $(\mathrm{dual}\,\alpha)_* : W_m \to W_n$.

Definition 2.75. An *undulating squiggle on* $n+1$ *lines* is a word $(v_0, \dots, v_m) \in W_n$ that starts and ends with A or B, and such that if $v_0 = A$, $v_0 \leq v_1 \geq v_2 \leq \dots v_m$ and if $v_0 = B$, $v_0 \geq v_1 \leq v_2 \geq \dots v_m$. Its *width* is $w(v_0, \dots, v_m) = m$. The *inner letters* of $(v_0, \dots, v_m)$ are the letters v_i for $0 < i < m$. We write S_n for the set of undulating squiggles on $n+1$ lines.

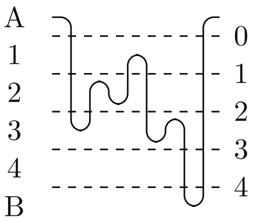


Figure 2.1: Undulating squiggle (A,3,2,2,1,3,3,B,A)

The collection S_n of these sets form a simplicial subset $S \subseteq W$. Indeed, simplicial operators are induced by maps of alphabets that are non decreasing. The information contained in an undulating squiggle can be represented by an actual squiggle, by using the letters of the word as turning points. For instance, the word $(A, 3, 2, 2, 1, 3, 3, B, A)$ can be represented as in Figure 2.1. We define an equivalence relation $\sim_n$ on S_n, as the equivalence relation generated by $(v_0, \ldots, v_m) \sim_n (v_0, \ldots, v_i, x, x, v_{i+1}, \ldots, v_m)$, and we define $S_n^< = S_n/_{\sim_n}$. Remark that the collection of $S_n^<$ inherits a simplicial set structure, because the structure maps factor through the quotient.

Definition 2.76. A *strictly undulating squiggle on* $n + 1$ *lines* is an undulating squiggle on $n + 1$ lines, satisfying the stronger condition obtained by replacing $\leq$ by $<$ and $\geq$ by $>$ in Definition 2.75.

Remark 2.77. Observe that there is a unique strictly undulating squiggle on $n + 1$ lines in each equivalence class for the relation $\sim_n$, since one can always form a strictly undulating squiggle out of an undulating squiggle, by locating successive letters that are equal and suppressing them. Remark that this process will end, since the words are finite, and will not change the first and the last letters, because A, B are the minimal and maximal elements, and does not depend on the order of suppressions. For instance, the strictly undulating squiggle associated to our example of undulating squiggle displayed in Figure 2.1 is $A31BA$, which is represented in Figure 2.2. One can thus identify $S_n^<$ with the set of strictly undulating squiggles on $n + 1$ lines, which is a subset of W_n. However, the collection $S_\bullet^<$ of these sets does not form a simplicial subset of $W_\bullet$, because if $\alpha : [n] \longrightarrow [m]$ is a simplicial operator, $v' = (\operatorname{dual}\alpha)_*(v)$ will only be undulating in general. We instead use the simplicial structure induced by the quotient construction above, which means that we also apply a reduction step to $(\operatorname{dual}\alpha)_*(v)$ to get a strictly undulating squiggle again. For instance, Figure 2.3 shows the process of applying the face map d_2 to $A423132BA$. We first suppress the line labeled 2 (we identify the rows labeled 2 and 3) and relabel the subsequent lines and rows, and then

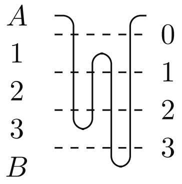


Figure 2.2: Strictly undulating squiggle (A,3,1,B,A)

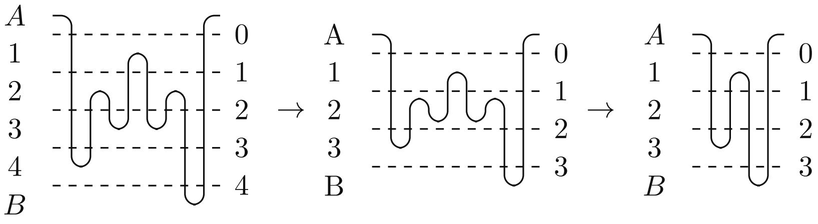


Figure 2.3: The d_2 face of (A,4,2,3,1,3,2,B,A)

maintain strict undulation.

Definition 2.78. The simplicially enriched category $\mathfrak{Adj}$ has morphisms and objects given by the simplicial graph

$$\begin{tikzcd} S^{<} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \{A, B\} \end{tikzcd}$$

where the source map s picks the last letter of a word and the target map t picks the first letter. Composition of n-morphisms is given as follows. If $v = (v_0, \ldots, v_m)$ and $v' = (v'_0, \cdots, v'_{m'})$ are strictly undulating squiggles with $v'_0 = v_m$, then $v \circ v' = (v_0, \ldots, v_m, v'_1, \ldots, v'_{m'})$. This composition is illustrated in Figure 2.4.

In their work, Riehl and Verity prove that this simplicial category is a 2-category: its homspaces are nerves of categories. Moreover they show that it has the same 2-universal property as Street and Schanuel's model **Adj**, given in [49] and reviewed in 2.3.1: 2-Functors $\mathfrak{Adj} \to \mathscr{C}$ are in bijective correspondence with adjunctions in $\mathscr{C}$. The adjunction data is encoded as follows.

- The left adjoint is $l = (A, B)$ while the right adjoint is $r = (B, A)$.
- The unit of the adjunction is $\eta = (B, 1, B)$.
- The counit of the adjunction is $\epsilon = (A, 1, A)$.

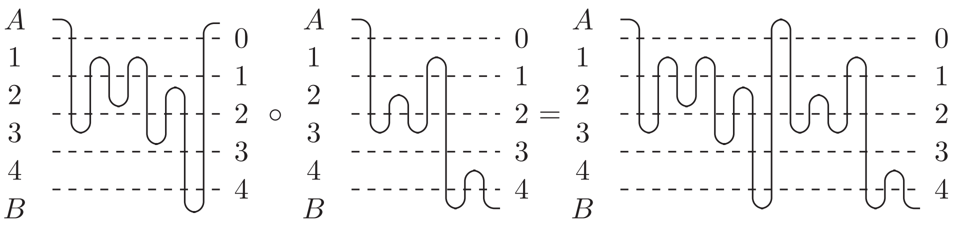
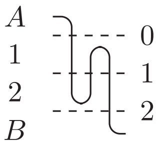


Figure 2.4: Example of composition

As an example, let us prove the triangle identity. The 2-morphism $(A, 2, 1, B)$



has for zeroth face ϵl, first face the identity over l and second face $l\eta$. Thus this 2-morphism encodes the first triangle identity. Similarly, $(B, 1, 2, A)$ encodes the second triangle identity.

2.3.3 The isomorphism $\mathfrak{Adj} \cong \mathbf{Adj}$

By Yoneda's lemma, $\mathfrak{Adj} \cong \mathbf{Adj}$. This is not quite enough to understand the interplay between the two models. In this part, we provide a new and explicit description of this isomorphism. Both models are not entirely straightforward. In the categorical model, composition at A is not completely obvious, whereas in the simplicial model the difficulty lies in the faces. We define a new simplicial category $\mathbb{A}\mathbf{dj}$, which somewhat "explains" both of them. Inspired by topological data analysis, we provide a quotient map $\mathbb{A}\mathbf{dj} \longrightarrow \mathbf{Adj}$. There is also an injective map $\operatorname{Mor} \mathfrak{Adj} \to \operatorname{Mor} \mathbb{A}\mathbf{dj}$ which preserves composition and units, but which fails to be simplicial. However, the composite is the isomorphism $\mathfrak{Adj} \cong \mathbf{Adj}$. We start by some preliminary definitions.

Definition 2.79. A function $l : [0, t] \longrightarrow I = [0, 1]$ is called a *step function* if there exist a integer n and real numbers $0 = a_0 < a_1 < \ldots < a_n = t$ such that l is constant on the intervals $[a_i, a_{i+1}[$ for $i = 0, \ldots n - 2$ and on $[a_{n-1}, a_n]$. We write Step_t for the set of step functions $[0, t] \to [0, 1]$.

Let $J : \mathrm{Step}_t \times [0, t] \to \{\text{closed subintervals of } [0, 1]\}$ be a function given as follows. For $l \in \mathrm{Step}_t$ as above and $x \in [0, t] \setminus \{a_1, \ldots a_{n-1}\}$ we define $J(l, x) =$

$\{l(x)\}$. For $i = 1, \ldots, n-1$ we define $J(l, a_i)$ to be the closed interval whose boundary is $\{l(a_{i-1}), l(a_i)\}$.

Definition 2.80. Let $l \in \mathrm{Step}_t$ and $m \in \mathrm{Step}_s$ for $s, t \in \mathbb{R}_+$. The concatenation of l and m is $l \star m \in \mathrm{Step}_{t+s}$ and is defined by

$$l \star m(x) = \begin{cases} l(x) & x \in [0, t[; \\ m(x-t) & x \in [t, t+s] \end{cases}$$

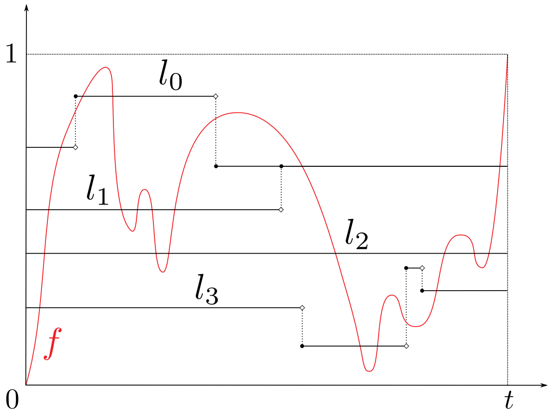
Remark 2.81. Let X be a set, and consider $\mathrm{Triv}(X)$, the trivial groupoid on X. Its set of objects is X, and there is a unique morphism $x \to y$ for all $x, y \in X$. Remark that its nerve $N\mathrm{Triv}(X)$ is a simplicial set with $(N\mathrm{Triv}(X))_n = X^{n+1}$, where the face map d_i deletes the $i+1$-th entry while the degeneracy map s_i doubles the $i+1$-th entry.

Definition 2.82. The simplicial category $\mathbb{A}\mathbf{dj}$ is defined as follows.

- $|\mathbb{A}\mathbf{dj}| = \{A, B\}$;
- The set of n-morphisms is

$$\mathrm{Mor}\,\mathbb{A}\mathbf{dj}_n = \left\{ (t, f, \overrightarrow{l}) : \begin{array}{l} t \in [0, \infty), \overrightarrow{l} \in (\mathrm{Step}_t)^{\times n+1}, \\ f : [0, t] \longrightarrow I \text{ continuous}, f(0), f(t) \in \{0, 1\}, \\ l_i(s) \geq l_{i+1}(s) \text{ for all } 0 \leq i \leq n, s \in [0, t] \\ |\{x \in [0, t] : f(x) = l_i(x)\}| < \infty, \forall\ 0 \leq i \leq n+1. \end{array} \right\};$$

  A typical 3-morphism is represented below.



- The simplicial structure is induced from the one on $N\mathrm{Triv}(\mathrm{Step}_t)$, that is, degeneracies are given by *literally* doubling a step function, while faces are given by *literally* omitting one.

- Let $\chi : \{0,1\} \longrightarrow \{A,B\}$ be the bijection such that $\chi(0) = B$. The target and source maps $\mathrm{Mor}\ \mathbb{A}\mathbf{dj}_n \longrightarrow \{A,B\}$ are given respectively by $(t,f,\overrightarrow{l}) \mapsto \chi f(0)$ and $(t,f,\overrightarrow{l}) \mapsto \chi f(t)$.
- The composition map $\mathrm{Mor}\ \mathbb{A}\mathbf{dj}_n \times_{\{A,B\}} \mathrm{Mor}\ \mathbb{A}\mathbf{dj}_n \longrightarrow \mathrm{Mor}\ \mathbb{A}\mathbf{dj}_n$ is given by concatenation of paths (without reparametrization). That is,
$$(t,f,\overrightarrow{l}) \circ (t',f',\overrightarrow{l'}) = (t+t', f\star f', \overrightarrow{l\star l'}).$$

For $(t,f,\overrightarrow{l}) \in \mathrm{Mor}\ \mathbb{A}\mathbf{dj}_n$, we define a sequence

$$\begin{tikzcd} X_0(t,f,\overrightarrow{l}) \arrow[r] & X_1(t,f,\overrightarrow{l}) \arrow[r] & \dots \arrow[r] & X_n(t,f,\overrightarrow{l}) \end{tikzcd}$$

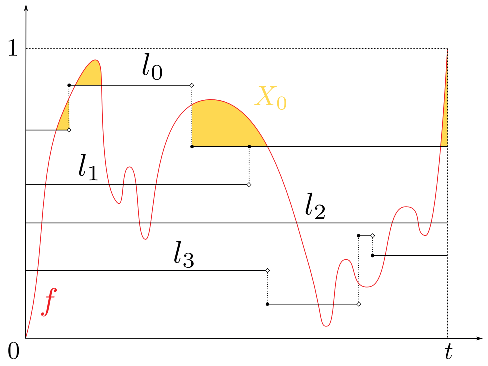
of topological spaces by

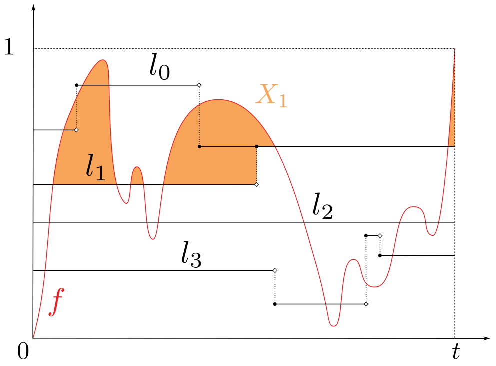
$$X_i(t,f,\overrightarrow{l}) = \{(x,y) \in [0,t]\times[0,1] : J(l_i,x) < y < f(x)\},$$

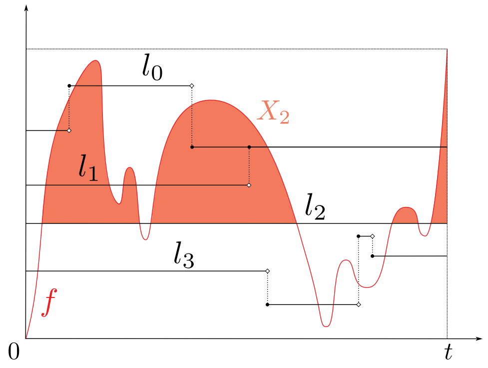
and the maps are given by inclusion. To shorten notation, we will write X_i for $X_i(t,f,\overrightarrow{l})$ whenever possible without ambiguity. Let us picture the result for our typical 3-morphism. The sequence

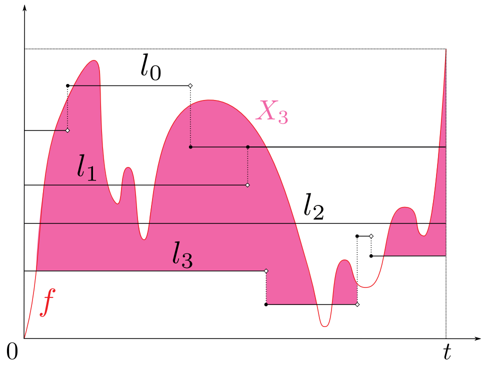
$$\begin{tikzcd} X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & X_3 \end{tikzcd}$$

is given by the inclusion of the spaces pictured below.









Remark that this gives a map of simplicial sets Mor $\mathbb{A}\mathbf{dj} \longrightarrow N(\mathbf{Top})$. (The notation $N(\mathbf{Top})$ is slightly abusive, since **Top** is not small. Observe that no harm is done since the image of that map is a simplicial set, and it is much more convenient to not specify this image). Let

$$D_i(t, f, \overrightarrow{l}) = \{x \in [0, t] : l_i(x) = f(x) \text{ or } f(x) \in J(l_i, x)\} \sqcup \{0, t\}.$$

Remark that $D_i(t, f, \overrightarrow{l})$ is a finite set since l_i has a finite number of steps and l_i and f coincide on a finite number of points.

Proposition 2.83. *Let* $(t, f, \overrightarrow{l}) : X \to Y$ *be an* m*-morphism of* $\mathbb{A}\mathbf{dj}$. *Write* $D_i(t, f, \overrightarrow{l}) = \{j_0, \ldots, j_n\}$ *such that* $j_s < j_{s+1}$ *for all* s.

- *if* $Y = A$, $\pi_0 X_i = \{\{(x, y) \in X_i : x \in]j_i, j_{i+1}[\} : i \in [0, n-1] \cap 2\mathbb{N}\}$;
- *if* $Y = B$, $\pi_0 X_i = \{\{(x, y) \in X_i : x \in]j_i, j_{i+1}[\} : i \in [0, n-1] \cap (2\mathbb{N} + 1)\}$.

Proof. We defer the proof to Appendix A. □

We postcompose our map Mor $\mathbb{A}\mathbf{dj} \longrightarrow N(\mathbf{Top})$ with $\pi_0 : N(\mathbf{Top}) \longrightarrow N(\mathbf{Set})$. The simplicial map P : Mor $\mathbb{A}\mathbf{dj} \longrightarrow N(\mathbf{Top}) \longrightarrow N(\mathbf{Set})$ we obtain factors through $N(\Delta_+)$ because the connected components are naturally ordered by the order on the first component of their representatives, and the inclusions preserve the order. We thus get a map of simplicial sets Mor $\mathbb{A}\mathbf{dj} \longrightarrow N(\Delta_+)$. By Proposition 2.83, if the domain of an n-morphism is A, the maps of sets one gets by the previous procedure will always preserve the maximal element. Similarly, if the codomain of an n-morphism is A, the maps of sets one gets will always preserve the minimal element. Thus, to check that the simplicial map Mor $\mathbb{A}\mathbf{dj} \to$ Mor **Adj** gives simplicial functor $P : \mathbb{A}\mathbf{dj} \to \mathbf{Adj}$ which is the identity on objects, it is enough to look at composition and units. For the units, we chose our convention accordingly. For the composition condition, let ϕ, γ be two composable n-morphisms. It's not hard to see that if we compose at B, $P(\phi) + P(\gamma) = P(\phi \circ_n \gamma)$. If we compose at A, the union of the maximal connected components of $X_i(\phi)$ and the minimal connected component of $X_i(\gamma)$ is exactly one connected component of $X_i(\phi \circ_n \gamma)$. Thus, P preserves composition.

One can define a map R : Mor $\mathfrak{Adj} \longrightarrow$ Mor $\mathbb{A}\mathbf{dj}$ using a standard representation of each strictly undulating squiggle $w \in S_n^<$, for instance by setting $R(w)$ to be the piecewise linear function $R(v) : [0, \frac{w(v)}{2}] \longrightarrow I$ such that

$$R(v)(i) = \begin{cases} 0 & v_i = B \\ 1 & v_i = A \\ \frac{1}{2(n+2)} + \frac{n+2-v_i}{n+2} & \text{otherwise.} \end{cases} \tag{2.6}$$

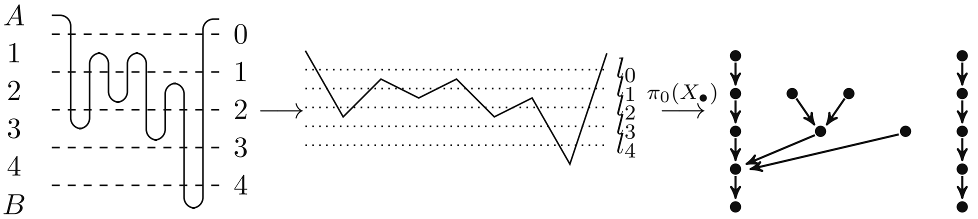


Figure 2.5: (A,3,1,2,1,3,2,B,A), its representation and the corresponding element of $N(\Delta_{-\infty,+\infty})$

and setting as lines l_i the constant function with value $\frac{n+1-i}{n+2}$. This map preserves composition, but is not simplicial because of the choice of some particular functional representations and lines l_i. Nevertheless the composition

$$\text{Mor } \mathfrak{Adj} \longrightarrow \text{Mor } \mathbb{A}\mathbf{dj} \longrightarrow \text{Mor } \mathbf{Adj}$$

is simplicial. The idea is that in the description of the faces of $\mathfrak{Adj}$, to keep a strictly undulating squiggle, we eliminated non-strict undulation. But non strict undulation has no effect on the connected components of the associated spaces. Similarly, for the degeneracies, doubling a line l_i has the same effect as doubling it and stretching the two copies away a little bit.

One can prove directly that this simplicial functor is full and faithful. This could be done by giving an algorithm to produce a squiggle that would be sent to a given element of **Adj**, and also showing that it is unique. We don't want to bother the reader further with technical details, thus this alternative proof is not included. We encourage nevertheless the reader to draw some pictures to convince himself. We illustrated the procedure in Figure 2.5.

Actually, the work of Schanuel and Street on the one hand and of Riehl and Verity on the other are enough to conclude. Indeed, the 2-categorical universal property of both models **Adj** and $\mathfrak{Adj}$ ensures that it is enough to check that the generating adjunction of $\mathfrak{Adj}$ is sent to the generating adjunction of **Adj**. This is obviously the case, because of the description of them given in Section 2.3.2 and in 2.3.1.

From now on, we no longer notationally distinguish the two models of **Adj** since we are provided with an explicit isomorphism between them.

2.3.4 The Eilenberg-Moore object of algebras as a weighted limit

We now develop another point of view on the 2-adjunction 2.36, which enables us to express the Eilenberg-Moore object of algebras as a weighted limit. The universal 2-category containing a monad, also called *the free monad* by Schanuel and Street in [49], is a 2-category **Mnd** such that there is a natural bijection between the monads in a 2-category $\mathscr{C}$ and the 2-functors $\mathbf{Mnd} \to \mathscr{C}$. It is the full 2-subcategory of **Adj** generated by the object B. In [3, Theorem 4.4], Auderset shows the following.

Proposition 2.84. *Let* $\mathcal{T}$ *be a monad over an object* B *in a* 2*-category* $\mathscr{C}$ *and* $\mathbb{T} : \mathbf{Mnd} \longrightarrow \mathscr{C}$ *the corresponding* 2*-functor. When* $\mathscr{C}$ *is complete, the enriched right Kan extension of the* 2*-functor* $\mathbb{T} : \mathbf{Mnd} \longrightarrow \mathscr{C}$ *along the inclusion* $j : \mathbf{Mnd} \longrightarrow \mathbf{Adj}$ *exists and is the free forgetful adjunction in* $\mathscr{C}$*:*

$$B \underset{u}{\overset{f}{\rightleftarrows}} \mathrm{Alg}(\mathcal{T}). \tag{2.7}$$

It follows that $\mathrm{Alg}(\mathcal{T})$ *can be expressed as the weighted limit* $\{\mathbf{Adj}(A, j(-)), \mathbb{T}\}$*.*

In the previous proposition, $\mathscr{C}$ is assumed to be complete, which is enough to ensure the existence of any enriched right Kan extension with target $\mathscr{C}$. It can be weakened to the existence of the only limits needed to construct that particular right Kan extension. We will not explore this aspect, but *finitely complete* is a sufficient hypothesis (see [28, page 92] for instance).

Example 2.85. Let $\mathscr{C} = \mathbf{Cat}$ and let $\mathscr{B} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{A}$ be an adjunction with $L \dashv R$, unit η and counit ϵ. This determines a 2-functor $\mathbb{A} : \mathbf{Adj} \to \mathbf{Cat}$, and thus also a monad $\mathbb{A} \circ j : \mathbf{Mnd} \to \mathbf{Cat}$, which is $\mathcal{T} = (\mathscr{B}, RL, R\epsilon L, \eta)$.

The object of algebras $\mathrm{Alg}(\mathcal{T})$ is the usual category of algebras over $\mathcal{T}$, and the adjunction (2.7) is the usual free-forgetful adjunction. Let us describe the 2-natural transformation $\begin{tikzcd} \mathbf{Adj} \arrow[r, bend left, "\mathbb{A}", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Ran}_j(\mathbb{A}\circ i)"', ""{name=D}] & \mathbf{Cat} \arrow[Rightarrow, from=U, to=D, "\alpha"'] \end{tikzcd}$ given by the universal property of the enriched right Kan extension. It is given by $\alpha_B = \mathrm{id}_{\mathscr{B}}$ and $\alpha_A = \mathrm{Can}_{\mathbb{A}} : \mathscr{A} \to \mathscr{B}^{\mathcal{T}}$, where $\mathrm{Can}_{\mathbb{A}}$ is the comparison functor defined by

- $\mathrm{Can}_{\mathbb{A}}(A) = (RA, R\epsilon)$ for all $A \in |\mathscr{A}|$;
- $\mathrm{Can}_{\mathbb{A}}(f) = Rf$ for all $f \in \mathrm{Mor}\ \mathscr{A}$.

2.4 ∞-Cosmoi and their homotopy 2-category

All material present in this part is due to Riehl and Verity and is taken from their series of articles [43], [45] and [44], with some minor adaptations. Let us recall the definition of an ∞-cosmos from Riehl and Verity [46].

Definition 2.86. An ∞-*cosmos* $\mathcal{K}$ is a simplicially enriched category with two classes of 0-arrows, $\mathscr{W}$, whose elements are called *weak equivalences* and $\mathscr{F}$, whose elements are called *isofibrations*, such that

(i) $\mathscr{W}$ and $\mathscr{F}$ contains all isomorphisms and are closed under composition;

(ii) $\mathscr{W}$ satisfies the 2-*out-of-*6 *property*, that is if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ is a diagram in $\mathcal{K}_0$, if gf and hg are in $\mathscr{W}$, so are f, g, h, and hgf.

(iii) The category $\mathcal{K}$ possesses an (enriched) terminal object 1, cotensors A^X of all objects A by all finitely presented simplicial sets X and (enriched) pullbacks of isofibrations along any 0-arrow;

(iv) For all $A \in |\mathcal{K}|$, $! : A \to 1$ is an isofibration;

(v) The classes $\mathscr{F}$ of isofibrations and $\mathscr{F} \cap \mathscr{W}$ of *trivial isofibrations* are stable under pullback along any 0-cell;

(vi) For any inclusion $i : X \to Y$ of finitely presented simplicial sets and isofibration $p : E \to B$, the Leibniz cotensor $\widehat{p^i}$ displayed in the diagram below is an isofibration, which is trivial if i is a weak equivalence in the Joyal model structure or p is trivial;

$$\begin{tikzcd}
E^Y \arrow[dr, dashed, "\widehat{p^i}"] \arrow[drr, bend left, "p^Y"] \arrow[ddr, bend right, "E^i"'] & & \\
& E^X \times_{B^X} B^Y \arrow[r] \arrow[d] & B^Y \arrow[d, "B^i"] \\
& E^X \arrow[r, "p^X"] & B^X
\end{tikzcd}$$

(vii) For all $A \in |\mathcal{K}|$, there exists a trivial isofibration $QA \to A$ where QA is *cofibrant* in the sense that it enjoys the left lifting property with respect to trivial isofibrations.

A consequence of the axioms is the existence of limits weighted by finite projective cell complexes, as shown in the following proposition.

Proposition 2.87. *An ∞-cosmos is closed under limits weighted by finite projective cell complexes.*

Proof. Let $\mathcal{D}$ be a small simplicial category, $\Gamma : \mathcal{D} \to \mathbf{sSet}$ be a finite projective cell complex and $\mathcal{K}$ an ∞-cosmos. In other words, there exists a filtration $\emptyset = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_n = \Gamma$ of Γ such that $\Gamma_i \to \Gamma_{i+1}$ can be expressed as a pushout of the form

$$\begin{tikzcd}
\partial\Delta[n] \times \mathcal{D}(D_i, -) \arrow[r] \arrow[d] & \Gamma_i \arrow[d] \\
\Delta[n] \times \mathcal{D}(D_i, -) \arrow[r] & \Gamma_{i+1}
\end{tikzcd}$$

for some $D_i \in \mathcal{D}$. We are going to show by induction on i that $\mathcal{K}$ is closed under limits weighted by Γ_i. Let $F : \mathcal{D} \to \mathcal{K}$ be an arbitrary simplicial functor. Remark that $\{\emptyset, F\}$ is the (enriched) terminal object which exists in $\mathcal{K}$ by Definition 2.86 (iii). We suppose now that the weighted limit $\{\Gamma_i, F\}$ exists in $\mathcal{K}$ and we are going to show that it is also the case for $\{\Gamma_{i+1}, F\}$. There is a pullback in $\mathbf{sSet}^{\mathcal{K}^{\mathrm{op}}}$

$$\begin{tikzcd}
{[\mathcal{D}, \mathbf{sSet}](\Gamma_{i+1}, \mathcal{K}(-, F(-)))} \arrow[r] \arrow[d] & {[\mathcal{D}, \mathbf{sSet}](\Delta[n] \times \mathcal{D}(D_i, -), \mathcal{K}(-, F(-)))} \arrow[d] \\
{[\mathcal{D}, \mathbf{sSet}](\Gamma_i, \mathcal{K}(-, F(-)))} \arrow[r] & {[\mathcal{D}, \mathbf{sSet}](\partial\Delta[n] \times \mathcal{D}(D_i, -), \mathcal{K}(-, F(-)))}
\end{tikzcd}$$

which can rewritten as

$$\begin{tikzcd}
{[\mathcal{D}, \mathbf{sSet}](\Gamma_{i+1}, \mathcal{K}(-, F(-)))} \arrow[r] \arrow[d] & \mathcal{K}(-, F(D_i)^{\Delta[n]}) \arrow[d] \\
\mathcal{K}(-, \{\Gamma_i, F\}) \arrow[r] & \mathcal{K}(-, F(D_i)^{\partial\Delta[n]}).
\end{tikzcd}$$

By Definition 2.86 (vi), $F(D_i)^j : F(D_i)^{\Delta[n]} \to F(D_i)^{\partial\Delta[n]}$ is an isofibration. By Definition 2.86 (iii), the following (enriched) pullback exists in $\mathcal{K}$.

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & F(D_i)^{\Delta[n]} \arrow[d] \\
\{\Gamma_i, F\} \arrow[r] & F(D_i)^{\partial\Delta[n]}
\end{tikzcd}$$

This presents P as the weighted limit $\{\Gamma_{i+1}, F\}$. □

Unfortunately, limits weighted by finite projective cell complex are not enough for our purposes. The weight $\mathbf{Adj}(A, j(-)) : \mathbf{Mnd} \to \mathbf{sSet}$ which plays a role in the construction of algebras for a homotopy coherent monad (see Proposition 2.97) is indeed a projective cell complex, but it is not finite. By the collage construction and the proof of 2.16, ([45, Proposition 3.3.3]) this would be the case if and only if coll $\mathbf{Adj}(A, j(-))$ had finitely many non-degenerate atomic arrows $A \to B$, which is not the case.

However, in the examples, most ∞-cosmoi are constructed as the subcategory of fibrant objects of a cofibrantly generated model category enriched over the Joyal model structure.

Examples 2.88. The following list of examples is provided by Riehl and Verity in [46].

- The category of quasi-categories (see [46, Example 2.1.4]).
- The category of categories (see [46, Example 2.2.4]).
- The category of complete Segal spaces (see [46, Example 2.2.5]).
- The category of Segal categories (see [46, Example 2.2.7]).
- The category of marked simplicial sets (see [46, Example 2.2.8]).
- The category of Θ_n-spaces(see [46, Example 2.2.10]).

Proposition 2.89. *Let $\mathscr{M}$ be a cofibrantly generated model category enriched over the Joyal model structure on simplicial set and $\mathcal{D}$ a simplicial category. For any relative projective cell complex*
$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\Gamma"{name=U}] \arrow[r, bend right, "\Lambda"'{name=L}] & \mathbf{sSet} \arrow[Rightarrow, from=U, to=L, "\alpha"] \end{tikzcd}$$
and level-wise isofibration
$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "E"{name=U}] \arrow[r, bend right, "B"'{name=L}] & \mathscr{M} \arrow[Rightarrow, from=U, to=L, "\pi"] \end{tikzcd}$$
, the Leibniz cotensor $\widehat{\{\alpha, \pi\}}$ displayed in the diagram below is a fibration, which is trivial if α is a levelwise weak equivalence in the Joyal model structure or π is level-wise trivial;

$$\begin{tikzcd}
\{\Lambda, E\} \arrow[dr, dashed, "\widehat{\{\alpha,\pi\}}"'] \arrow[drr, bend left, "\{\Lambda,\pi\}"] \arrow[ddr, bend right, "\{\alpha,E\}"'] & & \\
& \{\Gamma, E\} \times_{\{\Gamma,B\}} \{\Lambda, B\} \arrow[r] \arrow[d] & \{\Lambda, B\} \arrow[d, "\{\alpha,B\}"] \\
& \{\Gamma, E\} \arrow[r, "\{\Gamma,\pi\}"] & \{\Gamma, B\}
\end{tikzcd}$$

As a consequence, the full subcategory $\mathscr{M}_{\text{fib}}$ is closed under limits weighted by projectively cofibrant weights.

Proof. Since $\mathscr{M}$ is a cofibrantly generated $\mathbf{sSet}_J$-model category, the projective model structure on $\mathscr{M}^{\mathcal{D}}$ exists (See [22, Theorem 4.31, Remark 4.34]). We are going to show that $\{-,-\} : (\mathbf{sSet}^{\mathcal{D}})^{\mathrm{op}} \times \mathscr{M}^{\mathcal{D}} \to \mathscr{M}$ is a right Quillen bifunctor. Let $\gamma : \Gamma \to \Gamma'$ be a projective cofibration and $\alpha : F \to G$ a projective fibration. Let $i : M \to M'$ be a cofibration in $\mathscr{M}$, which is taken to be acyclic if γ and α are not acyclic. We need to show that $(i, \widehat{\{\gamma,\alpha\}})$ has the lifting property, where $\widehat{\{\gamma,\alpha\}}$ is the pullback product. By (bi)adjunction, we can check instead that $(\gamma, \widehat{\mathscr{M}(i,\alpha)})$ has the lifting property, that is, $\widehat{\mathscr{M}(i,\alpha)}$ is still a projective fibration which is acyclic if γ is not. Since projective fibrations are levelwise fibrations, this follows from $\mathscr{M}$ being enriched over $\mathbf{sSet}_J$. □

Riehl and Verity note in [46] that in all the examples listed in 2.88, all objects are cofibrant, and suggest that asking that all objects are cofibrant is a good simplifying assumption. In this thesis, our context is an ∞-cosmos where all objects are cofibrant (which is thus quasi-categorically enriched) and closed under limits weighted by projective cell complexes. More precisely, we strengthen the definition of ∞-cosmos as follows.

Definition 2.90. A simplicial category $\mathcal{K}$ is an ∞-*cosmos* if there exist a cofibrantly generated model category $\mathscr{M}$ enriched over $\mathbf{sSet}_J$ and with all objects cofibrant, such that $\mathcal{K} \cong \mathscr{M}_{\mathrm{fib}}$, where $\mathscr{M}_{\mathrm{fib}}$ denotes the full subcategory of fibrant objects.

This assumption might seem over restrictive, but in practice most relevant examples of ∞-cosmoi are of this sort, since all examples listed in 2.88 do. Observe also that an ∞-cosmos $\mathcal{K}$ is enriched in quasi-categories.

Example 2.91. Since $\mathrm{op} : \mathbf{sSet} \to \mathbf{sSet}$ preserves the Joyal model structure, observe that if $\mathscr{M}$ is a model category enriched over $\mathbf{sSet}_J$, so is $\mathscr{M}^{\mathrm{co}} := \mathrm{op}_*(\mathscr{M})$, obtained from $\mathscr{M}$ by keeping the same objects and applying op to the homspaces.

Definition 2.92. Let $\mathcal{K}$ be an ∞-cosmos. The *homotopy* 2*-category* of $\mathcal{K}$ is $\mathcal{K}_2 := h_*\mathcal{K}$.

Proposition 2.93. *Let $\mathcal{K}$ be an ∞-cosmos. A map $f : X \to Y$ in $\mathcal{K}$ is a weak equivalence if and only if it is an equivalence in $\mathcal{K}_2$.*

Proof. [46, Proposition 3.1.8]. □

In [41], Riehl and Verity defined *functors of* ∞*-cosmoi* and *weak equivalences of* ∞*-cosmoi* as follows.

Definition 2.94. Let $\mathcal{K}$ and $\mathcal{L}$ be two ∞-cosmoi. A simplicial functor $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ is said to be a *functor of ∞-cosmoi* if it preserves isofibrations and limits weighted by projectively cofibrant weights. It is a *weak equivalence of ∞-cosmoi* if it also satisfies the following conditions

- for all $K, K' \in |\mathcal{K}|$, $\mathbb{P}_{KK'} : \mathcal{K}(K, K') \to \mathcal{L}(\mathbb{P}K, \mathbb{P}K')$ is an equivalence of quasi-categories;
- for all $L \in |\mathcal{L}|$, there exists $K \in |\mathcal{K}|$ and a weak equivalence $\mathbb{P}K \xrightarrow{\sim} L$.

2.4.1 Homotopy coherent monads and adjunctions

Definition 2.95. A *homotopy coherent monad* in an ∞-cosmos $\mathcal{K}$ is a simplicial functor $\mathbf{Mnd} \to \mathcal{K}$.

Definition 2.96. A *homotopy coherent adjunction* in an ∞-cosmos $\mathcal{K}$ is a simplicial functor $\mathbf{Adj} \to \mathcal{K}$.

In [45, Lemma 6.1.8], Riehl and Verity show that $\mathbf{Adj}(A, j(-))$ is a projective cell complex, where $j : \mathbf{Mnd} \to \mathbf{Adj}$ is the inclusion of the full subcategory $\mathbf{Mnd}$ into $\mathbf{Adj}$. This implies that we can generalize the discussion of 2.3.4 to obtain the following result.

Proposition 2.97. *Let $\mathbb{T} : \mathbf{Mnd} \to \mathcal{K}$ be a homotopy coherent monad in an ∞-cosmos and let $X = \mathbb{T}(B)$. The enriched right Kan extension of $\mathbb{T}$ along $j : \mathbf{Mnd} \longrightarrow \mathbf{Adj}$ exists. It describes the free forgetful homotopy coherent adjunction in $\mathcal{K}$ induced by the homotopy coherent monad:*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] & \mathrm{Alg}(\mathbb{T}), \arrow[l, shift left, "u"] \end{tikzcd} \tag{2.8}$$

where $\mathrm{Alg}(\mathbb{T}) = \{\mathbf{Adj}(A, j(-)), \mathbb{T}\}$.

The weighted limit $\mathrm{Alg}(\mathbb{T}) = \{\mathbf{Adj}(A, j(-)), \mathbb{T}\}$ is called the *object of (homotopy coherent) $\mathbb{T}$-algebras.*

The universal property of the enriched right Kan extension implies that given a homotopy coherent adjunction $\mathbb{A} : \mathbf{Adj} \to \mathcal{K}$ with induced homotopy coherent monad $\mathbb{T} = \mathbb{A} \circ j$, there is a comparison map $\mathbb{A}(A) \to \mathrm{Alg}(\mathbb{T})$. We review rapidly in 2.4.3 the monadicity theorem of Riehl and Verity, which provides a criterion to determine when this comparison map is an equivalence.

2.4.2 Absolute left liftings and left exact transformations

Definition 2.98. Let $\mathscr{C}$ be a 2-category. A pair (l, λ) as in the diagram

$$\begin{tikzcd}
 & Y \arrow[d, "f"] \\
X \arrow[ur, "l"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3ex, "\lambda\Uparrow"] & Z
\end{tikzcd}$$

in a is said to be an *absolute left lifting of* g *through* f if for any diagram

$$\begin{tikzcd}
W \arrow[r, "h_1"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\omega\Uparrow"] & Y \arrow[d, "f"] \\
X \arrow[r, "g"'] & Z
\end{tikzcd}$$

there exists a unique 2-cell $\bar{\omega} : lh_2 \Rightarrow h_1$ such that

$$\begin{tikzcd}
W \arrow[r, "h_1"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\omega\Uparrow"] & Y \arrow[d, "f"] \arrow[r, phantom, "="] & W \arrow[r, "h_1"] \arrow[d, "h_2"'] \arrow[dr, phantom, shift left=2ex, "\bar{\omega}\Uparrow"] & Y \arrow[d, "f"] \\
X \arrow[r, "g"'] & Z & X \arrow[r, "g"'] \arrow[ur, "l" description] \arrow[ur, phantom, shift right=3ex, "\lambda\Uparrow"] & Z
\end{tikzcd}$$

In [41], Riehl and Verity show that the notion of colimit of a diagram $d : 1 \to \mathcal{K}^X$ in an ∞-cosmos $\mathcal{K}$ can be defined in terms of absolute left liftings in the homotopy 2-category of $\mathcal{K}$.

Definition 2.99. Let $\mathcal{K}$ be an ∞-cosmos, X a simplicial set and $K \in |\mathcal{K}|$. A diagram $d : 1 \to \mathcal{K}^X$ admits a colimit in K if and only if the diagram

$$\begin{tikzcd}
 & K \cong K^{\Delta[0]} \arrow[d, "K^!"] \\
1 \arrow[r, "d"] & K^X
\end{tikzcd}$$

admits an absolute left lifting of d through $K^!$ in $\mathcal{K}_2$, where $! : X \to \Delta[0]$ is the unique such map.

Indeed, when $\mathcal{K} = \mathbf{qCat}_\infty$, it is equivalent to the classical notion of colimit in a quasi-category that one can find for instance in [35].

Definition 2.100. Let $\mathscr{C}$ be a 2-category and

$$\begin{tikzcd}
 & & Y \arrow[d, "f"] \arrow[dr, "y"] & \\
X \arrow[rr, "g"'] \arrow[dr, "x"] & & Z \arrow[dr, "z"] & Y' \arrow[d, "f'"] \\
 & X' \arrow[rr, "g'"'] & & Z'
\end{tikzcd}$$

be a natural transformation between two diagrams admitting absolute left liftings $(l, \lambda), (l'\lambda')$. The natural transformation is said to be *left exact* if the 2-cell τ induced by the universal property of the second absolute left lifting in the following diagram

$$\begin{tikzcd}
 & & Y \arrow[d, "f"] \arrow[dr, "y"] & & & & & Y \arrow[dr, "y"] & \\
X \arrow[urr, "l"] \arrow[rr, "g"'] \arrow[dr, "x"] \arrow[ur, phantom, "\lambda\Uparrow"] & & Z \arrow[dr, "z"] \arrow[r, phantom, "\|"] \arrow[dl, phantom, "\|"] & Y' \arrow[d, "f'"] & = & X \arrow[urr, "l"] \arrow[dr, "x"] \arrow[rrr, phantom, "\tau\Uparrow"] & & & Y' \arrow[d, "f'"] \arrow[dll, phantom, "\lambda'\Uparrow"] \\
 & X' \arrow[rr, "g'"'] & & Z' & & & X' \arrow[urr, "l'"] \arrow[rr, "g'"'] & & Z'
\end{tikzcd}$$

is an isomorphism.

Remark 2.101. When $\mathscr{C} = \mathbf{qCat}_2$, a natural transformation is left exact if and only if it is pointwise left exact. In the condition of the Definition 2.100, the universal properties of the absolute left liftings implies directly that for all $a : \Delta[0] \to X$, $(la, \lambda a)$ is an absolute left lifting of ga through f, and $(l'xa, \lambda'xa)$ is an absolute left lifting of $g'xa$ through f'. Moreover, τ is an isomorphism if and only if for all $a : \Delta[0] \to X$, τa is an isomorphism. But τa is exactly the 2-cell which is an isomorphism if and only if $(\mathrm{id}_\Delta[0], y, z)$ is left exact with respect to the pairs of left liftings defined above.

Definition 2.102. The simplicial category $\mathbf{qCat}_{l\infty}^{\cdot\to\cdot\leftarrow\cdot}$ is defined to be the simplicial subcategory of $\mathbf{qCat}_{\infty}^{\cdot\to\cdot\leftarrow\cdot}$ whose objects are diagrams admitting an absolute left lifting and whose n-morphisms are the n-simplices of $\mathbf{qCat}_{\infty}^{\cdot\to\cdot\leftarrow\cdot}$ (see Definition 2.3) whose vertices are left exact natural transformations.

Riehl and Verity proved the following proposition, which is useful to show that a quasi-category which happens to be a limit weighted by a cofibrant weight has some type of colimits.

Proposition 2.103. *The simplicial subcategory* $\mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \rightarrow & \bullet \end{smallmatrix}}$ *is closed in* $\mathbf{qCat}_{\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \rightarrow & \bullet \end{smallmatrix}}$ *under limits weighted by projective cofibrant weights.*

Proof. See [44, Proposition 4.9]. ☐

Suppose that $X = \{\Gamma, F\}$ for a projectively cofibrant weight $\Gamma : \mathcal{D} \to \mathbf{sSet}$ and simplicial functor $F : \mathcal{D} \to \mathbf{qCat}_\infty$. Let A be a simplicial set. Observe, using universal properties, that $\{\Gamma, F\}^A = \{\Gamma, F(-)^A\}$. As a consequence, a diagram $\delta : 1 \to \{\Gamma, F\}^A$ corresponds to a map $1 \to \{\Gamma, F(-)^A\}$ and thus to a simplicial natural transformation $\bar{\delta} : \Gamma \to F(-)^A$. If for all $D \in |\mathcal{D}|$, $F(D)$ admits colimits of A-shaped diagrams, and for all 0-morphism $d : D \to D'$, $F(d)$ preserves those colimits, then there is a diagram $\mathcal{D} \to \mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \rightarrow & \bullet \end{smallmatrix}}$ given by

$$D \mapsto \begin{tikzcd} & F(D) \arrow[d, "F(D)^!"] \\ \Gamma(D) \arrow[r, "\bar{\delta}_D"] & F(D)^A \end{tikzcd}$$

Now, Proposition 2.103 implies that the weighted limit of this diagram

$$\begin{tikzcd} & \{\Gamma, F\} \arrow[d] \\ \{\Gamma, \Gamma\} \arrow[r, "\{\Gamma, \bar{\delta}\}"] & \{\Gamma, F\}^A \end{tikzcd}$$

belongs to $\mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \rightarrow & \bullet \end{smallmatrix}}$, and thus, so does

$$\begin{tikzcd} & & \{\Gamma, F\} \arrow[d] \\ 1 \arrow[r, "\mathrm{id}_\Gamma"] & \{\Gamma, \Gamma\} \arrow[r, "\{\Gamma, \bar{\delta}\}"] & \{\Gamma, F\}^A. \end{tikzcd}$$

But this horizontal composite is exactly δ.

2.4.3 Monadicity theorem

Let $\begin{tikzcd} \mathscr{X} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{Y} \arrow[l, shift left, "R"] \end{tikzcd}$ be an adjunction, $\mathbb{T} = (\mathscr{X}, T, \mu, \eta)$ be the associated monad on $\mathscr{X}$ and $\begin{tikzcd} \mathscr{X} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(\mathbb{T}) \arrow[l, shift left, "U"] \end{tikzcd}$ the free-forgetful adjunction. Recall that there exists a unique functor Can : $\mathscr{Y} \to \mathrm{Alg}(\mathbb{T})$ such that the following diagram commutes

$$\begin{tikzcd}
\mathscr{X} & \mathscr{Y} \arrow[l, "R"'] \arrow[d, dashed, "\mathrm{Can}"] & \mathscr{X} \arrow[r, "L"] \arrow[dr, "F"'] & \mathscr{Y} \arrow[d, dashed, "\mathrm{Can}"] \\
 & \mathrm{Alg}(\mathbb{T}) \arrow[ul, "U"] & & \mathrm{Alg}(\mathbb{T})
\end{tikzcd}$$

The monadicity theorem provides a necessary and sufficient condition under which this canonical functor is an equivalence. It relies heavily on the fact that every algebra over a monad $\mathbb{T}$ is the coequalizer of a canonical diagram of free algebras, which is split upon applying the forgetful functor. More precisely, if (X, m) is an algebra over $\mathbb{T}$,

$$\begin{tikzcd}
T^2X \arrow[r, shift left, "\mu"] \arrow[r, shift right, "Tm"'] & TX \arrow[r, "m"] & X
\end{tikzcd}$$

is a coequalizer in $\mathrm{Alg}(\mathbb{T})$ which is split in $\mathscr{X}$. Even more precisely,

$$\begin{tikzcd}
T^2X \arrow[r, shift left, "\mu"] \arrow[r, shift right, "Tm"'] & TX \arrow[l, bend right=50, "\eta_{TX}"'] \arrow[r, "m"'] & X \arrow[l, bend right=40, "\eta_X"']
\end{tikzcd}$$

is a split equalizer, but the curved arrows are not morphisms of algebras. In fact, both objects are truncation of wider objects. Indeed, there is a whole augmented simplicial object

$$\begin{tikzcd}
\cdots \arrow[r, shift left=6] \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[r, shift right=6] & T^3X \arrow[r, shift left=4, "\mu"] \arrow[r, leftarrow, shift left=2, "T\eta T" description] \arrow[r, "T\mu" description] \arrow[r, leftarrow, shift right=2, "T^2\eta" description] \arrow[r, shift right=4, "T^2m"'] & T^2X \arrow[r, shift left=2, "\mu"] \arrow[r, leftarrow, "T\eta" description] \arrow[r, shift right=2, "Tm"'] & TX \arrow[r, "m"] & X
\end{tikzcd}$$

which is actually given by

$$\Delta_+^{\mathrm{op}} = \mathbf{Adj}(A, A) \xrightarrow{\mathbb{A}_{AA}} \mathbf{Cat}(\mathrm{Alg}(\mathbb{T}), \mathrm{Alg}(\mathbb{T})) \xrightarrow{(X,m)^*} \mathbf{Cat}(*, \mathrm{Alg}(\mathbb{T})) = \mathrm{Alg}(\mathbb{T}) \; ,$$

where $\mathbb{A} : \mathbf{Adj} \to \mathbf{Cat}$ is the 2-functor associated to the free forgetful adjunction and $(X, m)^*$ denotes the precomposition by $* \to \mathrm{Alg}(\mathbb{T})$ picking the algebra (X, m). The augmentation is the canonical one obtained as the colimit of the simplicial object. Moreover, there is also a functor

$$\Delta_{+\infty} = \mathbf{Adj}(A, B) \xrightarrow{\mathbb{A}_{AB}} \mathbf{Cat}(\mathrm{Alg}(\mathbb{T}), \mathscr{X}) \xrightarrow{(X,m)^*} \mathbf{Cat}(*, \mathscr{X}) = \mathscr{X}$$

which is an augmented simplicial set admitting a splitting and given by

$$\begin{tikzcd}
\cdots \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & T^3X \arrow[l, bend right, shift right=10] \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=6, "\mu T"] \arrow[r, "T\mu"] \arrow[r, shift right=6, "T^2m"'] & T^2X \arrow[l, bend right, shift right=10, "\eta T^2"'] \arrow[l, shift left=3, "T\eta T"] \arrow[l, shift right=3, "T^2\eta"'] \arrow[r, shift left=3, "\mu"] \arrow[r, shift right=3, "Tm"'] & TX \arrow[l, bend right, shift right=8, "\eta T"'] \arrow[l, "T\eta"] \arrow[r, "m"] & X \arrow[l, bend right, shift right=6, "\eta"']
\end{tikzcd}$$

Since the diagram

$$\begin{tikzcd}
\mathbf{Adj}(A,A) \arrow[r, "\mathbb{A}_{AA}"] \arrow[d, "r_*"'] & \mathbf{Cat}(\mathrm{Alg}(\mathbb{T}),\mathrm{Alg}(\mathbb{T})) \arrow[d, "U_*"] \\
\mathbf{Adj}(A,B) \arrow[r, "\mathbb{A}_{AB}"] & \mathbf{Cat}(\mathrm{Alg}(\mathbb{T}),\mathscr{X})
\end{tikzcd}$$

is commutative, these wider objects assemble in a commutative diagram

$$\begin{tikzcd}
\Delta^{\mathrm{op}} \arrow[r, hook] \arrow[d, "\mathrm{dual}"] & \mathbf{Adj}(A,A) \arrow[r] \arrow[d, "r_*"'] & \mathrm{Alg}(\mathbb{T}) \arrow[d, "U"] \\
\Delta_{+\infty} \arrow[r, equal] & \mathbf{Adj}(A,B) \arrow[r] & \mathscr{X}
\end{tikzcd}$$

This analysis motivates the following definition.

Definition 2.104. Let $R : Y \to X$ be a map of quasi-categories An *R-split simplicial object* in Y is a commutative diagram

$$\begin{tikzcd}
\Delta^{\mathrm{op}} \arrow[r, "S"] \arrow[d, "\mathrm{dual}"'] & Y \arrow[d, "R"] \\
\Delta_{+\infty} \arrow[r, "\tilde{S}"] & X.
\end{tikzcd}$$

Moreover, in category theory, a split coequalizer is an absolute coequalizer. The analog of this is the following proposition.

Theorem 2.105. *For any quasi-category Q, the canonical diagram*

$$\begin{tikzcd}
 & Q \arrow[d, "Q!"] \\
Q^{\Delta_{+\infty}} \arrow[ur, "\mathrm{ev}_{\mathbf{0}}"] \arrow[r, "Q^{\mathrm{dual}}"] \arrow[r, phantom, shift left=4, "\lambda \Uparrow"] & Q^{\Delta^{\mathrm{op}}}
\end{tikzcd}$$

is an absolute left lifting diagram. Hence, given any simplicial object admitting an augmentation and a splitting, the augmented simplicial object defines a colimit cone over the original simplicial object. Furthermore, such colimits are preserved by any simplicial map.

Proof. See [43, Theorem 5.3.1]. ☐

Remark 2.106. The 2-cell λ appearing in the theorem above can be obtained as follows. We consider the diagram $\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, bend left, "\mathrm{dual}"] \arrow[r, bend right, "\Delta_{\mathbf{0}}"'] \arrow[r, phantom, "\Downarrow\bar{\lambda}"] & \Delta_{+\infty} \end{tikzcd}$ in **Cat**, where $\Delta_{\mathbf{0}}$ is the constant functor with value $\mathbf{0}$, and $\bar{\lambda}$ is the natural transformation given by projection onto the terminal object. This determines a simplicial map $\hat{\lambda} : \Delta[1] \times \Delta^{\mathrm{op}} \to \Delta_{+\infty}$. The 2-cell λ is given by the adjunct of the composite

$$Q^{\Delta_{+\infty}} \times \Delta[1] \times \Delta^{\mathrm{op}} \xrightarrow{1\times\hat{\lambda}} Q^{\Delta_{+\infty}} \times \Delta_{+\infty} \xrightarrow{\mathrm{ev}} Q\ .$$

Let $R : Y \to X$ be a functor of quasi-categories. We recall what it means for Y to admit colimits of R-split simplicial objects. First of all, there is a quasi-category $S(R)$ of R-split simplicial objects in Y, given by the following pullback

$$\begin{tikzcd}
S(R) \arrow[r] \arrow[d] & Y^{\Delta^{\mathrm{op}}} \arrow[d, "R_*"] \\
X^{\Delta_{+\infty}} \arrow[r, "X^{\mathrm{dual}}"'] & X^{\Delta^{\mathrm{op}}}
\end{tikzcd}$$

By Definition [43, 5.2.9], Y admits colimits of R-split simplicial objects if and only if there is an absolute left lifting

$$\begin{tikzcd}
 & Y \arrow[d, "\mathrm{const}"] \\
S(R) \arrow[ur, "\mathrm{colim}"] \arrow[r] \arrow[ur, phantom, "\lambda\Uparrow"', near end] & Y^{\Delta^{\mathrm{op}}}
\end{tikzcd}$$

in $\mathbf{qCat}_2$. The notion of R-split simplicial object is actually the correct generalization needed for the monadicity theorem in quasi-categories, whose statement is as follows.

