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Ill-Posedness of the quasilinear wave equation in the space $H^{7/4}(\ln H)^{-\beta}$ in $\mathbb{R}^{2+1}$

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Abstract

In this thesis, we study the optimal local well posedness of Quasi-linear wave equations. Motivated by the study of the Einstein equations in relativity theory, there are numerous works dedicated to the local well-posedness issue for this equation. The first works were oriented to the study of the lifespan and regularity of solutions of such equations, with more regular initial conditions. In 1980 for instance, Klainerman showed in [Kla80] the existence of a smooth and unique solution for a non-linear wave equation, provided that the initial data are regular enough and small enough. In our case, we are interested in rough initial data, and especially in the minimal regularity we have to impose on the initial data to ensure that the problem is well-posed. The best positive result in low dimensions for this class of equations was proved in 2005 by Tataru and Smith in [ST05]. More precisely, they show that quasilinear wave equation equation is locally well-posed provided that the initial data are in $H^{3+\varepsilon} \times H^{2+\varepsilon}$ *in dimension* $3+1$*, and in* $H^{11/4+\varepsilon} \times H^{7/4+\varepsilon}$ *in dimension* $2+1$*, for any* $\varepsilon > 0$*. In 1998, Lindblad ([Lin98]) gave a counter-example to local well-posedness for this class of equations in dimension* $3+1$ *by exhibiting a quasilinear wave equation and initial data in* $H^3 \times H^2$ *leading to an instantaneous blow up. This means that the index provided by Tataru and Smith is sharp in dimension* $3+1$*.*

In this thesis, we provide a counter-example for the dimension $2+1$*. Later, we study perturbations of the equation and show that the instantaneous blow up is preserved. Because the index corresponding to the dimension* $2+1$ *is* $7/4$*, hence not an integer, we deal with most of the fractional differentiations involved by using the convolution formula for* $|\Delta|^{s/2}$*. To show that the initial data are in the desired space, we make use of Fourier transform and characterization of the Sobolev spaces via the Fourier transform. We also introduce the logarithmically modified Sobolev spaces* $H^s(\ln H)^{-\beta}$*, in which our initial condition perfectly fits, and prove for these spaces some lemmas that are relevant to our situation. In order to prove the blow up, the characteristic method is used. When we study perturbations of the equation, we do not deal with the general case. However, a dependency to the* x_2 *variable is introduced, and we are no longer able* *to find an explicit formula for the characteristics nor the solution. We still use a method inspired by the characteristics found in the previous case, and show that the behaviour near the singularity has similarities, leading to a blow up.*

To ensure that the whole argument is sound, we first consider a regularization of the initial condition χ_ε and show that a blow up occurs when reaching a time t_ε that goes to 0 as ε goes to 0. By proving estimates on the domain and the Sobolev norm, we construct a solution leading to an instantaneous blow up, using a scaling and summing argument. We will show that for any $\lambda > 0$, $H^{11/4}(\ln H)^{-\beta} \subset H^{11/4-\lambda}$. This means that the index $11/4$ is sharp.

Résumé

Dans cette thèse, Nous nous intéressons au bien-fondé d'équations d'ondes quasilineaires. Motivés par l'étude des équations d'Einstein intervenant dans la théorie de la relativité, de nombreux travaux se penchent sur le bien-fondé de ce type d'equations. Les premiers travaux se penchèrent d'abord sur les questions de durée de vie et de régularité de solutions de telles équations, avec des conditions initiales régulières. En 1980, Klainerman montre dans [Kla80] l'existence d'une solution unique et régulière, sous la condition que les données initiales soient suffisamment régulières et petites. Dans notre cas, nous nous intéressons à des données initiales irrégulières, et en particulier à la régularité minimale que l'on doit imposer sur les conditions initiales pour garantir que le probleme est bien posé. Le résultat positif le plus précis en faible dimension pour cette classe d'équations a été prouvé en 2005 par Tataru et Smith dans [ST05]. Plus précisement, ils montrent que l'équation d'onde quasilinéaire est bien posée lorsque les données initiales appartiennent à $H^{3+\varepsilon} \times H^{2+\varepsilon}$ *en dimension* $3+1$*, et à* $H^{11/4+\varepsilon} \times H^{7/4+\varepsilon}$ *en dimension* $2+1$*, pour un* $\varepsilon > 0$*. En 1998, Lindblad ([Lin98]) donne un contre exemple au bon fondement en dimension* $3+1$*. Pour ce faire, il choisit des données initiales appartenant à* $H^3 \times H^2$ *qui mènent à une explosion instantanée de la solution. Ainsi, l'indice du résultat obtenu par Tataru and Smith est optimal en dimension* $3+1$*.*

Dans cette thèse, nous donnons un contre exemple en dimension $2+1$*. Par la suite, nous étudions la stabilité de l'explosion instantanée lorsque l'on modifie l'équation. Comme l'indice qui correspond a la dimension* $2+1$ *est* $7/4$*, et est donc non entier, nous utilisons une formule de type convolution pour* $|\Delta|^{s/2}$ *pour exprimer certaines dérivées fractionnaires. Afin de prouver que les données initiales appartiennent à l'espace souhaité, nous utilisons la transformée de Fourier ainsi que la caractérisation des espaces de Sobolev via la transformée de Fourier. Nous introduisons aussi une modification logarithmique des espaces de Sobolev* $H^s(\ln H)^{-\beta}$*, qui correspond parfaitement à notre condition initiale. Nous prouvons également des lemmes pertinents et nécessaires concernant ces espaces. Afin de prouver l'explosition, nous utilisons la méthode des caractéristiques.*

Résumé

Au cours de notre étude des modifications de cette équation, nous ne nous intéressons au cas général. Néanmoins, une dépendence en la variable x_2 est introduite, et nous ne sommes alors plus en mesure de calculer des formules explicites pour les caracteristiques ou la solution. Nous choisissons néanmoins une methode inspirée par la methode des caractéristiques du cas précédent, et nous montrons que le comportement de la solution proche de la singularité est similaire, produisant ainsi une explosion.

Afin de nous assurer que l'argument est formel et correct, nous considérons d'abord une régularisation de notre condition initiale χ_ε et montrons que l'explosion se produit lorsque l'on s'approche du temps t_ε, qui tend vers 0 quand ε tend vers 0. Nous avons dû ensuite controler la taille du domaine de dépendance ainsi que la norme Sobolev, ceci nous permet de construire une solution qui explose instanement. Pour construire cette solution, nous utilisons un changement de variables et des translations afin de creer une fonction qui fait intervenir une infinité de χ_ε a la fois. Nous montrons egalement que pour tout $\lambda > 0$, $H^{11/4}(\ln H)^{-\beta} \subset H^{11/4-\lambda}$. Ainsi, l'index $11/4$ est optimal.

Contents

Introduction

Introduction

We study the well-posedness of Quasi-linear wave equations. We will consider the following equation

$$\sum_{i,j=0}^{n} g^{ij}(u,u')\partial_{x^i}\partial_{x^j}u = F(u,u'), \quad (t,x)\in S_T = [0,T[\times\mathbb{R}^n, \tag{.0.1}$$

where $\partial_{x^0} = \partial_t$ and $G = (g^{ij})$ and F are smooth functions. Also we assume that g is close to the Minkowski metric m ; i.e.,

$$\sum_{i,j=0}^{n} |g^{ij} - m^{ij}| \leqslant 1/2. \tag{.0.2}$$

We will also define the corresponding Cauchy problem,

$$\begin{cases} \sum_{i,j=0}^{n} g^{ij}(u,u')\partial_{x^i}\partial_{x^j}u = F(u,u'), \quad (t,x)\in S_T = [0,T[\times\mathbb{R}^n, \\ (u,\partial_t u)_{|t=0} = (f,h), \end{cases} \tag{.0.3}$$

where $\partial_{x^0} = \partial_t$ and $G = (g^{ij})$ and F are smooth functions.

We focus on the well-posedness of such equations. Informally, the concept of well-posedness usually involves existence, uniqueness, and continuity with respect to the initial conditions. We are interested on the lowest regularity we have to impose on the Cauchy data in order to ensure that the problem is well-posed.

If we impose conditions on the form of the equation, one can obtain results in term of well-posedness. For instance, if we impose restrictions on the quadratic non-linearities, a condition known as the null condition (introduced by Christodoulou and Klainermann), global smooth small enough solutions exist in the case $n = 3$. This result uses very different method to what we will use. For the case $n = 2$, one can reach the same result but further restrictions must also be imposed on the cubic non-linearities. These topics are for instance discussed in [Kla86]. An application is for instance the global existence result in the case of the Einstein vacuum equation provided by Lindblad and Rodnianski in [LR05]. Such a result had already been proven in [CK93] by Christodoulou and Klainermann, but the work done in [LR05] gives the existence of a global solution under a weak null condition, and leads to a simplified proof of this result.

Looking now at local well-posedness for rough initial data, it has been shown in 1993 in [KM93] by Klainerman and MacHedon that the vectorial partial differential equation $\Box\phi^I = F^I(u, Du)$, where F satisfies a null condition (for its Du dependency), is locally well-posed if we assume that the initial condition is in $H^s \times H^{s-1}$ for $s > 2$ in dimension $3+1$. This result has later been improved by Yi Zhou in [Zho97], showing the well-posedness for $s > 7/4$ in dimension $3+1$.

However, some of the arguments are not valid if we consider general quasi-linear wave equations. For the question of the global well-posedness for smooth initial data, in [Fri83], J. Fritz provides equations for which every non-trivial solution with compactly supported Cauchy data blows up in finite time. In our situation, we consider the local well-posedness of the equation, meaning the existence of a time $T > 0$ such that the equation is well-posed on $[0, T[$. We will place ourselves in the context of low regularity functions and the derivatives will have to be understood in the weak sense.

More precisely, we are interested in the smallest possible s such that (.0.1) is well-posed in the case of the dimension $2+1$; meaning that if $(f, g) \in H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$, then there exists a unique local distributional proper solution (see A.1.4) of (.0.1) for some $T > 0$ satisfying

$$(u, \partial_t u) \in C\left([0, T[; H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)\right). \tag{.0.4}$$

Using Sobolev estimate, one can show that for $s \geqslant n+1$, the problem is well-posed.

An improvement have for instance been made by H. Bahouri and J. Y. Chemin in [BC99], and in parallel by D. Tataru in [Tat00], where they show the well-posedness of the equation for $s \geqslant \frac{n+1}{2} + \frac{1}{4}$, using mainly Strichartz estimates.

Further improvements have been made, and the best result for well-posedness in low dimension has been made by D. Tataru and H. F. Smith in [ST05] and is described by the next theorem.

Theorem .0.1. *[ST05]*

We consider the following Cauchy problem.

$$\begin{cases} \sum_{i,j} g^{ij}(u)\partial_{x^i}\partial_{x^j} u = \sum_{i,j} q^{ij}(u)\partial_{x^i} u \partial_{x^j} u, \quad (t,x) \in S_T = [0,T[\times \mathbb{R}^n, \\ (u, \partial_t u)_{|t=0} = (f,g), \end{cases} \tag{.0.5}$$

where $\partial_{x^0} = \partial_t$ and $G = (g^{ij})$ and $Q = (q^{ij})$ are smooth functions. Also we assume that g is close to the Minksowski metric m. Note that here, the metric is allowed to depend on u but not on its derivatives.

The Cauchy problem (.0.5) is locally well-posed in $H^s \times H^{s-1}$ provided that

$$\begin{aligned} s &> \frac{n}{2} + \frac{3}{4} = \frac{7}{4} \quad \text{for } n = 2, \\ s &> \frac{n+1}{2} \quad \text{for } n = 3,4,5. \end{aligned} \tag{.0.6}$$

Our aim is to provide a negative result. This is done by finding an equation and an initial condition such that the resulting Cauchy problem is not well-posed, meaning that there exists no time T such that the problem is well-posed on $[0,T[$. Such a phenomenon is known as instantaneous blow up. A negative result for $n = 3$ has already been done by H. Lindblad in [Lin98]. The corresponding index s for his problem is 3, he hence exhibits the sharpness of the criteria established in [ST05].

First, we need to compute the corresponding index when the functions G and Q are allowed to also depend on ∇u and not only on u.

Corollary .0.2. *Introducing the Cauchy problem with a ∇u dependency on G and Q as* *the following*

$$\begin{cases} \sum_{i,j} g^{ij}(u,\nabla u)\partial_{x^i}\partial_{x^j} u = \sum_{i,j} q^{ij}(u,\nabla u)\partial_{x^i} u \partial_{x^j} u, \quad (t,x) \in S_T = [0,T[\times \mathbb{R}^3, \\ (u,\partial_t u)_{|t=0} = (f,g), \end{cases} \tag{.0.7}$$

where $\partial_{x^0} = \partial_t$ and $G = (g^{ij})$ and $Q = (q^{ij})$ are smooth functions. Also we assume that g is close to the Minkowski metric m ;

The Cauchy problem (.0.7) is locally well-posed in $H^s \times H^{s-1}$ provided that

$$\begin{aligned} s &> \frac{n}{2} + \frac{3}{4} + 1 \quad \text{for } n = 2, \\ s &> \frac{n+1}{2} + 1 \quad \text{for } n = 3,4,5. \end{aligned} \tag{.0.8}$$

Remark .0.3. *By contraposition, if we find an initial condition for (.0.7) such that the problem is ill-posed for $s = 11/4$ in dimension $2+1$, it means that the problem (.0.5) is ill-posed for $s = 7/4$ in dimension $2+1$. Hence, we have the sharpness of the index provided in [ST05].*

Proof. (Of corollary .0.2)

We consider the problem given by (.0.7). Differentiating the equation with respect to x_k, we obtain

$$\sum_{i,j} \Big[\big(\partial_k u \partial_1 g^{ij}(u,\nabla u)\big) \partial_i \partial_j u + \sum_\alpha \big(\partial_{\alpha+1} g^{ij}(u,\nabla u)\big) \partial_k \partial_\alpha u \cdot \partial_i \partial_j u + \big(g^{ij}(u,\nabla u)\big) \partial_i \partial_j \partial_k u \Big]$$

$$= \sum_{i,j} \Bigg[\big(\partial_k u \partial_1 g^{ij}(u,\nabla u)\big) \partial_i u \partial_j u + \sum_\alpha \big(\partial_i u \partial_{\alpha+1} q^{ij}(u,\nabla u)\big) \partial_j u \cdot \partial_k \partial_\alpha u + q^{ij}(u,\nabla u)\partial_i \partial_k u \cdot \partial_j u + q^{ij}(u,\nabla u)\partial_i u \cdot \partial_j \partial_k u \Bigg] \tag{.0.9}$$

Consequently, the system (.0.9) for every k can be put in the form

$$\sum_{i,j} \tilde{g}_k^{ij}(v)\partial_i\partial_j v = \sum_{ij} \tilde{q}_k^{ij}(v)\partial_i v\partial_j v, \tag{.0.10}$$

where $v = (u, \nabla u)$. Now, $v \in H^{s-1}$ so (.0.10) is well-posed.

□

There are two mechanisms that can create a blow up. One of them is a space independent blow-up. In a nutshell, the blow up is caused by the underlying ODE that itself leads to a solution that blows up. Another kind of blow up is caused by the focusing of the characteristics to a single point, leading to an infinite increase of the derivatives. Informally, in the case of a blow up at $t = T$, if we denote by ϕ a characteristic function and u the solution of the equation, in the fist case, we see a phenomenon of the form (up to derivatives)

$$u(\phi) \xrightarrow[t\to T]{} \infty, \quad \phi \neq 0, \tag{.0.11}$$

happening, whereas in the second case, one observes a phenomenon of the form

$$\partial\phi(\nu) \xrightarrow[t\to T]{} 0, \tag{.0.12}$$

for a certain ν. This will also lead to a blow up of the derivatives using the chain rule. Those types of blow up has already been described by Alinhac for instance in [Ali95].

We create a counterexample for the dimension $1+2$, keeping in mind the intuition provided in [Lin98]. Our goal is to create a second type (geometric) blow up.

We consider the model equation and the corresponding problem (inspired by Lindblad's counterexample)

$$\begin{cases} \Box u = (Du)D^2 u, \\ (u, \partial_t u)_{|t=0} = (f, g), \end{cases} \tag{.0.13}$$

where $D = (\partial_{x_1} - \partial_t)$.

Note that (.0.13) is of the form (.0.1) with

$$g = \begin{bmatrix} 1-v & v & 0 \\ v & -1-v & 0 \\ 0 & 0 & -1 \end{bmatrix}, v = Du. \tag{.0.14}$$

Remark .0.4. *From a scaling argument, we can see that if $s < 2$, then the problem is ill-posed.*

Indeed, let u be a solution of (.0.13) that blows up at a time T. (In fact, we will show the existence of solutions that blow up in a more regular context later on in the thesis.) Define $u_\varepsilon(t,x) = \varepsilon u(t/\varepsilon, x/\varepsilon)$. Now,

$$\square u_\varepsilon(t,x) = \varepsilon^{-1}(\square u)(t/\varepsilon, x/\varepsilon) = (Du_\varepsilon)(D^2 u_\varepsilon) \tag{.0.15}$$

So u_ε be a solution of (.0.13) and has a lifespan of εT, and

$$||u_\varepsilon(0,\cdot)||_{\dot{H}^s} = \left(\int_{x\in\mathbb{R}^2} \varepsilon^{-2s}\varepsilon^2 \nabla^s u(x/\varepsilon, t/\varepsilon)^2 dx \right)^{1/2} = \varepsilon^{2-s}||u(0,\cdot)||_{\dot{H}^s} \tag{.0.16}$$

This quantity goes to 0 in ε when $s < 2$.

The counterexample we produce is in a slightly less regular space than $H^{11/4}$. Indeed we will consider the logarithmic perturbation of $H^{11/4}$, denoted by $H^{11/4}(\ln H)^{-\beta}$ as the set of functions f such that the L^2 norm of $|\xi|^{11/4}\left(1+|\ln(|\xi|)|^{-\beta}\right)\mathscr{F}(f)(\xi)$, where $\mathscr{F}$ denotes the Fourier transform, is finite. Our counterexample will belong to the set $H^{11/4}(\ln H)^{-\beta}$ with $\beta > 1/2$. For the functions that we consider, we will show that this set is located between $H^{11/4-\varepsilon}$ and $H^{11/4}$, for any $\varepsilon > 0$. Hence, this proves the optimality of the index $11/4$ in the context of usual Sobolev spaces. It is however interesting to notice that the function that we create can be defined as a function in $H^{11/4} \times H^{7/4}$, as it is done in Appendix A.1, for which we expect to witness the same behaviour, but the method we use to show the blow up can not be applied anymore. The argument that does not hold in this situation, and the reason why we need slightly less regularity for the proof to hold, lies in the fact that the point where the blow up occurs can not be proven to be in the domain of dependence anymore.

In the following, we construct a solution (.0.13) with initial data in $H^{11/4}(\ln H)^{-\beta} \times H^{7/4}(\ln H)^{-\beta}$, that blows up instantly at $t = 0^+$, as formulated in the following theorem.

Theorem .0.5. *We consider $\beta > 1/2$. There exists initial data $(f,g) \in \dot{H}^{11/4}(\ln H)^{-\beta} \times \dot{H}^{7/4}(\ln H)^{-\beta}$ supported on a compact set, with $||f||_{\dot{H}^{11/4}(\ln H)^{-\beta}} + ||g||_{\dot{H}^{7/4}(\ln H)^{-\beta}}$ arbitrarily small, such that (.0.13) does not have any proper solution u such that*

$$(u, \partial_t u) \in C\left([0,T[; \dot{H}^{11/4}(\ln H)^{-\beta}(\mathbb{R}^2) \times \dot{H}^{7/4}(\ln H)^{-\beta}(\mathbb{R}^2)\right) \text{ for any } T > 0.$$

In fact, we have a slightly stronger result, that is, for this initial condition, for all $\lambda > 0$ small enough, (.0.13) does not have any proper solution u such that $(u, \partial_t u) \in C\left([0,T[; \dot{H}^{11/4-\lambda}(\mathbb{R}^2) \times \dot{H}^{7/4-\lambda}(\mathbb{R}^2)\right)$ for any $T > 0$.

The fact that the index is not an integer in dimension $2+1$ raises several issues. We have to make use of fractional derivatives as well as the Fourier transform to characterize the belonging of a function to a specific Sobolev spaces. For the logarithmically defined Sobolev spaces, we have to use the characterization using the Fourier transform. In order to show that the initial condition belongs to the space $H^{11/4}(\ln H)^{-\beta} \times H^{7/4}(\ln H)^{-\beta}$, we will hence use the Fourier transform using a dyadic decomposition of the domain of definition in x. Because of the way the function is defined, small values of x corresponds in some sense to high values of ξ.

To show that the $H^{11/4}(\ln H)^{-\beta} \times H^{7/4}(\ln H)^{-\beta}$ norm of the function diverges, we will use a convolution integral expression to make our computations. The computations and ways to exhibit the pathological behavior have been made using the Caputo's derivative, the Riemann-Liouville derivative and the Grunwald-Letnikov formula for fractional derivative in mind. These derivatives involve some counter intuitive behaviors. Figure 1 shows the fractional differentiation of a symmetric function, that do not have any symmetry property. Figure 2 shows that the fractional derivative is not local. In our situation however, we will use the more common following fractional derivative formula that comes from Fourier analysis, i.e. that for α a multi-index and $\xi_x^\alpha = \prod_i \xi_{x_i}^{\alpha_i}$,

$$\frac{\partial^\alpha f}{\partial x^\alpha}(x) = \mathscr{F}^{-1}\left((2\pi i)^s \xi_x^\alpha \cdot \mathscr{F}(f)(\xi)\right)(x). \tag{.0.17}$$

Using the fact that $\hat{fg} = \hat{f} * \hat{g}$, we will find a more suitable expression in the next section to manipulate these concepts.

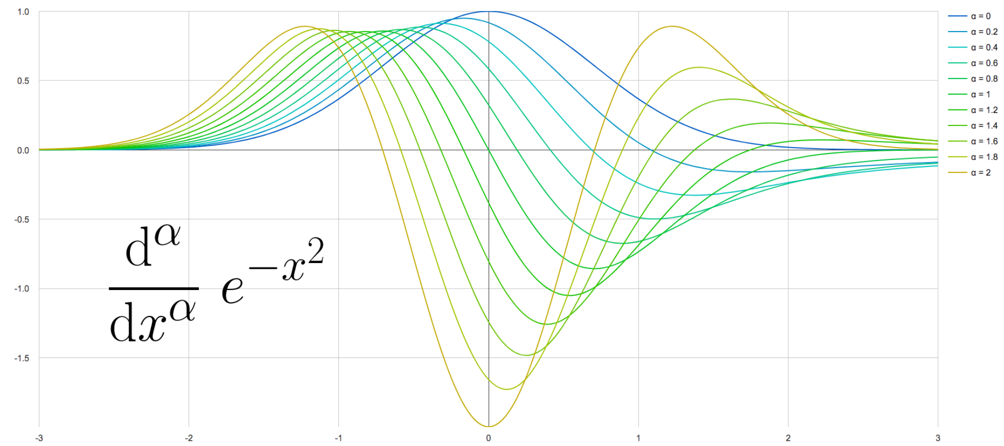

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Figure 1 – Asymmetric Grunwald-Letnikov derivatives of a symmetric function

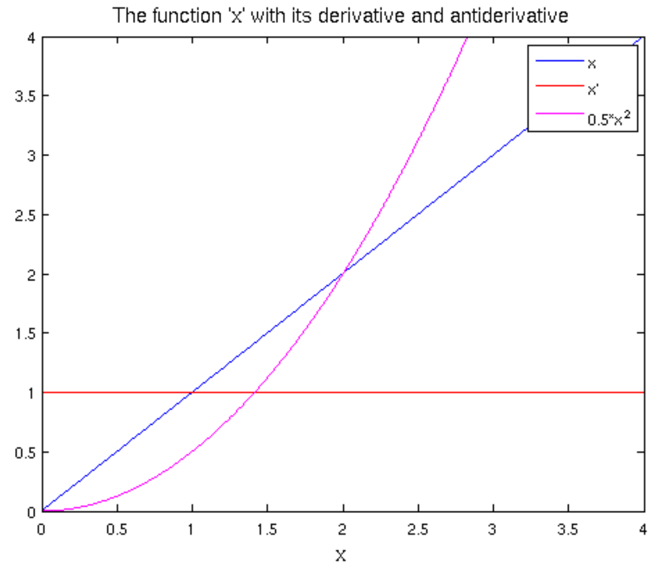

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Figure 2 – Non locality of the fractional derivative

Figure 3 – Fractional differentiation: counter-intuitive behaviours

We define the Riemann-Liouville fractional integral for $\alpha \in]0,1[$

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{.0.18}$$

We define the Riemann-Liouville fractional derivative using the ${}_aD_t^{-\alpha}$ operator. With

$\alpha \in]0,1[$,

$$ {}_aD_t^\alpha f(t) = \frac{d^n}{dt^n} {}_aD_t^{-(n-\alpha)} f(t), \tag{.0.19} $$

where n is the smallest integer bigger than α.

Next, we define the Caputo fractional derivative of order $\alpha > 0$:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)(\tau)}}{(t-\tau)^{\alpha+1-m}} d\tau. \tag{.0.20} $$

In, [IW13] for instance, they introduce weak fractional derivatives. The intuitions are relevant to our situation, however, because we regularize the Cauchy data, we will only consider $C^\infty(\mathbb{R}^2)$ functions.

We can observe on figure 1 that the Grunwald-Letnikov's derivative possesses a "shift to the left" behaviour, before catching up for every integer order. The link with the definition provided in (.0.17) is that the definition coming from (.0.17) is the average of the derivative with the shift to the left, and the derivative with the shift to the right (going from $-\infty$ to t). It is a bit more clear why the Sobolev spaces coincide under the right assumptions. The stated results come from [Li18], [GL18] and [AT11].

In our context, we need globally defined functions, as well as functions that vanish sufficiently fast at infinity, in order to define fractional derivatives. Again, we will use the definition of the fractional derivative provided by (.0.17).

In the next section, we introduce the notations and the definitions we will use.

Notations

First, for an integrable function $f : \mathbb{R}^n \to \mathbb{R}$, we denote by $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ the function

$$ \mathscr{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{-2\imath\pi <y\cdot\xi>} dy. \tag{.0.21} $$

We also define the Sobolev norm denoted by $||\cdot||_{H(\mathbb{R}^n)^s}$ as

$$||f(\cdot)||^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi, \tag{.0.22}$$

and the corresponding Sobolev space $\dot{H}^s(\mathbb{R}^n)$ of functions such that this norm is finite.

When s is an integer, the following holds

$$(2\pi)^s \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(t,\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\nabla^s f(t,x)|^2 \, dx, \tag{.0.23}$$

where $\nabla^s u = (\partial_1^s u, ..., \partial_n^s u)$.

We also define the notion of domain of dependence of the corresponding notations.

Definition .0.6. *Let $\Omega \subset \mathbb{R}_+ \times \mathbb{R}^2$ be an open set equipped with a Lorentzian metric $g_{j,k}$ satisfying (.0.2). It is a domain of dependence for g if the closure of the causal past $\Lambda_{t',x'}$ of each $(t',x') \in \Omega$ is contained in Ω, with $z \in \Lambda_{t',x'}$ iff it can be joined to (t',x') by a Lipschitz continuous curve $(t,x(t))$ satisfying*

$$\sum_{i,j=0}^{2} g_{i,j}(x) \frac{dx_i}{dt} \frac{dx_j}{dt} \geqslant 0, \tag{.0.24}$$

almost everywhere.

For a domain $\Omega \subseteq \mathbb{R}^{1+n}$, we denote by Ω_t the set

$$\Omega_t = \{(\tau, x) \in \Omega, \quad t = \tau\}. \tag{.0.25}$$

Also, for a set $\Omega \subseteq \mathbb{R}^{1+2}$, we denote (the dependence in Ω is not explicitly written)

$$a_t(x) = |\{y \in \mathbb{R}, \quad (t, x_1, x_2) \in \Omega\}|. \tag{.0.26}$$

Remark .0.7. *If we consider the Lorentzian metric that defines the linear wave equation, the causal past of a point is its associated light cone.*

We will make free use of the Huygens principle, meaning that for a solution u defined on a domain of dependence Ω corresponding to the Lorentzian metric involved the equation, the values of u on the set Ω_t only depend of the value of u on the set Ω_0. This will be useful as we will first solve the equation for a locally defined function, and later create an extension of the initial condition. The computation previously made will remain valid in the corresponding domain of dependence.

For α a multi-index, we will use the following definition for fractional derivative

$$\frac{\partial^\alpha f}{\partial x^\alpha}(x) = \mathscr{F}^{-1}\left((2\pi i)^s \xi_x^\alpha \cdot \mathscr{F}(f)(\xi)\right)(x). \tag{.0.27}$$

Part I

Ill-posedness of the quasilinear wave equation in two spatial dimensions in $H^{11/4}(\ln H)^{-\beta}$.

In this chapter, we provide an initial condition that shows the ill-posedness of equation (.0.1) in the local sense. We introduce a specific equation that is of the form (.0.1), and an initial condition χ. The equation that we will consider is (.0.13). To properly show the ill-posedness, we first regularize the initial condition and show that the corresponding solution blows up in the Sobolev norm when $t \to t_\varepsilon$ with $t_\varepsilon \to 0$, and then we create an initial data that leads to an instantaneous blow up by a scaling and summing argument.

I Strategy and control of the initial condition

In this chapter, we quickly explain the strategy of the proof, and we later show that the initial condition belongs to the desired space, i.e. $H^{11/4}(\ln H)^{-\beta}$. We also go through technical lemmas that we will need later to perform the proof of the blow up.

I.1 Explicit resolution and preliminary results

We will first solve the equation using the characteristic method.

We consider the equation (.0.13) and look at solutions of the form $u(t,x) = u_1(t,x_1)$.

The equation in one space dimension can be factored as the following.

$$\begin{aligned} \left(\left(\partial_t + \partial_{x_1}\right) + v\left(\partial_{x_1} - \partial_t\right)\right)\left(\partial_{x_1} - \partial_t\right) u = 0 \\ u(0,x_1) = 0, \quad \partial_t u(0,x_1) = -\chi(x_1) \end{aligned} \tag{I.1.1}$$

where $v = (\partial_{x_1} - \partial_t)u$; which is equivalent when $v \neq 1$ to

$$\left(\partial_t + \frac{1+v}{1-v}\partial_{x_1}\right)\left(\partial_{x_1} - \partial_t\right) u = 0. \tag{I.1.2}$$

Now, this partial differential equation can be explicitly solved. Introducing ϕ such that

$$\begin{cases} \phi(0,y) = y, \\ \partial_t \phi(t,y) = \dfrac{1+v(t,\phi(t,y))}{1-v(t,\phi(t,y)}, \end{cases} \tag{I.1.3}$$

we obtain

$$\frac{\partial}{\partial t}\left(v(t,\phi(t,y))\right) = 0 \;\Rightarrow\; v(t,\phi(t,y)) = \chi(y) \;\; \forall t. \tag{I.1.4}$$

Now, this gives us an explicit formula for ϕ,

$$\phi(t,y) = y + t\frac{1+\chi(y)}{1-\chi(y)}. \tag{I.1.5}$$

We will consider the Cauchy problem corresponding to the two following choices for the initial condition. First, we consider for $\alpha > 0 \in \mathbb{R}$ (the conditions that α has to satisfy will appear later through the proof)

$$v_0(x_1,x_2) = \chi(x_1) = -\int_0^{x_1} |\ln(s)|^\alpha ds, \tag{I.1.6}$$

which will be the initial value that corresponds to an instantaneous blow up of the solution. But we will work with the regularized initial condition

$$v_{0,\varepsilon}(x_1,x_2) = \chi_\varepsilon(x_1) = -\int_0^{x_1} \psi_\varepsilon(s)|\ln(s)|^\alpha ds, \tag{I.1.7}$$

where

$$\begin{cases} 1 > \psi_\varepsilon(x) > 0 \text{ for } \varepsilon/2 < x < \varepsilon, \\ \psi_\varepsilon(x) = 0 \text{ for } x < \varepsilon/2, \\ \psi_\varepsilon(x) = 1 \text{ for } x > \varepsilon \\ \exists C > 0, |\psi'_\varepsilon(x)| \leqslant \dfrac{C}{\varepsilon}. \end{cases} \tag{I.1.8}$$

This resolution holds whenever the solution depends only on x_1. We will consider initial conditions defined on $\mathbb{R}^2$ entirely, but that coincides with (I.1.6) or (I.1.7) on a set Ω.

We choose

$$\Omega_0 = \left\{ (x_1, x_2) | x_1 \geqslant 0, |x_2| \leqslant \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta} \right\} \bigcap \left[0; \frac{1}{2}\right] \times \left[0; \frac{1}{2}\right], \tag{I.1.9}$$

and we take Ω to be the largest domain of dependence (defined in Definition .0.6) for the metric whose inverse is given by

$$\sum_{i,j=0}^{2} g^{ij}(t,x)\partial_{x_i}\partial_{x_j} = \partial_t^2 - \sum_{i=1}^{2} \partial_{x_i}^2 - v(t,x_1)(\partial_t - \partial_{x_1})^2, \tag{I.1.10}$$

and such that $\Omega \cap \{t = 0\} = \Omega_0$. We correspondingly define $\Omega_t = \{(x_1, x_2) | (t, x_1, x_2) \in \Omega\}$ and $a_t(x_1)$ to be the width of Ω_t at x_1.

I.2 Introduction of the logarithmic perturbation of H^s and related lemmas

In this chapter, we will introduce logarithmic perturbations of the spaces $\dot{H}^{7/4}$ and $H^{7/4}$, namely $\dot{H}^{7/4}(\ln H)^{-\beta}$ and $H^{7/4}(\ln H)^{-\beta}$ that will contain functions slightly less (provided that $\beta \geqslant 0$) regular than $\dot{H}^{7/4}$ and $H^{7/4}$. We will show preliminary lemmas, and then we will show that we can find an extension of our function on $\mathbb{R}^2$ that has a uniformly bounded norm in this space. We define the space $\dot{H}^s(\ln H)^{-\beta}$ as the set of all functions such that the norm

$$\begin{aligned} ||f||^2_{\dot{H}^s(\ln H)^{-\beta}} &= ||\mathscr{F}(f) \cdot \frac{|\xi|^s}{(1 + |\ln(|\xi|)|)^\beta}||^2_{L^2} \\ &= \int\int_{\xi_1,\xi_2 \in \mathbb{R}^2} \left[\frac{|\xi|^s}{(1 + |\ln(|\xi|)|)^\beta} \int\int_{x_1,x_2 \in \mathbb{R}} e^{2i\pi\xi_1 x_1} e^{2i\pi\xi_2 x_2} f(x_1,x_2) dx_1 dx_2 \right]^2 d\xi_1 d\xi_2, \end{aligned} \tag{I.2.11}$$

is finite. Similarly, we define the space $H^{7/4}(\ln H)^{-\beta}$ as the set of all functions such that the norm

$$||f||^2_{H^s(\ln H)^{-\beta}} = ||\mathscr{F}(f)\cdot\frac{(1+|\xi|^2)^{s/2}}{1+|\ln(|\xi|)|^{\beta}}||^2_{L^2}$$
$$= \int\int_{\xi_1,\xi_2\in\mathbb{R}^2}\frac{(1+|\xi|^2)^s}{(1+|\ln(|\xi|)|^{\beta})^2}\left[\int\int_{x_1,x_2\in\mathbb{R}} e^{2i\pi\xi_1x_1}e^{2i\pi\xi_2x_2}f(x_1,x_2)dx_1dx_2\right]^2 d\xi_1 d\xi_2, \tag{I.2.12}$$

is finite.

We will first show that $\dot{H}^s \subseteq \dot{H}^s(\ln H)^{-\beta} \subseteq \dot{H}^{s-\lambda}$. This first lemma is for L^1 functions.

Lemma I.2.1. *Let f be a function in $L^1(\mathbb{R}^2)$. For any nonnegative s and β, for any small enough positive λ, we have the following properties.*

(i) $||f||_{\dot{H}^s} < \infty \Rightarrow ||f||_{\dot{H}^s(\ln H)^{-\beta}} < \infty$,

(ii) $||f||_{\dot{H}^s(\ln H)^{-\beta}} < \infty \Rightarrow ||f||_{\dot{H}^{s-\lambda}} < \infty$.

Proof. First, we note that because f is in $L^1(\mathbb{R}^2)$, $\mathscr{F}(f)$ is globally bounded, indeed,

$$|\mathscr{F}(f)(\xi)| = \left|\int\int_{x_1,x_2\in\mathbb{R}^2} e^{2i\pi(x_1\xi_1+x_2\xi_2)}f(x_1,x_2)dx_1dx_2\right| \leqslant \int\int_{x_1,x_2\in\mathbb{R}^2}|f(x_1,x_2)|$$
$$\leqslant ||f||_{L^1(\mathbb{R}^2)}. \tag{I.2.13}$$

Now, we show (i). Consider f such that $||f||_{\dot{H}^s} < \infty$. Because $1+|\ln(|\xi|)| \geqslant 1$ for any ξ, we have

$$\begin{aligned} ||f||^2_{\dot{H}^{7/4}(\ln H)^{-\beta}} &= \int\int_{\xi_1,\xi_2\in\mathbb{R}^2} \frac{|\xi|^{2s}}{(1+|\ln(|\xi|)|^{\beta})^2} \left(\mathscr{F}(f)(\xi_1,\xi_2)\right)^2 \\ &\leqslant \int\int_{\xi_1,\xi_2\in\mathbb{R}^2} |\xi|^{2s} \left(\mathscr{F}(f)(\xi_1,\xi_2)\right)^2 = ||f||^2_{\dot{H}^s}. \end{aligned} \tag{I.2.14}$$

Now, we show (ii). Consider f such that $||f||_{\dot{H}^s} < \infty$. Take $r(\lambda,\beta)$ such that $|\xi| > r$ implies $(1+|\ln(|\xi|)|^{\beta})^2 < |\xi|^{2\lambda}$. We then have

$$\begin{aligned} ||f||^2_{\dot{H}^{s-\lambda}} &= \int\int_{\xi_1,\xi_2\in\mathbb{R}^2} \frac{|\xi|^{2s}}{|\xi|^{2\lambda}} \left(\mathscr{F}(f)(\xi_1,\xi_2)\right)^2 \\ &= \int\int_{|\xi|<r(\lambda,\beta)} \frac{|\xi|^{2s}}{|\xi|^{2\lambda}} \left(\mathscr{F}(f)(\xi_1,\xi_2)\right)^2 + \int\int_{|\xi|>r(\lambda,\beta)} \frac{|\xi|^{2s}}{|\xi|^{2\lambda}} \left(\mathscr{F}(f)(\xi_1,\xi_2)\right)^2 \\ &\leqslant |B(0,r(\lambda,\beta))|\cdot|r(\lambda,\beta)|^{2s-2\lambda}||\mathscr{F}(f)||^2_{\infty} + \int\int_{|\xi|>r_0} \frac{|\xi|^{2s}}{(1+|\ln(|\xi|)|^{\beta})^2} \left(\mathscr{F}(f)(\xi_1,\xi_2)\right)^2 \\ &\leqslant C(\lambda,\beta)\cdot||\mathscr{F}(f)||_{\infty} + ||f||_{\dot{H}^s(\ln H)^{-\beta}} < \infty. \end{aligned} \tag{I.2.15}$$

□

We now state a second lemma, that is more relevant to our situation.

Lemma I.2.2. *Let $f \in L^2$. Let β and s be two nonnegative real numbers, λ a small enough real number and K a compact subset of $\mathbb{R}^2$. If f is supported in K, then we have the two following:*

(i) $||f||_{\dot{H}^s} < \infty \Rightarrow ||f||_{\dot{H}^s(\ln H)^{-\beta}} < \infty$,

(ii) $||f||_{\dot{H}^s(\ln H)^{-\beta}} < \infty \Rightarrow ||f||_{\dot{H}^{s-\lambda}} < \infty$.

Proof. The proof of (i) is identical to the proof we made for lemma I.2.1. The proof for (ii) will share similarities with the proof of Prop 1.55 in [BCD11] p. 39.