Theorem 2.107. *Let* $\mathbb{A} : \mathbf{Adj} \to \mathbf{qCat}_\infty$ *be a homotopy coherent adjunction with underlying homotopy coherent monad* $\mathbb{T}$, *and let* $\mathbb{A}(A) = Y$, $\mathbb{A}(B) = X$ $\mathbb{A}(r) = R$. *Suppose that*

- R *is conservative;*
- Y *admits colimits of* R*-split simplicial objects and* R *preserves them.*

Then, the comparison functor $Y \to \mathrm{Alg}(\mathbb{T})$ *is an equivalence.*

Proof. See [45, Theorem 7.2.4 and 7.2.7]. □

It is mostly routine to check that this theorem can be proven for any ∞-cosmos $\mathcal{K}$. However, the theorem above is enough for our needs.

The next result is a tool due to Riehl and Verity to check that a simplicial map is conservative.

Proposition 2.108. *Suppose $\mathscr{A}$ is a small simplicial category and that $i : V \to W$ in $\mathbf{sSet}^{\mathscr{A}}$ is a projective cofibration of projectively cofibrant weights with the property that for all objects $a \in A$ the simplicial map $i_a : V(a) \to W(a)$ is surjective on vertices. Then for any diagram $D \in \mathbf{qCat}_{\infty}^{\mathscr{A}}$, the functor*

$$\{i, D\} : \{W, D\} \to \{V, D\}$$

is conservative.

Proof. See [45, Proposition 6.2.2]. □

2.5 The homotopy coherent nerve

The homotopy coherent nerve was introduced by Cordier in [10]. It is a special case of a classical construction, attributed to Kan. If $\mathbf{X}[-] : \Delta \to \mathscr{C}$ is a cosimplicial object in a cocomplete category, the functor $\mathcal{N}_{\mathbf{X}} : \mathscr{C} \to \mathbf{sSet}$ given by, for all $C \in |\mathscr{C}|$, $\mathcal{N}_{\mathbf{X}}(C) = \mathscr{C}(\mathbf{X}[-], C)$, has a left adjoint given by the coend formula $\mathcal{C}_{\mathbf{X}}(Y) = \int^{n \in \Delta} Y_n \cdot \mathbf{X}[n]$, where $\cdot : \mathbf{Set} \times \mathscr{C} \to \mathscr{C}$ is the **Set**-tensor.

In the case of the homotopy coherent nerve, the cosimplicial object is a cosimplicial simplicial category $\mathcal{C}\Delta[-] : \Delta \to \mathbf{sCat}$. This simplicial category $\mathcal{C}\Delta[n]$ is actually a 2-category, and can be defined in various ways. In [35], Lurie defines the hom-category $\mathcal{C}\Delta[n](i, j)$, for $0 \leq i \leq j \leq n$, as the poset P_{ij} of subsets of $[i, j] = \{i, i+1, \ldots, j\}$ containing both endpoints and ordered by inclusion, with composition given by union.

An equivalent approach is described by Riehl in [42]. There is a monad on reflexive graphs (or graphs with a distinguished loop), which associates to a reflexive graph G the underlying reflexive graph of the free category on G. An algebra for this monad is a category, and thus to any category $\mathscr{C}$, one can associate a bar construction $\mathcal{C}\mathscr{C}$, which determines $\mathcal{C}\Delta[n]$ by applying this construction to the totally ordered sets $[n]$.

Both constructions make the description of the n-simplices of $\mathcal{C}\Delta[n]$ a little bit difficult to grasp. For instance, it is not obvious that $\mathcal{C}\Delta[n]$ is a simplicial computad, or to determine which n-simplices are non-degenerate or atomic. The goal of this section is to give a graphical model of $\mathcal{C}\Delta[-] : \Delta \to \mathbf{sCat}$ which is easier to understand. The inspiration partly comes from the sticky configurations of Andrade's thesis [1]. Remark that $\mathcal{C}\Delta[n]^{\mathrm{op}} \cong \mathcal{C}\Delta[n]$. For combinatorial reasons it is more convenient for us to use $\mathcal{C}\Delta[n]^{\mathrm{op}}$, and thus in this text we will denote by $\mathcal{C}\Delta[n]$ what other authors would denote $\mathcal{C}\Delta[n]^{\mathrm{op}}$.

Remark that if $i < j$, up to a shift from $[i, j]$ to $[0, j-i] = \mathbf{j} - \mathbf{i} + \mathbf{1}$, P_{ij} can identified with a slice category,

$$P_{ij} \cong \mathrm{inj}(\Delta^{\geq \mathbf{2}}_{-\infty,+\infty})/\mathbf{j} - \mathbf{i} + \mathbf{1},$$

where $\mathrm{inj}(\Delta^{\geq \mathbf{2}}_{-\infty,+\infty})$ denotes the category of finite ordinals greater or equal to $\mathbf{2}$, with injective, order-preserving maps that preserve minimal and maximal elements. Recall that by Stone duality for intervals, which we review in 2.3.1, $\Delta^{\geq \mathbf{2}}_{-\infty,+\infty} \cong \Delta^{\mathrm{op}}$ and this restricts and corestricts to an isomorphism

$$\mathrm{inj}(\Delta^{\geq \mathbf{2}}_{-\infty,+\infty}) \cong \mathrm{surj}(\Delta)^{\mathrm{op}}$$

where $\mathrm{surj}(\Delta)$ denotes the subcategory of Δ constituted of surjective maps. This implies that $\mathrm{inj}(\Delta^{\geq \mathbf{2}}_{-\infty,+\infty})/\mathbf{j} - \mathbf{i} + \mathbf{1} \cong \left(\mathbf{j} - \mathbf{i}/\mathrm{surj}(\Delta)\right)^{\mathrm{op}}$.

Let us define the category $\tilde{P}_{ij}$ as follows. Its objects are order-preserving, surjective maps $\phi : (i, j] \to \mathbf{n}$, where $(i, j] = \{i+1, i+2, \dots, j\}$. If $\psi : (i, j] \to \mathbf{m}$ is another such object, a morphism $\phi \to \psi$ is an order-preserving, surjective map $f : \mathbf{n} \to \mathbf{m}$ such that $f\phi = \psi$. It is clear that up to a relabeling, $\left(\mathbf{j} - \mathbf{i}/\mathrm{surj}(\Delta)\right) \cong \tilde{P}_{ij}$.

As a consequence, $\mathcal{C}\Delta[n](j, i) \cong N(\tilde{P}_{ij}^{\mathrm{op}})$. A subset

$$I = \{i = i_0 < \ldots < i_m = j\} \subseteq [i, j]$$

corresponds to the surjective map $\phi_I : (i, j] \to \mathbf{m}$ given by

$$\phi_I(x) = l \Leftrightarrow i_l < x \leq i_{l+1}$$

Conversely, a surjective map $\phi : (i, j] \to \mathbf{m}$ corresponds to the subset

$$I_\phi = \{-1 + \min \phi^{-1}(x) : x \in \mathbf{m}\} \cup \{j\}.$$

An m-simplex of $N(\tilde{P}_{ij}^{\mathrm{op}})$ corresponds to a forest of trees of height $m+1$ with a total number of $j - i$ leaves labeled linearly from $i+1$ to j. Indeed, it is exactly the same as a sequence of $m+1$ surjective and order-preserving maps (starting with the object $[i+1, j]$), and each such map can be seen as a forest of trees with height 1. For instance, the Figure 2.6 represents a 2-simplex in $\mathcal{C}\Delta[15](8, 1)$. This 2-simplex corresponds to the sequence of maps $[2, 8] \to \mathbf{5} \to \mathbf{2} \to \mathbf{1}$ which can be "read off" by going up in the tree. The i-th degeneracy is given by doubling the i-th line, while the i-th face is given by removing the i-th line, and appropriately

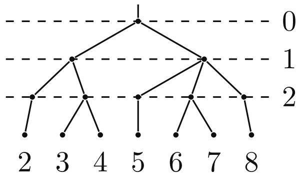


Figure 2.6: A 2-simplex of $\mathcal{C}\Delta[15](8,1)$

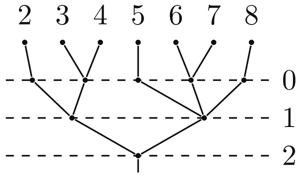


Figure 2.7: A 2-simplex of $\mathcal{C}\Delta[15]^{\mathrm{co}}(8,1)$

composing the corollas (appearing at height i and $i+1$) when $i > 0$ or deleting the roots when $i = 0$. Since in the nerve of a category, an m-simplex is degenerate at i if and only if the $i-1$-th map is an identity, it is clear that an m-simplex is degenerate at $i < m$ if and only if all the nodes on line i have valence 2. Composition is given by juxtaposition of forests. An m-simplex is thus atomic if and only if it is a tree.

In the rest of this thesis, we work with $\mathcal{C}\Delta[n]^{\mathrm{co}}$, whose elements can be represented in a similar manner by forests of trees, their root being on the bottom instead. This choice was previously made by Verity in [56], while constructing nerves of complicial Gray-categories. For instance, Figure 2.7 represents a 2-simplex of $\mathcal{C}\Delta[15]^{\mathrm{co}}(8,1)$.

In this situation, an m-simplex is still atomic if and only if it is a tree, but it is degenerate at $i < m$ if and only if all the nodes on line $i+1$ have valence 2. We take the opportunity to draw some illustrations of $\mathcal{C}\Delta[n]^{\mathrm{co}}$ for $n = 2$ in Figures 2.8 and for $n = 3$ in Figures 2.9 and 2.10.

Let us describe the cosimplicial structure of $\mathcal{C}\Delta[-]^{\mathrm{co}} : \Delta \to 2\text{-}\mathbf{Cat}$. Classically, given a simplicial operator $\phi : [n] \to [m]$, $\phi_* : \mathcal{C}\Delta[n] \to \mathcal{C}\Delta[m]$ is given

- on objects by $j \mapsto \phi^{\mathrm{op}}(j)$ for all $0 \leq j \leq n$;
- on hom-categories by

$$\begin{aligned} P_{ij} &\to P_{\phi^{\mathrm{op}}(i),\phi^{\mathrm{op}}(j)} \\ U &\mapsto \phi^{\mathrm{op}}(U). \end{aligned}$$

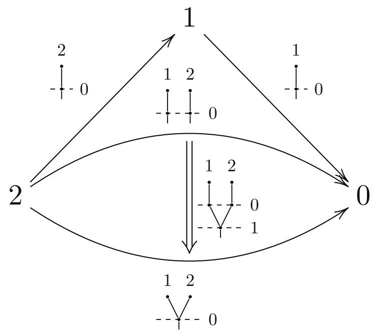


Figure 2.8: $\mathcal{C}\Delta[2]^{\mathrm{co}}$

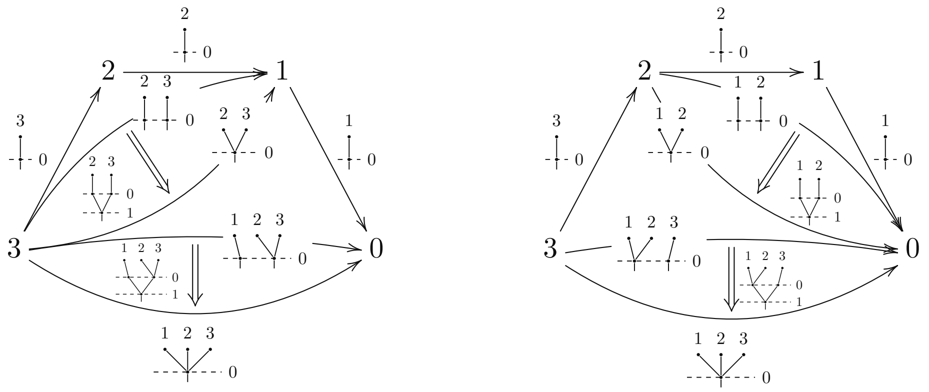


Figure 2.9: The four cofaces of $\mathcal{C}\Delta[3]^{\mathrm{co}}$

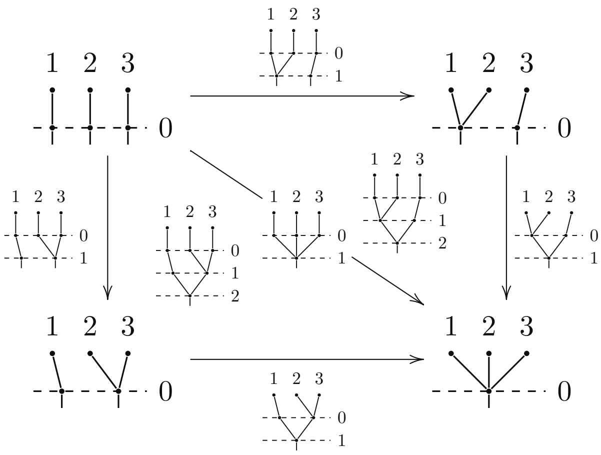


Figure 2.10: $\mathcal{C}\Delta[3]^{\mathrm{co}}(3,0)$

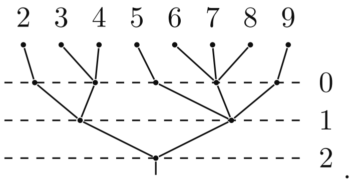
We provide a description of cofaces and codegeneracies in our model. In order to distinguish the cosimplicial functorial structure from the simplicial structure on homspaces in a neat way, we use double-struck letters for the cosimplicial structure. For $0 \leq k \leq n, \mathbb{d}^k : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ is given

- on objects by $j \mapsto d^{n-k}(j)$ for all $0 \leq j \leq n$;
- on hom-categories by

$$\tilde{P}_{ij} \to \tilde{P}_{d^{n-k}(i),d^{n-k}(j)}$$

$$(i,j] \twoheadrightarrow \mathbf{m} \mapsto (d^{n-k}(i), d^{n-k}(j)] \xrightarrow{s^{n-k}} (i,j] \twoheadrightarrow \mathbf{m}$$

Let us describe the action on a forest with labels in $(y,x]$. If $n-k \notin (y,x]$ the forest is unchanged but the leaves are relabeled. If $n-k \in (y,x]$ we glue a corolla with two leaves $\{n-k, n-k+1\}$ onto the leaf labeled $n-k$, and relabel accordingly. For instance, if $\underline{f} \in \mathcal{C}\Delta[15]^{\mathrm{co}}(8,1)$ is the 2-simplex of Figure 2.7, $\mathbb{d}^{10}(\underline{f})$ is obtained from $\underline{f}$ by inserting a corolla with two leaves $\{6,7\}$ into the leaf 6. We obtain the following element



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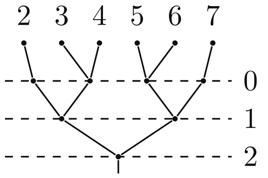
For $0 \leq k \leq n, \mathbb{s}^k : \mathcal{C}\Delta[n+1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ is given

- on objects by $j \mapsto s^{n-k}(j)$ for all $0 \leq j \leq n$;
- on hom-categories by

$$\tilde{P}_{ij} \to \tilde{P}_{s^{n-k}(i),s^{n-k}(j)}$$

$$p : (i,j] \twoheadrightarrow \mathbf{m} \mapsto \ (s^{n-k}(i), s^{n-k}(j)] \xrightarrow{d^{n-k+1}} (i,j] \xrightarrow{p} \mathrel{\mkern-14mu}\twoheadrightarrow \operatorname{im}(p \circ d^{n-k+1}) \subseteq \mathbf{m}$$

Let us describe the action on a forest with labels in $(y,x]$. If $n-k+1 \notin (y,x]$ the forest is unchanged but the leaves are relabeled. If $n-k+1 \in (y,x]$ we delete the branch with label $n-k+1$ (not only the leaf, but up to the first intersection) and relabel accordingly. For instance, if $\underline{f} \in \mathcal{C}\Delta[15]^{\mathrm{co}}(8,1)$ is the 2-simplex of Figure 2.7, $\mathbb{s}^{10}(\underline{f})$ is obtained from $\underline{f}$ by deleting the branch labeled 5. We obtain the following element



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Remark 2.109. Let $\mathcal{K}$ be a category enriched in quasi-categories. By [55, Example 57], to a quasi-category X can be associated a 1-trivial saturated weak complicial set $e(X)$ with same underlying simplicial set, by declaring thin the 1-simplices that are isomorphisms. The functor $e : \mathbf{qCat} \to \mathbf{wcSet}$ preserves products, and thus $e_*(\mathcal{K})$ is a category enriched in weak complicial sets (with cartesian product). Such enriched categories are called *complicial Gray-categories* by Verity in [56]. In that article, Verity add a stratification to the cosimplicial simplicial category $\mathcal{C}\Delta[n]^{\mathrm{co}}$, in order to produce a (stratified) nerve construction

$$\begin{tikzcd}\mathbf{Strat} \arrow[r, bend left, "\mathcal{C}^{\mathrm{co}}"] \arrow[r, phantom, "\perp"] & (\mathbf{Strat}, \circledast)\text{-}\mathbf{Cat} \arrow[l, bend left, "\mathrm{N}"]\end{tikzcd}$$

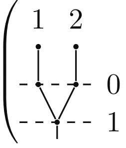
In our particular case,

$$\mathrm{N}(e_*(\mathcal{K}))_n = \mathbf{sCat}(\mathcal{C}\Delta[n]^{\mathrm{co}}, \mathcal{K}).$$

Moreover, $\mathrm{N}(e_*\mathcal{K})$ has a 2-trivial stratification, which is given as follows.

- A simplicial functor $\mathbb{F} : \mathcal{C}\Delta[2]^{\mathrm{co}} \to \mathcal{K}$ is declared to be thin if and only if

$\mathbb{F}\Bigg($



$: 2 \to 0\Bigg)$

is an isomorphism in $\mathcal{K}(\mathbb{F}(2), \mathbb{F}(0))$.

- A simplicial functor $\mathbb{F} : \mathcal{C}\Delta[1]^{\mathrm{co}} \to \mathcal{K}$ is declared to be thin if and only if $\mathbb{F}(1) \xrightarrow{\mathbb{F}\left(\begin{smallmatrix} 1 & \\ \bullet & \\ \vert & \\ \text{-}\bullet\text{-} & 0 \end{smallmatrix}\right)} \mathbb{F}(0)$ is an equivalence in $\mathcal{K}_2$.

By [56, Theorem 40], $\mathrm{N}(e_*\mathcal{K})$ is a weak complicial set. We claim that it is also saturated.

Proposition 2.110. *Let $\mathcal{K}$ be a category enriched in quasi-categories. The stratified simplicial set $\mathrm{N}(e_*\mathcal{K})$ is a 2-trivial and saturated weak complicial set.*

Proof. We defer the proof to the Appendix A. ☐

Chapter 3

The 2-category $\mathbf{Adj}^S_{\mathbf{hc}}[n]$

Let $S \subseteq \mathcal{C}\Delta[n]^{\mathrm{co}}$ be a 2-subcategory S of $\mathcal{C}\Delta[n]^{\mathrm{co}}$ such that S is a simplicial computad. Recall that $\mathcal{C}\Delta[n]^{\mathrm{co}}$ is reviewed in detail in Section 2.5. The goal of this chapter is to introduce explicitly the 2-category $\mathbf{Adj}^S_{\mathrm{hc}}[n]$. We prove in Chapter 5 that it satisfies the following universal property.

Corollary 5.1. *Let S be a convenient 2-subcategory of $\mathcal{C}\Delta[n]^{\mathrm{co}}$ and $\mathscr{C}$ a 2-category. There is a natural bijection*

$$2\text{-}\mathbf{Cat}(\mathbf{Adj}^S_{\mathrm{hc}}[n], \mathscr{C}) \cong 2\text{-}\mathbf{Cat}(S, \mathbf{Adj}_r(\mathscr{C})). \tag{5.1}$$

This motivates our construction. The 2-category $\mathbf{Adj}_r(\mathscr{C})$ of the right hand side is a 2-category of adjunctions in $\mathscr{C}$ which is reviewed in 2.32, and convenient 2-subcategories of $\mathcal{C}\Delta[n]^{\mathrm{co}}$ are defined in Subsection 3.2.4. One should think of S as providing a shape, and $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is the universal 2-category containing diagrams of adjunctions of shape S. It is particularly instructive to think about the following choices of S.

(i) $S = \mathcal{C}\Delta[0]$ the terminal 2-category.

(ii) $S = \mathcal{C}\Delta[1]$, the 2-category $\bullet \to \bullet$.

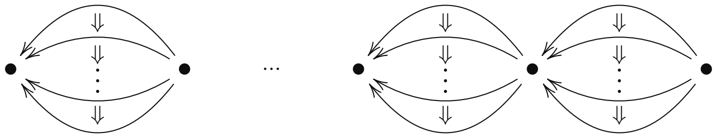
(iii) $S \subseteq C\Delta[2]^{\mathrm{co}}$ the full 2-subcategory generated by the objects 0 and 2. Observe that it encodes the shape $\begin{tikzcd}\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet\end{tikzcd}$ (see Figure 2.8).

(iv) $S \subseteq \mathcal{C}\Delta[3]^{\mathrm{co}}$, the full 2-subcategory generated by the objects 0 and 3 and the 1-cells appearing the in the upper triangle of Figure 2.10. Observe that

it encodes the shape
$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=N, below}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet
\arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=N, to=D]
\end{tikzcd}.$$

(v) $S = [n] \subseteq \mathcal{C}\Delta[n]^{\mathrm{co}}$, which is constituted of linear trees, and encode the shape $\bullet \longrightarrow \bullet \longrightarrow \dots \longrightarrow \bullet$.

(vi) More generally, let $\theta = \mathbf{n}(\mathbf{m}_1, \dots, \mathbf{m}_{n-1})$ be the free 2-category generated by



with n objects and respectively $m_1, \dots, m_{n-1}$ 1-cells between two consecutive objects. There is a convenient $S_\theta \subseteq \mathcal{C}\Delta[\sum_{i=1}^{n-1} m_i]^{\mathrm{co}}$ such that $S_\theta \cong \theta$. This is described in Subsection 3.2.4.

(vii) $S = \mathcal{C}\Delta[n]^{\mathrm{co}}$, constituted of all possible trees.

The 2-category $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is a generalization of $\mathbf{Adj}$, since $\mathbf{Adj}^{\mathcal{C}\Delta[0]}_{\mathrm{hc}}[0] = \mathbf{Adj}$. We construct it in two steps. The first step is to consider the simpler case where S is chosen to be $[n] \subseteq \mathcal{C}\Delta[n]^{\mathrm{co}}$, as in (v) of the list of example above. To make notation simpler, we denote by $\mathbf{Adj}[n]$ the 2-category $\mathbf{Adj}^{[n]}_{\mathrm{hc}}[n]$. The explicit description of this 2-category is the subject of Section 3.1. We also adapt the graphical calculus of squiggles of Riehl and Verity to describe it simplicially.

The second step is to appropriately combine S and $\mathbf{Adj}[n]$ to obtain $\mathbf{Adj}^S_{\mathrm{hc}}[n]$. This is carried out in Section 3.2.1. We study atomic morphisms of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ in 3.2.3, and show that it is "almost" a simplicial computad, in a sense made precise in Proposition 3.25.

The reader might be tempted to bypass the explicit construction of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ and use Corollary 5.1 as a definition, since it determines $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ up to isomorphism, by Yoneda's lemma. This is doable since when a 2-categorical structure is given by equations, like adjunctions, adjunction morphisms and adjunction transformations, the *free* or *universal* 2-category containing such a structure always exists. Indeed, existence can be proven easily by using presentation by computads, which is a standard 2-category theory argument that we provide for the structure of *n-composable adjunction morphisms* in Appendix B.

However, such a construction is *not* suited for studying $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ combinatorially, as we will need to do in Chapter 4. Indeed, it is difficult in general to know when two words of generators are equivalent and thus equal in the quotient, as the literature on word problems suggests. Thus, what might seems to be a shortcut is actually a detour.

3.1 The 2-category Adj$[n]$

3.1.1 The 2-categorical model

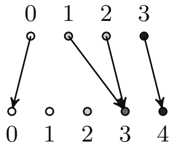
Let S, T be partially ordered sets. Recall that there is a partial order on the set of maps $S \to T$ given as follows. For $f, g : S \to T$, $f \leq g$ if and only if for all $s \in S$, $f(s) \leq g(s)$. We write $\cdot_{\mathbf{Adj}}$ for the horizontal composition in **Adj** and for two integers $y \leq x$, $[y, x]$ denotes the set $[y, x] = \{y, y + 1, \dots x\}$, which is naturally ordered. Let us introduce the explicit 2-categorical model for **Adj**$[n]$.

- The set of objects is $\{(X, k) : X \in |\mathbf{Adj}|, k \in \{0, \dots, n\}\}$.
- For the hom-categories, we set $\mathbf{Adj}[n]((X, x), (Y, y)) = \emptyset$ if $x < y$. A 1-cell $(X, x) \to (Y, y)$ is a non-decreasing map $\mu : \mathbf{m} \to [y, x]$, preserving the maximal element if $X = A$ and such that $\mathbf{m} : X \to Y$ constitutes a 1-cell in the categorical model of **Adj** (See Section 2.3.1). One can think of it as a colored ordinal, with colors belonging to the set $[y, x]$. For instance, the picture below shows a coloring of $\mathbf{5}$.

  A 2-cell from $\mu : \mathbf{m} \to [y, x]$ to $\mu' : \mathbf{m}' \to [y, x]$ is a non-decreasing map $f : \mathbf{m} \to \mathbf{m}'$ such that f represents a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "\mathbf{m}"{name=U}] \arrow[r, bend right, "\mathbf{m}'"'{name=D}] & Y \arrow[phantom, from=U, to=D, "\Downarrow f"] \end{tikzcd}$ in the categorical model of **Adj** and such that $\mu \leq \mu' f$.

  $$\begin{tikzcd} \mathbf{m} \arrow[r, "f"] \arrow[d, "\mu"'] & \mathbf{m}' \arrow[dl, "\mu'"] \\ {[y,x]} \arrow[ur, phantom, "\Rightarrow"] & \end{tikzcd}$$

- If $X, Y, Z \in |\mathbf{Adj}|$ and $x \geq y \geq z$, we define the composition

  $$\mathbf{Adj}[n]((Y, y), (Z, z)) \times \mathbf{Adj}[n]((X, x), (Y, y)) \longrightarrow \mathbf{Adj}[n]((X, x), (Z, z))$$

  as follows. Let

  $$f : (\mathbf{l}, \lambda) \to (\mathbf{l}', \lambda') \in \mathbf{Adj}[n]((Y, y), (Z, z))$$
  $$g : (\mathbf{m}, \mu) \to (\mathbf{m}', \mu') \in \mathbf{Adj}[n]((X, x), (Y, y))$$

be a pair of 2-cells. Observe that $f \cdot_{\mathbf{Adj}} g$ can be written as the composite

$$\mathbf{l} \cdot_{\mathbf{Adj}} \mathbf{m} \xrightarrow{(d^{l-1})^{\delta_{YA}}} \mathbf{l} + \mathbf{m} \xrightarrow{f+g} \mathbf{l}' + \mathbf{m}' \xrightarrow{(s^{l'-1})^{\delta_{YA}}} \mathbf{l}' \cdot_{\mathbf{Adj}} \mathbf{m}'$$

where δ_{YA} denotes the Kronecker delta: its value is 1 if $A = Y$ and 0 otherwise. By this, we mean that if $Y = A$, $(d^{l-1})^{\delta_{YA}} = d^{l-1}$ and is the identity otherwise, and similarly for $(s^{l'-1})^{\delta_{YA}}$.

We define $\lambda \cdot \mu$ to be the composite

$$\mathbf{l} \cdot_{\mathbf{Adj}} \mathbf{m} \xrightarrow{(d^{l-1})^{\delta_{YA}}} \mathbf{l} + \mathbf{m} \xrightarrow{\lambda+\mu} [z, x] = [z, y] \cup [y, x],$$

and let $(\mathbf{l}, \lambda) \cdot (\mathbf{m}, \mu) = (\mathbf{l} \cdot_{\mathbf{Adj}} \mathbf{m}, \lambda \cdot \mu)$.

The composition of 2-cells is given by $f \cdot g = f \cdot_{\mathbf{Adj}} g$. We have to check that

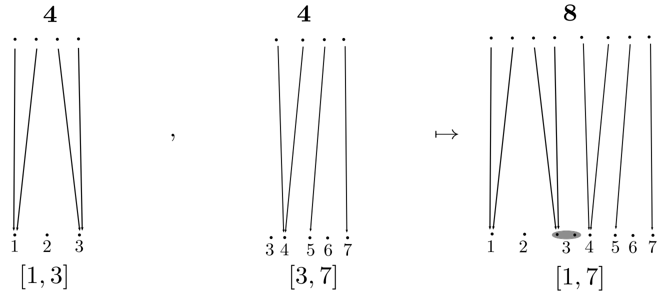
$$\lambda \cdot \mu \leq (\lambda' \cdot \mu') \circ (f \cdot_{\mathbf{Adj}} g).$$

To check the condition when $Y = A$, one can draw the following diagram

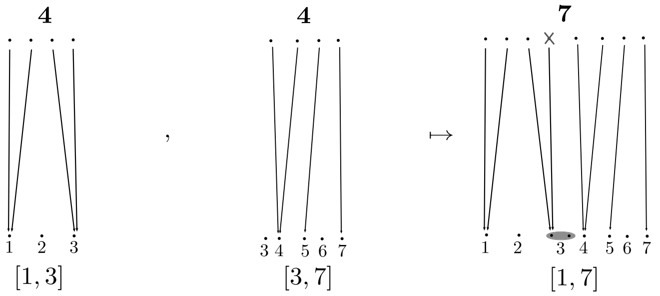
$$\begin{tikzcd}
\mathbf{l} + \mathbf{m} - \mathbf{1} \arrow[r, "d^{l-1}"] \arrow[ddr, bend right, "f \cdot_{\mathbf{Adj}} g"'] \arrow[dr, phantom, "="] & \mathbf{l} + \mathbf{m} \arrow[r, "\lambda+\mu"] \arrow[d, "f+g"] \arrow[dr, phantom, "\Leftarrow"] & {[z, x]} \\
& \mathbf{l}' + \mathbf{m}' \arrow[ur, "\lambda'+\mu'"'] \arrow[d, shift right, "s^{l'-1}"'] & \\
& \mathbf{l}' + \mathbf{m}' - \mathbf{1} \arrow[u, shift right, "d^{l'-1}"'] &
\end{tikzcd}$$

and remark that id $\leq d^{l'-1} s^{l'-1}$. One can consider the right triangle of the same diagram when $Y = B$. We provide below some examples of how 1-cells compose, first when $Y = B$ and then when $Y = A$.

$$\mathbf{Adj}[n]((B, 3), (A, 1)) \times \mathbf{Adj}[n]((A, 7), (B, 3)) \to \mathbf{Adj}[n]((A, 7), (A, 1))$$



$$\mathbf{Adj}[n]((A,3),(B,1))\times\mathbf{Adj}[n]((A,7),(A,3))\to\mathbf{Adj}[n]((A,7),(B,1))$$



The composition is basically given by ordinal sum, with the exception that the crossed point is discarded and the two points in the dark grey region are identified.

- One can easily check that the identities are $1_{(B,k)}=\mathbf{0}\to\{k\}$ and $1_{(A,k)}=\mathbf{1}\to\{k\}$, for all $k\in\{0,\ldots,n\}$.

Proposition 3.1. *The* 2*-category* $\mathbf{Adj}[n]$ *is well defined: its composition is associative.*

Proof. Let $(X,x),(Y,y),(Z,z),(W,w)\in|\mathbf{Adj}[n]|$ with $x\geq y\geq z\geq w$. Consider 1-cells $\rho:\mathbf{r}\to[w,z]$ $\mu:\mathbf{m}\to[z,y]$ $\pi:\mathbf{p}\to[y,x]$. Observe that $\rho\cdot(\mu\cdot\pi)$ is given by the composite

$$\mathbf{r}\cdot_{\mathbf{Adj}}(\mathbf{m}\cdot_{\mathbf{Adj}}\mathbf{p})\xrightarrow{(d^{r-1})^{\delta_{ZA}}}\mathbf{r}+\mathbf{m}\cdot_{\mathbf{Adj}}\mathbf{p}\xrightarrow{\rho+(\mu\cdot\pi)}[w,z]\cup[z,x]=[w,x].$$

which can be further decomposed into

$$\mathbf{r}\cdot_{\mathbf{Adj}}(\mathbf{m}\cdot_{\mathbf{Adj}}\mathbf{p})\xrightarrow{(d^{r-1})^{\delta_{ZA}}}\mathbf{r}+\mathbf{m}\cdot_{\mathbf{Adj}}\mathbf{p}\xrightarrow{(d^{r+m-1})^{\delta_{YA}}}\mathbf{r}+\mathbf{m}+\mathbf{p}\xrightarrow{\rho+\mu+\pi}[w,x].$$

On the other hand, $(\rho\cdot\mu)\cdot\pi$ is given by the composite

$$(\mathbf{r}\cdot_{\mathbf{Adj}}\mathbf{m})\cdot_{\mathbf{Adj}}\mathbf{p}\xrightarrow{(d^{r\cdot_{\mathbf{Adj}}m-1})^{\delta_{YA}}}(\mathbf{r}\cdot_{\mathbf{Adj}}\mathbf{m})+\mathbf{p}\xrightarrow{(\rho\cdot\mu)+\pi}[w,y]\cup[y,x]=[w,x].$$

which can be further decomposed in

$$(\mathbf{r}\cdot_{\mathbf{Adj}}\mathbf{m})\cdot_{\mathbf{Adj}}\mathbf{p}\xrightarrow{(d^{r\cdot_{\mathbf{Adj}}m-1})^{\delta_{YA}}}(\mathbf{r}\cdot_{\mathbf{Adj}}\mathbf{m})+\mathbf{p}\xrightarrow{(d^{r-1})^{\delta_{ZA}}}\mathbf{r}+\mathbf{m}+\mathbf{p}\xrightarrow{\rho+\mu+\pi}[w,x].$$

It is now an easy consequence of the cosimplicial identities that the composition is associative on 1-cells. It is associative on 2-cells because the composition of **Adj** is also associative. ☐

3.1.2 The simplicial model

We provide now a simplicial category isomorphic to $N_*\mathbf{Adj}[n]$ closely related to the simplicial model of **Adj** (see Section 2.3.2). The set of object is the same as in the 2-categorical model, that is $|\mathbf{Adj}[n]| = |\mathbf{Adj}| \times \{0, \dots, n\}$. We turn our attention to m-morphisms. Recall that m-morphisms of the simplicial category **Adj** are strictly undulating squiggle on $m + 1$ lines. We are going to decorate a strictly undulating squiggle on $m + 1$ lines so as to encode the extra data an m-morphism of $\mathbf{Adj}[n]$ should have.

Definition 3.2. Let v be a strictly undulating squiggle on $m + 1$ lines. The *i-th strip* of the strictly undulating squiggle is the subspace

$$\mathrm{Strip}(v, i) = \left\{ (x_1, x_2) \in \left[0, \frac{w(v)}{2}\right] \times [0, 1] : \begin{array}{l} J(l_i, x_1) < x_2 < f(x_1), \\ x_2 \leq J(l_{i-1}, x_1) \end{array} \right\}$$

of its standard representation $R(v) = (f, (l_i))$ in $\mathbb{A}\mathbf{dj}$ (see Equation (2.6)).

It is can be alternatively be described as $X_i(R(v)) \setminus X_{i-1}(R(v))$. It is the set of points of the square which are under the graph of the function f, above line l_i but under line l_{i-1}. It is straightforward to check that $\pi_0(\mathrm{Strip}(v, i)) = \pi_0 X_i(R(v))$. Indeed, if $(e_1, e_2) \in X_{i-1}(R(v))$ and h is chosen such that $h \in]l_i(e_1), l_{i-1}(e_1)]$, $(e_1, h) \in \mathrm{Strip}(w, i)$ and the vertical segment $[(e_1, e_2), (e_1, h)]$ lies in $X_i(R(v))$.

Thus, the partial order on $\mathrm{Strip}(v, i)$ given by $(x_1, x_2) \leq (t_1, t_2)$ if and only if $x_1 \leq t_1$, equips the quotient $\pi_0(\mathrm{Strip}(v, i))$ with a total order, as in Section 2.3.3. Recall that for two integers $y \leq x$, we write $[y, x]$ for the set $[y, x] = \{y, y+1, \dots x\}$, which is naturally ordered.

Definition 3.3. For $y, x \in \{0, \dots, n\}$, an m-morphism $\underline{v} : (X, x) \to (Y, y)$ exists in $\mathbf{Adj}[n]$ if and only if $y \leq x$. Such an m-morphism $\underline{v}$ is constituted of a strictly undulating squiggle $v = (v_0, \dots, v_r) \in W_m$, together with colorings of the connected components of its strips $c_i : \pi_0\mathrm{Strip}(v, i) \to [y, x]$ for $i = 0, \dots m$, such that the following conditions are satisfied.

(i) For all $i = 0, \dots m$, the coloring c_i is non-decreasing.

(ii) If $X = A$ then, for all $i = 0, \dots m$, $c_i(\max \pi_0(\mathrm{Strip}(v, i)) = x$.

(iii) For all $i = 0, \dots m - 1$, if $C \in \pi_0\mathrm{Strip}(v, i)$ shares a boundary with $C' \in \pi_0\mathrm{Strip}(v, i + 1)$, $c_{i+1}(Y) \geq c_i(X)$.

Such an m-morphism will often be referred to as a *colored squiggle on* $m + 1$ *lines*. To notationally distinguish between an m-morphism of $\mathbf{Adj}[n]$ and its underlying squiggle, we write $\underline{v}$ for the colored squiggle and v for its underlying squiggle.

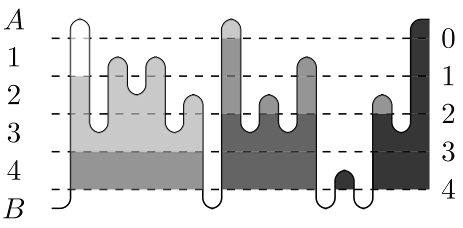


Figure 3.1: A 4-morphism from $(A,4)$ to $(B,0)$.

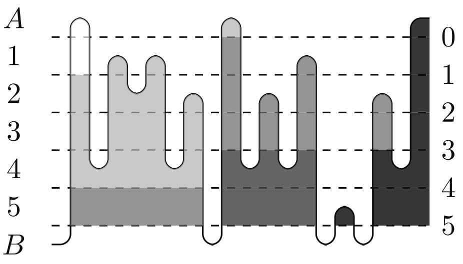


Figure 3.2: s_2 applied to the colored squiggle of Figure 3.1.

For instance, if we assign colors linearly so that 0 corresponds to white and n to black, one can picture such a morphism of **Adj**$[n]$ as in Figure 3.1. Let $\underline{v} : (X,x) \to (Y,y)$ be an m-morphism of **Adj**$[n]$ and let us describe how the colorings behave with respect to the simplicial structure.

- Remark that

$$\pi_0\mathrm{Strip}(s_i v, j) = \begin{cases} \pi_0\mathrm{Strip}(v,j) & j \leq i \\ \pi_0\mathrm{Strip}(v, j-1) & j \geq i+1 \end{cases} . \tag{3.1}$$

  The colorings of $s_i\underline{v}$ are induced by the colorings of $\underline{v}$ and Equation (3.1). An example can be found in Figure 3.2.

- Remark that

$$\pi_0\mathrm{Strip}(d_i v, j) = \begin{cases} \pi_0\mathrm{Strip}(v,j) & j < i \\ \pi_0\mathrm{Strip}(v, j+1) & j \geq i \end{cases} . \tag{3.2}$$

  The colorings of $d_i\underline{v}$ are induced by the colorings of $\underline{v}$ and Equation (3.2). An example can be found in Figure 3.3.

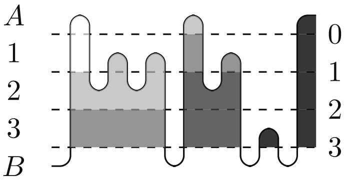


Figure 3.3: d_2 applied to the colored squiggle of Figure 3.1.

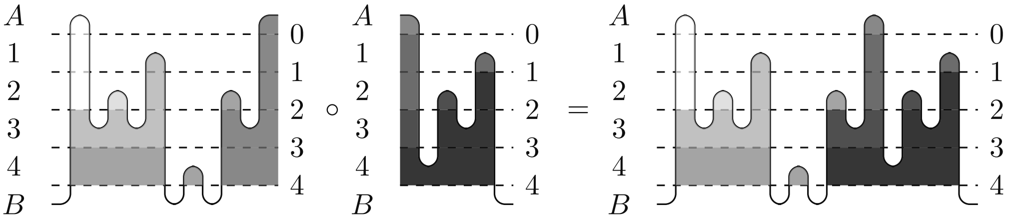


Figure 3.4: Composition at A

Finally, let $\underline{v'} : (Z, z) \to (X, x)$ be another m-morphism and let us describe the behavior of the colorings with respect to the composition of two colored squiggles.

- When $X = B$,

$$\pi_0\text{Strip}(v \circ_m v', j) = \pi_0\text{Strip}(v, j) \sqcup \pi_0\text{Strip}(v', j) \tag{3.3}$$

  for all j and the coloring of the composition is induced from the colorings of $\underline{v}$ and $\underline{v'}$.

- When $X = A$,

$$\pi_0\text{Strip}(v \circ_m v', j) = \pi_0\text{Strip}(v, j) \setminus \{\max \pi_0\text{Strip}(v, j)\} \sqcup \pi_0\text{Strip}(v', j), \tag{3.4}$$

  and the coloring of the composition is induced from the colorings of $\underline{v}$ and $\underline{v'}$. The maximal component of $\pi_0\text{Strip}(v, j)$ is merged with the minimal component of $\pi_0\text{Strip}(v', j)$ in $\pi_0\text{Strip}(v \circ_m v', j)$ and we choose to remember the color given by $\underline{v'}$. The heuristic reason for this choice is that the color of $\max \pi_0\text{Strip}(v, j)$ is x by definition. It does not encode any information and thus can be discarded. Such a composition is illustrated in Figure 3.4.

Notation 3.4. Let $\underline{v} : (X, x) \to (Y, y)$ be an m-morphism of $\mathbf{Adj}[n]$ and $v = (v_0, \ldots, v_r)$ its underlying squiggle. The *width* of $\underline{v}$ is $w(\underline{v}) = r$.

Remark 3.5. If $\underline{u}, \underline{v}$ are two composable m-morphisms, $w(\underline{u} \circ_m \underline{v}) = w(\underline{u}) + w(\underline{v})$.

Notation 3.6. Let us name the 0 and 1 morphisms corresponding to the adjunction in level i. Since they are morphisms $(X, i) \to (Y, i)$, the corresponding colorings are monochromatic, and thus we only need to specify the underlying squiggle.

- The left adjoint $(A, B) : (B, i) \to (A, i)$ is denoted l_i.
- The right adjoint $(B, A) : (A, i) \to (B, i)$ is denoted r_i.
- The unit of the adjunction $l_i \dashv r_i$ is $\eta_i = (B, 1, B)$.
- The counit of the adjunction $l_i \dashv r_i$ is $\epsilon_i = (A, 1, A)$.

Let us describe the 0-morphisms which constitute the adjunction morphisms.

- $(A) : (A, i) \to (A, i-1)$ with its only possible coloring is denoted a_i.
- $(B) : (B, i) \to (B, i-1)$ with the empty coloring is denoted b_i.

By using the same ideas as in the proof of the isomorphism between the two models of **Adj** provided in Section 2.3.3, it is not difficult to extend the discussion of 2.3.3 to understand that the simplicial and categorical models of $\mathbf{Adj}[n]$ do match. Indeed, if $\underline{v} : (X, x) \to (Y, y)$ is a colored squiggle on $m + 1$ lines, it corresponds to a diagram

$$\begin{tikzcd}(\mathbf{l_0}, \lambda_0) \arrow[r, "f_1"] & \dots \arrow[r, "f_m"] & (\mathbf{l_m}, \lambda_m)\end{tikzcd}$$

where the i-th object is (isomorphic to) $c_i : \pi_0\mathrm{Strip}(v, i) \to [y, x]$ and the j-th map corresponds to $\pi_0\mathrm{Strip}(v, j-1) = \pi_0 X_{j-1}(v) \to \pi_0 X_j(v) = \pi_0\mathrm{Strip}(v, j)$. We leave the details to the reader, and turn our attention to $\mathbf{Adj}^S_{\mathrm{hc}}[n]$.

3.2 Description of $\mathbf{Adj}^S_{\mathbf{hc}}[n]$

3.2.1 The simplicial and 2-categorical models

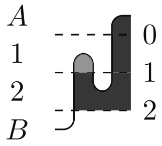
Definition 3.7. Let $\underline{v}$ be a colored squiggle $(X, x) \to (Y, y)$. A connected component of a strip of $\underline{v}$ is said to be *encoding* if $X \neq A$ or it is not the maximal connected component of its strip.

Said differently, a connected component is *not* encoding if and only if it contains a portion of the right boundary of the square.

Definition 3.8. Let $\underline{v} \in \mathbf{Adj}[n]$ be a colored squiggle $(X, x) \to (Y, y)$ on $m+1$ lines and $j < k \in (y, x]$. We say that a connected component C of the i-th strip of $\underline{v}$ *witnesses the separation of* j *and* k *at* i if and only if its coloring $c_i(C)$ satisfies $j \leq c_i(C) < k$. We say that j and k are *separated by* $\underline{v}$ *at* i if there is a witness of the separation of j and k at i in $\underline{v}$.

Observe that non-encoding components can never witness the separation of j and k, since their coloring is maximal by definition. For instance, the following colored squiggle

$\underline{v} =$



$: (A, 2) \to (A, 0)$

separates 1 and 2 at 1 but does not separate them at 0 or at 2 (the color of the grey region is 1, whereas the color of the black regions is 2).

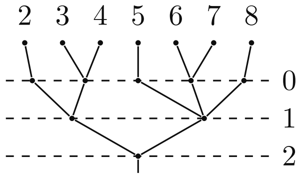
Proposition 3.9. *Let* $\underline{v} \in \mathbf{Adj}[n]_m$ *be a colored squiggle and* $\phi : [l] \to [m]$ *a simplicial operator. There is a witness of the separation of* j *and* k *at* i *in* $\phi^*(\underline{v}) \in \mathbf{Adj}[n]_l$ *if and only if there is a witness of the separation of* j *and* k *at* $\phi(i)$ *in* $\underline{v}$*.*

Proof. Use equations (3.2) and (3.1) to get the result for faces and degeneracies, and remark that the simplicial operators verifying the proposition are closed under composition. □

Corollary 3.10. *Let* $\underline{v} \in \mathbf{Adj}[n]_m$ *be a colored squiggle and* $\phi : [l] \to [m]$ *a simplicial operator. If* $\phi^*(\underline{v})$ *separates* j *and* k *at* i*, then* $\underline{v}$ *separates* j *and* k *at* $\phi(i)$*.*

Definition 3.11. Let $j < k \in (y, x]$ with $x, y \in \{0, \ldots, n\}$, $\underline{f} \in \mathcal{C}\Delta[n]^{\mathrm{co}}(x, y)_m$ and $0 \leq i \leq m$. We say that j and k are separated at i in $\underline{f}$ if either the leaves labeled j and k belongs to distinct trees, or if they belong to the same tree and the unique path going from one to the other crosses the dotted line $i+1$.

For instance, the 2-simplex of $\mathcal{C}\Delta[15](8, 1)^{\mathrm{co}}$ pictured below separates 4 and 5 at 0 and 1 but not at 2.



Proposition 3.12. *Let* $j < k \in [y, x]$ *with* $x, y \in \{0, \ldots, n\}$*,* $\underline{f} \in \mathcal{C}\Delta[n]^{\mathrm{co}}(x, y)_m$ *and* $\phi : [l] \to [m]$ *a simplicial operator. If* $\underline{f}$ *separates* j *and* k *at* $\phi(i)$*,* $\phi^*(\underline{f})$ *separates* j *and* k *at* i*.*

Proof. Remark that $\underline{f} \in C\Delta[n]^{\mathrm{co}}(x,y)$ separates j and k at i if and only if the i-th object $f_i : (y,x] \to \mathbf{m}$ of the corresponding element of $N(\tilde{P}_{yx})$ is such that $f_i(j) \neq f_i(k)$. Observe that the i-th object of $\phi^* \underline{f}$ is exactly the $\phi(i)$-th object of $\underline{f}$. The proposition follows. ☐

Definition 3.13. Let $\underline{v} \in \mathbf{Adj}[n]_m$ be a colored squiggle $(X,x) \to (Y,y)$, and $\underline{f} \in \mathcal{C}\Delta[n]^{\mathrm{co}}(x,y)_m$ a forest of trees. We say that $\underline{f}$ is *compatible* with $\underline{v}$ if for all pairs $j < k \in (y,x]$, if $\underline{v}$ separates j and k at i, so does $\underline{f}$.

Definition 3.14. We define a simplicial category $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ as follows.

- Its set of object is given by
$$|\mathbf{Adj}_{\mathrm{hc}}[n]| = |\mathbf{Adj}| \times |S|.$$
- Its hom-spaces are given by
$$\mathbf{Adj}_{\mathrm{hc}}[n]((X,x),(Y,y)) = \left\{ (\underline{v},\underline{f}) : \begin{array}{l} \underline{v} \in \mathbf{Adj}[n]((X,x),(Y,y)), \\ \underline{v} \text{ has colors in } |S|, \\ \underline{f} \in S(x,y), \\ \underline{f} \text{ is compatible with } \underline{v}. \end{array} \right\}.$$

Simplicial structure and composition are inherited from those of $\mathbf{Adj}[n]$ and $S \subseteq \mathcal{C}\Delta[n]^{\mathrm{co}}$. We will write $\mathbf{Adj}_{\mathrm{hc}}[n]$ for $\mathbf{Adj}^{\mathcal{C}\Delta[n]^{\mathrm{co}}}_{\mathrm{hc}}[n]$.

The hom-simplicial sets $\mathbf{Adj}^S_{\mathrm{hc}}[n]((X,x),(Y,y))$ are indeed subsimplical sets of $\mathbf{Adj}[n]((X,x),(Y,y)) \times S(x,y)$, by Corollary 3.10 and Proposition 3.12. It is clear that the composition is well defined, since the compatibility condition is only to be checked within each tree of the forest. It is even a 2-subcategory, as the following proposition states.

Proposition 3.15. *The simplicial category $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is a 2-category.*

Proof. Remark that $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is the 2-subcategory of $\mathbf{Adj}[n] \times S$ whose hom-category $\mathbf{Adj}^S_{\mathrm{hc}}[n]((X,x),(Y,y))$ is the full subcategory of $\mathbf{Adj}[n]((X,x),(Y,y)) \times S(x,y)$ generated by the objects consisting of pairs $(f : \mathbf{k} \to [y,x], p : (y,x] \twoheadrightarrow \mathbf{m})$ for which

- $f(\mathbf{k}) \subseteq |S|$;
- if $f^{-1}(i) \neq \emptyset$, then $p(y,i] \cap p[i+1,x] = \emptyset$.

☐

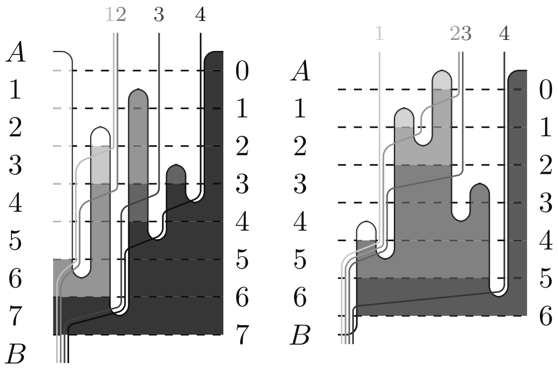


Figure 3.5: Morphisms of $\mathbf{Adj}[n]$

Now that the combinatorics of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is rigorously defined, we will give a representation of the combined morphism $(\underline{v}, \underline{f})$. Hopefully, this will make more natural the condition of compatibility between $\underline{v}$ and $\underline{f}$, and a more visual way to think about such an m-arrow.

Remark that given a colored squiggle $\underline{v} \in \mathbf{Adj}[n]$, one can draw color separation lines, as pictured in Figure 3.5. The line separating color $i-1$ from color i is drawn with color i. The set of color separation lines determines the colorings in the following way. A connected component which is totally on the left of the color separation line i has coloring strictly smaller than i. A region which is partially on the right of the separation line i has color greater or equal to i, for $i = y+1, \dots, x$. The fact that such lines can be drawn without intersections is a consequence of the conditions on the colorings in the definition of $\mathbf{Adj}[n]$. Observe that two color separation lines can be drawn at the same place in the i-th strip of $\underline{v}$ if and only if they are not separated by $\underline{v}$ at i. Given an element $(\underline{v}, \underline{f}) \in \mathbf{Adj}[n]((X, x), (Y, y))$, the compatibility condition between $\underline{v}$ and $\underline{f}$ ensures that one can identify some colors separations lines along the way in the representation of $\underline{v}$ so as to represent $\underline{f}$ itself, as in Figure 3.6. It is easier to use this type of representation while dealing with the combinatorics of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$, and every argument can then be made completely rigorous using the description in Definition 3.14.

Observe that there is a canonical inclusion functor $\iota : \mathbf{Adj}[n] \to \mathbf{Adj}_{\mathrm{hc}}[n]$. Indeed, given a colored squiggle on $m+1$ lines $\underline{v} : (X, x) \to (Y, y)$, consider $\underline{f} : x \to y$ the forest of linear trees of height $m+1$ with leaves labeled from $y+1$ to x. Define $\iota(\underline{v}) = (\underline{v}, \underline{f})$. Observe that in particular when $x = y$, $\underline{f}$ is the empty tree. As a consequence, we abuse notation and denote also by l_i, r_i, η_i and ϵ_i respectively $\iota(l_i), \iota(r_i), \iota(\eta_i)$ and $\iota(\epsilon_i)$.

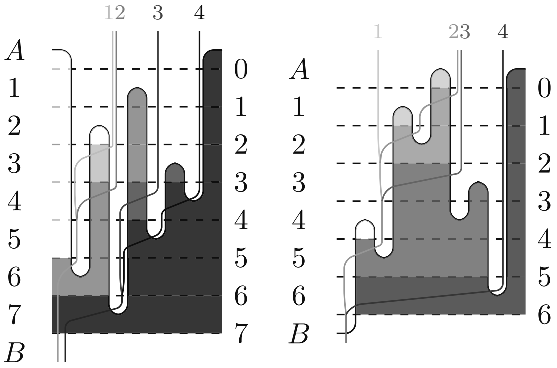


Figure 3.6: Morphisms of $\mathbf{Adj}_{\mathrm{hc}}[n]$

3.2.2 Non-degenerate morphisms of $\mathbf{Adj}^S_{\mathbf{hc}}[n]$

Definition 3.16. Let $\underline{v} \in \mathbf{Adj}[n]((X,x),(Y,y))$ be a colored squiggle on $m+1$ lines, $0 \leq k \leq m$ and $\tilde{c}_k : \pi_0\mathrm{Strip}(\underline{v},k) \to [y,x]$ be any coloring of its k-th strip. The coloring $\tilde{c}_k$ is said to be *admissible* if the strictly undulating squiggle with colorings $\underline{\tilde{v}}$, obtained from $\underline{v}$ by replacing its k-th coloring by $\tilde{c}_k$, is still a colored squiggle, which means it satisfies conditions (i), (ii) and (iii) of Definition 3.3.

We now provide a criterion to check if an m-morphism of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is non-degenerate.

Proposition 3.17. *Let $0 \leq k < m$ An m-morphism $(\underline{v},\underline{f}) : (X,x) \to (Y,y)$ of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is not degenerate at k if and only if one of the following condition holds.*

- *There is $0 < l < w(\underline{v})$ such that $v_l = k+1$.*
- *The coloring of the $k+1$-st strip of $\underline{v}$ is not minimal among admissible colorings of the $k+1$-st strip of $\underline{v}$.*
- *The forest $\underline{f}$ is not degenerate at k, that is there is a non-trivial branching on line $k+1$.*

Proof. Observe that a m-morphism $(\underline{v},\underline{f})$ is degenerate at k if and only if both $\underline{v}$ and $\underline{f}$ are. Thus, we have to show that $\underline{v}$ is not degenerate at k if and only if one of the first two condition is met.

Let

$$(\mathbf{l_0},\lambda_0) \xrightarrow{g_1} \dots \xrightarrow{g_m} (\mathbf{l_m},\lambda_m)$$

be the element of the nerve of the appropriate hom-category of the 2-categorical description of $\mathbf{Adj}[n]$ that $\underline{v}$ corresponds to. The map g_{k+1} is not an identity precisely if is not the identity map $\mathbf{l_k} \to \mathbf{l_{k+1}}$ or if it is the identity map but $\lambda_k < \lambda_{k+1}$.

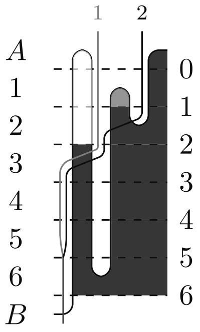
An ordered map is an identity if and only if it is a bijection. Observe that if there is a turn v_l in the $k+1$-st row, the map g_{k+1} is not surjective if $v_{l-1} > v_l$ and not injective if $v_{l-1} < v_l$. Thus, g_{k+1} is not the identity map $\mathbf{l_k} \to \mathbf{l_{k+1}}$ if and only if there is $0 < l < w(\underline{v})$ such that $v_l = k+1$.

Finally, suppose that there are no turns in the $k+1$-st row. The coloring of the $k+1$-st strip of $\underline{v}$ is not minimal among admissible colorings of the $k+1$-st strip of $\underline{v}$, if and only if $\lambda_k < \lambda_{k+1}$. □

We illustrate the previous proposition by a detailed example.

Example 3.18. Let us consider the 6-morphism

$(\underline{v}, \underline{f}) =$



$: (A, 2) \to (B, 0)$

Observe that

- it is not degenerate at 0, since $v_3 = 1$;
- it is not degenerate at 1, since $v_4 = 2$;
- it is not degenerate at 2, since the 3-rd coloring is not minimal among admissible colorings of the 3-rd strip (above the dotted line labeled 3).
- it is degenerate at 3;
- it is not degenerate at 4, since there is a branching of the tree $\underline{f}$ on line 5;
- it is not degenerate at 5, since $v_2 = 6$.

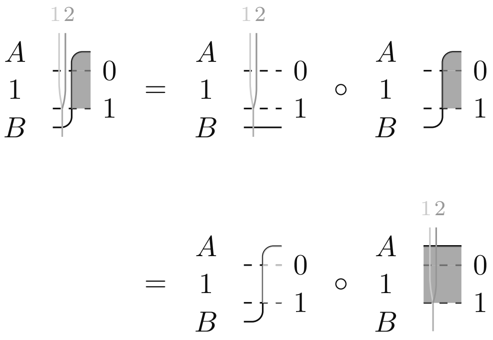


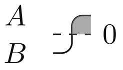
Figure 3.7: $(B \leftarrow A) \times S$ is not a simplicial computad

3.2.3 Atomic morphisms of $\mathbf{Adj}^S_{\mathbf{hc}}[n]$

We consider the simplicial subcategory of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ consisting of the m-morphisms $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ such that all connected components of strips of $\underline{v}$ are not encoding. Remark that such colored squiggles $\underline{v} : (X, x) \to (Y, y)$ are always degenerations of exactly one of the following 0-morphisms.

- 

$: (A, x) \to (A, y);$

- 

$: (A, x) \to (B, y);$

- 

$: (B, x) \to (B, y).$

Remark that any $\underline{f}$ is compatible with such a $\underline{v}$, because no pairs are ever separated in it. As a consequence, this simplicial subcategory is exactly $(B \leftarrow A) \times S$, where $B \leftarrow A$ denotes the category with two objects A and B and a unique non identity arrow $B \leftarrow A$. Remark that $(B \leftarrow A) \times S$ is not a simplicial computad. In Figure 3.7, we exhibit a 1-morphism with two distinct decompositions in atomic morphisms.

This is the only obstruction to $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ being a simplicial computad. Indeed, we are going to show that the inclusion $(B \leftarrow A) \times S \to \mathbf{Adj}^S_{\mathrm{hc}}[n]$ is a relative simplicial computad. To do so, we start by characterizing the atomic m-arrows of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$.

Definition 3.19. Let $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ be an m-morphism of $\mathbf{Adj}^S_{\mathbf{hc}}[n]$ which is not in $(B \leftarrow A) \times S$. Let $W \in \{A, B\}$ and $0 < k < w(\underline{v})$. We say that *the underlying squiggle of $\underline{v}$ passes by W at k between two forests which belong to S* if

- The letter v_k is W in the underlying squiggle of $(\underline{v}, \underline{f})$.
- $\underline{f}$ decomposes as $x \xrightarrow{f_1} w \xrightarrow{f_2} y$ in S such that
  - (i) All connected components of strips of $\underline{v}$ which lies entirely to the left of the turn v_k, have colors in $[y, w]$;
  - (ii) All connected components of $\underline{v}$ which lies partially to the right of the turn v_k have colors in $[w, x]$.

Proposition 3.20. *Let $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ be an m-morphism of $\mathbf{Adj}^S_{\mathbf{hc}}[n]$ which is not in $(B \leftarrow A) \times S$. It is atomic if and only if the following three conditions are satisfied.*

(i) Its underlying squiggle does not pass by A or B between two forests which belong to S.

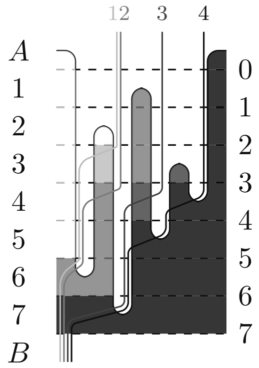
(ii) If x' is the label of the minimal leaf of the atomic forest (in S) containing the leaf labeled x in $\underline{f}$, there is a connected component of one of the strips of $\underline{v}$ which is encoding and with color greater or equal to x'.

(iii) If y' is the label of the maximal leaf of the atomic forest (in S) containing the leaf labeled $y+1$ in $\underline{f}$, there is a connected component of one of the strips of $\underline{v}$ which is encoding and with color strictly smaller to y'.

Examples 3.21. Before proving the proposition, we illustrate the criteria by providing examples. We let $n = 4$ and $S = \mathcal{C}\Delta[4]^{\mathrm{co}}$.

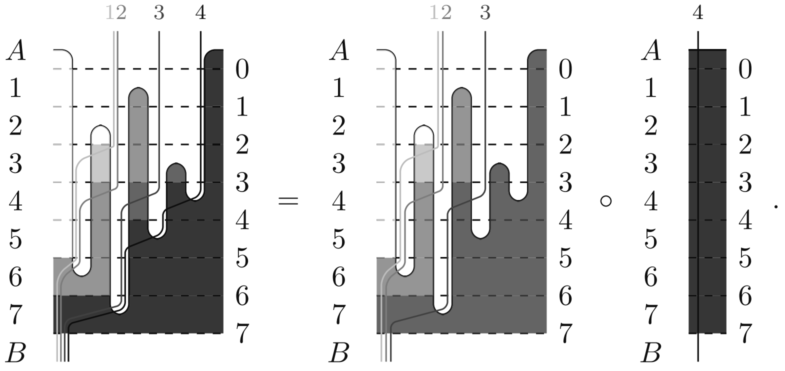
(i) We consider the morphism

$(\underline{v}, \underline{f}) =$



$: (A, 4) \to (A, 0).$

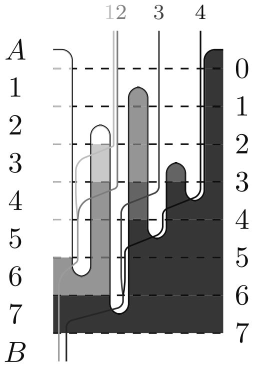
Let us check if $(\underline{v}, \underline{f})$ is atomic using the criteria of Proposition 3.20. It is clear from the underlying squiggle that condition (i) is met. The atomic forest containing the leaf labeled 4 is a linear tree, with only one leaf. Its minimal leaf is thus also labeled 4 and there are no encoding components with color greater or equal to 4 (black). As a consequence, $(\underline{v}, \underline{f})$ is not atomic. Indeed,



Observe that the condition (iii) is satisfied, since the atomic forest containing 1 is a linear tree, its maximal leaf is also 1 and there is an encoding connected component with color 0 (white). Thus, the morphism appearing on the left hand side in the decomposition of $(\underline{v}, \underline{f})$ above is now atomic.

(ii) We consider the morphism

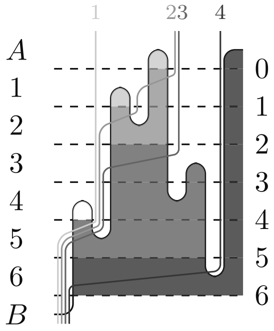
$(\underline{v}, \underline{f}) =$



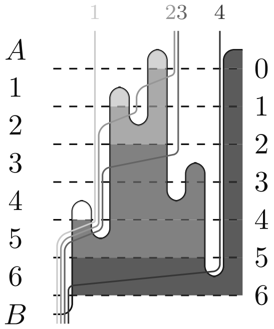
$: (A, 4) \to (A, 0).$

Let us check if $(\underline{v}, \underline{f})$ is atomic using the criteria of Proposition 3.20. It is clear from the previous example that the conditions (i), (iii) of the proposition are satisfied. The atomic forest containing the leaf labeled 4 has two leaves, its minimal leaf is labeled 3 and there is an encoding component with color 3. Thus, $(\underline{v}, \underline{f})$ is atomic.

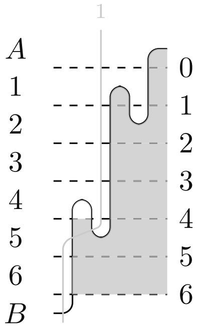
(iii) We consider

$(\underline{v}, \underline{f}) =$  $: (A, 4) \to (B, 0).$

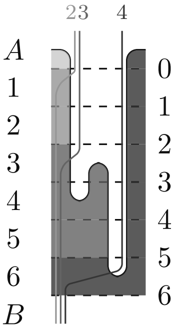
The underlying squiggle does contain a letter $v_5 = A$, and $\underline{f}$ can be decomposed as the composite of the first two linear trees followed by the last two. Moreover, all connected components which lie entirely to the left of the turn corresponding to v_5 have colors smaller or equal to 1, whereas all connected components appearing partially to the right of that turn have colors greater or equal to 1. As a consequence, $(\underline{v}, \underline{f})$ is not atomic. Indeed, it decomposes as



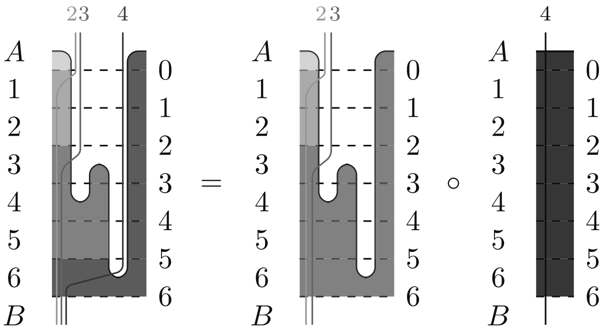
$=$



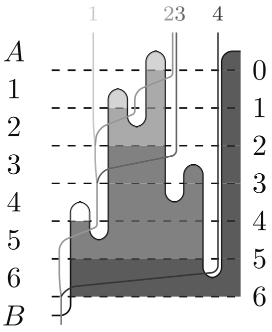
$\circ$



The morphism appearing on the left in the above decomposition is now atomic. Its underlying squiggle does not contain an inner letter A or B, and it contains encoding components with color 0 and 1. The right morphism is not atomic. It does not contain an encoding connected component with color 4. Thus, condition (ii) fails and it can be further decomposed as

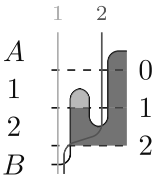


(iv) We consider

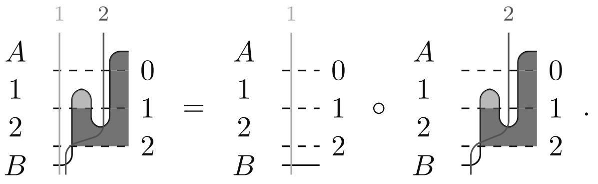
$(\underline{v}, \underline{f}) =$  $: (A, 4) \to (A, 0).$

We claim that the underlying squiggle no longer passes by A between two forests of S. Indeed, $\underline{f}$ is atomic and there is a connected component entirely to the left of the turn with color strictly greater than 0 and a connected component partially to the right of the turn with color strictly smaller than 4. Moreover, it satisfies conditions (ii) and (iii). Indeed, there is an encoding connected component with color 3 and $1 \leq 3 < 4$. Thus, $(\underline{v}, \underline{f})$ is atomic.

Example 3.22. Let $n = 2$ and S be the full subcategory of $\mathcal{C}\Delta[2]^{\text{co}}$ generated by the objects 2 and 0. Consider the morphism

$(\underline{v}, \underline{f}) =$  $: (A, 2) \to (B, 0).$

Observe that $(\underline{v}, \underline{f})$ is atomic in $\mathbf{Adj}^S_{\mathrm{hc}}[2]$, since $\underline{f}$ is atomic in S. However, it is not atomic in $\mathbf{Adj}_{\mathrm{hc}}[2]$ since in this simplicial category it can be written as the composite



This shows that atomicity of an m-morphism $(\underline{v}, \underline{f})$ does depend on the ambient simplicial category $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ it leaves in.

We start the proof of Proposition 3.20 by providing a lemma which clarifies the role of the first condition.

Lemma 3.23. *Let* $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ *be an* m*-morphism of* $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ *which is not in* $(B \leftarrow A) \times S$ *and* $0 < k < w(\underline{v})$*. The following are equivalent*

(i) The m*-morphism* $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ *can be written as a composite*

$$(X, x) \xrightarrow{(\underline{v_1}, \underline{f_1})} (W, w) \xrightarrow{(\underline{v_2}, \underline{f_2})} (Y, y) ,$$

with $w(\underline{v_2}) = k$*.*

(ii) The underlying squiggle of $(\underline{v}, \underline{f})$ *passes by* W *at* k *between two forests of* S*.*

Proof. Suppose $(\underline{v}, \underline{f})$ decomposes as in (i). Let $k = w(\underline{v_2})$.

- The connected components of $\underline{v}$ which lies entirely to the left of the turn $v_k = W$ are coming from $\underline{v_2}$, and thus have colors belonging to $[y, w]$;
- The connected components of $\underline{v}$ which lies partially to the right of the turn $v_k = W$ are coming from $\underline{v_1}$, and thus have colors belonging to $[w, x]$.

To conclude, observe that the set of indices of leaves of $\underline{f_2}$ is $[y + 1, w]$.

Conversely, suppose that the underlying squiggle of $(\underline{v}, \underline{f})$ passes by W at k between two forests which belong to S, and let $(t_0, \ldots, t_j) \cdot (u_0, \ldots, u_k)$ be the decomposition of $\underline{v}$ by breaking apart at the letter $v_k = W$ given by Definition 3.19, where

$$x \xrightarrow{\underline{f_1}} w \xrightarrow{\underline{f_2}} y ,$$

is the corresponding decomposition of $\underline{f}$. Remark that by Equations (3.3) and (3.4), the coloring of $\underline{v}$ determines colorings for $(t_0, \ldots, t_j)$ and $(u_0, \ldots, u_k)$. By the conditions on the colors of connected components of $\underline{v}$ on the left and right of the turning point of Definition 3.19, we get colored squiggles $\underline{t} : (W, w) \to (Y, y)$ and $\underline{u} : (X, x) \to (W, w)$. Then, $(\underline{v}, \underline{f}) = (\underline{t}, \underline{f_2}) \circ (\underline{u}, \underline{f_1})$. ☐

Proof of Proposition 3.20. We proceed by contraposition. We first suppose that $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ is not atomic, and let

$$(X, x) \xrightarrow{(\underline{v_1}, \underline{f_1})} (Z, z) \xrightarrow{(\underline{v_2}, \underline{f_2})} (Y, y)$$

be a decomposition where both morphisms are not identities. One of the following three cases holds.

(i) The underlying squiggles of both $\underline{v_1}$ and $\underline{v_2}$ are not identities. By Lemma 3.23, the underlying squiggle of $(\underline{v}, \underline{f})$ passes by Z between two forests of S.

(ii) $\underline{v_1}$ is a degeneracy of $b_{z+1} \cdots b_x$ or of $a_{z+1} \cdots a_x$, and $z < x$ (recall Notation 3.6). The atomic forest containing x has smallest index strictly bigger than z, and all encoding connected components of strips of $\underline{v}$ are coming from $\underline{v_2}$, and thus their colorings are smaller than z. As a consequence, the second condition of the proposition is not satisfied.

(iii) $\underline{v_2}$ is a degeneracy of $b_{y+1} \cdots b_z$ or $a_{y+1} \cdots a_z$ with $y < z$. The maximal leaf of the atomic forest containing $y+1$ in $\underline{f}$ is at most z. All encoding connected component of a strip of $\underline{v}$ comes from $\underline{v_1}$, and thus has color greater or equal to z. Thus, the third condition is not satisfied.

Let us prove the converse implication. By Lemma 3.23, it is enough to show that the negation of the second and third conditions implies that $(\underline{v}, \underline{f})$ is not atomic. We prove it for the second condition, the third one being completely similar. Let $\underline{g}$ be the atomic component of $\underline{f}$ containing x, $\underline{h}$ be the composite of all others atomic components of $\underline{f}$ and $\tilde{x}$ the maximal index of a leaf in $\underline{h}$. Then $(\underline{v}, \underline{f}) = (\underline{\tilde{v}}, \underline{h}) \cdot ((X), \underline{g})$, where $\underline{\tilde{v}} : (X, \tilde{x}) \to (Y, y)$ is obtained from $\underline{v}$ by changing the color of non encoding connected components of strips to $\tilde{x}$. The arrow $(\underline{\tilde{v}}, \underline{h})$ is not an identity since $\underline{\tilde{v}}$ contains an encoding connected component, and $\underline{g}$ is a non-trivial forest. ☐

Lemma 3.24. *Let $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ be an m-morphism of $\mathbf{Adj}_{\mathrm{hc}}^{S}[n]$ that decomposes as*

$$(X, x) \xrightarrow{(\underline{v_1}, \underline{f_1})} (Q, q) \xrightarrow{(\underline{v_2}, \underline{f_2})} (Y, y) .$$

Suppose that the underlying squiggle of $\underline{v}$ passes also by Z at k between two forests of S with $k < w(\underline{v_2})$.

The underlying squiggle of $\underline{v_2}$ passes by Z at k between two forests of S.

Proof. Let

$$(X,x) \xrightarrow{(\underline{u_1},\underline{g_1})} (Z,z) \xrightarrow{(\underline{u_2},\underline{g_2})} (Y,y)\ ,$$

with $w(\underline{u_2}) = k$, be the decomposition of $(\underline{v},\underline{f})$ given by Lemma 3.23. Since S is a simplicial computad, observe that one of the following two cases holds.

(i) There exists $\underline{\tilde{g}_1}$ such that $\underline{g_2} \cdot \underline{\tilde{g}_1} = \underline{f_2}$. This decomposition of $\underline{f_2}$ shows that the underlying squiggle of $\underline{v_2}$ passes by Z at k between two forests of S.

(ii) There exists $\underline{\tilde{g}_2} \neq \mathrm{id}$ such that $\underline{\tilde{g}_2} \cdot \underline{g_1} = \underline{f_1}$. This implies that $z > q$ and thus $\underline{v}$ has no encoding component completely to the left of the turn v_j and partially to the right of the turn v_k. Thus, $k = j-1$, $v_k = B$ and $v_j = A$. Observe that $\underline{f_2} = \underline{f_2} \cdot \mathrm{id}$ is a decomposition witnessing that the underlying squiggle of $\underline{v_2}$ passes by Z at k between two forests of S.

□

Proposition 3.25. *The simplicial functor $(B \leftarrow A) \times S \to \mathbf{Adj}^S_{\mathrm{hc}}[n]$ is a relative simplicial computad.*

Proof. We are going to show that this simplicial functor verifies the conditions of Lemma 2.8. Let $(\underline{v},\underline{f})$ be an m-morphism. Since the width of an m-morphism is finite, there are a finite number of connected components of strips and of color separating lines. By iterating the process of decomposing, the components will eventually become atomic or belong to $(B \leftarrow A) \times S$, thus existence is established.

Let $(\underline{v},\underline{f}) \in \mathbf{Adj}^S_{\mathrm{hc}}[n]((X,x),(Y,y))$, and let

$$\begin{tikzcd}
(X,x) \arrow[r, "c_{k+1}"] & (Z_k,z_k) \arrow[r, "(\underline{v_k},\underline{f_k})"] & (W_k,w_k) \arrow[r, "c_k"] & \dots \arrow[dll, rounded corners, "c_1"] \\
& (Z_0,z_0) \arrow[r, "(\underline{v_0},\underline{f_0})"] & (W_0,w_0) \arrow[r, "c_0"] & (Y,y)
\end{tikzcd}$$

be a decomposition of $(\underline{v},\underline{f})$ where $(\underline{v_i},\underline{f_i})$ is atomic and does not belong to $(B \leftarrow A) \times S$, and $c_i \in (B \leftarrow A) \times S$. We are going to show that this decomposition is uniquely determined by $(\underline{v},\underline{f})$ itself, by double induction, on $x-y$ and on $w(\underline{v})$. The first induction can start since **Adj** is a simplicial computad, whereas the second can always start since S is one.

Suppose that $(\underline{v},\underline{f}) : (X,x) \to (Y,y)$ has width $m > 0$. If it is atomic or belongs to $(B \leftarrow A) \times S$, there is nothing to prove. If $(\underline{v},\underline{f})$ is not atomic, it does not satisfy one of the conditions (i), (ii), (iii) of Proposition 3.20.

- If $(\underline{v}, \underline{f})$ does not satisfy (ii), let $\hat{\underline{f}} : x \to \tilde{x}$ be the composite of all atomic components of $\underline{f}$ with all leaves strictly greater than any color of an encoding component of a strip of $\underline{v}$, and $\check{\underline{f}} : \tilde{x} \to y$ such that $\underline{f} = \check{\underline{f}} \cdot \hat{\underline{f}}$. One of the following holds.

  $v_{w(\underline{v})} = B$ or ($v_{w(\underline{v})} = A$ and $v_{w(\underline{v})-1} \neq B$). Let $\tilde{\underline{v}}$ be the colored squiggle obtained from $\underline{v}$ by changing the color of all non-encoding connected components to $\tilde{x}$. Observe that $(\underline{v}, \underline{f}) = (\tilde{\underline{v}}, \check{\underline{f}}) \cdot ((X), \hat{\underline{f}})$ and $(\tilde{\underline{v}}, \check{\underline{f}})$ now satisfies (ii). Thus $c_{k+1} = ((X), \hat{\underline{f}})$. We can apply the induction hypothesis on $(\tilde{\underline{v}}, \check{\underline{f}})$ since $\tilde{x} - y < x - y$.

  $v_{w(\underline{v})} = A$ and $v_{w(\underline{v})-1} = B$. Let $\tilde{\underline{v}}$ be the unique colored squiggle such that $\underline{v} = \tilde{\underline{v}} \cdot r_x$. Observe that $(\underline{v}, \underline{f}) = (\tilde{\underline{v}}, \check{\underline{f}}) \cdot (r_x, \hat{\underline{f}})$ and $(\tilde{\underline{v}}, \check{\underline{f}})$ now satisfies (ii). Thus $c_{k+1} = (r_x, \hat{\underline{f}})$. We can apply the induction hypothesis on $(\tilde{\underline{v}}, \check{\underline{f}})$ to conclude, since $\tilde{x} - y < x - y$.

- If $(\underline{v}, \underline{f})$ does satisfy (ii) but does not satisfy (iii), we can apply the same procedure as in the first point to compute c_0 and conclude by induction on $x - y$. Indeed, the forest component of c_0 is the composite of all atomic components of $\underline{f}$ with all leaves smaller or equal to any color of an encoding component of a strip of $\underline{v}$, and its colored squiggle is either (A), (B) or (BA) depending on the first two letters of $\underline{v}$.

- Suppose that $(\underline{v}, \underline{f})$ does satisfy (ii) and (iii) but not (i). We already know from the two previous points that $c_{k+1} = \mathrm{id}$, $c_0 = \mathrm{id}$. Let j be the smallest integer such that the underlying squiggle of $(\underline{v}, \underline{f})$ passes by W at j between two forests of S. By Lemma 3.24, the underlying squiggle of $\underline{v_0}$ is $(v_0, \ldots, v_j)$. Let $\underline{u} = (v_j, \ldots v_{w(\underline{v})})$. Remark that by Equations (3.3) and (3.4), the coloring of $\underline{v}$ determines the colors of the non-encoding components of strips of $\underline{v_0}$ and of $\underline{u}$ such that $\underline{v_0} \cdot \underline{u} = \underline{v}$. Since $(\underline{v_0}, \underline{f_0})$ is atomic, by Proposition 3.20, $\underline{f_0} : \tilde{y} \to y$ is the composite of all atomic components of $\underline{f}$ that contain a leaf smaller or equal to the color of any encoding component of a strip of $\underline{v_0}$. Let $\tilde{\underline{f}} : x \to \tilde{y}$ be such that $\underline{f} = \underline{f_0} \cdot \tilde{\underline{f}}$. We can apply the induction hypothesis on $(\underline{u}, \tilde{\underline{f}})$ to conclude, since $w(\underline{u}) < w(\underline{v})$.

□

3.2.4 Convenient 2-subcategories of $\mathcal{C}\Delta[n]^{\mathrm{co}}$

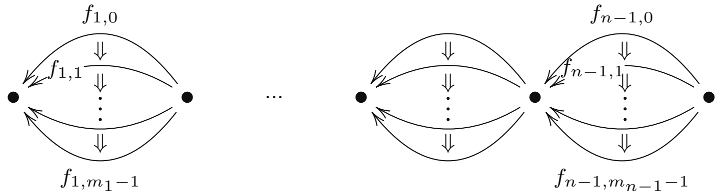
Definition 3.26. A 2-subcategory of $\mathcal{C}\Delta[n]^{\mathrm{co}}$ is said to be *convenient* if

(i) It is a simplicial computad;

(ii) If a forest $\underline{f} : x \to y$ is atomic in S and can be decomposed as $x \xrightarrow{\underline{f_1}} w \xrightarrow{\underline{f_2}} y$ in $\mathcal{C}\Delta[n]^{\mathrm{co}}$, then S does not contain the object w.

Remark that all 2-subcategories which also happen to be simplicial subcomputads are convenient. In particular, $[n]$ and $\mathcal{C}\Delta[n]^{\mathrm{co}}$ are convenient. It is also possible to encode globular shapes as follows.

Definition 3.27. Let $\theta = \mathbf{n}(\mathbf{m_1}, \dots, \mathbf{m_{n-1}})$ be the free 2-category generated by



with n objects and respectively $m_1, \dots, m_{n-1}$ 1-cells between two consecutive objects. The 2-subcategory $S_\theta \subseteq \mathcal{C}\Delta[m]^{\mathrm{co}}$, where $m = \sum_{i=1}^{n-1} m_i$, is defined as follows.

- Define $m^j = \sum_{i=1}^{j} m_i$. The object set is $|S_\theta| = \{m^j : 0 \leq j \leq n-1\}$.
- Hom-categories $S_\theta(x, y)$ are full subcategories of $\mathcal{C}\Delta[m]^{\mathrm{co}}(x, y)$ generated by 1-cells which are morphisms $p : (y, x] \twoheadrightarrow \mathbf{k}$ where
  (i) $|p^{-1}(r)| \neq 1 \Rightarrow p^{-1}(r) \cap |S_\theta| \neq \emptyset$, for all $r \in \mathbf{k}$.
  (ii) $p(r) \neq p(r+1)$ for all $r \in |S_\theta| \cap (y, x)$.

Proposition 3.28. *Let $\theta = \mathbf{n}(\mathbf{m_1}, \dots, \mathbf{m_{n-1}})$ be as in Definition 3.27. The 2-subcategory $S_\theta \subseteq \mathcal{C}\Delta[m]^{\mathrm{co}}$ is convenient. Moreover, $S_\theta \cong \theta$ as 2-categories.*

Proof. Let us show that S_θ is a simplicial computad. Let $\underline{f} \in S_\theta(x, y)$, and let

$$x \xrightarrow{\underline{f_1}} z_1 \xrightarrow{\underline{f_2}} z_2 \xrightarrow{\underline{f_3}} \dots \xrightarrow{\underline{f_k}} y$$

be its unique decomposition in $\mathcal{C}\Delta[m]^{\mathrm{co}}$. By condition (ii) of Definition 3.27, $|S_\theta| \cap (y, x) \subseteq \{z_1, \dots, z_{k-1}\}$. Let $(i_j)_{j=1}^{s}$ be the strictly increasing subsequence such that $|S_\theta| \cap (y, x) = \{z_{i_1}, \dots z_{i_s}\}$. Then, $\underline{f}$ is the composite

$$x \xrightarrow{\underline{f_{i_1}}\cdots\underline{f_1}} z_{i_1} \xrightarrow{\underline{f_{i_2}}\cdots\underline{f_{i_1+1}}} z_{i_2} \longrightarrow \dots \longrightarrow z_{i_s} \xrightarrow{\underline{f_k}\cdots\underline{f_{i_s+1}}} y \tag{3.5}$$

Observe that this decomposition lies in S_θ, and the arrows are atomic in S_θ because there are just no object to factor through in S_θ. This shows existence, uniqueness can be can be deduced from the uniqueness in the decomposition in $\mathcal{C}\Delta[m]^{\mathrm{co}}$. Remark that the decomposition (3.5) also shows that atomic arrows $x \to y$ in S_θ are atomic precisely if $(y, x) \cap |S_\theta| = \emptyset$, thus, it is convenient. We are now going to show the isomorphism of 2-categories by showing that S_θ satisfies the universal property of θ. Let $\mathscr{C}$ be a 2-category. Since 2-categories embed fully faithfully in simplicial categories, we can use Lemma 2.13. Thus, 2-functors $S_\theta \to \mathscr{C}$ are in bijective correspondence with the following data:

- Objects $X^j \in |\mathscr{C}|$ for $0 \leq j \leq n-1$;
- Simplicial maps $S_\theta(m^j, m^{j-1}) \to \mathscr{C}(X^j, X^{j-1})$ $1 \leq j < n$.

By condition (i) of Definition 3.27, a morphism $p : (m^{j-1}, m^j] \to \mathbf{t}$ of S_θ is uniquely determined by $0 < t \leq m_j$, since there is a unique map $p : (m^{j-1}, m^j] \to \mathbf{t}$ satisfying that condition for a fixed $\mathbf{t}$. This unique map is given by

$$p(s) = \begin{cases} s - m_j + 1 & s - m_j + 1 \leq t - 1 \\ t & \text{otherwise.} \end{cases} .$$

As a consequence, $S_\theta(m^j, m^{j-1}) = \mathbf{m_j}$. $\square$

Chapter 4

The lifting theorem

In this chapter, S always denotes a 2-subcategory of $\mathcal{C}\Delta[n]^{\mathrm{co}}$. The goal of this chapter is to prove the following theorem, which can be considered as the combinatorial technical core of this thesis.

Theorem A. *Let $\mathscr{A} \subseteq \mathscr{A}'$ be relative right parental subcomputads of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$, where S is a convenient 2-subcategory of $\mathcal{C}\Delta[n]^{\mathrm{co}}$, and $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ a local isofibration of quasi-categorically enriched categories. Suppose furthermore that $\mathscr{A}$ contains l_i and ϵ_i for all $i \in |S|$. Then, a lifting problem*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "\mathbb{F}"] \arrow[d] & \mathcal{K} \arrow[d, "\mathbb{P}"] \\
\mathscr{A}' \arrow[r] & \mathcal{L}
\end{tikzcd}$$

has a solution, provided that for all $i \in |S|$, $\mathbb{F}(\epsilon_i)$ is the counit of an adjunction $\mathbb{F}(l_i) \dashv \mathbb{F}(r_i)$ in $\mathcal{K}$. Moreover, if $\mathcal{K}$ is a 2-category, the lift is unique.

It turns out that quite a few of the results we obtain are more or less difficult corollaries of this theorem, such as Corollaries 5.1 and 5.13. Note that the author proved independently a version of Corollary 5.1 for $\mathbf{Adj}[n]$, before realizing that Theorem A actually implies it.

This theorem is a direct generalization of a theorem of Riehl and Verity [45, Theorem 4.3.8] to $\mathbf{Adj}^S_{\mathrm{hc}}[n]$. This entire chapter is deeply inspired by [45, Section 4], as we use similar techniques, which we had to slightly modify to deal with the colorings and the S-component.

The idea of the proof is to construct the inclusion $\mathscr{A} \subseteq \mathscr{A}'$ as a transfinite composition of pushouts of maps belonging to the set $L = L_1 \cup L_2$ with

- $L_1 = \{2[\Lambda^k[m]] \to 2[\Delta[m]], 0 < k < m\}$;

- $L_2 = \{\mathfrak{C}[\partial\Delta[m-1]] \to \mathfrak{C}[\Delta[m-1]], m > 0\}$.

This is done in Section 4.2. We want to warn the reader who is familiar with the notation of [45, Theorem 4.3.8] that the simplicial category that Riehl and Verity denote $\mathfrak{C}[X]$ is the dual of our $\mathfrak{C}[X]$. To avoid unnecessary notational difficulties, we decided to redefine $\mathfrak{C}[X]$ instead of adding op everywhere.

Solving the initial lifting problem can now be done by solving lifting problems along those special inclusions. Lifting against the first set of maps comes for free, since $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ is a local isofibration. Moreover, if $\mathcal{K}$ is a 2-category, the lifts will be unique. For the second set of maps, the idea is to use the universal property of the counits ϵ_i, more precisely that ϵ_i is terminal in an appropriate slice quasi-category (See [43, Proposition 4.4.8], [45, Proposition 4.3.6] and 4.32 for its dual form). Even though the proof of Proposition 4.32 is not original and mainly a dualized copy of the proof of [45, Proposition 4.3.6], we include it in order to make clear that when $\mathcal{K}$ is a 2-category, the lift is again unique. This is the content of subsection 4.3.1.

A pushout along any of the maps in L freely adds exactly two atomic arrows, one being a face (or an atomic component of a face) of the other one. Thus, we need to carefully pair atomic arrows in $\mathscr{A}'$ and find an order in which to add those pairs of atomic arrows. This is provided by the parent-child relation, as made precise in the next section. An m-morphism $(\underline{v}, \underline{f})$ is either *right fillable* and has a corresponding *non-fillable child* $(\underline{v}, \underline{f})^\diamond$, or is not right fillable and has a unique *right fillable parent* $(\underline{v}, \underline{f})^\dagger$.

4.1 Right fillability and parent-child relation

4.1.1 Right fillable morphism and its distinguished face

Definition 4.1. Let $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ be an m-morphism of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$. The *depth* of $(\underline{v}, \underline{f})$ is $k(\underline{v}) = v_{w(\underline{v})-1}$ the penultimate letter of the underlying squiggle. We say that $(\underline{v}, \underline{f})$ is *right fillable* if

- it does not belong to $(B \leftarrow A) \times S$, that is, at least one of the connected components of one of its strips is encoding;
- it is atomic and non-degenerate;
- $v_{w(\underline{v})} = A$, i.e., $X = A$;
- $v_i \neq k(\underline{v})$ for all $0 \leq i < w(\underline{v}) - 1$;
- its $k(\underline{v})$-th coloring is minimal among admissible colorings of the $k(\underline{v})$-th strip;

- $\underline{f}$ is degenerate at $k(\underline{v}) - 1$, that is, there is no branching on the dotted line labeled $k(\underline{v})$.

Remark 4.2. The previous definition is a direct generalization of [45, Definition 4.1.1]. A monochromatic m-arrow with identity S-component is right fillable in the sense of Definition 4.1 if and only if it is right fillable in the sense of Riehl and Verity. We dualized the fillability condition from left to right because we cannot use left fillability in our context. In **Adj**, left or right fillabilty is a matter of choice, since left fillability in **Adj** is the same as right fillability in $\mathbf{Adj}^{\mathrm{op}}$, and $\mathbf{Adj} \cong \mathbf{Adj}^{\mathrm{op}}$. Since in $\mathbf{Adj}[n]$, the squares involving right adjoints commute, this duality does not hold, and it turns out that it is right fillability that is the correct technical choice to prove the theorem.

In the light of Proposition 3.17, one can roughly express the last three conditions of Definition 4.1 as follows. "The last turn of $\underline{v}$ is the unique obstruction to $(\underline{v}, \underline{f})$ being degenerate at $k(\underline{v}) - 1$."

The Figures 4.1 and 4.2 provide examples and non-examples of right fillable morphisms. In each of them, the left morphism is not right fillable whereas the right morphism is right fillable.

Remark 4.3. Observe that if $(\underline{v}, \underline{f})$ is right fillable, $1 \leq k(\underline{v}) \leq m$, since otherwise it would decompose as $(\underline{\tilde{v}}, \underline{f}) \cdot (r_x, 1_x)$, and $(\underline{v}, \underline{f}) \neq (r_i, 1_x)$ since it is a non-degenerate m-arrow with $m > 0$. Moreover, the minimality condition implies that the $k(\underline{v})$-th coloring of a right fillable arrow $(\underline{v}, \underline{f})$ is uniquely determined by its $k(\underline{v}) - 1$-st coloring and its underlying strictly undulating squiggle.

As noted in [45, Observation 4.1.2], a right fillable m-morphism has a distinguished face $d_{k(\underline{v})}(\underline{v}, \underline{f})$. In the next lemmas we analyze this face.

Lemma 4.4. *Let $(\underline{v}, \underline{f})$ be a right fillable m-morphism and $k = k(\underline{v})$. The width of its colored squiggle is the same as that of its distinguished face. More precisely, $w(\underline{v}) = w(d_k(\underline{v}))$.*

Proof. The face $d_k(\underline{v})$ is obtained by removing the line right below the unique turning point in the k-th strip. The effect on the string of turning points $(v_0, \ldots, v_{w(\underline{v})})$ is to apply s^k to all its letters. In general, it could be the case that this string is no longer strictly undulating, and thus a reduction step would occur, diminishing the width. For this to happen, one would need that two consecutive letters of $\underline{v}$ have the same image under s^k. But, there is a unique letter k in this string, the penultimate. Since the ultimate letter is A and $\underline{v}$ is strictly undulating, the letter directly preceding the penultimate letter is strictly smaller than k. ☐

Lemma 4.5. *Let $(\underline{v}, \underline{f})$ be a right fillable m-morphism and $k = k(\underline{v})$. Its distinguished face $d_k(\underline{v}, \underline{f})$ is non-degenerate.*

Proof. Since $\mathbf{Adj}^{S}_{\mathrm{hc}}[n]$ is a 2-category (see Proposition 3.15), an m-morphism is degenerate if and only if one of its constituent morphisms is an identity. As a consequence, if $(\underline{v}, \underline{f})$ is non-degenerate, the only way its face $d_k(\underline{v}, \underline{f})$ can be degenerate is if the composite of the maps going to and leaving from the corresponding k-th object of $\underline{v}$ compose to an identity. Thus, if $d_k(\underline{v}, \underline{f})$ is degenerate, it is degenerate at $k-1$. But the k-th strip of $d_k(\underline{v})$ contains a turning point, thus $d_k(\underline{v})$ cannot be degenerate. ☐

Lemma 4.6. *Let $(\underline{v}, \underline{f})$ be a right fillable m-morphism and $k = k(\underline{v})$. Then, $s_{k-1}d_k\underline{f} = \underline{f}$.*

Proof. Since $\underline{f}$ is degenerate at $k-1$, there exists $\underline{f}'$ such that $\underline{f} = s_{k-1}\underline{f}'$. Then,

$$s_{k-1}d_k\underline{f} = s_{k-1}d_k s_{k-1}\underline{f}' = s_{k-1}\underline{f}' = \underline{f}.$$

☐

Observe that in particular, this implies that there is a correspondence between the decompositions in atomic forests of $\underline{f}$ and $d_k(\underline{f})$.

To further analyze the distinguished face of a right fillable m-morphism, we distinguish two cases, $k < m$ and $k = m$.

Lemma 4.7. *Let $(\underline{v}, \underline{f})$ be a right fillable m-morphism such that $k := k(\underline{v}) < m$. Its distinguished face $d_k(\underline{v}, \underline{f})$ is atomic and not right fillable.*

Proof. Let $\underline{u} = d_k(\underline{v})$. Observe that if $u_j \in \{A, B\}$, so does $v_j = u_j$. Since the $k(\underline{v})$-strip contains a unique turning point, there are no components of this strip that are not sharing a boundary with a component of the $k-1$ strip. As a consequence, by minimality if z is a color of an encoding component of the k-th strip, z is also the color of the component of the $k-1$ strip right above it. By Proposition 3.20, and Lemma 4.6, $d_k(\underline{v}, \underline{f})$ is atomic. Since $(\underline{v}, \underline{f})$ is not degenerate, it is in particular not degenerate at k. This leaves three options.

- The colored squiggle $\underline{v}$ is not degenerate at k because the underlying squiggle is not, which means there is a turning point in the $k+1$-st strip. This implies that $d_k(\underline{v})$ has two turning points in its k-th strip, and thus $d_k(\underline{v}, \underline{f})$ is not right fillable.
- The colored squiggle $\underline{v}$ is not degenerate at k, even though the underlying squiggle is. This means that there is an encoding connected component of the $k+1$-st strip of $\underline{v}$ which has a color strictly bigger than the color of the connected component of the k-th strip it shares a boundary with. This implies that the coloring of the k-th strip of $\underline{v}$, which is also admissible as

a coloring of the k-th strip of $d_k(\underline{v}, \underline{f})$, is strictly smaller than the actual coloring of the k-th strip of $d_k(\underline{v}, \underline{f})$. As a consequence, $d_k(\underline{v}, \underline{f})$ is not right fillable.

- The forest $\underline{f}$ is not degenerate at k. In particular, there is a corolla on line $k + 1$ with a least two leaves. As a consequence, when composed with the corollas of line k, the end result has also at least two leaves, and thus $d_k(\underline{f})$ is not degenerate at $k - 1$. Thus, $d_k(\underline{v}, \underline{f})$ is not right fillable.

□

Lemma 4.8. *Let* $(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ *be a right fillable* m*-morphism such that* $k := k(\underline{v}) = m$*. Its distinguished face* $d_k(\underline{v}, \underline{f})$ *is not atomic since it decomposes as*

$$\begin{tikzcd}(A, x) \arrow[r, "r_x"] & (B, x) \arrow[r, "(\underline{v},\underline{f})^\diamond"] & (Y, y)\end{tikzcd} .$$

The $m - 1$ *morphism* $(\underline{v}, \underline{f})^\diamond$ *is atomic, non-degenerate and not right fillable.*

Proof. Since r_x is degenerate and $d_k(\underline{v}, \underline{f})$ is not degenerate by Lemma 4.5, $(\underline{v}, \underline{f})^\diamond$ is not degenerate. Remark that the underlying squiggle of $(\underline{v}, \underline{f})^\diamond$ has a connected component of a strip that is encoding, since $\underline{v}$ does and r_x has no such component. By Proposition 3.20, and 4.6, $(\underline{v}, \underline{f})^\diamond$ is atomic since it contributes all the encoding connected components of $(\underline{v}, \underline{f})$, its inner letters are also inner letters of $(\underline{v}, \underline{f})$, and $(\underline{v}, \underline{f})$ is itself atomic. It is obviously not right fillable, since its domain is (B, x). □

4.1.2 Parent-child relation

We are now ready to describe the parent child-relation.

Definition 4.9. Let Fill_m denotes the set of right fillable m-morphisms and Atom_m the set of non-degenerate atomic m-morphisms of $\mathbf{Adj}^S_{\mathrm{hc}}[n] \setminus (B \leftarrow A) \times S$.

Definition 4.10. Let $(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ be a right fillable m-morphism and $k = k(\underline{v})$. Its *non-fillable child* $(\underline{v}, \underline{f})^\diamond$ is defined to be $d_k(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ if $k > m$, and $(\underline{v}, \underline{f})^\diamond : (B, x) \to (Y, y)$ as defined in Lemma 4.8 if $k = m$.

Proposition 4.11. *The map* $\diamond : \mathrm{Fill}_m \to \mathrm{Mor}(\mathbf{Adj}^S_{\mathrm{hc}}[n]_{m-1})$ *corestricts to a map* $\diamond : \mathrm{Fill}_m \to \mathrm{Atom}_{m-1} \setminus \mathrm{Fill}_{m-1}$.

Proof. This is a direct consequence of Lemmas 4.5, 4.7 and 4.8. □

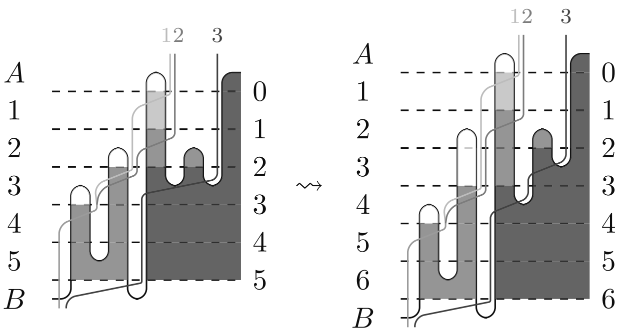


Figure 4.1: Right fillable parent of a morphism $(A, 3) \to (B, 0) \in \mathrm{Atom}_6 \setminus \mathrm{Fill}_6$

The goal of this part is to prove that $\diamond$ is actually a bijection. We now construct an inverse

$$\dagger : \mathrm{Atom}_{m-1} \setminus \mathrm{Fill}_{m-1} \to \mathrm{Fill}_m.$$

Let $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ be a member of Atom_{m-1} which is not right fillable. We separate two cases, $X = A$ or $X = B$.

Right fillable parent of a non-fillable morphism with domain (A, x)

Let $(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ be a member of Atom_{m-1} which is not right fillable. Let $\underline{v}^\dagger$ be the colored squiggle obtained from $\underline{v}$ by drawing a new line through the $k(\underline{v})$ strip of $\underline{v}$, leaving all turning points but the last one on the bottom, and coloring the new strip obtained using the minimal admissible coloring.

Definition 4.12. Let $(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ be a member of Atom_{m-1} which is not right fillable and let $k = k(\underline{v})$. Its *right fillable parent* $(\underline{v}, \underline{f})^\dagger$ is defined by $(\underline{v}, \underline{f})^\dagger = (\underline{v}^\dagger, s_{k-1}\underline{f})$.

An example of such a right fillable parent is provided in Figure 4.1.

Lemma 4.13. *Let* $(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ *be a member of* Atom_{m-1} *which is not right fillable. The pair* $(\underline{v}, \underline{f})^\dagger$ *is an* m*-morphism* $(A, x) \to (Y, y)$ *of* $\mathbf{Adj}^S_{\mathrm{hc}}[n]$.

Proof. Let $k = k(\underline{v})$. The two new strips of $\underline{v}^\dagger$ are the k-th and the $k+1$-st. Remark that for $i \neq k, k+1$,

$$\pi_0\mathrm{Strip}(\underline{v}^\dagger, i) = \pi_0\mathrm{Strip}(s_{k-1}\underline{v}, i)$$

and also the colorings match up. As a consequence, we only have to check that if, given $j < j' \in (y, x]$, if $\underline{v}^\dagger$ separates j and j' at k or $k+1$ then so does $s_{k-1}\underline{f}$.

Since we chose the minimal admissible coloring for the k-th strip of $\underline{v}^\dagger$, if j and j' are separated at k in $\underline{v}^\dagger$, they are also separated at $k-1$ in $\underline{v}$, and thus separated at $k-1 = s_{k-1}(k)$ in $\underline{f}$. By Proposition 3.12, $s_{k-1}(\underline{f})$ separates j and j' at k. If j and j' are separated at $k+1$ in $\underline{v}^\dagger$, they are also separated at $k = s_{k-1}(k+1)$ in $\underline{v}$ and thus in $\underline{f}$. By Proposition 3.12, j and j' are separated by $s_{k-1}(\underline{f})$ at $k+1$. □

Remark 4.14. Observe that by construction, $d_{k(\underline{v})}(\underline{v},\underline{f})^\dagger = (\underline{v},\underline{f})$.

Lemma 4.15. *Let* $(\underline{v},\underline{f}) : (A,x) \to (Y,y)$ *be a member of* Atom_{m-1} *which is not right fillable. The m-morphism* $(\underline{v},\underline{f})^\dagger$ *is non-degenerate.*

Proof. Let $k = k(\underline{v})$. Since $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is a 2-category, and $d_k(\underline{v},\underline{f})^\dagger = (\underline{v},\underline{f})$ is not degenerate, $(\underline{v},\underline{f})^\dagger$ can only be degenerate at $k-1$ and k. Since $\underline{v}^\dagger$ has a turn in the k-th strip, it is not degenerate at $k-1$. Moreover, $(\underline{v},\underline{f})$ is not right fillable, so there are three possibilities.

- The underlying squiggle of $\underline{v}$ has a second turning point in the k-th strip, which means $\underline{v}^\dagger$ has one in its k+1-st strip, and thus $(\underline{v},\underline{f})^\dagger$ is non-degenerate.
- The k-th coloring of $\underline{v}$ is not minimal among admissible colorings of the k-th strip, which means $\underline{v}^\dagger$ has different colorings for its k-th and $k+1$-st strips, and thus $(\underline{v},\underline{f})^\dagger$ is non-degenerate.
- The forest $\underline{f}$ is not degenerate at $k-1$, which means that $s_{k-1}\underline{f}$ is not degenerate at k, and thus $(\underline{v},\underline{f})^\dagger$ is non-degenerate.

□

Lemma 4.16. *Let* $(\underline{v},\underline{f}) : (A,x) \to (Y,y)$ *be a member of* Atom_{m-1} *which is not right fillable. The m-morphism* $(\underline{v},\underline{f})^\dagger$ *is right fillable.*

Proof. Since S is a simplicial computad, degeneracies of atomic forests are atomic, thus the decomposition into atomic forests of $s_{k-1}\underline{f}$ can be obtained from the one of $\underline{f}$ by degenerating all the atomic forests accordingly. Remark that all colors used in encoding connected components of $\underline{v}$ are also used in encoding connected components of $\underline{v}^\dagger$, and vice-versa. By Proposition 3.20, since $(\underline{v},\underline{f})$ is atomic, so is $(\underline{v},\underline{f})^\dagger$. Remark that all other conditions are satisfied by construction or by Lemma 4.15. □

Lemma 4.17. *Let* $(\underline{v},\underline{f}) : (A,x) \to (Y,y)$ *be a right fillable m-morphism with* $k(\underline{v}) < m$. *Then,* $(\underline{v},\underline{f})^{\diamond\dagger} = (\underline{v},\underline{f})$.

Proof. Let $k = k(\underline{v})$. By definition, $\underline{f}$ is degenerate at $k-1$, which means there exists $\underline{f'}$ such that $\underline{f} = s_{k-1}\underline{f'}$. As a consequence,

$$(\underline{v}, \underline{f})^{\diamond\dagger} = ((d_k\underline{v})^\dagger, s_{k-1}d_k s_{k-1}\underline{f'}) = ((d_k\underline{v})^\dagger, \underline{f}).$$

We thus have to show that $(d_k\underline{v})^\dagger = \underline{v}$. Let $(v_0, \dots, v_r)$ be the underlying squiggle of $\underline{v}$. By definition, the underlying squiggle of $d_k\underline{v}$ is $(s^k(v_0), \dots, s^k(v_r))$ (Recall that the width is unchanged by Lemma 4.4). By drawing a new line through the k-th strip, leaving all turning points but the last one on the bottom, we effectively apply d^k to all the letters but the last letter with value k, which we leave unchanged. Since $(\underline{v}, \underline{f})$ is right fillable, this last letter with value k is the penultimate, and thus the underlying squiggle of $(d_k\underline{v})^\dagger$ is $(v_0, \dots, v_{r-2}, k, v_r)$, which is exactly the underlying squiggle of $\underline{v}$. By minimality of the coloring of the k-th strip, the colorings of the k-th strip of $\underline{v}$ and $(d_k\underline{v})^\dagger$ match, whereas the coloring of the $k+1$-st strip agrees by construction. We have that for $i \neq k, k+1$,

$$\pi_0\mathrm{Strip}((d_k(\underline{v})^\dagger, i) = \pi_0\mathrm{Strip}(s_{k-1}d_k(\underline{v}), i) = \pi_0\mathrm{Strip}(\underline{v}, d^k s^{k-1} i) = \pi_0\mathrm{Strip}(\underline{v}, i)$$

and also the colorings match up. The first equality has already been used in the proof of Lemma 4.13, whereas the second one is a consequence of equations (3.1) and (3.2). □

Right fillable parent of a non-fillable morphism with domain (B, x)

We turn now our attention to $(m-1)$-morphisms $(\underline{v}, \underline{f}) : (B, x) \to (Y, y)$ which are members of Atom_{m-1} and thus not right fillable. Let $\underline{v}^\dagger : (A, x) \to (Y, y)$ be the colored squiggle obtained from $\underline{v}$ by drawing a new line through the B-row of $\underline{v} \cdot r_x$, only leaving the last turning point on top, and coloring the new strip with the minimal admissible coloring.

Definition 4.18. Let $(\underline{v}, \underline{f}) : (B, x) \to (Y, y)$ be a member of Atom_{m-1} which is not right fillable. Its *right fillable parent* $(\underline{v}, \underline{f})^\dagger : (A, x) \to (Y, y)$ is defined by $(\underline{v}, \underline{f})^\dagger = (\underline{v}^\dagger, s_{m-1}\underline{f})$.

An example of such a right fillable parent is provided in Figure 4.2.

Lemma 4.19. *Let* $(\underline{v}, \underline{f}) : (B, x) \to (Y, y)$ *be a member of* Atom_{m-1} *which is not right fillable. The pair* $(\underline{v}, \underline{f})^\dagger$ *is an* m*-morphism* $(A, x) \to (Y, y)$ *of* $\mathbf{Adj}^S_{\mathrm{hc}}[n]$.