We obtain using the Cauchy-Schwartz inequality as well as the Fourier-Plancherel formula that

$$|\hat{f}(\xi)| \leqslant ||f||_{L^1} \leqslant \sqrt{|K|}||f||_{L^2} \leqslant C\sqrt{|K|} \cdot ||\hat{f}||_{L^2}. \tag{I.2.16}$$

Now, for any positive ε, we have

$$\begin{aligned} ||\hat{f}||_{L^2} &\leqslant \int\int_{B(0,\varepsilon)} |\hat{f}(\xi)|^2 + \int\int_{\mathbb{R}^2\backslash B(0,\varepsilon)} |\hat{f}(\xi)|^2 \\ &\leqslant_{(I.2.16)} C|B(0,1)| \cdot \varepsilon^2 \cdot |K| \cdot ||\hat{f}||^2_{L^2} + \int\int_{\mathbb{R}^2\backslash B(0,\varepsilon)} \frac{|\xi|^{2s}(1+|\ln(\xi)|^\beta)^2}{|\xi|^{2s}(1+|\ln(\xi)|^\beta)^2} |\hat{f}(\xi)|^2 \\ &\leqslant C|B(0,1)| \cdot \varepsilon^2 \cdot |K| \cdot ||\hat{f}||^2_{L^2} + \int\int_{\mathbb{R}^2\backslash B(0,\varepsilon)} \frac{|\xi|^{2s}\left(\frac{1}{\varepsilon^s} + \frac{|\ln(\xi)|^\beta}{|\xi|^s}\right)^2}{(1+|\ln(\xi)|^\beta)^2} |\hat{f}(\xi)|^2. \end{aligned} \tag{I.2.17}$$

Now, the function $r \mapsto \frac{|\ln(r)|}{r^s}$ is decreasing on $]0,1[$ and bounded on $[1,\infty[$ which means that for any ξ such that $|\xi| \geqslant \varepsilon$, $\frac{|\ln(|\xi|)^\beta|}{|\xi|^s} \leqslant \max(C, \frac{|\ln(\varepsilon)|^\beta}{\varepsilon^s})$. Now, for ε such that $C \cdot |B(0,1)| \cdot \varepsilon^2 \cdot |K| = \frac{1}{2}$, we obtain

$$||\hat{f}||^2_{L^2} \leqslant C_{|K|}||f||_{\dot{H}^s(\ln H)^{-\beta}} < \infty, \tag{I.2.18}$$

and that $f \in L^1$.

From I.2.1, we obtain the desired result.

□

We now state one more lemma. Using this lemma, we will only have to compute the homogeneous logarithmically modified Sobolev norm as long as our functions are compactly supported. We will use this lemma later on to show that the initial condition we consider belongs to $H^{7/4}(\ln H)^{-\beta}$.

Lemma I.2.3. *Let s be a nonnegative real number, and $f \in L^1_{loc}$. If f belongs to $\dot{H}^s(\ln H)^{-\beta}$ and is compactly supported, then f belongs to $H^s(\ln H)^{-\beta}$.*

Proof. Let $f \in \dot{H}^s(\ln H)^{-\beta}$ and supported in K, a compact subset of $\mathbb{R}^2$. Then, by lemma I.2.2, $f \in \dot{H}^{s/2}$. Hence, we have that $f \in L^2$. (The proof of this can for instance be found in [BCD11] p. 39).

Now, we compute the non-homogeneous modified Sobolev norm.

$$\begin{aligned}
&\int\int_{\xi\in\mathbb{R}^2} \frac{(1+|\xi|^2)^s}{(1+|\ln(|\xi|)|^{\beta})^2}|\hat{f}(\xi)|^2 \\
&= \int\int_{\xi\in B(0,1)} \frac{(1+|\xi|^2)^s}{(1+|\ln(|\xi|)|^{\beta})^2}|\hat{f}(\xi)|^2 + \int\int_{\xi\in\mathbb{R}^2\setminus B(0,1)} \frac{(1+|\xi|^2)^s}{(1+|\ln(|\xi|)|^{\beta})^2}|\hat{f}(\xi)|^2 \\
&\leqslant 2^s||\hat{f}||_{L^2}^2 + \int\int_{\xi\in\mathbb{R}^2\setminus B(0,1)} \frac{(1+|\xi|^2)^s}{|\xi|^{2s}} \frac{|\xi|^{2s}}{(1+|\ln(|\xi|)|^{\beta})^2}|\hat{f}(\xi)|^2 \\
&\leqslant 2^s\cdot C||f||_{L^2} + 2^s\cdot C||f||_{\dot{H}^s(\ln H)^{-\beta}}.
\end{aligned} \tag{I.2.19}$$

□

Now, we will show a new lemma that we will use later. We express the Sobolev norm as a convolution-type integral. This type of integrals are widely used for differentiation of fractional order.

Used together with our previous lemma, it establishes a link between our logarithmically modified Sobolev spaces defined via Fourier transform and the fractional derivative.

Lemma I.2.4. *Let λ be a small, nonnegative number. Let $f:\mathbb{R}^2\to\mathbb{R}$ and $\omega\subseteq\mathbb{R}^2$ such that $f=0$ outside of ω.*

Then,

$$\begin{aligned}
||f||_{\dot{H}^{7/4-\lambda}_{x_1}(\mathbb{R}^2)} = C\int\int_{(x_1,x_2)\in\omega} \left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2) \\
\cdot\int_{y|\ (y,x_2)\in\omega} |x_1-y|^{-1/2+2\lambda}\left(\frac{\partial^2 f}{\partial x_1^2}\right)(y,x_2)\,dy\,dx_2\,dx_1
\end{aligned} \tag{I.2.20}$$

Proof.

$$
\begin{aligned}
||f||_{\dot H^{7/4-\lambda}_{x_1}(\mathbb{R}^2)} &= ||\frac{\partial^2 f}{\partial x_1^2}||_{\dot H^{-1/4-\lambda}} \\
&= \int\int_{(x_1,x_2)\in\mathbb{R}^2} |\nabla_{x_1}^{-1/4-\lambda}|\left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)\cdot|\nabla_{x_1}^{-1/4-\lambda}|\left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)dx \\
&= \int\int_{(x_1,x_2)\in\mathbb{R}^2} |\nabla_{x_1}^{-1/2-2\lambda}|\left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)\cdot\left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)dx \\
&=_{(*)} C\int\int_{(x_1,x_2)\in\mathbb{R}^2} \left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)\cdot\int_{y\in\mathbb{R}}\frac{\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2)}{|x_1-y|^{1/2-2\lambda}}dx \\
&= C\int\int_{(x_1,x_2)\in\omega}\left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)\cdot\int_{y|\ (y,x_2)\in\omega}|x_1-y|^{-1/2+2\lambda}\left(\frac{\partial^2 f}{\partial x_1^2}\right)(y,x_2)
\end{aligned}
\tag{I.2.21}
$$

For $(*)$, we used that

$$
(-\Delta)^{s/2}(f)(x) = \left((2\pi|\xi|)^s\widehat{f}(\xi)\right)^{\vee}(x), \tag{I.2.22}
$$

and that for $n > s > 0$,

$$
(2\pi)^{-s}\left(|\xi|^{-s}\right)^{\vee}(x) = (2\pi)^{-s}\frac{\pi^{\frac{s}{2}}\Gamma(\frac{n-s}{2})}{\pi^{\frac{n-s}{2}}\Gamma(\frac{s}{2})}|x|^{s-n}. \tag{I.2.23}
$$

In our case, because we integrate only with respect to x_1, we obtain from (I.2.22) and (I.2.23)

$$
|\nabla_{x_1}|^{-1/2-2\lambda}f(x) = C(\lambda)\int_{y\in\mathbb{R}}\frac{f(y)}{|x_1-y|^{1/2-2\lambda}}, \tag{I.2.24}
$$

where

$$
C(\lambda) = (2\pi)^{-1/2-2\lambda}\pi^{2\lambda}\frac{\Gamma(\frac{1-\frac{1}{2}-2\lambda}{2})}{\Gamma(\frac{1/2+2\lambda}{2})} \tag{I.2.25}
$$

□

I.3 Proof that $(\partial_t u)_{|t=0} \in H^{7/4}(\ln H)^{-\beta}$

We now introduce the main theorem of this chapter.

Theorem I.3.1. *Let ψ_ε be functions such that*

$$\begin{cases} \psi_\varepsilon(x) = 0, \quad x \in [0, \varepsilon/2], \\ \psi_\varepsilon(x) = 1, \quad x \in [\varepsilon, \infty], \\ \forall x, \ \psi_\varepsilon(x) \in [0,1], \\ |\partial_k \psi_\varepsilon(x)| \leqslant \dfrac{C(k)}{\varepsilon^k}. \end{cases} \tag{I.3.26}$$

We consider α, β, δ such that $2\alpha - 2\beta - \delta < -1$. For a fixed $\beta > 1/2$, it is possible to choose $\alpha > 0$ and $\delta > 0$ that satisfy this condition.

With $\chi_\varepsilon : (x_1, x_2) \in \Omega \mapsto -\int_0^{x_1} \psi_\varepsilon(s)|\ln(y)|^\alpha dy$, there exists $h_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ such that $h_{\varepsilon,|\Omega_0}(x_1, x_2) = \chi_\varepsilon(x_1)$ and $||h_\varepsilon||_{H^{7/4}(\ln H)^{-\beta}} < \infty$. Moreover, the bound on the norm can be chosen to be independent of ε.

Proof. First, we define χ_ε on $\mathbb{R}$ entirely by

$$\chi_\varepsilon : x_1 \in \mathbb{R} \mapsto \begin{cases} -\displaystyle\int_0^{x_1} \psi_\varepsilon(s)|\ln(y)|^\alpha dy, \text{ for } x_1 > 0, \\ 0, \text{ for } x_1 \leqslant 0. \end{cases} \tag{I.3.27}$$

We consider a smooth function $\psi : \mathbb{R} \to [0,1]$ such that

$$\begin{cases} \psi(x) = 1, \quad x \in [0, 1/4], \\ \psi(x) = 0, \quad x \geqslant 1/2. \end{cases} \tag{I.3.28}$$

ψ is defined on $\mathbb{R}^-$ by setting $\psi(x) = \psi(-x)$ for $x < 0$.

Define

$$h_\varepsilon(x_1,x_2) = \chi_\varepsilon(x_1)\cdot\psi\left(\frac{|\ln(x_1)|^\delta x_2}{\sqrt{x_1}}\right)\cdot\psi(x_1). \tag{I.3.29}$$

We multiply χ by a cutoff function in x_1 and x_2 that respects geometry of Ω, i.e. $\psi\left(\frac{|\ln(x_1)|^\delta x_2}{\sqrt{x_1}}\right) = 0$ when $x_2 \geqslant \frac{1}{2}\sqrt{x_1}|\ln(x_1)|^{-\delta}$; and we multiply χ by a simple cutoff function in x_1.

Lastly, we consider a dyadic partition of unity, and λ will denote dyadic numbers. Take a function $\zeta:\mathbb{R}\to[0,1]$ such that

$$\sum_{j\in\mathbb{Z}}\zeta_j(x_1) = \sum_{j\in\mathbb{Z}}\zeta\left(\frac{x_1}{2^j}\right) = \sum_\lambda\zeta\left(\frac{x_1}{\lambda}\right) = 1,\ \forall x_1\in\mathbb{R}, \tag{I.3.30}$$

$$Supp\,\zeta_\lambda \subseteq [\frac{1}{4}\cdot 2^j, 4\cdot 2^j] = [\frac{1}{4}\cdot\lambda, 4\cdot\lambda], \tag{I.3.31}$$

$$\forall x,\ \zeta_j(x)\in[0,1]. \tag{I.3.32}$$

Now, we define

$$h_{\lambda,\varepsilon}(x_1,x_2) = \zeta_\lambda(x_1)h_\varepsilon(x_1,x_2), \tag{I.3.33}$$

and we have

$$h_\varepsilon = \sum_\lambda h_{\lambda,\varepsilon} = \sum_{\lambda\leqslant 2^{j_0}} h_{\lambda,\varepsilon}. \tag{I.3.34}$$

In virtue of lemma I.2.3, we only have to study the homogeneous modified Sobolev norm, as our function is compactly supported.

Now, we will find an estimate for $||h_{\varepsilon,\lambda}||_{H^{7/4}(\ln H)^{-\beta}}$.

First, we compute $\mathscr{F}(h_{\varepsilon,\lambda}(\cdot,\cdot))(\xi_1,\xi_2)$.

$$\begin{aligned}\mathscr{F}\left(h_{\varepsilon,\lambda}(\cdot,\cdot)\right)(\xi_1,\xi_2) &= \int_{x_1}\int_{x_2} e^{-2i\pi x_1\xi_1} e^{-2i\pi x_2\xi_2}\psi(x_1)\zeta_\lambda(x_1)\chi_\varepsilon(x_1)\psi\left(\frac{|\ln(x_1)|^\delta}{\sqrt{x_1}}x_2\right)\\ &= \int_{x_1} e^{-2i\pi x i_1}\psi(x_1)\zeta_\lambda(x_1)\chi_\varepsilon(x_1)\int_{x_2} e^{-2i\pi\xi_2}\psi\left(\frac{|\ln(x_1)|^\delta}{\sqrt{x_1}}x_2\right)\\ &= \int_{x_1} e^{-2i\pi x i_1}\psi(x_1)\zeta_\lambda(x_1)\chi_\varepsilon(x_1)\left[\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\cdot\mathscr{F}(\psi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\xi_2\right)\right]\\ &= \int_{x_1=\frac{\lambda}{4}}^{4\lambda} e^{-2i\pi x i_1}\psi(x_1)\zeta_\lambda(x_1)\chi_\varepsilon(x_1)\left[\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\cdot\mathscr{F}(\psi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\xi_2\right)\right]. \end{aligned} \quad \text{(I.3.35)}$$

Because $\frac{1-C}{C} \to_{C\to 0^+} \infty$, we can chose $C_0 > 0$ such that $\frac{1-C_0}{C_0} \geqslant \frac{\alpha}{|\ln\frac{1}{2}|^\alpha}$.

Now, we have for $\varepsilon/2 \leqslant y \leqslant 1/2$, and $\alpha \leqslant 1$

$$|\ln(y)|^\alpha - \alpha|\ln(y)|^{\alpha-1} \geqslant C_0|\ln(y)|^\alpha. \quad \text{(I.3.36)}$$

Hence, we have for $x_1 \in [\lambda/4, 4\lambda]$,

$$\begin{aligned}|\chi_\varepsilon(x_1)| &= \left|\int_{s=\varepsilon/2}^{x_1}\psi_\varepsilon(s)|\ln(s)|^\alpha\right|\\ &\leqslant \frac{1}{C_0}\left|\int_{s=\varepsilon/2}^{x_1}|\ln(s)|^\alpha - \alpha|\ln(s)|^{\alpha-1}\right| \leqslant C\lambda|\ln(\lambda)|^\alpha. \end{aligned} \quad \text{(I.3.37)}$$

Now, we look at

$$||h_{\lambda,\varepsilon}||^2_{\dot{H}^{7/4}(\ln H)^{-\beta}} = \int_{\xi_1,\xi_2}\left(\frac{|\xi|^{7/4}}{(1+|\ln(|\xi|)|)^\beta}\cdot\mathscr{F}(h_{\lambda,\varepsilon})(\xi_1,\xi_2)\right)^2. \quad \text{(I.3.38)}$$

We will use the fact that $\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}\xi_2}{|\ln(x_1)|^\delta}\right)$ is rapidly decreasing when $\xi_2 >> \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}$ and that $\mathscr{F}(\zeta\lambda)(\xi_1) = \mathscr{F}(\zeta(\frac{\cdot}{\lambda}))(\xi_1)$ is rapidly decreasing when $\xi_1 >> \frac{1}{\lambda}$ to essentially reduce the integration domain to $[0, \frac{1}{\lambda}] \times [0, \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}]$.

First, we compute the following integral

$$\int_{\xi_1 \leqslant \lambda^{-1}} \int_{\xi_2 \leqslant \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}} \left(\frac{|\xi|^{7/4}}{(1+|\ln(|\xi|)|)^\beta} \cdot \mathscr{F}(h_{\lambda,\varepsilon})(\xi_1,\xi_2) \right)^2. \tag{I.3.39}$$

We have that

$$|\mathscr{F}(h_{\lambda,\varepsilon})(\xi_1,\xi_2)| \leqslant C \left| \int_{\lambda/4}^{4\lambda} \lambda |\ln(\lambda)|^\alpha \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta} \right| \leqslant C\lambda^{5/2} |\ln(\lambda)|^{\alpha-\delta}. \tag{I.3.40}$$

And so, we get

$$\begin{aligned} &\left| \int_{\xi_1 \leqslant \lambda^{-1}} \int_{\xi_2 \leqslant \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}} \left(\frac{|\xi|^{7/4}}{(1+|\ln(|\xi|)|)^\beta} \cdot \mathscr{F}(h_{\lambda,\varepsilon})(\xi_1,\xi_2) \right)^2 \right| \\ &\qquad \leqslant C\lambda^{-1} \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}} \frac{\lambda^{-7/2}}{|\ln(\lambda)|^{2\beta}} \lambda^5 |\ln(\lambda)|^{2\alpha-2\delta} = C|\ln(\lambda)|^{2\alpha-2\beta-\delta}. \end{aligned} \tag{I.3.41}$$

Now, we will make a precise argument to justify that integrating over the whole space $\mathbb{R}^2$ does not give a bigger term in λ.

First, we look at

$$\int_{\xi_2 \leqslant \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}} \int_{\xi_1 \geqslant \lambda^{-1}} \left(\frac{|\xi|^{7/4}}{(1+|\ln(|\xi|)|)^\beta} \cdot \mathscr{F}(h_{\lambda,\varepsilon})(\xi_1,\xi_2) \right)^2. \tag{I.3.42}$$

We can write

$$\begin{aligned}
&\int_{\lambda/4}^{4\lambda} e^{2i\pi\xi_1 x_1}\chi_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\\
&\qquad = C\frac{1}{\xi_1}\int_{\lambda/4}^{4\lambda} e^{2i\pi\xi_1 x_1}\Bigg[\chi'_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\\
&\qquad\qquad + \chi_\varepsilon(x_1)\zeta'_\lambda(x_1)\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\\
&\qquad\qquad + \chi_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{1}{2\sqrt{x_1}|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\\
&\qquad\qquad + \chi_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{+\delta}{x}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta+1}}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\\
&\quad + \chi_\varepsilon(x_1)\zeta_\lambda(x_1)\left(\frac{1}{2|\ln(x_1)|^{2\delta}}+\frac{\delta}{|\ln(x_1)|^{2\delta+1}}\right)(\mathscr{F}(\phi))'\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\Bigg]. \qquad \text{(I.3.43)}
\end{aligned}$$

Now, the $\frac{1}{\xi_1}$ we gain is going to be smaller than λ on our considered set. So now, we show that we lose at most λ when differentiating the involved functions. When we will integrate $\frac{1}{\xi_1^2}$ over the set $\xi_1 \geqslant 1/\lambda$, we will multiply by $\frac{1}{\lambda}$ which is not worse than the $\frac{1}{\lambda}$ we had in the first estimate because of the size of the set.

First,

$$\chi'_\varepsilon(x_1) \leqslant C|\ln(\lambda)|^\alpha, \qquad \text{(I.3.44)}$$

$$\zeta'_\lambda(x_1) = \frac{\partial}{\partial_{x_1}}\left(\zeta\left(\frac{x_1}{\lambda}\right)\right)(x_1) = \frac{1}{\lambda}\zeta'\left(\frac{x_1}{\lambda}\right) \leqslant \frac{C}{\lambda}. \qquad \text{(I.3.45)}$$

So, we have

$$\left|\chi'_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right| \leqslant C|\ln(\lambda)|^{\alpha-\delta}\sqrt{\lambda}, \qquad \text{(I.3.46)}$$

$$\left|\chi_\varepsilon(x_1)\zeta'_\lambda(x_1)\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right| \leqslant C|\ln(\lambda)|^{\alpha-\delta}\sqrt{\lambda}, \qquad \text{(I.3.47)}$$

$$\left|\chi_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{1}{2\sqrt{x_1}|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right| \leqslant C|\ln(\lambda)|^{\alpha-\delta}\sqrt{\lambda}, \tag{I.3.48}$$

$$\left|\chi_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{\delta}{\sqrt{x_1}|\ln(x_1)|^{\delta+1}}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right| \leqslant C|\ln(\lambda)|^{\alpha-\delta}\sqrt{\lambda}, \tag{I.3.49}$$

$$\Bigg|\chi_\varepsilon(x_1)\zeta_\lambda(x_1)\left(\frac{1}{2|\ln(x_1)|^{2\delta}}+\frac{\delta}{|\ln(x_1)|^{2\delta+1}}\right) \cdot (\mathscr{F}(\phi))'\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\Bigg| \leqslant C|\ln(\lambda)|^{\alpha-\delta}\sqrt{\lambda}. \tag{I.3.50}$$

By induction and Leibniz differentiation formula, we quickly obtain that

$$\frac{\partial^k}{\partial_{x_1}^k}\left(\chi_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right) \leqslant C_k|\ln(\lambda)|^{\alpha-\delta}\lambda^{\frac{3}{2}-k}. \tag{I.3.51}$$

Hence, we have

$$\left(\int_{\lambda/4}^{4\lambda} e^{2i\pi\xi_1 x_1}\chi_\varepsilon(x_1)\zeta_\lambda(x_1)\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right| \leqslant \frac{C_5}{\xi_1^5}|\ln(\lambda)|^{\alpha-\delta}\lambda^{-7/2}\cdot\lambda \tag{I.3.52}$$

Now, we obtain that

$$\begin{aligned}\int_{\xi_1 \geqslant \lambda^{-1}} & \left(\frac{|\xi|^{7/4}}{(1+|\ln(|\xi|)|)^{\beta}} \cdot \mathscr{F}(h_{\lambda,\varepsilon})(\xi_1,\xi_2) \right)^2 \\ &\leqslant \int_{\xi_1 \geqslant \lambda^{-1}} \frac{C_5}{\xi_1^{10}} |\ln(\lambda)|^{2\alpha-2\delta} \lambda^{-7} \frac{|\xi|^{7/2}}{|\ln(|\xi|)|^{2\beta}} \lambda^2 \\ &\leqslant \frac{C}{\lambda^5} \lambda^{11/2} |\ln(\lambda)|^{2\alpha-2\delta-2\beta} = C\lambda^{1/2} |\ln(\lambda)|^{2\alpha-2\beta-2\delta}. \quad \text{(I.3.53)}\end{aligned}$$

Lastly, we hence obtain

$$\int_{\xi_2=0}^{\frac{|\ln(\lambda)|^{\delta}}{\sqrt{\lambda}}} \int_{\xi_1} \int_{\lambda/4}^{4\lambda} e^{2i\pi\xi_1 x_1} \chi_\varepsilon(x_1) \zeta_\lambda(x_1) \frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}} \mathscr{F}(\phi) \left(\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}} \right) \leqslant C |\ln(\lambda)|^{2\alpha-2\beta-\delta}. \quad \text{(I.3.54)}$$

Now, we study the expression for $\xi_2 \geqslant 2\frac{|\ln(\lambda)|^{\delta}}{\sqrt{\lambda}}$. We notice that

$$\mathscr{F}(\phi) \left(\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}} \xi_2 \right) \leqslant \frac{C_k |\ln(x_1)|^{k\cdot\delta}}{x_1^{k/2} \xi_2^k}. \quad \text{(I.3.55)}$$

Here, we can for instance take $k=4$ and we first obtain, because $C|\xi_2| \geqslant |\xi| \geqslant C|\xi_2|$ on the considered set, and

$$\begin{aligned}\int_{\xi_1 \leqslant \lambda^{-1}} & \left(\frac{|\xi|^{7/4}}{(1+|\ln(\xi)|)^{\beta}} \int_{x_1=\lambda/4}^{4\lambda} e^{2i\pi\xi_1 x_1} \chi_\varepsilon(x_1) \zeta_\lambda(x_1) \frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}} \mathscr{F}(\phi) \left(\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}} \right) \right)^2 \\ &\leqslant C \int_{\xi_1 \leqslant \lambda^{-1}} \left(\frac{|\xi|^{7/4}}{(1+|\ln(\xi)|)^{\beta}} \frac{\lambda^2 |\ln(\lambda)|^{\alpha} \sqrt{\lambda}}{|\ln(\lambda)|^{\delta}} \frac{|\ln(\lambda)|^{4\delta}}{\lambda^2 \xi_2^4} \right)^2 \\ &\leqslant C |\ln(\lambda)|^{2\alpha-6\delta} \frac{1}{\xi_2^8} \frac{|\xi_2|^{7/2}}{|\ln(\xi_2)|^{2\beta}} \leqslant C |\ln(\lambda)|^{2\alpha-6\delta} \frac{1}{|\xi_2|^{9/2} |\ln(\lambda)|^{2\beta}}, \quad \text{(I.3.56)}\end{aligned}$$

and hence,

$$\int_{\xi_2 \geqslant 2\frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}} \int_{\xi_1 \leqslant \lambda^{-1}} \left(\frac{|\xi|^{7/2}}{(1+|\ln(|\xi|)|)^\beta} \int_{x_1=\lambda/4}^{4\lambda} e^{2i\pi\xi_1 x_1} \chi_\varepsilon(x_1)\zeta_\lambda(x_1) \right.$$
$$\left. \cdot \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta} \mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right)^2$$
$$\leqslant C \int_{\xi_2 \geqslant 2\frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}} C|\ln(\lambda)|^{2\alpha-6\delta} \frac{1}{|\xi_2|^{9/2}|\ln(\lambda)|^{2\beta}} \leqslant C\left(\frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}\right)^{-7/2} |\ln(\lambda)|^{2\alpha-6\delta-2\beta}$$
$$\leqslant C|\ln(\lambda)|^{2\alpha-\delta-2\beta}. \quad \text{(I.3.57)}$$

We now study the term

$$\int_{\xi_2 \geqslant 2\frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}} \int_{\xi_1 \geqslant \lambda^{-1}} \left(\int_{x_1=\lambda/4}^{4\lambda} e^{2i\pi\xi_1 x_1} \chi_\varepsilon(x_1)\zeta_\lambda(x_1) \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta} \mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right)^2. \quad \text{(I.3.58)}$$

Using both (I.3.52) and (I.3.55) and similar techniques, we obtain

$$\left(\int_{\lambda/4}^{4\lambda} e^{2i\pi\xi_1 x_1} \chi_\varepsilon(x_1)\zeta_\lambda(x_1) \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta} \mathscr{F}(\phi)\left(\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}\right)\right|$$
$$\leqslant C \frac{|\ln(\lambda)|^{\alpha-\delta}\lambda^{-5/2}}{\xi_1^5} \cdot \frac{|\ln(\lambda)|^{4\delta}}{\lambda^2 \xi_2^4}. \quad \text{(I.3.59)}$$

We first study the set where $\xi_1 \geqslant \xi_2$. We have the following :

$$\iint_{\xi_2 \geqslant \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}, \xi_1 \geqslant \lambda^{-1}, \xi_1 \geqslant \xi_2} \left(\frac{|\xi|^{7/4}}{(1+|\ln(|\xi|)|)^\beta} \cdot \mathscr{F}(h_{\lambda,\varepsilon})(\xi_1,\xi_2) \right)^2$$
$$\leqslant \iint_{\xi_2 \geqslant \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}, \xi_1 \geqslant \lambda^{-1}, \xi_1 \geqslant \xi_2} C \left(\frac{|\xi_1|^{7/4}}{|\ln(|\xi|)|^\beta} \frac{|\ln(\lambda)|^{\alpha-\delta} \lambda^{-5/2}}{\xi_1^5} \cdot \frac{|\ln(\lambda)|^{4\delta}}{\lambda^2 \xi_2^4} \right)^2$$
$$\leqslant C \iint_{\xi_2 \geqslant \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}, \xi_1 \geqslant \lambda^{-1}, \xi_1 \geqslant \xi_2} \frac{\lambda^{-9} |\ln(\lambda)|^{2\alpha-2\beta+6\delta}}{\xi_1^{10} \cdot \xi_2^8}$$
$$\leqslant C \int_{\xi_2 \geqslant \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}} \frac{|\ln(\lambda)|^{2\alpha-2\beta+6\delta}}{\xi_2^8} << |\ln(\lambda)|^{2\alpha-2\beta+6\delta}. \quad \text{(I.3.60)}$$

The second part of the integration domain leads to the same result.

Now, we have that

$$|| \frac{|\xi|^{7/4}}{(1+|\ln(|\xi|)|)^\beta} \mathscr{F}(\zeta_\lambda \cdot v_0)(\xi_1,\xi_2) ||^2_{L^2(\xi_1,\xi_2)} \leqslant C |\ln(\lambda)|^{2\alpha-2\beta-\delta}. \quad \text{(I.3.61)}$$

Taking the sum over the dyadic numbers $\lambda = 2^{-k}$, with $2\alpha - 2\beta - \delta < -1$, we obtain that $g_\varepsilon \in H^{7/4}(\ln H)^{-\beta}$. Also, because the constant C does not depend on ε, we obtain that the Sobolev norms of g_ε, for $\varepsilon \in (0,-1]$, are uniformly bounded.

□

I.4 Lower bound on the width of the domain near the singularity

Lastly, we will use the following result that gives an estimation of $a_t(\phi(t,y))$ for small values of y. Intuitively, the y factor obtained in dimension $1+3$ now becomes a $\frac{\sqrt{y}}{|\ln(y)|^\delta}$, but here we only need the fact that the width obtained at the singularity is strictly bigger than zero.

Using this result, we will be able to use a cutoff near the singularity in the next chapter. For this lemma, we consider the initial condition defined by (I.1.7) and with a cutoff as defined in theorem I.3.1. We consider t_ε to be the first value such that there exists y_ε

such that $\phi_y(t_\varepsilon, \nu_\varepsilon) = 0$. The fact that t_ε and ν_ε exist and $t_\varepsilon = O(\frac{1}{|\ln(\varepsilon)|^\alpha})$ will be shown in chapter II.2.

Proposition I.4.1. *Let $\nu_\varepsilon \in \mathbb{R}\backslash\{0\}$, and consider an interval of t of the form $J_\varepsilon = [0, C\frac{1}{|\ln(\varepsilon)|^\delta}[$. Then there exists a positive constants r such that*

$$a_0(\phi(0,y)) \sim_{y=0} 2\sqrt{2}\frac{\sqrt{y}}{|\ln(y)|^\delta}, \tag{I.4.62}$$

and for $t \in J_\varepsilon$,

$$B((t,\phi(t,\nu_\varepsilon),0),r) = \{(t,x_1,x_2) \in \mathbb{R}^3 | \sqrt{(\phi(t,\nu_\varepsilon)-x_1)^2 + x_2^2} \leqslant r\} \subset \Omega_t, \tag{I.4.63}$$

provided that the condition

$$\alpha > 2\delta \tag{I.4.64}$$

is satisfied.

Proof. (of (I.4.62)) (trivial) Because $\phi(0,y) = y$,

$$a_0(\phi(0,y)) = a_0(y) = \int_{|x_2| \leqslant \sqrt{2}\frac{\sqrt{y}}{|\ln(y)|^\delta}} dx_2 = 2\sqrt{2}\frac{\sqrt{y}}{|\ln(y)|^\delta} \tag{I.4.65}$$

Let us now prove I.4.63. We assume that $|v| < 1/100$, and that $t \leqslant C\frac{1}{|\ln(\varepsilon)|^\alpha}$ (This will be achieved whenever ε is small enough, which means we will only consider times $t \leqslant C\frac{1}{|\ln(\varepsilon)|^\alpha}$ very small, see part 3 for further details.)

We will distinguish three cases, first, we consider curves whose starting point has an abscissa strictly bigger than $x_0 = \frac{\varepsilon}{4}$.

It follows from definition .0.6 that $(t', x') \in \Omega$ if and only if $(t', x'_1) \in \Omega^1$ and all Lipschitz continuous curves from (t', x') that satisfy (.0.24) intersect the hyperplane $t = 0$ in the set $\{x | |x_2| \leqslant \frac{\sqrt{2x_1}}{|\ln(x_1)|^\delta}\}$.

Now, let $(t(s), x_1(s), x_2(s))$ be a Lipschitz continuous curve parameterized so that $t(s)+x_1(s)=s$. Note $q(s)=x_1(s)-t(s)$. Note that (.0.24) is equivalent to (using the fact that $\frac{dt(s)}{ds}+\frac{dx_1(s)}{ds}=1$),

$$R(s) \leqslant v(t(s), x_1(s)) - \frac{dq(s)}{ds}, \tag{I.4.66}$$

where $R(s)=\left(\frac{dx_2(s)}{ds}\right)^2$.

Now, using this set of new variables $s=x_1+t$, $q=x_1-t$ and $U(s,q)=u((s-q)/2,(s+q)/2)$, (I.1.1) becomes

$$\begin{cases} (\partial_s + V(s,q)\partial_q)\partial_q U(s,q) = 0, \ \ V(s,q) = 2\partial_q U(s,q), \\ \qquad\qquad\qquad\qquad U(y,y)=0, \ \ U_q(y,y) = \frac{1}{2}\chi(y) \end{cases} \tag{I.4.67}$$

The characteristics are given by $s = constant$ and $q=h(s,y)$ with

$$\frac{d}{ds}h(s,y) = V(s,h(s,y)), \ \ h(y,y)=y. \tag{I.4.68}$$

Thus, $s \mapsto V(s,h(s,y))$ is constant on the curve and is equal to $\chi(y)$.

Now assume that the curve is such that $q(a)=a$ and $q(b)=h(b,y)$. With $r(s)=\sqrt{x_2^2}$, assume in addition that $r(a)^2 = \frac{2a}{|\ln(a)|^{2\delta}}$. Note that because $q(a)=a$ is equivalent to $t(a)=0$, $r(a)^2=\frac{2a}{|\ln(a)|^{2\delta}}$ is simply that the point $(t(a),x_1(a),x_2(a)) = (0, a, \frac{\sqrt{2a}}{|\ln(a)|^{\delta}})$ is on the edge of the domain Ω_0. Also, the condition $q(b)=h(b,y)$ is equivalent to $x_1(b)=\phi(t(b),y)$.

Now,

$$\begin{aligned} |r(b)-r(a)| &\leqslant \int_a^b \sqrt{R(s)}ds \leqslant \sqrt{b-a}\sqrt{\int_a^b R(s)ds} \\ &\leqslant \sqrt{b-a}\sqrt{q(a)-h(b,y)} \quad \text{(Using } V_q \leqslant 0) \end{aligned} \tag{I.4.69}$$

Now,

$$q(a) - h(b,y) = a - y + h(y,y) - h(b,y) = a - y + \int_y^a V(s,h(s,y))ds$$
$$\leqslant \frac{101}{100}(a-y) \leqslant \frac{9}{8}a \quad \text{(I.4.70)}$$

Also,

$$b - y = 2t + h(b,y) - h(y,y) = 2t + \int_y^b V(s,h(s,y))ds \leqslant 2t + \frac{b-y}{100} \quad \text{(I.4.71)}$$

Hence $\sqrt{b-a} \leqslant \sqrt{b-y} \leqslant \sqrt{2}\cdot\sqrt{t}$, because $a \geqslant y$. (It comes from the expression of h using ϕ.)

Now, since $a = \frac{1}{2}r(a)^2|\ln(a)|^{2\delta}$, we obtain that $a \geqslant \varepsilon/4$ implies $|\ln(a)|^\delta \leqslant C|\ln(\varepsilon)|^\delta$. Hence, using condition (I.4.64), we obtain

$$|r(b) - r(a)| \leqslant C\sqrt{t}\cdot\sqrt{a} \leqslant Cr(a)\sqrt{t}\cdot|\ln(\varepsilon)|^\delta \leqslant r(a)|\ln(\varepsilon)|^{\delta-\alpha/2} \leqslant \frac{1}{2}r(a), \quad \text{(I.4.72)}$$

$$r(b) \geqslant r(a) - |r(a) - r(b)| \geqslant r(a) - 2/3r(a) \geqslant 1/2r(a). \quad \text{(I.4.73)}$$

Meaning that any curve starting with a abscissa bigger than x_0 does not reach the inside of a ball centered in $(t = t, x_1 = \phi(t,v_\varepsilon), x_2 = 0)$ of a certain radius δ_1, for a fixed v_ε only depending on ε. However, the radius δ_1 may depend on ε.

Now, we study the case where the starting abscissa is smaller than $x_0 = \varepsilon/4$.

We will consider the curve $\mathscr{C} = \{(x_1,x_2)|x_2 = \pm\frac{\sqrt{2x_1}}{|\ln(x_2)|^\delta}, x_0 \leqslant x_1 < t/2\}$.

The distance between the curve $\mathscr{C}$ and the point $(\phi(t_\varepsilon,v_\varepsilon),0) = (t,0)$ is given by

$$f(y) = \sqrt{(\phi(t_\varepsilon,v_\varepsilon) - y)^2 + \frac{2y}{|\ln(y)|^{2\delta}}}. \quad \text{(I.4.74)}$$

Now, because for any $t < t_\varepsilon$, for any y, $\partial_y\phi(t,y) \neq 0$, and $\partial_y\phi(t,0) > 0$, we obtain that for any $t < t_\varepsilon$, for any y, $\partial_y \phi(t,y) \geqslant 0$. Hence, we obtain the following inequality (because $v_\varepsilon \geqslant \varepsilon/2$)

$$\phi(t,v_\varepsilon) \geqslant \phi(t,\varepsilon/2) = \frac{\varepsilon}{2} + t\frac{1+\chi_\varepsilon(\varepsilon/2)}{1-\chi_\varepsilon(\varepsilon/2)} = \frac{\varepsilon}{2} + t, \tag{I.4.75}$$

and so we get $\phi(t_\varepsilon, v_\varepsilon) \geqslant \frac{\varepsilon}{2} + t_\varepsilon$. Now, because $y \leqslant \varepsilon/4$, we obtain that

$$f(y) \geqslant |\phi(t_\varepsilon, v_\varepsilon) - y| \geqslant \frac{\varepsilon}{4} + t. \tag{I.4.76}$$

Let us rewrite the condition (.0.6). The metric $(g^{i,j})$ is given by (.0.14), which means that the inverse is given by

$$(g_{i,j}) = \begin{bmatrix} 1+v & v & 0 \\ v & -1+v & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{I.4.77}$$

Now, (.0.6) becomes

$$(1+v)\cdot 1 + 2v\left(\frac{\partial x_1}{\partial_t}\right) + (-1+v)\left(\frac{\partial x_1}{\partial t}\right)^2 - \left(\frac{\partial x_2}{\partial t}\right)^2 \geqslant 0, \tag{I.4.78}$$

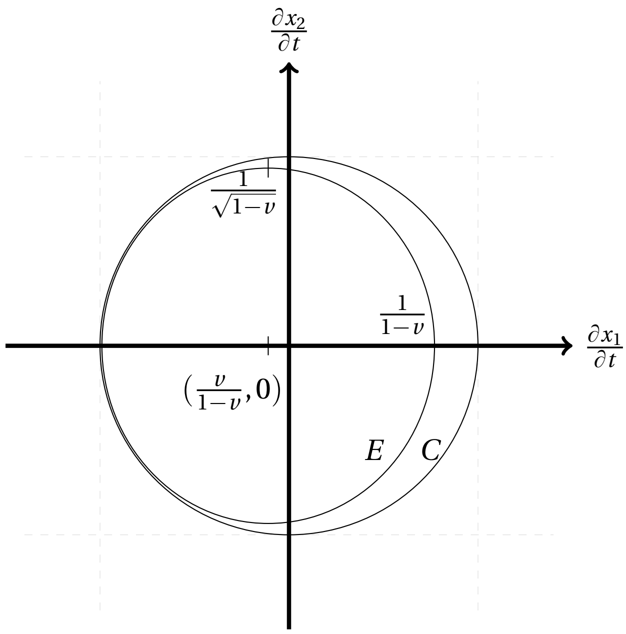
or after a few steps,

$$(1-v)^2\left(\frac{\partial x_1}{\partial t} - \frac{v}{1-v}\right)^2 + (1-v)\left(\frac{\partial x_2}{\partial t}\right)^2 \leqslant 1. \tag{I.4.79}$$

Call E the ellipse given by (I.4.79), and C the circle of center $(0,0)$ and radius 1. We will show that we are in the situation depicted in figure I.1.

Let us prove that E is in fact included in C.