Proof. Remark that for $i < m$,

$$\pi_0\mathrm{Strip}(\underline{v}^\dagger, i) = \pi_0\mathrm{Strip}(s_{m-1}(\underline{v} \cdot r_x), i)$$

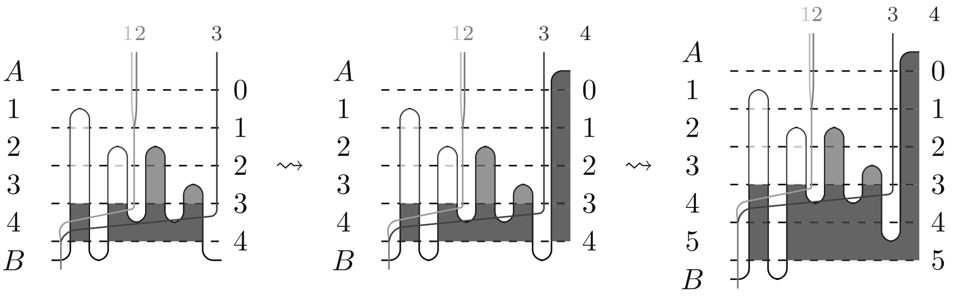


Figure 4.2: Right fillable parent of a morphism $(B, 3) \to (B, 0) \in \mathrm{Atom}_4 \setminus \mathrm{Fill}_4$

and also the colorings match up. Since the only encoding connected component of $s_{m-1}(\underline{v} \cdot r_x)$ are coming from $s_{m-1}\underline{v}$, if j and j' are separated at $i < m$ in $\underline{v}^\dagger$, they are separated at i in $s_{m-1}\underline{v}$ and thus in $s_{m-1}\underline{f}$. Moreover, if $\underline{v}^\dagger$ separates j from j' at m, $\underline{v}$ separates j from j' at $m-1$ and thus so does $\underline{f}$. As a consequence, $s_{m-1}\underline{f}$ separates j from j' at m. ☐

Remark 4.20. Observe that by construction, $d_m(\underline{v}, \underline{f})^\dagger = (\underline{v}, \underline{f}) \cdot r_x$.

Lemma 4.21. *Let* $(\underline{v}, \underline{f}) : (B, x) \to (Y, y)$ *be a member of* Atom_{m-1} *which is not right fillable. The* m*-morphism* $(\underline{v}, \underline{f})^\dagger$ *is non-degenerate.*

Proof. Since $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is a 2-category and $d_m(\underline{v}, \underline{f})^\dagger = (\underline{v}, \underline{f}) \cdot r_x$ is not degenerate, $(\underline{v}, \underline{f})^\dagger$ can only be degenerate at $m-1$. But the underlying squiggle of $(\underline{v}, \underline{f})^\dagger$ contains a turn in the m-th strip. ☐

Lemma 4.22. *Let* $(\underline{v}, \underline{f}) : (B, x) \to (Y, y)$ *be a member of* Atom_{m-1} *which is not right fillable. The* m*-morphism* $(\underline{v}, \underline{f})^\dagger$ *is right fillable.*

Proof. Since S is a simplicial computad, degeneracies of atomic forests are atomic, thus the decomposition into atomic forest of $s_{m-1}\underline{f}$ can be obtained from the one of $\underline{f}$ by degenerating all the atomic forest accordingly. Remark that all colors used in encoding connected components of $\underline{v}$ are also used in encoding connected components of $\underline{v}^\dagger$, and vice-versa. By Proposition 3.20, since $\underline{v}$ is atomic, so is $(\underline{v}, \underline{f})^\dagger$. Remark that all other conditions are met by construction or by Lemma 4.21. ☐

Lemma 4.23. *Let* $(\underline{v}, \underline{f}) : (B, x) \to (Y, y)$ *be a right fillable* m*-morphism with* $k(\underline{v}) = m$*. Then,* $(\underline{v}, \underline{f})^{\diamond\dagger} = (\underline{v}, \underline{f})$*.*

Proof. Let us write $(\underline{v}^\diamond, \underline{f}^\diamond)$ for the child $(\underline{v}, \underline{f})^\diamond$ of $(\underline{v}, \underline{f})$. By construction, $\underline{f}^\diamond = d_m \underline{f}$. By right fillability, $\underline{f}$ is degenerate at $m-1$, which means there exists $\underline{f}'$ such that $\underline{f} = s_{m-1}\underline{f}'$. As a consequence,

$$(\underline{v}, \underline{f})^{\diamond\dagger} = (\underline{v}^{\diamond\dagger}, s_{m-1} d_m s_{m-1} \underline{f}') = (\underline{v}^{\diamond\dagger}, \underline{f}).$$

We thus have to show that $\underline{v}^{\diamond\dagger} = \underline{v}$. Let $(v_0, \ldots, v_r)$ be the underlying squiggle of $\underline{v}$. By right fillability, the underlying squiggle of $d_m(\underline{v})$ is $(v_0, \ldots, v_{r-2}, B, A)$. By drawing a new line through the B-th strip, leaving only the last turning point on top, we effectively only change the penultimate letter of the underlying squiggle, and replace B by m. As a consequence, the underlying squiggles of $\underline{v}^{\diamond\dagger}$ and $\underline{v}$ are the same. We have that for $i < m$,

$$\pi_0 \mathrm{Strip}(\underline{v}^{\diamond\dagger}, i) = \pi_0 \mathrm{Strip}(s_{m-1} d_m \underline{v}, i) = \pi_0 \mathrm{Strip}(\underline{v}, d^m s^{m-1} i) = \pi_0 \mathrm{Strip}(\underline{v}, i)$$

and also the colorings match up. The colorings of the m-th strip are the same by minimality. ☐

We thus have proved the following proposition.

Proposition 4.24. *The map* $\diamond : \mathrm{Fill}_m \to \mathrm{Atom}_{m-1} \setminus \mathrm{Fill}_{m-1}$ *is a bijection with inverse* $\dagger : \mathrm{Atom}_{m-1} \setminus \mathrm{Fill}_{m-1} \to \mathrm{Fill}_m$.

Proof. Combine Lemmas 4.16 and 4.22 to construct the map $\dagger$. The equality $\dagger \circ \diamond = \mathrm{id}$ is a consequence of Lemmas 4.17 and 4.23. The equality $\diamond \circ \dagger = \mathrm{id}$ is implied by Remarks 4.14 and 4.20. ☐

4.1.3 Right parental relative subcomputads

Definition 4.25. We say that a simplicial subcategory $\mathscr{A}$ of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is a *relative subcomputad* of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ if it contains $(B \leftarrow A) \times S$ and if both inclusions $(B \leftarrow A) \times S \to \mathscr{A}$ and $\mathscr{A} \to \mathbf{Adj}^S_{\mathrm{hc}}[n]$ are relative simplicial computads.

Recall that by Lemma 2.8, atomic morphisms of a relative subcomputad $\mathscr{A}$ are atomic morphisms of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$. As a consequence, Lemma 2.8 implies that a relative subcomputad $\mathscr{A}$ of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is determined by the set its atomic morphisms $\mathrm{Atom}(\mathscr{A})$.

Moreover, any set J of atomic morphisms of $\mathbf{Adj}^S_{\mathrm{hc}}[n] \setminus (B \leftarrow A) \times S$ which satisfies the following properties

- if $(\underline{v}, \underline{f}) \in J$, the decomposition of $d_i(\underline{v}, \underline{f})$ given by Lemma 2.8, Equation (2.1) with respect to $(B \leftarrow A) \times S \subseteq \mathbf{Adj}^S_{\mathrm{hc}}[n]$ involves only elements of J and $(B \leftarrow A) \times S$,

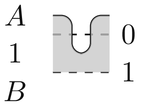


Figure 4.3: ϵ_i

- J is closed under degeneracies,

determines a relative subcomputad, consisting of those m-morphisms whose decomposition given by Lemma 2.8, Equation (2.1), involves only elements of J and $(B \leftarrow A) \times S$. Those two constructions are inverses of one another by Lemma 2.8.

Definition 4.26. We say that a relative subcomputad $\mathscr{A}$ of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is *right parental* if every non-degenerate atomic morphism $(\underline{v}, \underline{f})$ of $\mathscr{A}$ not in $(B \leftarrow A) \times S$ is either right fillable or its *right fillable parent* $(\underline{v}, \underline{f})^\dagger$ belongs to $\mathscr{A}$.

Remark that by the discussion above and since $\mathscr{A}$ is a relative subcomputad, if $(\underline{v}, \underline{f}) \in \mathscr{A}$ is right fillable, $(\underline{v}, \underline{f})^\diamond \in \mathscr{A}$. A right parental relative subcomputad thus corresponds to a set of atomic morphisms of $\mathbf{Adj}^S_{\mathrm{hc}}[n] \setminus (B \leftarrow A) \times S$ which in addition is closed under the parent-child relation. As an immediate consequence, the intersection of an arbitrary collection of relative subcomputad is a relative simplicial computad, which is parental if all the elements of the collection were right parental. We thus introduce the following definition.

Definition 4.27. Let I be a set of morphisms of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$.

- The *relative subcomputad generated by* I is the intersection of all relative subcomputads of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ containing I.
- The *right parental relative subcomputad generated by* I is the intersection of all right parental relative subcomputads of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ containing I.

Examples 4.28. We introduce the right parental subcomputads of $\mathbf{Adj}_{\mathrm{hc}}[n]$ that we use along Theorem A in this thesis.

(i) Let $\overline{\{\epsilon_i : i \in |S|\}}$ be the relative subcomputad of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ generated by $\{\epsilon_i : i \in |S|\}$, where $\epsilon_i : (A, i) \to (B, i)$ is the 2-morphism pictured in Figure 4.3 along with its faces (where the monochrome coloring has value i). Observe that the set of non-degenerate atomic morphisms of $\overline{\{\epsilon_i : i \in |S|\}}$ not belonging to $(B \leftarrow A) \times S$ is $\{l_i : i \in |S|\} \cup \{\epsilon_i : i \in |S|\}$. Moreover, ϵ_i is right fillable, and it is the fillable parent of l_i. When $n = 0$ and $S = \mathcal{C}\Delta[0]^{\mathrm{co}} = \star$, we write the corresponding subcomputad $\overline{\{\epsilon\}}$. These right parental subcomputads are used in Section 5.2.

(ii) Let Mono be the relative subcomputad of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ generated by all atomic morphisms $(X, i) \to (Y, i)$, $i \in |S|$. The notation reminds the reader that the generating atomic morphisms are those which are monochromatic. Observe that it is a relative parental subcomputad of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$, since † never changes the set of colors used on encoding connected components of strips. This right parental subcomputad is used to prove the 2-universal property of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ in Corollary 5.1.

(iii) Consider all non-degenerate atomic morphisms $(\underline{v}, \underline{f})$ of $\mathbf{Adj}_{\mathrm{hc}}[n]$ such that $\underline{f}$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[1]^{\mathrm{co}} \to \mathcal{C}\Delta[2]$ for $i \in [n]$. Since $\mathbb{d}^i$ is a simplicial functor, the relative subcomputad of $\mathbf{Adj}_{\mathrm{hc}}[n]$ generated by those atomic morphisms is closed under the parent-child relation and thus a right parental relative subcomputad. We write this simplicial subcomputad $\overline{\partial \mathbf{Adj}_{\mathrm{hc}}}[n]$. It is used in Chapter 6 to prove Theorem C.

(iv) Let $\overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n]$ be the relative subcomputad of $\mathbf{Adj}_{\mathrm{hc}}[n]$ generated by the non-degenerate atomic m-morphisms $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ such that $\underline{f}$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ for $i \in [n] \setminus \{k\}$. Since $\mathbb{d}^i$ is a simplicial functor, the relative subcomputad of $\mathbf{Adj}_{\mathrm{hc}}[2]$ generated by those atomic morphisms is closed under the parent-child relation and thus a right parental relative subcomputad. This right parental subcomputad is used in Chapter 6.

4.2 Decomposition as an L-cell complex of an inclusion of right parental relative subcomputads

Let $L = L_1 \cup L_2$ with

- $L_1 = \{\mathbb{2}[\Lambda^k[m]] \to \mathbb{2}[\Delta[m]], 0 < k < m\}$;
- $L_2 = \{\mathbb{3}[\partial\Delta[m-1]] \to \mathbb{3}[\Delta[m-1]], m > 0\}$.

The goal of this section is to show that any inclusion of right parental relative subcomputad can be expressed as an L-cell complex.

4.2.1 A single pushout against a morphism in L

In this part, we define and analyze the relative computad structure of maps of L_2. This enables us to determine under which circumstances an inclusion of right parental relative subcomputads can be expressed as a single pushout against a map of L.

Definition 4.29. Let $\mathfrak{Z}[-] : \mathbf{sSet} \to \mathbf{sCat}$ be a functor defined as follows. For X a simplicial set, the simplicial category $\mathfrak{Z}[X]$ has three objects denoted $0, 1, 2$ and

- $\mathfrak{Z}[X](0,1) = \mathfrak{Z}[X](i,i) = \Delta[0]$,
- $\mathfrak{Z}[X](1,2) = X$,
- $\mathfrak{Z}[X](0,2) = X \star \Delta[0]$,

where $\star$ denotes the join operation reviewed in 2.2.1. The only non-trivial composition $\mathfrak{Z}[X](1,2) \times \mathfrak{Z}[X](0,1) \to \mathfrak{Z}[X](0,2)$ is given by

$$\begin{tikzcd} X \times \Delta[0] \arrow[r] & X \arrow[r] & X \star \Delta[0] \end{tikzcd}$$

If $f : X \to Y$ is a simplicial map, $\mathfrak{Z}[f] : \mathfrak{Z}[X] \to \mathfrak{Z}[Y]$ acts by f on $\mathfrak{Z}[X](1,2)$ and by $f \star \Delta[0]$ on $\mathfrak{Z}[X](0,2)$.

Let us analyze $\mathfrak{Z}[\Delta[m]]$. By Remark 2.58, the nerve construction preserves joins. As a consequence, $\Delta[m] \star \Delta[0] \cong \Delta[m+1]$ and the inclusion $\Delta[m] \to \Delta[m] \star \Delta[0] = \Delta[m+1]$ corresponds after identification with the coface map d^{m+1}.

As a consequence, the non-degenerate atomic morphisms of $\mathfrak{Z}[\Delta[m]]$ are exactly the unique non degenerate 0-morphism $0 \to 1$, all injective maps $[k] \to [m] \in \mathfrak{Z}[\Delta[m]](1,2)$, and all injective maps $[k] \to [m+1] \in \mathfrak{Z}[\Delta[m]](0,2)$ with $m+1$ in their image. Remark that they generate freely $\mathfrak{Z}[\Delta[m]]$, showing that it is a simplicial computad. Moreover, all non-degenerate atomic morphisms of $\mathfrak{Z}[\Delta[m]]$ are one of the following:

- the unique 0-morphism $0 \to 1$;
- $1_{[m]} : 1 \to 2$ or an (iterated) face of this morphism;
- $1_{[m+1]} : 0 \to 2$ or an (iterated) face of this morphism.

This shows that simplicial functors $\mathfrak{Z}[\Delta[m]] \to \mathscr{C}$ are in bijective correspondence with the choice of

- three objects of $\mathscr{C}$, C_0, C_1, C_2;
- a 0-morphism $g : C_0 \to C_1$;
- an m-morphism $\beta : C_1 \to C_2$;
- an $m+1$-morphism $\alpha : C_0 \to C_2$;

such that $d_{m+1}\alpha = \beta \cdot g$. Thus, we will denote such a simplicial functor $\mathfrak{Z}[\Delta[m]] \to \mathscr{C}$ by (α, β, g).

We now turn our attention to the inclusion $\mathfrak{Z}[\partial\Delta[m]] \to \mathfrak{Z}[\Delta[m]]$. Since there is a coequalizer of augmented simplicial sets (with trivial augmentation)

$$\begin{tikzcd}
\coprod\limits_{0\leq i<j\leq m} \Delta[m-2] \arrow[r, shift left, "d^{j-1}"] \arrow[r, shift right, "d^{i}"'] & \coprod\limits_{k=0}^{m} \Delta[m-1] \arrow[r, "\sum_k d^k"] & \partial\Delta[m]
\end{tikzcd}$$

and $\star$ is cocontinous in each variable over augmented simplicial sets, there is a coequalizer

$$\begin{tikzcd}
\coprod\limits_{0\leq i<j\leq m} \Delta[m-1] \arrow[r, shift left, "d^{j-1}"] \arrow[r, shift right, "d^{i}"'] & \coprod\limits_{k=0}^{m} \Delta[m] \arrow[r, "\sum_k d^k"] & \partial\Delta[m] \star \Delta[0]
\end{tikzcd} .$$

This shows that after identification,

$$\partial\Delta[m] \star \Delta[0] = \Lambda^{m+1}[m+1] \subseteq \Delta[m+1] = \Delta[m] \star \Delta[0].$$

As a consequence, there are only three non-degenerate arrows of $\mathfrak{Z}[\Delta[m]]$ which are missing in $\mathfrak{Z}[\partial\Delta[m]]$, namely

- $\mathrm{id}_{[m]} : [m] \to [m] \in \mathfrak{Z}[\Delta[m]](1, 2)$;
- $\mathrm{id}_{[m+1]} : [m+1] \to [m+1] \in \mathfrak{Z}[\Delta[m]](0, 2)$;
- $d^{m+1} : [m] \to [m+1] \in \mathfrak{Z}[\Delta[m]](0, 2)$.

Remark that the following diagram

$$\begin{tikzcd}
0 \arrow[r, "\mathrm{id}_{[0]}"] \arrow[rr, bend right, "d^{m+1}"'] & 1 \arrow[r, "\mathrm{id}_{[m]}"] & 2
\end{tikzcd}$$

is commutative in $\mathfrak{Z}[\Delta[m]]$. Thus, there are only two non-degenerate atomic arrows missing in the simplicial subcategory $\mathfrak{Z}[\partial\Delta[n]]$. Therefore, $\mathfrak{Z}[\partial\Delta[n]]$ is an atom-complete simplicial subcategory of $\mathfrak{Z}[\Delta[m]]$ and thus $\mathfrak{Z}[\partial\Delta[m]] \to \mathfrak{Z}[\Delta[m]]$ is a relative simplicial computad by Corollary 2.12. As a consequence is a diagram

$$\begin{tikzcd}
\mathbb{2}[\partial\Delta[m]] \arrow[r, "\partial\mathrm{id}_{[m]}:1\to 2"] \arrow[d] & \mathbb{3}[\partial\Delta[m]] \arrow[d] \\
\mathbb{2}[\Delta[m]] \arrow[r] & \mathscr{W} \arrow[d] \\
\mathbb{2}[\partial\Delta[m+1]] \arrow[ur, "\partial\mathrm{id}_{[m+1]}:0\to 2"] \arrow[d] & \mathbb{3}[\Delta[m]] \\
\mathbb{2}[\Delta[m+1]] \arrow[ur] &
\end{tikzcd} \tag{4.1}$$

where both quadrilaterals are pushout squares.

Proposition 4.30. *Suppose that $\mathscr{A}$ is a relative subcomputad of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ which is right parental and $(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ is a right fillable m-morphism of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ not contained in $\mathscr{A}$, and such that $d_i(\underline{v}, \underline{f})$ belongs to $\mathscr{A}$ for all $i \neq k(\underline{v})$. Then, the relative subcomputad $\mathscr{A}'$ generated by $\mathscr{A} \cup \{(\underline{v}, \underline{f})\}$ is again right parental, and we may express the inclusion $\mathscr{A} \to \mathscr{A}'$ as a pushout*

$$\begin{tikzcd}
\mathbb{2}[\Lambda^{k(\underline{v})}[m]] \arrow[r] \arrow[d] & \mathscr{A} \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{2}[\Delta[m]] \arrow[r] \arrow[drr, bend right, "(\underline{v},\underline{f})"'] & \mathscr{A}' \arrow[dr, dashed] & \\
& & \mathbf{Adj}^S_{\mathrm{hc}}[n]
\end{tikzcd} \tag{4.2}$$

if $0 < k(\underline{v}) < m$ *and as a pushout*

$$\begin{tikzcd}
\mathbb{3}[\partial\Delta[m-1]] \arrow[r] \arrow[d] & \mathscr{A} \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{3}[\Delta[m-1]] \arrow[r] \arrow[drr, "((\underline{v},\underline{f}),(\underline{v},\underline{f})^{\diamond},r_x)"'] & \mathscr{A}' \arrow[dr, dashed] & \\
& & \mathbf{Adj}^S_{\mathrm{hc}}[n]
\end{tikzcd} \tag{4.3}$$

if $k(\underline{v}) = m$, and where the dotted arrow obtained by the universal property is the inclusion of $\mathscr{A}'$ into $\mathbf{Adj}^S_{\mathrm{hc}}[n]$.

Proof. Remark that if a relative subcomputad contains a right fillable m-morphism $(\underline{v}, \underline{f})$, it should contain $(\underline{v}, \underline{f})^\diamond$, either because it is a face of $(\underline{v}, \underline{f})$ or because it belongs to the decomposition given by Equation (2.1) of a face of $(\underline{v}, \underline{f})$. This shows that, the maps

$$(\underline{v}, \underline{f}) : \mathbb{2}[\Delta[m]] \to \mathbf{Adj}^S_{\mathrm{hc}}[n]$$

and

$$((\underline{v}, \underline{f}), (\underline{v}, \underline{f})^\diamond, r_x) : \mathbb{3}[\Delta[m-1]] \to \mathbf{Adj}^S_{\mathrm{hc}}[n]$$

factor through $\mathscr{A}'$. Moreover, the hypothesis of the proposition implies that their restrictions to $\mathbb{2}[\Lambda^{k(\underline{v})}[m]]$ in the first case and to $\mathbb{3}[\partial\Delta[m-1]]$ in the second one factor through $\mathscr{A}$. This defines the squares involved in the proposition.

It is clear that $\mathscr{A}$ does not contain $(\underline{v}, \underline{f})^\diamond$ since it is right parental. This implies that the maps from the pushouts

$$\mathbb{2}[\Delta[m]] \coprod_{\mathbb{2}[\Lambda^{k(\underline{v})}[m]]} \mathscr{A} \to \mathscr{A}'$$
$$\mathbb{3}[\Delta[m-1]] \coprod_{\mathbb{3}[\partial\Delta[m-1]]} \mathscr{A} \to \mathscr{A}'$$

are injective, and their images determine a relative subcomputad of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$. By minimality of $\mathscr{A}'$, the squares of the proposition are actually pushout squares. Moreover, $\mathscr{A}'$ is right parental, since $\mathscr{A}$ is right parental and if we consider non-degenerate atomic arrows, we have just added a pair parent-child for the correspondence given by $\diamond$ and $\dagger$. ☐

Now, we need to show that we can add all these parent-child pairs in a well chosen order, such that each inclusion satisfies the hypothesis of the previous proposition.

4.2.2 Transfinite composition

Theorem 4.31. *Suppose that $\mathscr{A} \subseteq \mathscr{A}'$ are right parental relative subcomputads of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$. Then, the inclusion $\mathscr{A} \to \mathscr{A}'$ can be expressed as a transfinite composition of pushouts of the form* (4.2) *or* (4.3).

Proof. Let $\mathbb{X}$ be the set of right fillable morphisms in $\mathscr{A}'$ but not in $\mathscr{A}$. Set an order on $\mathbb{X}$ as follows. Given $(\underline{v}, \underline{f}), (\underline{u}, \underline{g}) \in X$, let $(\underline{v}, \underline{f}) \leq (\underline{u}, \underline{g})$ if one of the following condition is satisfied

- $w(\underline{v}) < w(\underline{u})$;
- $w(\underline{v}) = w(\underline{u})$ and $k(\underline{v}) < k(\underline{u})$;

- $w(\underline{v}) = w(\underline{u})$, $k(\underline{v}) = k(\underline{u})$ and the dimension of $\underline{v}$ is less or equal to the dimension of $\underline{u}$.

Extend this order to a total order on $\mathbb{X}$.

Let $(\underline{v}, \underline{f})$ be a non degenerate m-morphism. A strip of $\underline{v}$ contains certainly fewer than $w(\underline{v})$ components. As a consequence, there are necessarily fewer than $w(\underline{v})^{n+1}$ possible colorings of a single strip and thus there are fewer than $w(\underline{v})^{n+1} + n - 1$ consecutive strips without a turn in the underlying squiggle of $(\underline{v}, \underline{f})$, since there are at most $n-1$ branching in a tree of $\mathcal{C}\Delta[n]^{\mathrm{op}}$.

This shows that there are less than $w(w^{n+1} + (n-1))$ non-degenerate m-morphisms of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ of a given width w. Therefore, an element always has a finite number of predecessors, which implies that there is an order-preserving bijection $(\underline{v_\bullet}, \underline{f_\bullet}) : \mathbb{N} \to \mathbb{X}$.

Let $\mathscr{A}_i$ be the relative subcomputad generated by $\mathscr{A} \cup \{(\underline{v_j}, \underline{f_j}) : 0 \leq j < i\}$. We will prove that

- $\mathscr{A}_i$ is right parental;
- the set of right fillable morphisms in $\mathscr{A}'$ but not in $\mathscr{A}_i$ is $\{(\underline{v_j}, \underline{f_j}) : j \geq i\}$;
- the inclusion $\mathscr{A}_i \to \mathscr{A}_{i+1}$ satisfies the hypothesis of Proposition 4.30;

by induction on i.

Let m be the dimension of $(\underline{v_i}, \underline{f_i}) : (A, x_i) \to (Y_i, y_i)$ and $0 \leq l \leq m$, with $l \neq k(\underline{v_i})$. We have to show that $d_l(\underline{v_i}, \underline{f_i}) \in \mathscr{A}_i$. Consider the unique decomposition of $d_l(\underline{v_i}, \underline{f_i})$ as

$$d_l(\underline{v_i}, \underline{f_i}) = c_{s+1} \cdot \sigma_s^*(\underline{u_i^s}, \underline{g_i^s}) \cdots c_1 \cdot \sigma_0^*(\underline{u_i^0}, \underline{g_i^0}) \cdot c_0$$

where $c_j \in (B \leftarrow A) \times S$ and $(\underline{u_i^j}, \underline{g_i^j})$ is a non-degenerate atomic morphism of $\mathscr{A}_{i+1}$ not in $(B \leftarrow A) \times S$, and $\sigma_i : [m_s] \to [m]$ is surjective. Since $\mathscr{A}_i$ is right parental subcomputad, it is enough to show that each $(\underline{u_i^j}, \underline{g_i^j})$ or its right fillable parent (if it is not right fillable) are strictly smaller in $\mathbb{X}$ than $(\underline{v_i}, \underline{f_i})$. This is automatically the case if $w(\underline{u_i^j}) \leq w(\underline{v_i}) - 2$ or if $w(\underline{u_i^j}) \leq w(\underline{v_i}) - 1$ and the domain of $(\underline{u_i^j}, \underline{g_i^j})$ is $(A, \tilde{x})$ for some $\tilde{x}$.

Remark that since every morphism $(\underline{u_i^j}, \underline{g_i^j})$ has an encoding connected component, it has a width greater or equal to 1 if its domain is $(B, \tilde{x})$ for some $\tilde{x}$ and greater or equal to 2 otherwise. Observe that either the underlying squiggle of c_0 is an identity and $(\underline{u_i^0}, \underline{g_i^0})$ has domain $(A, \tilde{x})$ or c_0 has width 1 and $w(d_k(\underline{v_i})) < w(\underline{v_i})$.

Suppose $s > 0$ and $0 \leq j \leq s$. We distinguish two cases.

- The domain of $(\underline{u_i^j}, \underline{g_i^j})$ is $(B, \tilde{x})$ for some $\tilde{x}$, and since $s > 0$, $w(\underline{u_i^j}) \leq w(\underline{v_i}) - 2$.

- The domain of $(\underline{u_i^j}, \underline{g_i^j})$ is $(A, \tilde{x})$ for some $\tilde{x}$, and since $s > 0$, $w(\underline{u_i^j}) \leq w(\underline{v_i}) - 1$.

As a consequence, $d_l(\underline{v_i}, \underline{f_i}) \in \mathscr{A}_i$. We are left with the case $s = 0$, which means $d_l(\underline{v_i}, \underline{f_i}) = c_1\sigma^*(\underline{u_i}, \underline{g_i})c_0$.

case $l \neq k(\underline{v_i}) - 1$**:** Remark that the underlying squiggle of c_0 is always an identity. Indeed,

$$k(d_l(\underline{v_i})) = \begin{cases} k(\underline{v_i}) & l \geq k(\underline{v_i}) + 1 \\ k(\underline{v_i}) - 1 & l \leq k(\underline{v_i}) - 2 \end{cases}$$

It is enough to show that $(\underline{u_i}, \underline{g_i})$ is right fillable. Indeed, since $k(\sigma^*(\underline{v})) \geq k(\underline{v})$ for all degeneracy operators σ, $k(\underline{u_i}) \leq k(\underline{v_i})$. Moreover, the dimension of $(\underline{u_i}, \underline{g_i})$ is strictly smaller than the one of $(\underline{v_i}, \underline{f_i})$. Thus $(\underline{u_i}, \underline{g_i}) < (\underline{v_i}, \underline{f_i})$.

Since $l \neq k(\underline{v_i}), k(\underline{v_i}) - 1$, the only occurrence of $k(\underline{v_i})$ in the underlying squiggle of $d_l(\underline{v_i})$ is the penultimate letter. As a consequence, the only occurrence of $k(\underline{u_i})$ in the underlying squiggle of $\underline{u_i}$ is also the penultimate letter. Moreover,

$$\pi_0\text{Strip}(\underline{u_i}, k(\underline{u_i})) = \pi_0\text{Strip}(\underline{v_i}, k(\underline{v_i}))$$

with the same colorings on encoding connected components. Moreover, one also have

$$\pi_0\text{Strip}(\underline{u_i}, k(\underline{u_i}) - 1) = \pi_0\text{Strip}(\underline{v_i}, k(\underline{v_i}) - 1)$$

with the same colorings on encoding connected components. The facts that $\underline{f_i}$ is degenerate at $k(\underline{v}) - 1$, $\mathcal{C}\Delta[n]^{\text{co}}$ is a 2-category and $l \neq k(\underline{v_i}) - 1$ imply that $d_l(\underline{f_i})$ is degenerate at $k(d_l\underline{v}) - 1$. As a consequence, $\underline{g_i}$ is degenerate at $k\underline{u_i} - 1$.

case $l = k(\underline{v_i}) - 1$**:** If in the process of forming the underlying squiggle of $d_l(\underline{v_i})$, a reduction step occurs, then $w(d_l(\underline{v_i})) \leq w(\underline{v_i}) - 2$. It is thus enough to consider the case where $w(d_l(\underline{v_i})) = w(\underline{v_i})$. Then, $k(d_l(\underline{v_i})) = k(\underline{v_i}) - 1$. Either $(\underline{u_i}, \underline{g_i})$ or its right fillable parent have a depth strictly smaller than the one of $\underline{v_i}$.

The relative subcomputad $\mathscr{A}_0 = \mathscr{A}$ is right parental. Thus, the induction can start at $i = 0$. Supposing that $\mathscr{A}_i$ is right parental, Proposition 4.30 and the previous argument implies that $\mathscr{A}_{i+1}$ is right parental, and the set of right fillable morphisms in $\mathscr{A}'$ but not in $\mathscr{A}_{i+1}$ is $\{(\underline{v_j}, \underline{f_j}) : j \geq i\} \setminus \{(\underline{v_i}, \underline{f_i})\}$.

Moreover, since $\mathscr{A}'$ is right parental, $\mathscr{A}' \subseteq \bigcup_{i \in \mathbb{N}} \mathscr{A}_i$. It is also clear that $\mathscr{A}_i \subseteq \mathscr{A}'$, which ends the proof. □

4.3 Proof of Theorem A

4.3.1 Lifting against a morphism of L_2

We now state a proposition which is the dual of [45, Proposition 4.3.6]. Roughly, the universal property of a counit enables us to solve lifting problems against $\mathfrak{C}[\partial\Delta[m]] \to \mathfrak{C}[\Delta[m]]$, as long as some particular morphisms are obtained from an adjunction. The lift is unique when the counit sits in a 2-category.

Proposition 4.32. *Let $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ be a local isofibration of quasi-categorically enriched categories, $B \mathrel{\substack{\xrightarrow{f} \\ \perp \\ \xleftarrow[u]{}}} A$ an adjunction in $\mathcal{K}$ with unit η and counit ϵ and $m \geq 1$. A lifting problem*

$$\begin{tikzcd}
\mathfrak{C}[\partial\Delta[m]] \arrow[r, "\mathbb{F}"] \arrow[d] & \mathcal{K} \arrow[d, "\mathbb{P}"] \\
\mathfrak{C}[\Delta[m]] \arrow[r] & \mathcal{L}
\end{tikzcd}$$

has a solution, provided that the diagram

$$\begin{tikzcd}
\mathfrak{C}[\Delta[0]] \arrow[r, "{\mathfrak{C}[(d^0)^m]}"] & \mathfrak{C}[\partial\Delta[m]] \arrow[r, "\mathbb{F}"] & \mathcal{K}
\end{tikzcd}$$

is given by

$$\begin{tikzcd}
 & & B \arrow[dl, "f"'] & \\
\mathbb{F}(2) & A \arrow[l, "g"] & & A \arrow[ul, "u"'] \arrow[ll, bend right, "fu"', ""{name=U, below}] \arrow[ll, bend left, "1", ""{name=D, above}] \arrow[from=U, to=D, Rightarrow, "\epsilon"]
\end{tikzcd}$$

for a $g : A \to \mathbb{F}(2)$ (ϵ is seen as a 1-morphism $A \to A$ in $\mathcal{K}$).

Moreover, if $\mathcal{K}$ is a 2-category, the lift is unique.

Before proving the proposition, we recall [45, Lemma 4.3.5], which is key to the proof. We add a condition which imply uniqueness of the lift, and prove this part only. We also state and prove another lemma, which makes the proof of Proposition 4.32 more streamlined.

Lemma 4.33. *Suppose that E and B are quasi-categories which possess terminal objects and that $p : E \twoheadrightarrow B$ is an isofibration which preserves terminal objects, in the sense that if t is terminal in E then pt is terminal in B. Then any lifting* *problem*

$$\begin{tikzcd}
\partial\Delta[m] \arrow[r, "f"] \arrow[d] & E \arrow[d, two heads] \\
\Delta[m] \arrow[r] & B
\end{tikzcd}$$

with $m > 0$ has a solution so long as f carries the vertex $\{m\}$ to a terminal object in E. Moreover, if E is a category, the lift is unique.

Proof. We only prove that when E is a category, the lift is unique, and refer to [45, Lemma 4.3.5] for existence. We use the adjunction $h \dashv N$. When $m \geq 3$, $h(\partial\Delta[m]) = h(\Delta[m])$ and thus uniqueness is true without hypothesis. The map $h(\partial\Delta[2]) \to h(\Delta[2])$ is an epimorphism, and thus uniqueness is also true without hypothesis. Given e and t two objects of E, with t terminal, the universal property of t implies the uniqueness of a morphism $e \to t$. This settles uniqueness of the lift when $m = 1$. ☐

Lemma 4.34. *Let $\mathbf{sSet}_J^{\bullet\to\bullet}$ be the category of arrows of $\mathbf{sSet}_J$, seen as a category of functors and endowed with the projective model structure. There is a Quillen pair*

$$\begin{tikzcd}
\mathbf{sSet}_J \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & (\emptyset \to \Delta[0]) \downarrow \mathbf{sSet}_J^{\bullet\to\bullet} \arrow[l, shift left, "R"]
\end{tikzcd}$$

where L and R are given by

- $L(X) = \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[0] \arrow[r] & X \star \Delta[0] \end{tikzcd}$;
- $R\left(\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & W \arrow[d, "p"] \\ \Delta[0] \arrow[r, "z"] & Z \end{tikzcd}\right) = p_{/z}$ *(See Definition 2.61).*

Proof. Consider a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & W \arrow[d, "p"] \\
X \star \Delta[0] \arrow[r, "f"] & Z
\end{tikzcd}$$

under $\emptyset \to \Delta[0]$, and let $z : \Delta[0] \to Z$ be the base point of Z. The morphism f (under $\Delta[0]$) corresponds to a morphism $\bar{f} : X \to Z_{/z}$ such that the following diagram is commutative

$$\begin{tikzcd}
X \arrow[r, "\bar{f}"] \arrow[d, hook] & Z_{/z} \arrow[d] \arrow[rd, bend left, "\phi"] & \\
X \star \Delta[0] \arrow[r, "\bar{f}\star\Delta[0]"] \arrow[rr, bend right, "f"'] & Z_{/z} \star \Delta[0] \arrow[r, "\epsilon_z^{\Delta[0]}"] & Z,
\end{tikzcd}$$

where $\phi : Z_{/z} \to Z$ is the forgetful map. As a consequence, there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\bar{f}"] & W \arrow[d, "p"] \\
Z_{/z} \arrow[r, "\phi"] & Z
\end{tikzcd}$$

which induces a map $X \to p_{/z}$. This association is the bijection needed to establish the adjunction. Since $- \star \Delta[0] : \mathbf{sSet}_J \to \Delta[0] \downarrow \mathbf{sSet}_J$ is a left Quillen functor (see 2.60), it is enough to show that the image of a cofibration through L is a cofibration. Since L preserves colimits, it is enough to show that that the map

$$\begin{tikzcd}
\partial\Delta[m] \arrow[r] \arrow[d] & \Delta[m] \arrow[d] \\
\partial\Delta[m] \star \Delta[0] \arrow[r] & \Delta[m] \star \Delta[0]
\end{tikzcd}$$

is a projective cofibration under $\emptyset \to \Delta[0]$. Recall that $\partial\Delta[m]\star\Delta[0] = \Lambda^{m+1}[m+1]$. Moreover, the top and bottom squares of the following commutative diagrams

$$\begin{tikzcd}
\partial\Delta[m] \arrow[rr, "1"] \arrow[d] \arrow[rdd] & & \partial\Delta[m] \arrow[d] \arrow[rdd] & \\
\Delta[0] \sqcup \partial\Delta[m] \arrow[rr, "\{m+1\}+d^{m+1}"] \arrow[rdd] & & \Lambda^{m+1}[m+1] \arrow[rdd] & \\
& \Delta[m] \arrow[rr, "1"] \arrow[d] & & \Delta[m] \arrow[d] \\
& \Delta[0] \sqcup \Delta[m] \arrow[rr, "\{m+1\}+d^{m+1}"] & & \partial\Delta[m+1]
\end{tikzcd}$$

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[d] \arrow[ddr] & & \Delta[m] \arrow[d] \arrow[ddr] & \\
\Delta[0] \sqcup \partial\Delta[m+1] \arrow[rr, "\{m+1\}+1"] \arrow[ddr] & & \partial\Delta[m+1] \arrow[ddr] & \\
& \emptyset \arrow[rr] \arrow[d] & & \Delta[m] \arrow[d] \\
& \Delta[0] \sqcup \Delta[m+1] \arrow[rr, "\{m+1\}+1"] & & \Delta[m+1]
\end{tikzcd}$$

are pushout squares, and thus those two diagrams are pushouts in $\mathbf{sSet}_J^{\bullet\to\bullet}$ (under $\emptyset \to \Delta[0]$)). The left faces of the diagrams are projective cofibrations, and thus so are the right faces. $\square$

Proof of Proposition 4.32. Let $X = \mathbb{F}(2) \in |\mathcal{K}|$. Observe that simplicial functors $\mathbb{G} : \mathfrak{Z}[Y] \to \mathcal{K}$ such that $\mathbb{G}(0) = A$, $\mathbb{G}(1) = B$, $\mathbb{G}(2) = X$ and $\mathbb{G}(* : 0 \to 1) = u$ are in bijective correspondence with commutative diagrams

$$\begin{tikzcd}
Y \arrow[r, "\mathbb{G}_{12}"] \arrow[d] & \mathcal{K}(B,X) \arrow[d, "\mathcal{K}(u,X)"] \\
Y \star \Delta[0] \arrow[r, "\mathbb{G}_{02}"] & \mathcal{K}(A,X)
\end{tikzcd}$$

under $(\emptyset \to \Delta[0])$, where the composite $\begin{tikzcd} \Delta[0] \arrow[r] \arrow[rr, bend left, "g"] & Y \star \Delta[0] \arrow[r, "\mathbb{G}_{02}"] & \mathcal{K}(A,X) \end{tikzcd}$ is the base point of $\mathcal{K}(A,X)$. By the adjunction of Lemma 4.34, these commutative diagrams are in bijective correspondence with maps $Y \to \mathcal{K}(u,X)_{/g}$. Therefore, the original lifting problem is equivalent to the lifting problem

$$\begin{tikzcd}
\partial\Delta[m] \arrow[r, "\gamma"] \arrow[d] & \mathcal{K}(u,X)_{/g} \arrow[d] \\
\Delta[m] \arrow[r] & \mathcal{L}(\mathbb{P}u, \mathbb{P}X)_{/\mathbb{P}g}.
\end{tikzcd}$$

Remark that by assumption, $\gamma(m) = g\epsilon : gfu \to g$. There is an adjunction

$$\begin{tikzcd}
\mathcal{K}(B,X) \arrow[r, shift left, "\mathcal{K}(u,X)"] \arrow[r, phantom, "\perp"] & \mathcal{K}(A,X) \arrow[l, shift left, "\mathcal{K}(f,X)"]
\end{tikzcd}$$

in $\mathbf{qCat}_\infty$ with unit $\mathcal{K}(\eta, X)$ and counit $\mathcal{K}(\epsilon, X)$. By [43, Proposition 4.4.8], since $g : A \to X$ is a 0-arrow in $\mathcal{K}(A,X)$, the counit $g\epsilon : gfu \to g$ is an object of the slice quasi-category $\mathcal{K}(u, X)_{/g}$ which is terminal. There is also an adjunction $\begin{tikzcd}\mathbb{P}B \arrow[r, shift left, "\mathbb{P}f"] \arrow[r, phantom, "\perp"] & \mathbb{P}A \arrow[l, shift left, "\mathbb{P}u"]\end{tikzcd}$ in $\mathcal{L}$ with unit $\mathbb{P}\eta$ and counit $\mathbb{P}\epsilon$. Since $\mathbb{P}$ is a local isofibration, it induces a fibration

$$\begin{tikzcd}
\mathcal{K}(B, X) \arrow[r, "\mathbb{P}_{BX}"] \arrow[d, "\mathcal{K}(u,X)"] & \mathcal{L}(\mathbb{P}B, \mathbb{P}X) \arrow[d, "\mathcal{L}(\mathbb{P}u,\mathbb{P}X)"] \\
\mathcal{K}(A, X) \arrow[r, "\mathbb{P}_{AX}"] & \mathcal{L}(\mathbb{P}A, \mathbb{P}X)
\end{tikzcd}$$

with base points g and $\mathbb{P}g$ in $(\emptyset \to \Delta[0]) \downarrow \mathbf{sSet}_J^{\bullet\to\bullet}$. By Lemma 4.34, $\mathcal{K}(u, X)_{/g} \to \mathcal{L}(\mathbb{P}u, \mathbb{P}X)_{/\mathbb{P}g}$ is an isofibration. As a consequence, by Lemma 4.33 there is a lift in any diagram

$$\begin{tikzcd}
\partial\Delta[m] \arrow[r, "\gamma"] \arrow[d] & \mathcal{K}(u, X)_{/g} \arrow[d] \\
\Delta[m] \arrow[r] & \mathcal{L}(\mathbb{P}u, \mathbb{P}X)_{/\mathbb{P}g}
\end{tikzcd}$$

which is unique when $\mathcal{K}$ is a 2-category. □

4.3.2 The proof

We are ready to prove Theorem A, which we restate below for the reader's convenience.

Theorem A. *Let $\mathscr{A} \subseteq \mathscr{A}'$ be relative right parental subcomputads of $\mathbf{Adj}_{\mathrm{hc}}^{S}[n]$, where S is a convenient 2-subcategory of $\mathcal{C}\Delta[n]^{\mathrm{co}}$, and $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ a local isofibration of quasi-categorically enriched categories. Suppose furthermore that $\mathscr{A}$ contains l_i and ϵ_i for all $i \in |S|$. Then, a lifting problem*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "\mathbb{F}"] \arrow[d] & \mathcal{K} \arrow[d, "\mathbb{P}"] \\
\mathscr{A}' \arrow[r] & \mathcal{L}
\end{tikzcd}$$

has a solution, provided that for all $i \in |S|$, $\mathbb{F}(\epsilon_i)$ is the counit of an adjunction $\mathbb{F}(l_i) \dashv \mathbb{F}(r_i)$ in $\mathcal{K}$. Moreover, if $\mathcal{K}$ is a 2-category, the lift is unique.

Proof. By Theorem 4.31, it is enough to show the result for a pushout of the form (4.2) and (4.3). The first case follows easily from the fact that $\Lambda^{k(\underline{v})}[m]$ is an inner horn and $\mathbb{P}$ is a local isofibration. Let $(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ be a right fillable m-arrow with $k(\underline{v}) = m$, $m \geq 2$ and consider the associated diagram

$$\begin{tikzcd}
\mathfrak{Z}[\partial\Delta[m-1]] \arrow[rr, "\delta_{(\underline{v},\underline{f})}"] \arrow[d] & & \mathscr{A} \arrow[r, "\mathbb{F}"] \arrow[d] & \mathcal{K} \arrow[d, "\mathbb{P}"] \\
\mathfrak{Z}[\Delta[m-1]] \arrow[rr, "{((\underline{v},\underline{f}),(\underline{v},\underline{f})^\diamond,r_x)}"'] & & \mathscr{A}' \arrow[r] & \mathcal{L}
\end{tikzcd}$$

where the left square is a pushout. We have to check that $\mathbb{F}\delta_{(\underline{v},\underline{f})}$ verifies the conditions of Proposition 4.32 with respect to the adjunction $\mathbb{F}(l_x) \dashv \mathbb{F}(r_x)$ given by hypothesis. By construction, $\mathbb{F} \cdot \delta_{(\underline{v},\underline{f})} \cdot \mathfrak{Z}[(d_0)^{m-1}]$ is the diagram

$$\begin{tikzcd}
& \mathbb{F}(B,x) \arrow[dl, "\mathbb{F}((d_0)^{m-1}(\underline{v},\underline{f})^\diamond)"'] & \\
\mathbb{F}(Y,y) & & \mathbb{F}(A,x) \arrow[ul, "\mathbb{F}(r_x)"'] \arrow[ll, bend right=10, "\mathbb{F}((d_0)^{m-1}d_m(\underline{v},\underline{f}))"', ""{name=U}] \arrow[ll, bend left=10, "\mathbb{F}((d_0)^{m}(\underline{v},\underline{f}))", ""{name=D}'] \arrow[from=U, to=D, phantom, "\Downarrow \mathbb{F}((d_0)^{m-1}(\underline{v},\underline{f}))"]
\end{tikzcd}$$

Remark that since $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ is a 2-category, $(d_0)^{m-1}(\underline{v}, \underline{f})$ is obtained from $(\underline{v}, \underline{f})$ by looking at the two last strips of $(\underline{v}, \underline{f})$. By right fillability of $(\underline{v}, \underline{f})$, there is a unique turn in the last strip of $\underline{v}$. Let

$$z_0 \xleftarrow[g_1]{} z_1 \xleftarrow[g_2]{} \ldots \longleftarrow z_{t-1} \xleftarrow[g_t]{} z_t = x$$

be the decomposition in $\mathcal{C}\Delta[n]^{\mathrm{co}}$ of the atomic component of $\underline{f}$ (in S) containing x. Let c be the maximal color of an encoding connected component of a strip of $\underline{v}$. Since $(\underline{v}, \underline{f})$ is atomic and by Proposition 3.20,$c > z_0$. Since the last strip has minimal coloring, and since color can only increase while going down or right, the last encoding connected component of the penultimate row has color c. Finally, since $\underline{f}$ is degenerate at $m - 1$, and $\underline{f}$ is compatible with $\underline{v}$, there is no witness of the separation of $z_i + 1$ from z_{i+1} in the penultimate row. Thus, $c \in \{z_1, \ldots, z_t\}$. Since S is convenient, $z_1, \ldots, z_{t-1} \notin |S|$. As a consequence, $c = x$.

Let $\breve{g} = (d^0)^m(\underline{v}, \underline{f})$ and $g = \mathbb{F}(\breve{g})$. Therefore, $(d_0)^{m-1}(\underline{v}, \underline{f}) = \breve{g} \cdot \epsilon_x$, and thus $\mathbb{F}((d_0)^{m-1}(\underline{v}, \underline{f})) = gF(\epsilon_x)$. This also implies that $\mathbb{F}((d^0)^{m-1}(\underline{v}, \underline{f})^\diamond) = g\mathbb{F}(l_x)$.

We still have to solve the case $m = 1$. We claim that there is no right fillable 1-morphism in $\mathscr{A}' \setminus \mathscr{A}$. Let $(\underline{v}, \underline{f}) : (A, x) \to (Y, y)$ be a right fillable 1-morphism. For the same reason as before, observe that $(\underline{v}, \underline{f}) = d_0(\underline{v}, \underline{f}) \cdot \epsilon_x$. Since $(\underline{v}, \underline{f})$ is atomic, $(\underline{v}, \underline{f}) = \epsilon_x$ and therefore belongs to $\mathscr{A}$. □

Chapter 5

Homotopy coherent diagrams

In this chapter, we fix a convenient 2-subcategory $S \subseteq \mathcal{C}\Delta[n]^{\mathrm{co}}$ which plays the role of a shape. The reader can consult the introduction of Chapter 3 for a list of possible choices for S.

In Section 5.1, we obtain the 2-universal property of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ as a consequence of Theorem A. We also determine the 2-universal property of the full 2-subcategory of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ generated by the set of objects $\{B\} \times |S|$, which is denoted $\mathbf{Mnd}^S_{\mathrm{hc}}[n]$ and is related to diagrams of monads. This motivates the definition of homotopy coherent diagrams of (homotopy coherent) monads and of (homotopy coherent) adjunctions in an ∞-cosmos $\mathcal{K}$, respectively as simplicial functors $\mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathcal{K}$ and $\mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathcal{K}$.

In Section 5.2, we strengthen Theorem A. Instead of mere existence, we show that an appropriate space of lifts is contractible, which should be interpreted as essential uniqueness of the lift. We study specifically the case where the inclusion of relative parental subcomputads is $\mathscr{A} \subseteq \mathbf{Adj}^S_{\mathrm{hc}}[n]$, where $\mathscr{A}$ is as small as Theorem A permits. This enables us to determine the least data needed to identify a homotopy coherent diagram of adjunctions up to equivalence (See Corollary 5.14). Strikingly, Section 5.2 is a formal consequence of Theorem A itself, together with some similar results of Riehl and Verity for **Adj** [45].

The last section of this chapter is devoted to the relationship between homotopy coherent diagrams of adjunctions and of monads. It is obvious from the definition that a homotopy coherent diagram of adjunctions induces a homotopy coherent diagram of monads by restriction. Conversely, given a homotopy coherent diagram of monads $\mathbb{T} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathcal{K}$, the enriched right Kan extension $\mathrm{Ran}_{j_S}\mathbb{T} : \mathbf{Adj}_{\mathrm{hc}}[n] \to \mathcal{K}$ along the inclusion provides a way to construct a *free* homotopy coherent diagram of adjunctions associated to a homotopy coherent diagram of monads. We show in Corollary 5.25 that the comparison maps $\mathrm{Ran}_{j_S}\mathbb{T}(A, k) \to \mathrm{Alg}(\mathbb{T}|_k)$ are equivalences, where $\mathbb{T}|_k$ denotes the homotopy coherent monad of level k, obtained by the appropriate precomposition $\mathbf{Mnd} \longrightarrow \mathbf{Mnd}^S_{\mathrm{hc}}[n] \xrightarrow{\mathbb{T}} \mathcal{K}$.

5.1 Universal property of $\mathbf{Adj}^S_{\mathbf{hc}}[n]$ and of $\mathbf{Mnd}^S_{\mathbf{hc}}[n]$

In this section, we prove the 2-universal property of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$, as a corollary of Theorem A. We also determine the 2-universal property of the full 2-subcategory of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ generated by the set of objects $\{B\} \times |S|$.

Corollary 5.1. *Let S be a convenient 2-subcategory of $\mathcal{C}\Delta[n]^{\mathrm{co}}$ and $\mathscr{C}$ a 2-category. There is a natural bijection*

$$2\text{-}\mathbf{Cat}(\mathbf{Adj}^S_{\mathrm{hc}}[n], \mathscr{C}) \cong 2\text{-}\mathbf{Cat}(S, \mathbf{Adj}_r(\mathscr{C})). \tag{5.1}$$

Proof. Let $F : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathscr{C}$ be a 2-functor. Remark that there is an adjunction $\begin{tikzcd}(B,j) \arrow[r, shift left, "l_j"] \arrow[r, phantom, "\perp"] & (A,j) \arrow[l, shift left, "r_j"]\end{tikzcd}$ in $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ for all $j \in |S|$. Moreover, if $\underline{f} : i \to j$ is a 1-cell of S, the pair $\Big(((B), \underline{f}), ((A), \underline{f})\Big)$ constitutes a 1-cell

$$\begin{tikzcd}(B,i) \arrow[r, shift left, "l_i"] \arrow[r, phantom, "\perp"] & (A,i) \arrow[l, shift left, "r_i"] \arrow[r] & (B,j) \arrow[r, shift left, "l_j"] \arrow[r, phantom, "\perp"] & (A,j) \arrow[l, shift left, "r_j"]\end{tikzcd}$$

Finally, if $\underline{g} : i \to j$ is a 2-cell $d_1\underline{g} \Rightarrow d_0\underline{g}$ of S, the pair $\Big(((B), \underline{g}), ((A), \underline{g})\Big)$ constitutes an adjunction transformation

$$\Big(((B), d_1\underline{g}), ((A), d_1\underline{g})\Big) \Rightarrow \Big(((B), d_0\underline{g}), ((A), d_0\underline{g})\Big)$$

We have defined a 2-functor $i_S : S \to \mathbf{Adj}_r(\mathbf{Adj}^S_{\mathrm{hc}}[n])$. By 2-functoriality of $\mathbf{Adj}_r(-)$, F induces a 2-functor

$$\bar{F} : \mathbf{Adj}_r(\mathbf{Adj}^S_{\mathrm{hc}}[n]) \to \mathbf{Adj}_r(\mathscr{C}).$$

We associate to F the 2-functor $F^\flat := \bar{F} \circ i_S : S \to \mathbf{Adj}_r(\mathcal{C})$.

Conversely, let $G : S \to \mathbf{Adj}_r(\mathcal{C})$ be a 2-functor. Let us consider Mono (see Examples (ii)) the relative simplicial computad generated by monochromatic nondegenerate atomic arrows of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$. Recall that it is right parental since † never changes the set of colors used on encoding connected component of strips. Remark that there is a forgetful 2-functor $\mathbf{Adj}_r(\mathcal{C}) \to 2\text{-}\mathbf{Cat}(\bullet \to \bullet, \mathcal{C})$ obtained by forgetting all adjunction data but the right adjoints. As a consequence, we can extract from G a 2-functor $\bar{G} : (B \leftarrow A) \times S \to \mathcal{C}$. By Lemma 2.13, in order to extend $\bar{G}$ to a simplicial functor $\hat{G} : \mathrm{Mono} \to \mathcal{C}$, it is enough to specify the image of the monochromatic non-degenerate atomic arrows $(X, i) \to (Y, i)$ for all $i \in |S|$ in a coherent fashion. But since $\mathbf{Adj}$ is the universal 2-category containing an adjunction, we can use the classifying 2-functor $\mathbb{A}_i : \mathbf{Adj} \to \mathcal{C}$ of the adjunction $G(i)$ to define the images of the monochromatic arrows of color i. By construction, $\hat{G}$ satisfies the hypothesis of Theorem A with respect to the inclusion $\mathrm{Mono} \to \mathbf{Adj}_{\mathrm{hc}}[n]$. We define $G^\sharp$ to be the unique lift $G^\sharp : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathscr{C}$ of $\hat{G}$. Remark also that by construction, $(G^\sharp)^\flat = G$.

Let $F : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathscr{C}$ be a 2-functor. Observe that F is a lift of $\hat{F^\flat} : \mathrm{Mono} \to \mathscr{C}$, and thus by uniqueness in Theorem A, $(F^\flat)^\sharp = F$. ☐

Notation 5.2. We denote by $\zeta_S : 2\text{-}\mathbf{Cat}(\mathbf{Adj}^S_{\mathrm{hc}}[n], \mathcal{C}) \to 2\text{-}\mathbf{Cat}(S, \mathbf{Adj}_r(\mathcal{C}))$ the natural bijection of Corollary 5.1.

Definition 5.3. Let $\mathbf{Mnd}^S_{\mathrm{hc}}[n]$ be the 2-category determined up to isomorphism by the existence of a natural bijection

$$\xi_S : 2\text{-}\mathbf{Cat}(\mathbf{Mnd}^S_{\mathrm{hc}}[n], \mathscr{C}) \to 2\text{-}\mathbf{Cat}(S, \mathbf{Mnd}(\mathscr{C}))$$

The existence of $\mathbf{Mnd}^S_{\mathrm{hc}}[n]$ can be obtained by constructing it using a presentation by computads. We will instead determine a concrete model for $\mathbf{Mnd}^S_{\mathrm{hc}}[n]$. Recall that when $\mathscr{C}$ admits the construction of algebras, there is a 2-adjunction $\begin{tikzcd}\mathbf{Adj}_r(\mathscr{C}) \arrow[r, shift left, "M"] \arrow[r, phantom, "\perp"] & \mathbf{Mnd}(\mathscr{C}) \arrow[l, shift left, "\mathrm{Alg}"]\end{tikzcd}$, where the counit is an identity. As a consequence, there is an induced 2-adjunction

$$\begin{tikzcd}2\text{-}\mathbf{Cat}(S, \mathbf{Adj}_r(\mathscr{C})) \arrow[r, shift left, "M_*"] \arrow[r, phantom, "\perp"] & 2\text{-}\mathbf{Cat}(S, \mathbf{Mnd}(\mathscr{C})) \arrow[l, shift left, "\mathrm{Alg}_*"]\end{tikzcd} .$$

which can be replaced by

$$\begin{tikzcd}2\text{-}\mathbf{Cat}(\mathbf{Adj}^S_{\mathrm{hc}}[n], \mathscr{C}) \arrow[r, shift left, "\xi_S^{-1} M_* \zeta_S"] \arrow[r, phantom, "\perp"] & 2\text{-}\mathbf{Cat}(\mathbf{Mnd}^S_{\mathrm{hc}}[n], \mathscr{C}) \arrow[l, shift left, "\zeta_S^{-1} \mathrm{Alg}_* \xi_S"]\end{tikzcd} .$$

Let $j_S : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{Adj}^S_{\mathrm{hc}}[n]$ be $\xi_S^{-1} M_* \zeta_S(1_{\mathbf{Adj}^S_{\mathrm{hc}}[n]})$ and consider a 2-functor $F : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathbf{Cat}$. By naturality, $F j_S = F_* \xi_S^{-1} M_* \zeta_S(1_{\mathbf{Adj}^S_{\mathrm{hc}}[n]}) = \xi_S^{-1} M_* \zeta_S(F)$. As a consequence, the left adjoint of the adjunction

$$\begin{tikzcd}2\text{-}\mathbf{Cat}(\mathbf{Adj}^S_{\mathrm{hc}}[n], \mathbf{Cat}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & 2\text{-}\mathbf{Cat}(\mathbf{Mnd}^S_{\mathrm{hc}}[n], \mathbf{Cat}) \arrow[l, shift left]\end{tikzcd} .$$

is j_S^*, and thus the right adjoint is given by the enriched right Kan extension $\mathrm{Ran}_{j_S}(-)$. The counit is an isomorphism. But the component of the counit at a 2-functor $\mathbb{T} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{Cat}$ evaluated at an object X is given by the morphism of weighted limits

$$\{\mathbf{Adj}^S_{\mathrm{hc}}[n](j_S(X), j_S(-)), \mathbb{T}\} \to \{\mathbf{Mnd}^S_{\mathrm{hc}}[n](X, -), \mathbb{T}\}$$

induced by the morphisms of weights given by j_S. By Corollary 2.40 and the Yoneda lemma, j_S induces a natural isomorphism

$$j_{S X-} : \mathbf{Mnd}^S_{\mathrm{hc}}[n](X, -) \longrightarrow \mathbf{Adj}^S_{\mathrm{hc}}[n](j_S(X), j_S(-)).$$

Thus j_S is full and faithful. It is not hard to check that it is actually injective on objects. This exhibits $\mathbf{Mnd}^S_{\mathrm{hc}}[n]$ as the full 2-subcategory of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ generated by the set of objects $\{B\} \times |S|$.

Notation 5.4. We denote by $\mathbf{Mnd}_{\mathrm{hc}}[n]$ the 2-category $\mathbf{Mnd}^{\mathcal{C}\Delta[n]^{\mathrm{co}}}_{\mathrm{hc}}[n]$. We denote by $j : \mathbf{Mnd} \to \mathbf{Adj}$ the inclusion $\mathbf{Mnd}_{\mathrm{hc}}[0] \to \mathbf{Adj}_{\mathrm{hc}}[0]$. We also write j_n for the inclusion $\mathbf{Mnd}_{\mathrm{hc}}[n] \to \mathbf{Adj}_{\mathrm{hc}}[n]$.

The 2-universal properties of $\mathbf{Mnd}^S_{\mathrm{hc}}[n]$ and $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ motivate the following definitions.

Definition 5.5. A *homotopy coherent diagram of monads of shape S in an ∞-cosmos $\mathcal{K}$* is a simplicial functor $\mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathcal{K}$.

Definition 5.6. A *homotopy coherent diagram of adjunctions of shape S in an ∞-cosmos $\mathcal{K}$* is a simplicial functor $\mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathcal{K}$.

5.2 Minimal data defining a homotopy coherent diagram of adjunctions

In this part, we use Theorem A to describe the minimal data needed to specify a homotopy coherent diagram of adjunctions in an ∞-cosmos $\mathcal{K}$, up to a zigzag of natural weak equivalences. If we apply Theorem A to the inclusion of relative right parental subcomputads $\overline{\{\epsilon_i : i \in |S|\}} \subseteq \mathbf{Adj}^S_{\mathrm{hc}}[n]$ (See Examples 4.28 (i)) and local isofibration $\mathcal{K} \to 1$, we obtain that any simplicial functor

$$\mathbb{A} : \overline{\{\epsilon_i : i \in |S|\}} \to \mathcal{K}$$

sending ϵ_i to the counit of an adjunction in $\mathcal{K}$ can be extended to a simplicial functor $\mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathcal{K}$. Observe that by Corollary 2.13, providing a simplicial functor $\mathbb{A} : \overline{\{\epsilon_i : i \in |S|\}} \to \mathcal{K}$ is the same as providing a simplicial functor

$$(B \leftarrow A) \times S \to \mathcal{K}$$

encoding the diagram of right adjoints, together with choices of left adjoints and counits for each right adjoint. We show that this diagram of *right adjoints* is enough to determine the full homotopy coherent diagram up to a contractible space of choices (see Corollary 5.13). This implies that the full homotopy coherent diagram is determined up to zig-zag of natural weak equivalences (see Corollary 5.14). We start by providing an explicit meaning for this *space of lifts*, following Riehl and Verity [45].

Definition 5.7. Let $\mathcal{K}$ be an ∞-cosmos and $X \in |\mathbf{sSet}|$. Since $(-)^X$ preserves products, there is a simplicial category $\mathcal{K}^X$ with the same objects as $\mathcal{K}$ and $\mathcal{K}^X(K,L) = \mathcal{K}(K,L)^X$. The simplicial categories $\mathcal{K}^\Delta[n]$ assemble into a simplicial object $\mathcal{K}^{\Delta[-]}$. Let

$$\mathrm{Icon}(\mathcal{K},\mathcal{L}) = \mathbf{sCat}(\mathcal{K},\mathcal{L}^{\Delta[-]})$$

In [45], the authors motivate the notation by the relationship with the 2-categorical icons defined by Lack in [30],

> The notation *icon* is chosen here because a 1-simplex in $\mathrm{Icon}(\mathcal{K},\mathcal{L})$ should be thought of as analogous to an identity component oplax natural transformation in 2-category theory, as defined by Lack [30]. In particular, the simplicial functors $\mathcal{K} \to \mathcal{L}$ serving as the domain and the codomain of a 1-simplex in $\mathrm{Icon}(\mathcal{K},\mathcal{L})$ agree on objects.

Observe that when X is a *connected* simplicial set, there is a bijective correspondence between simplicial maps $X \to \mathrm{Icon}(\mathcal{K},\mathcal{L})$ and simplicial functors $\mathcal{K} \to \mathcal{L}^X$. Indeed, a map $X \to \mathrm{Icon}(\mathcal{K},\mathcal{L})$ corresponds to natural transformation

$$\begin{tikzcd} \Delta \downarrow X \arrow[r, bend left, "{\Delta[-]}"{name=U}] \arrow[r, bend right, "\Delta_{\mathrm{Icon}(\mathcal{K},\mathcal{L})}"'{name=D}] & \mathbf{sSet} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

, where $\Delta \downarrow X$ denotes the category of simplices of X and Δ_W stands for the constant functor with value W. This natural transformation α corresponds to a natural transformation

$$\begin{tikzcd} \Delta \downarrow X \arrow[r, bend left, "\Delta_{\mathcal{K}}"{name=U}] \arrow[r, bend right, "\mathcal{L}^{\Delta[-]}"'{name=D}] & \mathbf{sCat} \arrow[from=U, to=D, phantom, "\Downarrow\bar{\alpha}"] \end{tikzcd}$$

and thus to a map $\mathcal{K} \to \lim_{\Delta\downarrow X} \mathcal{L}^{\Delta[-]}$. If X is connected, so is $\Delta \downarrow X$ and $\lim_{\Delta\downarrow X} \mathcal{L}^{\Delta[-]} = \mathcal{L}^X$. The assumption that X is connected is actually needed. Indeed, this correspondence is wrong when $X = \partial\Delta[1]$. This is related to the fact that the limit of a constant diagram with value X is X if and only if the index category is connected.

If we restrict our attention to maps $X \to \mathrm{Icon}(\mathcal{K},\mathcal{L})$ such that the composite $X \to \mathrm{Icon}(\mathcal{K},\mathcal{L}) \to \mathbf{Set}(|\mathcal{K}|,|\mathcal{L}|)$ is constant, then the assumption that X is connected can be removed. Observe also that this composite is always constant as long as X is connected.

Remark 5.8. By [45, Lemma 4.4.2], the simplicial enrichment described in Definition 5.7 behaves well with respect to relative simplicial computads and local isofibrations. As a consequence,

$$\mathrm{Icon}(\mathscr{A}', \mathcal{K}) \to \mathrm{Icon}(\mathscr{A}, \mathcal{K})$$

is an isofibration if $\mathscr{A} \subseteq \mathscr{A}'$ is an inclusion of relative subcomputads of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$.

Theorem 5.9. *Let $\mathscr{A} \subseteq \mathscr{A}'$ be relative right parental subcomputads of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$, where S is a convenient 2-subcategory of $\mathcal{C}\Delta[n]^{\mathrm{co}}$. Let $\mathcal{K}$ be a category enriched in quasi-categories. If* Mono $\subseteq \mathscr{A}$*, the isofibration*

$$\mathrm{Icon}(\mathscr{A}', \mathcal{K}) \to \mathrm{Icon}(\mathscr{A}, \mathcal{K})$$

is trivial.

Proof. We show that the isofibration has the right lifting property with respect to $\partial\Delta[n] \to \Delta[n]$, for $n \geq 0$. The right lifting property of our isofibration with respect to $\emptyset \to *$ is a consequence of Theorem A. Its hypotheses are verified since Mono $\subseteq \mathscr{A}$, and thus for all $i \in |S|$, we have a simplicial functor $\mathbb{A}_i : \mathbf{Adj} \to$ Mono $\subseteq \mathscr{A} \to \mathcal{K}$ which guarantees that the images of r_i, l_i and ϵ_i are part of an adjunction in $\mathcal{K}$. We thus suppose that $n \geq 1$ and consider a lifting problem

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{Icon}(\mathscr{A}', \mathcal{K}) \arrow[d] \\
\Delta[n] \arrow[r] & \mathrm{Icon}(\mathscr{A}, \mathcal{K}).
\end{tikzcd} \tag{5.2}$$

Since $\Delta[n]$ is connected,

$$\Delta[n] \to \mathrm{Icon}(\mathscr{A}, \mathcal{K}) \to \mathbf{Set}(\{A, B\} \times |S|, |\mathcal{K}|)$$

is constant and so is

$$\partial\Delta[n] \to \mathrm{Icon}(\mathscr{A}', \mathcal{K}) \to \mathbf{Set}(\{A, B\} \times |S|, |\mathcal{K}|).$$

As a consequence, the lifting problem (5.2) is equivalent to the lifting problem given by

$$\begin{tikzcd}
\mathscr{A}' \arrow[r] & \mathcal{K}^{\partial\Delta[n]} \\
\mathscr{A} \arrow[r] \arrow[u] & \mathcal{K}^{\Delta[n]}. \arrow[u]
\end{tikzcd}$$

Observe that $\mathcal{K}^X$ is a category enriched in quasi-categories for any simplicial set X. Moreover, the right-hand vertical map is a local isofibration, since for every

$X, Y \in |\mathcal{K}|$, $\mathcal{K}(X, Y)$ is a quasi-category and $\partial\Delta[n] \to \Delta[n]$ is a cofibration. Again, the hypotheses of Theorem A hold since Mono $\subseteq \mathscr{A}$, and Theorem A completes the proof. □

Observe that by Proposition 2.13, there are pullback diagrams

$$\begin{tikzcd}
\mathrm{Icon}(\mathrm{Mono}, \mathcal{K}) \arrow[r] \arrow[d] & \mathrm{Icon}(\overline{\{\epsilon_i : i \in |S|\}}, \mathcal{K}) \arrow[r] \arrow[d] & \mathrm{Icon}((B \leftarrow A) \times S, \mathcal{K}) \arrow[d] \\
\prod\limits_{i \in |S|} \mathrm{Icon}(\mathbf{Adj}, \mathcal{K}) \arrow[r] & \prod\limits_{i \in |S|} \mathrm{Icon}(\overline{\{\epsilon\}}, \mathcal{K}) \arrow[r] & \prod\limits_{i \in |S|} \mathrm{Icon}(B \leftarrow A, \mathcal{K}),
\end{tikzcd} \tag{5.3}$$

where the vertical maps are induced by suitable restrictions of the evident inclusion $\coprod_{i \in |S|} \mathbf{Adj} \to \mathbf{Adj}^{S}_{\mathrm{hc}}[n]$. Remark that since $\overline{\{\epsilon\}}$ and $B \leftarrow A$ are actually simplicial computads, the simplicial sets appearing in the bottom line of the previous diagram are quasi-categories. Following Riehl and Verity [45], we make the following definitions.

Definition 5.10. Let counit($\mathcal{K}$) be the maximal Kan complex of the full sub-quasi-category of Icon($\overline{\{\epsilon\}}, \mathcal{K}$) generated by those simplicial functors $\overline{\{\epsilon\}} \to \mathcal{K}$ which map respectively l, r, ϵ to a left adjoint f, a right adjoint u and a counit $\tau : fu \to 1$.

Definition 5.11. Let radj($\mathcal{K}$) be the maximal Kan complex of the full sub-quasi-category of Icon($B \leftarrow A, \mathcal{K}$) generated by those simplicial functors $(B \leftarrow A) \to \mathcal{K}$ which map r to a right adjoint.

By [45, Propositions 4.4.12, 4.4.17], the bottom row of the diagram displayed in (5.3) corestricts to a sequence of trivial isofibrations

$$\prod_{i \in |S|} \mathrm{Icon}(\mathbf{Adj}, \mathcal{K}) \longrightarrow \prod_{i \in |S|} \mathrm{counit}(\mathcal{K}) \longrightarrow \prod_{i \in |S|} \mathrm{radj}(\mathcal{K}).$$

Consider the following two pullbacks

$$\begin{tikzcd}
\mathrm{counits}^{S}(\mathcal{K}) \arrow[r] \arrow[d] & \mathrm{Icon}(\overline{\{\epsilon_i : i \in |S|\}}, \mathcal{K}) \arrow[d] & \mathrm{radjs}^{S}(\mathcal{K}) \arrow[r] \arrow[d] & \mathrm{Icon}((B \leftarrow A) \times S, \mathcal{K}) \arrow[d] \\
\prod\limits_{i \in |S|} \mathrm{counit}(\mathcal{K}) \arrow[r] & \prod\limits_{i \in |S|} \mathrm{Icon}(\overline{\{\epsilon\}}, \mathcal{K}) & \prod\limits_{i \in |S|} \mathrm{radj}(\mathcal{K}) \arrow[r] & \prod\limits_{i \in |S|} \mathrm{Icon}(B \leftarrow A, \mathcal{K}).
\end{tikzcd}$$

There is an induced diagram

$$\begin{tikzcd}
\mathrm{Icon}(\mathrm{Mono},\mathcal{K}) \arrow[r] \arrow[d] & \mathrm{counits}^S(\mathcal{K}) \arrow[r] \arrow[d] & \mathrm{radjs}^S(\mathcal{K}) \arrow[d] \\
\prod_{i\in|S|} \mathrm{Icon}(\mathbf{Adj},\mathcal{K}) \arrow[r] & \prod_{i\in|S|} \mathrm{counit}(\mathcal{K}) \arrow[r] & \prod_{i\in|S|} \mathrm{radj}(\mathcal{K}),
\end{tikzcd}$$

and by the Pullback Lemma, the involved squares are also pullbacks. Thus the upper row is also constituted of trivial isofibrations. Together with Theorem 5.9, this analysis established the following result.

Theorem 5.12. *The map*

$$\mathrm{Icon}(\mathbf{Adj}^S_{\mathrm{hc}}[n],\mathcal{K}) \to \mathrm{radjs}^S(\mathcal{K})$$

is a trivial isofibration.

Corollary 5.13. *Let $\mathcal{K}$ be an ∞-cosmos. Suppose $\mathbb{A} : (B \leftarrow A) \times S \to \mathcal{K}$ is a simplicial functor such that $\mathbb{A}(r_i)$ is a right adjoint in $\mathcal{K}$ for all $i \in |S|$. The (homotopy) fiber $F_{\mathbb{A}}$ of the map*

$$\mathrm{Icon}(\mathbf{Adj}^S_{\mathrm{hc}}[n],\mathcal{K}) \to \mathrm{radjs}^S(\mathcal{K})$$

at $\mathbb{A}$ is a trivial Kan complex.

Proof. Observe that there is a pullback diagram

$$\begin{tikzcd}
F_{\mathbb{A}} \arrow[r] \arrow[d] & \mathrm{Icon}(\mathbf{Adj}^S_{\mathrm{hc}}[n],\mathcal{K}) . \arrow[d] \\
* \arrow[r, "\mathbb{A}"] & \mathrm{radjs}^S(\mathcal{K})
\end{tikzcd}$$

Since the right vertical map is a trivial isofibration, so is the left vertical map. Trivial isofibrations are trivial fibrations in the Joyal model structure on simplicial sets. Since the Joyal and the Quillen model structure on simplicial sets share the same set of generating cofibrations, they have the same trivial fibrations. Thus, $F_{\mathbb{A}}$ is a contractible Kan complex. □

Thus, there is an essentially unique lift of a simplicial functor $\mathbb{A} : (B \leftarrow A) \times S \to \mathcal{K}$ such $\mathbb{A}(r_i)$ is a right adjoint in $\mathcal{K}$ for all $i \in |S|$, to a simplicial functor $\mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathcal{K}$. The next corollary is a consequence of this fact, and an immediate generalization of an unpublished result of Riehl and Verity which was communicated to the author in response to one of his questions.

Corollary 5.14. *Let $\mathcal{K}$ be an ∞-cosmos. Suppose $\mathbb{A} : (B \leftarrow A) \times S \to \mathcal{K}$ is a simplicial functor such that $\mathbb{A}(r_i)$ is a right adjoint in $\mathcal{K}$ for all $i \in |S|$, and consider $\overline{\mathbb{A}_1}, \overline{\mathbb{A}_2} : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathcal{K}$ two lifts of $\mathbb{A}$ to homotopy coherent diagrams of adjunctions of shape S. Then, there is a zigzag of natural weak equivalences*

$$\begin{tikzcd}
\mathbf{Adj}^S_{\mathrm{hc}}[n] \arrow[r, bend left=50, "\overline{\mathbb{A}_1}"] \arrow[r, bend right=50, "\overline{\mathbb{A}_2}"'] \arrow[r] \arrow[r, phantom, shift left=2.5ex, "\Uparrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \mathcal{K}
\end{tikzcd}$$

Proof. Observe that by Corollary 5.13 and by Proposition 2.52, we have a map $\mathbb{I} \to \mathrm{Icon}(\mathbf{Adj}^S_{\mathrm{hc}}[n], \mathcal{K})$ whose vertices are precisely $\overline{\mathbb{A}_1}$ and $\overline{\mathbb{A}_2}$. This corresponds to a map $\overline{\mathbb{A}} : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathcal{K}^{\mathbb{I}}$ such that postcomposing with the two obvious projections $p_1, p_2 : \mathcal{K}^{\mathbb{I}} \to \mathcal{K}$ yields respectively $\overline{\mathbb{A}_1}$ and $\overline{\mathbb{A}_2}$. Remark that there is a third simplicial functor $u^{\mathbb{I}} : \mathcal{K}^{\mathbb{I}} \to \mathcal{K}$ which maps an object $X \in |\mathcal{K}|$ to its cotensor $X^{\mathbb{I}}$. One can describe explicitly the action on the homspaces, but this simplicial functor can also be obtained as follows. There is a simplicial monad $(-)^{\mathbb{I}} : \mathcal{K} \to \mathcal{K}$ whose (simplicial) Kleisli category is exactly $\mathcal{K}^{\mathbb{I}}$. The simplicial functor we are describing is the right adjoint $\mathcal{K}^{\mathbb{I}} \to \mathcal{K}$. It is now enough to observe that there is a diagram of natural weak equivalences

$$\begin{tikzcd}
\mathcal{K}^{\mathbb{I}} \arrow[r, bend left=50, "p_1"] \arrow[r, bend right=50, "p_2"'] \arrow[r, "u^{\mathbb{I}}" description] \arrow[r, phantom, shift left=2.5ex, "\Uparrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \mathcal{K}.
\end{tikzcd}$$

□

5.3 Induced homotopy coherent diagrams of free-forgetful adjunctions

We fix an ∞-cosmos $\mathcal{K}$. In this section, we study enriched right Kan extensions of homotopy coherent diagrams of monads $\mathbb{T} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathcal{K}$ along the inclusion

$$j_S : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{Adj}^S_{\mathrm{hc}}[n].$$

We use results from Riehl and Verity's articles [45] and [44], of which we provided a brief overview in 2.4.2 and 2.4.3. In Subsection 5.3.1, we prove that the enriched right Kan extension does exist, by showing that the weights involved in the weighted limits which define it are projectively cofibrant (See Proposition 2.42). In Subsection 5.3.2, we apply the monadicity theorem of Riehl and Verity to show that the comparison maps $\mathrm{Ran}_{j_S}\mathbb{T}(A,k) \to \mathrm{Alg}(\mathbb{T}|_k)$ are equivalences, where $\mathbb{T}|_k$ denotes the homotopy coherent monad of level k, obtained by the appropriate precomposition $\mathbf{Mnd} \longrightarrow \mathbf{Mnd}^S_{\mathrm{hc}}[n] \xrightarrow{\mathbb{T}} \mathcal{K}$. This implies that up to equivalence, homotopy coherent diagrams of monads encode homotopy coherent diagrams between their respective objects of algebras. This relationship will be made more explicit in Theorem 6.30, in the case $S = \bullet \to \bullet$.

5.3.1 A projective cell-complex

In this part, we show that $\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,z), j_S(-)) : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{sSet}$ is a projective cell complex. In order to do so, we study the collage construction

$$\mathcal{C}_z = \mathrm{coll}\,\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,z), j_S(-)).$$

It can be identified with the simplicial subcategory of $\mathbf{Adj}^S_{\mathrm{hc}}[n]$ whose object set is

$$|\mathcal{C}_z| = \left|\mathbf{Mnd}^S_{\mathrm{hc}}[n]\right| \sqcup \{(A,z)\}$$

and whose non-empty homspaces are

- for all $(X,x) \in |\mathcal{C}_z|$ and $y \in |S|$, $\mathcal{C}_z((X,x),(B,y)) = \mathbf{Adj}^S_{\mathrm{hc}}[n]((X,x),(B,y))$;
- $\mathcal{C}_z((A,z),(A,z)) = \{\mathrm{id}\}$.

In [45, Lemma 6.1.8], Riehl and Verity prove that the simplicial category $\mathrm{coll}\,\mathbf{Adj}(A, j(-))$ is a simplicial computad. Its atomic morphisms are the squiggles which do not contain an instance of the letter B in their interior. Our immediate goal is to generalize this result to $\mathcal{C}_z$.

Proposition 5.15. *Let $(\underline{v}, \underline{f}) : (X,x) \to (B,y)$ be an m-morphism of $\mathcal{C}_z$ with $w(\underline{v}) > 0$. It is atomic if and only if the following three conditions are fulfilled.*

(i) Its underlying squiggle does not pass by B between two forests which belongs to S.

(ii) If x' is the minimal leaf of the atomic forest (in S) containing x in $\underline{f}$, there is a connected component of one of the strips of $\underline{v}$ with color greater or equal to x' (non-necessarily encoding).

(iii) If y' is the maximal leaf of the atomic forest (in S) containing $y+1$ in $\underline{f}$, there is a connected component of one of the strips of $\underline{v}$ with color strictly smaller to y' (non-necessarily encoding).

Proof. We proceed by contraposition. We first suppose that $(\underline{v},\underline{f}) : (X,x) \to (Y,y)$ is not atomic, and let

$$(X,x) \xrightarrow{(\underline{v_1},\underline{f_1})} (B,w) \xrightarrow{(\underline{v_2},\underline{f_2})} (Y,y)$$

be a decomposition where both morphisms are not identities. One of the following three cases holds.

(i) The underlying squiggles of both $\underline{v_1}$ and $\underline{v_2}$ are not identities. By Lemma 3.23, the underlying squiggle of $(\underline{v},\underline{f})$ passes by B between two forests of S.

(ii) $\underline{v_1}$ is a degeneracy of $b_{w+1}\cdots b_x$, and $w < x$. The atomic forest containing x has smallest index strictly bigger than w, and all connected components of strips of $\underline{v}$ are coming from $\underline{v_2}$, and thus their colorings are smaller than w. As a consequence, the second condition of the proposition is unmet.

(iii) $\underline{v_2}$ is a degeneracy of $b_{y+1}\cdots b_w$, with $y < w$. The maximal leaf of the atomic forest containing $y+1$ in $\underline{f}$ is at most w. All connected components of a strip of $\underline{v}$ comes from $\underline{v_1}$, and thus has color greater or equal to w. Thus, the third condition is unmet.

Let us prove the converse implication. By Lemma 3.23, it is enough to show that the negation of the second and third conditions implies that $(\underline{v},\underline{f})$ is not atomic. We prove it for the second condition, the third one being completely similar. Remark that the condition implies that $X \neq A$. Let $\underline{g}$ be the atomic component of $\underline{f}$ containing x, $\underline{h}$ be the composite of all others atomic components of $\underline{f}$, and $\tilde{x}$ the maximal index of a leaf in $\underline{h}$. Then $(\underline{v},\underline{f}) = (\underline{\tilde{v}},\underline{h}) \cdot ((B),\underline{g})$, where $\underline{\tilde{v}} : (B,\tilde{x}) \to (Y,y)$ is obtained from $\underline{v}$ by changing its source. The arrow $(\underline{\tilde{v}},\underline{h})$ is not an identity since $w(\underline{\tilde{v}}) > 0$, and $\underline{g}$ is a non-trivial forest. □

Proposition 5.16. *The simplicial category $\mathcal{C}_z$ is a simplicial computad.*

Proof. The existence of the decomposition in atomic morphisms is obvious from the fact that every m-morphism of $\mathcal{C}_z$ has an underlying squiggle which is finite, and an underlying forest which is also finite. We prove uniqueness of the decomposition of a morphism $(\underline{v},\underline{f}) : (X,x) \to (B,y)$ by double induction, on $x-y$ and on $w(\underline{v})$. The first induction can start since coll $\mathbf{Adj}(A, i(-))$ is a simplicial computad, whereas the second can always start since S is one.

Suppose now that $(\underline{v},\underline{f}) : (X,x) \to (B,y)$ has width $m > 0$. If it is atomic, there is nothing to prove. Otherwise, consider a decomposition

$$(X,x) \xrightarrow{(\underline{v_1},\underline{f_1})} (B,z_1) \xrightarrow{(\underline{v_2},\underline{f_2})} \dots \xrightarrow{(\underline{v_k},\underline{f_k})} (B,y)$$

in atomic morphisms of $\mathcal{C}_z$ of it. Since $(\underline{v}, \underline{f})$ is not atomic, it does not satisfy one of the conditions (i), (ii), (iii) of Proposition 5.15.

- If $(\underline{v}, \underline{f})$ does not satisfy (ii), then $X = B$ and $(\underline{v_1}, \underline{f_1}) = ((B), \underline{f_1})$, where $\underline{f_1} : x \to z_1$ is the atomic component of $\underline{f}$ containing the leaf with index x. Moreover,
$$(\underline{v_k}, \underline{f_k}) \cdots (\underline{v_2}, \underline{f_2}) = (\underline{v}, \underline{f_k} \cdots \underline{f_2}) : (B, z_1) \to (B, y)$$
has a unique decomposition by induction hypothesis, since $z_1 - y < x - y$.
- If $(\underline{v}, \underline{f})$ does not satisfy (iii), then $(\underline{v_k}, \underline{f_k}) = ((B), \underline{f_k})$, where $\underline{f_k} : z_{k-1} \to y$ is the atomic component of $\underline{f}$ containing the leaf with index $y+1$. Moreover,
$$(\underline{v_{k-1}}, \underline{f_{k-1}}) \cdots (\underline{v_1}, \underline{f_1}) = (\underline{v}, \underline{f_{k-1}} \cdots \underline{f_1}) : (B, x) \to (B, z_{k-1})$$
has a unique decomposition by induction hypothesis, since $x - z_{k-1} < x - y$.
- If $(\underline{v}, \underline{f})$ does satisfy (ii) and (iii) but not (i), let j be the smallest integer such that the underlying squiggle of $(\underline{v}, \underline{f})$ passes by B at j between two forest of S. By Lemmas 3.24 and 3.23, $\underline{v_k} = (v_0, \ldots, v_j)$ and $\underline{v_{k-1}} \cdots \underline{v_1} = (v_j, \ldots v_{w(\underline{v})})$. By Equation (3.3), the colorings of both morphisms of $\mathbf{Adj}[n]$ are determined by the coloring of $\underline{v}$. Moreover, since $(\underline{v_k}, \underline{f_k})$ is atomic, $\underline{f_k}$ is the composite of all atomic components of $\underline{f}$ that contain a leaf smaller or equal to the color of any connected component of a strip of $\underline{v_k}$, whereas $\underline{f_{k-1}} \cdots \underline{f_1}$ is the composite of all other atomic components of $\underline{f}$. Thus, $(\underline{v_k}, \underline{f_k})$ is uniquely determined by $(\underline{v}, \underline{f})$ and so does $(\underline{v_{k-1}}, \underline{f_{k-1}}) \cdots (\underline{v_1}, \underline{f_1})$. By the second induction hypothesis, the latter morphism has a unique decomposition since $w(\underline{v_{k-1}} \cdots \underline{v_1}) < w(\underline{v})$.

□

Proposition 5.17. *For all $z \in |S|$, the natural transformation*
$$r_z^* : \mathbf{Mnd}^S_{\mathrm{hc}}[n]((B, z), -) \to \mathbf{Adj}^S_{\mathrm{hc}}[n]((A, z), j_S(-))$$
is a relative projective cell complex, which is levelwise bijective on vertices.

Proof. We start by proving that the natural transformation is levelwise bijective on vertices. Let $y \in |S|$. From the categorical description of both $\mathbf{Adj}[n]$ and $\mathcal{C}\Delta[n]^{\mathrm{co}}$, the vertices of the simplicial set $\mathbf{Adj}^S_{\mathrm{hc}}((X, x), (B, y))$ corresponds to pairs
$$(f : \mathbf{k} \to [y, x], p : (y, x] \twoheadrightarrow \mathbf{m}) \in |\mathbf{Adj}[n]((X, x), (B, y))| \times |S(x, y)|$$
for which

- $f(\mathbf{k}) \subseteq |S|$;
- if $f^{-1}(i) \neq \emptyset$, then $p(y,i] \cap p[i+1,x] = \emptyset$.

We only need to address the case where $y \leq x$ since otherwise both simplicial sets are empty. Observe that there is an adjunction $\begin{tikzcd} \mathbf{Adj}(B,B) \arrow[r, shift left, "r^*"] \arrow[r, phantom, "\perp"] & \mathbf{Adj}(A,B) \arrow[l, shift left, "l^*"] \end{tikzcd}$.

More explicitly, this is adjunction can be written as $\begin{tikzcd} \Delta_+ \arrow[r, shift left, "-+\mathbf{1}"] \arrow[r, phantom, "\perp"] & \Delta_{+\infty} \arrow[l, shift left, "u"] \end{tikzcd}$, where u is the identity on objects and maps. The bijection between homsets induced by this adjunction implies that there is a bijection between order-preserving maps $\mathbf{k} \to [y,x] \cap |S|$ and order-preserving maps $\mathbf{k}+\mathbf{1} \to [y,x] \cap |S|$ which preserve the maximal element. Thus, r_z^* is levelwise bijective on vertices.

By Proposition 2.16, it is now enough to show that

$$\operatorname{coll}\mathbf{Mnd}^S_{\mathrm{hc}}[n]((B,z),-) \to \operatorname{coll}\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,z),j_S(-)) = \mathcal{C}_z$$

is a relative simplicial computad. By Proposition 5.16, $\mathcal{C}_z$ is a simplicial computad. Since r_z^* is levelwise injective, we can identify $\operatorname{coll}\mathbf{Mnd}^S_{\mathrm{hc}}[n]((B,z),-)$ with its image in $\mathcal{C}_z$. This simplicial subcategory has homspaces given by

$$\operatorname{coll}\mathbf{Mnd}^S_{\mathrm{hc}}[n]((B,z),-)((A,z),(B,y)) = \mathbf{Adj}^S_{\mathrm{hc}}[n]((B,z),(B,y)) \cdot r_z$$
$$\operatorname{coll}\mathbf{Mnd}^S_{\mathrm{hc}}[n]((B,z),-)((B,x),(B,y)) = \mathcal{C}_z((B,x),(B,y)).$$

The simplicial subcategory $\operatorname{coll}\mathbf{Mnd}^S_{\mathrm{hc}}[n]((B,z),-)$ of $\mathcal{C}_z$ is thus atom-complete, which ends the proof by Corollary 2.12. ☐

Corollary 5.18. *The simplicial category* $\mathbf{Mnd}^S_{\mathrm{hc}}[n]$ *is a simplicial computad.*

Proof. It is an atom-complete subcategory of $\mathcal{C}_z$ (for any z). ☐

Corollary 5.19. *Let X be an object of* $\mathbf{Adj}^S_{\mathrm{hc}}[n]$*. The composite*

$$\mathbf{Mnd}^S_{\mathrm{hc}}[n] \xrightarrow{j_S} \mathbf{Adj}^S_{\mathrm{hc}}[n] \xrightarrow{\mathbf{Adj}^S_{\mathrm{hc}}[n](X,-)} \mathbf{sSet}$$

is a projective cell complex.

Proof. Observe that $\mathbf{Mnd}^S_{\mathrm{hc}}[n]((B,x),-) = \mathbf{Adj}^S_{\mathrm{hc}}[n]((B,x),j_S(-))$ is trivially a projective cell-complex. We conclude by Proposition 5.17. ☐

Corollary 5.20. *Let $\mathbb{T} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathcal{K}$ be a homotopy coherent diagram of monads of shape S. The simplicially enriched right Kan extension*

$$\mathbb{A} = \operatorname{Ran}_{j_S}\mathbb{T} : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathcal{K}$$

exists. Moreover, the map $\mathbb{A}(r_k) : \mathbb{A}(A,k) \to \mathbb{A}(B,k) = \mathbb{T}(B,k)$ is an isofibration.

Proof. Let $\mathscr{M}$ be the cofibrantly generated model category enriched over the Joyal model structure such that $\mathcal{K} = \mathscr{M}_{\mathrm{fib}}$. By Proposition 2.89, the weighted limit functor

$$\{-,-\} : (\mathbf{sSet}_J^{\mathbf{Mnd}^S_{\mathrm{hc}}[n]})^{\mathrm{op}} \times \mathscr{M}^{\mathbf{Mnd}^S_{\mathrm{hc}}[n]} \to \mathscr{M}$$

is a right Quillen bifunctor, where the categories of enriched functors are endowed with the projective model structure. Observe that $\mathbb{T} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathcal{K} \subseteq \mathscr{M}$ is projectively fibrant. The weights involved in the construction of the enriched right Kan extension are all projectively cofibrant. Thus, the weighted limits involved in the construction of the enriched right Kan extension $\mathbb{A}$ exist in $\mathcal{K}$, and thus $\mathbb{A}$ does exist. Finally, since $\mathbb{A}(r_k) = \{r_k^*, \mathbb{T}\}$ and r_k^* is a projective cofibration, $\mathbb{A}(r_k)$ is an isofibration. □

Corollary 5.21. *Let $\mathbb{T} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{qCat}_\infty$ be a homotopy coherent diagram of monads of shape S in $\mathbf{qCat}_\infty$. The morphism*

$$\mathrm{Ran}_{j_S}\mathbb{T}(r_k) : \mathrm{Ran}_{j_S}\mathbb{T}(A,k) \to \mathrm{Ran}_{j_S}\mathbb{T}(B,k)$$

is a conservative isofibration.

Proof. Apply Proposition 2.108 together with Proposition 5.17. □

Definition 5.22. Let $f : K \to L$ be a 0-morphism of an ∞-cosmos $\mathcal{K}$. We say that f is *conservative* if and only if $f_* : \mathcal{K}(X,K) \to \mathcal{K}(X,L)$ is conservative for every $X \in |\mathcal{K}|$.

Corollary 5.23. *Let $\mathbb{T} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathcal{K}$ be a homotopy coherent diagram of monads of shape S in an ∞-cosmos $\mathcal{K}$. The morphism*

$$\mathrm{Ran}_{j_S}\mathbb{T}(r_k) : \mathrm{Ran}_{j_S}\mathbb{T}(A,k) \to \mathrm{Ran}_{j_S}\mathbb{T}(B,k)$$

is a conservative isofibration.

Proof. Let W be an object of $\mathcal{K}$. Observe that

$$\mathrm{Ran}_{j_S}\mathbb{T}(X,k) = \{\mathbf{Adj}^S_{\mathrm{hc}}[n]((X,k), j_S(-)), \mathbb{T}\}$$

and thus

$$\mathcal{K}(W, \mathrm{Ran}_{j_S}\mathbb{T}(X,k)) \cong \{\mathbf{Adj}^S_{\mathrm{hc}}[n]((X,k), j_S(-)), \mathcal{K}(W, \mathbb{T}(-))\}.$$

This means that $\mathcal{K}(W, \mathrm{Ran}_{j_S}\mathbb{T}(-)) : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathbf{qCat}_\infty$ is the enriched right Kan extension of $\mathcal{K}(W, \mathbb{T}(-))$ along j_S. Conclude with 5.20 and 5.21. □

5.3.2 Identifying the domain of the right adjoints

In this part, we use the monadicity theorem 2.107 due to Riehl and Verity in order to study enriched right Kan extensions along $j_S : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{Adj}^S_{\mathrm{hc}}[n]$.

Theorem 5.24. *Let $\mathbb{T} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{qCat}_\infty$ be a homotopy coherent diagram of monads of shape S and $\mathbb{T}|_k = \mathbb{T} \circ i_k$, where $i_k : \mathbf{Mnd} \to \mathbf{Mnd}^S_{\mathrm{hc}}[n]$ corresponds to the monad on the object (B, k). If $\mathbb{A} = \mathrm{Ran}_{j_S}\mathbb{T} : \mathbf{Adj}^S_{\mathrm{hc}}[n] \to \mathbf{qCat}_\infty$ is its enriched right Kan extension along j_S, the comparison map $\mathbb{A}(A, k) \to \mathrm{Alg}(\mathbb{T}|_k)$ is an equivalence.*

Proof. Since $j_S : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \longrightarrow \mathbf{Adj}^S_{\mathrm{hc}}[n]$ is an embedding, we can choose $\mathbb{A} = \mathrm{Ran}_{j_S}\mathbb{T}$ so that $\mathbb{A} \circ j_S = \mathbb{T}$. Write : $m_k : \mathbf{Adj} \longrightarrow \mathbf{Adj}^S_{\mathrm{hc}}[n]$ for the 2-functor induced by the adjunction $l_k \dashv r_k$ in $\mathbf{Adj}^S_{\mathrm{hc}}[n]$.

The simplicial functor $\mathbb{A}|_k = \mathbb{A} \circ m_k : \mathbf{Adj} \longrightarrow \mathbf{qCat}_\infty$ is a homotopy coherent adjunction, its associated homotopy coherent monad is exactly $\mathbb{T}|_k$. We are going to show that $\mathbb{A}|_k$ satisfies the hypothesis of the monadicity theorem 2.107. By Corollary 5.21, we already know that $\mathbb{A}(r_k) : \mathbb{A}(A, k) \to \mathbb{A}(B, k)$ is conservative. It remains to show that $\mathbb{A}(A, k)$ admits colimits of $\mathbb{A}(r_k)$-split simplicial objects and that $\mathbb{A}(r_k)$ preserves them. The quasi-category $S(\mathbb{A}(r_k))$ of $\mathbb{A}(r_k)$-split simplicial objects is given by the pullback

$$\begin{tikzcd}
S(\mathbb{A}(r_k)) \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbb{A}(A, k)^{\Delta^{\mathrm{op}}} \arrow[d, "\mathbb{A}(r_k)_*"] \\
\mathbb{A}(B, k)^{\Delta_{+\infty}} \arrow[r] & \mathbb{A}(B, k)^{\Delta^{\mathrm{op}}}
\end{tikzcd}$$

We need to show that there is an absolute left lifting

$$\begin{tikzcd}
 & \mathbb{A}(A, k) \arrow[d, "c"] \\
S(\mathbb{A}(r_k)) \arrow[ur, "\mathrm{colim}"] \arrow[r, "p_1"'] \arrow[ur, phantom, shift left=6, "\Uparrow\lambda"'] & \mathbb{A}(A, k)^{\Delta^{\mathrm{op}}}.
\end{tikzcd}$$

In [43, Proposition 5.2.11], Riehl and Verity shows that a family of diagrams $k : K \to A^B$ has (co)limits if and only if all its vertices $\Delta[0] \xrightarrow{d} K \xrightarrow{k} A^X$ admit (co)limits. As a consequence, it is enough to show that for every object X of $S(\mathbb{A}(r_k))$, there is an absolute left lifting

$$\begin{tikzcd}
 & \mathbb{A}(A, k) \arrow[d] \\
\Delta[0] \arrow[ur] \arrow[r, "p_1 \circ X"'] \arrow[ur, phantom, shift left=6, "\Uparrow\lambda_X"'] & \mathbb{A}(A, k)^{\Delta^{\mathrm{op}}}.
\end{tikzcd}$$

Since $\mathbb{A}(A,k) = \{\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k), j_S(-)), \mathbb{T}\}$, we have a **sSet**-natural isomorphism

$$\mathbf{sSet}(Z, \mathbb{A}(A,k)) \cong [\mathbf{Mnd}^S_{\mathrm{hc}}[n], \mathbf{sSet}]\big(\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k), j_S(-)), \mathbf{sSet}(Z, \mathbb{T}(-))\big)$$

and similarly

$$\mathbf{sSet}(Z, \mathbb{A}(B,k)) \cong [\mathbf{Mnd}^S_{\mathrm{hc}}[n], \mathbf{sSet}]\big(\mathbf{Adj}^S_{\mathrm{hc}}[n]((B,k), j_S(-)), \mathbf{sSet}(Z, \mathbb{T}(-))\big)$$

Thus, the object X corresponds to a pair of simplicial natural transformations

$$\begin{aligned} \alpha^X &: \mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k), j_S(-)) \longrightarrow \mathbb{T}(-)^{\Delta^{\mathrm{op}}} \\ \beta^X &: \mathbf{Adj}^S_{\mathrm{hc}}[n]((B,k), j_S(-)) \longrightarrow \mathbb{T}(-)^{\Delta_{+\infty}} \end{aligned}$$

that fits in a commutative diagram

$$\begin{tikzcd} \mathbf{Adj}^S_{\mathrm{hc}}[n]((B,k), j_S(-)) \arrow[r, "\beta^X"] \arrow[d, "r_k^*"] & \mathbb{T}(-)^{\Delta_{+\infty}} \arrow[d, "\mathrm{dual}^*"] \\ \mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k), j_S(-)) \arrow[r, "\alpha^X"] & \mathbb{T}(-)^{\Delta^{\mathrm{op}}} \end{tikzcd}$$

By Theorem 2.105 and since absolute left lifting diagrams are closed under precomposition by a 1-cell, for each object $(B,s) \in \left|\mathbf{Mnd}^S_{\mathrm{hc}}[n]\right|$,

$$\begin{tikzcd} & & \mathbb{T}(B,s) \arrow[d] \\ \mathbf{Adj}^S_{\mathrm{hc}}[n]((B,k), j_S(B,s)) \arrow[r, "\beta^X_{(B,s)}"'] & \mathbb{T}(B,s)^{\Delta_{+\infty}} \arrow[r, "\mathrm{dual}^*"'] & \mathbb{T}(B,s)^{\Delta^{\mathrm{op}}} \end{tikzcd}$$

admits an absolute left lifting through the vertical map. Since

- β^X is natural;
- the colimit given by Theorem 2.105 is preserved by any simplicial map;
- a natural transformation is left exact if and only if it is pointwise left exact, by Remark 2.101;

this can be promoted to a diagram $\mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix}}$

$$\begin{tikzcd}
& & \mathbb{T}(-) \arrow[d] \\
\mathbf{Adj}^S_{\mathrm{hc}}[n]((B,k), j_S(-)) \arrow[r, "\beta^X"'] & \mathbb{T}(-)^{\Delta_{+\infty}} \arrow[r, "\mathrm{dual}^*"'] & \mathbb{T}(-)^{\Delta^{\mathrm{op}}}.
\end{tikzcd}$$

Since the map $(r_k)^* : \mathbf{Adj}[n]((B,k),(B,s)) \longrightarrow \mathbf{Adj}[n]((A,k),(B,s))$ is surjective on objects on each component, again by [43, Proposition 5.2.11], the diagram

$$\begin{tikzcd}
& \mathbb{T}(B,s) \arrow[d] \\
\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k),(B,s)) \arrow[r, "\alpha^X"'] & \mathbf{sSet}(\Delta^{\mathrm{op}}, \mathbb{T}(B,s))
\end{tikzcd}$$

admits an absolute left lifting through the vertical map. Moreover, since left exactness can be checked pointwise, this can be promoted to a diagram

$$\mathbb{D} : \mathbf{Mnd}^S_{\mathrm{hc}}[n] \to \mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix}}.$$

By Proposition 2.103, the subcategory $\mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix}}$ has limits weighted by cofibrants weights and those are preserved by the inclusion in $\mathbf{qCat}_{\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix}}$. In particular, we can choose the weight to be $\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k), j_S(-))$ by Proposition 5.19. Since weighted limits are computed levelwise in $\mathbf{qCat}_{\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix}}$, we obtain that the cospan $\{\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k), j_S(-)), \mathbb{D}\}$ is

$$\begin{tikzcd}
& \mathbb{A}(A,k) \arrow[d] \\
\{\mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k),-), \mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k),-)\} \arrow[r, "(\alpha^X)_*"'] & \mathbb{A}(A,k)^{\Delta^{\mathrm{op}}}
\end{tikzcd}$$

in $\mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix}}$. It is not hard to see that $(\alpha^X)_*(\mathrm{id}) = p_1 X$, and thus $\mathbb{A}(A,k)$ has colimits of $\mathbb{A}(r_k)$-split simplicial object. To check that $\mathbb{A}(r_k)$ also preserves them, consider the morphism of $\mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix}}$ induced by the morphism of weights

$$r_k^* : \mathbf{Adj}^S_{\mathrm{hc}}[n]((B,k),-) \to \mathbf{Adj}^S_{\mathrm{hc}}[n]((A,k),-)$$

But this morphism is exactly

$$\begin{tikzcd}
 & \mathbb{A}(A,k) \arrow[d] \arrow[dr, "\mathbb{A}(r_k)"] & & . \\
\{\mathbf{Adj}_{\mathrm{hc}}^{S}[n]((A,k),-), \mathbf{Adj}_{\mathrm{hc}}^{S}[n]((A,k),-)\} \arrow[r, "(\alpha^X)_*"'] \arrow[d] & \mathbb{A}(A,k)^{\Delta^{\mathrm{op}}} \arrow[dr, "\mathbb{A}(r_k)_*"] & \mathbb{A}(B,k) \arrow[d] & \\
\mathbf{Adj}_{\mathrm{hc}}^{S}[n]((A,k),(B,k)) \arrow[rr, "\alpha^X_{(B,k)}"'] & & \mathbb{A}(B,k)^{\Delta^{\mathrm{op}}} &
\end{tikzcd}$$

The fact that this diagram lies in $\mathbf{qCat}_{l\infty}^{\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \rightarrow & \bullet \end{smallmatrix}}$ means that this is a left exact natural transformation. As a consequence $\mathbb{A}(r_k)$ preserves pointwise the colimit of $\mathbb{A}(r_k)$-split simplicial objects, and thus also globally by Remark 2.101.

□

Corollary 5.25. *Let $\mathcal{K}$ be an ∞-cosmos, $\mathbb{T} : \mathbf{Mnd}_{\mathrm{hc}}^{S}[n] \to \mathcal{K}$ a homotopy coherent diagram of monads of shape S and $\mathbb{T}|_k = \mathbb{T} \circ i_k$, where $i_k : \mathbf{Mnd} \to \mathbf{Mnd}_{\mathrm{hc}}^{S}[n]$ corresponds to the monad on the object (B,k). If $\mathbb{A} = \mathrm{Ran}_{j_S}\mathbb{T} : \mathbf{Adj}_{\mathrm{hc}}^{S}[n] \to \mathcal{K}$ is its enriched right Kan extension along j_S, the comparison map to the object of $\mathbb{T}|_k$-algebras displayed below is a weak equivalence*

$$\mathbb{A}(A,k) \to \mathrm{Alg}(\mathbb{T}|_k).$$

Proof. Write $m_k : \mathbf{Adj} \longrightarrow \mathbf{Adj}_{\mathrm{hc}}^{S}[n]$ for the 2-functor induced by the adjunction $l_k \dashv r_k$ and let as before $\mathbb{A}|_k = \mathbb{A} \circ m_k$. Recall that $\mathcal{K}(W, \mathbb{A}(-)) : \mathbf{Adj}_{\mathrm{hc}}^{S}[n] \to \mathbf{qCat}_\infty$ is the enriched right Kan extension of $\mathcal{K}(W, \mathbb{T}(-))$ along j_S. By Theorem 5.24, the comparison map

$$\mathcal{K}(W, \mathbb{A}(A,k)) \to \mathrm{Alg}(\mathcal{K}(W, \mathbb{T}|_k(-))) \cong \mathcal{K}(W, \mathrm{Alg}(\mathbb{T}|_k))$$

is an equivalence. But this map is given exactly by postcomposition by the comparison map $\mathbb{A}(A,k) \to \mathrm{Alg}(\mathbb{T}|_k)$, and thus the comparison map is a representable equivalence in $\mathcal{K}_2$. Therefore, it is an equivalence by Proposition 2.29 and consequently a weak equivalence by Proposition 2.93. □

Chapter 6

Towards an $(\infty, 2)$-category of homotopy coherent monads

Let $\mathcal{K}$ be an ∞-cosmos. In this chapter, we build a 2-trivial stratified simplicial set $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ whose objects are homotopy coherent monads in $\mathcal{K}$. Our main goal is to study the following conjecture.

Conjecture B. *Let $\mathcal{K}$ be an ∞-cosmos. The stratified simplicial set $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ is a 2-trivial and saturated weak complicial set.*

We reduce the conjecture to a horn-filling problem, which we partially solve using Theorem A.

If $\mathcal{K}$ and $\mathcal{L}$ are ∞-cosmoi which are appropriately equivalent, the stratified simplicial sets of homotopy coherent monads in $\mathcal{K}$ and in $\mathcal{L}$ are also equivalent. This model-independence is established in Theorem 6.15.

If conjecture B holds, we can obtain a global homotopy theory for homotopy coherent monads in $\mathcal{K}$, in the form of a quasi-category $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1$, made from the underlying simplicial set of $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ by forgetting all simplices that have non-thin faces of dimension 2. This is the content of Corollary 6.23.

Let $\mathrm{Iso} : \mathbf{Cat} \to \mathbf{Set}$ be the functor sending a category $\mathscr{D}$ to the set $\mathrm{Iso}(\mathscr{D})$ of isomorphism classes of objects of $\mathscr{D}$. Observe that Iso preserves products. Thus, we can associate to a 2-category $\mathscr{C}$ a category $\mathrm{Iso}_*(\mathscr{C})$ obtained by applying Iso to the hom-categories. Remark that an isomorphism in $\mathrm{Iso}_*(\mathscr{C})$ is exactly an equivalence in $\mathscr{C}$. The operation $(-)|_1$ we describe above should be understood as an analog of $\mathrm{Iso}_* : 2\text{-}\mathbf{Cat} \to \mathbf{Cat}$ in our context.

Studying Conjecture B suggested the existence of an $(\infty, 2)$-category of homotopy coherent adjunctions $\mathbf{Adj}_r(\mathcal{K})$ related to $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$. This is made precise in the following theorem.

Theorem C. *Let $\mathcal{K}$ be an ∞-cosmos. The stratified simplicial set $\mathbf{Adj}_r(\mathcal{K})$ is a 2-trivial and saturated weak complicial set. Moreover, if $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ is a weak equivalence of ∞-cosmoi, $\mathbf{Adj}_r(\mathbb{P}) : \mathbf{Adj}_r(\mathcal{K}) \to \mathbf{Adj}_r(\mathcal{L})$ is a weak equivalence of weak complicial sets.*

We call the 1-simplicies of $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ *homotopy coherent monad morphisms.* We have shown in Section 5.3 that homotopy coherent monad morphisms induce morphisms on the level of the objects of algebras. We classify these homotopy coherent monad morphisms up to homotopy. In Theorem 6.30 we prove that $h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1)$ is a full reflective subcategory of $h(\mathbf{Adj}_r(\mathcal{K})|_1)$. This enables us to classify equivalences of homotopy coherent monads in $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$, which is done in Corollary 6.33.