Figure I.1



Let us compute $E \cap C$,

$$\begin{cases} (1-v)^2\left(x-\dfrac{v}{1-v}\right)^2+(1-v)\,(y)^2=1 \\ x^2+y^2=1 \end{cases} \tag{I.4.80}$$

Assuming by symmetry $y \geqslant 0$,

$$(I.4.80) \Rightarrow \begin{cases} (1-v)^2\left(x-\dfrac{v}{1-v}\right)^2+(1-v)\,(y)^2=1 \\ y=-\sqrt{1-x^2} \end{cases} \tag{I.4.81}$$

and hence $(1-v)^2(x-\frac{v}{1-v})^2+(1-v)(1-x^2)=1$. Now, the discriminant of this equation in x is

$$\Delta=\frac{4v^2}{(1-v)^2}-4\frac{v}{1-v}\frac{v}{1-v}=0, \tag{I.4.82}$$

and the only solution we find is $(\frac{\frac{2v}{1-v}}{\frac{-2v}{1-v}},0)=(-1,0)$. This means that the ellipse E is included in the circle C. This means that a curve satisfying (.0.6), satisfies

$$\left(\frac{\partial x_1}{\partial t}\right)^2+\left(\frac{\partial x_2}{\partial t}\right)^2\leqslant 1 \tag{I.4.83}$$

and hence, any curve satisfying (.0.6) also satisfies

$$||(x_1(t),x_2(t))-(x_1(0),x_2(0))||_2\leqslant t. \tag{I.4.84}$$

Because $d((x_1(t),x_2(t)),C)\geqslant t+\frac{\varepsilon}{4}$, there exists a positive number δ_2 such that a ball of radius δ_2 and centered in $(\phi(t,v_\varepsilon),0)$ cannot be reached.

□

Reducing the domain to $x_1\leqslant\frac{1}{|\ln(\varepsilon)|^{\alpha/2}}$.

In this chapter, we additionally multiply our initial condition by a cutoff in x_1. The goal is to cut the function way after the point around which the phenomenon occurs, but to still have a domain that becomes as small as we want when $\varepsilon\to 0$. We will need the function to still be in $H^{7/4}(\ln H)^{-\beta}$, and we also need that the point $(v_\varepsilon,0)$ is in the interior of the domain of dependence at the time $t=t_\varepsilon$.

For the second part of the requirements, a rough estimate is to notice that the speed of the information in our problem is at most 1, and t_ε satisfies $t_\varepsilon\leqslant\frac{1}{|\ln(\varepsilon)|^{\alpha}}$. This means that if the cutoff modifies the function only for $x_1\geqslant\frac{1}{|\ln(\varepsilon)|^{\alpha/2}}$, thanks to (I.4.84), we obtain our desired result for ε small enough.

Hence, we consider the cutoff (with the previous notations) $l_\varepsilon(x_1)=l\left(x_1\cdot|\ln(\varepsilon)|^{\alpha/2}\right)$, where l is a C^∞ functions on $\mathbb{R}$ such that

$$\begin{aligned}&l(x)=1 \text{ for } x\leqslant 1,\\&l(x)=0 \text{ for } x\geqslant 2,\\&l(x)\in[0,1] \text{ always.}\end{aligned} \tag{I.4.85}$$

Note that we then have

$$\begin{aligned}
&l_\varepsilon(x) = 1 \text{ for } x \leqslant \frac{1}{|\ln(\varepsilon)|^{\alpha/2}}, \quad (i)\\
&l(x) = 0 \text{ for } x \geqslant \frac{2}{|\ln(\varepsilon)|^{\alpha/2}}, \quad (ii)\\
&l_\varepsilon(x) \in [0,1] \text{ always}, \quad (iii)\\
&\exists C_k > 0, \left|\frac{\partial^k l_\varepsilon(x_1)}{\partial x_1^k}\right| \leqslant \frac{C_k}{|\ln(\varepsilon)|^{\frac{k\alpha}{2}}}. \quad (iv)
\end{aligned} \tag{I.4.86}$$

Now, thanks to (i) and the previous remark, we have that $(v_\varepsilon, 0)$ belongs to the interior of the domain of dependence for $t = t_\varepsilon$.

Now, in the proof of chapter I.3, because of (i) of (I.4.86), the estimation of

$$\int_{\xi_1 \leqslant \lambda^{-1}} \int_{\xi_2 \leqslant \frac{|\ln(\lambda)|^\delta}{\sqrt{\lambda}}} \left(\frac{|\xi|^{7/4}}{(1+|\ln(|\xi|)|)^\beta} \cdot \mathscr{F}(g_{\lambda,\varepsilon})(\xi_1,\xi_2) \right)^2, \tag{I.4.87}$$

is the same, since $l_\varepsilon = 1$ on this domain. Lastly, the estimation of the Sobolev norm for the whole set is also the same, since l_ε also satisfies (I.3.44) and (I.3.45).

II Blow up of the solution to the regularized problem in $H^{11/4-\lambda}(\mathbb{R}^2)$.

In this chapter, we consider a regularized version of the initial condition, so the equation is well-posed and the method we use to compute its expression are sound. We then construct a counterexample using the statements we have made.

II.1 Strategy

In this chapter, we will consider the initial condition g_ε whose existence and definition are provided in theorem I.3.1 (it was denoted by h_ε in the proof of the theorem).

$$g_\varepsilon(x_1, x_2) = \chi_\varepsilon(x_1) \cdot \psi\left(\frac{|\ln(x_1)|^\delta x_2}{\sqrt{x_1}}\right) \cdot \psi(x_1), \tag{II.1.1}$$

where

$$\chi_\varepsilon(x) = -\int_0^x \psi_\varepsilon(s)\,|\ln(s)|^\alpha\,ds, \tag{II.1.2}$$

First, we recall that the initial condition is in $\dot{H}^{7/4}(\ln H)^{-\beta}$, with a norm that can be bounded uniformly with respect to ε. Also, we define

$$\kappa(x_1, x_2) = \psi\left(\frac{|\ln(x_1)|^\delta x_2}{\sqrt{x_1}}\right) \cdot \psi(x_1). \tag{II.1.3}$$

The Cauchy problem

$$\begin{cases} \square u = (Du)D^2 u, \\ (u, \partial_t u)_{|t=0} = (0, -g_\varepsilon), \end{cases} \tag{II.1.4}$$

is now well-posed on some interval of the form $[0, t_\varepsilon^1[$. Then, we define some time t_ε for which we start to observe the concentration of the characteristics described in the first chapter, for some point v_ε. By this, we mean that $\phi_{\varepsilon,y}(t_\varepsilon, v_\varepsilon) = 0$. The Sobolev norm of the solution will be proven to be unbounded as $t \to t_\varepsilon$; besides the time t_ε is going to 0 as $\varepsilon \to 0$. Next, using a scaling argument, we put together a sequence of these solutions for which the lifespan is going to 0, and such that total initial Sobolev norm is still finite. For any time $t > 0$, t will be beyond the lifespan of one of those solutions, thus leading to an infinite Sobolev norm.

II.2 Blow up when $t \to t_\varepsilon$ and control of t_ε with respect to ε.

Let u_ε be the solution for $t < t_\varepsilon$ of the Cauchy problem:

$$\begin{cases} \square u_\varepsilon = v_\varepsilon D v_\varepsilon, \\ (u, \partial_t u)_{|t=0} = (0, -g_\varepsilon), \end{cases} \tag{II.2.5}$$

with $v_\varepsilon = Du_\varepsilon$. We will prove the following theorem:

Theorem II.2.1. *Let u_ε be a solution of (II.2.5), and $\delta_\varepsilon > 0$ (conditions on δ_ε will be precised later).*

Let $\psi_\varepsilon^1 : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying

$$\begin{cases} \psi_\varepsilon^1(x) = 1 \textit{ for } \phi(t_\varepsilon, v_\varepsilon) - \delta_\varepsilon < x < \phi(t_\varepsilon, v_\varepsilon) + \delta_\varepsilon \\ \psi_\varepsilon^1(x) = 0 \textit{ for } \phi(t_\varepsilon, v_\varepsilon) + 2\delta_\varepsilon < x \textit{ or } x < \phi(t_\varepsilon, v_\varepsilon) - 2\delta_\varepsilon \\ 0 < \psi_\varepsilon^1(x) < 1 \textit{ elsewhere,} \end{cases} \tag{II.2.6}$$

and $\psi^2_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying

$$\begin{cases} \psi^2_\varepsilon(x) = 1 \textit{ for } -\delta_\varepsilon < x < \delta_\varepsilon \\ \psi^2_\varepsilon(x) = 0 \textit{ for } 2\delta_\varepsilon < x \textit{ or } x < -2\delta_\varepsilon \\ 0 < \psi^2_\varepsilon(x) < 1 \textit{ elsewhere,} \end{cases} \tag{II.2.7}$$

so that $h_\varepsilon : (t, x_1, x_2) \mapsto v_\varepsilon(t, x_1, x_2)\psi^1_\varepsilon(x_1)\psi^2_\varepsilon(x_2)$ is localized in a square of width $4\delta_\varepsilon$, cut in half by $x_1 = \phi_\varepsilon(t_\varepsilon, v_\varepsilon)$; and such that $h_\varepsilon = v_\varepsilon$ in a square of width $2\delta_\varepsilon$, cut in half by $x_1 = v_{t,\varepsilon}$.

We have, for any $\lambda > 0$ small enough,

$$||h_\varepsilon(t)||_{H^{7/4-\lambda}_{x_1}} \to \infty \quad \textit{as } t \to t_\varepsilon. \tag{II.2.8}$$

Preliminary work

Now, the computations made in the first part are still valid in Ω, the domain of dependence such that $\Omega \cap \{t = 0\} = \{(x_1, x_2) | \chi_\varepsilon \kappa = \chi\}$. We also only consider the domain in time before the blow up. Or, with ϕ_ε computed as previously,

$$\Omega = \{(x_1, x_2, t) | \phi_\varepsilon(t, \varepsilon) \leqslant x_1 \leqslant \phi(t, 3), |x_2| \leqslant a_t(x_1), t < t_\varepsilon\}. \tag{II.2.9}$$

On this domain, we have that

$$v(t, \phi_\varepsilon(y)) = \chi_\varepsilon(y), \tag{II.2.10}$$

where

$$\phi_\varepsilon(t, y) = y + t\frac{1 + \chi_\varepsilon(y)}{1 - \chi_\varepsilon(y)}. \tag{II.2.11}$$

In the following, we show that $||v_\varepsilon(t)||_{H^{3/4}_{x_1}} \to \infty$ as $t \to t_\varepsilon$. This will be used to construct the counterexample. Now, we give the expressions of the terms that will be used.

Differentiating II.2.11 with respect to y and t we compute the derivatives that we will need,

$$\begin{aligned}
\phi_{\varepsilon,y}(t,y) &= 1+2t\frac{\chi_\varepsilon'(y)}{(1-\chi_\varepsilon(y))^2},\\
\phi_{\varepsilon,ty}(t,y) &= 2\frac{\chi_\varepsilon'(y)}{(1-\chi_\varepsilon(y))^2},\\
\phi_{\varepsilon,yy} &= 2t\frac{\chi_\varepsilon''(y)(1-\chi_\varepsilon(y))+2\chi_\varepsilon'(y)^2}{(1-\chi_\varepsilon(y))^3},\\
\phi_{\varepsilon,tyy}(t,y) &= 2\frac{\chi_\varepsilon''(y)(1-\chi_\varepsilon(y))+2\chi_\varepsilon'(y)^2}{(1-\chi_\varepsilon(y))^3}.
\end{aligned} \tag{II.2.12}$$

Now $y\mapsto \frac{|\chi_\varepsilon'(y)|}{(1-\chi_\varepsilon(y))^2}$ is a continuous function on a compact set, it reaches its maximum M_ε at $y=\nu_\varepsilon$. Call $t_\varepsilon=\frac{1}{M_\varepsilon}$, we have the following properties:

$$\begin{aligned}
&\phi_{\varepsilon,y}(t,y)\neq 0 \text{ for } t<t_\varepsilon,\\
&\phi_{\varepsilon,y}(t_\varepsilon,\nu_\varepsilon)=0,\\
&t_\varepsilon\leqslant \frac{C}{|\ln(\varepsilon)|^\alpha}.
\end{aligned} \tag{II.2.13}$$

We have that for $y>\varepsilon$, $\phi_{\varepsilon,yy}>0$ which means that $\nu_\varepsilon<\varepsilon$. Note that $\phi_\varepsilon(t,\cdot)$ is an injection. We choose ψ_ε such that ν_ε is unique and such that $\phi_{\varepsilon,yyy}(t_\varepsilon,\nu_\varepsilon)\neq 0$. Note that we hence have at $t=t_\varepsilon$ the following properties for y close enough to ν_ε (we can choose the δ_ε such that this is satisfied, since it only depends on ε and not on t).

$$\begin{aligned}
&\phi_{\varepsilon,yy}(t_\varepsilon,\nu_\varepsilon)=0,\\
&\exists C_{1,\varepsilon},C_{2,\varepsilon}>0,\ \ C_{1,\varepsilon}(y-\nu_\varepsilon)\leqslant\phi_{\varepsilon,yy}(t_\varepsilon,y)\leqslant C_{2,\varepsilon}(y-\nu_\varepsilon),\\
&\exists C_{1,\varepsilon},C_{2,\varepsilon}>0,\ \ C_{1,\varepsilon}(y-\nu_\varepsilon)^2\leqslant\phi_{\varepsilon,y}(t_\varepsilon,y)\leqslant C_{2,\varepsilon}(y-\nu_\varepsilon)^2.
\end{aligned} \tag{II.2.14}$$

Remark II.2.2. *The fact that $\phi_{\varepsilon,yy}$ is not of constant sign really is an issue for the estimation of the Sobolev norm, but we will explain later how we address this issue. $\phi_{\varepsilon,y}$ however, is of constant sign.*

Also, we recall

$$\exists M_\varepsilon < 0,\;\; 0 > \chi_\varepsilon(y) > M_\varepsilon,\;\; 0 > \chi'_\varepsilon(y) > M_\varepsilon,\;\; |\chi''_\varepsilon(y)| \leqslant |M_\varepsilon| \tag{II.2.15}$$

Now, using taylor expansions and the expressions given by (II.2.14) and (II.2.15) we write the following inequalities

$$\begin{aligned} &C_{1,\varepsilon}(\nu_\varepsilon - y)^2 + C_{1,\varepsilon}\phi_{\varepsilon,ty}(t_\varepsilon,\nu_\varepsilon)(t-t_\varepsilon) \\ &\qquad \leqslant \phi_{\varepsilon,y}(t,y) \leqslant C_{2,\varepsilon}(\nu_\varepsilon - y)^2 + C_{2,\varepsilon}\phi_{\varepsilon,ty}(t_\varepsilon,\nu_\varepsilon)(t-t_\varepsilon),\ (i) \\ &\qquad\qquad C_{1,\varepsilon}(\nu_\varepsilon - y) \leqslant \phi_{\varepsilon,yy}(t,y) \leqslant C_{2,\varepsilon}(\nu_\varepsilon - y),\ (ii) \end{aligned} \tag{II.2.16}$$

where (ii) comes from the fact that $\phi_{\varepsilon,yy}(t,y) = \frac{t}{t_\varepsilon}\phi_{\varepsilon,yy}(t_\varepsilon,y)$. Now, using (II.2.14) and (II.2.16), because $\phi_{\varepsilon,ty} < 0$ and $(t - t_\varepsilon) < 0$, both components have the same sign, and we can write

$$C_{1,\varepsilon}|\nu_\varepsilon - y|^2 + C_{1,\varepsilon}(t_\varepsilon - t) \leqslant |\phi_{\varepsilon,y}(t,y)| \leqslant C_{2,\varepsilon}|\nu_\varepsilon - y|^2 + C_{2,\varepsilon}(t_\varepsilon - t)\,, \tag{II.2.17}$$

and $\phi_{\varepsilon,y} < 0$ everywhere for $t < t_\varepsilon$.

Informally, we recall that our goal is to obtain lower and upper bounds for the integral (we omitted the independent variable)

$$\int_{x_1}\int_y \frac{1}{|x_1 - y|^{1/2-2\lambda}} \frac{\partial^2 h_\varepsilon}{\partial^2_{x_1}}(t,x_1)\frac{\partial^2 h_\varepsilon}{\partial^2_{x_1}}(t,y), \tag{II.2.18}$$

We obtain an estimation for the remaining terms.

$$
\begin{aligned}
&h_\varepsilon(t,x_1,x_2) = \psi^1_\varepsilon(x_1)\psi^2_\varepsilon(x_2)v_\varepsilon(t,x),\\
&\frac{\partial h_\varepsilon}{\partial_{x_1}}(t,x_1,x_2) = \psi^2_\varepsilon(x_2)\Big[\psi^{1,'}_\varepsilon(x_1)v_\varepsilon(t,x_1)+\psi^1_\varepsilon(x_1)v_{\varepsilon,x_1}(t,x_1)\Big],\\
&\frac{\partial^2 h_\varepsilon}{\partial^2_{x_1}}(t,x_1,x_2) = \psi^2_\varepsilon(x_2)\Big[\psi^{1,''}_\varepsilon(x_1)v_\varepsilon(t,x_1)+2\psi^{1,'}_\varepsilon(x_1)v_{\varepsilon,x_1}(t,x_1)+\psi^1_\varepsilon(x_1)v_{\varepsilon,x_1x_1}(t,x_1)\Big],\\
&\Rightarrow \exists C_{1,\varepsilon},C_{2,\varepsilon}>0,\ \ C_{1,\varepsilon}\psi^2_\varepsilon(x_2)v_{\varepsilon,x_1x_1}(t,x_1)\leqslant \frac{\partial^2 h_\varepsilon}{\partial^2_{x_1}}(t,x_1,x_2)\leqslant C_{2,\varepsilon}\psi^2_\varepsilon(x_2)v_{\varepsilon,x_1x_1}(t,x_1)\ (i)\\
&\Rightarrow \exists C_{1,\varepsilon}>0,\ \ \left|\frac{\partial^2 h_\varepsilon}{\partial^2_{x_1}}(t,x_1,x_2)\right|\leqslant C_{1,\varepsilon}\left|\psi^2_\varepsilon(x_2)v_{\varepsilon,x_1x_1}(t,x_1)\right|,\ (ii)
\end{aligned}
\tag{II.2.19}
$$

where (i) is valid when $x_1\in[\phi_\varepsilon(t_\varepsilon,\nu_\varepsilon)-\delta_\varepsilon,\phi_\varepsilon(t_\varepsilon,\nu_\varepsilon)+\delta_\varepsilon]$ and (ii) is valid everywhere.

Now we are ready to start the proof of theorem II.2.1. We will use the formula obtained in lemma I.2.4 for the expression of the Sobolev norm.

Proof. First, with $I_\varepsilon = \int_{x_2=-2\delta_\varepsilon}^{2\delta_\varepsilon} \left(\psi_\varepsilon^2(x_2)\right)^2 dx_2$, we define the following three integrals.

$$
\begin{aligned}
I_\varepsilon(t) = &\int_{x_1=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} \int_{x_2=-2\delta_\varepsilon}^{2\delta_\varepsilon} \Bigg[\left(\frac{\partial^2(v\psi_\varepsilon^1\psi_\varepsilon^2)}{\partial x_1^2} \right)(t,x_1,x_2) \\
&\qquad \cdot \int_{y=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda} \left(\frac{\partial^2(v\psi_\varepsilon^1\psi_\varepsilon^2)}{\partial x_1^2} \right)(t,y,x_2)dy \Bigg] dx_2 dx_1 \\
= I_\varepsilon &\int_{x_1=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} \Bigg[\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2} \right)(t,x_1) \\
&\qquad \cdot \int_{y=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2} \right)(t,y)dy \Bigg] dx_1 \\
= I_\varepsilon &\int_{x_1=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-\delta_\varepsilon} \Bigg[\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2} \right)(t,x_1) \\
&\qquad \cdot \int_{y=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2} \right)(t,y)dy \Bigg] dx_1 \\
+ I_\varepsilon &\int_{x_1=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+\delta_\varepsilon} \Bigg[\left(\frac{\partial^2 v}{\partial x_1^2} \right)(t,x_1) \\
&\qquad \cdot \int_{y=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda} \left(\frac{\partial^2 v}{\partial x_1^2} \right)(t,y)dy \Bigg] dx_1 \\
+ I_\varepsilon &\int_{x_1=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} \Bigg[\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2} \right)(t,x_1) \\
&\qquad \cdot \int_{y=\phi_\varepsilon(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi_\varepsilon(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2} \right)(t,y)dy \Bigg] dx_1 \\
= I_\varepsilon &(I_\varepsilon^1(t) + I_\varepsilon^2(t) + I_\varepsilon^3(t))
\end{aligned}
\tag{II.2.20}
$$

The strategy of this proof will be to find upper bounds for $|I_\varepsilon^1|$ and $|I_\varepsilon^3|$ (as $t \to t_\varepsilon$) while finding a lower bound for I_ε^2 (as $t \to t_\varepsilon$) going to $+\infty$ faster than the bounds on $|I_\varepsilon^1|$ and $|I_\varepsilon^3|$ are increasing, thus giving the convergence of $I_\varepsilon(t)$ to $+\infty$ as $t \to t_\varepsilon$.

Because of (II.2.19), we can study the integrals where $\frac{\partial^2(v\psi_\varepsilon^1)}{\partial_{x_1}^2}$ is replaced by $\frac{\partial^2 v}{\partial_{x_1}^2}$. Also, we make the change of variable (not explicitly relabelled) $x_1 = \phi_\varepsilon(t,x_1)$ and $y = \phi_\varepsilon(t,y)$, which makes the term $\phi_{\varepsilon,y}$ appear twice.

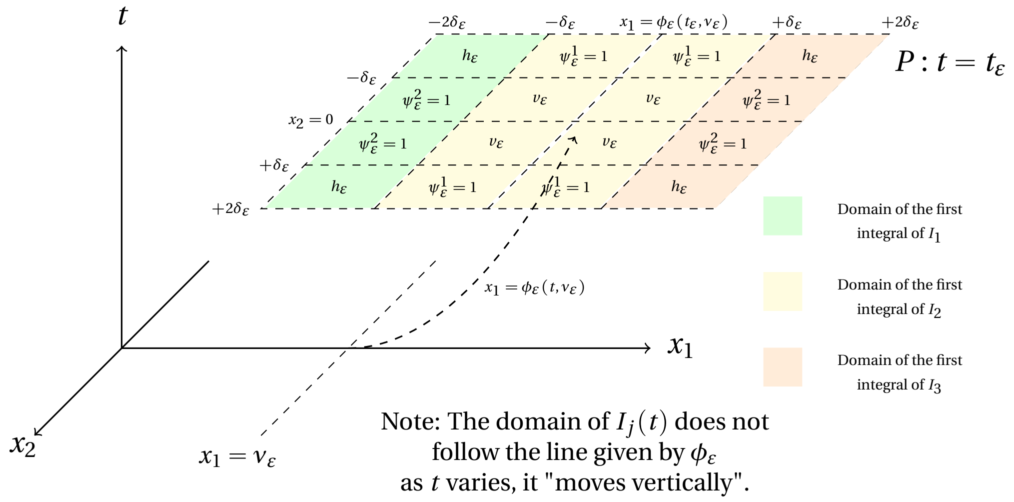


Figure II.1 – Definition of $\psi^1_\varepsilon, \psi^2_\varepsilon, h_\varepsilon, I^1_\varepsilon, I^2_\varepsilon$ and I^3_ε.

Since

$$v_\varepsilon(t, \phi_\varepsilon(t, y)) = \chi_\varepsilon(y), \tag{II.2.21}$$

then,

$$\begin{aligned} v_{\varepsilon,x}(t, \phi_\varepsilon(t, y)) &= \frac{\chi'_\varepsilon(y)}{\phi_{\varepsilon,y}(t, y)} \\ v_{\varepsilon,xx}(t, \phi_\varepsilon(t, y)) &= \frac{\chi''_\varepsilon(y)\phi_{\varepsilon,y}(t, y) - \chi'_\varepsilon(y)\phi_{\varepsilon,yy}(t, y)}{\phi^3_{\varepsilon,y}(t, y)} \end{aligned} \tag{II.2.22}$$

Now by continuity, choose δ_ε so that for $z \in]\phi_\varepsilon(t_\varepsilon, \nu_\varepsilon) - 2\delta_\varepsilon, \phi_\varepsilon(t_\varepsilon, \nu_\varepsilon) + 2\delta_\varepsilon[$, then $\phi^{-1}_{\varepsilon,t}(z) \in]\varepsilon - \eta_\varepsilon, \varepsilon + \eta_\varepsilon[$, for a certain range of t close enough to t_ε, of the form $]t^0_\varepsilon, t_\varepsilon[$. Hence, we can choose the δ_ε such that η_ε is small enough, and all the results obtained in the preliminaries hold.

For clarity, call

$$
\begin{aligned}
\zeta_\varepsilon^1(t) &= \phi_{\varepsilon,t}^{-1}(\phi_\varepsilon(t_\varepsilon, \nu_\varepsilon) - 2\delta_\varepsilon)),\\
\zeta_\varepsilon^2(t) &= \phi_{\varepsilon,t}^{-1}(\phi_\varepsilon(t_\varepsilon, \nu_\varepsilon) - \delta_\varepsilon)),\\
\zeta_\varepsilon^3(t) &= \phi_{\varepsilon,t}^{-1}(\phi_\varepsilon(t_\varepsilon, \nu_\varepsilon) + \delta_\varepsilon)),\\
\zeta_\varepsilon^4(t) &= \phi_{\varepsilon,t}^{-1}(\phi_\varepsilon(t_\varepsilon, \nu_\varepsilon) + 2\delta_\varepsilon)),
\end{aligned}
\tag{II.2.23}
$$

and note that for t_ε^1 close enough to t_ε,

$$
\nu_\varepsilon - \eta_\varepsilon < \zeta_\varepsilon^1(t) < \zeta_\varepsilon^2(t) < \nu_\varepsilon < \zeta_\varepsilon^3(t) < \zeta_\varepsilon^4(t) < \nu_\varepsilon + \eta_\varepsilon. \tag{II.2.24}
$$

Also by continuity, for t close enough to t_ε, there exist two constants ζ_ε^{2+} and ζ_ε^{3-} such that

$$
\nu_\varepsilon - \eta_\varepsilon < \zeta_\varepsilon^1(t) < \zeta_\varepsilon^2(t) < \zeta_\varepsilon^{2+} < \nu_\varepsilon < \zeta_\varepsilon^{3-} < \zeta_\varepsilon^3(t) < \zeta_\varepsilon^4(t) < \nu_\varepsilon + \eta_\varepsilon. \tag{II.2.25}
$$

Let us now consider I_ε^1 (the case of I_ε^3 is similar).

$$
\begin{aligned}
|I_\varepsilon^1(t)| &\leqslant C_\varepsilon \int_{x_1=\zeta_\varepsilon^1(t)}^{\zeta_\varepsilon^2(t)} \int_{y=\zeta_\varepsilon^1(t)}^{\zeta_\varepsilon^4(t)} \frac{|\phi_{\varepsilon,y}(t,x_1)\phi_{\varepsilon,y}(t,y)|}{|\phi_\varepsilon(t,x_1) - \phi_\varepsilon(t,y)|^{1/2-2\lambda}} |v_{\varepsilon,xx}(t,x_1) v_{\varepsilon,xx}(t,y)|\\
&\leqslant C_\varepsilon \int_{x_1=\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2+}} |\phi_{\varepsilon,y}(t,x_1) v_{\varepsilon,xx}(t,x_1)| \int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\nu_\varepsilon+\eta_\varepsilon} \frac{|\phi_{\varepsilon,y}(t,y)|}{|\phi_\varepsilon(t,x_1) - \phi_\varepsilon(t,y)|^{1/2-2\lambda}} |v_{\varepsilon,xx}(t,y)|
\end{aligned}
\tag{II.2.26}
$$

We study the inner integral of (II.2.26) for $x_1 \in [\nu_\varepsilon - \eta_\varepsilon, \zeta_\varepsilon^{2+}]$. Now, both $\frac{1}{|\phi_\varepsilon(t,x_1) - \phi_\varepsilon(t,y)|^{1/2-2\lambda}}$ and $v_{\varepsilon,yy}(t,y)$ are unbounded in the second integral, but the regions where they are unbounded are uniformly disjoint in t because of (II.2.25), so we split again the domain.

$$\int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\nu_\varepsilon+\eta_\varepsilon} \frac{1}{|\phi_\varepsilon(t,x_1)-\phi_\varepsilon(t,y)|^{1/2-2\lambda}} v_{\varepsilon,xx}(t,y) \leqslant_{(*1)} C_\varepsilon \int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2+}} \frac{1}{|\phi_\varepsilon(t,x_1)-\phi_\varepsilon(t,y)|^{1/2-2\lambda}}$$
$$+ C_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} |\phi_{\varepsilon,y}(t,y) v_{\varepsilon,xx}(t,\phi_\varepsilon(t,y))| + C_\varepsilon \int_{y=\zeta_\varepsilon^{3-}}^{\nu_\varepsilon+\eta_\varepsilon} \frac{1}{|\phi_\varepsilon(t,x_1)-\phi_\varepsilon(t,y)|^{1/2-2\lambda}}$$
$$\leqslant C_\varepsilon\left[(i)+(ii)+(iii)\right] \quad \text{(II.2.27)}$$

In $(*1)$, we used (II.2.14), (II.2.16) and (II.2.22) and the fact that $|\nu_\varepsilon - y| \geqslant C_\varepsilon$ on the set $[\nu_\varepsilon-\eta_\varepsilon, \zeta_\varepsilon^{2+}]$.

Now we give a majoration for (i) in (II.2.27). The case (iii) is trivial.

Remark II.2.3. *For $I_\varepsilon^3(t)$, the case of (i) is trivial, the (ii) works the same as (ii) for $I_\varepsilon^1(t)$, and (iii) works the same as (i) for $I_\varepsilon^1(t)$ that we do now.*

Now, using the mean value theorem as well as (II.2.16), and $|c-\nu_\varepsilon| \geqslant |\zeta_\varepsilon^{2+}-\nu_\varepsilon|$, we obtain

$$(i) \leqslant C_\varepsilon \int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2+}} \frac{1}{|x_1-y|^{1/2-2\lambda}} \frac{1}{|\phi_{\varepsilon,y}(t,c)|^{1/2-2\lambda}} \leqslant C_\varepsilon \int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2+}} \frac{1}{|x_1-y|^{1/2-2\lambda}} \leqslant C_\varepsilon. \quad \text{(II.2.28)}$$

Now, we study (ii). Using the expression of $v_{\varepsilon,xx}$ given by (II.2.22), and the results obtained in the preliminary work, it is not clear if the bigger term is $\frac{1}{\phi_{\varepsilon,y}(t,x)^2}$ or $\frac{\phi_{\varepsilon,yy}(t,x)}{\phi_{\varepsilon,y}(t,x)^3}$. (The other factors being uniformly bounded for a fixed ε). Indeed, the root of $\phi_{\varepsilon,y}$ is of order 2 in x instead of the root being of order 1 in x for $\phi_{\varepsilon,yy}$, but the inequalities also involve a term depending on t in $\phi_{\varepsilon,y}$. So we make the computations for both.

Using the expressions from (II.2.22) and (II.2.27) as well as the inequalities obtained in (II.2.15) and (II.2.16), we obtain

$$
\begin{aligned}
(ii) \leqslant C_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{1}{|\phi_{\varepsilon,y}(t,y)|^2} + C_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{|\phi_{\varepsilon,yy}(t,y)|}{|\phi_{\varepsilon,y}(t,y)|^3} \leqslant \\
C_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{1}{(|y-\nu_\varepsilon|^2+(t_\varepsilon-t))} + C_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{|y-\nu_\varepsilon|}{(|y-\nu_\varepsilon|^2+(t_\varepsilon-t))^2}
\end{aligned}
\tag{II.2.29}
$$

Now, because we have the following primitives

$$
\int \frac{1}{(x^2+a)} = \frac{\arctan\left(\frac{x}{\sqrt{a}}\right)}{\sqrt{a}}, \quad \int \frac{x}{(x^2+a)^2} = -\frac{1}{2(x^2+a)}, \tag{II.2.30}
$$

We obtain that

$$
|(i)+(ii)+(iii)| \leqslant C_\varepsilon \frac{1}{(t_\varepsilon - t)}. \tag{II.2.31}
$$

Finally, we have the following upper bound for $I_\varepsilon^1(t)$, using $|x_1-\nu_\varepsilon| \geqslant C_\varepsilon$,

$$
|I_\varepsilon^1(t)| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)} \int_{x_1=\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2-}} \left[\frac{1}{|\phi_{\varepsilon,y}(t,x_1)|} + \frac{|\phi_{\varepsilon,yy}(t,x_1)|}{|\phi_{\varepsilon,yy}(t,x_1)|^2} \right] \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)}. \tag{II.2.32}
$$

By symmetry, we also have $I_\varepsilon^3(t) \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)}$.

Now, we do the estimation for $I_\varepsilon^2(t)$.

This one is more complicated because we do not have an upper bound for $\frac{\chi''(y)}{(\phi_{\varepsilon,y}(t,x))^2} - \frac{\chi'(y)\phi_{\varepsilon,yy}(t,x)}{(\phi_{\varepsilon,y}(t,y))^3}$. Indeed, the first term is of constant sign whereas the second term is changing sign when x is lower or bigger than ν_ε. Also, the second term is not smaller than the first one. The idea we will use is the following one. In case of an integer Sobolev norm, the term is squared and is of constant sign. In the case of this fractional Sobolev norm, the kernel is not a Dirac but concentrates at $y = x_1$ as $\frac{1}{|\phi_{\varepsilon,y}(t,c)(x_1-y)|^{1/2-2\lambda}}$.

From (II.2.16), we have

$$C_{1,\varepsilon}(\nu_\varepsilon - y) \leqslant \phi_{\varepsilon,yy}(t,y) \leqslant C_{2,\varepsilon}(\nu_\varepsilon - y).$$

We can write from (II.2.22)

$$\begin{aligned}
&\int_{x_1=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{1}{|\phi_\varepsilon(t,x_1)-\phi_\varepsilon(t,y)|^{1/2-2\lambda}} \phi_{\varepsilon,y}(t,x_1)v_{xx}(t,x_1)\phi_{\varepsilon,y}(t,y)v_{xx}(t,y) \\
&= \int_{x_1=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{1}{|\phi_\varepsilon(t,x_1)-\phi_\varepsilon(t,y)|^{1/2-2\lambda}} \Bigg[\frac{\chi''(x_1)}{\phi_{\varepsilon,y}(t,x_1)} \frac{\chi''(y)}{\phi_{\varepsilon,y}(t,y)} - \frac{\chi''(x_1)}{\phi_{\varepsilon,y}(t,x_1)} \frac{\chi'(y)\phi_{\varepsilon,yy}(t,y)}{\phi_{\varepsilon,y}(t,y)^2} \\
&- \frac{\chi''(y)}{\phi_{\varepsilon,y}(t,y)} \frac{\chi'(x_1)\phi_{\varepsilon,yy}(t,x_1)}{\phi_{\varepsilon,y}(t,x_1)^2} + \frac{\chi'(x_1)\phi_{\varepsilon,yy}(t,x_1)}{\phi_{\varepsilon,y}(t,x_1)^2} \frac{\chi'(y)\phi_{\varepsilon,yy}(t,y)}{\phi_{\varepsilon,y}(t,y)^2} \Bigg] = (i)+(ii)+(iii)+(iv)
\end{aligned} \tag{II.2.33}$$

We will show that $(iv) >> |(i)|+|(ii)|+|(iii)|$ as $t \to t_\varepsilon$. Note that we do not study the integrals corresponding to $(x_1,y) \in [\zeta_\varepsilon^{2+}, \zeta_\varepsilon^{3-}] \times [\nu_\varepsilon - \eta_\varepsilon, \zeta_\varepsilon^{2+}]$ and $(x_1,y) \in [\zeta_\varepsilon^{2+}, \zeta_\varepsilon^{3-}] \times [\zeta_\varepsilon^{3-}, \nu_\varepsilon + \eta_\varepsilon]$ because the method is identical to the one we used for I_1.

First, we look at (iv). In this case, we have to keep the two integrals together. Because $\int_y \frac{\chi'(y)\phi_{\varepsilon,yy}(t,y)}{\phi_{\varepsilon,y}(t,y)^2}$ is going to be smaller than $\int_y \frac{\chi''(y)}{\phi_{\varepsilon,y}(t,y)^1}$ because of ϕ_{yy} anti-symmetric properties (with ν_ε as the center). But the weight function will be higher when $(x-\nu_\varepsilon)$ and $(y-\nu_\varepsilon)$ are of the same sign.
Denote $i(x_1,y) = \frac{\chi'(x_1)\phi_{\varepsilon,yy}(t,x_1)}{\phi_{\varepsilon,y}(t,x_1)^2} \frac{\chi'(y)\phi_{\varepsilon,yy}(t,y)}{\phi_{\varepsilon,y}(t,y)^2} \frac{1}{|\phi_\varepsilon(t,x_1)-\phi_\varepsilon(t,y)|^{1/2-2\lambda}}$. In figure II.2, every pole of i is displayed as a thick line. We see that the part corresponding to α and δ where $i \geqslant 0$ is going to be bigger than the part corresponding to β and γ where $i \leqslant 0$.
We also denote

$$\begin{aligned}
\alpha &= \{(x_1,y) \in [\zeta_\varepsilon^{2+}, \nu_\varepsilon] \times [\zeta_\varepsilon^{2+}, \nu_\varepsilon]\}, \quad \beta = \{(x_1,y) \in [\nu_\varepsilon, \zeta_\varepsilon^{3-}] \times [\zeta_\varepsilon^{2+}, \nu_\varepsilon]\} \\
\gamma &= \{(x_1,y) \in [\zeta_\varepsilon^{2+}, \nu_\varepsilon] \times [\nu_\varepsilon, \zeta_\varepsilon^{3-}]\}, \quad \delta = \{(x_1,y) \in [\nu_\varepsilon, \zeta_\varepsilon^{3-}] \times [\nu_\varepsilon, \zeta_\varepsilon^{3-}]\}
\end{aligned} \tag{II.2.34}$$

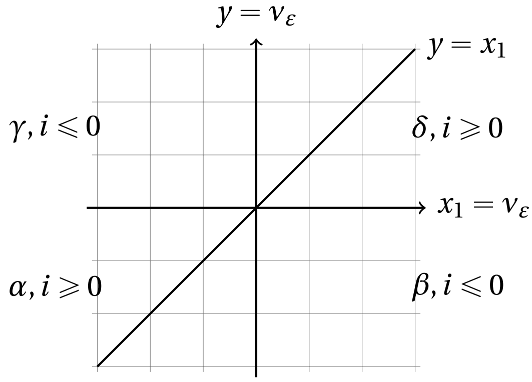


Figure II.2 – Definition of α, β, γ and δ

And we have

$$(iv) = \int\int_\alpha i(x_1, y) + \int\int_\beta i(x_1, y) + \int\int_\gamma i(x_1, y) + \int\int_\delta i(x_1, y). \tag{II.2.35}$$

We regroup the term δ and β together. We will show that this term is non-negative, and also provide a lower bound for it. Without relabelling, we can choose ζ_ε^{2+} and ζ_ε^{3-} to be symmetric with respect to ν_ε. We denote $\kappa_\varepsilon = \zeta_\varepsilon^{3-} - \nu_\varepsilon$.