6.1 Several stratified nerve constructions

The 2-universal properties of the 2-categories $\mathbf{Mnd}_{\mathrm{hc}}[n]$ and $\mathbf{Adj}_{\mathrm{hc}}[n]$ imply that we can assemble them into cosimplicial 2-categories $\mathbf{Mnd}_{\mathrm{hc}}[-], \mathbf{Adj}_{\mathrm{hc}}[-] : \Delta \to$ 2-$\mathbf{Cat}$. Let us describe the action of a cosimplicial operator $\phi : [n] \to [m]$ on a colored squiggle first. The simplicial functor $\phi_* : \mathbf{Adj}[n] \to \mathbf{Adj}[m]$ is defined by

- $\phi_*(X, j) = (X, \phi^{\mathrm{op}}(j))$.
- $\phi_*(\underline{v})$ is obtained from $\underline{v}$ by applying ϕ^{op} to its colorings.

The cosimplicial operator $\phi : [n] \to [m]$ acts on an s-morphism $(\underline{v}, \underline{f}) \in \mathbf{Adj}_{\mathrm{hc}}[n]$ component-wise, that is by $\phi_*(\underline{v}, \underline{f}) = (\phi_*\underline{v}, \phi_*\underline{f})$. The cosimplicial structure of $\mathcal{C}\Delta[-]^{\mathrm{co}}$ is described in Section 2.5. We use double-struck letters to denote the cofaces and codegeneracies of the cosimplicial objects $\mathbf{Mnd}[-], \mathbf{Adj}[-]$ and $\mathbf{Mnd}_{\mathrm{hc}}[-], \mathbf{Adj}_{\mathrm{hc}}[-]$, as we do for $\mathcal{C}\Delta[-]^{\mathrm{co}}$.

Applying Kan's construction (briefly reviewed in 2.5) to the cosimplicial objects

$$\mathbf{Mnd}_{\mathrm{hc}}[-], \mathbf{Adj}_{\mathrm{hc}}[-], (B \leftarrow A) \times \mathcal{C}\Delta[-]^{\mathrm{co}} : \Delta \to \mathbf{sCat},$$

we obtain adjunctions

$$\begin{tikzcd}\mathbf{sSet} \arrow[r, shift left, "\mathcal{C}_{\mathbf{Mnd}}"] \arrow[r, phantom, "\perp"] & \mathbf{sCat} \arrow[l, shift left, "\mathcal{N}_{\mathbf{Mnd}}"]\end{tikzcd}, \quad \begin{tikzcd}\mathbf{sSet} \arrow[r, shift left, "\mathcal{C}_{\mathbf{Adj}}"] \arrow[r, phantom, "\perp"] & \mathbf{sCat} \arrow[l, shift left, "\mathcal{N}_{\mathbf{Adj}}"]\end{tikzcd}, \quad \begin{tikzcd}\mathbf{sSet} \arrow[r, shift left, "(B\leftarrow A)\times\mathcal{C}^{\mathrm{co}}"] \arrow[r, phantom, "\perp"] & \mathbf{sCat} \arrow[l, shift left, "\mathcal{N}_{B\leftarrow A}"]\end{tikzcd}.$$

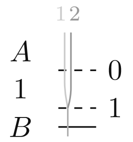
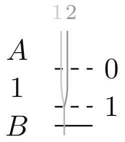
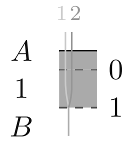
The right adjoints are given, for $\mathcal{K}$ a simplicial category, by

$$\begin{aligned}\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K}) &= \mathbf{sCat}(\mathbf{Mnd}_{\mathrm{hc}}[-], \mathcal{K})\\ \mathcal{N}_{\mathbf{Adj}}(\mathcal{K}) &= \mathbf{sCat}(\mathbf{Adj}_{\mathrm{hc}}[-], \mathcal{K})\\ \mathcal{N}_{B\leftarrow A}(\mathcal{K}) &= \mathbf{sCat}((B \leftarrow A) \times \mathcal{C}\Delta[-]^{\mathrm{co}}, \mathcal{K}).\end{aligned}$$

Since there are natural inclusions

$$\mathbf{Mnd}_{\mathrm{hc}}[n] \longrightarrow \mathbf{Adj}_{\mathrm{hc}}[n] \longleftarrow (B \leftarrow A) \times \mathcal{C}\Delta[n]^{\mathrm{co}} ,$$

there are natural maps $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K}) \longleftarrow \mathcal{N}_{\mathbf{Adj}}(\mathcal{K}) \longrightarrow \mathcal{N}_{B\leftarrow A}(\mathcal{K})$. These nerves can be stratified as follows.

- For $n \geq 3$, an n-simplex of $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$, $\mathcal{N}_{\mathbf{Adj}}(\mathcal{K})$ or $\mathcal{N}_{B\leftarrow A}(\mathcal{K})$ is always declared to be thin.
- A 2-simplex $\mathbb{T} : \mathbf{Mnd}_{\mathrm{hc}}[2] \to \mathcal{K}$ is declared to be thin if and only if $\mathbb{T}$ ( $: (B,2) \to (B,0)$) is a 2-cell isomorphism in $\mathcal{K}_2$.
- A 2-simplex $\mathbb{A} : \mathbf{Adj}_{\mathrm{hc}}[2] \to \mathcal{K}$ or $\mathbb{A} : (B \leftarrow A) \times \mathcal{C}\Delta[n]^{\mathrm{co}} \to \mathcal{K}$ is declared to be thin if and only if $\mathbb{A}$ ( $: (B,2) \to (B,0)$) and $\mathbb{A}$ ( $: (A,2) \to (A,0)$) are 2-cell isomorphisms in $\mathcal{K}_2$.
- A 1-simplex of $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$, $\mathcal{N}_{\mathbf{Adj}}(\mathcal{K})$ or $\mathcal{N}_{B\leftarrow A}(\mathcal{K})$ is declared to be thin if and only if it is an equivalence with respect to the thin 2-simplices defined above.

The nerve $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ is called is called *the homotopy coherent monadic nerve* of the ∞-cosmos $\mathcal{K}$. Let us explain the motivation for the stratifications we have just defined.

Remark 6.1. For $n \geq 2$, an n-simplex $\mathbb{T} : \mathbf{Mnd}_{\mathrm{hc}}[n] \to \mathcal{K}$ of $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ is thin if and only if the composite

$$\mathcal{C}\Delta[n]^{\mathrm{co}} \xhookrightarrow{u_n} \mathbf{Mnd}_{\mathrm{hc}}[n] \xrightarrow{\mathbb{T}} \mathcal{K}$$

is thin in Verity's nerve of complicial Gray-categories $\mathrm{N}(e_*(\mathcal{K}))$, defined in [56] and analyzed in 2.109. This is the motivation for the choice of stratification for $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$.

Definition 6.2. Let $\mathcal{K}$ be an ∞-cosmos. We define a stratified simplicial set $\mathbf{Adj}_r(\mathcal{K})$ as follows. Its underlying simplicial set is the subsimplicial set of $\mathcal{N}_{\mathbf{Adj}}(\mathcal{K})$ of those homotopy coherent diagrams of adjunctions $\mathbb{H} : \mathbf{Adj}_{\mathrm{hc}}[n] \to \mathcal{K}$ such that $\mathbb{H}(r_x)$ is an isofibration for all $x \in [n]$. A n-simplex $\mathbb{H} : \mathbf{Adj}_{\mathrm{hc}}[n] \to \mathcal{K}$ of $\mathbf{Adj}_r(\mathcal{K})$ is thin if and only if it is thin in $\mathcal{N}_{\mathbf{Adj}}(\mathcal{K})$.

Observe that $\mathbf{Adj}_r(\mathcal{K})$ is not functorial with respect to simplicial functors $\mathcal{K} \to \mathcal{L}$ which do not preserve isofibrations, but it is functorial with respect to functors of ∞-cosmoi (see Definition2.94).

Remark 6.3. Let $\mathcal{K}$ be an ∞-cosmos and $\mathscr{M}$ a model category enriched over $\mathbf{sSet}_J$, with all objects being cofibrant, such that $\mathcal{K} = \mathscr{M}_{\mathrm{fib}}$. The arrow category $\bullet \to \bullet$ has a Reedy structure such that the induced Reedy model category $\mathscr{M}^{\bullet \to \bullet}$ has

- fibrations between fibrant objects as fibrant objects and
- maps with cofibrant domain as cofibrant objects.

By [2], $\mathscr{M}^{\bullet \to \bullet}$ is an $\mathbf{sSet}_J$-enriched model structure. As a consequence, its subcategory of fibrant objects is an ∞-cosmos which we denote by $\mathcal{K}^{\bullet \to \bullet}$. There is also a category $\mathcal{K}^{\bullet \to \bullet}_{\mathbf{Adj}}$ enriched in quasi-categories, whose objects are homotopy coherent adjunctions $\mathbb{A} : \mathbf{Adj} \to \mathcal{K}$ such that $\mathbb{A}(r)$ is an isofibration and whose homspaces are given by

$$\mathcal{K}^{\bullet \to \bullet}_{\mathbf{Adj}}(\mathbb{A}, \mathbb{B}) = \mathcal{K}^{\bullet \to \bullet}(\mathbb{A}(r), \mathbb{B}(r)).$$

Observe that there is a pullback diagram

$$\begin{tikzcd}
\mathbf{Adj}_r(\mathcal{K}) \arrow[r] \arrow[d] & \mathcal{N}_{\mathbf{Adj}}(\mathcal{K}) \arrow[d] \\
\mathrm{N}(e_*(\mathcal{K}^{\bullet \to \bullet}_{\mathbf{Adj}})) \arrow[r] & \mathcal{N}_{B \leftarrow A}(\mathcal{K}).
\end{tikzcd}$$

It is straightforward to see that this diagram is a pullback of the underlying simplicial sets. To enhance this pullback to a pullback of stratified simplicial sets, we need to show that the left vertical map preserves thinness. By construction, for $n \geq 2$, an n-simplex $\mathbb{A} : \mathbf{Adj}_{\mathrm{hc}}[n] \to \mathcal{K}$ of $\mathbf{Adj}_r(\mathcal{K})$ is thin if and only if the adjoint $\mathcal{C}\Delta[n]^{\mathrm{co}} \to \mathcal{K}^{\bullet \to \bullet}$ of the composite

$$\begin{tikzcd}
(B \leftarrow A) \times \mathcal{C}\Delta[n]^{\mathrm{co}} \arrow[r, hook] & \mathbf{Adj}_{\mathrm{hc}}[n] \arrow[r, "\mathbb{A}"] & \mathcal{K}
\end{tikzcd}$$

is thin in Verity's nerve of complicial Gray-categories $\mathrm{N}(e_*(\mathcal{K}^{\bullet \to \bullet}))$. For 1-simplicies, it is enough to observe that any simplicial set map which preserves thin 2-simplicies preserve equivalences. This is the motivation for the stratifications of $N_{\mathbf{Adj}}(\mathcal{K})$ and $N_{B \leftarrow A}(\mathcal{K})$.

6.2 Discussion about Conjecture B

We restate Conjecture B below for the reader's convenience, before reducing it to a horn-filling problem.

Conjecture B. *Let $\mathcal{K}$ be an ∞-cosmos. The stratified simplicial set $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ is a 2-trivial and saturated weak complicial set.*

By [55, Theorem 56], it is enough to show that $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ is almost an inner weak complicial set. More precisely, we have to show that it has the right lifting property with respect to the inner complicial horn extensions $\Lambda^k[n] \to \Delta^k[n]$ for $0 < k < n$ and $n \geq 2$ and the inner complicial thinness extensions $\Delta^k[n]' \to \Delta^k[n]''$ for $n > 2$ and $0 < k < n$. The right lifting property of $N_{\mathbf{Mnd}}(\mathcal{K})$ against the inner complicial thinness extensions is a direct consequence of Remark 6.1 and the fact that $\mathrm{N}(e_*(\mathcal{K}))$ is a weak complicial set, which is proven in [56, Theorem 40].

Observe also $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ is saturated if and only if $\mathrm{N}(e_*(\mathcal{K}))$ is saturated, which is the case by Proposition 2.110.

In order to prove that $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ is a 2-trivial and saturated weak complicial set, the crux is thus to show that it satisfies the right lifting property with respect to *inner* complicial horn extensions. Our next goal is thus to compute $\mathcal{C}_{\mathbf{Mnd}}(\Lambda^k[n])$, for $0 < k < n$. Let

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, shift left, "\mathcal{C}^{\mathrm{co}}"] \arrow[r, phantom, "\perp"] & \mathbf{sCat} \arrow[l, shift left, "\mathcal{N}_{\mathrm{hc}}"] \end{tikzcd}$$

be the adjunction associated by Kan's construction to the cosimplicial simplicial category $\mathcal{C}\Delta[-]^{\mathrm{co}} : \Delta \to \mathbf{sCat}$. The left adjoint $\mathcal{C}^{\mathrm{co}} : \mathbf{sSet} \to \mathbf{sCat}$ carries inclusions of simplicial sets to inclusions of simplicial categories, and this allows us to compute $\mathcal{C}^{\mathrm{co}}(\Lambda^k[n])$, as explained in [56, Observation 19]. There is a coequalizer

$$\begin{tikzcd} \coprod\limits_{\substack{0\leq i<j\leq n\\ i,j\neq k}} \Delta[n-2] \arrow[r, shift left, "\sum_{i<j} d^i"] \arrow[r, shift right, "\sum_{i<j} d^{j-1}"'] & \coprod\limits_{\substack{i=0,\ldots,n\\ i\neq k}} \Delta[n-1] \arrow[r] \arrow[rr, bend left, "\sum_i d^i"] & \Lambda^k[n] \arrow[r, hook] & \Delta[n] \end{tikzcd}, \tag{6.1}$$

and since $\mathcal{C}^{\mathrm{co}}$ preserves colimits and inclusions, there is a coequalizer

$$\begin{tikzcd} \coprod\limits_{\substack{0\leq i<j\leq n\\ i,j\neq k}} \mathcal{C}\Delta[n-2]^{\mathrm{co}} \arrow[r, shift left, "\sum_{i<j} \mathbb{d}^i"] \arrow[r, shift right, "\sum_{i<j} \mathbb{d}^{j-1}"'] & \coprod\limits_{\substack{i=0,\ldots,n\\ i\neq k}} \mathcal{C}\Delta[n-1]^{\mathrm{co}} \arrow[r] \arrow[rr, bend left, "\sum_i \mathbb{d}^i"] & \mathcal{C}^{\mathrm{co}}(\Lambda^k[n]) \arrow[r, hook] & \mathcal{C}\Delta[n]^{\mathrm{co}} \end{tikzcd}$$

Thus, $\mathcal{C}^{\mathrm{co}}(\Lambda^k[n])$ is the smallest simplicial subcategory of $\mathcal{C}\Delta[n]^{\mathrm{co}}$ containing the images of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ for $i = 0, \ldots, n$ with $i \neq k$.

Remark 6.4. An m-morphism $\underline{f} : x \to y$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ if and only if $n-i \notin [y, x]$ or the unique path between leaves $n-i$ and $n-i+1$ in $\underline{f}$ has length 2. Said differently, the leaves $n-i$ and $n-i+1$ meet at line 0 in the representation of $\underline{f}$.

As a consequence, $\mathcal{C}^{\mathrm{co}}(\Lambda^k[n])$ is the simplicial subcomputad of $\mathcal{C}\Delta[n]^{\mathrm{co}}$ such that

- $\mathcal{C}^{\mathrm{co}}(\Lambda^k[n])(x, y) = \mathcal{C}\Delta[n]^{\mathrm{co}}(x, y)$ if $(x, y) \neq (n, 0)$; and
- an atomic m-morphism $\underline{f} : n \to 0$ belongs to $\mathcal{C}^{\mathrm{co}}(\Lambda^k[n])(n, 0)$ if and only there is $i \neq n-k$ such that the leaves $i, i+1$ meet on line 0 in the representation of the tree $\underline{f}$. Said differently, the length of the unique path from leaf i to leaf $i+1$ is 2 (the smallest possible).

Definition 6.5. We define two subcomputads of $\mathbf{Mnd}_{\mathrm{hc}}[n]$.

- The simplicial category $\partial\mathbf{Mnd}_{\mathrm{hc}}[n]$ is the subcomputad of $\mathbf{Mnd}_{\mathrm{hc}}[n]$ generated by the non-degenerate atomic m-morphisms $(\underline{v}, \underline{f}) : (B, x) \to (B, y)$ of $\mathbf{Mnd}_{\mathrm{hc}}[n]$ such that $\underline{f}$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ for $i \in [n]$.
- Let $0 < k < n$. The simplicial category $\mathbf{Mnd}^k_{\mathrm{hc}}[n]$ is the subcomputad of $\mathbf{Mnd}_{\mathrm{hc}}[n]$ generated by the non-degenerate atomic m-morphisms $(\underline{v}, \underline{f}) : (B, x) \to (B, y)$ of $\mathbf{Mnd}_{\mathrm{hc}}[n]$ such that $\underline{f}$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ for $i \in [n] \setminus \{k\}$.

Lemma 6.6. *An m-morphism $(\underline{v}, \underline{f}) : (B, x) \to (B, y) \in \mathbf{Mnd}_{\mathrm{hc}}[n]$ is in the image of*

$$\mathbb{d}^i : \mathbf{Mnd}_{\mathrm{hc}}[n-1] \to \mathbf{Mnd}_{\mathrm{hc}}[n]$$

if and only if $\underline{f}$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$.

Proof. The direct implication is obvious. Suppose that there is $\underline{\tilde{f}} : \tilde{x} \to \tilde{y}$ such that $\mathbb{d}^i(\underline{\tilde{f}}) = \underline{f}$. Then

$$\mathbb{d}^i\mathbb{s}^i(\underline{v}, \underline{f}) = \mathbb{d}^i\mathbb{s}^i(\underline{v}, \mathbb{d}^i(\underline{\tilde{f}})) = (\mathbb{d}^i\mathbb{s}^i(\underline{v}), \underline{f}).$$

Observe that $\mathbb{d}^i\mathbb{s}^i(\underline{v}) = \underline{v}$ if the color $n-i$ is not used in $\underline{v}$. By Remark 6.4, since $\underline{f}$ is in the image of $\mathbb{d}^i$, either $n-i \notin [y, x]$, and we are done, or $n-i, n-i+1$ are never separated in $\underline{f}$. As a consequence, the color $n-i$ is not used on a connected component of $\underline{v}$, since $\underline{f}$ is compatible with $\underline{v}$. □

Lemma 6.7. *Let $(\underline{v}, \underline{f}) : (B, x) \to (B, y)$ be a non-degenerate atomic m-morphism of $\mathbf{Mnd}_{\mathrm{hc}}[n-1]$ and $0 \leq i \leq n$. Then, $\mathbb{d}^i(\underline{v}, \underline{f})$ is also a non-degenerate atomic m-morphism.*

Proof. Suppose $\mathbb{d}^i(\underline{v}, \underline{f}) = (\underline{v_2}, \underline{f_2}) \cdot (\underline{v_1}, \underline{f_1})$. Then

$$(\underline{v}, \underline{f}) = \mathbb{s}^i \mathbb{d}^i(\underline{v}, \underline{f}) = \mathbb{s}^i(\underline{v_2}, \underline{f_2}) \cdot \mathbb{s}^i(\underline{v_1}, \underline{f_1}).$$

Since $(\underline{v}, \underline{f})$ is atomic, there is j such that $\mathbb{s}^i(\underline{v_j}, \underline{f_j}) = \mathrm{id}$. But the only non-trivial m-morphism that $\mathbb{s}^i$ sends to an identity is $b_{n-i} : (B, n-i) \to (B, n-i-1)$, since the domain of $(\underline{v_1}, \underline{f_1})$ cannot be $n-i$, $(\underline{v_2}, \underline{f_2}) = b_{n-i}$. But this implies that $\mathbb{d}^i(\underline{v}, \underline{f})$ does not satisfy the conditions of Remark 6.4. □

Proposition 6.8. *Let* $0 < k < n$. *The simplicial category* $\mathcal{C}_{\mathbf{Mnd}}(\partial\Delta[n])$ *is isomorphic to* $\partial\mathbf{Mnd}_{\mathrm{hc}}[n]$.

Proof. Since there is a coequalizer

$$\begin{tikzcd}
\coprod\limits_{0 \le i < j \le n} \Delta[n-2] \arrow[r, shift left, "\sum_{i<j} d^i"] \arrow[r, shift right, "\sum_{i<j} d^{j-1}"'] & \coprod\limits_{i=0,\dots,n} \Delta[n-1] \arrow[r, "\sum_i d^i"] & \partial\Delta[n]
\end{tikzcd}$$

and $\mathcal{C}_{\mathbf{Mnd}}$ preserves colimits, $\mathcal{C}_{\mathbf{Mnd}}(\partial\Delta[n])$ can be presented as the coequalizer of the parallel pair of morphisms in the diagram displayed in (6.2). We are going to show that $\partial\mathbf{Mnd}_{\mathrm{hc}}[n]$ is also a coequalizer of the same pair. Let

$$\begin{tikzcd}
\coprod\limits_{0 \le i < j \le n} \mathbf{Mnd}_{\mathrm{hc}}[n-2] \arrow[r, shift left, "\sum_{i<j} \mathbb{d}^i"] \arrow[r, shift right, "\sum_{i<j} \mathbb{d}^{j-1}"'] & \coprod\limits_{i=0,\dots,n} \mathbf{Mnd}_{\mathrm{hc}}[n-1] \arrow[r, "\sum_i \mathbb{d}^i"] \arrow[dr, "\mathbb{Q}"] & \partial\mathbf{Mnd}_{\mathrm{hc}}[n] \\
& & \mathcal{D}
\end{tikzcd} \tag{6.2}$$

be a commutative diagram of simplicial categories. We have to show that there is a unique $\mathbb{F} : \partial\mathbf{Mnd}_{\mathrm{hc}}[n] \to \mathcal{D}$ such that $\mathbb{F} \circ (\sum_i \mathbb{d}^i) = \mathbb{Q}$. The simplicial functor $\mathbb{F}$ is uniquely determined by its action on the atomic morphisms. Let us write $\mathbb{Q}_j : \mathbf{Mnd}_{\mathrm{hc}}[n-1] \to \mathcal{D}$ for the restriction of $\mathbb{Q}$ to the j-th component. Let $(\underline{v}, \underline{f}) : (B, x) \to (B, y)$ be a non-degenerate atomic m-arrow of $\partial\mathbf{Mnd}_{\mathrm{hc}}[n]$. By 6.6, there is $(\underline{\tilde{v}}, \underline{\tilde{f}}) : (B, \tilde{x}) \to (B, \tilde{y}) \in \mathbf{Mnd}_{\mathrm{hc}}[n-1]$ and i such that $\mathbb{d}^i(\underline{\tilde{v}}, \underline{\tilde{f}}) = (\underline{v}, \underline{f})$. As a consequence, if $\mathbb{F} \circ (\sum_i \mathbb{d}^i) = \mathbb{Q}$, then $\mathbb{F}(\underline{v}, \underline{f}) = \mathbb{Q}_i(\underline{\tilde{v}}, \underline{\tilde{f}})$. This shows in particular the uniqueness of $\mathbb{F}$. We prove now that defining $\mathbb{F}$ this way on atomic morphisms is not ambiguous. Indeed, suppose $(\underline{v}, \underline{f}) = \mathbb{d}^i(\underline{\tilde{v}}, \underline{\tilde{f}}) = \mathbb{d}^j(\underline{\hat{v}}, \underline{\hat{f}})$ with $i < j$. Because of the cosimplicial identities, remark that $(\underline{\tilde{v}}, \underline{\tilde{f}}) = \mathbb{s}^{i-1}(\underline{v}, \underline{f})$ and $(\underline{\hat{v}}, \underline{\hat{f}}) = \mathbb{s}^j(\underline{v}, \underline{f})$. The cosimplicial identities together with the commutativity of (6.2) imply that

$$\begin{aligned}
\mathbb{Q}_j(\underline{\hat{v}}, \underline{\hat{f}}) &= \mathbb{Q}_j(\mathbb{s}^j(\underline{v}, \underline{f})) \\
&= \mathbb{Q}_j\left(\mathbb{s}^j \mathbb{d}^i(\underline{\tilde{v}}, \underline{\tilde{f}})\right) \\
&= \mathbb{Q}_j\left(\mathbb{d}^i \mathbb{s}^{j-1}(\underline{\tilde{v}}, \underline{\tilde{f}})\right) \\
&= \mathbb{Q}_i\left(\mathbb{d}^{j-1} \mathbb{s}^{j-1} \mathbb{s}^{i-1}(\underline{v}, \underline{f})\right) \\
&= \mathbb{Q}_i\left(\mathbb{s}^{i-1} \mathbb{d}^j \mathbb{s}^j(\underline{v}, \underline{f})\right) \\
&= \mathbb{Q}_i\left(\mathbb{s}^{i-1}(\underline{v}, \underline{f})\right) \\
&= \mathbb{Q}_i(\underline{\tilde{v}}, \underline{\tilde{f}}).
\end{aligned}$$

Thus $\mathbb{F}$ is well defined on atomic morphisms. By Proposition 2.13, its extension to $\partial\mathbf{Mnd}_{\mathrm{hc}}[n]$ is well defined if whenever $(\underline{v}, \underline{f})$ is an atomic m-arrow of $\partial\mathbf{Mnd}_{\mathrm{hc}}[n]$ and $s \in \{0, \ldots, m\}$, $d_s\mathbb{F}(\underline{v}, \underline{f}) = \mathbb{F}(d_s(\underline{v}, \underline{f}))$. Suppose $(\underline{v}, \underline{f}) = \mathbb{d}^i(\underline{\tilde{v}}, \underline{\tilde{f}})$ as above and let

$$d_s(\underline{\tilde{v}}, \underline{\tilde{f}}) = (\underline{u_k}, \underline{g_k}) \circ \cdots \circ (\underline{u_1}, \underline{g_1})$$

be the decomposition in atomic morphisms of the s-th face of $(\underline{\tilde{v}}, \underline{\tilde{f}})$. Then,

$$\mathbb{d}^i(d_s(\underline{\tilde{v}}, \underline{\tilde{f}})) = \mathbb{d}^i(\underline{u_k}, \underline{g_k}) \circ \cdots \circ \mathbb{d}^i(\underline{u_1}, \underline{g_1})$$

is the decomposition in atomic morphisms of $d_s(\underline{v}, \underline{f})$ by Lemma 6.7. Thus,

$$\begin{aligned}
\mathbb{F}(d_s(\underline{v}, \underline{f})) &= \mathbb{F}(\mathbb{d}^i(\underline{u_k}, \underline{g_k})) \circ \cdots \circ \mathbb{F}(\mathbb{d}^i(\underline{u_1}, \underline{g_1})) \\
&= \mathbb{Q}_i(\underline{u_k}, \underline{g_k}) \circ \cdots \circ \mathbb{Q}_i(\underline{u_1}, \underline{g_1}) \\
&= \mathbb{Q}_i(d_s(\underline{\tilde{v}}, \underline{\tilde{f}})) \\
&= d_s\mathbb{F}(\underline{v}, \underline{f}).
\end{aligned}$$

□

Proposition 6.9. *Let $0 < k < n$. The simplicial category $\mathcal{C}_{\mathbf{Mnd}}(\Lambda^k[n])$ is isomorphic to $\mathbf{Mnd}^k_{\mathrm{hc}}[n]$.*

Proof. Observe that by Proposition 6.8 and since $\mathcal{C}_{\mathbf{Mnd}}$ preserves colimits, $\mathcal{C}_{\mathbf{Mnd}}$ sends inclusions of simplicial sets to relative subcomputads between simplicial computads. Since the diagram displayed in (6.1) is a coequalizer, we get a coequalizer

$$\begin{tikzcd}
\coprod\limits_{\substack{0\leq i<j\leq n\\ i,j\neq k}} \mathbf{Mnd}_{\mathrm{hc}}[n-2] \arrow[r, shift left, "\sum_{i<j} \mathbb{d}^i"] \arrow[r, shift right, "\sum_{i<j} \mathbb{d}^{j-1}"'] & \coprod\limits_{\substack{i=0,\dots,n\\ i\neq k}} \mathbf{Mnd}_{\mathrm{hc}}[n-1] \arrow[r] \arrow[rd, bend right, "\sum_i \mathbb{d}^i"'] & \mathcal{C}_{\mathbf{Mnd}}(\Lambda^k[n]) \arrow[d, hook] \\
 & & \mathbf{Mnd}_{\mathrm{hc}}[n]
\end{tikzcd}$$

As a consequence, $\mathcal{C}_{\mathbf{Mnd}}(\Lambda^k[n])$ is the smallest subcomputad of $\mathbf{Mnd}_{\mathrm{hc}}[n]$ which contains the image of $\mathbb{d}^i$ for $i \in [n] \setminus \{k\}$. This subcomputad is $\mathbf{Mnd}^k_{\mathrm{hc}}[n]$ by Lemma 6.6 and 6.7. $\square$

Observe that $\Lambda^k[n]$ has no non-degenerate thin simplicies of dimension 1, since it is an inner horn. Therefore, by Remark 6.1, a stratified simplicial set map $h^k : \Lambda^k[n] \to \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ corresponds to a simplicial functor $\mathbb{h}^k : \mathbf{Mnd}^k_{\mathrm{hc}}[n] \to \mathcal{K}$ such that the composite

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{co}}(\Lambda^k[n]) \arrow[r, hook] & \mathbf{Mnd}^k_{\mathrm{hc}}[n] \arrow[r, "\mathbb{h}^k"] & \mathcal{K}
\end{tikzcd}$$

corresponds to a stratified simplicial set map $\Lambda^k[n] \to \mathrm{N}(e_*(\mathcal{K}))$. Since $\mathrm{N}(e_*(\mathcal{K}))$ is a weak complicial set by [56, Theorem 40], there is a lift in the diagram

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & \mathrm{N}(e_*(\mathcal{K})) \\
\Delta^k[n] \arrow[ru, dashed] &
\end{tikzcd} \tag{6.3}$$

which by adjunction implies the existence of a lift

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{co}}(\Lambda^k[n]) \arrow[r, hook] \arrow[d] & \mathbf{Mnd}^k_{\mathrm{hc}}[n] \arrow[r, "\mathbb{h}^k"] & \mathcal{K} \\
\mathcal{C}\Delta[n]^{\mathrm{co}} \arrow[rru, dashed] & &
\end{tikzcd}$$

and thus of a map $\widetilde{\mathbb{h}^k} : \mathbf{Mnd}^k_{\mathrm{hc}}[n] \coprod_{\mathcal{C}^{\mathrm{co}}(\Lambda^k[n])} \mathcal{C}\Delta[n]^{\mathrm{co}} \to \mathcal{K}$.

Definition 6.10. Let $0 < k < n$. The simplicial category $\widetilde{\mathbf{Mnd}^k_{\mathrm{hc}}}[n]$ is defined as the subcomputad of $\mathbf{Mnd}_{\mathrm{hc}}[n]$ generated by the non-degenerate atomic m-morphisms $(\underline{v}, \underline{f}) : (B, x) \to (B, y)$ of $\mathbf{Mnd}_{\mathrm{hc}}[n]$ such that $\underline{f}$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ for $i \in [n] \setminus \{k\}$, or such that $\underline{v} = (B)$ with empty coloring.

Proposition 6.11. *The square*

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{co}}(\Lambda^k[n]) \arrow[r, hook] \arrow[d, hook] & \mathbf{Mnd}^k_{\mathrm{hc}}[n] \arrow[d, hook] \\
\mathcal{C}\Delta[n]^{\mathrm{co}} \arrow[r, hook] & \widetilde{\mathbf{Mnd}}^k_{\mathrm{hc}}[n]
\end{tikzcd}$$

is a pushout.

Proof. By Proposition 2.13, an extension of a simplicial functor $\mathbb{T} : \mathbf{Mnd}^k_{\mathrm{hc}}[n] \to \mathcal{K}$ to a simplicial functor $\mathbb{T} : \widetilde{\mathbf{Mnd}}^k_{\mathrm{hc}}[n] \to \mathcal{K}$ is uniquely determined by specifying, for all atomic non-degenerate morphisms $((B), \underline{f})$ of $\widetilde{\mathbf{Mnd}}^k_{\mathrm{hc}}[n] \setminus \mathbf{Mnd}^k_{\mathrm{hc}}[n]$ a morphism $\mathbb{T}((B), \underline{f})$ of $\mathcal{K}$ such that $d_i\mathbb{T}((B), \underline{f}) = \mathbb{T}((B), d_i\underline{f})$. By Proposition 2.13, this is exactly the same as providing a simplicial functor $\mathcal{C}\Delta[n]^{\mathrm{co}} \to \mathcal{K}$ extending the composite $\mathcal{C}^{\mathrm{co}}(\Lambda^k[n]) \hookrightarrow \mathbf{Mnd}^k_{\mathrm{hc}}[n] \xrightarrow{\mathbb{T}} \mathcal{K}$. □

Definition 6.12. Let $\overline{\mathbf{Adj}}^k_{\mathrm{hc}}[n]$ be the relative subcomputad of $\mathbf{Adj}_{\mathrm{hc}}[n]$ generated by the non-degenerate atomic m-morphisms $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$ such that $\underline{f}$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ for $i \in [n] \setminus \{k\}$. This is a right-parental relative subcomputad of $\mathbf{Adj}_{\mathrm{hc}}[n]$ (See 4.28 (iv)). Let $\overline{\mathbf{Mnd}}^k_{\mathrm{hc}}[n]$ be the full subcategory of $\overline{\mathbf{Adj}}^k_{\mathrm{hc}}[n]$ generated by the objects (B, j) for $j \in [n]$.

The simplicial category $\overline{\mathbf{Adj}}^k_{\mathrm{hc}}[n]$ contains by definition $(B \leftarrow A) \times \mathcal{C}\Delta[n]^{\mathrm{co}}$. Thus, there is an inclusion $u : \widetilde{\mathbf{Mnd}}^k_{\mathrm{hc}}[n] \to \overline{\mathbf{Mnd}}^k_{\mathrm{hc}}[n]$. We assume that

$$\widetilde{\mathbb{h}}^k : \widetilde{\mathbf{Mnd}}^k_{\mathrm{hc}}[n] \to \mathcal{K}$$

can be extended along u to a simplicial functor $\overline{\mathbb{h}}^k : \overline{\mathbf{Mnd}}^k_{\mathrm{hc}}[n] \to \mathcal{K}$. This is the obstacle which remains to prove Conjecture B, as the following discussion shows.

We will use an enriched right Kan extension along $\iota : \overline{\mathbf{Mnd}}^k_{\mathrm{hc}}[n] \to \overline{\mathbf{Adj}}^k_{\mathrm{hc}}[n]$. Fortunately, the following proposition holds.

Proposition 6.13. *Let (X, x) be an object of $\mathbf{Adj}_{\mathrm{hc}}[n]$. The composite*

$$\begin{tikzcd}
\overline{\mathbf{Mnd}}^k_{\mathrm{hc}}[n] \arrow[r, "\iota"] & \overline{\mathbf{Adj}}^k_{\mathrm{hc}}[n] \arrow[rr, "{\overline{\mathbf{Adj}}^k_{\mathrm{hc}}[n]((X,x),-)}"] & & \mathbf{sSet}
\end{tikzcd}$$

is a projective cell complex.

Proof. Representables are trivially projective cell complexes. As a consequence, $\overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n]((B,z),\iota(-))$ is a projective cell complex. By Proposition 2.16, we have to show that $\operatorname{coll}\overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n]((A,z),\iota(-))$ is a simplicial computad. Remark that $\operatorname{coll}\overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n]((A,z),\iota(-)) \subseteq \operatorname{coll}\mathbf{Adj}_{\mathrm{hc}}[n]((A,z),j_n(-)) = \mathcal{C}_z$ and the latter is a simplicial computad by Proposition 5.16. It is thus enough to show that $\mathcal{C}^k_z :=$ $\operatorname{coll}\overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n]((A,z),\iota(-))$ is an atom-complete subcategory of $\mathcal{C}_z$. The inclusion $\mathcal{C}^k_z((X,x),(B,y)) \subseteq \mathcal{C}_z((X,y),(B,y))$ is an identity unless $x=n$ and $y=0$. When $X=B$ it is

$$\overline{\mathbf{Mnd}^k_{\mathrm{hc}}}[n]((B,n),(B,0)) \subseteq \mathbf{Mnd}_{\mathrm{hc}}[n]((B,n),(B,0))$$

and if $z=n$ and $X=A$, it is

$$\overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n]((A,n),(B,0)) \subseteq \mathbf{Adj}_{\mathrm{hc}}[n]((A,n),(B,0)).$$

As a consequence, since $\overline{\mathbf{Mnd}^k_{\mathrm{hc}}}[n]$ is a subcomputad of $\mathbf{Mnd}_{\mathrm{hc}}[n]$, we are left with verifying that if $(\underline{v},\underline{f}) : (A,n) \to (B,0)$ is an atomic morphism appearing in the decomposition of a morphism $(\underline{u},\underline{g})$ of $\mathcal{C}^k_z$ in $\mathcal{C}_z$, $(\underline{v},\underline{f})$ also belong to C^k_z. Observe that $(\underline{u},\underline{g}) = (\underline{w},\mathrm{id}_0)\cdot(\underline{v},\underline{f})$ and thus $\underline{g}=\underline{f}$. If $(\underline{v},\underline{f})$ factors through (A,j) with $0<j<n$, then it belongs to $\mathcal{C}^k_z$. Otherwise, $(\underline{v},\underline{f}) = (\underline{w}_2,\mathrm{id})\cdot(\underline{\tilde{v}},\underline{f})\cdot(\underline{w}_1,\mathrm{id})$, with $(\underline{\tilde{v}},\underline{f})$ atomic in $\mathbf{Adj}_{\mathrm{hc}}[n]$. Since $(\underline{u},\underline{g})$ is in $\mathcal{C}^k_z$, $\underline{g}=\underline{f}$ is in the image of $\mathbb{d}^i : \mathcal{C}\Delta[n-1]^{\mathrm{co}} \to \mathcal{C}\Delta[n]^{\mathrm{co}}$ for $i \in [n]\setminus\{k\}$ and thus $(\underline{v},\underline{f}) \in C^k_z$. □

Corollary 6.14. *Let $\mathcal{K}$ be an ∞-cosmos and $\overline{\mathbb{T}} : \overline{\mathbf{Mnd}^k_{\mathrm{hc}}}[n] \to \mathcal{K}$. The simplicially enriched right Kan extension $\overline{\mathbb{A}} : \overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n] \to \mathcal{K}$ of $\overline{\mathbb{T}}$ along ι exists and can be chosen such that $\overline{\mathbb{A}} \circ \iota = \overline{\mathbb{T}}$.*

Proof. Since $\mathcal{K}$ is closed under limits weighted by projectively cofibrant cell complex, Proposition 6.13 implies that the enriched right Kan extension exists. Moreover, since ι is an embedding, we can choose the enriched right Kan extension $\overline{\mathbb{A}} : \overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n] \to \mathcal{K}$ such that $\overline{\mathbb{A}} \circ \iota = \overline{\mathbb{T}}$. □

Thus, a lifting problem

$$\begin{tikzcd}
\overline{\mathbf{Mnd}^k_{\mathrm{hc}}}[n] \arrow[r, "\overline{\mathbb{h}^k}"] \arrow[d] & \mathcal{K} \\
\mathbf{Mnd}_{\mathrm{hc}}[n] \arrow[ru, dashed] &
\end{tikzcd}$$

can be solved if and only if the lifting problem

$$\begin{tikzcd}
\overline{\mathbf{Mnd}^k_{\mathrm{hc}}}[n] \arrow[r, "\iota"] \arrow[d] & \overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n] \arrow[r, "\overline{\mathbb{H}}"] \arrow[d] & \mathcal{K} \\
\mathbf{Mnd}_{\mathrm{hc}}[n] \arrow[r, "j_n"] & \mathbf{Adj}_{\mathrm{hc}}[n] \arrow[ur, dashed] &
\end{tikzcd} \tag{6.4}$$

can be solved, where $\overline{\mathbb{H}}$ is the enriched right Kan extension of $\overline{\mathbb{h}^k}$ along ι. But $\overline{\mathbf{Adj}^k_{\mathrm{hc}}}[n]$ is a right parental relative subcomputad (4.28, (iv)). Moreover, it contains all adjunction data, since it certainly contains monochromatic squiggles. As a consequence, Theorem A implies that the lifting problem (6.4) can be solved, if $\mathcal{K}$ is quasi-categorically enriched.

Let $\mathbb{T} : \mathbf{Mnd}_{\mathrm{hc}}[n] \to \mathcal{K}$ be a lift of $\overline{\mathbb{h}^k}$. This corresponds to a simplicial map

$$h : \Delta[n] \to \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$$

lifting $h^k : \Lambda^k[n] \to \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$. This map can be promoted to a stratified simplicial map

$$h : \Delta^k[n] \to \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$$

if and only if $\mathbb{T} \circ u_n : \mathcal{C}\Delta[n]^{\mathrm{co}} \to \mathcal{K}$ corresponds to a stratified map

$$\Delta^k[n] \to \mathrm{N}(e_*(\mathcal{K})).$$

But this map is exactly the lift obtained in display 6.3.

Let us briefly discuss why it seems reasonable that $\widetilde{\mathbb{h}^k}$ extends along $u : \widetilde{\mathbf{Mnd}^k_{\mathrm{hc}}}[n] \to \overline{\mathbf{Mnd}^k_{\mathrm{hc}}}[n]$. The atomic morphisms $(\underline{v}, \underline{f}) : (B, n) \to (B, 0)$ of $\overline{\mathbf{Mnd}^k_{\mathrm{hc}}}[n] \setminus \widetilde{\mathbf{Mnd}^k_{\mathrm{hc}}}[n]$ factor in $\mathbf{Adj}_{\mathrm{hc}}[n]$ through (A, j) for some $0 < j < n$. One can use $\mathrm{Ran}_{j_{n-1}}(\widetilde{\mathbb{h}^k}\mathbb{d}^i)$ for $i = 0, n$ to map both legs to morphisms

$$\widetilde{\mathbb{h}^k}(B, n) \to \mathrm{Ran}_{j_{n-1}}(\widetilde{\mathbb{h}^k}\mathbb{d}^n)(A, j)$$
$$\mathrm{Ran}_{j_{n-1}}(\widetilde{\mathbb{h}^k}\mathbb{d}^0)(A, j) \to \widetilde{\mathbb{h}^k}(B, 0)$$

in $\mathcal{K}$. However the codomain of the first is not equal to the domain of the second one, but only equivalent. Each individual morphism can be lifted, but doing so coherently is the main obstruction. This ends our discussion about Conjecture B.

We close this section by proving that the homotopy coherent monadic nerve of an ∞-cosmos is a model-invariant construction, as the following proposition shows.

Theorem 6.15. *Let $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ be a simplicial functor between ∞-cosmoi, such that*

- *for all $K, K' \in |\mathcal{K}|$, $\mathbb{P}_{KK'} : \mathcal{K}(K, K') \to \mathcal{L}(\mathbb{P}K, \mathbb{P}K')$ is an equivalence of quasi-categories;*
- *for all $L \in |\mathcal{L}|$, there exists $K \in |\mathcal{K}|$ and a weak equivalence $\mathbb{P}K \xrightarrow{\sim} L$.*

Then, $\mathcal{N}_{\mathbf{Mnd}}(\mathbb{P}) : \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K}) \to \mathcal{N}_{\mathbf{Mnd}}(\mathcal{L})$ is a weak equivalence.

Proof. There is a model structure on the category of simplicial categories and simplicial functors due to Lurie [35, Proposition A.3.2.4] such that a simplicial functor $\mathbb{P}$ satisfying the hypothesis of the theorem is a weak equivalence, and categories enriched in quasi-categories are fibrant (Use $S = \mathbf{sSet}_J$). We write this model category $\mathbf{sSet}_J - \mathbf{Cat}$. By Ken Brown's lemma, we can suppose without loss of generality that $\mathbb{P}$ is additionally a fibration in this model structure. In particular, we are going to show that $\mathcal{N}_{\mathbf{Mnd}}(\mathbb{P})$ satisfies the right lifting property with respect to the cofibrations of the model structure on weak complicial sets. Every cofibration can be obtained as a transfinite composition of maps $\partial\Delta[n] \to \Delta[n], n \geq 0$ and $\Delta[n] \to \Delta[n]_t, n \geq 1$, where $\Delta[n]_t$ is obtained from $\Delta[n]$ by adding $1_{[n]} : [n] \to [n]$ to the thin simplicies. It is thus enough to solve lifting problems

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d] & \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K}) \arrow[d, "\mathcal{N}_{\mathbf{Mnd}}(\mathbb{P})"] & \Delta[n] \arrow[r] \arrow[d] & \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K}) \arrow[d, "\mathcal{N}_{\mathbf{Mnd}}(\mathbb{P})"] \\
\Delta[n] \arrow[r] & \mathcal{N}_{\mathbf{Mnd}}(\mathcal{L}) & \Delta[n]_t \arrow[r] & \mathcal{N}_{\mathbf{Mnd}}(\mathcal{L}).
\end{tikzcd}$$

The lifting problem on the left is equivalent to the lifting problem

$$\begin{tikzcd}
\mathcal{C}_{\mathbf{Mnd}}(\partial\Delta[n]) \arrow[r] \arrow[d] & \mathcal{K} \arrow[d, "\mathbb{P}"] \\
\mathbf{Mnd}_{\mathrm{hc}}[n] \arrow[r] & \mathcal{L}.
\end{tikzcd} \tag{6.5}$$

By Proposition 6.8, the left vertical map in the diagram displayed in (6.5) is a cofibration in the model structure on enriched categories in which $\mathbb{P}$ is a trivial fibration, and thus this lifting problem has a solution.

The lifting problem

$$\begin{tikzcd}
\Delta[n] \arrow[r] \arrow[d] & \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K}) \arrow[d, "\mathcal{N}_{\mathbf{Mnd}}(P)"] \\
\Delta[n]_t \arrow[r] & \mathcal{N}_{\mathbf{Mnd}}(\mathcal{L})
\end{tikzcd}$$

is equivalent to the assertion that if a n-simplex $\mathbb{F} : \mathbf{Mnd}_{\mathrm{hc}}[n] \to \mathcal{K}$ is such that $\mathbb{P} \circ \mathbb{F}$ is thin, so is $\mathbb{F}$. This is always the case for $n > 2$ since the homotopy coherent monadic nerves are 2-trivial. Since $\mathbb{P}$ is a trivial fibration, for every pair of objects K, K' of $\mathcal{K}$, $\mathbb{P}_{KK'} : \mathcal{K}(K, K') \to \mathcal{L}(\mathbb{P}K, \mathbb{P}K')$ is a trivial isofibration between quasi-categories, and thus it reflects isomorphisms. This settles the case $n = 2$. The case $n = 1$ is a consequence of the case $n = 2$ and the right lifting property against $\partial\Delta[1] \to \Delta[1]$ and $\partial\Delta[2] \to \Delta[2]$. ☐

6.3 Proof of Theorem C

We restate Theorem C for the reader's convenience.

Theorem C. *Let $\mathcal{K}$ be an ∞-cosmos. The stratified simplicial set $\mathbf{Adj}_r(\mathcal{K})$ is a 2-trivial and saturated weak complicial set. Moreover, if $\mathbb{P} : \mathcal{K} \to \mathcal{L}$ is a weak equivalence of ∞-cosmoi, $\mathbf{Adj}_r(\mathbb{P}) : \mathbf{Adj}_r(\mathcal{K}) \to \mathbf{Adj}_r(\mathcal{L})$ is a weak equivalence of weak complicial sets.*

In order to prove Theorem C, we have to compute $\mathcal{C}_{\mathbf{Adj}}(\partial\Delta[n])$.

Definition 6.16. Let $(\underline{v}, \underline{f}) : (X, x) \to (Y, y)$. Recall that by the proof of Proposition 3.25, $(\underline{v}, \underline{f})$ has a unique decomposition

$$\begin{tikzcd}
(X,x) \arrow[r, "c_{k+1}"] & (Z_k, z_k) \arrow[r, "(\underline{v_k}, \underline{f_k})"] & (W_k, w_k) \arrow[r, "c_k"] & \dots \arrow[dll, rounded corners, "c_1"] \\
 & (Z_0, z_0) \arrow[r, "(\underline{v_0}, \underline{f_0})"] & (W_0, w_0) \arrow[r, "c_0"] & (Y, y)
\end{tikzcd}$$

where $(\underline{v_i}, \underline{f_i})$ is atomic and does not belong to $(B \leftarrow A) \times \mathcal{C}\Delta[n]^{\mathrm{co}}$, while c_i belongs to $(B \leftarrow A) \times \mathcal{C}\Delta[n]^{\mathrm{co}}$. For $n \geq 1$, the simplicial category $\partial\mathbf{Adj}_{\mathrm{hc}}[n]$ is defined to be the simplicial subcategory of $\mathbf{Adj}_{\mathrm{hc}}[n]$ consisting of those m-morphisms whose unique decomposition as above is such that for all j, $\underline{f_j}$ is in the image of $\mathbb{d}^i$ for some $i \in [n]$, and for all j, $c_j \in (B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n])$.

Proposition 6.17. *Let $n \geq 1$. The simplicial category $\mathcal{C}_{\mathbf{Adj}}(\partial\Delta[n])$ is isomorphic to $\partial\mathbf{Adj}_{\mathrm{hc}}[n]$.*

Proof. We need to show that there is a coequalizer

$$\coprod_{0 \leq i < j \leq n} \mathbf{Adj}_{\mathrm{hc}}[n-2] \overset{\sum_{i<j} \mathbb{d}^i}{\underset{\sum_{i<j} \mathbb{d}^{j-1}}{\rightrightarrows}} \coprod_{i=0,\dots,n} \mathbf{Adj}_{\mathrm{hc}}[n-1] \xrightarrow{\sum_i \mathbb{d}^i} \partial\mathbf{Adj}_{\mathrm{hc}}[n] \,. \tag{6.6}$$

Let $\mathbb{Q} : \coprod_{i=0,\dots,n} \mathbf{Adj}_{\mathrm{hc}}[n-1] \to \mathcal{D}$ be a simplicial functor that coequalizes the parallel pair of the diagram displayed in (6.6). Since $(B \leftarrow A) \times - : \mathbf{sCat} \to \mathbf{sCat}$ is a left adjoint, there is a coequalizer

$$\begin{tikzcd}
\coprod\limits_{0\leq i<j\leq n} (B \leftarrow A) \times \mathcal{C}\Delta[n-2]^{\mathrm{co}} \arrow[r, shift left, "\sum_{i<j} \mathbb{d}^i"] \arrow[r, shift right, "\sum_{i<j} \mathbb{d}^{j-1}"'] & \coprod\limits_{i=0,\dots,n} (B \leftarrow A) \times \mathcal{C}\Delta[n-1]^{\mathrm{co}} \arrow[d, "\sum_i \mathbb{d}^i"] \\
 & (B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n])
\end{tikzcd}$$

and thus $\mathbb{Q}$ induces a simplicial functor $\mathbb{Q}' : (B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n]) \to \mathcal{D}$. By construction and Lemma 2.8, $(B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n]) \to \partial\mathbf{Adj}_{\mathrm{hc}}[n]$ is a relative simplicial computad. By Proposition 2.13, a simplicial functor $\mathbb{F} : \partial\mathbf{Adj}_{\mathrm{hc}}[n] \to \mathcal{D}$ is uniquely determined by its action on $(B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n])$ and on atomic morphisms not belonging to it. Such a simplicial functor $\mathbb{F}$ satisfies $\mathbb{F} \circ (\sum_i \mathbb{d}^i) = \mathbb{Q}$ if and only if

- its action on $(B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n])$ is given by $\mathbb{Q}'$;
- for all atomic morphisms $(\underline{v}, \underline{f})$ not belonging to $(B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n])$ and such that $(\underline{v}, \underline{f}) = \mathbb{d}^i(\underline{v}', \underline{f}')$ for some $(\underline{v}'\underline{f}') \in \mathbf{Adj}_{\mathrm{hc}}[n-1]$, $\mathbb{F}(\underline{v}, \underline{f}) = \mathbb{Q}_i(\underline{v}', \underline{f}')$, where $\mathbb{Q}_i$ is the restriction of $\mathbb{Q}$ to the i-th component.

But the analogs of Lemma 6.6 and 6.7 for **Adj** imply that all atomic arrows of $\mathbf{Adj}_{\mathrm{hc}}[n]$ not belonging to $(B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n])$ satisfy the condition above. Thus, the functor $\mathbb{F}$, if it exists, is uniquely determined by $\mathbb{Q}$ and the equation $\mathbb{F} \circ (\sum_i \mathbb{d}^i) = \mathbb{Q}$. Moreover, an argument similar to that of the proof of Proposition 6.9 shows that the definition above is not ambiguous and satisfies the hypothesis of Proposition 2.13. □

Proposition 6.18. *Let $n > 0$ and $0 < k < m$. There is a pushout square*

$$\begin{tikzcd}
(B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n]) \arrow[r] \arrow[d] & \partial\mathbf{Adj}_{\mathrm{hc}}[n] \arrow[d] \\
(B \leftarrow A) \times \mathcal{C}\Delta[n]^{\mathrm{co}} \arrow[r] & \overline{\partial\mathbf{Adj}_{\mathrm{hc}}}[n].
\end{tikzcd}$$

where $\overline{\partial\mathbf{Adj}_{\mathrm{hc}}}[n]$ is defined as in Examples 4.28 (iii).

Proof. This is a consequence of Proposition 2.13. □

We are now ready to prove Theorem C.

Proof. We will show that the map $\mathbf{Adj}_r(\mathcal{K}) \to \mathrm{N}(e_*(\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}}))$ is a trivial fibration in the model structure for 2-trivial and saturated weak complicial sets. Observe that by construction, it has the right lifting property with respect to $\partial\Delta[0] \to \Delta[0]$.

We prove now that the map $\mathcal{N}_{\mathbf{Adj}}(\mathcal{K}) \to \mathcal{N}_{B\leftarrow A}(\mathcal{K})$ has the right lifting property with respect to $\partial\Delta[n] \to \Delta[n]$ for all $n \geq 1$. This will prove the right lifting property of $\mathbf{Adj}_r(\mathcal{K}) \to \mathrm{N}(e_*(\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}}))$ with respect to $\partial\Delta[n] \to \Delta[n]$, because of the pullback of Remark 6.3.

A commutative diagram

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d] & \mathcal{N}_{\mathbf{Adj}}(\mathcal{K}) \arrow[d] \\
\Delta[n] \arrow[r] & \mathcal{N}_{B\leftarrow A}(\mathcal{K})
\end{tikzcd}$$

corresponds to a commutative diagram

$$\begin{tikzcd}
(B \leftarrow A) \times \mathcal{C}^{\mathrm{co}}(\partial\Delta[n]) \arrow[r] \arrow[d] & \partial\mathbf{Adj}_{\mathrm{hc}}[n] \arrow[r] & \mathcal{K} \\
(B \leftarrow A) \times \mathcal{C}\Delta[n]^{\mathrm{co}} \arrow[urr] & &
\end{tikzcd}$$

and thus to a simplicial functor $\overline{\partial\mathbf{Adj}_{\mathrm{hc}}}[n] \to \mathcal{K}$ by Proposition 6.18. This simplicial functor can be extended to a simplicial functor $\mathbf{Adj}_{\mathrm{hc}}[n] \to \mathcal{K}$ by Theorem A.

Observe that $\mathcal{N}_{\mathbf{Adj}}(\mathcal{K}) \to \mathcal{N}_{B\leftarrow A}(\mathcal{K})$ has the right lifting property with respect to $\Delta[n] \to \Delta[n]_t$ for $n \geq 2$, and thus so does $\mathbf{Adj}_r(\mathcal{K}) \to \mathrm{N}(e_*(\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}}))$. For $n = 1$, suppose that $\mathbb{A} : \mathbf{Adj}_{\mathrm{hc}}[1] \to \mathcal{K}$ is a 1-simplex of $\mathbf{Adj}_r(\mathcal{K})$ whose restriction $(B \leftarrow A) \times \mathcal{C}\Delta[1]^{\mathrm{co}} \to \mathcal{K}$ is thin in $\mathrm{N}(e_*(\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}}))$. This means that there are two thin 2-simplicies $\mathbb{H}_1, \mathbb{H}_2 : (B \leftarrow A) \times \mathcal{C}\Delta[2]^{\mathrm{co}} \to \mathcal{K}$ of $\mathrm{N}(e_*(\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}}))$ witnessing that the restriction of $\mathbb{A}$ is an equivalence. Using the lifting property with respect to $\partial\Delta[n] \to \Delta[n]$ and $\Delta[2] \to \Delta[2]_t$, one obtains two thin 2-simplicies of $\mathbf{Adj}_r(\mathcal{K})$ which witness that $\mathbb{A}$ is an equivalence and thus thin.

It is now enough, by the 2-out-of-3 property, to show that $\mathbb{P}$ induces a weak equivalence $\mathrm{N}(e_*(\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}})) \to \mathrm{N}(e_*(\mathcal{L}^{\bullet\to\bullet}_{\mathbf{Adj}}))$. But the homotopy coherent nerve preserves weak equivalences between fibrant objects, and thus it is enough to show that $\mathbb{P}$ induces a weak equivalence $\mathbb{P}^{\bullet\to\bullet}_{\mathbf{Adj}} : \mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}} \to \mathcal{L}^{\bullet\to\bullet}_{\mathbf{Adj}}$. If $f : X \to Y, g : X' \to Y'$ are objects of $\mathcal{K}^{\bullet\to\bullet}$, the quasi-categories $\mathcal{K}^{\bullet\to\bullet}(f, g)$ and $\mathcal{L}^{\bullet\to\bullet}(\mathbb{P}f, \mathbb{P}g)$ are the pullbacks

$$\begin{tikzcd}
\mathcal{K}^{\bullet\to\bullet}(f, g) \arrow[r] \arrow[d] & \mathcal{K}(X, X') \arrow[d, "g_*"'] \\
\mathcal{K}(Y, Y') \arrow[r, "f^*"'] & \mathcal{K}(X, Y')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}^{\bullet\to\bullet}(\mathbb{P}f, \mathbb{P}g) \arrow[r] \arrow[d] & \mathcal{L}(\mathbb{P}X, \mathbb{P}X') \arrow[d, "\mathbb{P}g_*"'] \\
\mathcal{L}(\mathbb{P}Y, \mathbb{P}Y') \arrow[r, "\mathbb{P}f^*"'] & \mathcal{L}(\mathbb{P}X, \mathbb{P}Y')
\end{tikzcd}$$

Since the right vertical maps are isofibrations and all objects are fibrant, those pullbacks are homotopy pullbacks and thus the simplicial functor $\mathbb{P}^{\bullet\to\bullet}_{\mathbf{Adj}}$ induced by $\mathbb{P}$ is locally a weak equivalence, provided that $\mathbb{P}$ is one. Finally, we need to show that $\mathbb{P}^{\bullet\to\bullet}_{\mathbf{Adj}}$ is essentially surjective as long as $\mathbb{P}$ is a weak equivalence of ∞-cosmoi. This is the case if and only if $\mathbb{P}^{\bullet\to\bullet} : \mathcal{K}^{\bullet\to\bullet} \to \mathcal{L}^{\bullet\to\bullet}$ is essentially surjective, by Theorem A and by [41, Theorem 3.6.6.].

To prove this, let $f : L \to L'$ be an isofibration between fibrant objects of $\mathcal{L}$. Since $\mathbb{P}$ is essentially surjective, there are fibrant objects K, K' and weak equivalences $h : \mathbb{P}K \to L$, $h' : \mathbb{P}K' \to L'$. Since $\mathbb{P}$ induces a weak equivalence of quasi-categories $\mathcal{K}(K, K') \to \mathcal{L}(\mathbb{P}K, \mathbb{P}K')$, there is a morphism $\alpha : K \to K'$ such that $h' \circ \mathbb{P}(\alpha) \cong f \circ h$. We can factorize α as a trivial cofibration $j : K \to \hat{K}$ followed by a fibration $\hat{f} : \hat{K} \to K'$. Since weak equivalences are equivalences in the homotopy 2-category, they are preserved by $\mathbb{P}$. As a consequence, the following diagram commutes up to isomorphism,

$$\begin{tikzcd}
\mathbb{P}(\hat{K}) \arrow[r, "\mathbb{P}(\hat{f})"] \arrow[d, "h \cdot \overline{\mathbb{P}j}"] & \mathbb{P}(K') \arrow[d, "h'"] \\
L \arrow[r, "f"] & L'
\end{tikzcd}$$

where $\overline{\mathbb{P}j}$ is an equivalence inverse of $\mathbb{P}j$. Let $H : \mathbb{I} \to \mathcal{L}(\mathbb{P}\hat{K}, L')$ be the extension of an isomorphism filling the diagram above (See Proposition 2.52). Since f is an isofibration, there is a lift in the diagram

$$\begin{tikzcd}
\Delta[0] \arrow[r, "h \cdot \overline{\mathbb{P}j}"] \arrow[d] & \mathcal{L}(\mathbb{P}\hat{K}, L) \arrow[d, "f_*"] \\
\mathbb{I} \arrow[r, "H"] & \mathcal{L}(\mathbb{P}\hat{K}, L')
\end{tikzcd}$$

which provides an isomorphism $\hat{h} \cong h \cdot \overline{\mathbb{P}j}$ such that $f\hat{h} = h'\mathbb{P}(\hat{f})$. □

6.4 Classification results

The goal of this section is to classify homotopy coherent monad morphisms up to homotopy. Homotopy coherent monad and adjunction morphisms are defined as follows.

Definition 6.19. Let $\mathcal{K}$ be an ∞-cosmos and $\mathbb{T}, \mathbb{S} : \mathbf{Mnd} \to \mathcal{K}$ be two homotopy coherent monads. A *homotopy coherent monad morphism* $\mathbb{T} \to \mathbb{S}$ is a 1-simplex $\mathbb{T} \to \mathbb{S}$ in $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$. More explicitly, it is a simplicial functor $\mathbb{M} : \mathbf{Mnd}_{\mathrm{hc}}[1] \to \mathcal{K}$ such that $\mathbb{M}\mathbb{d}^1 = \mathbb{T}$ and $\mathbb{M}\mathbb{d}^0 = \mathbb{S}$.

Definition 6.20. Let $\mathcal{K}$ be an ∞-cosmos and $\mathbb{A}, \mathbb{B} : \mathbf{Adj} \to \mathcal{K}$ be two homotopy coherent adjunctions. An *homotopy coherent adjunction morphism* $\mathbb{A} \to \mathbb{B}$ is a simplicial functor $\mathbb{M} : \mathbf{Adj}_{\mathrm{hc}}[1] \to \mathcal{K}$ such that $\mathbb{M}\mathbb{d}^1 = \mathbb{A}$ and $\mathbb{M}\mathbb{d}^0 = \mathbb{B}$.

Following Verity, [55, Notation 13], let $\mathrm{sp}_1 : \mathbf{Strat} \to \mathbf{Strat}$ be the functor that associates to a stratified simplicial set X the simplicial subset of those simplices whose faces of dimension greater or equal to 2 are all thin.

Definition 6.21. Let $\mathcal{K}$ be an ∞-cosmos. We define

- $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1$ to be the underlying simplicial set of the stratified simplicial set $\mathrm{sp}_1\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$;
- $\mathbf{Adj}_r(\mathcal{K})|_1$ to be the underlying simplicial set of the stratified simplicial set $\mathrm{sp}_1\mathbf{Adj}_r(\mathcal{K})$.

Observe that by [55, Lemma 25] and Theorem C, $\mathrm{sp}_1(\mathbf{Adj}_r(\mathcal{K}))$ is a 1-trivial weak complicial set, which is saturated by construction. By [55, Corollary 114], its underlying simplicial set is a quasi-category. Thus, the following corollary holds.

Corollary 6.22. *Let $\mathcal{K}$ be an ∞-cosmos. The simplicial set $\mathbf{Adj}_r(\mathcal{K})|_1$ is a quasi-category.*

Similarly, if Conjecture B holds, $\mathrm{sp}_1(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K}))$ is a 1-trivial weak complicial set, which is saturated by construction. By [55, Corollary 114], its underlying simplicial set is a quasi-category. Thus, the following corollary holds.

Corollary 6.23. *Let $\mathcal{K}$ be an ∞-cosmos. Assuming that Conjecture B holds, the simplicial set $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1$ is a quasi-category.*

In Theorem 6.30, we prove that $h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1)$ is a full reflective subcategory of $h(\mathbf{Adj}_r(\mathcal{K})|_1)$. In particular, this enables us to classify equivalences of homotopy coherent monads in Corollary 6.33.

We start this section by describing $h(\mathbf{Adj}_r(\mathcal{K})|_1)$.

6.4.1 Description of $h(\mathbf{Adj}_r(\mathcal{K})|_1)$

Definition 6.24. Let $\mathscr{C}$ be a 2-category. We consider the 2-category $\mathrm{PsArr}(\mathscr{C})$ defined as follows.

- Its objects are 1-cells $r : Y \to X$.

- Its 1-cells $r \to r'$ are diagrams

$$\begin{tikzcd}
X \arrow[d, "\alpha_0"'] & Y \arrow[l, "r"'] \arrow[d, "\alpha_1"] \arrow[dl, phantom, "\overset{\Nwarrow}{\alpha}"] \\
X' & Y' \arrow[l, "r'"]
\end{tikzcd}$$

where α is an isomorphism, with vertical pasting as composition.

- Its 2-cells $\begin{tikzcd} r \arrow[r, bend left, "{(\alpha_0,\alpha_1,\alpha)}"] \arrow[r, bend right, "{(\beta_0,\beta_1,\beta)}"'] \arrow[r, phantom, "\Downarrow"] & r' \end{tikzcd}$ are pairs of 2-cells $\tau_0 : \alpha_0 \Rightarrow \beta_0, \tau_1 : \alpha_1 \Rightarrow \beta_1$ such that

$$\begin{tikzcd}
X \arrow[d, bend right, "\beta_0"'] \arrow[d, bend left, "\alpha_0"] \arrow[d, phantom, "\underset{\tau_0}{\Leftarrow}"] & Y \arrow[l, "r"'] \arrow[d, "\alpha_1"] \arrow[dl, phantom, "\overset{\Nwarrow}{\alpha}"] \\
X' & Y' \arrow[l, "r'"]
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[d, "\beta_0"'] & Y \arrow[l, "r"'] \arrow[d, bend right, "\beta_1"'] \arrow[d, bend left, "\alpha_1"] \arrow[d, phantom, "\underset{\tau_1}{\Leftarrow}"] \arrow[dl, phantom, "\overset{\Nwarrow}{\beta}"] \\
X' & Y' \arrow[l, "r'"]
\end{tikzcd}. \tag{6.7}$$

The experienced reader will recognize that $\mathrm{PsArr}(\mathscr{C})$ is the classical 2-category of strict 2-functors $(\bullet \to \bullet) \to \mathscr{C}$, pseudo-natural transformations and modifications.

Definition 6.25. Let $\mathcal{K}$ be an ∞-cosmos. We define a category $\mathrm{IPA}(\mathcal{K})$ as follows.

- Its objects are homotopy coherent adjunctions $\mathbb{A} : \mathbf{Adj} \to \mathcal{K}$ such that $\mathbb{A}(r) : \mathbb{A}(A) \to \mathbb{A}(B)$ is an isofibration.

- Its morphisms are given by

$$\mathrm{IPA}(\mathcal{K})(\mathbb{A}, \mathbb{B}) = \mathrm{Iso}_*\mathrm{PsArr}(\mathcal{K}_2)(\mathbb{A}(r), \mathbb{B}(r)).$$

More explicitly, the morphisms $\mathbb{A} \to \mathbb{B}$ in $\mathrm{IPA}(\mathcal{K})$ are equivalence classes of diagrams

$$\begin{tikzcd}
\mathbb{A}(B) \arrow[d, "\alpha_0"'] & \mathbb{A}(A) \arrow[l, "\mathbb{A}(r)"'] \arrow[d, "\alpha_1"] \arrow[dl, phantom, "\overset{\Nwarrow}{\alpha}"] \\
\mathbb{B}(B) & \mathbb{B}(A), \arrow[l, "\mathbb{B}(r)"]
\end{tikzcd}$$

where α an isomorphism. Two such diagrams are equivalent if they are isomorphic in $\mathrm{PsArr}(\mathcal{K}_2)(\mathbb{A}(r), \mathbb{B}(r))$.

Remark 6.26. Let $\mathbb{A}, \mathbb{B} : \mathbf{Adj} \to \mathcal{K}$ be two homotopy coherent adjunctions. Observe that any class in $\mathrm{IPA}(\mathcal{K})(\mathbb{A}, \mathbb{B})$ can be by represented by a commutative diagram

$$\begin{tikzcd}
\mathbb{A}(B) \arrow[d, "\alpha_0"] & \mathbb{A}(A) \arrow[l, "\mathbb{A}(r)"'] \arrow[d, "\hat{\alpha}_1"] \arrow[dl, equal, shorten=14pt] \\
\mathbb{B}(B) & \mathbb{B}(A) \arrow[l, "\mathbb{B}(r)"']
\end{tikzcd} . \tag{6.8}$$

Indeed, if

$$\begin{tikzcd}
\mathbb{A}(B) \arrow[d, "\alpha_0"] & \mathbb{A}(A) \arrow[l, "\mathbb{A}(r)"'] \arrow[d, "\alpha_1"] \\
\mathbb{B}(B) & \mathbb{B}(A) \arrow[l, "\mathbb{B}(r)"'] \arrow[ul, Rightarrow, shorten=14pt, "\alpha"]
\end{tikzcd}$$

is an 1-cell belonging to $\mathrm{PsArr}(\mathcal{K}_2)(\mathbb{A}(r), \mathbb{B}(r))$, let $\hat{\alpha} : \mathbb{I} \to \mathcal{K}(\mathbb{A}(A), \mathbb{B}(B))$ be the extension of the isomorphism α given by Proposition 2.52. Since $\mathbb{B}(r)$ is an isofibration, there is a lift in the commutative diagram

$$\begin{tikzcd}
* \arrow[r, "\alpha_1"] \arrow[d] & \mathcal{K}(\mathbb{A}(A), \mathbb{B}(A)) \arrow[d, "\mathcal{K}(\mathbb{A}(A), \mathbb{B}(r))"] \\
\mathbb{I} \arrow[r, "\hat{\alpha}"'] & \mathcal{K}(\mathbb{A}(A), \mathbb{B}(B))
\end{tikzcd}$$

since the right vertical map is an isofibration, and the left vertical map is a trivial cofibration. The lift provides the desired map $\hat{\alpha}_1 : \mathbb{A}(A) \to \mathbb{B}(A)$ such that $\mathbb{B}(r)\hat{\alpha}_1 = \alpha_0\mathbb{A}(r)$ together with an isomorphism $\tau : \hat{\alpha}_1 \Rightarrow \alpha_1$, which makes the pair $(1, \tau)$ the witness that the two diagrams are equivalent.

Moreover, suppose that there is a commutative diagram

$$\begin{tikzcd}
\mathbb{A}_0(B) \arrow[dr, "b_2"] \arrow[dd, "b_1"'] & & \mathbb{A}_0(A) \arrow[ll, "\mathbb{A}_0(r)"'] \arrow[dr, "a_2"] \arrow[dd, "a_1"'] & \\
 & \mathbb{A}_1(B) \arrow[l, Rightarrow, "\sigma_B"'] \arrow[dl, "b_0"] & & \mathbb{A}_1(A) \arrow[l, Rightarrow, "\sigma_A"'] \arrow[dl, "a_0"] \arrow[ll, bend left, "\mathbb{A}_1(r)"] \\
\mathbb{A}_2(B) & & \mathbb{A}_2(A) \arrow[ll, "\mathbb{A}_2(r)"] &
\end{tikzcd} \tag{6.9}$$

such that $\mathbb{A}_2(r) \cdot \sigma_A = \sigma_B \cdot \mathbb{A}_0(r)$ in $\mathcal{K}_2$. Let s_B and s_A be representatives respectively of the classes σ_B, σ_A. The fact that $\mathbb{A}_2(r) \cdot \sigma_A = \sigma_B \cdot \mathbb{A}_0(r)$ in $\mathcal{K}_2$ is witnessed by a 2-simplex $H : \Delta[2] \to \mathcal{K}(\mathbb{A}_0(A), \mathbb{A}_2(B))$ such that $d_0H = \mathbb{A}_2(r) \cdot s_A$,

$d_1 H = s_B \cdot \mathbb{A}_0(r)$ and $d_2 H$ is degenerate. Since $\mathbb{A}_2(r)$ is an isofibration, there is a lift in the diagram

$$\begin{tikzcd}
\Lambda^1[2] \arrow[r] \arrow[d] & \mathcal{K}(\mathbb{A}_0(A), \mathbb{A}_2(A)) \arrow[d, "\mathcal{K}(\mathbb{A}_0(A), \mathbb{A}_2(r))"] \\
\Delta[2] \arrow[r, "H"'] \arrow[ur, dashed, "\bar{H}"] & \mathcal{K}(\mathbb{A}_0(A), \mathbb{A}_2(B))
\end{tikzcd}$$

Observe that $d_1 \bar{H}$ provides a 1-simplex s'_A such that $\mathbb{A}_2(r) \cdot s'_A = s_B \cdot \mathbb{A}_0(r)$ in $\mathcal{K}$ and $[s_A] = [s'_A]$ in $\mathcal{K}_2(\mathbb{A}_0(A), \mathbb{A}_2(A))$.

As a consequence, $\mathrm{IPA}(\mathcal{K}) \cong h(\mathrm{N}(e_* \mathcal{K}^{\bullet \to \bullet}_{\mathbf{Adj}})|_1)$. Since $\mathbf{Adj}_r(\mathcal{K}) \to \mathrm{N}(e_* \mathcal{K}^{\bullet \to \bullet}_{\mathbf{Adj}})$ is a trivial isofibration (by the proof of Theorem C) and is bijective on objects, $h(\mathbf{Adj}_r(\mathcal{K})|_1) \cong h(\mathrm{N}(e_* \mathcal{K}^{\bullet \to \bullet}_{\mathbf{Adj}})|_1)$.

6.4.2 A reflective subcategory

The previous section implies that the map $\mathbf{Adj}_r(\mathcal{K}) \to \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$ induces a functor

$$\Lambda : \mathrm{IPA}(\mathcal{K}) \to h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1)$$

such that for every homotopy coherent adjunction $\mathbb{A} \in |\mathrm{IPA}(\mathcal{K})|$, $\Lambda(\mathbb{A}) = \mathbb{A}j$. We build now its right adjoint.

Recall that if $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1$ is not assumed to be a quasi-category, its homotopy category is the quotient of the free category on the graph

$$\begin{tikzcd}
\mathbf{sCat}(\mathbf{Mnd}_{\mathrm{hc}}[1], \mathcal{K}) \arrow[r, shift left] \arrow[r, shift right] & \mathbf{sCat}(\mathbf{Mnd}, \mathcal{K})
\end{tikzcd}$$

by the relations $\mathbb{T}\mathbbm{s}^0 \sim 1_{\mathbb{T}}$ for every homotopy coherent monad $\mathbb{T}$ and $\mathbb{H}\mathbbm{d}^1 \sim \mathbb{H}\mathbbm{d}^0 \circ \mathbb{H}\mathbbm{d}^2$ for every thin 2-simplex $\mathbb{H} : \mathbf{Mnd}_{\mathrm{hc}}[2] \to \mathcal{K}$.

Theorem 6.27. *Let $\mathcal{K}$ be an ∞-cosmos. There is a functor*

$$\Gamma : h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1) \to \mathrm{IPA}(\mathcal{K})$$

such that for every homotopy coherent monad $\mathbb{T}$, $\Gamma(\mathbb{T}) = \mathrm{Ran}_j \mathbb{T}$, the associated free-forgetful homotopy coherent adjunction.

We recall the following standard 2-categorical lemma, see for instance the proof of [28, Proposition 2.3].

Lemma 6.28. *Let $\mathscr{C}$ be a 2-category and $\kappa : X \to Y$ a 1-cell that has two right adjoints $\lambda, \lambda' : Y \to X$, with counits ϵ and ϵ' respectively. Then, there is an isomorphism $\sigma : \lambda \to \lambda'$ such that the following triangle is commutative*

$$\begin{tikzcd}
\kappa\lambda \arrow[r, Rightarrow, "\epsilon"] \arrow[d, Rightarrow, "\kappa\sigma"'] & 1_Y \\
\kappa\lambda' \arrow[ur, Rightarrow, "\epsilon'"'] &
\end{tikzcd}$$

Proof. Choose σ to be the mate of the identity 1_κ. More explicitly, if η and η' are respectively the units of the adjunctions $\kappa \dashv \lambda, \kappa \dashv \lambda'$,

$$\sigma = \begin{tikzcd} Y \arrow[r, "\lambda"] \arrow[dr, "1"'] & X \arrow[d, "\kappa"] \arrow[dr, "1"] \arrow[dl, phantom, "\overset{\epsilon}{\swarrow}"] & \\ & Y \arrow[r, "\lambda'"'] \arrow[ur, phantom, "\overset{\eta'}{\swarrow}"] & X \end{tikzcd} , \quad \sigma^{-1} = \begin{tikzcd} Y \arrow[r, "\lambda'"] \arrow[dr, "1"'] & X \arrow[d, "\kappa"] \arrow[dr, "1"] \arrow[dl, phantom, "\overset{\epsilon'}{\swarrow}"] & \\ & Y \arrow[r, "\lambda"'] \arrow[ur, phantom, "\overset{\eta}{\swarrow}"] & X \end{tikzcd} .$$

$\square$

Lemma 6.29. *Let $\mathbb{A} : \mathbf{Adj}_{\mathrm{hc}}[1] \to \mathcal{K}$ be an adjunction morphism in a ∞-cosmos $\mathcal{K}$, and let $\mathbb{T} = (\mathbb{A}\mathbb{d}^1)j, \mathbb{S} = (\mathbb{A}\mathbb{d}^0)j$ be the two homotopy coherent monads associated to the source and target of $\mathbb{A}$. Let also*

$$\begin{tikzcd} \mathbb{A}(B,1) \arrow[r, shift left, "f_{\mathbb{T}}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(\mathbb{T}) \arrow[l, shift left, "u_{\mathbb{T}}"] \end{tikzcd} , \qquad \begin{tikzcd} \mathbb{A}(B,0) \arrow[r, shift left, "f_{\mathbb{S}}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(\mathbb{S}) \arrow[l, shift left, "u_{\mathbb{S}}"] \end{tikzcd}$$

be the corresponding free-forgetful adjunctions. Suppose that the two comparison maps $\kappa_1 : \mathbb{A}(A, 1) \to \mathrm{Alg}(\mathbb{T})$, $\kappa_0 : \mathbb{A}(A, 0) \to \mathrm{Alg}(\mathbb{S})$ are equivalences, and let $\bar{\kappa}_1$ be an equivalence inverse to κ_1 with counit $\rho : \kappa_1\bar{\kappa}_1 \to 1_{\mathrm{Alg}(\mathbb{T})}$. Then,

$$\begin{tikzcd} \mathbb{A}(B,1) \arrow[d, "\mathbb{A}(b_1)"'] & \mathrm{Alg}(\mathbb{T}) \arrow[l, "u_{\mathbb{T}}"'] \arrow[d, "\kappa_0\mathbb{A}(a_1)\bar{\kappa}_1"] \arrow[dl, phantom, "\overset{\nwarrow}{\mathbb{A}(b_1)u_{\mathbb{T}}\rho}"] \\ \mathbb{A}(B,0) & \mathrm{Alg}(\mathbb{S}) \arrow[l, "u_{\mathbb{S}}"] \end{tikzcd}$$

is a 1-cell $u_{\mathbb{T}} \to u_{\mathbb{S}}$ of $\mathrm{PsArr}(\mathcal{K}_2)$ *whose isomorphism class does not depend on the choice of the equivalence inverse $\bar{\kappa}_1$ and counit ρ.*

Proof. Observe that

$$\begin{aligned} u_{\mathbb{S}}(\kappa_0\mathbb{A}(a_1)\bar{\kappa}_1) &= \mathbb{A}(r_0)\mathbb{A}(a_1)\bar{\kappa}_1 \\ &= \mathbb{A}(b_1)\mathbb{A}(r_1)\bar{\kappa}_1 \\ &= \mathbb{A}(b_1)u_{\mathbb{T}}(\kappa_1\bar{\kappa}_1) \end{aligned}$$

and thus $\mathbb{A}(b_1)u_{\mathbb{T}}\rho$ is indeed an isomorphism $u_{\mathbb{S}}(\kappa_0\mathbb{A}(a_1)\bar{\kappa}_1) \to \mathbb{A}(b_1)u_{\mathbb{T}}$.

Let $\bar{\kappa}_1'$ be another right adjoint equivalence inverse with corresponding counit ρ'. By Lemma 6.28, there is an isomorphism $\sigma : \bar{\kappa}_1 \to \bar{\kappa}_1'$ which induces an isomorphism

$$\begin{tikzcd} \mathrm{Alg}(\mathbb{T}) \arrow[r, bend left, "\bar{\kappa}_1"] \arrow[r, bend right, "\bar{\kappa}_1'"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{A}(A,1) \arrow[r, "\mathbb{A}(a_1)"] & \mathbb{A}(A,0) \arrow[r, "\kappa_0"] & \mathrm{Alg}(\mathbb{S}) \end{tikzcd} .$$

We compute

$$\begin{aligned}(\mathbb{A}(b_1)u_{\mathbb{T}}\rho') \circ (u_{\mathbb{S}}\kappa_0\mathbb{A}(a_1)\sigma) &= (\mathbb{A}(b_1)u_{\mathbb{T}}\rho') \circ (\mathbb{A}(r_0)\mathbb{A}(a_1)\sigma)\\ &= (\mathbb{A}(b_1)u_{\mathbb{T}}\rho') \circ (\mathbb{A}(b_1)\mathbb{A}(r_1)\sigma)\\ &= \mathbb{A}(b_1)u_{\mathbb{T}}(\rho' \circ \kappa_1\sigma)\\ &= \mathbb{A}(b_1)u_{\mathbb{T}}\rho,\end{aligned}$$

where the last equality is a consequence of Lemma 6.28. As a consequence, the pair $(1, \kappa_0\mathbb{A}(a_1)\sigma)$ witnesses that the two 1-cells $u_{\mathbb{T}} \to u_{\mathbb{S}}$ of $\mathrm{PsArr}(\mathcal{K}_2)$ are isomorphic. □

Proof of Theorem 6.27. The functor Γ acts on objects by $\Gamma(\mathbb{T}) = \mathrm{Ran}_j\mathbb{T}$, associating to a homotopy coherent monad $\mathbb{T}$ its free-forgetful homotopy coherent adjunction. Observe that $\mathrm{Ran}_j\mathbb{T}(r)$ is an isofibration since the natural transformation of weights $\mathbf{Adj}(B, j(-)) \to \mathbf{Adj}(A, j(-))$ which induces it is a relative projective cell-complex.

Let $\mathbb{M} : \mathbb{T} \to \mathbb{S}$ be a homotopy coherent monad morphism. By Corollary 5.25 its enriched right Kan extension $\mathrm{Ran}_{j_1}\mathbb{M} : \mathbf{Adj}_{\mathrm{hc}}[1] \to \mathcal{K}$ is such that the comparison maps $\kappa_1 : \mathrm{Ran}_{j_1}\mathbb{M}(A, 1) \to \mathrm{Alg}(\mathbb{T})$, and $\kappa_0 : \mathrm{Ran}_{j_1}\mathbb{M}(A, 0) \to \mathrm{Alg}(\mathbb{S})$ are equivalences. As a consequence, it satisfies the conditions of Lemma 6.29 and thus it determines a well defined class $\Gamma(\mathbb{M})$ of $\mathrm{IPA}(\mathcal{K})(\Gamma(\mathbb{T}), \Gamma(\mathbb{S}))$ represented by

$$\begin{tikzcd}
\mathrm{Ran}_{j_1}\mathbb{M}(B,1) \arrow[d, "\mathrm{Ran}_{j_1}\mathbb{M}(b_1)"'] & \mathrm{Alg}(\mathbb{T}) \arrow[l, "u_{\mathbb{T}}"'] \arrow[d, "\kappa_0\mathrm{Ran}_{j_1}\mathbb{M}(a_1)\bar{\kappa}_1"] \arrow[dl, phantom, "\overset{\Nwarrow}{\mathrm{Ran}_{j_1}\mathbb{M}(b_1)u_{\mathbb{T}}\rho}"] \\
\mathrm{Ran}_{j_1}\mathbb{M}(B,0) & \mathrm{Alg}(\mathbb{S}) \arrow[l, "u_{\mathbb{S}}"']
\end{tikzcd}$$

where $\bar{\kappa}_1$ is an equivalence inverse of κ_1 with associated counit $\rho : \kappa_1\bar{\kappa}_1 \Rightarrow 1_{\mathrm{Alg}(\mathbb{T})}$. To prove that this construction extends to a functor $\Gamma : h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1) \to \mathrm{IPA}(\mathcal{K})$, we need to show that

- given a 0-simplex $\mathbb{T} : \mathbf{Mnd} \to \mathcal{K}$, $\Gamma(\mathbb{T}\mathbb{s}^0) = 1_{\Gamma(\mathbb{T})}$ in $\mathrm{IPA}(\mathcal{K})$;
- given a thin 2-simplex $\mathbb{H} : \mathbf{Mnd}_{\mathrm{hc}}[2] \to \mathcal{K}$, $\Gamma(\mathbb{H}\mathbb{d}^1) = \Gamma(\mathbb{H}\mathbb{d}^0) \circ \Gamma(\mathbb{H}\mathbb{d}^2)$ in $\mathrm{IPA}(\mathcal{K})$.

Consider a thin 2-simplex $\mathbb{H} : \mathbf{Mnd}_{\mathrm{hc}}[2] \to \mathcal{K}$. Let $\mathbb{T}^i = \mathbb{H}\mathbb{d}^i\mathbb{d}^1$ and $\mathbb{S}^i = \mathbb{H}\mathbb{d}^i\mathbb{d}^0$ for $0 \leq i \leq 2$. We have the following diagram of homotopy coherent monad morphisms.

$$\begin{tikzcd}
\mathbb{T}^1 = \mathbb{T}^2 \arrow[dd, "\mathbb{H}\mathbb{d}^1"'] \arrow[dr, "\mathbb{H}\mathbb{d}^2"] & \\
& \mathbb{S}^2 = \mathbb{T}^0 \arrow[dl, "\mathbb{H}\mathbb{d}^0"] \\
\mathbb{S}^1 = \mathbb{S}^0 &
\end{tikzcd} \tag{6.10}$$

Let us fix $i \in [2]$. There are two ways to induce a morphism $u_{\mathbb{T}^i} \to u_{\mathbb{S}^i}$ in $\mathrm{PsArr}(\mathcal{K}_2)$, since by Corollary 5.25,

$$(\mathrm{Ran}_{j_2}\mathbb{H})\mathbb{d}^i, \mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i) : \mathbf{Adj}_{\mathrm{hc}}[1] \to \mathcal{K}$$

both satisfy the hypothesis of Lemma 6.29. Let

$$\begin{aligned}
\kappa_1^i &: (\mathrm{Ran}_{j_2}\mathbb{H})\mathbb{d}^i(A,1) \to \mathrm{Alg}(\mathbb{T}^i)\\
\kappa_0^i &: (\mathrm{Ran}_{j_2}\mathbb{H})\mathbb{d}^i(A,0) \to \mathrm{Alg}(\mathbb{S}^i)\\
\lambda_1^i &: \mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(A,1) \to \mathrm{Alg}(\mathbb{T}^i)\\
\lambda_0^i &: \mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(A,0) \to \mathrm{Alg}(\mathbb{S}^i)
\end{aligned}$$

be the respective comparison maps. Let also $\bar{\kappa}_1^i$ and $\bar{\lambda}_1^i$ be equivalence inverses respectively of κ_1^i and λ_1^i, with respective counits $\rho^i : \kappa_1^i\bar{\kappa}_1^i \Rightarrow 1_{\mathrm{Alg}(\mathbb{T}^i)}$, $\psi^i : \lambda_1^i\bar{\lambda}_1^i \Rightarrow 1_{\mathrm{Alg}(\mathbb{S}^i)}$. By Lemma 6.29, we have two distinct objects of $\mathrm{PsArr}(\mathcal{K}_2)(\mathbb{T}^i, \mathbb{S}^i)$ displayed below.

$$\begin{tikzcd}
\mathbb{H}\mathbb{d}^i(B,1) \arrow[d, "\mathbb{H}\mathbb{d}^i(b_1)"'] & \mathrm{Alg}(\mathbb{T}^i) \arrow[l, "u_{\mathbb{T}^i}"'] \arrow[d, "\kappa_0^i[(\mathrm{Ran}_{j_2}\mathbb{H})\mathbb{d}^i(a_1)]\bar{\kappa}_1^i"] \arrow[dl, phantom, "\overset{\Nwarrow}{\mathbb{H}\mathbb{d}^i(b_1)u_{\mathbb{T}^i}\rho^i}"] \\
\mathbb{H}\mathbb{d}^i(B,0) & \mathrm{Alg}(\mathbb{S}^i) \arrow[l, "u_{\mathbb{S}^i}"']
\end{tikzcd} \tag{6.11}$$

$$\begin{tikzcd}
\mathbb{H}\mathbb{d}^i(B,1) \arrow[d, "\mathbb{H}\mathbb{d}^i(b_1)"'] & \mathrm{Alg}(\mathbb{T}^i) \arrow[l, "u_{\mathbb{T}^i}"'] \arrow[d, "\lambda_0^i\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(a_1)\bar{\lambda}_1^i"] \arrow[dl, phantom, "\overset{\Nwarrow}{\mathbb{H}\mathbb{d}^i(b_1)u_{\mathbb{T}^i}\psi^i}"] \\
\mathbb{H}\mathbb{d}^i(B,0) & \mathrm{Alg}(\mathbb{S}^i) \arrow[l, "u_{\mathbb{S}^i}"']
\end{tikzcd} \tag{6.12}$$

We are going to show that these objects are isomorphic. By the universal property of the enriched right Kan extension $\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)$, there are also comparison maps $\tau_j^i : (\mathrm{Ran}_{j_2}\mathbb{H})\mathbb{d}^i(A,j) \to \mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(A,j)$ for $j = 1, 0$. The universal property of the free-forgetful homotopy coherent adjunction implies that $\lambda_j^i\tau_j^i = \kappa_j^i$ for $j = 1, 0$. As a consequence, τ_j^i is an equivalence. Let $\bar{\tau}_1^i$ be an equivalence inverse of τ_1^i with corresponding counit $\phi^i : \tau_1^i\bar{\tau}_1^i \Rightarrow 1$. Observe that we can freely choose that $\bar{\kappa}_1^i = \bar{\tau}_1^i\bar{\lambda}_1^i$ with counit ρ^i being $\lambda_1^i\tau_1^i\bar{\tau}_1^i\bar{\lambda}_1^i \xrightarrow{\lambda_1^i\phi^i\bar{\lambda}_1^i} \lambda_1^i\bar{\lambda}_1^i \xrightarrow{\psi^i} 1$. Now,

$$\begin{aligned} \kappa_0^i[(\mathrm{Ran}_{j_2}\mathbb{H})\mathbb{d}^i(a_1)]\bar{\kappa}_1^i &= \lambda_0^i\tau_0^i[(\mathrm{Ran}_{j_2}\mathbb{H})\mathbb{d}^i(a_1)]\bar{\tau}_1^i\bar{\lambda}_1^i \\ &= \lambda_0^i\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(a_1)\tau_1^i\bar{\tau}_1^i\bar{\lambda}_1^i, \end{aligned}$$

and thus $\lambda_0^i\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(a_1)\phi^i\bar{\lambda}_1^i$ is an isomorphism

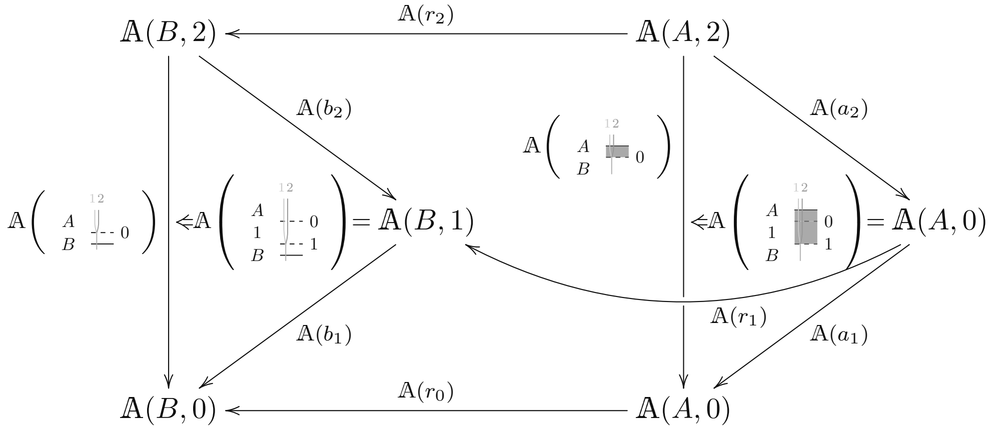
$$\kappa_0^i[(\mathrm{Ran}_{j_2}\mathbb{H})\mathbb{d}^i(a_1)]\bar{\kappa}_1^i \to \lambda_0^i\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(a_1)\bar{\lambda}_1^i.$$

We compute

$$\begin{aligned} &(\mathbb{H}\mathbb{d}^i(b_1)u_{\mathbb{T}^i}\psi^i) \circ (u_{\mathbb{S}^i}\lambda_0^i\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(a_1)\phi^i\bar{\lambda}_1^i) \\ &= (\mathbb{H}\mathbb{d}^i(b_1)u_{\mathbb{T}^i}\psi^i) \circ (\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(r_0)\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(a_1)\phi^i\bar{\lambda}_1^i) \\ &= (\mathbb{H}\mathbb{d}^i(b_1)u_{\mathbb{T}^i}\psi^i) \circ (\mathbb{H}\mathbb{d}^i(b_1)\mathrm{Ran}_{j_1}(\mathbb{H}\mathbb{d}^i)(r_1)\phi^i\bar{\lambda}_1^i) \\ &= \mathbb{H}\mathbb{d}^i(b_1)u_{\mathbb{T}^i}(\psi^i \circ \lambda_1^i\phi^i\bar{\lambda}_1^i) \\ &= \mathbb{H}\mathbb{d}^i(b_1)u_{\mathbb{T}^i}\rho^i. \end{aligned}$$

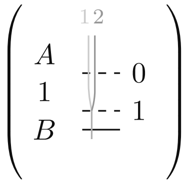
Thus, the two objects of $\mathrm{PsArr}(\mathcal{K}_2)(\mathbb{T}^i, \mathbb{S}^i)$ displayed in (6.11) and (6.12) are isomorphic.

Observe that $\mathbb{A} = \mathrm{Ran}_{j_2}\mathbb{H}$ provides a diagram

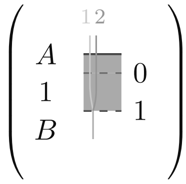


(6.13)

Recall that $\mathbb{A}(r_0)$ is conservative (see Corollary 5.23). Since $\mathbb{H}$ is thin, the class of $\mathbb{A}$



is an isomorphism in $\mathcal{K}_2$, and thus the class of $\mathbb{A}$



also. Consider the diagram displayed in (6.10).

- Let $\mathbb{T} = \mathbb{T}^1 = \mathbb{T}^2$ and remark that $\kappa_1^1 = \kappa_1^2 : \mathbb{A}(A,2) \to \mathrm{Alg}(\mathbb{T})$.
- Let $\mathbb{S} = \mathbb{S}^1 = \mathbb{S}^0$ and remark that $\kappa_0^1 = \kappa_0^0 : \mathbb{A}(A,0) \to \mathrm{Alg}(\mathbb{S})$.
- Let $\mathbb{R} = \mathbb{S}^2 = \mathbb{T}^0$ and remark that $\kappa_0^2 = \kappa_1^0 : \mathbb{A}(A,1) \to \mathrm{Alg}(\mathbb{R})$.

Let η_1^0 be the unit of the adjunction $\kappa_1^0 \dashv \bar{\kappa}_1^0$. The vertical pasting of the squares displayed in (6.11) for $i = 2, 0$ represents the class $\Gamma(\mathbb{Hd}^0) \circ \Gamma(\mathbb{Hd}^2)$. It is displayed in the following diagram

$$\begin{tikzcd}[column sep=large]
\mathbb{A}(B,2) \arrow[d, "\mathbb{A}(b_2)"'] & \mathrm{Alg}(\mathbb{T}) \arrow[l, "u_{\mathbb{T}}"'] \arrow[d, "\kappa_1^0\mathbb{A}(a_2)\bar{\kappa}_1^1"] \arrow[dd, bend left=70, "\kappa_0^1\mathbb{A}(a_1)\mathbb{A}(a_2)\bar{\kappa}_1^1", ""{name=C, left}] \arrow[dl, phantom, "\Nwarrow\ \mathbb{A}(b_2)u_{\mathbb{T}}\rho^1"] \\
\mathbb{A}(B,1) \arrow[d, "\mathbb{A}(b_1)"'] & \mathrm{Alg}(\mathbb{R}) \arrow[l, "u_{\mathbb{R}}"'] \arrow[d, "\kappa_0^1\mathbb{A}(a_1)\bar{\kappa}_1^0"] \arrow[dl, phantom, "\Nwarrow\ \mathbb{A}(b_1)u_{\mathbb{R}}\rho^0"] \arrow[from=C, Rightarrow, "\zeta"] \\
\mathbb{A}(B,0) & \mathrm{Alg}(\mathbb{S}), \arrow[l, "u_{\mathbb{S}}"']
\end{tikzcd}$$

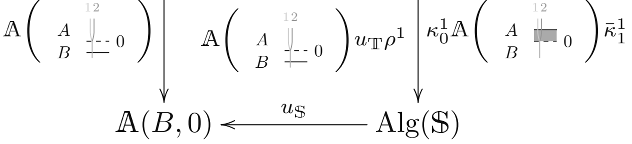
where $\zeta = \kappa_0^1\mathbb{A}(a_1)\eta_1^0\mathbb{A}(a_2)\bar{\kappa}_1^1$. By the triangle identity for $\kappa_1^0 \dashv \bar{\kappa}_1^0$,

$$\begin{aligned}
&(\mathbb{A}(b_1)u_{\mathbb{R}}\rho^0\kappa_1^0\mathbb{A}(a_2)\bar{\kappa}_1^1) \circ (u_{\mathbb{S}}\zeta) \\
&= (\mathbb{A}(b_1)u_{\mathbb{R}}\rho^0\kappa_1^0\mathbb{A}(a_2)\bar{\kappa}_1^1) \circ (\mathbb{A}(r_0)\mathbb{A}(a_1)\eta_1^0\mathbb{A}(a_2)\bar{\kappa}_1^1) \\
&= \mathbb{A}(b_1) \cdot (u_{\mathbb{R}}\rho^0\kappa_1^0 \circ \mathbb{A}(r_1)\eta_1^0) \cdot (\mathbb{A}(a_2)\bar{\kappa}_1^1) \\
&= (\mathbb{A}(b_1)u_{\mathbb{R}}) \cdot (\rho^0\kappa_1^0 \circ \kappa_1^0\eta_1^0) \cdot (\mathbb{A}(a_2)\bar{\kappa}_1^1) \\
&= 1.
\end{aligned}$$

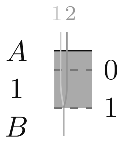
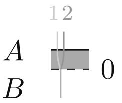
Thus, the class of $\Gamma(\mathbb{Hd}^0) \circ \Gamma(\mathbb{Hd}^2)$ is equal to the class of the diagram

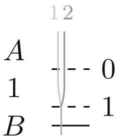
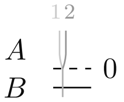
$$\begin{tikzcd}
\mathbb{A}(B,2) \arrow[d, "\mathbb{A}(b_1)\mathbb{A}(b_2)"'] & \mathrm{Alg}(\mathbb{T}) \arrow[l, "u_{\mathbb{T}}"'] \arrow[d, "\kappa_0^1\mathbb{A}(a_1)\mathbb{A}(a_2)\bar{\kappa}_1^1"] \arrow[dl, phantom, "\Nwarrow\ \mathbb{A}(b_1)\mathbb{A}(b_2)u_{\mathbb{T}}\rho^1"] \\
\mathbb{A}(B,0) & \mathrm{Alg}(\mathbb{S}) \arrow[l, "u_{\mathbb{S}}"']
\end{tikzcd}$$

whereas the class of $\Gamma(\mathbb{Hd}^1)$ is equal to the class of the diagram

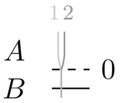
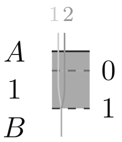


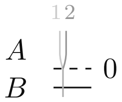
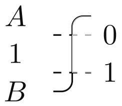
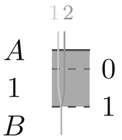
Consider the isomorphisms

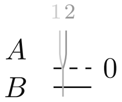
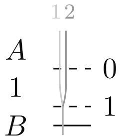
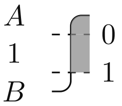
$\kappa_0^1\mathbb{A}\Bigg($  $\Bigg)\bar{\kappa}_1^1 : \kappa_0^1\mathbb{A}(a_1)\mathbb{A}(a_2)\bar{\kappa}_1^1 \to \kappa_0^1\mathbb{A}\Bigg($  $\Bigg)\bar{\kappa}_1^1,$

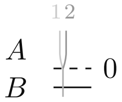
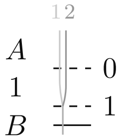
$\mathbb{A}\Bigg($  $\Bigg) : \mathbb{A}(b_1)\mathbb{A}(b_2) \to \mathbb{A}\Bigg($  $\Bigg).$

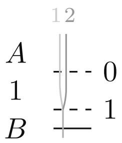
Observe that

$\Bigg(\mathbb{A}\Bigg($  $\Bigg)u_{\mathbb{T}}\rho^1\Bigg) \circ \Bigg(u_{\$}\kappa_0^1\mathbb{A}\Bigg($  $\Bigg)\bar{\kappa}_1^1\Bigg)$

$= \Bigg(\mathbb{A}\Bigg($  $\Bigg)u_{\mathbb{T}}\rho^1\Bigg) \circ \Bigg(\mathbb{A}\Bigg($  $\Bigg)\mathbb{A}\Bigg($  $\Bigg)\bar{\kappa}_1^1\Bigg)$

$= \Bigg(\mathbb{A}\Bigg($  $\Bigg)u_{\mathbb{T}}\rho^1\Bigg) \circ \Bigg(\mathbb{A}\Bigg($  $\Bigg)\mathbb{A}\Bigg($  $\Bigg)\bar{\kappa}_1^1\Bigg)$

$= \Bigg(\mathbb{A}\Bigg($  $\Bigg)u_{\mathbb{T}}\rho^1\Bigg) \circ \Bigg(\mathbb{A}\Bigg($  $\Bigg)u_{\mathbb{T}}\kappa_1^1\bar{\kappa}_1^1\Bigg)$

$= \Bigg(\mathbb{A}\Bigg($  $\Bigg) \circ u_{\mathbb{T}}\Bigg) \circ (\mathbb{A}(b_1)\mathbb{A}(b_2)u_{\mathbb{T}}\rho^1),$

where the last equality holds because of the interchange law in $\mathcal{K}_2$. As a consequence, $\Gamma(\mathbb{H}\mathbb{d}^0) \circ \Gamma(\mathbb{H}\mathbb{d}^2) = \Gamma(\mathbb{H}\mathbb{d}^1)$.

Let $\mathbb{T} : \mathbf{Mnd} \to \mathcal{K}$ be a homotopy coherent monad. Let us check that $\Gamma(\mathbb{T}\mathbb{s}^0)$ is the identity class in $\mathrm{IPA}(\mathcal{K})(\Gamma(\mathbb{T}), \Gamma(\mathbb{T}))$. The universal property of $\mathrm{Ran}_{j_1}(\mathbb{T}\mathbb{s}^0)$ implies the existence of a simplicial natural transformation $\alpha : (\mathrm{Ran}_j\mathbb{T})\mathbb{s}^0 \to \mathrm{Ran}_{j_1}(\mathbb{T}\mathbb{s}^0)$. Observe that by the universal property of $\mathrm{Ran}_j\mathbb{T}$, there are comparison maps

$$\kappa_i : \mathrm{Ran}_{j_1}(\mathbb{T}\mathbb{s}^0)(A, i) \to \mathrm{Alg}(\mathbb{T})$$

which are equivalences by Corollary 5.25. Observe that by the universal property of $\mathrm{Ran}_j\mathbb{T}$, the equivalence inverse of κ_i can be chosen to be $\alpha_{(A,i)}$, and moreover $\kappa_i \circ \alpha_{(A,i)} = 1_{\mathrm{Alg}(\mathbb{T})}$. As a consequence,

$$\begin{aligned}
\kappa_0 \mathrm{Ran}_{j_1}(\mathbb{T}\mathbb{s}^0)(a_1)\bar{\kappa}_1 &= \kappa_0 \mathrm{Ran}_{j_1}(\mathbb{T}\mathbb{s}^0)(a_1)\alpha_{(A,1)} \\
&= \kappa_0 \alpha_{(A,0)} \\
&= 1
\end{aligned}$$

We thus have finished proving that our construction provides a functor

$$\Gamma : h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1) \to \mathrm{IPA}(\mathcal{K}).$$

$\square$

The next theorem should be thought of as an analog of Proposition 2.36.

Theorem 6.30. *Let $\mathcal{K}$ be an ∞-cosmos. There is an adjunction*

$$\begin{tikzcd}
\mathrm{IPA}(\mathcal{K}) \arrow[r, shift left, "\Lambda"] \arrow[r, phantom, "\perp"] & h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1) \arrow[l, shift left, "\Gamma"]
\end{tikzcd}$$

which witnesses that $h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1)$ is a full reflective subcategory of $\mathrm{IPA}(\mathcal{K})$*.*

Proof. Let us prove first that $\Lambda\Gamma = 1$. Observe that $\Lambda\Gamma(\mathbb{T}) = (\mathrm{Ran}_j\mathbb{T})j = \mathbb{T}$ for every homotopy coherent monad $\mathbb{T}$. For morphisms, it is enough to prove the result on generators. Let $\mathbb{M} : \mathbb{T} \to \mathbb{S}$ be a homotopy coherent monad morphism and $\mathrm{Ran}_{j_1}\mathbb{M}$ its enriched right Kan extension. Let

$$\begin{aligned}
\kappa_1 &: \mathrm{Ran}_{j_1}\mathbb{M}(A, 1) \to \mathrm{Alg}(\mathbb{T}) \\
\kappa_0 &: \mathrm{Ran}_{j_1}\mathbb{M}(A, 0) \to \mathrm{Alg}(\mathbb{S})
\end{aligned}$$

be the comparison morphisms. By construction, $\Gamma(\mathbb{M})$ is the class of

$$\begin{tikzcd}
\mathbb{M}(B,1) \arrow[d, "\mathbb{M}(b_1)"'] & \mathrm{Alg}(\mathbb{T}) \arrow[l, "u_{\mathbb{T}}"'] \arrow[d, "\kappa_0 \mathrm{Ran}_{j_1}\mathbb{M}(a_1)\bar{\kappa}_1"] \arrow[dl, phantom, "\overset{\Nwarrow}{\mathbb{M}(b_1)u_{\mathbb{T}}\rho}"] \\
\mathbb{M}(B,0) & \mathrm{Alg}(\mathbb{S}) \arrow[l, "u_{\mathbb{S}}"']
\end{tikzcd} \quad , \qquad (6.14)$$

the diagram obtained by applying Lemma 6.29 with equivalence inverse $\bar{\kappa}_1$, counit $\rho : \kappa_1\bar{\kappa}_1 \Rightarrow 1$ with corresponding unit $\eta : 1 \Rightarrow \bar{\kappa}_1\kappa_1$.