To do our estimate, we will need the following refined approximation for $\phi_{\varepsilon,yy}(t, y)$, that comes from a Taylor's expansion.

$$C(\nu_\varepsilon - x_1) - C_1(\nu_\varepsilon - x_1)^2 \leqslant \phi_{\varepsilon,yy}(t, x_1) \leqslant C(\nu_\varepsilon - x_1) + C_2(\nu_\varepsilon - x_1)^2. \tag{II.2.36}$$

$$\begin{aligned} J &= \int\int_{\delta\cup\beta} \frac{1}{|\phi_\varepsilon(t,x_1) - \phi_\varepsilon(t,y)|^{1/2-2\lambda}} \frac{\chi'(x_1)\phi_{\varepsilon,yy}(t,x_1)}{\phi_{\varepsilon,y}(t,x_1)^2} \frac{\chi'(y)\phi_{\varepsilon,yy}(t,y)}{\phi_{\varepsilon,y}(t,y)^2} \\ &= \int_{x_1=0}^{\kappa_\varepsilon}\int_{y=0}^{\kappa_\varepsilon} \Bigg[\frac{\chi'(\nu_\varepsilon + y)\phi_{\varepsilon,yy}(t,\nu_\varepsilon + y)\chi'(\nu_\varepsilon + x_1)\phi_{\varepsilon,yy}(t,\nu_\varepsilon + x_1)}{|\phi_\varepsilon(t,\nu_\varepsilon + x_1) - \phi_\varepsilon(t,\nu_\varepsilon + y)|^{1/2-2\lambda}\phi_{\varepsilon,y}(t,\nu_\varepsilon + x_1)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon + y)^2} \\ &\quad + \frac{\chi'(\nu_\varepsilon - y)\phi_{\varepsilon,yy}(t,\nu_\varepsilon - y)\chi'(\nu_\varepsilon + x_1)\phi_{\varepsilon,yy}(t,\nu_\varepsilon + x_1)}{|\phi_\varepsilon(t,\nu_\varepsilon + x_1) - \phi_\varepsilon(t,\nu_\varepsilon - y)|^{1/2-2\lambda}\phi_{\varepsilon,y}(t,\nu_\varepsilon - y)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon + x_1)^2} \Bigg] \end{aligned} \tag{II.2.37}$$

Now, we write J defined in (II.2.37) as a difference term, using the approximation

(II.2.36).

$$J = J_1 + J_2 + J_3 + J_4, \tag{II.2.38}$$

There exists f a bounded function such that

$$\begin{aligned} J_1 = C\int_{x_1=0}^{\kappa_\varepsilon}\int_{y=0}^{\kappa_\varepsilon}\Bigg[&\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1\cdot\chi'(\nu_\varepsilon+y)\cdot y}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)|^{1/2-2\lambda}\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon+x_1)^2} \\ &-\frac{\chi'(\nu_\varepsilon-y)\cdot y\cdot\chi'(\nu_\varepsilon+x_1)\cdot x_1}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon-y)|^{1/2-2\lambda}\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon+x_1)^2}\Bigg], \end{aligned} \tag{II.2.39}$$

$$\begin{aligned} J_2 = C\int_{x_1=0}^{\kappa_\varepsilon}\int_{y=0}^{\kappa_\varepsilon}\Bigg[&\frac{\chi'(\nu_\varepsilon+y)\cdot y\cdot\chi'(\nu_\varepsilon+x_1)\cdot f(\nu_\varepsilon+x_1)\cdot x_1^2}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)|^{1/2-2\lambda}\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon+x_1)^2} \\ &-\frac{\chi'(\nu_\varepsilon-y)\cdot y\cdot\chi'(\nu_\varepsilon+x_1)\cdot f(\nu_\varepsilon+x_1)\cdot x_1^2}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon-y)|^{1/2-2\lambda}\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon+x_1)^2}\Bigg], \end{aligned} \tag{II.2.40}$$

$$\begin{aligned} J_3 = C\int_{x_1=0}^{\kappa_\varepsilon}\int_{y=0}^{\kappa_\varepsilon}\Bigg[&\frac{\chi'(\nu_\varepsilon+y)\cdot f(\nu_\varepsilon+y)\cdot y^2\cdot\chi'(\nu_\varepsilon+x_1)\cdot x_1}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)|^{1/2-2\lambda}\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon+x_1)^2} \\ &-\frac{\chi'(\nu_\varepsilon-y)\cdot f(\nu_\varepsilon+y)\cdot y^2\cdot\chi'(\nu_\varepsilon+x_1)\cdot x_1}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon-y)|^{1/2-2\lambda}\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon+x_1)^2}\Bigg], \end{aligned} \tag{II.2.41}$$

and

$$J_4 = C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=0}^{\kappa_\varepsilon} \left[\frac{\chi'(\nu_\varepsilon + y) \cdot f(\nu_\varepsilon + y) \cdot y^2 \cdot \chi'(\nu_\varepsilon + x_1) \cdot f(\nu_\varepsilon + x_1) \cdot x_1^2}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon + y)|^{1/2-2\lambda} \phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2 \phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \right.$$
$$\left. - \frac{\chi'(\nu_\varepsilon - y) \cdot f(\nu_\varepsilon + y) \cdot y^2 \cdot \chi'(\nu_\varepsilon + x_1) \cdot f(\nu_\varepsilon + x_1) \cdot x_1^2}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon - y)|^{1/2-2\lambda} \phi_{\varepsilon,y}(t, \nu_\varepsilon - y)^2 \phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \right]. \tag{II.2.42}$$

The main contribution will come from J_1. We will show later that the contributions from J_2, J_3 and J_4 are smaller. Let us, for now, focus only on J_1. We decompose the integrand as follows.

$$\frac{1}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon + y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2}$$
$$- \frac{1}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon - y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon - y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon - y)^2}$$
$$= \left[\frac{1}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon + y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \right.$$
$$\left. - \frac{1}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon - y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \right]$$
$$+ \left[\frac{1}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon - y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \right.$$
$$\left. - \frac{1}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon - y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon - y)^2} \right]$$
$$+ \left[\frac{1}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon - y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon - y)^2} \right.$$
$$\left. - \frac{1}{|\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon - y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon - y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon - y)^2} \right]$$
$$= D_1 + D_2 + D_3. \tag{II.2.43}$$

To shorten a bit the notations, we will denote

$$\alpha_\pm = |\phi_\varepsilon(t, \nu_\varepsilon + x_1) - \phi_\varepsilon(t, \nu_\varepsilon \pm y)|. \tag{II.2.44}$$

For D_1, we write

$$\begin{aligned}&\frac{1}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)|^{1/2-2\lambda}}-\frac{1}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon-y)|^{1/2-2\lambda}}\\ &\qquad=\frac{\alpha_--\alpha_+}{\alpha_+^{1/2-2\lambda}\alpha_-^{1/2-2\lambda}\left(\alpha_+^{1/2+2\lambda}+\alpha_-^{1/2+2\lambda}\right)}+\frac{\alpha_+^{4\lambda}-\alpha_-^{4\lambda}}{\left(\alpha_+^{1/2+2\lambda}+\alpha_-^{1/2+2\lambda}\right)}\end{aligned} \tag{II.2.45}$$

It is clear that D_1 is nonnegative when $x_1 \geqslant y$. We now provide a lower bound for $D_{1,x_1\leqslant y}$. We will show that it is positive up to a smaller order term, and provide a lower bound for the positive term. We hence now consider $x_1 \leqslant y$.

We write,

$$\begin{aligned}&\alpha_--\alpha_+=|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon-y)|-|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)|\\ &=2\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)-\phi_\varepsilon(t,\nu_\varepsilon-y)=\int_{s=-y}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)-\int_{s=x_1}^{y}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)\\ &\qquad=\int_{s=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)+\int_{s=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-s)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)\right).\end{aligned} \tag{II.2.46}$$

We will use this idea to provide a lower bound for D_1.

$$\frac{1}{\alpha_+^{1/2-2\lambda}} - \frac{1}{\alpha_-^{1/2-2\lambda}} = \int_{s=\alpha_-}^{\alpha_+} -(\frac{1}{2}-2\lambda)\frac{1}{s^{3/2-2\lambda}} = C\int_{s=\alpha_+}^{\alpha_-} s^{-3/2+2\lambda}$$
$$= C\int_{s=\alpha_+}^{\alpha_+ + \int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)} s^{-3/2+2\lambda}$$
$$+ C\int_{s=\alpha_+ + \int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)}^{\alpha_+ + \int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z) + \int_{z=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)} s^{-3/2+2\lambda} = (i)+(ii), \tag{II.2.47}$$

where (i) is nonnegative and (ii) is small. First, we make the following upper bound for $|(ii)|$. If $\int_{x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon - s) - \phi_{\varepsilon,y}(t,\nu_\varepsilon + s)) > 0$, then (ii) is nonnegative. Otherwise, we have $\alpha_+ + \int_{-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s) + \int_{x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon - s) - \phi_{\varepsilon,y}(t,\nu_\varepsilon + s)) \leqslant \alpha_+ + \int_{-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)$. We then proceed as follows

$$\left|\int_{s=\alpha_+ + \int_{-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)}^{\alpha_+ + \int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z) + \int_{z=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)} s^{-3/2-2\lambda}\right|$$
$$= \left|\int_{s=\alpha_-}^{\alpha_- - \int_{z=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)} s^{-3/2+2\lambda}\right|$$
$$\leqslant \left|\int_{z=x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon - z) - \phi_{\varepsilon,y}(t,\nu_\varepsilon+z))\right| \cdot \alpha_-^{-3/2+2\lambda}. \tag{II.2.48}$$

From the Taylor expansion, we obtain for x, y small enough (only depending on ε),

$$\left|\int_{z=x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon - s) - \phi_{\varepsilon,y}(t,\nu_\varepsilon+s))\right| \leqslant C_1\left(y^4 - x_1^4 + (t_\varepsilon - t)^2 + (t-t_\varepsilon)(y^2 - x_1^2)\right). \tag{II.2.49}$$

This means that we obtain (up to a nonnegative contribution)

$$|(ii)| \leqslant C\left(y^4 + x_1^4 + (t_\varepsilon - t)\left(x_1^2 + y^2\right) + (t_\varepsilon - t)^2\right) \cdot \frac{1}{\alpha_-^{3/2-2\lambda}}. \tag{II.2.50}$$

We now go on with the lower bound for (i). Using the expression of (i) provided by (II.2.47) as well as the mean value theorem, we obtain

$$\begin{aligned}(i) &= \int_{s=\alpha_+}^{\alpha_+ + \int_{z=-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon+z)} s^{-3/2+2\lambda} \\ &\geqslant \left(\int_{z=-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)\cdot \frac{1}{(\phi_\varepsilon(t,\nu_\varepsilon+y)-\phi_\varepsilon(t,\nu_\varepsilon-x_1))^{3/2-2\lambda}}. \\ &\geqslant C x_1 \phi_{\varepsilon,y}(t,c_1)\cdot \frac{1}{(y+x_1)^{3/2-2\lambda}\cdot \phi_{\varepsilon,y}(t,c_2)^{3/2-2\lambda}}. \end{aligned} \quad \text{(II.2.51)}$$

Now, since we have $x_1 \leqslant y$, we obtain from (II.2.16)

$$\phi_{\varepsilon,y}(t,c_1) \geqslant C(t_\varepsilon - t), \quad \text{(II.2.52)}$$

and

$$\phi_{\varepsilon,y}(t,c_2) \leqslant C\left((t_\varepsilon - t) + y^2\right). \quad \text{(II.2.53)}$$

Using (III.2.65) and (III.2.66) inside of (III.2.64), we obtain

$$(i) \geqslant C \frac{x_1(t_\varepsilon - t)}{(y+x_1)^{3/2-2\lambda}\cdot((t_\varepsilon - t)+y^2)^{3/2-2\lambda}}. \quad \text{(II.2.54)}$$

Now, we obtain for D_1,

$$
\begin{aligned}
D_1 &= D_{x_1 \geqslant y} + D_{x_1 \leqslant y} \geqslant D_{x_1 \leqslant y} \\
&\geqslant C \iint_{x_1 \leqslant y} \frac{x_1(t_\varepsilon - t)}{(y+x_1)^{3/2-2\lambda} \cdot ((t_\varepsilon - t) + y^2)^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{x^4}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{y^4}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t) \cdot x_1^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t) \cdot y^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t)^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&= A_1 - B_1 - B_2 - B_3 - B_4 - B_5. \qquad \text{(II.2.55)}
\end{aligned}
$$

Note that the proof also works for $\lambda = 0$. We will now show that $A_1 \to \infty$, and that J_2, J_3, J_4, D_2, D_3, B_1, B_2, B_3, B_4 and B_5 are of a smaller order. We first consider A_1.

Using $|\phi_{\varepsilon,y}(t, c(x_1, y))| \leqslant M_\varepsilon \left((x_1)^2 + (t_\varepsilon - t)\right)$, on $y > x_1$, we have from (II.2.55) and the new change of variable $(x, y) = (r - z, r + z)$,

$$
\begin{aligned}
A_1 &\geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2} \int_{z=0}^{r} \frac{(r-z)(t_\varepsilon - t)}{r^{3/2-2\lambda}} \frac{r+z}{\left((r+z)^2 + (t_\varepsilon - t)\right)^{7/2-2\lambda}} \frac{r-z}{\left((r-z)^2 + (t_\varepsilon - t)\right)^2} \\
&\geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2} \frac{1}{r^{1/2-2\lambda}} \frac{(t_\varepsilon - t)}{\left((2r)^2 + (t_\varepsilon - t)\right)^{11/2-2\lambda}} \int_{z=0}^{r} (r-z)^2 \\
&\geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2} \frac{(t_\varepsilon - t) \cdot r^{5/2+2\lambda}}{\left((2r)^2 + (t_\varepsilon - t)\right)^{11/2-2\lambda}} \geqslant \frac{C_\varepsilon \cdot (t_\varepsilon - t)}{(t_\varepsilon - t)^{15/4-3\lambda}} \\
&\geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{11/4-3\lambda}}. \qquad \text{(II.2.56)}
\end{aligned}
$$

We now proceed with B_1. The case of B_2 is similar. We have since $\phi_{\varepsilon,y}(t,x) \geqslant (t_\varepsilon - t)$, with the mean value theorem

$$\begin{aligned}
|B_1| &\leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \int_{z=0}^{r} \frac{(r-z)^4}{r^{3/2-2\lambda}} \\
&\qquad \cdot \frac{r+z}{\left((r+z)^2 + (t_\varepsilon - t)\right)^2} \frac{r-z}{\left((r-z)^2 + (t_\varepsilon - t)\right)^2} \\
&\leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^4}{r^{1/2-2\lambda}(r^2 + (t_\varepsilon - t))^2} \int_{z=0}^{r} \frac{r-z}{((r-z)^2 + (t_\varepsilon - t))^2}.
\end{aligned} \tag{II.2.57}$$

Now, because

$$\int_{s=0}^{\infty} \frac{s}{(s^2 + (t_\varepsilon - t))^2} \leqslant \frac{C}{(t_\varepsilon - t)}, \tag{II.2.58}$$

(II.2.57) yields

$$|B_1| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{5/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^{7/2+2\lambda}}{(r^2 + (t_\varepsilon - t))^2} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{10/4-2\lambda}}. \tag{II.2.59}$$

We now proceed with B_3. The case of B_4 is similar. We have since $\phi_{\varepsilon,y}(t,x) \geqslant (t_\varepsilon - t)$, with the mean value theorem

$$\begin{aligned}
|B_3| &\leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \int_{z=0}^{r} \frac{(t_\varepsilon - t)\cdot(r-z)^2}{r^{3/2-2\lambda}} \\
&\qquad \cdot \frac{r+z}{\left((r+z)^2 + (t_\varepsilon - t)\right)^2} \frac{r-z}{\left((r-z)^2 + (t_\varepsilon - t)\right)^2} \\
&\leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^2}{r^{1/2-2\lambda}(r^2 + (t_\varepsilon - t))^2} \int_{z=0}^{r} \frac{r-z}{((r-z)^2 + (t_\varepsilon - t))^2}.
\end{aligned} \tag{II.2.60}$$

Now, because

$$\int_{s=0}^{\infty} \frac{s}{\left(s^2 + (t_\varepsilon - t)\right)^2} \leqslant \frac{C}{(t_\varepsilon - t)}, \tag{II.2.61}$$

(II.2.60) yields

$$|B_3| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^{3/2+2\lambda}}{\left(r^2 + (t_\varepsilon - t)\right)^2} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2+3/4-3\lambda}} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}. \tag{II.2.62}$$

Lastly, we provide the upper bound for B_5. We have since $\phi_{\varepsilon,y}(t,x) \geqslant (t_\varepsilon - t)$, with the mean value theorem

$$\begin{aligned} |B_5| \leqslant{}& \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \int_{z=0}^{r} \frac{(t_\varepsilon - t)^2}{r^{3/2-2\lambda}} \\ & \cdot \frac{r+z}{\left((r+z)^2 + (t_\varepsilon - t)\right)^2} \frac{r-z}{\left((r-z)^2 + (t_\varepsilon - t)\right)^2} \\ \leqslant{}& C_\varepsilon \cdot (t_\varepsilon - t)^{1/2+2\lambda} \int_{r=0}^{\kappa_\varepsilon/2} \frac{1}{r^{1/2-2\lambda}\left(r^2 + (t_\varepsilon - t)\right)^2} \int_{z=0}^{r} \frac{r-z}{\left((r-z)^2 + (t_\varepsilon - t)\right)^2}. \end{aligned} \tag{II.2.63}$$

Now, because

$$\int_{s=0}^{\infty} \frac{s}{\left(s^2 + (t_\varepsilon - t)\right)^2} \leqslant \frac{C}{(t_\varepsilon - t)}, \tag{II.2.64}$$

and

$$\int_{s=0}^{\infty} \frac{s^{-1/2+2\lambda}}{\left(s^2 + (t_\varepsilon - t)\right)^2} \leqslant \frac{C}{(t_\varepsilon - t)^{7/4}}, \tag{II.2.65}$$

(II.2.63) yields

$$|B_5| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^{-1/2+2\lambda}}{(r^2+(t_\varepsilon-t))^2} \leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{1/2+7/4-3\lambda}} \leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{9/4-3\lambda}}. \quad \text{(II.2.66)}$$

This concludes the proof for the first difference term D_1. Overall, we obtain

$$D_1 \geqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{11/4-3\lambda}}. \quad \text{(II.2.67)}$$

Now, we study D_2 and D_3.

For D_2, we first study the term

$$\frac{1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)^2} - \frac{1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)^2} = \frac{(\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)+\phi_{\varepsilon,y}(t,\nu_\varepsilon-y))\cdot(\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+y))}{\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)^2\phi_y(t,\nu_\varepsilon-y)^2}. \quad \text{(II.2.68)}$$

A Taylor expansion of $\phi_{\varepsilon,y}$ gives that $\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+y) = f_1(t,y)y^3 + f_2(t,y)(t-t_\varepsilon)^2$ where f_1 and f_2 are bounded. Because $\phi_y \geqslant 0$, we have that

$$\frac{1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)^2} - \frac{1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)^2} = \frac{f_1(t,y)y^3+f_2(t,y)(t_\varepsilon-t)^2}{\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)^2} + \frac{f_1(t,y)y^3+f_2(t,y)(t_\varepsilon-t)^2}{\phi_{\varepsilon,y}(t,\nu_\varepsilon+y)^2\phi_{\varepsilon,y}(t,\nu_\varepsilon-y)}. \quad \text{(II.2.69)}$$

The two involved terms are similar. We consider only the first one. We obtain

$$|D_2| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa/2} \int_{z=0}^{r} \frac{1}{z^{1/2-2\lambda}} \cdot \frac{(r+z)(r-z)^4}{\left((r+z)^2 + (t_\varepsilon - t)\right)^2 \cdot \left((r-z)^2 + (t_\varepsilon - t)\right) \cdot \left((z-r)^2 + (t_\varepsilon - t)\right)^2}. \quad \text{(II.2.70)}$$

Again, we split the domain of z in two parts.

$$\begin{aligned}|D_2| \leqslant{}& \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa/2} \int_{z=0}^{r/2} \frac{1}{z^{1/2-2\lambda}} \cdot \frac{(r+z)(r-z)^4}{\left((r+z)^2 + (t_\varepsilon - t)\right)^2 \cdot \left((r-z)^2 + (t_\varepsilon - t)\right) \cdot \left((z-r)^2 + (t_\varepsilon - t)\right)^2} \\ &+ \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa/2} \int_{z=r/2}^{r} \frac{1}{z^{1/2-2\lambda}} \cdot \frac{(r+z)(r-z)^4}{\left((r+z)^2 + (t_\varepsilon - t)\right)^2 \cdot \left((r-z)^2 + (t_\varepsilon - t)\right) \cdot \left((z-r)^2 + (t_\varepsilon - t)\right)^2}. \quad \text{(II.2.71)}\end{aligned}$$

For the first term, we obtain

$$\begin{aligned}&\frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa/2} \frac{r^5}{\left(r^2 + (t_\varepsilon - t)\right)^5} \int_{z=0}^{r/2} \frac{1}{z^{1/2-2\lambda}} \\ &\qquad \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa/2} \frac{r^{11/2+2\lambda}}{\left(r^2 + (t_\varepsilon - t)\right)^5} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}, \quad \text{(II.2.72)}\end{aligned}$$

and for the second term

$$\begin{aligned}&\frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa/2} \frac{1}{r^{1/2-2\lambda}} \frac{r}{\left(r^2 + (t_\varepsilon - t)\right)^2} \int_{s=0}^{r/2} \frac{s^4}{\left(s^2 + (t_\varepsilon - t)\right)^3} \\ &\qquad \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}. \quad \text{(II.2.73)}\end{aligned}$$

Overall, we obtain that $|D_2| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}$. Lastly, for D_3, we have to study the term $\chi'(v_\varepsilon - y) - \chi'(v_\varepsilon + y)$. We obtain

$$|\chi'(v_\varepsilon - y) - \chi'(v_\varepsilon + y)| \leqslant C_\varepsilon \cdot |y|. \tag{II.2.74}$$

With similar computations, the additional y leads to a gain of order $(t_\varepsilon - t)^{1/2}$. Lastly, for the terms J_2, J_3, J_4, the additional terms x_1, y, y and $x_1 \cdot y$ converts into (respectively) a gain of order $(t_\varepsilon - t)^{1/2}$, $(t_\varepsilon - t)^{1/2}$, $(t_\varepsilon - t)^{1/2}$ and $(t_\varepsilon - t)^1$.

Overall, we obtain that

$$\int\int_{\beta\cup\delta} i(x_1, y) \geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{11/4-3\lambda}} \tag{II.2.75}$$

and hence

$$(iv) \geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{11/4-3\lambda}}. \tag{II.2.76}$$

For (i), using the lower bound on ϕ_ε provided by (II.2.16), we obtain with the same method (we make the same change of variable and do not separate the domain as previously)

$$(i) \leqslant \int_x\int_y \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \frac{M_\varepsilon |x|^{-1/4}}{x^2 + (t_\varepsilon - t)} \frac{M_\varepsilon |y|^{-1/4}}{y^2 + (t_\varepsilon - t)} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{14/8-2\lambda}} \tag{II.2.77}$$

Also, for (ii), we obtain

$$\begin{aligned}(ii) \leqslant \int_x\int_y \frac{C}{(t_\varepsilon - t)^{1/2-2\lambda}} \frac{M_\varepsilon |x|^{-1/4}}{x^2 + (t_\varepsilon - t)} \frac{M_\varepsilon |y|^{3/4}}{(y^2 + (t_\varepsilon - t))^2} &\leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2+5/8+9/8-2\lambda}} \\ &\leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-2\lambda}} \end{aligned} \tag{II.2.78}$$

The case of (iii) is similar. Now, using (II.2.56), (II.2.76), (II.2.77) and (II.2.78), we obtain that

$$I_\varepsilon^2(t) = (i) + (ii) + (iii) + (iv) \geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{11/4-3\lambda}} \tag{II.2.79}$$

Because of (II.2.32), we get the desired result.

$$I_\varepsilon(t) = I_\varepsilon^1(t) + I_\varepsilon^2(t) + I_\varepsilon^3(t) \xrightarrow[t\to t_\varepsilon]{} \infty. \tag{II.2.80}$$

□

Remark II.2.4. *We also obtained a lower bound for the speed at which* $||v||_{H^{7/4}} \to \infty$.

Theorem II.2.5. $t_\varepsilon \to 0$ *as* $\varepsilon \to 0$.

Proof.

$$\max\left\{ h_\varepsilon(y) = \frac{|\chi_\varepsilon'(y)|}{(1-\chi_\varepsilon(y))^2} \right\} \geqslant h_\varepsilon(\varepsilon) \geqslant |\ln(\varepsilon)|^\alpha \to \infty. \tag{II.2.81}$$

By definition of t_ε, we thus have that $t_\varepsilon \to 0$ as $\varepsilon \to 0$.

□

II.3 A scaling argument to construct the solution

In the following, we use a scaling argument to create a sequence of solutions with summable $H^{11/4}$ norms, and we choose a ε parameter such that the lifespan goes to 0.

Let $v(t,x)$ be a solution of (.0.13). We define for $\omega, \gamma \in \mathbb{R}$,

$$v_\lambda(t,x) = \lambda^\omega v(\lambda^\gamma t, \lambda^\gamma x). \tag{II.3.82}$$

Now,

$$\begin{aligned} (\Box v_\lambda)(t,x) &= \lambda^\omega \lambda^{2\gamma} \Box v(\lambda^\gamma t, \lambda^\gamma x) \\ (Dv_\lambda D^2 v_\lambda)(t,x) &= \lambda^{2\omega}\lambda^{3\gamma}(Dv)(\lambda^\gamma t, \lambda^\gamma x)(D^2 v)(\lambda^\gamma t, \lambda^\gamma x); \end{aligned} \tag{II.3.83}$$

so v_λ is also a solution, provided that

$$\omega + \gamma = 0. \tag{II.3.84}$$

Now, we will make an estimation on the modification of the Sobolev norm due to the rescaling. Note that without the logarithmic modification, we have (with the change in variable in x and ξ)

$$||v_\lambda||_{\dot{H}^{11/4}} = \lambda^\omega (\lambda^\gamma)^{11/4-2/2} ||v||_{\dot{H}^{11/4}}. \tag{II.3.85}$$

Now, we estimate $||v_\lambda||_{\dot{H}^{11/4}(\ln H)^{-\beta}}$. We have by definition

$$\begin{aligned} ||v_\lambda||^2_{H^{11/4}(\ln H)^{-\beta}} &= \int_{\xi\in\mathbb{R}^2} |\lambda|^{2\gamma}|\lambda|^{-4\gamma} \frac{|\lambda|^{2\omega}|\lambda^\gamma\xi|^{11/2}}{(1+|\ln(\lambda^\gamma|\xi|)|)^{2\beta}} \mathscr{F}(v)(\xi)^2 \\ &\leqslant (\lambda)^{2\omega}(\lambda^\gamma)^{11/2-2} \int_{\xi\in\mathbb{R}^2} \frac{|\xi|^{11/2}}{(1+|\ln(|\lambda^\gamma\xi|)|)^{2\beta}} \mathscr{F}(v)(\xi)^2, \end{aligned} \tag{II.3.86}$$

We have the following properties for $\lambda < 1$,

$$\begin{aligned} &|\ln(|\lambda^\gamma\xi|)| = |\ln(\lambda^\gamma) + \ln(|\xi|)| \geqslant |\ln(\xi)| \text{ for } |\xi| \in (0,1], \\ &\text{for } |\xi| \in [1, \lambda^{-\gamma/2}],\ |\ln(\lambda^\gamma) + \ln(|\xi|)| \geqslant \frac{1}{2}|\ln(\lambda^\gamma)| \geqslant |\ln(|\xi|)|, \\ &\text{for } |\xi| \in [\lambda^{-2\gamma}, \infty),\ |\ln(\lambda^\gamma) + \ln(|\xi|)| \geqslant \frac{1}{2}|\ln(|\xi|)|. \end{aligned} \tag{II.3.87}$$

This means that we can already establish

$$\begin{aligned} &\int_{|\xi|<\frac{1}{\lambda^{\gamma/2}} \bigcup |\xi|>\frac{1}{\lambda^{2\gamma}}} \lambda^{2\gamma}|\lambda|^{-4\gamma} \frac{|\lambda|^{2\omega}|\lambda^\gamma\xi|^{11/2}}{(1+|\ln(\lambda^\gamma|\xi|)|)^{2\beta}} \mathscr{F}(v)(\xi)^2 \\ &\qquad\qquad \leqslant 2(\lambda)^{2\omega}(\lambda^\gamma)^{11/2-2} ||v||_{\dot{H}^{11/4}(\ln H)^{-\beta}}. \end{aligned} \tag{II.3.88}$$

For the last part, we have

$$\begin{aligned}\int_{\lambda^{-\gamma/2}<|\xi|<\lambda^{-2\gamma}} &\lambda^{2\gamma}\lambda^{-4\gamma}\frac{\lambda^{2\omega}|\lambda^{\gamma}\xi|^{11/2}}{(1+|\ln(\lambda^{\gamma}|\xi|)|)^{2\beta}}\mathscr{F}(v)(\xi)^2 \\ &\leqslant \int_{\lambda^{-\gamma/2}<|\xi|<\lambda^{-2\gamma}} \lambda^{2\gamma}\lambda^{-4\gamma}\lambda^{2\omega}|\lambda^{\gamma}\xi|^{11/2}\mathscr{F}(v)(\xi)^2 \\ &\leqslant \lambda^{-2\gamma}\lambda^{11/2\gamma}\lambda^{2\omega}(1+\left|\ln(\lambda^{2\gamma})\right|)^{2\beta}\int_{\lambda^{-\gamma/2}<|\xi|<\lambda^{-2\gamma}}\frac{|\xi|^{11/2}}{(1+|\ln(|\xi|)|)^{2\beta}}\mathscr{F}(v)(\xi)^2 \\ &\leqslant \lambda^{2\omega}\lambda^{7/2\gamma}(1+\left|\ln(\lambda^{2\gamma})\right|)^{2\beta}||v||_{\dot{H}^{11/4}(\ln H)^{-\beta}}. \end{aligned} \tag{II.3.89}$$

Overall, we obtain from (II.3.88) and (II.3.89),

$$||v_\lambda||_{\dot{H}^{11/4}(\ln H)^{-\beta}} \leqslant 2\lambda^{\omega}|\lambda|^{\frac{7}{4}\gamma}(1+2\,|\ln(\lambda^{\gamma})|)^{\beta}||v||_{\dot{H}^{11/4}(\ln H)^{-\beta}}. \tag{II.3.90}$$

Now, we will choose the following values for the parameters and apply this result to the solution u_ε defined in the previous chapter as the solution of the Cauchy problem (II.2.5).

Now, applying this to the solution u_ε defined in the previous chapter, define

$$\begin{cases} \omega = -1, \\ \gamma = 1, \\ \lambda = n^{-4}, \\ \varepsilon = \min\left(e^{-n^5}, e^{-n^{\frac{6}{\alpha}}}\right), \end{cases} \tag{II.3.91}$$

and (u_n) the corresponding sequence of solutions.

First, the rescaling in time shortens the lifespan of u_ε and we have that the lifespan of $u_{\varepsilon,\lambda}$, satisfies $t^{\lambda,\varepsilon} = \lambda^{-\gamma} t_\varepsilon$. Hence, we have by (II.2.81)

$$t_n \leqslant \frac{n^4}{|\ln(\varepsilon)|^{\alpha}} \leqslant \frac{1}{n} \to 0. \tag{II.3.92}$$

Now, we can choose any parameter $\beta > 1/2$ in our construction. So we can assume for instance that we have $\beta < 3/4$.

$$||u_{\lambda,\varepsilon}||_{H^{11/4}(\ln H)^{-\beta}} \leqslant 2\frac{1}{n^3}(1+2\ln(n^{-4}))^{\beta}||u_{0,\varepsilon}||_{H^{11/4}(\ln H)^{-\beta}}, \tag{II.3.93}$$

which is in $l^1(\mathbb{Z})$.

Now, define $\tilde{u}_n$ as a translation of u_n in x_1 such that $\mathrm{Supp}(\tilde{u}_i) \cap \mathrm{Supp}(\tilde{u}_j) = \varnothing$ for $i \neq j$ (and the domain of dependence do not intersect either), and finally define

$$Ł(t,x) = \sum_{n=0}^{\infty} \tilde{u}_n(t,x). \tag{II.3.94}$$

Each function $\tilde{u}_n$ is a translation of a function of which the support is included in $x_1 \in [\varepsilon/2, \frac{2}{|\ln(\varepsilon)|^{\alpha/2}}] \subseteq [\frac{1}{2}e^{-n^5}, \frac{2}{n^3}]$. Because the sequence $\frac{2}{n^3} - \frac{1}{2}e^{-n^5}$ is in $l^1(\mathbb{N})$, we can find a sequence of translations such that the projection of the support of Ł on the x_1 axis is bounded. Now, the width of the domain (the projection on the x_2 axis) is bounded by a constant because it is bounded by $\sqrt{\frac{1}{|\ln(\varepsilon_n)|^{\alpha/2}}} \cdot \left|\ln\left(\frac{1}{|\ln(\varepsilon_n)^{\alpha/2}}\right)\right|^{-\delta} \to 0$ as $n \to \infty$. This means that Ł has a compact support. In virtue of lemma I.2.3, we hence obtain

Theorem II.3.1. Ł *satisfies*

$$\begin{aligned} ||Ł_{|t=0}||_{H^{11/4}_{x_1}} &< \infty \\ ||\frac{\partial Ł_{|t=0}}{\partial t}||_{H^{7/4}_{x_1}} &< \infty \\ \forall t > 0, \quad ||\frac{\partial}{\partial x}(DŁ)(t,\cdot)||_{H^{3/4}_{x_1}} &= \infty \end{aligned} \tag{II.3.95}$$

Proof. It directly follows from the fact that $t_n \to 0$ as $n \to \infty$. □

Remark II.3.2. *The function* Ł *that we created to show the ill-posedness of the equation, is also of compact support.*

Part II

Stability of the blow up with respect to modifications of the equation or the initial condition

In this chapter, we study the stability of the instantaneous blow up phenomenon exhibited in the previous chapter. Note that such a phenomenon has already been exhibited for instance by Lindblad in [Lin98] in the case of the third dimension, but its stability is yet to be studied. In the previous chapter, we created an initial condition such that the following Cauchy problem is ill-posed.

$$\begin{cases} \Box u = DuD^2 u, \\ u_{|t=0} = 0, \ \dfrac{\partial u}{\partial t}_{|t=0} = -\chi, \end{cases} \quad (MQLW) \tag{II.3.96}$$

where $D = \partial_{x_1} - \partial_t$. More precisely, with $-\chi = Ł$ defined in chapter II.3 as (II.3.94), there exists no time $T > 0$ and solution u such that
$(u, \partial_t u) = C^0\left([0, T[, H^{11/4-\lambda}(\mathbb{R}^2) \times H^{7/4-\lambda}(\mathbb{R}^2)\right)$ for $\lambda > 0$. Since our initial condition belongs to $H^{11/4}(\ln H)^{-\beta} \times H^{7/4}(\ln H)^{-\beta}$ (where $H^s(\ln H)^{-\beta}$ is defined as in (I.2.12)), and as stated in (ii) of lemma I.2.1, we have $H^s(\ln H)^{-\beta} \subseteq H^{s-\lambda}$ for any $\lambda > 0$, this reveals the sharpness of the index found in [ST05] by Tataru and Smith.

In this chapter, we are interested in modified versions of our model equation that also lead to a blow up. In chapter III, we will look at a perturbation that modifies the underlying ODE we find using the characteristic method. More precisely, we introduce a right hand side of the form $c(\partial_{x_1} - \partial_t)$. This method would of course generalize to any modification of $(MQLW)$ such that the characteristics are the same, and the underlying ODE satisfies some conditions.

Later, in chapter IV, we introduce a source term with a x_2 and ∇u dependency. The characteristic method does not longer apply, and we need more refined techniques to show the instantaneous blow up. We are no longer able to give an explicit formula for the solution, and we need to characterize the fact that the solution still behaves in a pathological way, that shares similarities with the singularity that we observed for the unperturbed equation. The proof is more tedious, but is more interesting in several ways. First, we see how we can show the blow up using a characteristic method without an explicit formula for the characteristics nor the value of the function. Also, because of the x_2 dependency introduced in the f function, we need to show the blow up of the function $u(t, x_1, x_2)$ also depending on x_2. Hence, this paves the way for further generalizations. Lastly, having more control on the way the function blows up as x_2 varies would allow us to replace f by other operators using a frequency cutoff on x_2.

III A first modification of the equation that leads to an instantaneous blow up

In this chapter, we consider an other Cauchy problem. We will show that this Cauchy problem also leads to a blow up at time $t = 0^+$. The common point between this example and the previous one is the behaviour of the characteristic curves. However, this time the characteristics seen as a function of t for a fixed initial starting point $(0, x_1, x_2)$ will not be an affine function.

III.1 Definition of the new problem and preliminary results

We consider the Cauchy problem

$$\begin{cases} \left(\partial_t + \dfrac{1+v}{1-v}\partial_{x_1}\right)(\partial_t - \partial_{x_1})\, u = c\,(\partial_t - \partial_{x_1})\, u, \\ \dfrac{\partial u_{|t=0}}{\partial t} = -\chi,\, u_{|t=0} = 0. \end{cases} \tag{III.1.1}$$

With $v = (\partial_{x_1} - \partial_t)\, u \neq 1$.

We define χ_ε as

$$\chi_\varepsilon(y) = -\int_{s=0}^{y} \psi_\varepsilon |\ln(s)|^\alpha ds. \tag{III.1.2}$$

Where $\psi_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions.

$$\begin{cases} \psi_\varepsilon(s) = 0, \ \ \forall s < \dfrac{\varepsilon}{2}, \\ \psi_\varepsilon(s) = 1, \ \ \forall s > \varepsilon, \\ \psi_\varepsilon(s) \in [0,1], \ \ \forall s \in \mathbb{R}^+, \\ |\psi'_\varepsilon|(s) \leqslant \dfrac{2}{\varepsilon}, \ \ \forall s \in \mathbb{R}^+. \end{cases} \tag{III.1.3}$$

From now on, we will not always write explicitly the dependency of all functions with respect to ε. We consider χ to be the extension of χ to $\mathbb{R}^2$ (without explicit relabelling) as it has been done in theorem I.3.1. Of course, the computations for χ done in chapter I hold in our case as it is the same initial condition, and as the computations did not depend on the equation. We only do the part II again.

Now, if we set the condition

$$\phi_t = \frac{1+v}{1-v} \text{ and } \phi(0,x) = x, \tag{III.1.4}$$

Using the chain rule and (III.1.1), we obtain

$$\partial_t \left[v(t,\phi(t,x)) \right] = c v(t,\phi(t,x)). \tag{III.1.5}$$

And hence,

$$v(t,\phi(t,x)) = e^{ct} v(0,x) = e^{ct} \chi(x). \tag{III.1.6}$$

Note that for $c = 0$, we obtain (II.2.10). This result combined with (III.1.4) leads to the following equation for ϕ, the characteristic function.

$$\partial_t \phi(t,x) = \frac{1+e^{ct}\chi(x)}{1-e^{ct}\chi(x)}. \tag{III.1.7}$$

Integrating (III.1.7) with respect to t leads to the following explicit formula for ϕ.

$$\phi(t,x) = x + \int_{s=0}^{t} \frac{1+e^{cs}\chi(x)}{1-e^{cs}\chi(x)} ds = x + \int_{u=1}^{e^{ct}} \frac{1+u\chi(x)}{1-u\chi(x)} \frac{du}{cu}$$
$$= x + \int_{u=1}^{e^{ct}} \left[\frac{1}{cu} + \frac{2\chi(x)}{c(1-u\chi(x))} \right] du = x + t + \frac{2}{c} \ln\left(\frac{1-\chi(x)}{1-e^{ct}\chi(x)} \right). \quad \text{(III.1.8)}$$

Remark III.1.1. *For comparison sake, note that in the previous case, we could also write ϕ as*

$$\phi(t,x) = x + t + 2t \frac{\chi(x)}{1-\chi(x)}. \quad \text{(III.1.9)}$$

Also, taking the limit $c \to 0$, we have that

$$x + t + \frac{2}{c} \ln\left(\frac{1-\chi(x)}{1-e^{ct}\chi(x)} \right) = x + t + \frac{2}{c} \ln\left(\frac{1}{1 - \frac{ct\chi(x)}{1-\chi(x)} + o(c)} \right)$$
$$= x + t + \frac{2}{c} \ln\left(1 + \frac{ct\chi(x)}{1-\chi(x)} + o(c) \right) = x + t + 2t \frac{\chi(x)}{1-\chi(x)} + o(c). \quad \text{(III.1.10)}$$

Hence, we obtain the previous expression of ϕ, given by (III.1.9).

Now, we will state the main theorem of this chapter.

Theorem III.1.2. *There exists an initial condition, such that the solution u of the corresponding Cauchy problem (III.1.1) satisfies*

$$\begin{cases} u_{|t=0} \in H^{11/4}(\ln H)^{-\beta}(\mathbb{R}^2), \\ u(t,\cdot) \notin H^{11/4}(\ln H)^{-\beta}(\mathbb{R}^2), \ \ \forall t > 0. \end{cases} \quad \text{(III.1.11)}$$

We will proceed in a similar way as in chapter II. We define $\chi_\varepsilon(y) = -\int_{s=0}^{y} \psi_\varepsilon(s)|\ln(s)|^\alpha ds$ and the corresponding regularized problem

$$\begin{cases} \left(\partial_t + \frac{1+v}{1-v}\partial_{x_1}\right)(\partial_t - \partial_{x_1})\, u = c\,(\partial_t - \partial_{x_1})\, u, \\ \frac{\partial u_{|t=0}}{\partial t} = -\tilde{\chi}_\varepsilon, u_{|t=0} = 0. \end{cases} \quad \text{(III.1.12)}$$

First, we prove the following lemma.

Lemma III.1.3. *There exists $t_\varepsilon > 0$ such that for any $t < t_\varepsilon$, and for any $y \in \Omega_t$, $\phi_y(t,y) \neq 0$, and there exists ν_ε such that $\phi_y(t_\varepsilon,\nu_\varepsilon) = 0$. Furthermore, $t_\varepsilon \to 0$ as $\varepsilon \to 0$.*

And we will also show the following preliminary results.

Lemma III.1.4.

$$\begin{aligned}
\phi_{ty}(t,y) &= \frac{2\chi'(y)e^{ct}}{(1-e^{ct}\chi(y))^2}, \quad \phi_{tyy}(t,y) = \frac{2e^{ct}\left[\chi''(y)(1-e^{ct}\chi(y)) + 2\chi'(y)^2 e^{ct}\right]}{(1-e^{ct}\chi(y))^3} \\
\phi_y(t,y) &= 1 + \frac{2}{c}\frac{(e^{ct}-1)\chi'(y)}{(1-\chi(y))(1-e^{ct}\chi(y))}, \\
\phi_{yy}(t,y) &= \frac{2(e^{ct}-1)}{c}\frac{\left[1+e^{ct}-2e^{ct}\chi(y)\right]\chi'^2(y) + (1-\chi(y))(1-e^{ct}\chi(y))\chi''(y)}{(1-\chi(y))^2(1-e^{ct}\chi)}
\end{aligned} \tag{III.1.13}$$

$$\begin{aligned}
&\forall y, \forall t < t_\varepsilon, \;\; \chi(y) \leqslant 0, \;\; \chi'(y) \leqslant 0, \;\; \phi_y(t,y) > 0 \\
&\forall y \geqslant \varepsilon, \forall t < t_\varepsilon, \chi''(y) > 0, \;\; \phi_{yy}(t,y) > 0, \;\; \phi_{tyy}(t,y) > 0.
\end{aligned} \tag{III.1.14}$$

$$\phi_y(t_\varepsilon,\nu_\varepsilon) = 0, \;\; \phi_{yy}(t_\varepsilon,\nu_\varepsilon) = 0, \;\; \phi_{tyy}(t_\varepsilon,\nu_\varepsilon) \neq 0, \;\; \phi_{yyy}(t_\varepsilon,\nu_\varepsilon) \neq 0. \tag{III.1.15}$$

Proof. The expressions of ϕ_{ty}, ϕ_{yy}, ϕ_{tyy} are obtained by differentiation of (III.1.4) with respect to t, y and ty. The expression of ϕ_y is obtained by differentiation of (III.1.8).

Now, using this expression for ϕ_{ty}, we obtain

$$\phi_{ty}(t,\varepsilon) = \frac{-2|\ln(\varepsilon)|^\alpha e^{ct}}{(1-\chi^2(\varepsilon))^2} < 1/5|\ln(\varepsilon)|^\alpha, \tag{III.1.16}$$

because

$$|\chi(\varepsilon)| \leqslant \int_{s=\varepsilon/2}^{\varepsilon} |\ln(\varepsilon/2)|^\alpha \leqslant 1/10, \tag{III.1.17}$$

for ε small enough, which leads to (III.1.16). Also, this means that for $t < 1$, and for ε

small enough,

$$(1-\chi(y)) > 4/5, \ \ (1-e^{ct}\chi(y)) > 4/5. \tag{III.1.18}$$

Now, we show the existence of t_ε as in Lemma III.1.3.

$$\phi_y(t,\varepsilon)_{|y=0} = 1, \ \ \phi_{ty}(t,\varepsilon) < -\frac{1}{5}|\ln(\varepsilon)|^\alpha. \tag{III.1.19}$$

Now that means that there exists $\tilde{t}_\varepsilon \leqslant 5\frac{1}{|\ln(\varepsilon)|^\alpha}$ such that $\phi_y(\tilde{t}_\varepsilon,\varepsilon) = 0$.

Hence, the set $\{t < 1|\exists y \ \ \phi_y(t,y) \neq 0\}$ is not empty. By continuity, consider t_ε its minimum, and call v_ε the corresponding y. Then we have

$$t_\varepsilon \leqslant 5\frac{1}{|\ln(\varepsilon)|^\alpha}, \ \ \phi_y(t_\varepsilon,v_\varepsilon) = 0, \ \ \forall t < t_\varepsilon, \phi_y(t,y) > 0. \tag{III.1.20}$$

Now, we show (III.1.14).

$$\chi(y) = \int_{s=0}^{y} -\psi_\varepsilon(s)|\ln(s)|^\alpha ds \leqslant 0. \ \ \chi'(y) = -\psi_\varepsilon(y)|\ln(y)|^\alpha \leqslant 0. \tag{III.1.21}$$

$$\phi_y(t,y)_{|t=0} = 1 > 0, \ \ \phi_y(t,y) \neq 0 \ \text{ when } t < t_\varepsilon \Rightarrow \phi_y(t,y) > 0 \ \text{ when } t < t_\varepsilon. \tag{III.1.22}$$

$$\forall y \geqslant \varepsilon, \ \ \chi''(y) = -\left[\frac{-\alpha}{y}(-\ln(y))^{\alpha-1}\right] > 0. \tag{III.1.23}$$

$$\forall y \geqslant \varepsilon, \ \ \phi_{yy}(t,y) > 0, \tag{III.1.24}$$

Indeed, we plug in (III.1.23) in the expression of ϕ_{yy} given by (III.1.13), as well as (III.1.18), and we obtain the result. We do the same for ϕ_{tyy} and obtain $\phi_{tyy} > 0$.

Lastly, we prove (III.1.15).

By definition of $(t_\varepsilon, v_\varepsilon)$, it is clear that $\phi_y(t_\varepsilon, v_\varepsilon) = 0$. Now, if there exists y_ε such that $\phi_y(t_\varepsilon, y_\varepsilon) < 0$, by continuity we would have the existence of t'_ε and y'_ε such that $\phi_y(t'_\varepsilon, y'_\varepsilon) = 0$ which would contradict the definition of t_ε. Hence, $\phi_y(t_\varepsilon, v_\varepsilon)$ is a maximum of ϕ_y (for a fixed t) and we have $\phi_{yy}(t_\varepsilon, v_\varepsilon) = 0$.

Now, we consider a root $\tilde{v}_\varepsilon$ of $\phi_{tyy}(t_\varepsilon, \cdot)$. we have that

$$\phi_{tyy}(t_\varepsilon, \tilde{v}_\varepsilon) = 0 \Rightarrow \chi''(\tilde{v}_\varepsilon) = \frac{-2\chi'(\tilde{v}_\varepsilon)^2 e^{ct_\varepsilon}}{(1 - e^{ct_\varepsilon}\chi(\tilde{v}_\varepsilon))}. \tag{III.1.25}$$

Plugging (III.1.25) in the expression of ϕ_{yy} given by (III.1.13), we obtain

$$\phi_{yy}(t_\varepsilon, \tilde{v}_\varepsilon) = -\frac{2(e^{ct_\varepsilon} - 1)^2}{c} \frac{\chi'^2(\tilde{v}_\varepsilon)}{(1 - \chi(\tilde{v}_\varepsilon))^2 (1 - e^{ct_\varepsilon}\chi(\tilde{v}_\varepsilon))} \tag{III.1.26}$$

Using (III.1.18) again, we obtain $\phi_{yy}(t_\varepsilon, \tilde{v}_\varepsilon) \neq 0$. Then, using $\phi_{tyy} = 0 \Rightarrow \phi_{yy} \neq 0$, we obtain by contraposite $\phi_{yy} = 0 \Rightarrow \phi_{tyy} \neq 0$.

Now, because for $y > \varepsilon$, $\phi_{yy} > 0$, it means that $v_\varepsilon < \varepsilon$. We can choose ψ_ε such that $\phi_{yyy}(t_\varepsilon, v_\varepsilon) \neq 0$.

□

Remark III.1.5. *Note that these are the key ingredients that were needed to show theorem II.2.1. We should also in theory show an equivalent of proposition I.4.1, but this proof is tedious and also works because* $e^{ct} - 1 \sim_{t \to 0} ct$*, which means* t_ε *decreases to* 0 *at the same speed as in prop. I.4.1. The rest is unchanged because* χ *has been chosen to be the same.*

Lastly, our last preliminary work will be to prove the following estimates, that will hold when $y - v_\varepsilon$ is small enough, and t is close enough to t_ε.

$$\begin{aligned}
&\exists C^1_\varepsilon, C^2_\varepsilon > 0, \;\; -C^1_\varepsilon (y - v_\varepsilon)^2 \leqslant \phi_y(t_\varepsilon, y) \leqslant C^2_\varepsilon (y - v_\varepsilon)^2 \\
&\exists C^1_\varepsilon, C^2_\varepsilon > 0, \;\; C^1_\varepsilon (y - v_\varepsilon)^2 + C^1_\varepsilon (t_\varepsilon - t) \leqslant \phi_y(t, y) \leqslant C^2_\varepsilon (y - v_\varepsilon)^2 + C^2_\varepsilon (t_\varepsilon - t) \\
&\exists C^1_\varepsilon, C^2_\varepsilon > 0, \;\; C^1_\varepsilon (v_\varepsilon - y) \leqslant \phi_{yy}(t_\varepsilon, y) \leqslant C^2_\varepsilon (v_\varepsilon - y)
\end{aligned} \tag{III.1.27}$$

This results are quickly obtained using Taylor expansions and (III.1.14). Now, we do not have a lower bound for ϕ_{yy}. In that situation, we do not have an equivalent of the property $\phi_{yy}(t,y) = \frac{t}{t_\varepsilon}\phi_{yy}(t_\varepsilon, y)$, because the expression of ϕ_{yy} is different.

On the right interval, we have that

$$C^1_\varepsilon(v_\varepsilon - y) - C^1_\varepsilon(t_\varepsilon - t) \leqslant \phi_{yy}(t,y) \leqslant C^2_\varepsilon(v_\varepsilon - y) - C^2_\varepsilon(t_\varepsilon - t), \tag{III.1.28}$$

this will provide an upper bound for the norm but no lower bound, since the sign of the two expressions are different.

III.2 Proof of the blow up as $t \to t_\varepsilon$

Our next goal is to prove theorem III.1.2.

We will proceed by using a cutoff around the x corresponding to v_ε, i.e. $x = \phi(t_\varepsilon, v_\varepsilon)$.

First, we start by defining h_ε, as well as the cutoff functions.

Definition III.2.1. *Let $\psi^1_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying*

$$\begin{cases} \psi^1_\varepsilon(x) = 1 \text{ for } \phi(t_\varepsilon, v_\varepsilon) - \delta_\varepsilon < x < \phi(t_\varepsilon, v_\varepsilon) + \delta_\varepsilon \\ \psi^1_\varepsilon(x) = 0 \text{ for } \phi(t_\varepsilon, v_\varepsilon) + 2\delta_\varepsilon < x \text{ or } x < \phi(t_\varepsilon, v_\varepsilon) - 2\delta_\varepsilon \\ 0 < \psi^1_\varepsilon(x) < 1 \text{ elsewhere,} \end{cases} \tag{III.2.29}$$

and $\psi^2_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying

$$\begin{cases} \psi^2_\varepsilon(x) = 1 \text{ for } -\delta_\varepsilon < x < \delta_\varepsilon \\ \psi^2_\varepsilon(x) = 0 \text{ for } 2\delta_\varepsilon < x \text{ or } x < -2\delta_\varepsilon \\ 0 < \psi^2_\varepsilon(x) < 1 \text{ elsewhere,} \end{cases} \tag{III.2.30}$$

so that $h_\varepsilon : (x_1, x_2) \mapsto v_\varepsilon(t, x_1, x_2)\psi^1_\varepsilon(x_1)\psi^2_\varepsilon(x_2)$ is localized in a square of width $4\delta_\varepsilon$, cut in half by $x_1 = \phi(t_\varepsilon, v_\varepsilon)$; and such that $h_\varepsilon = v_\varepsilon$ in a square of width $2\delta_\varepsilon$, cut in half by $x_1 = v_{t,\varepsilon}$.

Also, we now define $I_\varepsilon(t)$, the integral that will diverge at $t = t_\varepsilon$, thus proving the blow up of $||u_\varepsilon(t,\cdot)||_{H^{11/4}(\mathbb{R}^2)}$.

$$I_\varepsilon(t) = ||h_\varepsilon(t,\cdot)||_{\dot{H}^{7/4}_{x_1}} = \int_{x_1=\phi(t_\varepsilon,\nu_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,\nu_\varepsilon)+2\delta_\varepsilon} \int_{x_2=-2\delta_\varepsilon}^{2\delta_\varepsilon} \left(\frac{\partial^2(v\psi^1_\varepsilon\psi^2_\varepsilon)}{\partial x_1^2}\right)(t,x_1,x_2)\cdot$$
$$\left[\int_{y=\phi(t_\varepsilon,\nu_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,\nu_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda} \left(\frac{\partial^2(v\psi^1_\varepsilon\psi^2_\varepsilon)}{\partial x_1^2}\right)(t,y,x_2)dy\right] dx_2 dx_1. \quad \text{(III.2.31)}$$

We compute the involved derivatives of v.

$$\begin{aligned} v(t,\phi(t,y)) &= \chi(y)e^{ct} \\ v_x(t,\phi(t,y)) &= \frac{e^{ct}\chi'(y)}{\phi_y(t,y)} \\ v_{xx}(t,\phi(t,y)) &= \frac{e^{ct}\left(\chi''(y)\phi_y(t,y)-\chi'(y)\phi_{yy}(t,y)\right)}{(\phi_y(t,y))^3} \end{aligned} \quad \text{(III.2.32)}$$

We have the following estimation

$$\begin{aligned} & h_\varepsilon(t,x_1,x_2) = \psi^1_\varepsilon(x_1)\psi^2_\varepsilon(x_2)v_\varepsilon(t,x), \\ & \frac{\partial h_\varepsilon}{\partial_{x_1}}(t,x_1,x_2) = \psi^2_\varepsilon(x_2)\Big[\psi^{1,'}_\varepsilon(x_1)v_\varepsilon(t,x_1) + \psi^1_\varepsilon(x_1)v_{\varepsilon,x}(t,x_1),\Big] \\ & \frac{\partial^2 h_\varepsilon}{\partial^2_{x_1}}(t,x_1,x_2)\psi^2_\varepsilon(x_2)\Big[\psi^{1,''}_\varepsilon(x_1)v_\varepsilon(t,x_1) + 2\psi^{1,'}_\varepsilon(x_1)v_{\varepsilon,x}(t,x_1) + \psi^1_\varepsilon(x_1)v_{\varepsilon,xx}(t,x_1)\Big], \\ & \Rightarrow \exists C_1,C_2>0,\ \ C_1\psi^2_\varepsilon(x_2)v_{\varepsilon,xx}(t,x_1) \leqslant \frac{\partial^2 h_\varepsilon}{\partial^2_{x_1}}(t,x_1,x_2) \leqslant C_2\psi^2_\varepsilon(x_2)v_{\varepsilon,xx}(t,x_1)\ (i) \\ & \Rightarrow \exists C_1>0,\ \ \left|\frac{\partial^2 h_\varepsilon}{\partial^2_{x_1}}(t,x_1,x_2)\right| \leqslant C_1\left|\psi^2_\varepsilon(x_2)v_{\varepsilon,xx}(t,x_1)\right|,\ (ii) \end{aligned}$$
$$\text{(III.2.33)}$$

Now, with $K_\varepsilon = \int_{x_2=-2\delta_\varepsilon}^{2\delta_\varepsilon} (\psi_\varepsilon(x_2))^2\,dx_2$,

$$I_\varepsilon(t) = K_\varepsilon \int_{x_1=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,x_1,x_2)\cdot \left[\int_{y=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda}\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,y,x_2)dy\right]dx_2dx_1. \quad \text{(III.2.34)}$$

We now split the domain of the first integral into three, corresponding to the three following integration domains.

$$\begin{aligned}I_\varepsilon(t) =& K_\varepsilon \int_{x_1=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)-\delta_\varepsilon} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,x_1)\\ &\cdot \int_{y=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda}\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,y)dydx_1\\ &+K_\varepsilon \int_{x_1=\phi(t_\varepsilon,v_\varepsilon)-\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+\delta_\varepsilon} \left(\frac{\partial^2 v}{\partial x_1^2}\right)(t,x_1)\\ &\cdot \int_{y=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda}\left(\frac{\partial^2 v}{\partial x_1^2}\right)(t,y)dydx_1\\ &+K_\varepsilon \int_{x_1=\phi(t_\varepsilon,v_\varepsilon)+\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,x_1)\\ &\cdot \int_{y=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2+2\lambda}\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,y)dydx_1.\end{aligned} \quad \text{(III.2.35)}$$

The first difference with the previous argument is that the expressions of ϕ, ϕ_y and ϕ_{yy} are not the same since they depend on the value of v, which itself depends on the underlying ODE. The second difference, is that the value of v is not the same because of the underlying ODE, which will have an impact on the computations when we will show the blow up. Fortunately, it does not make a big difference in the computations.

In the following computations, we use the function $\zeta_\varepsilon^i(t)$ that will be defined in the coming lemma.

$$\begin{aligned}
I_\varepsilon(t) = K_\varepsilon \int_{y_1=\zeta_\varepsilon^1(t)}^{\zeta_\varepsilon^2(t)} &\phi_{\varepsilon,y}(t,y_1)\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,\phi_\varepsilon(t,y_1)) \\
&\cdot \int_{y_2=\zeta_\varepsilon^1(t)}^{\zeta_\varepsilon^4(t)} |\phi_\varepsilon(t,y_1)-\phi_\varepsilon(t,y_2)|^{-1/2+2\lambda}\phi_{\varepsilon,y}(t,y_2)\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,\phi_\varepsilon(t,y_2))dydx_1 \\
+K_\varepsilon \int_{y_1=\zeta_\varepsilon^2(t)}^{\zeta_\varepsilon^3(t)} &\phi_{\varepsilon,y}(t,y_1)\left(\frac{\partial^2(v)}{\partial x_1^2}\right)(t,\phi_\varepsilon(t,y_1)) \\
&\cdot \int_{y_2=\zeta_\varepsilon^1(t)}^{\zeta_\varepsilon^4(t)} |\phi_\varepsilon(t,y_1)-\phi_\varepsilon(t,y_2)|^{-1/2+2\lambda}\phi_{\varepsilon,y}(t,y_2)\left(\frac{\partial^2(v)}{\partial x_1^2}\right)(t,\phi_\varepsilon(t,y_2))dydx_1 \\
+K_\varepsilon \int_{y_1=\zeta_\varepsilon^3(t)}^{\zeta_\varepsilon^4(t)} &\phi_{\varepsilon,y}(t,y_1)\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,\phi_\varepsilon(t,y_1)) \\
&\cdot \int_{y_2=\zeta_\varepsilon^1(t)}^{\zeta_\varepsilon^4(t)} |\phi_\varepsilon(t,y_1)-\phi_\varepsilon(t,y_2)|^{-1/2+2\lambda}\phi_{\varepsilon,y}(t,y_2)\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,\phi_\varepsilon(t,y_2))dydx_1
\end{aligned}$$

$$= K_\varepsilon(I_\varepsilon^1(t)+I_\varepsilon^2(t)+I_\varepsilon^3(t)). \tag{III.2.36}$$

Again, we will show that $|I_2(t)| >> |I_1(t)|+|I_3(t)|$ and $I_2(t)\to+\infty$ as $t\to t_\varepsilon$. We will use all the following results without proving it, as it is very similar to what we obtained in (II.2.24) and (II.2.25) in the previous chapter.

Lemma III.2.2. *On an interval centered at* $(t_\varepsilon, \nu_\varepsilon)$, *we have the following.*

If we define

$$\begin{aligned}
\zeta_\varepsilon^1(t) &= \phi_t^{-1}(\phi(t_\varepsilon,\nu_\varepsilon)-2\delta_\varepsilon), \\
\zeta_\varepsilon^2(t) &= \phi_t^{-1}(\phi(t_\varepsilon,\nu_\varepsilon)-\delta_\varepsilon), \\
\zeta_\varepsilon^3(t) &= \phi_t^{-1}(\phi(t_\varepsilon,\nu_\varepsilon)+\delta_\varepsilon), \\
\zeta_\varepsilon^4(t) &= \phi_t^{-1}(\phi(t_\varepsilon,\nu_\varepsilon)+2\delta_\varepsilon),
\end{aligned} \tag{III.2.37}$$

we can choose δ_ε and t_ε^1 such that for $t \in]t_\varepsilon^1, t_\varepsilon[$, we have

$$\nu_\varepsilon - \eta_\varepsilon < \zeta_\varepsilon^1(t) < \zeta_\varepsilon^2(t) < \nu_\varepsilon < \zeta_\varepsilon^3(t) < \zeta_\varepsilon^4(t) < \nu_\varepsilon + \eta_\varepsilon. \tag{III.2.38}$$

There exists ζ_ε^{2+} and ζ_ε^{3-} such that for δ_ε small enough, and t close enough to t_ε,

$$\nu_\varepsilon - \eta_\varepsilon < \zeta_\varepsilon^1(t) < \zeta_\varepsilon^2(t) < \zeta_\varepsilon^{2+} < \nu_\varepsilon < \zeta_\varepsilon^{3-} < \zeta_\varepsilon^3(t) < \zeta_\varepsilon^4(t) < \nu_\varepsilon + \eta_\varepsilon. \tag{III.2.39}$$

Let us first do the work for $I_1(t)$. The case of $I_3(t)$ is similar.

Using the estimation given by (III.2.33) and the expression of v_{xx} given by (III.2.32), as well as (III.2.38), we obtain

$$\left|I_\varepsilon^1(t)\right| \leqslant \int_{x_1=\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2+}} \left|\frac{\chi''(x_1)}{\phi_y(t,x_1)} - \frac{\chi'(x_1)\phi_{yy}(t,x_1)}{\phi_y(t,x_1)^2}\right| \left|\int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\nu_\varepsilon+\eta_\varepsilon} \right| M_\varepsilon^2 \frac{e^{2ct}}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}} \cdot \left[\frac{\chi''(y)}{\phi_y(t,y)} - \frac{\chi'(y)\phi_{yy}(t,y)}{\phi_y(t,y)^2}\right]\Bigg| \tag{III.2.40}$$

We study the inner integral of (II.2.26) for $x \in [\nu_\varepsilon - \eta_\varepsilon, \zeta_\varepsilon^{2+}]$. Now, both $\frac{1}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}}$ and $v_{\varepsilon,yy}(t,y)$ are unbounded in the second integral, but the regions where they are unbounded are uniformly disjoint in t because of (II.2.25), so we split again the domain, and use $e^{ct} \leqslant M_\varepsilon$.

$$\begin{aligned}\int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\nu_\varepsilon+\eta_\varepsilon} \frac{e^{2ct}}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}} v_{\varepsilon,yy}(t,y) &\leqslant_{(*1)} M_\varepsilon \int_{\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2+}} \frac{1}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}} \\ &+ M_\varepsilon \int_{\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} |\phi_y(t,y) v_{\varepsilon,xx}(t,\phi(t,y))| + M_\varepsilon \int_{\zeta_\varepsilon^{3+}}^{\nu_\varepsilon+\eta_\varepsilon} \frac{1}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}} \\ &\leqslant M_\varepsilon [(i)+(ii)+(iii)]\end{aligned} \tag{III.2.41}$$

In $(*1)$, we used (III.2.32), (III.1.27) and (III.1.28) and the fact that $|\nu_\varepsilon - y| \geqslant C_\varepsilon$ on the set $[\nu_\varepsilon - \eta_\varepsilon, \zeta_\varepsilon^{2+}]$.

Now we give a majoration for (i) in (III.2.41). The case (iii) is trivial.

Now, using the mean value theorem as well as (III.1.27), and $|c-\nu_\varepsilon| \geqslant |\zeta_\varepsilon^{2+}-\nu_\varepsilon|$, we obtain

$$|(i)| \leqslant M_\varepsilon \int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2+}} \frac{1}{|x_1-y|^{1/2-2\lambda}} \frac{1}{|\phi_y(t,c)|^{1/2-2\lambda}} \leqslant M_\varepsilon \int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\zeta_\varepsilon^{2+}} \frac{1}{|x_1-y|^{1/2-2\lambda}} \leqslant M_\varepsilon. \tag{III.2.42}$$

Now, we study (ii) using the expression of $v_{\varepsilon,xx}$ given by (III.2.32), and the results obtained in the preliminary work. We need to compute the estimates for the two terms separately.

Using the expressions from (III.2.32) and (III.2.41) as well as the inequalities obtained in (III.1.27) and (III.1.28), we obtain

$$\begin{aligned} |(ii)| \leqslant M_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{1}{|\phi_y(t,y)|^2} + M_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{|\phi_{yy}(t,y)|}{|\phi_y(t,y)|^3} \leqslant \\ M_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{1}{(|y-\nu_\varepsilon|^2+(t_\varepsilon-t))} + M_\varepsilon \int_{y=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \frac{|y-\nu_\varepsilon|+(t_\varepsilon-t)}{(|y-\nu_\varepsilon|^2+(t_\varepsilon-t))^2} \end{aligned} \tag{III.2.43}$$

Now, because we have the following primitives

$$\int \frac{1}{(x^2+a)} = \frac{\arctan\left(\frac{x}{\sqrt{a}}\right)}{\sqrt{a}}, \quad \int \frac{x}{(x^2+a)^2} = -\frac{1}{2\,(x^2+a)}, \tag{III.2.44}$$

We obtain that

$$|(i)+(ii)+(iii)| \leqslant C_\varepsilon \frac{1}{(t_\varepsilon-t)}. \tag{III.2.45}$$

Finally, we have the following upper bound for $I^1_\varepsilon(t)$, using $|x_1 - v_\varepsilon| \geqslant C_\varepsilon$,

$$|I^1_\varepsilon(t)| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)} \int_{x_1 = v_\varepsilon - \eta_\varepsilon}^{\zeta_\varepsilon^{2-}} \left[\frac{1}{|\phi_y(t,x_1)|} + \frac{|\phi_{yy}(t,x_1)|}{|\phi_{yy}(t,x_1)|^2} \right] \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)}. \tag{III.2.46}$$

By symmetry, we also have $I^3_\varepsilon(t) \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)}$.

Now, we exhibit a lower bound for $I^2_\varepsilon(t)$.

We can write from (III.2.32) and (III.2.33),

$$\begin{aligned}
&\int_{x_1=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \int_{y=v_\varepsilon-\eta_\varepsilon}^{v_\varepsilon+\eta_\varepsilon} \frac{1}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}} \phi_y(t,x_1) v_{xx}(t,x_1) \phi_y(t,y) v_{xx}(t,y) \\
&= \int_{x_1=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \int_{y=v_\varepsilon-\eta_\varepsilon}^{v_\varepsilon+\eta_\varepsilon} \frac{1}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}} \Bigg[\frac{\chi''(x_1)}{\phi_y(t,x_1)} \frac{\chi''(y)}{\phi_y(t,y)} - \frac{\chi''(x_1)}{\phi_y(t,x_1)} \frac{\chi'(y)\phi_{yy}(t,y)}{\phi_y(t,y)^2} \\
&- \frac{\chi''(y)}{\phi_y(t,y)} \frac{\chi'(x_1)\phi_{yy}(t,x_1)}{\phi_y(t,x_1)^2} + \frac{\chi'(x_1)\phi_{yy}(t,x_1)}{\phi_y(t,x_1)^2} \frac{\chi'(y)\phi_{yy}(t,y)}{\phi_y(t,y)^2} \Bigg] = (i)+(ii)+(iii)+(iv)
\end{aligned} \tag{III.2.47}$$

We will show that $(iv) >> |(i)| + |(ii)| + |(iii)|$ as $t \to t_\varepsilon$.

We will have to do additional steps in comparison with chapter II. Indeed, from III.1.27 and doing a method similar to what we did with ϕ_y, we can not infer any lower bound for the norm of ϕ_{yy}.

First, we perform a Taylor expansion of ϕ_{yy} as a function of $\mathbb{R}^2$ near $(t_\varepsilon, v_\varepsilon)$.

$$\begin{aligned}
\phi_{yy}(t,y) &= C^1_\varepsilon(y - v_\varepsilon) + C^2_\varepsilon(t_\varepsilon - t) + (y - v_\varepsilon)^2 f_1(t,y) + (y - v_\varepsilon)(t_\varepsilon - t) f_2(t,y) \\
&\quad + (t_\varepsilon - t)^2 f_3(t,y) = C^1_\varepsilon(y - v_\varepsilon) + C^2_\varepsilon(t_\varepsilon - t) + g(t,y),
\end{aligned} \tag{III.2.48}$$

where $f_1, f_2, f_3 \in L^\infty(w)$, for some open set w containing $(t_\varepsilon, v_\varepsilon)$ and depending only on ε.

First, we look at (iv). Again, because of $(y-\nu_\varepsilon)$ antisymmetric properties but the fact that $\frac{1}{\sqrt{|\phi(t,x_1)-\phi(t,y)|}}$ will concentrate the weight near $x_1 = y$, we have to split the domain into four parts, using the same idea as in chapter II.

$$\begin{aligned}\alpha &= \{(x_1,y)\in[\zeta_\varepsilon^{2+},\nu_\varepsilon]\times[\nu_\varepsilon-\eta_\varepsilon,\nu_\varepsilon]\}, \;\; \beta = \{(x_1,y)\in[\nu_\varepsilon,\zeta_\varepsilon^{3-}]\times[\nu_\varepsilon-\eta_\varepsilon,\nu_\varepsilon]\}\\ \gamma &= \{(x_1,y)\in[\zeta_\varepsilon^{2+},\nu_\varepsilon]\times[\nu_\varepsilon,\nu_\varepsilon+\eta_\varepsilon]\}, \;\; \delta = \{(x_1,y)\in[\nu_\varepsilon,\zeta_\varepsilon^{3-}]\times[\nu_\varepsilon,\nu_\varepsilon+\eta_\varepsilon]\}\end{aligned} \tag{III.2.49}$$

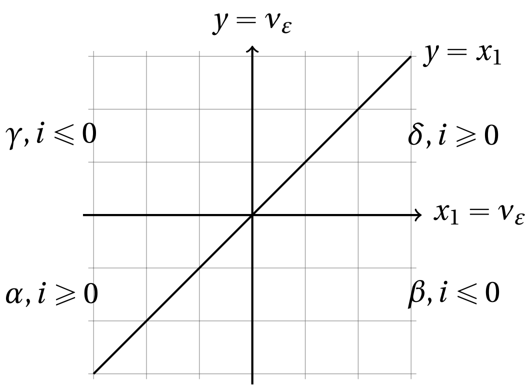


Figure III.1 – Definition of α, β, γ and δ

And we have

$$(iv) = \int\int_\alpha i(x_1,y) + \int\int_\beta i(x_1,y) + \int\int_\gamma i(x_1,y) + \int\int_\delta i(x_1,y). \tag{III.2.50}$$

We first consider the first two terms of ϕ_{yy} in (III.2.48) only, and look at the contribution of g later on. Call $\tilde{i}(x_1,y)$ the corresponding integrand. This also defines the corresponding integrals $(iv)_1$ and $(iv)_2$.

We will regroup the integral corresponding to δ and β. The symmetric case (γ and α) is identical. First, we remark that by (III.2.48), we have the two following inequalities (following a change of variables)

$$\begin{aligned}\int\int_{\delta\cup\beta}\tilde{i}(x_1,y) \geqslant C_\varepsilon \int\int_{\delta\cup\beta} & \frac{1}{|\phi(t,v_\varepsilon+x_1)-\phi(t,v_\varepsilon+y)|^{1/2-2\lambda}} \\ & \cdot \frac{x_1}{((x_1)^2+(t_\varepsilon-t))^2}\frac{y}{(y^2+(t_\varepsilon-t))^2} \\ + \int\int_{\delta\cup\beta} & \frac{2C_\varepsilon}{|\phi(t,v_\varepsilon+x_1)-\phi(t,v_\varepsilon+y)|^{1/2-2\lambda}}\frac{(t_\varepsilon-t)x_1}{(x_1^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2} \\ + \int\int_{\delta\cup\beta} & \frac{2C_\varepsilon}{|\phi(t,v_\varepsilon+x_1)-\phi(t,v_\varepsilon+y)|^{1/2-2\lambda}}\frac{(t_\varepsilon-t)y}{(x_1^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2} \\ + \int\int_{\delta\cup\beta} & \frac{2C_\varepsilon}{|\phi(t,v_\varepsilon+x_1)-\phi(t,v_\varepsilon+y)|^{1/2-2\lambda}}\frac{(t_\varepsilon-t)^2}{(x_1^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2},\end{aligned} \tag{III.2.51}$$

and

$$\begin{aligned}\int\int_{\delta\cup\beta}\tilde{i}(x_1,y) \leqslant C_\varepsilon \int\int_{\delta\cup\beta} & \frac{1}{|\phi(t,v_\varepsilon+x_1)-\phi(t,v_\varepsilon+y)|^{1/2-2\lambda}} \\ & \cdot \frac{x_1}{((x_1)^2+(t_\varepsilon-t))^2}\frac{y}{(y^2+(t_\varepsilon-t))^2} \\ + \int\int_{\delta\cup\beta} & \frac{2C'_\varepsilon}{|\phi(t,v_\varepsilon+x_1)-\phi(t,v_\varepsilon+y)|^{1/2-2\lambda}}\frac{(t_\varepsilon-t)x_1}{(x_1^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2} \\ + \int\int_{\delta\cup\beta} & \frac{2C'_\varepsilon}{|\phi(t,v_\varepsilon+x_1)-\phi(t,v_\varepsilon+y)|^{1/2-2\lambda}}\frac{(t_\varepsilon-t)y}{(x_1^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2} \\ + \int\int_{\delta\cup\beta} & \frac{2C'_\varepsilon}{|\phi(t,v_\varepsilon+x_1)-\phi(t,v_\varepsilon+y)|^{1/2-2\lambda}}\frac{(t_\varepsilon-t)^2}{(x_1^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2}.\end{aligned} \tag{III.2.52}$$

Hence, we will considerate separately the first term involved in (III.2.51) and (III.2.52), and the three others terms. The aforementioned term will give a lower bound and will be the term of the highest order. The three other terms will be of a smaller order. We first proceed with a upper bound of the three last terms.

Using $|\phi(t,c(x,y))| \geqslant C_\varepsilon(t_\varepsilon - t)$,

$$
\begin{aligned}
&\int\int_{\delta\cup\beta} \frac{2C_\varepsilon}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}} \frac{(t_\varepsilon-t)|x_1|}{(x_1^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2} \\
&\quad\leqslant \int\int_{\delta\cup\beta} \frac{2C_\varepsilon(t_\varepsilon-t)}{(t_\varepsilon-t)^{1/2-2\lambda}} \frac{1}{|x-y|^{1/2-2\lambda}} \frac{|x|}{(x^2+(t_\varepsilon-t))^2} \frac{1}{(y^2+(t_\varepsilon-t))^2} \\
&\leqslant_{H\ddot{o}lder} C_\varepsilon (t_\varepsilon-t)^{1/2+2\lambda} \left(\int\int_\alpha \frac{1}{|x-y|^{3/4-3\lambda}}\right)^{2/3} \left(\int_0^\infty \frac{x^3}{(x^2+(t_\varepsilon-t))^6}\right)^{1/3} \\
&\qquad\cdot \left(\int_0^\infty \frac{1}{(y^2+(t_\varepsilon-t))^6}\right)^{1/3} \\
&\qquad\qquad \leqslant C_\varepsilon \frac{(t_\varepsilon-t)^{1/2-2\lambda}}{(t_\varepsilon-t)^{4/3+11/6}} \leqslant C_\varepsilon \frac{1}{(t_\varepsilon-t)^{8/3-2\lambda}} \qquad \text{(III.2.53)}
\end{aligned}
$$

We will show later on that our first term is greater than $C_\varepsilon/(t_\varepsilon-t)^{11/4-C\lambda}$. Note that our margin is only $(t_\varepsilon-t)^{1/12}$. For the second cross term, one can replace x by y in (III.2.53). Now we do the last term of (III.2.51) and (III.2.52).

$$
\begin{aligned}
&\int\int_{\beta\cup\delta} \frac{2C_\varepsilon}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}} \frac{(t_\varepsilon-t)^2}{(x_1^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2} \\
&\quad\leqslant \int\int_{\beta\cup\delta} \frac{2C_\varepsilon(t_\varepsilon-t)^2}{(t_\varepsilon-t)^{1/2-2\lambda}} \frac{1}{|x-y|^{1/2-2\lambda}} \frac{1}{(x^2+(t_\varepsilon-t))^2} \frac{1}{(y^2+(t_\varepsilon-t))^2} \\
&\leqslant_{H\ddot{o}lder} C_\varepsilon (t_\varepsilon-t)^{1.5+2\lambda} \left(\int\int_{\beta\cup\delta} \frac{1}{|x-y|^{3/4-3\lambda}}\right)^{2/3} \left(\int_0^\infty \frac{1}{(x^2+(t_\varepsilon-t))^6}\right)^{1/3} \\
&\qquad\cdot \left(\int_0^\infty \frac{1}{(y^2+(t_\varepsilon-t))^6}\right)^{1/3} \\
&\qquad\qquad \leqslant C_\varepsilon \frac{(t_\varepsilon-t)^{3/2+2\lambda}}{(t_\varepsilon-t)^{4/3+11/6}} \leqslant_{(*1)} C_\varepsilon \frac{1}{(t_\varepsilon-t)^{13/6-2\lambda}} \qquad \text{(III.2.54)}
\end{aligned}
$$

Remark III.2.3. *Note that for estimates such as* $(*1)$*, we explicitly compute the anti-derivative. We do not write it because it is long and not so relevant. For instance, a primitive of* $\frac{1}{(x^2+a)^6}$ *is*

$$\frac{63\arctan\left(\frac{x}{\sqrt{a}}\right)}{256a^{\frac{11}{2}}}+\frac{315x^9+1470ax^7+2688a^2x^5+2370a^3x^3+965a^4x}{1280a^5x^{10}+6400a^6x^8+12800a^7x^6+12800a^8x^4+6400a^9x^2+1280a^{10}},$$

and we use

$$\int_0^{\zeta_\varepsilon^{3-}-\nu_\varepsilon}\frac{1}{(x^2+(t_\varepsilon-t))^6}\leqslant\int_0^{\infty}\frac{1}{(x^2+(t_\varepsilon-t))^6}=\frac{63\pi}{512(t_\varepsilon-t)^{\frac{11}{2}}}.$$

For the integrals that can not be computed explicitly, we proceed with a change of variable $x=\frac{x}{\sqrt{(t_\varepsilon-t)}}$.

Now we deal with the first term of (III.2.51) and (III.2.52). We hence consider the term

$$\begin{aligned}
J=&\int\int_{\delta\cup\beta}\frac{1}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}}\frac{\chi'(x_1)x_1}{\phi_y(t,x_1)^2}\frac{\chi'(y)y}{\phi_y(t,y)^2}\\
=&\int_{x_1=0}^{\kappa_\varepsilon}\int_{y=0}^{\kappa_\varepsilon}\Bigg[\frac{\chi'(\nu_\varepsilon+y)\cdot y\cdot\chi'(\nu_\varepsilon+x_1)\cdot x_1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}\phi_y(t,\nu_\varepsilon+x_1)^2\phi_y(t,\nu_\varepsilon+y)^2}\\
&-\frac{\chi'(\nu_\varepsilon-y)\cdot y\cdot\chi'(\nu_\varepsilon+x_1)\cdot x_1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}\phi_y(t,\nu_\varepsilon-y)^2\phi_y(t,\nu_\varepsilon+x_1)^2}\Bigg]
\end{aligned}\tag{III.2.55}$$

From now on, the computations will be identical to a part that have been done for (iv) in the previous chapter. We have included it in order to show that the computations still hold.

We similarly decompose the integrand as follows.

$$
\begin{aligned}
&\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}}\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2}\frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon+y)^2}\\
&\quad-\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}}\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2}\frac{\chi'(\nu_\varepsilon-y)\cdot y}{\phi_y(t,\nu_\varepsilon-y)^2}\\
&\quad=\Bigg[\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}}\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2}\frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon+y)^2}\\
&\quad-\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}}\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2}\frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon+y)^2}\Bigg]\\
&\quad+\Bigg[\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}}\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2}\frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon+y)^2}\\
&\quad-\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}}\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2}\frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon-y)^2}\Bigg]\\
&\quad+\Bigg[\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}}\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2}\frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon-y)^2}\\
&\quad-\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}}\frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2}\frac{\chi'(\nu_\varepsilon-y)\cdot y}{\phi_y(t,\nu_\varepsilon-y)^2}\Bigg]\\
&\quad=D_1+D_2+D_3. \qquad \text{(III.2.56)}
\end{aligned}
$$

To shorten a bit the notations, we will denote

$$
\alpha_\pm=|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon\pm y)|. \qquad \text{(III.2.57)}
$$

For D_1, we write

$$\frac{1}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)|^{1/2-2\lambda}}-\frac{1}{|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon-y)|^{1/2-2\lambda}}$$
$$=\frac{\alpha_--\alpha_+}{\alpha_+^{1/2-2\lambda}\alpha_-^{1/2-2\lambda}\left(\alpha_+^{1/2+2\lambda}+\alpha_-^{1/2+2\lambda}\right)}+\frac{\alpha_+^{4\lambda}-\alpha_-^{4\lambda}}{\left(\alpha_+^{1/2+2\lambda}+\alpha_-^{1/2+2\lambda}\right)} \quad \text{(III.2.58)}$$

It is clear that D_1 is nonnegative when $x_1 \geqslant y$. We now provide a lower bound for $D_{1,x_1\leqslant y}$. We will show that it is positive up to a smaller order term, and provide a lower bound for the positive term. We hence now consider $x_1 \leqslant y$.