By the 2-universal property of $\mathbf{Adj}_{\mathrm{hc}}[1]$, there is a 2-functor

$$\mathbf{Adj}_{\mathrm{hc}}[1] \to \mathbf{Adj} \times (\bullet \to \bullet),$$

since the 2-category $\mathbf{Adj} \times (\bullet \to \bullet)$ contains a strict adjunction morphism (See Corollary 5.1). Moreover,

$$\begin{aligned}(1, \kappa_1) &: (\mathrm{Ran}_{j_1}\mathbb{M})\mathbb{d}^1 \to \Gamma(\mathbb{T}),\\ (1, \kappa_0) &: (\mathrm{Ran}_{j_1}\mathbb{M})\mathbb{d}^0 \to \Gamma(\mathbb{S})\end{aligned}$$

determine simplicial natural transformations and thus simplicial functors

$$\mathbf{Adj} \times (\bullet \to \bullet) \to \mathcal{K}.$$

Let $\sigma_i : \mathbf{Adj}_{\mathrm{hc}}[1] \to \mathcal{K}$ be the homotopy coherent adjunction morphism corresponding to $(1, \kappa_i)$. Observe that the restrictions $\sigma_i j_1 : \mathbf{Mnd}_{\mathrm{hc}}[1] \to \mathcal{K}$ for $i = 1, 2$ are respectively the degeneracies of $\mathbb{T}$ and $\mathbb{S}$. We have a diagram of homotopy coherent adjunctions and homotopy coherent adjunction morphism

$$\begin{tikzcd}
\mathrm{Ran}_j\mathbb{T} \arrow[d, "\mathbb{F}"'] & (\mathrm{Ran}_{j_1}\mathbb{M})\mathbb{d}^1 \arrow[l, "\sigma_1"] \arrow[d, "\mathrm{Ran}_{j_1}\mathbb{M}"] \\
\mathrm{Ran}_j\mathbb{S} & (\mathrm{Ran}_{j_1}\mathbb{M})\mathbb{d}^0 \arrow[l, "\sigma_0"]
\end{tikzcd}$$

where $\mathbb{F}$ is the homotopy coherent adjunction morphism inducing $\Lambda\Gamma(\mathbb{M})$. Build as before a homotopy coherent adjunction morphism $\mathbb{G} : (\mathrm{Ran}_{j_1}\mathbb{M})\mathbb{d}^1 \to \mathrm{Ran}_j\mathbb{S}$ as a lift of the corresponding homotopy coherent adjunctions and the commutative diagram

$$\begin{tikzcd}
\mathbb{M}(B, 1) \arrow[d, "\mathbb{M}(b_1)"] & \mathrm{Ran}_{j_1}\mathbb{M}(A, 1) \arrow[l, "\mathrm{Ran}_{j_1}\mathbb{M}(r_1)"] \arrow[d, "\mathrm{Ran}_{j_1}\mathbb{M}(a_1)"] \\
\mathbb{M}(B, 0) & \mathrm{Ran}_{j_1}\mathbb{M}(A, 0) \arrow[l, "\mathrm{Ran}_{j_1}\mathbb{M}(r_0)"] \arrow[d, "\kappa_0"] \\
& \mathrm{Alg}(\mathbb{S}) \arrow[ul, "u_{\mathbb{S}}"]
\end{tikzcd}$$

Since $\mathbf{Adj}_r(\mathcal{K})|_1 \to \mathrm{N}(e_*\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}})|_1$ is a trivial fibration, the bottom triangle in the following diagram is filled with a thin 2-simplex $\mathbf{Adj}_{\mathrm{hc}}[2] \to \mathcal{K}$.

$$\begin{tikzcd}
\mathrm{Ran}_j\mathbb{T} \arrow[d, "\mathbb{F}"'] & (\mathrm{Ran}_{j_1}\mathbb{M})\mathbb{d}^1 \arrow[l, "\sigma_1"] \arrow[d, "\mathrm{Ran}_{j_1}\mathbb{M}"] \arrow[dl, "\mathbb{G}" description] \\
\mathrm{Ran}_j\mathbb{S} & (\mathrm{Ran}_{j_1}\mathbb{M})\mathbb{d}^0 \arrow[l, "\sigma_0"]
\end{tikzcd} \tag{6.15}$$

Moreover filling the top triangle with a thin 2-simplex is equivalent to furnishing a commutative diagram

$$\begin{tikzcd}
\mathbb{M}(B,1) \arrow[dr, "1"] \arrow[dd, "\mathbb{M}(b_1)"'] & & \mathrm{Ran}_{j_1}\mathbb{M}(A,1) \arrow[ll, "\mathrm{Ran}_{j_1}\mathbb{M}(r_1)"'] \arrow[dr, "\kappa_1"] \arrow[dd, "\kappa_0\mathrm{Ran}_{j_1}\mathbb{M}(a_1)"'] & \\
 & \mathbb{M}(B,1) \arrow[l, Rightarrow, "\sigma_B"'] \arrow[dl, "\mathbb{M}(b_1)"] & & \mathrm{Alg}(\mathbb{T}) \arrow[l, Rightarrow, "\sigma_A"'] \arrow[ll, bend left=10, "u_{\mathbb{T}}"] \arrow[dl, "\mathbb{F}(a_1)"] \\
\mathbb{M}(B,0) & & \mathrm{Alg}(\mathbb{S}) \arrow[ll, "u_{\mathbb{S}}"] &
\end{tikzcd}$$

in $\mathcal{K}_2$, with σ_A, σ_B isomorphisms. Recall that by construction, $\mathbb{F}(a_1)$ is obtained by first replacing the diagram (6.14) with a strictly commutative diagram. This involves lifting the isomorphism $\mathbb{M}(b_1)u_{\mathbb{T}}\rho$ through the isofibration $u_{\mathbb{S}}$. Let $\tau : \kappa_0\mathrm{Ran}_{j_1}\mathbb{M}(a_1)\kappa_1 \to \mathbb{F}(a_1)$ be this lifted isomorphism, which satisfies $u_{\mathbb{S}}\tau = \mathbb{M}(b_1)u_{\mathbb{T}}\rho$. Let

$$\sigma_A = (\kappa_0\mathrm{Ran}_{j_1}\mathbb{M}(a_1)\eta^{-1}) \circ (\tau^{-1}\kappa_1).$$

We compute

$$\begin{aligned}
&u_{\mathbb{S}}(\kappa_0\mathrm{Ran}_{j_1}\mathbb{M}(a_1)\eta^{-1}) \circ u_{\mathbb{S}}(\tau^{-1}\kappa_1) \\
&= (\mathrm{Ran}_{j_1}\mathbb{M}(r_0)\mathrm{Ran}_{j_1}\mathbb{M}(a_1)\eta^{-1}) \circ (\mathbb{M}(b_1)u_{\mathbb{T}}\rho^{-1}\kappa_1) \\
&= \mathbb{M}(b_1)(\mathrm{Ran}_{j_1}\mathbb{M}(r_1)\eta^{-1} \circ u_{\mathbb{T}}\rho^{-1}\kappa_1) \\
&= \mathbb{M}(b_1)u_{\mathbb{T}}(\kappa_1\eta^{-1} \circ \rho^{-1}\kappa_1). \\
&= 1.
\end{aligned}$$

Since $u_{\mathbb{S}}$ is conservative, σ_A is an isomorphism and σ_B can be chosen to be the identity.

The diagram displayed in (6.15) determines a map $\Delta[1] \times \Delta[1] \to \mathbf{Adj}_r(\mathcal{K})|_1$. Post-composing it with $\mathbf{Adj}_r(\mathcal{K})|_1 \to \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1$, we obtain that $\Lambda\Gamma(\mathbb{M}) \circ 1 = 1 \circ \mathbb{M}$ and thus $\Lambda\Gamma(\mathbb{M}) = \mathbb{M}$ in $h(\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1)$.

We finally construct a unit η of the adjunction, the counit being the identity. Let $\mathbb{A} \in |\mathrm{IPA}(\mathcal{K})|$, and $\mathbb{S} : \mathbf{Mnd} \to \mathcal{K}$ a homotopy coherent monad. Consider the commutative square

$$\begin{tikzcd}
\mathbb{A}(B) \arrow[d, "1"'] & \mathbb{A}(A) \arrow[l] \arrow[d, "\theta_{\mathbb{A}}"] \\
\mathbb{A}(B) & \mathrm{Alg}(\mathbb{A}j), \arrow[l]
\end{tikzcd}$$

where $\theta_{\mathbb{A}}$ is the comparison map, and let $\eta_{\mathbb{A}} : \mathbb{A} \to \Gamma\Lambda(\mathbb{A})$ be the corresponding map. By construction, $\eta_{\Gamma(\mathbb{S})} = \mathrm{id}_{\Gamma(\mathbb{S})}$. Observe that the homotopy coherent adjunction morphism induced by the strict natural transformation $\begin{tikzcd}\mathbf{Adj} \arrow[r, bend left, "\mathbb{A}"] \arrow[r, bend right, "\mathrm{Ran}_j(\mathbb{A}j)"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{K}\end{tikzcd}$ is a lift of $\eta_{\mathbb{A}}$, and thus $\Lambda(\eta_{\mathbb{A}}) = \mathrm{id}_{\Lambda(\mathbb{A})}$. Thus the two triangle equalities are satisfied. It is now enough to check that η is natural. Let $\alpha : \mathbb{A} \to \mathbb{B}$ be a 1-morphism of $\mathrm{IPA}(\mathcal{K})$ represented, without loss of generality, by a commutative diagram

$$\begin{tikzcd}
\mathbb{A}(B) \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\backslash\!\backslash"] & \mathbb{A}(A) \arrow[l, "\mathbb{A}(r)"'] \arrow[d, "\hat{\alpha}_1"] \\
\mathbb{B}(B) & \mathbb{B}(A). \arrow[l, "\mathbb{B}(r)"']
\end{tikzcd}$$

Consider $\mathbb{F} : \mathbf{Adj}_{\mathrm{hc}}[1] \to \mathcal{K}$, a lift of this diagram and of the homotopy coherent adjunctions $\mathbb{A}, \mathbb{B}$. The class of $\Gamma\Lambda(\alpha) \circ \eta_{\mathbb{A}}$ is represented by the left diagram displayed below, whereas $\eta_{\mathbb{B}} \circ \alpha$ is represented by the right one,

$$\begin{tikzcd}
\mathbb{A}(B) \arrow[d, "1"'] \arrow[dr, phantom, "\backslash\!\backslash"] & \mathbb{A}(A) \arrow[l, "\mathbb{A}(r)"'] \arrow[d, "\theta_{\mathbb{A}}"] \\
\mathbb{A}(B) \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\overset{\Nwarrow}{\alpha_0 u_{\mathbb{A}j}\rho}"] & \mathrm{Alg}(\mathbb{A}j) \arrow[l, "u_{\mathbb{A}j}"'] \arrow[d, "\kappa_0\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(a_1)\bar{\kappa}_1"] \\
\mathbb{B}(B) & \mathrm{Alg}(\mathbb{B}j) \arrow[l, "u_{\mathbb{B}j}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}(B) \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\backslash\!\backslash"] & \mathbb{A}(A) \arrow[l, "\mathbb{A}(r)"'] \arrow[d, "\hat{\alpha}_1"] \\
\mathbb{B}(B) \arrow[d, "1"'] \arrow[dr, phantom, "\backslash\!\backslash"] & \mathbb{B}(A) \arrow[l, "\mathbb{B}(r)"'] \arrow[d, "\theta_{\mathbb{B}}"] \\
\mathbb{B}(B) & \mathrm{Alg}(\mathbb{B}j) \arrow[l]
\end{tikzcd}
\tag{6.16}$$

where $\kappa_i : \mathrm{Ran}_{j_1}(\mathbb{F}j_1)(A, i) \to \mathrm{Alg}(\mathbb{F}j_1|_i)$ is the comparison map and $\bar{\kappa}_1$ a right adjoint equivalence inverse of κ_1, with counit ρ. Let ϕ be the unit of the adjunction $\kappa_1 \dashv \bar{\kappa}_1$. Let $\tau_i : \mathbb{F}(A, i) \to \mathrm{Ran}_{j_1}(\mathbb{F}j_1)(A, i)$ be the comparison map induced by the universal property of the enriched right Kan extension.

Observe that

$$\begin{aligned}
\theta_{\mathbb{B}}\hat{\alpha}_1 &= \kappa_0\tau_0\mathbb{F}(a_1) \\
&= \kappa_0\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(a_1)\tau_1.
\end{aligned}$$

and

$$\kappa_0\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(a_1)\bar{\kappa}_1\theta_{\mathbb{A}} = \kappa_0\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(a_1)\bar{\kappa}_1\kappa_1\tau_1.$$

Thus, $\kappa_0\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(a_1)\phi\tau_1$ is an isomorphism $\theta_{\mathbb{B}}\hat{\alpha}_1 \to \kappa_0\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(a_1)\bar{\kappa}_1\theta_{\mathbb{A}}$.

Moreover,

$$\begin{aligned}
&(\alpha_0 u_{\mathbb{A}j}\rho\theta_{\mathbb{A}}) \circ (u_{\mathbb{B}j}\kappa_0 \mathrm{Ran}_{j_1}(\mathbb{F}j_1)(a_1)\phi\tau_1) \\
&= (\alpha_0 u_{\mathbb{A}j}\rho\kappa_1\tau_1) \circ (\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(r_0)\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(a_1)\phi\tau_1) \\
&= \alpha_0((u_{\mathbb{A}j}\rho\kappa_1\tau_1) \circ (\mathrm{Ran}_{j_1}(\mathbb{F}j_1)(r_1)\phi\tau_1)) \\
&= \alpha_0((u_{\mathbb{A}j}\rho\kappa_1\tau_1) \circ (u_{\mathbb{A}j}\kappa_1\phi\tau_1)) \\
&= \alpha_0 u_{\mathbb{A}j}(\rho\kappa_1 \circ \kappa_1\phi)\tau_1 \\
&= 1.
\end{aligned}$$

Thus, the two diagrams displayed in (6.16) are isomorphic.

□

Proposition 6.31. *Let $\mathcal{K}$ be an ∞-cosmos. A morphism $f : X \to Y$ of $\mathcal{K}$ is an equivalence in $\mathrm{N}(e_*\mathcal{K})$ if and only if it is a weak equivalence.*

Proof. The morphism $f : X \to Y$ is a weak equivalence if and only if it is an equivalence in $\mathcal{K}_2$, by Proposition 2.93. By definition, it is an equivalence in $\mathcal{K}_2$ if and only if there is a map $g : Y \to X$ and isomorphisms $fg \cong 1_Y$, $gf \cong 1_X$. This is equivalent to f being an equivalence in $\mathrm{N}(e_*\mathcal{K})$. □

Corollary 6.32. *Let $\mathcal{K}$ be an ∞-cosmos and $\alpha : \mathbb{A} \to \mathbb{B}$ be a representative of a morphism of $\mathrm{IPA}(\mathcal{K})$ displayed below.*

$$\begin{tikzcd}
\mathbb{A}(B) \arrow[d, "\alpha_0"'] & \mathbb{A}(A) \arrow[l, "\mathbb{A}(r)"'] \arrow[d, "\alpha_1"] \\
\mathbb{B}(B) \arrow[ur, phantom, "\overset{\Nwarrow}{\alpha_2}"] & \mathbb{B}(A), \arrow[l, "\mathbb{B}(r)"]
\end{tikzcd}$$

The following propositions are equivalent.

(i) α_0, α_1 *are weak equivalences in* $\mathcal{K}$.

(ii) The class of α *is an isomorphism in* $\mathrm{IPA}(\mathcal{K})$.

Proof. Since $\mathbb{B}(r)$ is an isofibration, recall that we can choose a class representative where the diagram is strictly commutative by replacing α_1 by $\hat{\alpha}_1 : \mathbb{A}(A) \to \mathbb{B}(A)$. Observe that $\hat{\alpha}_1$ is a weak equivalence if and only if α_1 is. It is now enough to apply the previous proposition to $(\alpha_0, \hat{\alpha}_1) : \mathbb{A}(r) \to \mathbb{B}(r)$ as a morphism of $\mathcal{K}^{\bullet \to \bullet}$. □

Corollary 6.33. *Let $\mathbb{T}$ and $\mathbb{S}$ be two homotopy coherent monads in an ∞-cosmos $\mathcal{K}$. The following propositions are equivalent,*

(i) *The homotopy coherent monads* $\mathbb{T}$ *and* $\mathbb{S}$ *are equivalent in* $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$.

(ii) *The homotopy coherent monads* $\mathbb{T}$ *and* $\mathbb{S}$ *are isomorphic in* $h\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})|_1$.

(iii) *There is an equivalence* $\mathrm{Alg}(\mathbb{T}) \to \mathrm{Alg}(\mathbb{S})$ *and an equivalence* $\mathbb{T}(B) \to \mathbb{S}(B)$ *which makes the following diagram commutative up to isomorphism*

$$\begin{tikzcd}
\mathbb{T}(B) \arrow[d] & \mathrm{Alg}(\mathbb{T}) \arrow[l, "u_{\mathbb{T}}"'] \arrow[d] \\
\mathbb{S}(B) & \mathrm{Alg}(\mathbb{S}). \arrow[l, "u_{\mathbb{S}}"']
\end{tikzcd} \tag{6.17}$$

Proof. The fact that (i) implies (ii) is an easy observation. Since Γ is a functor, it sends isomorphisms to isomorphisms, and thus (ii) implies (iii) by Corollary 6.32. Finally, if (iii) holds, $\mathrm{Ran}_j\mathbb{T}$ and $\mathrm{Ran}_j\mathbb{S}$ are equivalent in $\mathrm{N}(e_*\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}})$. Since $\mathbf{Adj}_r(\mathcal{K}) \to \mathrm{N}(e_*\mathcal{K}^{\bullet\to\bullet}_{\mathbf{Adj}})$ is a trivial fibration (by the proof of Theorem C), they are also equivalent in $\mathbf{Adj}_r(\mathcal{K})$. By post-composing by the map $\mathbf{Adj}_r(\mathcal{K}) \to \mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$, we obtain that $\mathbb{T}$ and $\mathbb{S}$ are equivalent in $\mathcal{N}_{\mathbf{Mnd}}(\mathcal{K})$.

□

Observe that if one of the equivalent conditions of Corollary 6.32 is satisfied, the homotopy coherent monads $\mathbb{A}j$ and $\mathbb{B}j$ satisfy the equivalent conditions of Corollary 6.33.

Appendix A

Proofs of background results

For the reader's convenience, we recall the statement of Lemma 2.8.

Lemma 2.8. *Let $\mathbb{F} : \mathcal{C} \to \mathcal{D}$ be a simplicial functor which is bijective on objects. Then, the following propositions are equivalent.*

(i) $\mathbb{F}$ *is a relative simplicial computad;*

(ii)
- $\mathbb{F}$ *is faithful;*
- *for every m-morphism h of $\mathcal{D}$ there exist a unique integer $k \in \mathbb{N} \cup \{-1\}$ and m-morphisms $c_i \in \mathbb{F}(\mathcal{C})$ and $h_i \in \mathcal{D} \setminus \mathbb{F}(\mathcal{C})$ such that*

  $$h = c_{k+1} \cdot h_k \cdots h_1 \cdot c_1 \cdot h_0 \cdot c_0 \tag{2.1}$$

  and the h_i are atomic;
- *atomic morphisms of $\mathcal{D}$ not in the image of $\mathcal{C}$ are closed under degeneracies.*

Moreover, under these conditions, images of atomic morphisms of $\mathcal{C}$ are atomic in $\mathcal{D}$.

Before diving into the proof, let us introduce some terminology which makes the proof easier to formulate.

Definition A.1. Let X be a simplicial set and $x \in X_m$. The *essential degree* of x is the integer n such that there exists $x' \in X_n$ non-degenerate and $\sigma : [n] \to [m]$ surjective such that $x = \sigma^* x'$. This is well defined by the Eilenberg-Zilber lemma.

Proof. Let $I' = \{2[\partial\Delta[n]] \to 2[\Delta[n]] : n \in \mathbb{N}\}$. We suppose (ii) and construct the bijective on object and faithful functor $\mathbb{F}$ as a relative I'-cell complex, by induction on n. Up to renaming the objects of $\mathcal{D}$, we can assume that $\mathbb{F}$ is the identity on objects. Our induction hypothesis is the existence of a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathbb{F}"] \arrow[d, "\mathbb{i}^n"'] & \mathcal{D} \\
\mathcal{C}^n \arrow[ur, "\mathbb{F}^n"'] &
\end{tikzcd}$$

where $\mathbb{i}^n : \mathcal{C} \to \mathcal{C}^n$ is a I'-cell complex, $\mathbb{F}^n : \mathcal{C}^n \to \mathcal{D}$ is bijective on objects, an isomorphism on degrees $k < n$ and faithful on degrees $k \geq n$, and such that for all $c \in \mathcal{C}^n$, if $\mathbb{F}^n(c) = c_{k+1} \cdot h_k \cdots h_1 \cdot c_1 \cdot h_0 \cdot c_0$ is the decomposition of $\mathbb{F}^n(c)$ as in equation (2.1), then the essential degree of h_i is at most $n-1$.

We set $\mathcal{C}^0 = \mathcal{C}$, $\mathbb{i}_0 = \mathrm{id}_{\mathcal{C}}$ and $\mathbb{F}^0 = \mathbb{F}$.

Suppose now that $\mathbb{i}^n, \mathbb{F}^n$ exists as in the induction hypothesis. Let X_n be the set of degree n and non degenerate atomic arrows of $\mathcal{D}$ which are not in $\mathbb{F}(\mathcal{C})$. For $h : D \to D' \in X_n$, there is an associated map $h : \Delta[n] \to \mathcal{D}(D, D')$. Since $\mathbb{F}^n_k$ is an isomorphism in degrees $k < n$, the composite $\partial\Delta[n] \longrightarrow \Delta[n] \xrightarrow{h} \mathcal{D}(D, D')$ factors (uniquely) through

$$\mathbb{F}^n : \mathcal{C}^n(D, D') \to \mathcal{D}(D, D')$$

Let us write $\partial h : \partial\Delta[n] \to \mathcal{C}^n(D, D')$ for the induced map. Remark that ∂h induces a functor $\partial h : \mathbb{2}[\partial\Delta[n]] \to \mathcal{C}^n$.

We consider the following pushout

$$\begin{tikzcd}
\coprod_{h \in X_n} \mathbb{2}[\partial\Delta[n]] \arrow[r, "\sum_{h \in X_n} \partial h"] \arrow[d] & \mathcal{C}^n \arrow[d, "\mathbb{i}_n^{n+1}"] \arrow[ddr, bend left, "\mathbb{F}^n"] & \\
\coprod_{h \in X_n} \mathbb{2}[\Delta[n]] \arrow[r] \arrow[drr, bend right=10, "\sum_{h \in X_n} h"'] & \mathcal{C}^{n+1} \arrow[dr, dashed, "\mathbb{F}^{n+1}"] & \\
 & & \mathscr{D}
\end{tikzcd}$$

and the induced simplicial functor $\mathbb{F}^{n+1}$. We define $\mathbb{i}^{n+1} = \mathbb{i}_n^{n+1} \circ \mathbb{i}^n$, which is thus a relative I'-cell complex. Simplicial categories can be identified with functors $\Delta^{\mathrm{op}} \to \mathbf{Cat}$ with the structure maps being identity on objects. We can thus compute colimits level-wise. We have that $\mathbb{2}[\partial\Delta[n]]_k = \mathbb{2}[\Delta[n]]_k$ for $k < n$, thus $\mathbb{i}_n^{n+1}$ is an isomorphism in degrees $k < n$. As a consequence, $\mathbb{F}^{n+1}$ is also an isomorphism in degrees $k < n$.

Let us show that $\mathbb{F}^{n+1}$ is full on degree n. Let h be a n-morphism of $\mathcal{D}$, and let

$$h = c_{l+1} \cdot h_l \cdots h_1 \cdot c_1 \cdot h_0 \cdot c_0$$

be the unique decomposition given by the hypothesis. Remark that if h_i is degenerate, then there exists a unique $e_i \in \mathcal{C}^n$ such that $h_i = \mathbb{F}^n(e_i) = \mathbb{F}^{n+1}(\mathbb{i}_n^{n+1}(e_i))$, since $\mathbb{F}^n$ is an isomorphism in degree $n-1$. Otherwise, $h_i \in X_n$ and thus there exists also a unique $e_i \in \mathcal{C}^{n+1}$ with $\mathbb{F}^{n+1}(e_i) = h_i$. This shows that $\mathbb{F}^{n+1}$ is full in degree n. Let us show now that $\mathbb{F}^{n+1}$ is faithful in degree m for $m \geq n$. Since the diagram

$$\begin{tikzcd}
\coprod_{h\in X_n} \mathbb{2}[\partial\Delta[n]] \arrow[r, "\sum_{h\in X_n} \partial h"] \arrow[d] & \mathcal{C}^n \arrow[d, "\mathbb{i}_n^{n+1}"] \arrow[dd, bend left, "\mathbb{F}^n"] \\
\coprod_{h\in X_n} \mathbb{2}[\Delta[n]] \arrow[r] \arrow[rd, bend right=15, "\sum_{h\in X_n} d"'] & \mathcal{C}^{n+1} \arrow[d, dashed, "\mathbb{F}^{n+1}"] \\
& \mathcal{D}
\end{tikzcd}$$

in degree m is a category pushout, we can describe $\mathcal{C}_m^{n+1}$ as follows.

- Objects are the same as in $\mathcal{C}_m^n$.
- Morphisms are equivalence classes of paths in a graph. The graph is obtained from the underlying graph of $\mathcal{C}_m^n$ by adding the degeneracies of X_n as a set of edges, that is the edge set is

  $$\mathrm{Mor}\mathcal{C}_m^n \cup \{\sigma^*(h) : h \in X_n, \sigma : [n] \to [m] \text{ surjective}\}.$$

  The equivalence relation is generated by the elementary equivalences given by composing two successive morphisms in $\mathcal{C}_m^n$ and removing identities.

For $i = 1, 2$, let $\lambda_i = \tilde{c}_{i,w_i+1} \cdot \sigma_{i,w_i}^*(\tilde{h}_{i,w_i}) \cdots \sigma_{i,0}^*(\tilde{h}_{i,0}) \cdot \tilde{c}_{i,0}$ be such a path with $\tilde{c}_{i,s} \in \mathcal{C}_m^n$ and $\tilde{h}_{i,s} \in X_n$, with $\sigma_{i,s} : [n] \to [m]$ a surjective morphism. Remark that by (ii), there is a unique decomposition $\mathbb{F}^n(\tilde{c}_{i,j}) = c_{i,j}^{k_{i,j}+1} \cdot h_{i,j}^{k_{i,j}} \cdots h_{i,j}^0 \cdot c_{i,j}^0$ with $c_{i,j}^s \in \mathbb{F}(\mathcal{C})$ and $h_{i,j}^s \notin \mathbb{F}(\mathcal{C})$, and where $h_{i,j}^s$ are atomic arrows of $\mathcal{D}$ of essential degree at most $n-1$ (by induction hypothesis). Composing decompositions, one gets

$$\begin{aligned}
\mathbb{F}^{n+1}(\lambda_i) &= \mathbb{F}^n(\tilde{c}_{i,w_i+1}) \cdot \sigma_{i,w_i}^*(\tilde{h}_{i,w_i}) \cdots \sigma_{i,0}^*(\tilde{h}_{i,0}) \cdot \mathbb{F}^n(\tilde{c}_{i,0}) \\
&= (c_{i,w_i+1}^{k_{i,w_i+1}+1} \cdot h_{i,w_i+1}^{k_{i,w_i+1}} \cdots h_{i,w_i+1}^0 \cdot c_{i,w_i+1}^0) \cdot \sigma_{i,w_i}^*(\tilde{h}_{i,w_i}) \\
&\cdots \sigma_{i,0}^*(\tilde{h}_{i,0}) \cdot (c_{i,0}^{k_{i,0}+1} \cdot h_{i,0}^{k_{i,0}} \cdots h_{i,0}^0 \cdot c_{i,0}^0)
\end{aligned}$$

which is the unique decomposition of $\mathbb{F}^{n+1}(\lambda_i)$ as in Equation (2.1), since $\sigma_i^*(\tilde{h}_i)$ is atomic and does not belong to $\mathbb{F}(\mathcal{C})$. As a consequence, if $\mathbb{F}^{n+1}(\lambda_1) = \mathbb{F}^{n+1}(\lambda_2)$, those decompositions with $i = 1, 2$ are equal. Remark that $\tilde{h}_{i,s}$ are the unique atomic morphism of this decomposition which belong to $\mathcal{D} \setminus F(\mathcal{C})$ with essential degree n. As a consequence, $w_1 = w_2$, $\tilde{h}_{1,s} = \tilde{h}_{2,s}$ for all s, and also $\mathbb{F}^n(\tilde{c}_{1,s}) = \mathbb{F}^n(\tilde{c}_{i,s})$. Since $\mathbb{F}^n$ is faithful, $\lambda_1 = \lambda_2$.

To conclude, remark that $\mathbb{F} : \mathscr{C} \to \mathscr{D}$ is the transfinite composite of the $\mathbb{i}_n^{n+1}$, since eventually for each m, $\mathbb{F}^n_m$ eventually become an isomorphism when n grows.

We prove now that (ii) implies that atomic morphisms of $\mathcal{C}$ are atomic in $\mathcal{D}$. Let c be an atomic morphism of $\mathcal{C}$ and let $\mathbb{F}(c) = h_2 \cdot h_1$ be a decomposition. By (ii), one can decompose both arrows as

$$h_i = c_i^{k_i+1} \cdot h_i^{k_i} \cdot c_i^{k_i} \cdots c_i^1 \cdot h_i^0 \cdot c_i^0,$$

with $k_i \in \mathbb{N} \cup \{-1\}$, $c_i^j \in \mathbb{F}(\mathcal{C})$ and h_i^j an atomic morphisms of $\mathcal{D}$ not in $\mathbb{F}(\mathcal{C})$. Thus, one gets a second decomposition of $\mathbb{F}(c)$,

$$\mathbb{F}(c) = c_2^{k_2+1} \cdot h_2^{k_2} \cdots h_2^0 \cdot (c_2^0 c_1^{k_1+1}) \cdot \cdot h_1^{k_1} \cdots h_1^0 \cdot c_1^0.$$

By uniqueness of decomposition in (ii), $k_1, k_2 = -1$ and thus $h_1, h_2 \in \mathbb{F}(\mathscr{C})$. By faithfulness, c is also atomic in $\mathcal{D}$.

We finally show that (i) implies (ii). Remark that simplicial functors which are bijective on object and which satisfies proposition (ii) are closed under composition, since (ii) implies that the image of an atomic arrow in $\mathscr{C}$ is atomic in $\mathscr{D}$. Moreover, they are also closed under transfinite composition. Moreover, a pushout along $\mathbb{2}[\partial\Delta[n]] \to \mathbb{2}[\Delta[n]]$ is bijective on objects and satisfy (ii). □

We prove now Proposition 2.13, that we recall below.

Proposition 2.13. *Let $\mathbb{J} : \mathcal{C} \to \mathcal{D}$ be a relative simplicial computad which is bijective on objects and $\mathcal{E}$ a simplicial category. An extension of a simplicial functor $\mathbb{F} : \mathcal{C} \to \mathcal{E}$ to a simplicial functor $\mathcal{D} \to \mathcal{E}$ is uniquely specified by choosing for each non-degenerate atomic morphism $g : x \to y$ of $\mathcal{D}_m \setminus \mathbb{J}(\mathcal{C})$, a morphism $\mathbb{F}(g) : \mathbb{F}x \to \mathbb{F}y$ in $\mathcal{E}_m$; such that if $d_i g = \mathbb{J}(c_{k+1}) \cdot \sigma_k^*(g_k) \cdots \sigma_0^*(g_0) \cdot \mathbb{J}(c_0)$ is the decomposition of $d_i g$ as in* (2.2)*, then*

$$d_i\mathbb{F}(g) = \mathbb{F}(c_{k+1}) \cdot \sigma_k^*\mathbb{F}(g_k) \cdots \sigma_0^*\mathbb{F}(g_0) \cdot \mathbb{F}(c_0).$$

Proof. Let $h : d \to d'$ be a m-morphism of $\mathcal{D}$. Since $\mathbb{J} : \mathcal{C} \to \mathcal{D}$ is a relative simplicial computad, h as a unique decomposition

$$h = \mathbb{J}(c_{k+1}) \cdot \sigma_k^*(g_k) \cdots \sigma_0^*(g_0) \cdot \mathbb{J}(c_0)$$

as in (2.2).

We define $\mathbb{F}(h) = \mathbb{F}(c_{k+1}) \cdot \sigma_k^* F(g_k) \cdots \sigma_0^* F(g_0) \cdot F(c_0)$. By construction, $\mathbb{F}$ is level-wise functorial and compatible with degeneracies. We have to check that it commutes with faces, and by functoriality, it is enough to check it for atomic arrows of $\mathcal{D}$ not in the image $\mathbb{J}(\mathcal{C})$.

So suppose g is atomic in $\mathscr{D}$ and not in $\mathbb{J}(\mathscr{C})$. There exist a (unique) non-degenerate atomic arrow $\tilde{g}$ of $\mathscr{D}_n$ and degeneracy operator $\sigma : [m] \to [n]$ such that $g = \sigma^*(\tilde{g})$. We will show the result by induction on $m - n$.

- If $m - n = 0$, then g is non-degenerate and thus the result follow by hypothesis.

- Suppose that for all i, $\mathbb{F}(d_i g) = d_i \mathbb{F}(g)$ when g is an atomic morphism such that its degree minus its essential degree is less than k. Let $g = \sigma^*(\tilde{g})$ with $\sigma : [n+k] \to [n]$ surjective, with $\tilde{g}$ non-degenerate and atomic. Remark that since codegeneracies generates surjective morphisms, $\sigma = \sigma' \cdot s^l$ with $\sigma' : [n+k-1] \to [n]$. Now, $d_i g = (\sigma' \cdot s^l \cdot d^i)^* \tilde{g}$. There are two cases. If $i = l, l+1$, then $\sigma d^i = \sigma'$ and

$$\mathbb{F}(d_i g) = \sigma'^* \mathbb{F}(\tilde{g}) = d_i \sigma^* \mathbb{F}(\tilde{g}) = d_i \mathbb{F}(g).$$

  Otherwise, $s^l \cdot d^i = d^{i'} s^{l'}$ and by the induction hypothesis

$$\begin{aligned} \mathbb{F}(d_i g) &= \mathbb{F}(s_{l'} d_{i'} (\sigma')^* \tilde{g}) \\ &= s_{l'} \mathbb{F}(d_{i'} (\sigma')^* \tilde{g}) \\ &= s_{l'} d_{i'} \mathbb{F}((\sigma')^* \tilde{g}) \\ &= d_i s_l \mathbb{F}((\sigma') \tilde{g}) \\ &= d_i \mathbb{F}(g). \end{aligned}$$

□

We prove Proposition 2.83, after restating it and providing a lemma.

Proposition 2.83. *Let $(t, f, \overrightarrow{l}) : X \to Y$ be an m-morphism of $\mathbb{A}\mathbf{dj}$. Write $D_i(t, f, \overrightarrow{l}) = \{j_0, \ldots, j_n\}$ such that $j_s < j_{s+1}$ for all s.*

- *if $Y = A$, $\pi_0 X_i = \{\{(x,y) \in X_i : x \in]j_i, j_{i+1}[\} : i \in [0, n-1] \cap 2\mathbb{N}\}$;*
- *if $Y = B$, $\pi_0 X_i = \{\{(x,y) \in X_i : x \in]j_i, j_{i+1}[\} : i \in [0, n-1] \cap (2\mathbb{N}+1)\}$.*

Lemma A.2. *Let $l \in \mathrm{Step}_t$ be a step function, and for $i = 0, 1$, $(x_i, y_i) \in \mathbb{R}^2$ such that $0 \leq x_i \leq t$ and $y_i \in J(l, x_i)$. There is a simple curve $\phi : [0,1] \to \mathbb{R}^2$ such that $\phi(i) = (x_i, y_i)$ and $\operatorname{im} \phi \subseteq \bigcup_{x \in [x_0, x_1]} \{x\} \times J(l, x)$.*

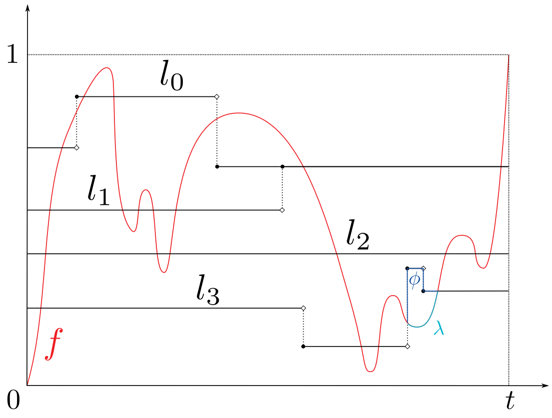
Proof. We prove the lemma by induction on the number of discontinuities of l. If there are none, one can reparametrize the graph of l to obtain the desired result. Suppose the result true for any step function with $n-1$ discontinuities, and let l be a step function with $n+1$ discontinuities, let $0 = a_0 < \ldots < a_{n+1} = t$ be the real numbers such that l is constant on $[a_i, a_{i+1}[$. By induction, we can assume $x_0 \leq a_1$ and $x_1 \geq a_n$. Remark that it is enough to suppose $y_1 = l(x_1)$, up to post concatenating with a parametrization of the segment $[(x_1, l(x_1)), (x_1, y_1)]$. Similarly we can assume $l(x_0) = y_0$.

Let ϕ be the simple curve obtained by induction such that $\phi(0) = (a_1, l(a_1))$, $\phi(1) = (x_1, y_1)$, and $\operatorname{im}\phi \subseteq \bigcup_{x\in[a_1,x_1]}\{x\} \times J(l,x)$. We also consider the curve ψ induced by the graph of l such that $\psi(0) = (x_0, y_0)$, $\psi(1) = (a_1, l(x_0))$ and $\operatorname{im}\psi \subseteq \bigcup_{x\in[x_0,a_1[}\{x\} \times J(l,x) \cup \{(a_1, l(x_0))\}$. Finally, the segment $[(a_1, l(x_0)), (a_1, l(a_1))] = J(l, a_1)$ is easily parametrized by a simple curve τ. The concatenation $\psi \star \tau \star \phi$ is the desired simple curve. □

Proof of Proposition 2.83. Let $s \in [0, n-1]$. We consider two curves λ, ϕ from $(j_s, f(j_s))$ to $(j_{s+1}, f(j_{s+1}))$. Let λ be the simple curve given by the graph of f and ϕ induced by l_i and the previous lemma. Remark that $\bar{\lambda} \star \phi$ is a simple closed curve, thus by Jordan's curve theorem, the complement of the curve in the plane has two distinct non empty connected components, whose boundaries are the given curve. By the intermediate value theorem, either $f(]j_s, j_{s+1}[) < l_i(]j_s, j_{s+1}[)$ and thus

$$X_i \cap \{(x,y) \in \mathbb{R}^2 : x \in [j_s, j_{s+1}]\} = \emptyset$$

or $f(]j_s, j_{s+1}[) > l_i(]j_s, j_{s+1}[)$ and thus this same space is non-empty and connected. The situation is illustrated by the following picture for $i = 3$ and $s = 4$.



By the intermediate value theorem, $X_i \cap \{(x,y) \in \mathbb{R}^2 : x \in [j_s, j_{s+1}]\} \simeq *$ on even s when $Y = A$ and on odd s when $Y = B$. □

Finally, we prove Proposition 2.110. We restate the Proposition for the reader's convenience.

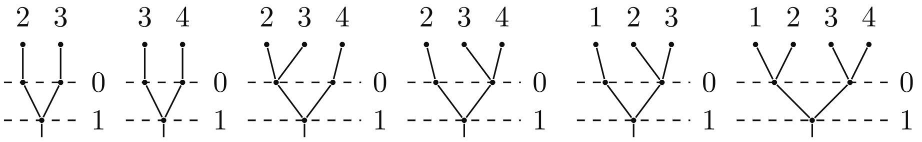
Proposition 2.110. *Let $\mathcal{K}$ be a category enriched in quasi-categories. The stratified simplicial set* $\mathrm{N}(e_*\mathcal{K})$ *is a* 2*-trivial and saturated weak complicial set.*

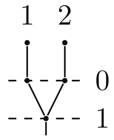
Proof. By construction, $\mathrm{N}(e_*\mathcal{K})$ has the right lifting property with respect to $\Delta[3]_{\simeq} \to \Delta[3]_{\sharp}$ since equivalences satisfy the 2-out-of-6 property. Moreover, if $\mathrm{N}(e_*\mathcal{K})$ has the right lifting property with respect to $\Delta[0] \star \Delta[3]_{\simeq} \to \Delta[0] \star \Delta[3]_{\sharp}$ if and only if $\mathrm{N}(e_*\mathcal{K})^{\mathrm{op}} = \mathrm{N}(e_*(\mathcal{K}^{\mathrm{op}}))$ has the right lifting property with respect to $\Delta[3]_{\simeq} \star \Delta[0] \to \Delta[3]_{\sharp} \star \Delta[0]$. It is thus enough to show the right lifting property of $\mathrm{N}(e_*\mathcal{K})$ with respect to $\Delta[3]_{\simeq} \star \Delta[0] \to \Delta[3]_{\sharp} \star \Delta[0]$. Let $f : \Delta[3]_{\simeq} \star \Delta[0] \to \mathrm{N}(e_*\mathcal{K})$ be a stratified simplicial set map. The simplicial set map f corresponds to a simplicial functor $\mathbb{F} : \mathcal{C}\Delta[4]^{\mathrm{co}} \to \mathcal{K}$. We analyze now what preserving the stratification means.

There are ten non-degenerate 1-simplices in $\Delta[3]_{\simeq} \star \Delta[0]$, six of them are thin and four are not. Since f preserve thin simplicies, the simplicial functors

$$\begin{tikzcd} \mathcal{C}\Delta[2]^{\mathrm{co}} \arrow[r] & \mathcal{C}\Delta[4]^{\mathrm{co}} \arrow[r, "\mathbb{F}"] & \mathcal{K} \end{tikzcd}$$

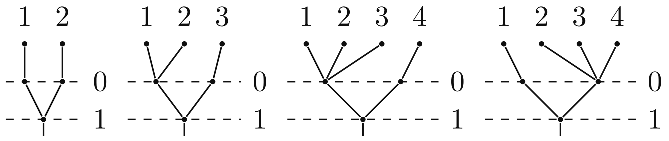
corresponding to these six thin simplicies are thin. As a consequence, the images by $\mathbb{F}$ of the following 1-morphisms are isomorphisms in $\mathcal{K}$.



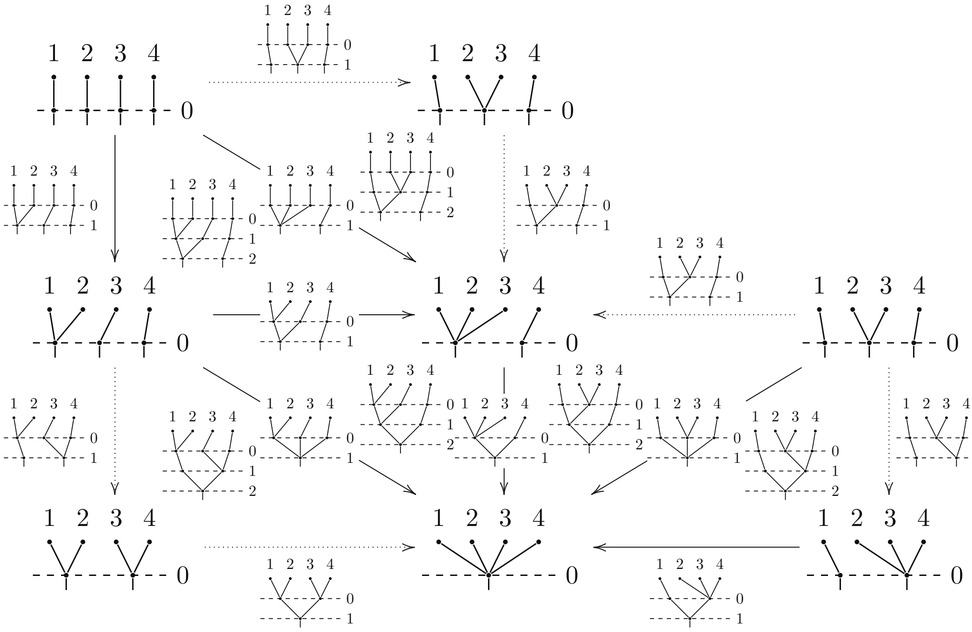
The 1-morphisms above correspond respectively to the image of  under the inclusions $\mathcal{C}\Delta[2]^{\mathrm{co}} \to \mathcal{C}\Delta[4]^{\mathrm{co}}$ corresponding to the non-degenerate 2-simplicies $(1 < 2 < 3)$, $(0 < 1 < 2)$, $(0 < 1 < 3)$, $(0 < 2 < 3)$, $(1 < 3 < 4)$ and $(0 < 2 < 4)$ of $\Delta[4]$. It is now enough to show that the simplicial functors

$$\begin{tikzcd} \mathcal{C}\Delta[2]^{\mathrm{co}} \arrow[r] & \mathcal{C}\Delta[4]^{\mathrm{co}} \arrow[r, "\mathbb{F}"] & \mathcal{K} \end{tikzcd}$$

corresponding to the 2-simplicies not thin in $\Delta[3]_{\simeq} \star \Delta[0]$ but thin in $\Delta[3]_{\sharp} \star \Delta[0]$ are actually thin. The simplicial functor corresponding respectively to the 2-simplicies $(2 < 3 < 4), (1 < 2 < 4), (0 < 1 < 4)$ and $(0 < 3 < 4)$ is thin if and only if the image of respectively the 1-morphism



is an isomorphism in $\mathcal{K}$. Consider the following squares in $\mathcal{C}\Delta[4]^{\mathrm{co}}(4,0)$.



Since the dotted arrows are sent to isomorphisms in $\mathcal{K}(\mathbb{F}(4),\mathbb{F}(0))$ and isomorphisms satisfy the 2-out-of-6 property, all the 1-morphisms of the above diagram are sent to isomorphisms in $\mathcal{K}(\mathbb{F}(4),\mathbb{F}(0))$. Since all 0-morphisms which do not contain the label 1 are sent to equivalences in $\mathcal{K}_2$, we have established the result. □

Appendix B

Presentation of a 2-category by computads

We want to construct an analogue of a free category on a graph, but for 2-categories. Computads will play the role of graphs. Since 2-categories are categories enriched in categories, computads will be graphs enriched in graphs. The material of this Appendix is taken from [51].

Definition B.1. A *computad* $\mathscr{G}$ is a graph $\mathrm{Gr}\,\mathscr{G}$ together with, for each pair of vertices A, B of $\mathrm{Gr}\,\mathscr{G}$, another graph $\mathscr{G}(A, B)$ whose vertex set is a subset of the set of paths in the graph $\mathrm{Gr}\,\mathscr{G}$ from A to B.

Any small 2-category $\mathscr{C}$ can be seen as a computad $U(\mathscr{C})$ by defining $\mathrm{Gr}\,U(\mathscr{C})$ to be the underlying graph of the underlying category of $\mathscr{C}$ and with graphs $U(\mathscr{C})(A, B)$ given by $U(\mathscr{C})(A, B)\big((f_1, \dots, f_n), (g_1, \dots, g_m)\big) = U\mathscr{C}(A, B)(f_1 \circ \dots \circ f_n, g_1 \circ \dots \circ g_m)$. This underlying functor also has a left adjoint, which is given in the following definition.

Definition B.2. For a computad $\mathscr{G}$, the 2-category $\mathcal{F}\mathscr{G}$ is constructed as follows.

- The set of objects is the vertex set of $\mathrm{Gr}\mathscr{G}$.
- For A, B two vertices, we define a pair of graphs $D^2\mathscr{G}(A, B), D^1\mathscr{G}(A, B)$ as follows. The vertex sets of both graphs are the set of paths in $\mathrm{Gr}\,\mathscr{G}$ which start at A and end at B. A diagram $\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"'] & X \arrow[r, "g"] & Y \arrow[r, bend left, ""{name=U2, below}] \arrow[r, bend right, ""{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"'] & Z \arrow[r, "h"] & B \end{tikzcd}$, where f, g, h are paths in $|\mathscr{G}|$ and α, β are arrows in $\mathscr{G}(W, X)$ and $\mathscr{G}(Y, Z)$ respectively, is an arrow of $D^2\mathscr{G}(A, B)$, from the top path to the bottom one. On the other hand, a diagram $\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"'] & X \arrow[r, "g"] & B \end{tikzcd}$ of a similar form is an arrow in $D^1\mathscr{G}(A, B)$ from the top path to the bottom one.

The category $\mathcal{F}\mathscr{G}(A,B)$ is the coequalizer in **Cat** of the diagram

$$\begin{tikzcd} \mathcal{F}(D^2\mathscr{G}(A,B)) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{F}(D^1\mathscr{G}(A,B)) \arrow[r] & \mathcal{F}\mathscr{G}(A,B) \end{tikzcd}$$

where the two arrows correspond to decomposing a diagram

$$\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, bend left, "a_0"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \arrow[r, "g"] & Y \arrow[r, bend left, "b_0"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\beta\Downarrow"] & Z \arrow[r, "h"] & B \end{tikzcd}$$

either as the composite

$$\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, bend left, "a_0"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \arrow[r, "g"] & Y \arrow[r, "b_0"] & Z \arrow[r, "h"] & B \end{tikzcd}$$

$$\circ$$

$$\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, "a_1"] & X \arrow[r, "g"] & Y \arrow[r, bend left, "b_0"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\beta\Downarrow"] & Z \arrow[r, "h"] & B \end{tikzcd}$$

or

$$\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, "a_0"] & X \arrow[r, "g"] & Y \arrow[r, bend left, "b_0"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\beta\Downarrow"] & Z \arrow[r, "h"] & B \end{tikzcd}$$

$$\circ$$

$$\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, bend left, "a_0"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \arrow[r, "g"] & Y \arrow[r, "b_1"] & Z \arrow[r, "h"] & B. \end{tikzcd}$$

More generally, the 2-cells in $F\mathscr{G}$ can be thought of as being built up from the ones in $\mathscr{G}$, by the operation of pasting (see [51]). For instance, the diagram

$$\begin{tikzcd} & A_2 \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_3"'] \arrow[r, phantom, "\Downarrow\alpha_1"] \arrow[d, phantom, "\Downarrow\alpha_2"] & A_3 \arrow[r, "f_6"] \arrow[dr, phantom, "\Downarrow\alpha_3"] & A_4 \arrow[dr, "f_7"] & \\ A_1 \arrow[ur, "f_1"] \arrow[urr, bend right, "f_4"'] \arrow[rrrr, "f_5"'] & {} & & & A_5 \end{tikzcd}$$

represents the composite

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] & A_2 \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_3"'] \arrow[r, phantom, "\Downarrow\alpha_1"] & A_3 \arrow[r, "f_7f_6"] & A_5
\end{tikzcd}$$

$$\circ$$

$$\begin{tikzcd}
A_1 \arrow[r, "\emptyset"] & A_1 \arrow[r, bend left, "f_3f_1"] \arrow[r, bend right, "f_4"'] \arrow[r, phantom, "\Downarrow\alpha_2"] & A_3 \arrow[r, "f_7f_6"] & A_5
\end{tikzcd}$$

$$\circ$$

$$\begin{tikzcd}
A_1 \arrow[r, "\emptyset"] & A_1 \arrow[r, bend left, "f_7f_6f_4"] \arrow[r, bend right, "f_5"'] \arrow[r, phantom, "\Downarrow\alpha_3"] & A_5 \arrow[r, "\emptyset"] & A_5.
\end{tikzcd}$$

The category of $\mathcal{V}$-categories is cocomplete as long as $\mathcal{V}$ is cocomplete, as first established in [59]. This implies in particular that 2-**Cat** is cocomplete. We describe now a very particular kind of coequalizer in this category.

Definition B.3. A *presentation of a* 2*-category* $\mathscr{C}$ *by computads* is a pair of computads $\mathscr{G}, \mathscr{H}$ with same object set and a coequalizer $\begin{tikzcd} \mathcal{F}\mathscr{G} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{F}\mathscr{H} \arrow[r] & \mathscr{C} \end{tikzcd}$ in 2-**Cat**, where F, G are identities on objects.

We now proceed to construct a 2-category $\mathscr{Q}$ satisfying the 2-universal property stated for **Adj**$[n]$ in Equation 5.1. We define a computad Adj$[n]$ as follows. Its graph Gr Adj$[n]$ is given by

$$\begin{tikzcd}
(B,n) \arrow[r, shift left, "L_n"] \arrow[d, "B_n"] & (A,n) \arrow[l, shift left, "R_n"] \arrow[d, "A_n"] \\
(B,n-1) \arrow[r, shift left, "L_{n-1}"] \arrow[d, "B_{n-1}"] & (A,n-1) \arrow[l, shift left, "R_{n-1}"] \arrow[d, "A_{n-1}"] \\
\vdots \arrow[d, "B_1"] & \vdots \arrow[d, "A_1"] \\
(B,0) \arrow[r, shift left, "L_0"] & (A,0) \arrow[l, shift left, "R_0"]
\end{tikzcd}$$

and non-trivial graphs Adj$[n](X,Y)$ are given by

$$\text{Adj}[n]((B,i),(B,i)) = \begin{tikzcd} (B,i) \arrow[r, bend left, "\emptyset"] \arrow[r, bend right, "R_iL_i"'] \arrow[r, phantom, "\eta_i\Downarrow"] & (B,i) \end{tikzcd}$$

and

$$\mathrm{Adj}[n]((A,i),(A,i)) = \begin{tikzcd}(A,i) \arrow[r, bend left, "L_iR_i"{name=U}] \arrow[r, bend right, "\emptyset"'{name=D}] & (A,i)\end{tikzcd} \arrow[Rightarrow, from=U, to=D, "\epsilon_i"'] .$$

The computad $\mathrm{Rel}[n]$ has underlying graph

$$\begin{tikzcd}
(B,n) \arrow[r, shift left, "L_n"] \arrow[d, "B_n"] & (A,n) \arrow[l, shift left, "R_n"] \arrow[d, "A_n"] \arrow[dl, "C_n"] \\
(B,n-1) \arrow[r, shift left, "L_{n-1}"] \arrow[d, "B_{n-1}"] & (A,n-1) \arrow[l, shift left, "R_{n-1}"] \arrow[d, "A_{n-1}"] \arrow[dl, "C_{n-1}"] \\
\vdots \arrow[d, "B_1"] & \vdots \arrow[d, "A_1"] \arrow[dl, "C_1"] \\
(B,0) \arrow[r, shift left, "L_0"] & (A,0) \arrow[l, shift left, "R_0"]
\end{tikzcd}$$

and non-trivial graphs $\mathrm{Rel}[n](X,Y)$ are given by

$$\mathrm{Rel}[n]((B,i),(A,i)) = \begin{tikzcd}(B,i) \arrow[r, bend left, "L_i"{name=U}] \arrow[r, bend right, "L_i"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\lambda_i"'] & (A,i)\end{tikzcd}$$

and

$$\mathrm{Rel}[n]((A,i),(B,i)) = \begin{tikzcd}(A,i) \arrow[r, bend left, "R_i"{name=U}] \arrow[r, bend right, "R_i"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\rho_i"'] & (B,i)\end{tikzcd} .$$

We define two 2-functors $M, N : \mathcal{F}(\mathrm{Rel}[n]) \to \mathcal{F}(\mathrm{Adj}[n])$ that are identities on objects, by $M(S) = N(S) = S$ for $S \in \{L_i, R_i, A_i, B_i\}$, $M(C_i) = B_i \cdot R_i$, $N(C_i) = R_{i-1} \cdot A_i$, $M(\rho_i) = (\epsilon_i R_i) \circ (R_i \eta_i)$, $N(\rho_i) = 1_{R_i}$, $M(\lambda_i) = (L_i \epsilon_i) \circ (\eta_i L_i)$, $N(\lambda_i) = 1_{L_i}$. The 2-category $\mathscr{Q}$ is the coequalizer (in 2-**Cat**) in the diagram

$$\begin{tikzcd}\mathcal{F}(\mathrm{Rel}[n]) \arrow[r, shift left, "M"] \arrow[r, shift right, "N"'] & \mathcal{F}(\mathrm{Adj}[n]) \arrow[r] & \mathscr{Q}.\end{tikzcd} \tag{B.1}$$

The universal properties of the free 2-category on a computad and of the coequalizer imply that $\mathscr{Q}$ satisfies the 2-universal property we were looking for.

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Education

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Award for 3rd best average mark for complete Master studies at EPFL.

2007–2011 **BSc in Mathematics**, *Ecole Polytechnique Fédérale de Lausanne*, Switzerland.

Publications

Zaganidis Dimitri, **The universal monad and adjunction morphisms**, *Submitted on April 20, 2016.*

Zaganidis Dimitri, **Homotopy coherent monad morphisms**, *In preparation.*

Talks

July 4, 2016 **Homotopy coherent monad morphisms**, *Young Topologists Meeting 2016*, University of Copenhagen, Denmark.

June 15, 2016 **The quasi-category of homotopy coherent monads in an $(\infty, 2)$-category**, *Higher Structures in Geometry and Physics: Advances*, MATRIX, The University of Melbourne at Creswick, Australia.

Sept 11, 2015 **The universal 2-category containing n-monad morphisms**, *Workshop in Category Theory and Algebraic Topology*, Université catholique de Louvain, Louvain-la-Neuve, Belgium.

Nov. 8, 2013 **A 2-categorical approach to quasi-categories**, *Topology Seminar*, EPFL, Lausanne, Switzerland.

Professional experience

2017-Now **Mathematician at INAIT**, *Innovation Park, EPFL, Lausanne*.
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2014-Now **Lecturer in Cours Euler**, *a mathematics program for high potential children*.
Geometry, 1st semester of a university level lecture.

2013 **Teaching assistant**, *EPFL, Lausanne*.
Topology, Algebraic K-theory.

2009–2013 **Student assistant**, *EPFL, Lausanne*.
Algebra, Linear algebra, Discrete mathematics, Computer science.

Other mathematical activities

Organisation of the Seminaire Burlet 2013-2016, *a working mountain camp for EPFL undergraduate students.*

Organisation of the Young Topologists Meeting 2015, *EPFL, Lausanne.*

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