We write,

$$\alpha_--\alpha_+=|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon-y)|-|\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)|$$
$$=2\phi_\varepsilon(t,\nu_\varepsilon+x_1)-\phi_\varepsilon(t,\nu_\varepsilon+y)-\phi_\varepsilon(t,\nu_\varepsilon-y)=\int_{s=-y}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)-\int_{s=x_1}^{y}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)$$
$$=\int_{s=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)+\int_{s=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-s)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)\right). \quad \text{(III.2.59)}$$

We will use this idea to provide a lower bound for D_1.

$$\frac{1}{\alpha_+^{1/2-2\lambda}}-\frac{1}{\alpha_-^{1/2-2\lambda}}=\int_{s=\alpha_-}^{\alpha_+}-(\frac{1}{2}-2\lambda)\frac{1}{s^{3/2-2\lambda}}=C\int_{s=\alpha_+}^{\alpha_-}s^{-3/2+2\lambda}$$
$$=C\int_{s=\alpha_+}^{\alpha_++\int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)}s^{-3/2+2\lambda}$$
$$+C\int_{s=\alpha_++\int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)}^{\alpha_++\int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)+\int_{z=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)}s^{-3/2+2\lambda}=(i)+(ii),$$
$$\text{(III.2.60)}$$

where (i) is nonnegative and (ii) is small. First, we make the following upper bound for $|(ii)|$. If $\int_{x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon-s)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+s))>0$, then (ii) is nonnegative. Otherwise, we have $\alpha_++\int_{s=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)+\int_{s=x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon-s)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+s))\leqslant$

$\alpha_+ + \int_{s=-x_1}^{x_1} \phi_{\varepsilon,y}(t, \nu_\varepsilon + s)$. We then proceed as follows

$$\left| \int_{s=\alpha_+ + \int_{-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon+s)}^{\alpha_+ + \int_{z=-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon+z) + \int_{z=x_1}^{y} \left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z) - \phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)} s^{-3/2-2\lambda} \right|$$
$$= \left| \int_{s=\alpha_-}^{\alpha_- - \int_{z=x_1}^{y} \left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z) - \phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)} s^{-3/2+2\lambda} \right|$$
$$\leqslant \left| \int_{z=x_1}^{y} \left(\phi_{\varepsilon,y}(t,\nu_\varepsilon - z) - \phi_{\varepsilon,y}(t,\nu_\varepsilon + z)\right) \right| \cdot \alpha_-^{-3/2+2\lambda}. \quad \text{(III.2.61)}$$

From the Taylor expansion, we obtain for x, y small enough (only depending on ε),

$$\left| \int_{z=x_1}^{y} \left(\phi_{\varepsilon,y}(t,\nu_\varepsilon - s) - \phi_{\varepsilon,y}(t,\nu_\varepsilon + s)\right) \right| \leqslant C_1 \left(y^4 - x_1^4 + (t_\varepsilon - t)^2 + (t_\varepsilon - t)(y^2 - x_1^2)\right). \quad \text{(III.2.62)}$$

This means that we obtain (up to a nonnegative contribution)

$$|(ii)| \leqslant C \left(y^4 + x_1^4 + (t_\varepsilon - t)\left(x_1^2 + y^2\right) + (t_\varepsilon - t)^2\right) \cdot \frac{1}{\alpha_-^{3/2-2\lambda}}. \quad \text{(III.2.63)}$$

We now go on with the lower bound for (i). Using the expression of (i) provided by (III.2.60) as well as the mean value theorem, we obtain

$$(i) = \int_{s=\alpha_+}^{\alpha_+ + \int_{z=-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon+z)} s^{-3/2+2\lambda}$$
$$\geqslant \left(\int_{z=-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon + z) \right) \cdot \frac{1}{\left(\phi_\varepsilon(t,\nu_\varepsilon + y) - \phi_\varepsilon(t,\nu_\varepsilon - x_1)\right)^{3/2-2\lambda}}.$$
$$\geqslant C x_1 \phi_{\varepsilon,y}(t,c_1) \cdot \frac{1}{(y + x_1)^{3/2-2\lambda} \cdot \phi_{\varepsilon,y}(t,c_2)^{3/2-2\lambda}}. \quad \text{(III.2.64)}$$

Now, since we have $x_1 \leqslant y$, we obtain from (II.2.16)

$$\phi_{\varepsilon,y}(t, c_1) \geqslant C(t_\varepsilon - t), \tag{III.2.65}$$

and

$$\phi_{\varepsilon,y}(t, c_2) \leqslant C\left((t_\varepsilon - t) + y^2\right). \tag{III.2.66}$$

Using (III.2.65) and (III.2.66) inside of (III.2.64), we obtain

$$(i) \geqslant C \frac{x_1(t_\varepsilon - t)}{(y + x_1)^{3/2-2\lambda} \cdot ((t_\varepsilon - t) + y^2)^{3/2-2\lambda}}. \tag{III.2.67}$$

Now, we obtain for D_1,

$$\begin{aligned}
D_1 &= D_{x_1 \geqslant y} + D_{x_1 \leqslant y} \geqslant D_{x_1 \leqslant y} \\
&\geqslant C \iint_{x_1 \leqslant y} \frac{x_1(t_\varepsilon - t)}{(y + x_1)^{3/2-2\lambda} \cdot ((t_\varepsilon - t) + y^2)^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{x^4}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{y^4}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t) \cdot x_1^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t) \cdot y^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&\quad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t)^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1) \cdot x_1}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y) \cdot y}{\phi_{\varepsilon,y}(t, \nu_\varepsilon + y)^2} \\
&= A_1 - B_1 - B_2 - B_3 - B_4 - B_5.
\end{aligned} \tag{III.2.68}$$

Note that the proof also works for $\lambda = 0$. We will now show that $A_1 \to \infty$, and that J_2,

J_3, J_4, D_2, D_3, B_1, B_2, B_3, B_4 and B_5 are of a smaller order. We first consider A_1.

Using $|\phi_{\varepsilon,y}(t,c(x_1,y))| \leqslant M_\varepsilon\left((x_1)^2+(t_\varepsilon-t)\right)$, on $y>x_1$, we have from (III.2.68) and the new change of variable $(x,y)=(r-z,r+z)$,

$$\begin{aligned}
A_1 &\geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2}\int_{z=0}^{r} \frac{(r-z)(t_\varepsilon-t)}{r^{3/2-2\lambda}} \frac{r+z}{\left((r+z)^2+(t_\varepsilon-t)\right)^{7/2-2\lambda}} \frac{r-z}{\left((r-z)^2+(t_\varepsilon-t)\right)^2} \\
&\geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2} \frac{1}{r^{1/2-2\lambda}} \frac{(t_\varepsilon-t)}{\left((2r)^2+(t_\varepsilon-t)\right)^{11/2-2\lambda}} \int_{z=0}^{r} (r-z)^2 \\
&\geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2} \frac{(t_\varepsilon-t)\cdot r^{5/2+2\lambda}}{\left((2r)^2+(t_\varepsilon-t)\right)^{11/2-2\lambda}} \geqslant \frac{C_\varepsilon\cdot(t_\varepsilon-t)}{(t_\varepsilon-t)^{15/4-3\lambda}} \\
&\geqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{11/4-3\lambda}}. \qquad \text{(III.2.69)}
\end{aligned}$$

We now proceed with B_1. The case of B_2 is similar. We have since $\phi_{\varepsilon,y}(t,x)\geqslant(t_\varepsilon-t)$, with the mean value theorem

$$\begin{aligned}
|B_1| &\leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2}\int_{z=0}^{r} \frac{(r-z)^4}{r^{3/2-2\lambda}} \\
&\qquad \cdot \frac{r+z}{\left((r+z)^2+(t_\varepsilon-t)\right)^2} \frac{r-z}{\left((r-z)^2+(t_\varepsilon-t)\right)^2} \\
&\leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^4}{r^{1/2-2\lambda}(r^2+(t_\varepsilon-t))^2} \int_{z=0}^{r} \frac{r-z}{((r-z)^2+(t_\varepsilon-t))^2}. \qquad \text{(III.2.70)}
\end{aligned}$$

Now, because

$$\int_{s=0}^{\infty} \frac{s}{(s^2+(t_\varepsilon-t))^2} \leqslant \frac{C}{(t_\varepsilon-t)}, \qquad \text{(III.2.71)}$$

(III.2.70) yields

$$|B_1| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{5/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^{7/2+2\lambda}}{(r^2+(t_\varepsilon - t))^2} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{10/4-2\lambda}}. \tag{III.2.72}$$

We now proceed with B_3. The case of B_4 is similar. We have since $\phi_{\varepsilon,y}(t,x) \geqslant (t_\varepsilon - t)$, with the mean value theorem

$$\begin{aligned} |B_3| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \int_{z=0}^{r} & \frac{(t_\varepsilon - t)\cdot(r-z)^2}{r^{3/2-2\lambda}} \\ & \cdot \frac{r+z}{\big((r+z)^2+(t_\varepsilon - t)\big)^2} \frac{r-z}{\big((r-z)^2+(t_\varepsilon - t)\big)^2} \\ \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} & \frac{r^2}{r^{1/2-2\lambda}(r^2+(t_\varepsilon - t))^2} \int_{z=0}^{r} \frac{r-z}{((r-z)^2+(t_\varepsilon - t))^2}. \end{aligned} \tag{III.2.73}$$

Now, because

$$\int_{s=0}^{\infty} \frac{s}{(s^2+(t_\varepsilon - t))^2} \leqslant \frac{C}{(t_\varepsilon - t)}, \tag{III.2.74}$$

(III.2.73) yields

$$\begin{aligned} |B_3| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^{3/2+2\lambda}}{(r^2+(t_\varepsilon - t))^2} & \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2+3/4-3\lambda}} \\ & \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}. \end{aligned} \tag{III.2.75}$$

Lastly, we provide the upper bound for B_5. We have since $\phi_{\varepsilon,y}(t,x) \geqslant (t_\varepsilon - t)$, with the mean value theorem

$$\begin{aligned}|B_5| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \int_{z=0}^{r} \frac{(t_\varepsilon - t)^2}{r^{3/2-2\lambda}} \\ \cdot \frac{r+z}{\left((r+z)^2 + (t_\varepsilon - t)\right)^2} \frac{r-z}{\left((r-z)^2 + (t_\varepsilon - t)\right)^2} \\ \leqslant C_\varepsilon \cdot (t_\varepsilon - t)^{1/2+2\lambda} \int_{r=0}^{\kappa_\varepsilon/2} \frac{1}{r^{1/2-2\lambda}\,(r^2 + (t_\varepsilon - t))^2} \int_{z=0}^{r} \frac{r-z}{((r-z)^2 + (t_\varepsilon - t))^2}.\end{aligned} \tag{III.2.76}$$

Now, because

$$\int_{s=0}^{\infty} \frac{s}{(s^2 + (t_\varepsilon - t))^2} \leqslant \frac{C}{(t_\varepsilon - t)}, \tag{III.2.77}$$

and

$$\int_{s=0}^{\infty} \frac{s^{-1/2+2\lambda}}{(s^2 + (t_\varepsilon - t))^2} \leqslant \frac{C}{(t_\varepsilon - t)^{7/4}}, \tag{III.2.78}$$

(III.2.76) yields

$$\begin{aligned}|B_5| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^{-1/2+2\lambda}}{(r^2 + (t_\varepsilon - t))^2} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2+7/4-3\lambda}} \\ \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}.\end{aligned} \tag{III.2.79}$$

This concludes the proof for the first difference term D_1. Overall, we obtain

$$D_1 \geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{11/4-3\lambda}}. \tag{III.2.80}$$

Now, we study D_2 and D_3.

For D_2, we first study the term

$$\frac{1}{\phi_y(t,v_\varepsilon+y)^2}-\frac{1}{\phi_y(t,v_\varepsilon-y)^2}
=\frac{(\phi_y(t,v_\varepsilon+y)+\phi_y(t,v_\varepsilon-y))\cdot(\phi_y(t,v_\varepsilon-y)-\phi_y(t,v_\varepsilon+y))}{\phi_y(t,v_\varepsilon+y)^2\phi_y(t,v_\varepsilon-y)^2}. \quad \text{(III.2.81)}$$

A Taylor expansion of ϕ_y gives that $\phi_y(t,v_\varepsilon-y)-\phi_y(t,v_\varepsilon+y)=f_1(t,y)y^3+f_2(t,y)(t_\varepsilon-t)^2$ where f_1 and f_2 are bounded. Because $\phi_y \geqslant 0$, we have that

$$\frac{1}{\phi_y(t,v_\varepsilon+y)^2}-\frac{1}{\phi_y(t,v_\varepsilon-y)^2}=\frac{f_1(t,y)y^3+f_2(t,y)(t_\varepsilon-t)^2}{\phi_y(t,v_\varepsilon+y)\phi_y(t,v_\varepsilon-y)^2}
+\frac{f_1(t,y)y^3+f_2(t,y)(t_\varepsilon-t)^2}{\phi_y(t,v_\varepsilon+y)^2\phi_y(t,v_\varepsilon-y)}. \quad \text{(III.2.82)}$$

The two involved terms are similar. We consider only the first one. We obtain

$$|D_2|\leqslant\frac{C_\varepsilon}{(t_\varepsilon-t)^{1/2-2\lambda}}\int_{r=0}^{\kappa_\varepsilon/2}\int_{z=0}^{r}\frac{1}{z^{1/2-2\lambda}}
\cdot\frac{(r+z)(r-z)^4}{\left((r+z)^2+(t_\varepsilon-t)\right)^2\cdot\left((r-z)^2+(t_\varepsilon-t)\right)\cdot\left((z-r)^2+(t_\varepsilon-t)\right)^2}. \quad \text{(III.2.83)}$$

Again, we split the domain of z in two parts.

$$|D_2|\leqslant\frac{C_\varepsilon}{(t_\varepsilon-t)^{1/2-2\lambda}}\int_{r=0}^{\kappa_\varepsilon/2}\int_{z=0}^{r/2}\frac{1}{z^{1/2-2\lambda}}
\cdot\frac{(r+z)(r-z)^4}{\left((r+z)^2+(t_\varepsilon-t)\right)^2\cdot\left((r-z)^2+(t_\varepsilon-t)\right)\cdot\left((z-r)^2+(t_\varepsilon-t)\right)^2}
+\frac{C_\varepsilon}{(t_\varepsilon-t)^{1/2-2\lambda}}\int_{r=0}^{\kappa_\varepsilon/2}\int_{z=r/2}^{r}\frac{1}{z^{1/2-2\lambda}}
\cdot\frac{(r+z)(r-z)^4}{\left((r+z)^2+(t_\varepsilon-t)\right)^2\cdot\left((r-z)^2+(t_\varepsilon-t)\right)\cdot\left((z-r)^2+(t_\varepsilon-t)\right)^2}. \quad \text{(III.2.84)}$$

For the first term, we obtain

$$\begin{aligned} &\frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^5}{(r^2+(t_\varepsilon - t))^5} \int_{z=0}^{r/2} \frac{1}{z^{1/2-2\lambda}} \\ &\qquad \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^{11/2-2\lambda}}{(r^2+(t_\varepsilon - t))^5} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}, \end{aligned} \tag{III.2.85}$$

and for the second term

$$\begin{aligned} &\frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{1}{r^{1/2-2\lambda}} \frac{r}{(r^2+(t_\varepsilon - t))^2} \int_{s=0}^{r/2} \frac{s^4}{(s^2+(t_\varepsilon - t))^3} \\ &\qquad \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4}}. \end{aligned} \tag{III.2.86}$$

Overall, we obtain that $|D_2| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4}}$. Lastly, for D_3, we have to study the term $\chi'(\nu_\varepsilon - y) - \chi'(\nu_\varepsilon + y)$. We obtain

$$|\chi'(\nu_\varepsilon - y) - \chi'(\nu_\varepsilon + y)| \leqslant C_\varepsilon \cdot |y|. \tag{III.2.87}$$

With similar computations, the additional y leads to a gain of order $(t_\varepsilon - t)^{1/2}$.

Overall, we obtain that

$$(iv)_1 \geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{11/4-3\lambda}}. \tag{III.2.88}$$

The last tasks now are to show that $(i), (ii)$ and (iii) are smaller than $(iv)_1$, and lastly that the contribution of g, i.e. $(iv)_2$ is of a smaller order in $(t_\varepsilon - t)^{-1}$. We make the same change of variable again, so now we have $(x, y) \in [\zeta^{2-} - \nu_\varepsilon, \zeta^{3+} - \nu_\varepsilon]^2 = \iota$. For (i), we have

$$
\begin{aligned}
|(i)| &= \left| \int\int_\iota \frac{1}{|\phi(t,\nu_\varepsilon+x)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}} \frac{\chi''(\nu_\varepsilon+x)\chi''(\nu_\varepsilon+y)}{(\phi_y(t,\nu_\varepsilon+x)\phi_y(t,\nu_\varepsilon+y))} \right| \\
&\leqslant C_\varepsilon \int\int_\iota \frac{1}{(t_\varepsilon-t)^{1/2-2\lambda}|x-y|^{1/2-2\lambda}} \frac{1}{(x^2+(t_\varepsilon-t))} \frac{1}{(y^2+(t_\varepsilon-t))} \\
&\leqslant_{Hölder} \frac{C_\varepsilon}{(t_\varepsilon-t)^{1/2-2\lambda}} \left(\int\int_\iota \frac{1}{|x-y|^{3/4-3\lambda}}\right)^{2/3} \left(\int_0^\infty \frac{1}{(x^2+(t_\varepsilon-t))^3}\right)^{1/3} \\
&\cdot \left(\int_0^\infty \frac{1}{(y^2+(t_\varepsilon-t))^3}\right)^{1/3} \leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{1/2+5/6+5/6-2\lambda}} \leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{13/6-2\lambda}}. \qquad \text{(III.2.89)}
\end{aligned}
$$

For (ii), we obtain using the triangular inequality on (III.1.28) for the term involving ϕ_{yy},

$$
\begin{aligned}
|(ii)| &= \left| \int\int_\iota \frac{1}{|\phi(t,\nu_\varepsilon+x)-\phi(t,\nu_\varepsilon+y|^{1/2-2\lambda}} \frac{\chi''(\nu_\varepsilon+x)\chi'(\nu_\varepsilon+y)\phi_{yy}(t,\nu_\varepsilon+y)}{\phi_y(t,\nu_\varepsilon+x)\phi_y(t,\nu_\varepsilon+y)^2} \right| \\
&\leqslant C_\varepsilon \int\int_\iota \frac{1}{(t_\varepsilon-t)^{1/2-2\lambda}|x-y|^{1/2-2\lambda}} \frac{1}{(x^2+(t_\varepsilon-t))} \frac{|y|+(t_\varepsilon-t)}{(y^2+(t_\varepsilon-t))^2} \\
&\leqslant_{Hölder} \frac{C_\varepsilon}{(t_\varepsilon-t)^{1/2-2\lambda}} \left(\int\int_\iota \frac{1}{|x-y|^{3/4-3\lambda}}\right)^{2/3} \left(\int_0^\infty \frac{1}{(x^2+(t_\varepsilon-t))^3}\right)^{1/3} \\
&\cdot \left(\int_0^\infty \frac{(y+(t_\varepsilon-t))^3}{(y^2+(t_\varepsilon-t))^6}\right)^{1/3} \leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{1/2+5/6+4/3-2\lambda}} \leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{8/3-2\lambda}}. \qquad \text{(III.2.90)}
\end{aligned}
$$

The case of (iii) is identical.

Now, using the expression of g provided by (III.2.48), we obtain

$$
|g(t,y)| \leqslant C_\varepsilon \left(|y-\nu_\varepsilon|^2 + (t_\varepsilon-t)\cdot|y-\nu_\varepsilon| + (t_\varepsilon-t)^2\right). \qquad \text{(III.2.91)}
$$

Hence, from (III.2.91) we get

$$
|g(t,x)g(t,y)| \leqslant \sum_{k_1+k_2=2, k_3+k_4=2} C_\varepsilon |y-\nu_\varepsilon|^{k_1} |x-\varepsilon|^{k_3} (t_\varepsilon-t)^{k_2+k_4}. \qquad \text{(III.2.92)}
$$

Hence, for the second part of (iv), we obtain with the same change of variable

$$\begin{aligned}|(iv)_2| &= \left| \int_{x_1=\zeta_\varepsilon^{2-}}^{y=\zeta_\varepsilon^{3-}} \int_{\zeta_\varepsilon^{2-}}^{\zeta_\varepsilon^{3-}} \frac{1}{|\phi(t,x)-\phi(t,y)|^{1/2-2\lambda}} \frac{g(t,x)g(t,y)\chi'(x)\chi'(y)}{\phi_y(t,x)^2\phi_y(t,y)^2} \right| \\ &\leqslant \sum_{k_1+k_2=2, k_3+k_4=2} \frac{(t_\varepsilon-t)^{k_2+k_4}}{(t_\varepsilon-t)^{1/2-2\lambda}} \int\int \frac{1}{|x-y|^{1/2-2\lambda}} \frac{C_\varepsilon x^{k_1} y^{k_3}}{(x^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2}.\end{aligned} \tag{III.2.93}$$

Now, using the same techniques again, (III.2.93) gives

$$\begin{aligned}&\sum_{k_1+k_2=2, k_3+k_4=2} \frac{(t_\varepsilon-t)^{k_2+k_4}}{(t_\varepsilon-t)^{1/2-2\lambda}} \int\int \frac{1}{|x-y|^{1/2-2\lambda}} \frac{C_\varepsilon x^{k_1} y^{k_3}}{(x^2+(t_\varepsilon-t))^2(y^2+(t_\varepsilon-t))^2} \\ &\qquad \leqslant C_\varepsilon \sum_{k_1+k_2=2, k_3+k_4=2} \frac{(t_\varepsilon-t)^{k_2+k_4}}{(t_\varepsilon-t)^{1/2-2\lambda}} \left(\int\int \frac{1}{|x-y|^{3/4-3\lambda}} \right)^{2/3} \\ &\qquad\qquad \cdot \left(\int_{x=0}^{\infty} \frac{x^{3k_1}}{(x^2+(t_\varepsilon-t))^3} \right)^{1/3} \left(\int_{y=0}^{\infty} \frac{y^{3k_3}}{(y^2+(t_\varepsilon-t))^3} \right)^{1/3}\end{aligned} \tag{III.2.94}$$

Note that $x^i \leqslant C_\varepsilon x^j$ when $i \leqslant j$.

Now, we distinguish three cases.

If $k_2+k_4 \geqslant 2$, then we obtain from (III.2.94)

$$\begin{aligned}|(iv)_2| &\leqslant C_\varepsilon (t_\varepsilon-t)^{1.5+2\lambda} \left(\int\int \frac{1}{|x-y|^{3/4-3\lambda}} \right)^{2/3} \\ &\quad \cdot \left(\int_{x=0}^{\infty} \frac{1}{(x^2+(t_\varepsilon-t))^3} \right)^{1/3} \left(\int_{y=0}^{\infty} \frac{1}{(y^2+(t_\varepsilon-t))^3} \right)^{1/3} \leqslant \frac{C_\varepsilon}{(t_\varepsilon-t)^{14/6}}.\end{aligned} \tag{III.2.95}$$

Now, if $k_2 + k_4 = 1$, then either $k_1 \geqslant 1$ or $k_3 \geqslant 1$. Then, (III.2.94) gives

$$|(iv)_2| \leqslant C_\varepsilon (t_\varepsilon - t)^{0.5+2\lambda} \left(\int\int \frac{1}{|x-y|^{3/4-3\lambda}} \right)^{2/3} \cdot \left(\int_{x=0}^{\infty} \frac{x^3}{(x^2+(t_\varepsilon - t))^3} \right)^{1/3} \left(\int_{y=0}^{\infty} \frac{1}{(y^2+(t_\varepsilon - t))^3} \right)^{1/3} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{15/6}}. \quad \text{(III.2.96)}$$

Lastly, if $k_2 + k_4 = 0$, then $k_1 = k_3 = 2$ and

$$|(iv)_2| \leqslant C_\varepsilon (t_\varepsilon - t)^{-0.5+2\lambda} \left(\int\int \frac{1}{|x-y|^{3/4-3\lambda}} \right)^{2/3} \cdot \left(\int_{x=0}^{\infty} \frac{x^6}{(x^2+(t_\varepsilon - t))^3} \right)^{1/3} \left(\int_{y=0}^{\infty} \frac{y^6}{(y^2+(t_\varepsilon - t))^3} \right)^{1/3} \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{13/6}}. \quad \text{(III.2.97)}$$

Now, we finally have from (III.2.88), (III.2.95), (III.2.96) and (III.2.97),

$$(iv) \geqslant (iv)_1 - |(iv)_2| \geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{11/4}}. \quad \text{(III.2.98)}$$

And thus the corresponding lower bound for $I_\varepsilon^2(t)$

$$I_\varepsilon^2(t) \geqslant (iv)_1 - |(iv)_2| \geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{11/4}}. \quad \text{(III.2.99)}$$

We can later use a scaling and summing argument similar to the one we use in chapter II.3. Doing so, we obtain an initial condition such that the solution instantly blows up at $t = 0+$.

IV Introducing a source term with a x_2 dependency

In this chapter, we discuss the stability of the blow up with respect to the addition of a source term in the equation. This case is a study of the stability with respect to the equation, but can also be seen as a next step toward the stability of the phenomenon by perturbations depending on x_2 as well. Indeed, using spectral cutoffs of u with respect to its second space variable, it may be possible that we can reduce the general stability to this case, for a fixed range of frequencies. This case is more tedious as we no longer have an explicit resolution, but it also deeper as we show that the function has to behave in a pathological way. We will also see that we can, in some sense, show that the solution will locally share similarities with the function that we previously computed for the problem without the x_2 dependency.

IV.1 Definition of the new problem and preliminary results

We now consider the following modified Cauchy problem.

$$\begin{cases} \Box_{x_1} u(t, x_1, x_2) = DuD^2 u + f(t, x_1, x_2, Du) \\ \dfrac{\partial u}{\partial t}_{|t=0} = -\chi, \quad u_{|t=0} = 0. \end{cases} \tag{IV.1.1}$$

Remark IV.1.1. *Here, we study the case where f depends on Du but not on u because it is the most difficult case. The case where f both depends on u and Du is really similar and does not bring more difficulties.*

We also define

$$\chi(y) = -\int_{s=0}^{y} \psi_\varepsilon(s)|\ln(s)|^\alpha ds. \tag{IV.1.2}$$

Where $\psi_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions.

$$\begin{cases} \psi_\varepsilon(s) = 0, \ \ \forall s < \dfrac{\varepsilon}{2}, \\ \psi_\varepsilon(s) = 1, \ \ \forall s > \varepsilon, \\ \psi_\varepsilon(s) \in [0,1], \ \ \forall s \in \mathbb{R}^+, \\ |\psi'_\varepsilon|(s) \leqslant \dfrac{2}{\varepsilon}, \ \ \forall s \in \mathbb{R}^+. \end{cases} \tag{IV.1.3}$$

We consider χ to be the extension of χ to $\mathbb{R}^2$ (without explicit relabelling) as it has been done in theorem I.3.1. Lastly, we also assume

$$\forall \alpha, \exists C, \ \left| \frac{\partial^\alpha}{\partial x^\alpha} f \right| \leqslant C. \tag{IV.1.4}$$

Of course, the computations done in chapter I hold in our case at it is the same initial condition, and as the computations did not depend on the equation. We only do the part II again because it is the only part that depends on the equation.

Now, rewriting (IV.1.1), we obtain with $v = Du \neq 1$,

$$\begin{cases} \left(\partial_t + \dfrac{1+v}{1-v}\partial_{x_1}\right) v = \dfrac{f(t,x_1,x_2,v)}{1-v} = g(t,x_1,x_2,v). \\ \dfrac{\partial u}{\partial t}_{|t=0} = -\chi, \ \ u_{|t=0} = 0, \end{cases} \tag{IV.1.5}$$

where g also satisfies (IV.1.4) as $v < 0$. We call C_g the involved constant.

We now state the main theorem of the chapter.

Theorem IV.1.2. *Let u_ε be the solution of problem (IV.1.1). There exists a time t_ε such that*

$$\begin{cases} ||u_\varepsilon(0,\cdot)||_{H^{11/4}(\ln H)^{-\beta}} < \infty \\ ||u_\varepsilon(t,\cdot)||_{H^{11/4}(\ln H)^{-\beta})} \to \infty \quad \textit{as } t \to t_\varepsilon. \end{cases} \tag{IV.1.6}$$

Also, we have that

$$t_\varepsilon \to 0 \quad as\, \varepsilon \to 0. \tag{IV.1.7}$$

We now do some preliminary work, later we will prove a technical lemma, and then we will be able to make the proof of the theorem.

Preliminary work

First, we define ϕ as

$$\begin{cases} \phi(0,x_1,x_2) = x_1 \\ \partial_t \phi(t,x_1,x_2) = \dfrac{1+v(t,\phi(t,x_1,x_2),x_2)}{1-v(t,\phi(t,x_1,x_2),x_2)}. \end{cases} \tag{IV.1.8}$$

Now, using the chain rule, we obtain from (IV.1.8) and (IV.1.5),

$$\frac{\partial}{\partial_t}\left(v(t,\phi(t,x_1,x_2),x_2)\right) = g(t,\phi(t,x_1,x_2),x_2,v(t,\phi(t,x_1,x_2),x_2)). \tag{IV.1.9}$$

From now on, for clarity's sake, we will not always specify the variables when there is no ambiguity. Most of the time, we only precise if the first space variable is x_1 or $\phi(t,x_1,x_2)$. Now, we obtain from (IV.1.9).

$$v(\phi) = \chi(x_1) + \int_{\tau=0}^{t} g(\tau,\phi)d\tau. \tag{IV.1.10}$$

Differentiating (IV.1.10) leads to the following expressions for the derivatives of v.

$$\begin{aligned} \partial_{x_1}\phi(x_1)\partial_{x_1}v(\phi) &= \chi'(x_1) + \int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_1 g(\tau,x_1) + \int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_{x_1}v(\tau,x_1)\partial_3 g(\tau,x_1) \\ \Rightarrow \partial_{x_1}v(\phi) &= \frac{\chi'(x_1) + \int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_1 g(\tau,\phi) + \int_\tau \partial_{x_1}\phi(x_1)\partial_{x_1}v(\tau,x_1)\partial_3 g(\tau,x_1)}{\partial_{x_1}\phi(\tau,x_1)}. \end{aligned} \tag{IV.1.11}$$

$$
\begin{aligned}
\partial^2_{x_1} v(\phi) = \frac{1}{(\partial_{x_1}\phi(x_1))^2}\Bigg[\chi''(x_1) + \int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\,\partial^2_1 g(\tau,\phi) + \int_\tau \partial^2_{x_1}\phi(\tau,x_1)\partial_1 g(\tau,\phi) \\
+ \int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\,\partial_{x_1} v(\tau,\phi)\partial_3\partial_1 g(\tau,\phi) + \int_\tau \partial^2_{x_1}\phi(\tau,x_1)\partial_3 g(\tau,\phi)\partial_{x_1} v(\tau,\phi) \\
+ \int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\,\partial_1\partial_3 g(\tau,\phi)\partial_{x_1} v(\tau,\phi) + \int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\,(\partial_{x_1} v(\tau,\phi))^2\,\partial^2_3 g(\tau,\phi) \\
+ \int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\,\partial^2_{x_1} v(\tau,\phi)\partial_3 g(\tau,\phi)\Bigg] \\
- \frac{\partial^2_{x_1}\phi(x_1)}{(\partial_{x_1}\phi(x_1))^3}\left[\chi'(x_1) + \int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_1 g(\tau,\phi) + \int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_3 g(\tau,\phi)\partial_{x_1} v(\tau,\phi)\right] \\
= A - B. \quad \text{(IV.1.12)}
\end{aligned}
$$

We also compute the derivatives of ϕ that we will need. From (IV.1.8), we get

$$
\phi_{tx_1}(x_1) = \frac{2\phi_{x_1} v_{x_1}(\phi)}{(1-v(\phi))^2}. \quad \text{(IV.1.13)}
$$

$$
\phi_{tx_1x_1}(x_1) = \frac{2(v(\phi))_{x_1x_1}(1-v(\phi)) + 2(v(\phi))^2_{x_1}}{(1-v(\phi))^3}. \quad \text{(IV.1.14)}
$$

At this stage, we will assume that $\phi_{x_2x_1x_1}(t_\varepsilon, v_\varepsilon)$ is not 0. This is an important assumption, as assuming $\phi_{x_2x_1x_1}(t_\varepsilon, v_\varepsilon) = 0$ leads to different computations. However, we consider only this case because the other case is much more similar that the situations that we have previously studied.

We start with a bootstrap argument to control the function ϕ_x and show estimates. We choose ψ_ε to be nondecreasing with respect to x. We call c_0 the number (implicitly depending on ε) such that $\psi_\varepsilon \in [0, \frac{1}{10}]$ for $x \in [\varepsilon/2, c_0]$. We consider only times such that $\phi_x \neq 0$ and $t < 1/|\ln|\ln(\varepsilon)||$.

We can now state the technical lemma.

Lemma IV.1.3. *We consider $x \in [c_0, \varepsilon]$. There exists a time t_ε such that the following* *properties are verified.*

$$t_\varepsilon \to 0 \quad as\, \varepsilon \to 0. \tag{IV.1.15}$$

$$\begin{aligned} &\phi_{x_1} > 0 \ \ \forall t < t_\varepsilon \\ &\exists (v_\varepsilon^1, v_\varepsilon^2) \ \ s.t. \ \ \phi_{x_1}(t_\varepsilon, v_\varepsilon^1, v_\varepsilon^2) = 0 \end{aligned} \tag{IV.1.16}$$

If $|\phi_{x_1}(t, x_1, x_2)| \leqslant 2$ *for* $t \in [0, t_1] \subseteq [0, t_\varepsilon]$ *and* $x \in [c_0, \varepsilon]$*, then* $\phi_{x_1}(t, x_1, x_2) \leqslant 1$ *for* $t \in [0, t_1]$ *and* $x \in [c_0, \varepsilon]$*.*

$$\begin{aligned} &0 < \phi_{x_1} < 1, \ \phi_{t,x_1} < 0 \ \ , \forall t < t_\varepsilon, \\ &\exists C_\varepsilon, \ \ |\phi_{t,x_1,x_1}| \leqslant C_\varepsilon |\ln(t_\varepsilon - t)|. \\ &v_{x_1} < 0, \ \ v < 0 \\ &\exists C_\varepsilon, \ \ |v_{x_1}(\phi)| \leqslant \frac{C_\varepsilon(\chi'(x_1) + C_\varepsilon)}{|\phi_{x_1}|}. \end{aligned} \tag{IV.1.17}$$

If x_1*,* x_2 *and* t *are sufficiently close to* $(v_\varepsilon^1, v_\varepsilon^2, t_\varepsilon)$*, we have the following estimates.*

$$\begin{aligned} \exists C_\varepsilon^1, C_\varepsilon^2 > 0, \ \ \frac{C_\varepsilon^1}{(x_1 - v_\varepsilon^1)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)} &\leqslant \frac{1}{\phi_{x_1}(x_1)} \\ &\leqslant \frac{C_\varepsilon^2}{(x_1 - v_\varepsilon^1)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)} \\ \exists C_\varepsilon^1, C_\varepsilon^2, C_\varepsilon^3, \ \ \phi_{x_1 x_1}(x_1) &= C_\varepsilon^1(x_1 - v_\varepsilon^1) + C_\varepsilon^2(x_2 - v_\varepsilon^2) + C_\varepsilon^3(t_\varepsilon - t) \\ &+ \sum_{i+j+k=2} (x_1 - v_\varepsilon^1)^i (x_2 - v_\varepsilon^2)^j (t_\varepsilon - t)^k f_{i,j,k}(t, x_1, x_2), \end{aligned} \tag{IV.1.18}$$

where all the involved f *functions are bounded near* $(t, v_\varepsilon^1, v_\varepsilon^2)$*. We also have*

$$|A| \leqslant \frac{C_\varepsilon |\ln(t_\varepsilon - t)|}{|\phi_{x_1}|^2}, \tag{IV.1.19}$$

where A is the A involved in (IV.1.12).

Lastly, for ε small enough, we have

$$\begin{aligned} \frac{C_\varepsilon \phi_{x_1 x_1}(x_1)}{(\phi_{x_1}(x_1))^3}\frac{\chi'}{2} \leqslant B \leqslant \frac{C_\varepsilon \phi_{x_1 x_1}(x_1)}{(\phi_{x_1}(x_1))^3} 2\chi', \\ \frac{C_\varepsilon \phi_{x_1 x_1}(x_1)}{(\phi_{x_1}(x_1))^3} 2\chi' \leqslant B \leqslant \frac{C_\varepsilon \phi_{x_1 x_1}(x_1)}{(\phi_{x_1}(x_1))^3}\frac{\chi'}{2}. \end{aligned} \tag{IV.1.20}$$

The first inequality being verified when $\phi_{x_1 x_1} \leqslant 0$, and the second being verified when $\phi_{x_1 x_1} \geqslant 0$.

Proof. We hence assume $|\phi_{x_1}| \leqslant 2$.

We have from (IV.1.10) and (IV.1.4) that

$$|v(\phi(t, x_1))| \leqslant |\chi(x_1)| + Ct \leqslant C_\varepsilon. \tag{IV.1.21}$$

Also, we have from (IV.1.10) that for $t < t_1$, where t_1 only depends on f, that

$$v(t, x_1, x_2) \leqslant 1/2. \tag{IV.1.22}$$

Hence we have from (IV.1.8), (IV.1.22) and (IV.1.21) that

$$|\phi_t| = \frac{|1 + v(t,\phi)|}{|1 - v(t,\phi)|} \leqslant C_\varepsilon. \tag{IV.1.23}$$

We will make use of Grönwall's inequality in (IV.1.11), we have

$$|\phi_{x_1} v_{x_1}| \leqslant |\psi_\varepsilon(x_1)||\ln(x_1)|^\alpha + 2\frac{1}{|\ln(\ln(\varepsilon))|} C_g + \int_\tau C_g |\phi_{x_1} v_{x_1}| \tag{IV.1.24}$$

and

$$|v_{x_1}\phi_{x_1}| \leqslant (|\ln(\varepsilon)|^{\alpha}+C)\, e^{\int_\tau |\partial_3 g|} \leqslant C\,|\ln(\varepsilon)|^{\alpha}\, e^{\frac{1}{|\ln(|\ln(\varepsilon)||}}. \tag{IV.1.25}$$

Now, looking at (IV.1.11) again, we obtain

$$\begin{aligned}
&-\psi_\varepsilon(x_1)|\ln(x_1)|^{\alpha}+\int_\tau \phi_{x_1}(\tau,x_1,x_2)\partial_1 g+\int_\tau \phi_{x_1} v_{x_1}\partial_3 g\\
&\leqslant -\frac{|\ln(\varepsilon)|^{\alpha}}{10}+2C_g\frac{1}{|\ln(|\ln(\varepsilon)|)|}+C_g\frac{1}{|\ln(|\ln(\varepsilon)|)|}C\,|\ln(\varepsilon)|^{\alpha}\, e^{\frac{1}{|\ln(|\ln(\varepsilon)|)|}} \leqslant 0,
\end{aligned} \tag{IV.1.26}$$

for ε small enough. From (IV.1.13), we hence get that $\phi_{t,x_1} \leqslant 0$. Because we only consider times such that $\phi_{x_1} \neq 0$, (IV.1.8) gives by continuity

$$\phi_{x_1} > 0. \tag{IV.1.27}$$

Hence $\phi_{x_1} \leqslant 1$ on the considered interval. This is the estimate we wanted for ϕ_{x_1} to make our boostrap argument on ϕ_{x_1} work. We go on with the other estimates.

Using the same reasoning as for (IV.1.26), we obtain

$$2\frac{\chi'(x_1)}{\phi_{x_1}} \leqslant v_{x_1} \leqslant \frac{\chi'(x_1)}{2}\frac{1}{\phi_{x_1}} \leqslant 0. \tag{IV.1.28}$$

Now, from (IV.1.11), (IV.1.13) and (IV.1.28)

$$\phi_{tx_1} \leqslant \frac{\chi'(x_1)}{2(1-v)^2} < 0. \tag{IV.1.29}$$

This leads us to the following two conclusions. From (IV.1.2) and (IV.1.3), we have that $\chi'(\varepsilon) = -|\ln(\varepsilon)|^{\alpha}$.

Hence, for ε small enough, there exists a time t_ε satisfying (IV.1.15) and (IV.1.16). We also have

$$t_\varepsilon \leqslant \frac{4}{|\ln(\varepsilon)|^\alpha}. \tag{IV.1.30}$$

Since we have that $t_\varepsilon << \frac{1}{|\ln(|\ln(\varepsilon)|)|}$, our time t_ε belongs to the set in which we made our estimate.

Now, because, by a continuity argument, we have that $\phi_{x_1}(t_\varepsilon, v_\varepsilon^1, v_\varepsilon^2)$ is a minimum in (x_1, x_2), we have that $\phi_{x_1 x_1} = \phi_{x_1 x_2} = 0$. A Taylor expansion leads to

$$\begin{aligned}\phi_{x_1}(t_\varepsilon, x_1, x_2) = C_{\varepsilon,1}(x_1 - v_\varepsilon^1)^2 + C_{\varepsilon,2}(x_2 - v_\varepsilon^2)^2 + C_{\varepsilon,3}(x_1 - v_\varepsilon^1)(x_2 - v_\varepsilon^2)\\ + o(d((x_1, x_2), (v_\varepsilon^1, v_\varepsilon^2))^2)\\ C_{\varepsilon,3}^2 \leqslant C_{\varepsilon,1}C_{\varepsilon,2}, \quad C_\varepsilon^1, C_\varepsilon^2 > 0.\end{aligned} \tag{IV.1.31}$$

Now, from (IV.1.31) and using $\sqrt{ab} \leqslant \frac{a+b}{2}$, we obtain

$$\begin{aligned}\frac{C_{\varepsilon,3}^2}{C_{\varepsilon,1}C_{\varepsilon,2}} \leqslant \delta < 1 \Rightarrow \left|C_{\varepsilon,3}(x_1 - v_\varepsilon^1)(x_2 - v_\varepsilon^2)\right| \leqslant \delta \left|\sqrt{C_{\varepsilon,1}}(x_1 - v_\varepsilon^1)\sqrt{C_{\varepsilon,2}}(x_2 - v_\varepsilon^2)\right|\\ \leqslant \delta \left[\frac{C_{\varepsilon,1}(x_1 - v_\varepsilon^2)^2 + C_{\varepsilon,2}(x_2 - v_\varepsilon^2)^2}{2}\right].\end{aligned} \tag{IV.1.32}$$

Now, from (IV.1.31) and (IV.1.32), we obtain

$$\begin{aligned}\frac{2-\delta}{2}C_{\varepsilon,1}(x_1 - v_\varepsilon^1)^2 + \frac{2-\delta}{2}C_{\varepsilon,2}(x_2 - v_\varepsilon^2)^2\\ \leqslant C_{\varepsilon,1}(x_1 - v_\varepsilon^1)^2 + C_{\varepsilon,2}(x_2 - v_\varepsilon^2)^2 + C_{\varepsilon,3}(x_1 - v_\varepsilon^1)(x_2 - v_\varepsilon^2)\\ \leqslant \frac{2+\delta}{2}C_{\varepsilon,1}(x_1 - v_\varepsilon^1)^2 + \frac{2+\delta}{2}C_{\varepsilon,2}(x_2 - v_\varepsilon^2)^2.\end{aligned} \tag{IV.1.33}$$

Hence, using the sign of ϕ_{tx_1}, we obtain estimate (IV.1.18).

Now, we deal with the terms A and B involved in (IV.1.12). We will go through each term one by one. With $0 < \phi_{x_1} \leqslant 1$, we obtain

$$\left| \int_\tau (\phi_{x_1}(\tau, x_1))^2 \, \partial_1^2 g(\tau, \phi) \right| \leqslant \int_\tau C \leqslant C. \tag{IV.1.34}$$

Using (IV.1.25) and $0 < \phi_{x_1} \leqslant 1$, we get

$$\left| \int_\tau (\phi_{x_1})^2 v_{x_1}(\tau, \phi) \partial_3 \partial_1 g \right| \leqslant C \int_\tau (\phi_{x_1})\,(\phi_{x_1} v_{x_1}) \leqslant C \int_\tau C(C + |\chi'|) \leqslant C(C + |\chi'|) \tag{IV.1.35}$$

$$\left| \int_\tau (\phi_{x_1})^2 \, (v_{x_1})^2 \, \partial_3^2 g \right| \leqslant \int_\tau |C(C + \chi')|^2 \leqslant C_\varepsilon. \tag{IV.1.36}$$

$$\left| \int_\tau \phi_{x_1 x_1} v_{x_1} \partial_3 g \right| \leqslant C_\varepsilon \int_\tau \frac{C(C + |\chi'|)}{\phi_{x_1}} \leqslant C_\varepsilon |\ln(t_\varepsilon - t)|. \tag{IV.1.37}$$

Indeed, (IV.1.18) provides in particular that $\frac{1}{\phi_{x_1}} \geqslant \frac{1}{(t_\varepsilon - t)}$.

We hence obtain a first estimate for A :

$$\begin{aligned} |A| &\leqslant \frac{1}{(\phi_{x_1})^2} \left[C_\varepsilon + C_\varepsilon |\ln(t_\varepsilon - t)| + \int_\tau |\phi_{x_1}^2 v_{x_1 x_1} \partial_3 g| \right] \\ &\leqslant \frac{1}{(\phi_{x_1})^2} \left[C_\varepsilon + C_\varepsilon |\ln(t_\varepsilon - t)| + C_\varepsilon \int_\tau |\phi_{x_1}|^2 |v_{x_1 x_1}| \right] \end{aligned} \tag{IV.1.38}$$

We look at the terms involved in B. Using (IV.1.25) and $0 < \phi_{x_1} \leqslant 1$ again, we obtain

$$\left| \int_\tau \phi_{x_1} \partial_1 g \right| \leqslant C_\varepsilon, \quad \left| \int_\tau \phi_{x_1} v_{x_1} \partial_3 g \right| \leqslant C_\varepsilon. \tag{IV.1.39}$$

This means that

$$|v_{x_1 x_1}| \leqslant \frac{1}{\phi_{x_1}^2} \left[C_\varepsilon |\ln(t_\varepsilon - t)| + C_\varepsilon \int_\tau |v_{x_1 x_1}| \phi_{x_1}^2 \right] + \frac{C_\varepsilon}{\phi_{x_1}^3}. \tag{IV.1.40}$$

Using Grönwall once again in (IV.1.40), we obtain

$$|v_{x_1x_1}| \leqslant \frac{C_\varepsilon}{\phi_{x_1}^3} e^{\frac{C_\varepsilon}{\phi_{x_1}^2}\int_\tau \phi_{x_1}^2(\tau)} \leqslant \frac{C_\varepsilon}{\phi_{x_1}^3} e^{C_\varepsilon \int_\tau 1} \leqslant \frac{C_\varepsilon}{\phi_{x_1}^3}. \tag{IV.1.41}$$

Now, inserting (IV.1.41) back in (IV.1.38), we obtain

$$|A| \leqslant \frac{C_\varepsilon}{\phi_{x_1}^2}\left[\ln(t_\varepsilon - t)| + \int_\tau C_\varepsilon \frac{1}{\phi_{x_1}}\right] \leqslant \frac{C_\varepsilon |\ln(t_\varepsilon - t)|}{\phi_{x_1}^2}, \tag{IV.1.42}$$

and hence we have proven (IV.1.19).

Now, we want to be more precise about the term involved in B. Using (IV.1.30), and $0 < \phi_x \leqslant 1$, we obtain

$$\left|\int_\tau \phi_{x_1}\partial_1 g\right| \leqslant C\int_\tau |\phi_{x_1}| \leqslant C\frac{1}{|\ln(\varepsilon)|^\alpha} \to_\varepsilon 0. \tag{IV.1.43}$$

$$\left|\int_\tau \phi_{x_1} v_{x_1}\partial_3 g\right| \leqslant C\int C(C+|\chi'|) \leqslant \frac{C(C+|\chi'|)}{|\ln(\varepsilon)|^\alpha} = o_\varepsilon(|\chi'|). \tag{IV.1.44}$$

Hence, for ε small enough (the constants involved in (IV.1.43) and (IV.1.44) being absolute constants only depending on the equation), estimates (IV.1.20) holds.

□

Now, we present a lemma that removes the assumption $x_1 \in [c_0, \varepsilon]$. Essentially, this will mean that $\nu_\varepsilon \in [c_0, \varepsilon]$. In fact, we need slightly more, we need that there is an open set $O =]\nu_\varepsilon - \delta_\varepsilon, \nu_\varepsilon + \delta_\varepsilon[$ such that $O \cap [\varepsilon/2, c_0] = \varnothing$. This will ensure that all our estimates are valid when we will do the cutoff in the next step.

Lemma IV.1.4. *For x_1 such that $\psi_\varepsilon(x_1) \leqslant \frac{1}{9}$, then $\nu_\varepsilon \neq x_1$.*

Proof. To show this, we will use the estimate (IV.1.30) and show that $\phi_{x_1}(x_1)$ can not reach 0 during that time. We consider x_1 such that $\psi_\varepsilon(x_1) \leqslant \frac{1}{9}$. By the same Grönwall estimate as in the previous proof, we obtain

$$|\phi_{x_1} v_{x_1}| \leqslant (C + \frac{|\ln(\varepsilon)|^\alpha}{9}) e^{\int_\tau \partial_3 g} \leqslant \frac{|\ln(\varepsilon)|^\alpha}{8}, \tag{IV.1.45}$$

for ε small enough. Now, because $(\phi_{x_1})_{|t=0} = 1$, we have that for all $t \leqslant \frac{4}{|\ln(\varepsilon)|^\alpha}$,

$$\phi_{x_1}(x_1) \geqslant 1 - t\frac{|\ln(\varepsilon)|^\alpha}{8} \geqslant 1/2. \tag{IV.1.46}$$

□

Remark IV.1.5. *Informally, we now have that*

$$v_{x_1 x_1} \simeq \frac{C_\varepsilon |\ln(t_\varepsilon - t)|}{\phi_{x_1}^2} - \frac{C_\varepsilon \chi' \phi_{x_1 x_1}}{\phi_{x_1}^3}.$$

The $\simeq$ symbols being a majoration for the first term, and a control of the behaviour for the second term. The situation is hence closer to the situation we had in the two previous cases. However, the estimates that we will now use for $\phi_{x_1 x_1}$ and ϕ_{x_1} now depends on x_2, and the estimation will be harder, especially the control of the sign.

IV.2 Proof of the blow up as $t \to t_\varepsilon$

We now consider the following cutoffs in x_1 and x_2.

Let $\psi_\varepsilon^1 : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying

$$\begin{cases} \psi_\varepsilon^1(x) = 1 \text{ for } \phi(t_\varepsilon, v_\varepsilon) - \delta_\varepsilon < x < \phi(t_\varepsilon, v_\varepsilon) + \delta_\varepsilon \\ \psi_\varepsilon^1(x) = 0 \text{ for } \phi(t_\varepsilon, v_\varepsilon) + 2\delta_\varepsilon < x \text{ or } x < \phi(t_\varepsilon, v_\varepsilon) - 2\delta_\varepsilon \\ 0 < \psi_\varepsilon^1(x) < 1 \text{ elsewhere,} \end{cases} \tag{IV.2.47}$$

and $\psi_\varepsilon^2 : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying

$$\begin{cases} \psi_\varepsilon^2(x) = 1 \text{ for } -\delta_\varepsilon < x < \delta_\varepsilon \\ \psi_\varepsilon^2(x) = 0 \text{ for } 2\delta_\varepsilon < x \text{ or } x < -2\delta_\varepsilon \\ 0 < \psi_\varepsilon^2(x) < 1 \text{ elsewhere.} \end{cases} \tag{IV.2.48}$$

We define $h(t,x_1,x_2) = \psi_\varepsilon^1(x_1)\psi_\varepsilon^2(x_2)v(t,x_1,x_2)$. We will show that $||h||_{\dot{H}^{7/4}_{x_1}(\mathbb{R}^2)} \to \infty$ as $t \to t_\varepsilon$, and use the fact that

$$I_\varepsilon(t) = ||\psi_\varepsilon^1\psi_\varepsilon^2 v||_{\dot{H}^{7/4}_{x_1}(\ln H)^{-\lambda}} = \int_{x_2}\int_{x_1}\int_y \frac{1}{|x_1-y|^{1/2-2\lambda}} h_{x_1x_1}(t,x_1,x_2)h_{x_1x_1}(t,y,x_2). \tag{IV.2.49}$$

This will show theorem IV.1.2.

Proof. We first split the integral in x_1 in three domains,

$$\begin{aligned} I_\varepsilon(t) = &\int_{x_2=v_\varepsilon^2-2\delta_\varepsilon}^{v_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon^2(x_2) \int_{x_1=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)-\delta_\varepsilon} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,x_1) \\ &\cdot \int_{y=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2-2\lambda} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,y)\,dy\,dx_1 \\ &+ \int_{x_2=v_\varepsilon^2-2\delta_\varepsilon}^{v_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon^2(x_2) \int_{x_1=\phi(t_\varepsilon,v_\varepsilon)-\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+\delta_\varepsilon} \left(\frac{\partial^2 v}{\partial x_1^2}\right)(t,x_1) \\ &\cdot \int_{y=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2-2\lambda} \left(\frac{\partial^2 v}{\partial x_1^2}\right)(t,y)\,dy\,dx_1 \\ &+ \int_{x_2=v_\varepsilon^2-2\delta_\varepsilon}^{v_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon^2(x_2) \int_{x_1=\phi(t_\varepsilon,v_\varepsilon)+\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,x_1) \\ &\cdot \int_{y=\phi(t_\varepsilon,v_\varepsilon)-2\delta_\varepsilon}^{\phi(t_\varepsilon,v_\varepsilon)+2\delta_\varepsilon} |x_1-y|^{-1/2-2\lambda} \left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,y)\,dy\,dx_1. \\ = &\int_{x_2=v_\varepsilon^2-2\delta_\varepsilon}^{v_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon^2(x_2) I_\varepsilon^1(t,x_2) + \int_{x_2=v_\varepsilon^2-2\delta_\varepsilon}^{v_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon^2(x_2) I_\varepsilon^2(t,x_2) + \int_{x_2=v_\varepsilon^2-2\delta_\varepsilon}^{v_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon^2(x_2) I_\varepsilon^3(t,x_2). \end{aligned} \tag{IV.2.50}$$

We now consider a fixed x_2 and study the integrals $I_\varepsilon^1(t,x_2)$, $I_\varepsilon^2(t,x_2)$ and $I_\varepsilon^3(t,x_2)$.

Let us make the change of variable $x_1 = \phi(t,x_1,x_2)$ (not explicitly relabelled).

$$\begin{aligned}
I_\varepsilon(t,x_2) = &\int_{x_1=\zeta_\varepsilon^1(t,x_2)}^{\zeta_\varepsilon^2(t,x_2)} \phi_{x_1}(t,x_1,x_2)\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,\phi(t,x_1,x_2)) \\
&\cdot \int_{y=\zeta_\varepsilon^1(t,x_2)}^{\zeta_\varepsilon^4(t,x_2)} |\phi(t,x_1)-\phi(t,y,x_2)|^{-1/2+2\lambda}\phi_{x_1}(t,y,x_2)\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,\phi(t,y,x_2))dx_1dy \\
+ &\int_{x_1=\zeta_\varepsilon^2(t,x_2)}^{\zeta_\varepsilon^3(t,x_2)} \phi_{x_1}(t,x_1,x_2)\left(\frac{\partial^2(v)}{\partial x_1^2}\right)(t,\phi(t,x_1,x_2)) \\
&\cdot \int_{y=\zeta_\varepsilon^1(t,x_2)}^{\zeta_\varepsilon^4(t,x_2)} |\phi(t,y,x_2)-\phi(t,y,x_2)|^{-1/2+2\lambda}\phi_{x_1}(t,y,x_2)\left(\frac{\partial^2(v)}{\partial x_1^2}\right)(t,\phi(t,y,x_2))dx_1dy \\
+ &\int_{x_1=\zeta_\varepsilon^3(t,x_2)}^{\zeta_\varepsilon^4(t,x_2)} \phi_{x_1}(t,x_1,x_2)\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,\phi(t,x_1,x_2)) \\
&\cdot \int_{y=\zeta_\varepsilon^1(t,x_2)}^{\zeta_\varepsilon^4(t,x_2)} |\phi(t,x_1)-\phi(t,y,x_2)|^{-1/2+2\lambda}\phi_{x_1}(t,y,x_2)\left(\frac{\partial^2(v\psi_\varepsilon^1)}{\partial x_1^2}\right)(t,\phi(t,y,x_2))dx_1dy \\
&= I_\varepsilon^1(t,x_2) + I_\varepsilon^2(t,x_2) + I_\varepsilon^3(t,x_2).
\end{aligned} \tag{IV.2.51}$$

We will use all the following results without proving it, as it is very similar to what we obtained in the previous chapter.

Lemma IV.2.1. *The following holds on an interval centered at $(t_\varepsilon, v_\varepsilon^1, \varepsilon^2)$.*

If we define

$$\begin{aligned}
\zeta_\varepsilon^1(t,x_2) &= \phi_t^{-1}(\phi(t_\varepsilon, v_\varepsilon^1, v_\varepsilon^2) - 2\delta_\varepsilon, x_2), \\
\zeta_\varepsilon^2(t,x_2) &= \phi_t^{-1}(\phi(t_\varepsilon, v_\varepsilon^1, v_\varepsilon^2) - \delta_\varepsilon, x_2), \\
\zeta_\varepsilon^3(t,x_2) &= \phi_t^{-1}(\phi(t_\varepsilon, v_\varepsilon^1, v_\varepsilon^2) + \delta_\varepsilon, x_2), \\
\zeta_\varepsilon^4(t,x_2) &= \phi_t^{-1}(\phi(t_\varepsilon, v_\varepsilon^1, v_\varepsilon^2) + 2\delta_\varepsilon, x_2),
\end{aligned} \tag{IV.2.52}$$

For any η_ε, we can choose δ_ε and t^1_ε such that for $t \in]t^1_\varepsilon, t_\varepsilon[$, we have

$$\nu_\varepsilon - \eta_\varepsilon < \zeta^1_\varepsilon(t,x_2) < \zeta^2_\varepsilon(t,x_2) < \nu_\varepsilon < \zeta^3_\varepsilon(t,x_2) < \zeta^4_\varepsilon(t,x_2) < \nu_\varepsilon + \eta_\varepsilon. \tag{IV.2.53}$$

There exists ζ^{2+}_ε and ζ^{3-}_ε such that for δ_ε small enough, and t close enough to t_ε,

$$\nu_\varepsilon - \eta_\varepsilon < \zeta^1_\varepsilon(t,x_2) < \zeta^2_\varepsilon(t,x_2) < \zeta^{2+}_\varepsilon < \nu_\varepsilon < \zeta^{3-}_\varepsilon < \zeta^3_\varepsilon(t,x_2) < \zeta^4_\varepsilon(t,x_2) < \nu_\varepsilon + \eta_\varepsilon. \tag{IV.2.54}$$

We first study $I^1_\varepsilon(t,x_2)$. The case of $I^3_\varepsilon(t,x_2)$ is similar. Using the expression of $I^1_\varepsilon(t,x_2)$ provided in (IV.2.51), as well as (IV.1.19), (IV.1.18) and (IV.1.20), we obtain

$$\begin{aligned}
\left|I^1_\varepsilon(t,x_2)\right| &\leqslant \int_{x_1=\nu^1_\varepsilon-\eta_\varepsilon}^{\zeta^{2+}_\varepsilon} \phi_{x_1}(t,x_1,x_2)\left(|A|+|B|\right)(x_1) \int_{y=\nu^1_\varepsilon-\eta_\varepsilon}^{\nu^1_\varepsilon+\eta_\varepsilon} \frac{\phi_{x_1}(t,y,x_2)\left(|A|+|B|\right)(y)}{|\phi(t,x_1,x_2)-\phi(t,y,x_2)|^{1/2-2\lambda}}.\\
\left|I^1_\varepsilon(t,x_2)\right| &\leqslant \int_{x_1=\nu^1_\varepsilon-\eta_\varepsilon}^{\zeta^{2+}_\varepsilon} \left(\frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{\phi_{x_1}(x_1)} + \frac{C_\varepsilon|\phi_{x_1x_1}(t,x_1,x_2)|}{\phi^2_{x_1}(x_1)}\right)\\
&\quad\cdot \int_{y=\nu^1_\varepsilon-\eta_\varepsilon}^{\nu^1_\varepsilon+\eta_\varepsilon} \left(\frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{\phi_{x_1}(y)} + \frac{C_\varepsilon|\phi_{x_1x_1}(t,y,x_2)|}{\phi^2_{x_1}(y)}\right)\\
&\quad\cdot \frac{1}{|\phi(t,x_1,x_2)-\phi(t,y,x_2)|^{1/2-2\lambda}}.
\end{aligned} \tag{IV.2.55}$$

On the considered domain for x_1, there is a constant depending only on ε such that

$$\frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{\phi_{x_1}(x_1)} + \frac{C_\varepsilon|\phi_{x_1x_1}(t,x_1,x_2)|}{\phi^2_{x_1}(x_1)} \leqslant C_\varepsilon|\ln(t_\varepsilon-t)|. \tag{IV.2.56}$$

Now, we split the domain of the second integral involved in (IV.2.55) into two parts. The first domain corresponding to the unboundedness of $\frac{1}{|\phi(t,x_1,x_2)-\phi(t,y,x_2)|^{1/2-2\lambda}}$, the second to the unboundedness of $\frac{1}{\phi_{x_1}}$.

$$|I^1_\varepsilon(t,x_2)| \leqslant M_\varepsilon \int_{x_1=v^1_\varepsilon-\eta_\varepsilon}^{\zeta^{2+}_\varepsilon} \int_{y=v^1_\varepsilon-\eta_\varepsilon}^{\zeta^{2+}_\varepsilon} \frac{|\ln(t_\varepsilon - t)|^2}{|\phi(t,x_1,x_2)-\phi(t,y,x_2)|^{1/2-2\lambda}}$$
$$+ M_\varepsilon \int_{x_1=v^1_\varepsilon-\eta_\varepsilon}^{\zeta^{2+}_\varepsilon} \int_{y=\zeta^{2+}_\varepsilon}^{v^1_\varepsilon+\eta_\varepsilon} \frac{|\ln(t_\varepsilon - t)|^2}{(y-v^1_\varepsilon)^2+(x_2-v^2_\varepsilon)^2+(t_\varepsilon-t)}$$
$$+ M_\varepsilon \int_{x_1=v^1_\varepsilon-\eta_\varepsilon}^{\zeta^{2+}_\varepsilon} \int_{y=\zeta^{2+}_\varepsilon}^{v^1_\varepsilon+\eta_\varepsilon} \frac{|\ln(t_\varepsilon - t)|((t_\varepsilon-t)+|x_2-v^2_\varepsilon|+|x_1-v^1_\varepsilon|)}{\left((y-v^1_\varepsilon)^2+(x_2-v^2_\varepsilon)^2+(t_\varepsilon-t)\right)^2}$$
$$\leqslant (i)+(ii)+(iii). \quad \text{(IV.2.57)}$$

Now, using the mean value theorem and (IV.1.18), we obtain (for a $c \in [x_1, y] \subseteq [v^1_\varepsilon - \eta_\varepsilon, \zeta^{2+}_\varepsilon]$) for (i)

$$(i) \leqslant M_\varepsilon \int_{x_1=v^1_\varepsilon-\eta_\varepsilon}^{\zeta^{2+}_\varepsilon} \int_{y=v^1_\varepsilon-\eta_\varepsilon}^{\zeta^{2+}_\varepsilon} \frac{|\ln(t_\varepsilon - t)|^2}{|\phi_{x_1}(t,c,x_2)|^{1/2-2\lambda}|x_1-y|^{1/2-2\lambda}} \leqslant M_\varepsilon \frac{|\ln(t_\varepsilon - t)|^2}{(t_\varepsilon-t)^{1/2-2\lambda}}. \quad \text{(IV.2.58)}$$

Now, for (ii), we obtain

$$(ii) \leqslant M_\varepsilon |\ln(t_\varepsilon - t)|^2 \int_{x_1=v^1_\varepsilon}^{\zeta^{2+}_\varepsilon} \frac{1}{\left((x_2-v^2_\varepsilon)^2+(t_\varepsilon-t)\right)^{1/2}} \leqslant M_\varepsilon \frac{|\ln(t_\varepsilon - t)|^2}{\left((x_2-v^2_\varepsilon)^2+(t_\varepsilon-t)\right)^{1/2}}. \quad \text{(IV.2.59)}$$

For (iii), we get

$$(iii) \leqslant M_\varepsilon |\ln(t_\varepsilon - t)| \int_{x_1=v^1_\varepsilon}^{\zeta^{2+}_\varepsilon} \frac{1}{\left((x_2-v^2_\varepsilon)^2+(t_\varepsilon-t)\right)^{2}} \leqslant M_\varepsilon \frac{|\ln(t_\varepsilon - t)|}{\left((x_2-v^2_\varepsilon)^2+(t_\varepsilon-t)\right)^{3/2}}. \quad \text{(IV.2.60)}$$

Now, using (IV.2.48), we get

$$\left|\int_{x_2=\nu_\varepsilon^2-\eta_\varepsilon}^{\nu_\varepsilon^2+\eta_\varepsilon} \psi_\varepsilon^2(x_2) I_\varepsilon^1(t,x_2)\right| \leqslant M_\varepsilon \int_{x_2=\nu_\varepsilon^2-\eta_\varepsilon}^{\nu_\varepsilon^2+\eta_\varepsilon} \left[|\ln(t_\varepsilon-t)|^2 + \frac{|\ln(t_\varepsilon-t)|^2}{\sqrt{(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)}} + \frac{|\ln(t_\varepsilon-t)|}{\left((x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^{3/2}}\right] \leqslant M_\varepsilon \frac{|\ln(t_\varepsilon-t)|^2}{(t_\varepsilon-t)}. \quad \text{(IV.2.61)}$$

By symmetry, we also have that

$$\left|\int_{x_2=\nu_\varepsilon^2-\eta_\varepsilon}^{\nu_\varepsilon^2+\eta_\varepsilon} \psi_\varepsilon^2(x_2) I_\varepsilon^3(t,x_2)\right| \leqslant M_\varepsilon \int_{x_2=\nu_\varepsilon^2-\eta_\varepsilon}^{\nu_\varepsilon^2+\eta_\varepsilon} \left[|\ln(t_\varepsilon-t)|^2 + \frac{|\ln(t_\varepsilon-t)|^2}{\sqrt{(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)}} + \frac{|\ln(t_\varepsilon-t)|}{\left((x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^{3/2}}\right] \leqslant M_\varepsilon \frac{|\ln(t_\varepsilon-t)|^2}{(t_\varepsilon-t)}. \quad \text{(IV.2.62)}$$

Now, we exhibit a lower bound for $I_\varepsilon^2(t,x_2)$. Using (IV.1.19). We consider the integral

$$\begin{aligned} I_\varepsilon^2(t,x_2) &= \int_{x_1=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\nu_\varepsilon+\eta_\varepsilon} \frac{(A(x_1)-B(x_1))(A(y)-B(y))}{|\phi(t,x_1,x_2)-\phi(t,y,x_2)|^{1/2-2\lambda}} \\ &= \int_{x_1=\zeta_\varepsilon^{2+}}^{\zeta_\varepsilon^{3-}} \int_{y=\nu_\varepsilon-\eta_\varepsilon}^{\nu_\varepsilon+\eta_\varepsilon} \frac{1}{|\phi(t,x_1,x_2)-\phi(t,y,x_2)|^{1/2-2\lambda}} \Big[A(x_1)A(y)+A(x_1)B(y) \\ &\qquad + B(x_1)A(y)+B(x_1)B(y)\Big] = (i)+(ii)+(iii)+(iv). \end{aligned} \quad \text{(IV.2.63)}$$

We will show that

$$\int_{x_2=\nu_\varepsilon^2-2\delta_\varepsilon}^{\nu_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon(x_2)(iv) >> \int_{x_2=\nu_\varepsilon^2-2\delta_\varepsilon}^{\nu_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon(x_2)|(i)| + \int_{x_2=\nu_\varepsilon^2-2\delta_\varepsilon}^{\nu_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon(x_2)|(ii)| + \int_{x_2=\nu_\varepsilon^2-2\delta_\varepsilon}^{\nu_\varepsilon^2+2\delta_\varepsilon} \psi_\varepsilon(x_2)|(iii)|, \quad \text{(IV.2.64)}$$

as $t \to t_\varepsilon$.

We now make the additional change of variable $(x_1, y) = (x_1 - v_\varepsilon^1, y - v_\varepsilon^1)$, and thus the new domain is $\iota = [\zeta_\varepsilon^{2+} - v_\varepsilon^1, \zeta_\varepsilon^{3-} - v_\varepsilon^1] \times [-\eta_\varepsilon, \eta_\varepsilon]$. We first consider the term (i) of (IV.2.63). We obtain from (IV.1.18) and (IV.1.19),

$$
\begin{aligned}
|(i)| &\leqslant \int\int_\iota \frac{1}{|\phi(t, v_\varepsilon^1 + x_1, x_2) - \phi(t, v_\varepsilon + y, x_2)|^{1/2-2\lambda}} \frac{C_\varepsilon |\ln(t_\varepsilon - t)|^2}{\phi_{x_1}(t, v_\varepsilon^1 + x_1, x_2)\phi_{x_1}(t, v_\varepsilon^1 + y, x_2)} \\
&\leqslant C_\varepsilon \frac{|\ln(t_\varepsilon - t)|^2}{(t_\varepsilon - t)^{1/2-2\lambda}} \int\int_\iota \frac{1}{|x_1 - y|^{1/2-2\lambda}} \frac{1}{x_1^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)} \frac{1}{y^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)} \\
&\leqslant_{Hölder} \frac{C_\varepsilon |\ln(t_\varepsilon - t)|^2}{(t_\varepsilon - t)^{1/2-2\lambda}} \left(\int\int_\iota \frac{1}{|x_1 - y|^{3/4-3\lambda}} \right)^{2/3} \left(\int_{x_1=0}^\infty \frac{1}{\left(x_1^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3} \right)^{1/3} \\
&\qquad \cdot \left(\int_{y=0}^\infty \frac{1}{\left(y^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3} \right)^{1/3} \\
&\qquad\qquad \leqslant \frac{C_\varepsilon |\ln(t_\varepsilon - t)|^2}{(t_\varepsilon - t)^{1/2-2\lambda}} \frac{1}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{10/6}}. \qquad \text{(IV.2.65)}
\end{aligned}
$$

Now, using (IV.2.48) and the inequality (IV.2.65), we obtain

$$
\begin{aligned}
\left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2)(i) \right| &\leqslant \frac{C_\varepsilon |\ln(t_\varepsilon - t)|^2}{(t_\varepsilon - t)^{1/2-2\lambda}} 2 \int_{x_2=0}^\infty \frac{1}{\sqrt{x_2}(x_2 + (t_\varepsilon - t))^{5/3}} \\
&\leqslant \frac{C_\varepsilon |\ln(t_\varepsilon - t)|^2}{(t_\varepsilon - t)^{5/3-2\lambda}}. \qquad \text{(IV.2.66)}
\end{aligned}
$$

Remark IV.2.2. *Here, we used that*

$$
\int_{x=0}^\infty \frac{1}{\sqrt{x}} \frac{1}{(x+a)^{5/3}} = \frac{B(\frac{1}{2}, \frac{7}{6})}{a^{7/6}},
$$

where B is the Euler integral of the first kind.

For (ii), we obtain using the triangular inequality in (IV.1.18) as well as (IV.1.19) and (IV.1.20),

$$\begin{aligned}
|(ii)| &\leqslant \int\int_\iota \frac{1}{|\phi(t,v^1_\varepsilon+x_1,x_2)-\phi(t,v^2_\varepsilon+y,x_2)|^{1/2-2\lambda}} \frac{C_\varepsilon|\ln(t_\varepsilon-t)||\phi_{x_1x_1}(t,v^1_\varepsilon+y,x_2)|}{\phi_{x_1}(t,v^1_\varepsilon+x_1,x_2)\phi^2_{x_1}(t,v^1_\varepsilon+y,x_2)}\\
&\leqslant \frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{(t_\varepsilon-t)^{1/2-2\lambda}}\int\int_\iota \frac{1}{|x_1-y|^{1/2-2\lambda}}\frac{1}{x_1^2+(x_2-v^2_\varepsilon)^2+(t_\varepsilon-t)}\frac{|y|+|x_2-v^2_\varepsilon|+(t_\varepsilon-t)}{(y^2+(x_2-v^2_\varepsilon)^2+(t_\varepsilon-t))^2}\\
&\leqslant_{H\ddot{o}lder} \frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{(t_\varepsilon-t)^{1/2-2\lambda}}\left(\int\int_\iota \frac{1}{|x_1-y|^{3/4-3\lambda}}\right)^{2/3}\left(\int_{x_1=0}^{\infty}\frac{1}{(x_1^2+(x_2-v^2_\varepsilon)^2+(t_\varepsilon-t))^3}\right)^{1/3}\\
&\qquad\cdot\left(\int_{y=0}^{\infty}\frac{(y+|v^2_\varepsilon-x_2|+(t_\varepsilon-t))^3}{(y^2+(x_2-v^2_\varepsilon)^2+(t_\varepsilon-t))^6}\right)^{1/3}\\
&\leqslant \frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{(t_\varepsilon-t)^{1/2-2\lambda}}\frac{1}{((x_2-v^2_\varepsilon)^2+(t_\varepsilon-t))^{5/6}}\frac{1}{((x_2-v^2_\varepsilon)^2+(t_\varepsilon-t))^{4/3}}. \quad \text{(IV.2.67)}
\end{aligned}$$

Now, integrating (IV.2.67) with respect to x_2 and using (IV.2.48) leads to

$$\begin{aligned}
\left|\int_{x_2=v^2_\varepsilon-2\delta_\varepsilon}^{v^2_\varepsilon+2\delta_\varepsilon}\psi^2_\varepsilon(x_2)(ii)\right| &\leqslant \frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{(t_\varepsilon-t)^{1/2-2\lambda}}2\int_{x_2=0}^{\infty}\frac{1}{(x_2^2+(t_\varepsilon-t))^{13/6}}\\
&\leqslant \frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{(t_\varepsilon-t)^{1/2-2\lambda}}\int_{x_2=0}^{\infty}\frac{1}{\sqrt{x_2}(x_2+(t_\varepsilon-t))^{13/6}} \leqslant \frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{(t_\varepsilon-t)^{13/6-2\lambda}}. \quad \text{(IV.2.68)}
\end{aligned}$$

Remark IV.2.3. *Here, we used that*

$$\int_{x=0}^{\infty}\frac{1}{\sqrt{x}}\frac{1}{(x+a)^{13/6}}=\frac{B(\frac{1}{2},\frac{5}{3})}{a^{5/3}},$$

where B is the Euler integral of the first kind.

The case of (iii) is identical, and we obtain

$$\left|\int_{x_2=v^2_\varepsilon-2\delta_\varepsilon}^{v^2_\varepsilon+2\delta_\varepsilon}\psi^2_\varepsilon(x_2)(iii)\right| \leqslant \frac{C_\varepsilon|\ln(t_\varepsilon-t)|}{(t_\varepsilon-t)^{13/6}}. \quad \text{(IV.2.69)}$$

Now, we look at (iv) and exhibit a lower bound. We split the integrand A into two parts, the first part corresponds to terms of order one of $\phi_{x_1x_1}$, the second corresponds to the terms of superior orders, as expressed in (IV.1.18). We denote :

$$
\begin{aligned}
(iv)_1 &= \int\int_\iota \frac{C_\varepsilon^1 x_1 + C_\varepsilon^2(x_2 - v_\varepsilon^2) + C_\varepsilon^3(t_\varepsilon - t)}{|\phi(t, v_\varepsilon^1 + x_1, x_2) - \phi(t, v_\varepsilon^2 + y, x_2)|^{1/2-2\lambda}} \\
&\qquad \cdot \frac{C_\varepsilon^1 y + C_\varepsilon^2(x_2 - v_\varepsilon^2) + C_\varepsilon^3(t_\varepsilon - t)}{\phi_{x_1}(y)^2 \phi_{x_1}(x_1)^2} \\
(iv)_2 &= \int\int_\iota \frac{\sum_{i+j+k=2}(x_1)^i (x_2 - v_\varepsilon^2)^j (t_\varepsilon - t)^k f_{i,j,k}(t, x_1, x_2)}{|\phi(t, v_\varepsilon^1 + x_1, x_2) - \phi(t, v_\varepsilon^2 + y, x_2)|^{1/2-2\lambda}} \\
&\qquad \cdot \frac{C_\varepsilon^1 y + C_\varepsilon^2(x_2 - v_\varepsilon^2) + C_\varepsilon^3(t_\varepsilon - t)}{\phi_{x_1}(y)^2 \phi_{x_1}(x_1)^2} \\
(iv)_3 &= \int\int_\iota \frac{C_\varepsilon^1 x_1 + C_\varepsilon^2(x_2 - v_\varepsilon^2) + C_\varepsilon^3(t_\varepsilon - t)}{|\phi(t, v_\varepsilon^1 + x_1, x_2) - \phi(t, v_\varepsilon^2 + y, x_2)|^{1/2-2\lambda}} \\
&\qquad \cdot \frac{\sum_{i+j+k=2}(y)^i (x_2 - v_\varepsilon^2)^j (t_\varepsilon - t)^k f_{i,j,k}(t, x_1, x_2)}{\phi_{x_1}(y)^2 \phi_{x_1}(x_1)^2} \\
(iv)_4 &= \int\int_\iota \frac{\sum_{i+j+k=2}(x_1)^i (x_2 - v_\varepsilon^2)^j (t_\varepsilon - t)^k f_{i,j,k}(t, x_1, x_2)}{|\phi(t, v_\varepsilon^1 + x_1, x_2) - \phi(t, v_\varepsilon^2 + y, x_2)|^{1/2-2\lambda}} \\
&\qquad \cdot \frac{\sum_{i+j+k=2}(y)^i (x_2 - v_\varepsilon^2)^j (t_\varepsilon - t)^k f_{i,j,k}(t, x_1, x_2)}{\phi_{x_1}(y)^2 \phi_{x_1}(x_1)^2}
\end{aligned}
\tag{IV.2.70}
$$

We will show that

$$
\begin{aligned}
&\left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2)(iv)_1 \right| >> \left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2)(iv)_2 \right| \\
&\qquad + \left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2)(iv)_3 \right| + \left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2)(iv)_4 \right|, \quad \text{as } t \to t_\varepsilon.
\end{aligned}
\tag{IV.2.71}
$$

First, we look at $(iv)_1$.

We more precisely consider the integral

$$(iv)_1^1 = \int\int_\iota \frac{x_1}{|\phi(t, v_\varepsilon^1 + x_1, x_2) - \phi(t, v_\varepsilon^2 + y, x_2)|^{1/2-2\lambda}} \cdot \frac{y}{\phi_{x_1}(y)^2 \phi_{x_1}(x_1)^2} \qquad \text{(IV.2.72)}$$

Again, because of $(y - v_\varepsilon)$ antisymmetric properties but the fact that $\frac{1}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}}$ will concentrate the weight near $x_1 = y$, we have to split the domain into four parts, using the same idea as in chapter II.

$$\begin{aligned}
&\alpha_{x_2} = \{(x_1, y) \in [\zeta_\varepsilon^{2+}, v_\varepsilon] \times [v_\varepsilon - \eta_\varepsilon, v_\varepsilon]\},\ \ \beta_{x_2} = \{(x_1, y) \in [v_\varepsilon, \zeta_\varepsilon^{3-}] \times [v_\varepsilon - \eta_\varepsilon, v_\varepsilon]\} \\
&\gamma_{x_2} = \{(x_1, y) \in [\zeta_\varepsilon^{2+}, v_\varepsilon] \times [v_\varepsilon, v_\varepsilon + \eta_\varepsilon]\},\ \ \delta_{x_2} = \{(x_1, y) \in [v_\varepsilon, \zeta_\varepsilon^{3-}] \times [v_\varepsilon, v_\varepsilon + \eta_\varepsilon]\} \\
&\alpha = \alpha_{x_2} \times [v_\varepsilon^2 - 2\delta_\varepsilon, v_\varepsilon^2 + 2\delta_\varepsilon],\ \ \beta = \beta_{x_2} \times [v_\varepsilon^2 - 2\delta_\varepsilon, v_\varepsilon^2 + 2\delta_\varepsilon], \\
&\gamma = \gamma_{x_2} \times [v_\varepsilon^2 - 2\delta_\varepsilon, v_\varepsilon^2 + 2\delta_\varepsilon],\ \ \delta = \delta_{x_2} \times [v_\varepsilon^2 - 2\delta_\varepsilon, v_\varepsilon^2 + 2\delta_\varepsilon]
\end{aligned} \qquad \text{(IV.2.73)}$$

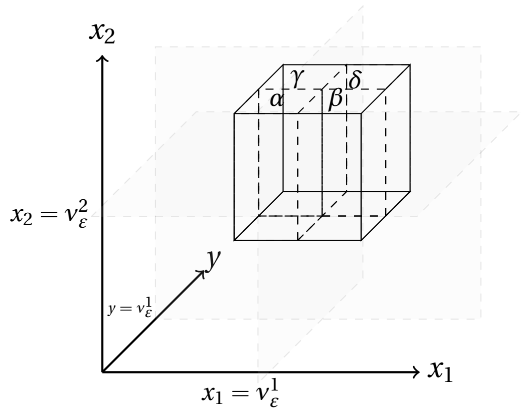


Figure IV.1 – Definition of α, β, γ and δ

And we have

$$(iv)_1^1 = \int\int_\alpha i(x_1, y) + \int\int_\beta i(x_1, y) + \int\int_\gamma i(x_1, y) + \int\int_\delta i(x_1, y). \qquad \text{(IV.2.74)}$$

We also make the change of variable $(x_1, y) = (x_1 - v_\varepsilon^1, y - v_\varepsilon^1)$ without relabelling the set nor the variables.

We will regroup the integral corresponding to δ and β. The symmetric case (γ and α) is identical. Because it is very similar to the computation we did in the previous chapters, we will skip a few details. We consider the term

$$\begin{aligned} J(x_2) &= \int\int_{\delta\cup\beta} \frac{C_\varepsilon}{|\phi(t,x_1)-\phi(t,y)|^{1/2-2\lambda}} \frac{x_1}{\phi_y(t,x_1)^2} \frac{y}{\phi_y(t,y)^2} \\ &= \int_{x_1=0}^{\kappa_\varepsilon}\int_{y=0}^{\kappa_\varepsilon} \left[\frac{y\cdot x_1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}\phi_y(t,\nu_\varepsilon+x_1)^2\phi_y(t,\nu_\varepsilon+y)^2} \right. \\ &\quad \left. - \frac{y\cdot x_1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}\phi_y(t,\nu_\varepsilon-y)^2\phi_y(t,\nu_\varepsilon+x_1)^2} \right] \end{aligned} \tag{IV.2.75}$$

We similarly decompose the integrand as follows.

$$\begin{aligned} &\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2} \frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon+y)^2} \\ &\quad - \frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2} \frac{\chi'(\nu_\varepsilon-y)\cdot y}{\phi_y(t,\nu_\varepsilon-y)^2} \\ &= \left[\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon+y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2} \frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon+y)^2} \right. \\ &\quad \left. - \frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2} \frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon+y)^2} \right] \\ &\quad + \left[\frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2} \frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon+y)^2} \right. \\ &\quad \left. - \frac{1}{|\phi(t,\nu_\varepsilon+x_1)-\phi(t,\nu_\varepsilon-y)|^{1/2-2\lambda}} \frac{\chi'(\nu_\varepsilon+x_1)\cdot x_1}{\phi_y(t,\nu_\varepsilon+x_1)^2} \frac{\chi'(\nu_\varepsilon+y)\cdot y}{\phi_y(t,\nu_\varepsilon-y)^2} \right] \\ &= D_1 + D_2. \end{aligned} \tag{IV.2.76}$$

To shorten a bit the notations, we will denote

$$\alpha_\pm = |\phi_\varepsilon(t,\nu_\varepsilon + x_1) - \phi_\varepsilon(t,\nu_\varepsilon \pm y)|. \tag{IV.2.77}$$

For D_1, we write

$$\begin{aligned}
&\frac{1}{|\phi_\varepsilon(t,\nu_\varepsilon + x_1) - \phi_\varepsilon(t,\nu_\varepsilon + y)|^{1/2-2\lambda}} - \frac{1}{|\phi_\varepsilon(t,\nu_\varepsilon + x_1) - \phi_\varepsilon(t,\nu_\varepsilon - y)|^{1/2-2\lambda}} \\
&\qquad = \frac{\alpha_- - \alpha_+}{\alpha_+^{1/2-2\lambda}\alpha_-^{1/2-2\lambda}\left(\alpha_+^{1/2+2\lambda} + \alpha_-^{1/2+2\lambda}\right)} + \frac{\alpha_+^{4\lambda} - \alpha_-^{4\lambda}}{\left(\alpha_+^{1/2+2\lambda} + \alpha_-^{1/2+2\lambda}\right)}
\end{aligned} \tag{IV.2.78}$$

It is clear that D_1 is nonnegative when $x_1 \geqslant y$. We now provide a lower bound for $D_{1,x_1\leqslant y}$. We will show that it is positive up to a smaller order term, and provide a lower bound for the positive term. We hence now consider $x_1 \leqslant y$.

We write,

$$\begin{aligned}
&\alpha_- - \alpha_+ = |\phi_\varepsilon(t,\nu_\varepsilon + x_1) - \phi_\varepsilon(t,\nu_\varepsilon - y)| - |\phi_\varepsilon(t,\nu_\varepsilon + x_1) - \phi_\varepsilon(t,\nu_\varepsilon + y)| \\
&= 2\phi_\varepsilon(t,\nu_\varepsilon + x_1) - \phi_\varepsilon(t,\nu_\varepsilon + y) - \phi_\varepsilon(t,\nu_\varepsilon - y) = \int_{s=-y}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon + s) - \int_{s=x_1}^{y} \phi_{\varepsilon,y}(t,\nu_\varepsilon + s) \\
&\qquad = \int_{s=-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon + s) + \int_{s=x_1}^{y} \left(\phi_{\varepsilon,y}(t,\nu_\varepsilon - s) - \phi_{\varepsilon,y}(t,\nu_\varepsilon + s)\right).
\end{aligned} \tag{IV.2.79}$$

We will use this idea to provide a lower bound for D_1.

$$\frac{1}{\alpha_+^{1/2-2\lambda}} - \frac{1}{\alpha_-^{1/2-2\lambda}} = \int_{s=\alpha_-}^{\alpha_+} -(\frac{1}{2}-2\lambda)\frac{1}{s^{3/2-2\lambda}} = C\int_{s=\alpha_+}^{\alpha_-} s^{-3/2+2\lambda}$$
$$= C\int_{s=\alpha_+}^{\alpha_+ + \int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)} s^{-3/2+2\lambda}$$
$$+ C\int_{s=\alpha_+ + \int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)}^{\alpha_+ + \int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z) + \int_{z=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)} s^{-3/2+2\lambda} = (i)+(ii), \tag{IV.2.80}$$

where (i) is nonnegative and (ii) is small. First, we make the following upper bound for $|(ii)|$. If $\int_{x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon - s) - \phi_{\varepsilon,y}(t,\nu_\varepsilon+s)) > 0$, then (ii) is nonnegative. Otherwise, we have $\alpha_+ + \int_{s=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s) + \int_{s=x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon - s) - \phi_{\varepsilon,y}(t,\nu_\varepsilon+s)) \leqslant \alpha_+ + \int_{s=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)$. We then proceed as follows

$$\left|\int_{s=\alpha_+ + \int_{-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+s)}^{\alpha_+ + \int_{z=-x_1}^{x_1}\phi_{\varepsilon,y}(t,\nu_\varepsilon+z) + \int_{z=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)} s^{-3/2-2\lambda}\right|$$
$$= \left|\int_{s=\alpha_-}^{\alpha_- - \int_{z=x_1}^{y}\left(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+z)\right)} s^{-3/2+2\lambda}\right|$$
$$\leqslant \left|\int_{z=x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon-z)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+z))\right| \cdot \alpha_-^{-3/2+2\lambda}. \tag{IV.2.81}$$

From the Taylor expansion, we obtain for x, y small enough (only depending on ε),

$$\left|\int_{z=x_1}^{y}(\phi_{\varepsilon,y}(t,\nu_\varepsilon-s)-\phi_{\varepsilon,y}(t,\nu_\varepsilon+s))\right| \leqslant C_1\left(y^4 - x_1^4 + (t_\varepsilon - t)^2 + (t-t_\varepsilon)(y^2-x_1^2)\right). \tag{IV.2.82}$$

This means that we obtain (up to a nonnegative contribution)

$$|(ii)| \leqslant C\left(y^4 + x_1^4 + (t_\varepsilon - t)\left(x_1^2+y^2\right) + (t_\varepsilon - t)^2\right)\cdot\frac{1}{\alpha_-^{3/2-2\lambda}}. \tag{IV.2.83}$$

We now go on with the lower bound for (i). Using the expression of (i) provided by (IV.2.80) as well as the mean value theorem, we obtain

$$
\begin{aligned}
(i) &= \int_{s=\alpha_+}^{\alpha_+ + \int_{z=-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon+z)} s^{-3/2+2\lambda} \\
&\geqslant \left(\int_{z=-x_1}^{x_1} \phi_{\varepsilon,y}(t,\nu_\varepsilon+z) \right) \cdot \frac{1}{\left(\phi_\varepsilon(t,\nu_\varepsilon+y)-\phi_\varepsilon(t,\nu_\varepsilon-x_1)\right)^{3/2-2\lambda}}. \\
&\geqslant C x_1 \phi_{\varepsilon,y}(t,c_1) \cdot \frac{1}{(y+x_1)^{3/2-2\lambda} \cdot \phi_{\varepsilon,y}(t,c_2)^{3/2-2\lambda}}. \qquad \text{(IV.2.84)}
\end{aligned}
$$

Now, since we have $x_1 \leqslant y$, we obtain from (II.2.16)

$$
\phi_{\varepsilon,y}(t,c_1) \geqslant C(t_\varepsilon - t), \qquad \text{(IV.2.85)}
$$

and

$$
\phi_{\varepsilon,y}(t,c_2) \leqslant C\left((t_\varepsilon - t) + y^2\right). \qquad \text{(IV.2.86)}
$$

Using (IV.2.85) and (IV.2.86) inside of (IV.2.84), we obtain

$$
(i) \geqslant C \frac{x_1(t_\varepsilon - t)}{(y+x_1)^{3/2-2\lambda} \cdot ((t_\varepsilon - t) + y^2 + (x_2 - \nu_\varepsilon^2)^2)^{3/2-2\lambda}}. \qquad \text{(IV.2.87)}
$$

Now, we obtain for D_1,

$$
\begin{aligned}
D_1 &= D_{x_1 \geqslant y} + D_{x_1 \leqslant y} \geqslant D_{x_1 \leqslant y} \\
&\geqslant C \iint_{x_1 \leqslant y} \frac{x_1(t_\varepsilon - t)}{(y+x_1)^{3/2-2\lambda} \cdot ((t_\varepsilon - t) + y^2)^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1)\cdot x_1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y)\cdot y}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + y)^2} \\
&\qquad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{x^4}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1)\cdot x_1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y)\cdot y}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + y)^2} \\
&\qquad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{y^4}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1)\cdot x_1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y)\cdot y}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + y)^2} \\
&\qquad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t)\cdot x_1^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1)\cdot x_1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y)\cdot y}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + y)^2} \\
&\qquad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t)\cdot y^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1)\cdot x_1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y)\cdot y}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + y)^2} \\
&\qquad - C \int_{x_1=0}^{\kappa_\varepsilon} \int_{y=x_1}^{\kappa_\varepsilon} \frac{(t_\varepsilon - t)^2}{\alpha_-^{3/2-2\lambda}} \frac{\chi'(\nu_\varepsilon + x_1)\cdot x_1}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + x_1)^2} \frac{\chi'(\nu_\varepsilon + y)\cdot y}{\phi_{\varepsilon,y}(t,\nu_\varepsilon + y)^2} \\
&= A_1 - B_1 - B_2 - B_3 - B_4 - B_5. \qquad \text{(IV.2.88)}
\end{aligned}
$$

Note that the proof also works for $\lambda = 0$. We will now show that $A_1 \to \infty$, and that J_2, J_3, J_4, D_2, B_1, B_2, B_3, B_4 and B_5 are of a smaller order. We first consider A_1. Here, we will consider A_1 because it gives the order of the main contribution, and showing that the other terms are of a smaller order is similar to what have been previously done. We will also consider the term B_1 to show how we deal with the smaller order terms. We will not consider B_i for $i > 1$ because it is similar to what have been previously done.

Using $|\phi_y(t, c(x_1, y))| \leqslant M_\varepsilon\left((x_1 - \nu_\varepsilon^1)^2 + (x_2 - \nu_\varepsilon^2)^2 + (t_\varepsilon - t)\right)$, on $x_1 > y$, we have from (IV.2.88) and the new change of variable $(x, y) = (r + z, r - z)$,

$$A_1 \geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2} \int_{z=0}^{r} \frac{(r-z)(t_\varepsilon - t)}{r^{3/2-2\lambda}} \frac{r+z}{\left((r+z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{7/2-2\lambda}} \cdot \frac{r-z}{\left((r-z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2}$$
$$\geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2} \frac{1}{r^{1/2-2\lambda}} \frac{(t_\varepsilon - t)}{\left((2r)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{11/2-2\lambda}} \int_{z=0}^{r} (r-z)^2$$
$$\geqslant C_\varepsilon \int_{r=0}^{\kappa_\varepsilon/2} \frac{(t_\varepsilon - t) \cdot r^{5/2+2\lambda}}{\left((2r)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{11/2-2\lambda}} \geqslant \frac{C_\varepsilon \cdot (t_\varepsilon - t)}{(t_\varepsilon - t)^{15/4-3\lambda}}$$
$$\geqslant \frac{C_\varepsilon}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{11/4-3\lambda}}. \quad \text{(IV.2.89)}$$

Hence, integrating (IV.2.89) with respect to x_2 and using the properties of ψ_ε^2 given in (III.2.30) yield

$$\int_{x_2=v_\varepsilon^2-2\delta_\varepsilon}^{v_\varepsilon^2+2\delta_\varepsilon} \int\int_\alpha \psi_\varepsilon^2(x_2) i(x_1, y, x_2) \geqslant \int_{x_2=v_\varepsilon^2-\delta_\varepsilon}^{v_\varepsilon^2+\delta_\varepsilon} \int\int_\alpha i(x_1, y, x_2)$$
$$\geqslant \int_{x_2=v_\varepsilon^2-\delta_\varepsilon}^{v_\varepsilon^2+\delta_\varepsilon} \frac{M_\varepsilon}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{11/4-3\lambda}} \geqslant \frac{M_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}. \quad \text{(IV.2.90)}$$

We now proceed with B_1. The case of B_2 is similar. We have since $\phi_{\varepsilon,y}(t,x) \geqslant (t_\varepsilon - t)$, with the mean value theorem

$$
\begin{aligned}
|B_1| \leqslant{}& \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \int_{z=0}^{r} \frac{(r-z)^4}{r^{3/2-2\lambda}} \\
&\cdot \frac{r+z}{\left((r+z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \cdot \frac{r-z}{\left((r-z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
\leqslant{}& \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^4}{r^{1/2-2\lambda}\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
&\cdot \int_{z=0}^{r} \frac{r-z}{\left((r-z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2}. \qquad \text{(IV.2.91)}
\end{aligned}
$$

Now, because

$$
\int_{s=0}^{\infty} \frac{s}{\left(s^2 + (t_\varepsilon - t)\right)^2} \leqslant \frac{C}{(t_\varepsilon - t)}, \qquad \text{(IV.2.92)}
$$

(IV.2.91) yields

$$
\begin{aligned}
|B_1| \leqslant{}& \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \cdot \frac{1}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)} \int_{r=0}^{\kappa_\varepsilon/2} \frac{r^{7/2+2\lambda}}{\left(r^2 + (t_\varepsilon - t)\right)^2} \\
\leqslant{}& \frac{C_\varepsilon}{(t_\varepsilon - t)^{3/2-2\lambda}} \cdot \frac{1}{\left((t_\varepsilon - t) + (x_2 - v_\varepsilon^2)^2\right)}. \qquad \text{(IV.2.93)}
\end{aligned}
$$

Overall, we obtain

$$
B_1(x_2) \leqslant \frac{C_\varepsilon}{\left((t_\varepsilon - t) + (x_2 - v_\varepsilon^2)^2\right)^{4/4}} \cdot \frac{1}{(t_\varepsilon - t)^{6/4-2\lambda}}. \qquad \text{(IV.2.94)}
$$

Hence, integrating with respect to x_2 and using the properties of ψ_ε^2 given in (III.2.30) yield

$$
\int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} B_1(x_2) \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^2}. \qquad \text{(IV.2.95)}
$$

At the end, we obtain

$$\int_{x_2}\int\int_{\beta\cup\delta} i(x_1,x_2,y) \geqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4-3\lambda}}. \tag{IV.2.96}$$

We only studied $(iv)_1^1$. We now need to do an estimation for all the other terms that arise when we expand $(iv)_1$ whose expression is given by (IV.2.70).

First, we look at

$$(iv)_1^2 = \int\int \frac{y(x_2 - v_\varepsilon^2)}{|\phi(t,x_1,x_2) - \phi(t,y,x_2)|^{1/2-2\lambda}} \frac{1}{\phi_{x_1}^2(x_1)\phi_{x_1}^2(y)}. \tag{IV.2.97}$$

We will use the fact that the integrand is "almost" odd in $(x - v_\varepsilon^1, y - v_\varepsilon^1)$. We will now consider four domains $\iota^{+,+}$, $\iota^{+,-}$, $\iota^{-,-}$ and $\iota^{-,+}$ (for the new variables $y = y - v_\varepsilon^1$, $x = x - v_\varepsilon^1$.) They are defined as

$$\begin{aligned} &\iota_{+,+} = \{x_1, y > 0\}, \;\; \iota_{+,-} = \{x_1 > 0, y < 0\}, \\ &\iota_{-,-} = \{x_1, y < 0\}, \;\; \iota_{-,+} = \{x_1 < 0, y > 0\}. \end{aligned} \tag{IV.2.98}$$

We will consider the two integrals

$$\begin{aligned} I_1 &= \int\int_{\iota^{+,+}\bigcup\iota^{-,-}} \frac{y(x_2 - v_\varepsilon^2)}{\phi_{x_1}(x_1)^2\phi_{x_1}(y)^2|\phi(x_1) - \phi(y)|^{1/2-2\lambda}} \\ I_2 &= \int\int_{\iota^{+,-}\bigcup\iota^{-,+}} \frac{y(x_2 - v_\varepsilon^2)}{\phi_{x_1}(x_1)^2\phi_{x_1}(y)^2|\phi(x_1) - \phi(y)|^{1/2-2\lambda}}. \end{aligned} \tag{IV.2.99}$$

It will be easier to deal with I_2. For I_1 however, we will need some preliminaries. We first perform the change of variable for the part $x_1 < 0, y < 0$ of the form $x = -x, y = -y$. We hence obtain for I_1

$$I_1 = \int\int_{\iota^{+,+}} y(x_2 - v_\varepsilon^2)$$
$$\cdot \left[\frac{1}{\phi_{x_1}(x_1)^2 \phi_{x_1}(y_1)^2 |\phi(x_1) - \phi(y)|^{1/2-2\lambda}} - \frac{1}{\phi_{x_1}(-x_1)^2 \phi_{x_1}(-y_1)^2 |\phi(-x_1) - \phi(-y)|^{1/2-2\lambda}} \right]. \tag{IV.2.100}$$

We simplify (we add and substract $\frac{1}{\phi_{x_1}(x_1)^2\phi_{x_1}(y)^2|\phi(-x)-\phi(-y)|^{1/2-2\lambda}}$) and obtain

$$I_1 = \int\int_{\iota^{+,+}} \frac{y(x_2 - v_\varepsilon)}{\phi_{x_1}(x_1)^2 \phi_{x_1}(y)^2} \left[\frac{1}{|\phi(x_1) - \phi(y)|^{1/2-2\lambda}} - \frac{1}{|\phi(-x) - \phi(-y)|^{1/2-2\lambda}} \right]$$
$$+ \int\int_{\iota^{+,+}} \frac{y(x_2 - v_\varepsilon^2)}{|\phi(-x) - \phi(-y)|^{1/2-2\lambda}} \left[\frac{1}{\phi_{x_1}(x_1)^2 \phi_{x_1}(y)^2} - \frac{1}{\phi_{x_1}(-x_1)^2 \phi_{x_1}(-y)^2} \right]$$
$$= J_1 + J_2. \tag{IV.2.101}$$

We denote for simplicity

$$\beta_\pm = |\phi(\pm x) - \phi(\pm y)|. \tag{IV.2.102}$$

We consider the following simplification inside J_1. Also, we work on the set where $y > x$ by symmetry. Call this set $\iota_y^{+,+}$.

$$\frac{1}{\beta_+^{1/2-2\lambda}} - \frac{1}{\beta_-^{1/2-2\lambda}} = \frac{\beta_- - \beta_+}{\beta_-^{1/2-2\lambda} \beta_+^{1/2-2\lambda} \left(\beta_-^{1/2+2\lambda} + -\beta_+^{1/2+2\lambda} \right)} + \frac{\beta_+^{4\lambda} - \beta_-^{4\lambda}}{\beta_+^{1/2+2\lambda} + \beta_-^{1/2+2\lambda}}. \tag{IV.2.103}$$

Now, because of the symmetry of the problem, $\beta_- - \beta_+$ is of a smaller order. Indeed, considering that the Taylor expansion of ϕ_{x_1} around v_ε^1 is of the form (for the old variable x_1)

$$\begin{aligned}\phi_{x_1}(t, v_\varepsilon^1 + y, x_2) = C_1 y^2 + C_2(x_2 - v_\varepsilon^2)^2 + C_3(t_\varepsilon - t)\\ + f_1(t,y,x_2)(x_2 - v_\varepsilon^2)^3 + f_2(t,y,x_2)y^3 + f_3(t,y,x_2)(t_\varepsilon - t)y\\ + f_4(t,y,x_2)(t_\varepsilon - t)(x_2 - v_\varepsilon^2) + f_5(t,y,x_2)y(v_\varepsilon - x_2)^2 + f_6(t,y,x_2)(t_\varepsilon - t)^2\\ + f_7(t,y,x_2)(v_\varepsilon - x_2)y^2. \quad \text{(IV.2.104)}\end{aligned}$$

We now focus exclusively on the first term of (IV.2.103). Call J_1^1 the corresponding integral. We obtain that the term of order 2 in the space variable and in order 1 in the time variable cancel out. Indeed, for the first part, we have

$$\frac{\beta_- - \beta_+}{\beta_-^{1/2-2\lambda}\beta_+^{1/2-2\lambda}\left(\beta_-^{1/2+2\lambda} + -\beta_+^{1/2+2\lambda}\right)} = \frac{\int_{s=\min(x,y)}^{\max(x,y)}(\phi_{x_1}(s) - \phi_{x_1}(-s))}{\beta_-^{1/2-2\lambda}\beta_+^{1/2-2\lambda}\left(\beta_-^{1/2+2\lambda} + -\beta_+^{1/2+2\lambda}\right)} \quad \text{(IV.2.105)}$$

and since we have, (we assume $x \leqslant y$ by symmetry), by (IV.2.104),

$$\begin{aligned}\left|\int_{s=x}^{y}(\phi_{x_1}(s) - \phi_{x_1}(-s))\right| \leqslant C(y-x)|x_2 - v_\varepsilon^2|^3 + C(y-x)y^3 + C(y-x)(t_\varepsilon - t)^2\\ + C(y-x)y(t_\varepsilon - t) + C(y-x)|x_2 - v_\varepsilon^2|(t_\varepsilon - t). \quad \text{(IV.2.106)}\end{aligned}$$

Plugging (IV.2.105) and (IV.2.106) into the definition of J_1^1 gives

$$\begin{aligned}J_1^1 \simeq \int\int_{\iota^{+,+}} \frac{Cy(x_2 - v_\varepsilon^2)(x-y)^4\phi_{x_1}(x_1)^{-2}\phi_{x_1}(y)^{-2}}{|\phi(-y) - \phi(-x)|^{1/2-2\lambda}|\phi(-x) - \phi(-y)|^{1/2-2\lambda}}\\ \frac{1}{\left(|\phi(y) - \phi(x)|^{1/2+2\lambda} + |\phi(-x) - \phi(-y)|^{1/2+2\lambda}\right)}. \quad \text{(IV.2.107)}\end{aligned}$$

By symmetry, we again assume that $y \geqslant 0$, and perform the change of variable $x = r - z$, $y = r + z$. ($z \in [0, r]$).

Now, we have

$$
\begin{aligned}
|J_1^1| &\leqslant C|x_2 - \nu_\varepsilon^2| \int\int \frac{(r+z)(2z)^4}{(2z)^{1.5-2\lambda}\phi_{x_1}(y)^{3.5-2\lambda}\phi_{x_1}(x_1)^2} \\
&\leqslant \int_r \int_{z=0}^{r} \frac{C|x_2-\nu_\varepsilon^2|(r+z)(2z)^4}{(2z)^{1.5-2\lambda}\left((r+z)^2+(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^{3.5-2\lambda}\left((r-z)^2+(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^2} \\
&\leqslant \int_r \int_{z=0}^{r/2} \frac{C|x_2-\nu_\varepsilon^2|r(2z)^4}{(2z)^{1.5-2\lambda}\left((r)^2+(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^{5.5-2\lambda}} \\
&+ \int_r \int_{z=r/2}^{r} \frac{C|x_2-\nu_\varepsilon^2|(r)(z)^4}{(r)^{1.5-2\lambda}\left((r)^2+(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^{3.5-2\lambda}\left((r-z)^2+(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^2} \\
&= (i) + (ii). \qquad \text{(IV.2.108)}
\end{aligned}
$$

We obtain for (i),

$$
\begin{aligned}
(i) &\leqslant \int_r C|x_2-\nu_\varepsilon^2| r \int_{z=0}^{r/2} \frac{C(2z)^{2.5+2\lambda}}{\left((r)^2+(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^{5.5-2\lambda}} \\
&\leqslant \int_r |x_2-\nu_\varepsilon^2| r^{4.5+2\lambda} \frac{C}{\left((r)^2+(x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^{5.5-2\lambda}} \\
&\leqslant |x_2-\nu_\varepsilon^2| \frac{C}{\left((x_2-\nu_\varepsilon^2)^2+(t_\varepsilon-t)\right)^{11/4-\lambda}}. \qquad \text{(IV.2.109)}
\end{aligned}
$$

Now, integrating with respect to x_2 yields

$$
\int_{x_2} \psi_\varepsilon^2(x_2)|(i)| \leqslant \frac{C}{(t_\varepsilon-t)^{7/4}}, \qquad \text{(IV.2.110)}
$$

which is indeed of a smaller order.

Going on with (ii), we get

$$
\begin{aligned}
(ii) &\leqslant \int_r \frac{C|x_2 - v_\varepsilon^2|(r)^5}{(r)^{1.5-2\lambda}\left((r)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{3.5-2\lambda}} \\
&\qquad\qquad \cdot \int_{z=0}^{r/2} \frac{1}{\left((z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
&\leqslant \int_r \frac{C|x_2 - v_\varepsilon^2|(r)^5}{(r)^{1.5-2\lambda}\left((r)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{3.5-2\lambda}} \cdot \frac{C}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)} \\
&\leqslant \frac{C}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{11/4-\lambda}}.
\end{aligned} \tag{IV.2.111}
$$

Now, integrating with respect to x_2 yields

$$
\int_{x_2} \psi_\varepsilon^2(x_2)|(ii)| \leqslant \frac{C}{(t_\varepsilon - t)^{7/4}}, \tag{IV.2.112}
$$

which is indeed of a smaller order.

We go on with J_1^2, which corresponds to the second term in (IV.2.103). Again, we will use the fact that

$$
\beta_+^{4\lambda} - \beta_-^{4\lambda} = \int_{\beta_-}^{\beta_+} 4\lambda s^{4\lambda-1}, \tag{IV.2.113}
$$

and that $|\beta_+ - \beta_-|$ is of a smaller order, as we have previously shown. Indeed, we get

$$
\left|\beta_-^{4\lambda} - \beta_+^{4\lambda}\right| \leqslant 4\lambda|\beta_- - \beta_+| \cdot \left(|\beta_-|^{4\lambda-1} + |\beta_+|^{4\lambda-1}\right) \tag{IV.2.114}
$$

The proof of this is very similar to what has been done to find the lower bound of I_2 so we will not do it explicitly here. At the end, the term is of a smaller order when $\lambda > 0$.

Now, we go on with the term J_2 defined in (IV.2.101). We will introduce the two difference terms

$$\frac{1}{\phi_{x_1}(x_1)^2\phi_{x_1}(y)^2} - \frac{1}{\phi_{x_1}(-x_1)^2\phi_{x_1}(-y)^2} = \left(\frac{1}{\phi_{x_1}(x_1)^2\phi_{x_1}(y)^2} - \frac{1}{\phi_{x_1}(x_1)^2\phi_{x_1}(-y)^2}\right) + \left(\frac{1}{\phi_{x_1}(x_1)^2\phi_{x_1}(-y)^2} - \frac{1}{\phi_{x_1}(-x_1)^2\phi_{x_1}(-y)^2}\right) = j_2^1 + j_2^2, \quad \text{(IV.2.115)}$$

and define J_2^1 and J_2^2 as the two corresponding integrals.

Now for J_2^1, we write the simplication

$$\frac{1}{\phi_{x_1}(x_1)^2\phi_{x_1}(y)^2} - \frac{1}{\phi_{x_1}(x_1)^2\phi_{x_1}(-y)^2} = \frac{(\phi_{x_1}(-y)-\phi_{x_1}(y))\,(\phi_{x_1}(-y)+\phi_{x_1}(y))}{\phi_{x_1}(-y)^2\phi_{x_1}(y)^2\phi_{x_1}(x_1)^2}. \quad \text{(IV.2.116)}$$

Now, using (IV.2.104) inside (IV.2.116), we obtain that

$$\left|J_2^1\right| \leqslant \int\int_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{|\phi_{x_1}(-y)-\phi_{x_1}(y)|}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} + \int\int_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{|\phi_{x_1}(-y)-\phi_{x_1}(y)|}{\phi_{x_1}(-y)\phi_{x_1}(y)\phi_{x_1}(x_1)^2}. \quad \text{(IV.2.117)}$$

Because the two terms are similar, we only consider the first term of (IV.2.117). We get

$$\begin{aligned}
&\iint_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{|\phi_{x_1}(-y)-\phi_{x_1}(y)|}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} \\
&\qquad \leqslant \iint_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{C_1 y^3}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} \\
&\qquad + \iint_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{C_2 y^2|x_2-v_\varepsilon^2|}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} \\
&\qquad + \iint_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{C_3 y|x_2-v_\varepsilon|^2}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} \\
&\qquad + \iint_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{C_4 |x_2-v_\varepsilon^2|^3}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} \\
&\qquad + \iint_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{C_5 (t_\varepsilon-t)y}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} \\
&\qquad + \iint_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{C_6 (t_\varepsilon-t)|x_2-v_\varepsilon^2|}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} \\
&\qquad + \iint_{\iota^{+,+}} \frac{y(x_2-v_\varepsilon^2)}{|\phi(-x)-\phi(-y)|^{1/2-2\lambda}} \frac{C_7 (t_\varepsilon-t)^2}{\phi_{x_1}(-y)^2\phi_{x_1}(y)\phi_{x_1}(x_1)^2} \\
&\qquad\qquad = (i)+(ii)+(iii)+(iv)+(v)+(vi)+(vii). \qquad \text{(IV.2.118)}
\end{aligned}$$

Now, the case of (i), (ii), (iii) and (iv) are similar, also (v) and (vi) are similar. This means that we will only consider (i), (v) and (vii).

For (i), we consider the same change of variable as previously. We obtain

$$\begin{aligned}
&(i) \leqslant \frac{C}{(t_\varepsilon-t)^{1/2-2\lambda}} \\
&\cdot \int_r \int_{z=0}^{r} \frac{(r+z)^4|x_2-v_\varepsilon^2|}{z^{1/2-2\lambda}\left((r+z)^2+(x_2-v_\varepsilon^2)^2+(t_\varepsilon-t)\right)^3\left((r-z)^2+(x_2-v_\varepsilon^2)^2+(t_\varepsilon-t)\right)^2} \\
&\qquad \leqslant \frac{C}{(t_\varepsilon-t)^{1/2-2\lambda}} \int_r \int_{z=0}^{r/2} \frac{r^4|x_2-v_\varepsilon^2|}{z^{1/2-2\lambda}\left(r^2+(x_2-v_\varepsilon^2)^2+(t_\varepsilon-t)\right)^5} \\
&+ \frac{C}{(t_\varepsilon-t)^{1/2-2\lambda}} \int_r \int_{z=r/2}^{r} \frac{r^4|x_2-v_\varepsilon|}{r^{1/2-2\lambda}\left(r^2+(x_2-v_\varepsilon^2)^2+(t_\varepsilon-t)\right)^3\left((r-z)^2+(x_2-v_\varepsilon^2)^2+(t_\varepsilon-t)\right)^2} \\
&\qquad\qquad = (i)_1+(i)_2. \qquad \text{(IV.2.119)}
\end{aligned}$$

For $(i)_1$, we get

$$(i)_1 \leqslant \frac{C}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_r \frac{r^{4.5+2\lambda}|x_2 - v_\varepsilon|}{(r^2 + (v_\varepsilon^2 - x_2)^2 + (t_\varepsilon - t))^5}$$
$$\leqslant \frac{C}{(t_\varepsilon - t)^{1/2-2\lambda}} \frac{1}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{9/4-3\lambda}}. \quad \text{(IV.2.120)}$$

Now, integrating (IV.2.120) with respect to x_2 yields

$$\int_{x_2} |\psi_\varepsilon^2(x_2)||(i)_1| \leqslant \frac{C}{(t_\varepsilon - t)^{7/4-3\lambda}}, \quad \text{(IV.2.121)}$$

which is of a smaller order.

For $(i)_2$, we obtain

$$(i)_2 \leqslant \frac{C}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_r \frac{r^{3.5+2\lambda}|x_2 - v_\varepsilon^2|}{\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3}$$
$$\cdot \int_{z=0} \frac{1}{\left(z^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2}$$
$$\leqslant \frac{C}{(t_\varepsilon - t)^{1/2-2\lambda}} \int_r \frac{r^{3.5+2\lambda}|x_2 - v_\varepsilon^2|}{\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3} \cdot \frac{C}{((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t))^{3/2}}$$
$$\leqslant \frac{C}{(t_\varepsilon - t)^{1/2-2\lambda}} \frac{1}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{9/4-3\lambda}}. \quad \text{(IV.2.122)}$$

Now, integrating (IV.2.122) with respect to x_2 yields

$$\int_{x_2} |\psi_\varepsilon^2(x_2)||(i)_2| \leqslant \frac{C}{(t_\varepsilon - t)^{7/4-3\lambda}}, \quad \text{(IV.2.123)}$$

which is of a smaller order. The same result holds for (ii), (iii) and (iv). We go on and consider now (v). We consider the same change of variable as previously. We obtain

$$
\begin{aligned}
(v) \leqslant &\frac{C}{(t_\varepsilon - t)^{1/2-2\lambda}} \\
&\cdot \int_r \int_{z=0}^r \frac{(r+z)^2|x_2 - v_\varepsilon^2|(t_\varepsilon - t)}{z^{1/2-2\lambda}\left((r+z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3 \left((r-z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
&\leqslant C(t_\varepsilon - t)^{1/2+2\lambda} \int_r \int_{z=0}^{r/2} \frac{r^2|x_2 - v_\varepsilon^2|}{z^{1/2-2\lambda}\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^5} \\
&+ C(t_\varepsilon - t)^{1/2+2\lambda} \int_r \int_{z=r/2}^r \frac{r^2|x_2 - v_\varepsilon|}{r^{1/2-2\lambda}\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3 \left((r-z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
&= (v)_1 + (v)_2. \qquad \text{(IV.2.124)}
\end{aligned}
$$

For $(v)_1$, we get

$$
\begin{aligned}
(v)_1 &\leqslant C(t_\varepsilon - t)^{1/2+2\lambda} \int_r \frac{r^{2.5+2\lambda}|x_2 - v_\varepsilon|}{(r^2 + (v_\varepsilon^2 - x_2)^2 + (t_\varepsilon - t))^5} \\
&\leqslant C(t_\varepsilon - t)^{1/2+2\lambda} \frac{1}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{13/4-\lambda}}. \qquad \text{(IV.2.125)}
\end{aligned}
$$

Now, integrating (IV.2.125) with respect to x_2 yields

$$
\int_{x_2} |\psi_\varepsilon^2(x_2)||(v)_1| \leqslant \frac{C}{(t_\varepsilon - t)^{7/4-3\lambda}}, \qquad \text{(IV.2.126)}
$$

which is of a smaller order.

For $(v)_2$, we obtain

$$
\begin{aligned}
(v)_2 \leqslant C(t_\varepsilon - t)^{1/2+2\lambda} \int_r \frac{r^{1.5+2\lambda}|x_2 - v_\varepsilon^2|}{\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3} \\
\cdot \int_{z=0} \frac{1}{\left(z^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
\leqslant C(t_\varepsilon - t)^{1/2+2\lambda} \int_r \frac{r^{1.5+2\lambda}|x_2 - v_\varepsilon^2|}{\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3} \cdot \frac{C}{((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t))^{3/2}} \\
\leqslant C(t_\varepsilon - t)^{1/2+2\lambda} \frac{1}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{13/4}}. \quad \text{(IV.2.127)}
\end{aligned}
$$

Now, integrating (IV.2.127) th respect to x_2 yields

$$
\int_{x_2} |\psi_\varepsilon^2(x_2)||(v)_2| \leqslant \frac{C}{(t_\varepsilon - t)^{7/4-3\lambda}}, \quad \text{(IV.2.128)}
$$

which is of a smaller order. Lastly, we go on with (vii).

$$
\begin{aligned}
(vii) \leqslant \frac{C}{(t_\varepsilon - t)^{1/2-2\lambda}} \\
\cdot \int_r \int_{z=0}^r \frac{|x_2 - v_\varepsilon^2|(t_\varepsilon - t)^2}{z^{1/2-2\lambda}\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3 \left((r-z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
\leqslant C(t_\varepsilon - t)^{3/2+2\lambda} \int_r \int_{z=0}^{r/2} \frac{|x_2 - v_\varepsilon^2|}{z^{1/2-2\lambda}\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^5} \\
+ C(t_\varepsilon - t)^{3/2+2\lambda} \int_r \int_{z=r/2}^r \frac{|x_2 - v_\varepsilon|}{r^{1/2-2\lambda}\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3 \left((r-z)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
= (vii)_1 + (vii)_2. \quad \text{(IV.2.129)}
\end{aligned}
$$

For $(v)_1$, we get

$$
\begin{aligned}
(vii)_1 \leqslant C(t_\varepsilon - t)^{3/2+2\lambda} \int_r \frac{r^{0.5+2\lambda}|x_2 - v_\varepsilon|}{(r^2 + (v_\varepsilon^2 - x_2)^2 + (t_\varepsilon - t))^5} \\
\leqslant C(t_\varepsilon - t)^{3/2+2\lambda} \frac{1}{((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t))^{17/4-\lambda}}.
\end{aligned} \tag{IV.2.130}
$$

Now, integrating (IV.2.130) with respect to x_2 yields

$$
\int_{x_2} |\psi_\varepsilon^2(x_2)||(vii)_1| \leqslant \frac{C}{(t_\varepsilon - t)^{13/4-3\lambda-3/2}} \leqslant \frac{C}{(t_\varepsilon - t)^{7/4-3\lambda}}, \tag{IV.2.131}
$$

which is of a smaller order.

For $(vii)_2$, we obtain

$$
\begin{aligned}
(vii)_2 \leqslant C(t_\varepsilon - t)^{3/2+2\lambda} \int_r \frac{|x_2 - v_\varepsilon^2|}{r^{1/2-2\lambda}\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3} \\
\cdot \int_{z=0} \frac{1}{\left(z^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^2} \\
\leqslant C(t_\varepsilon - t)^{3/2+2\lambda} \int_r \frac{r^{-1/2+2\lambda}|x_2 - v_\varepsilon^2|}{\left(r^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^3} \cdot \frac{C}{((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t))^{3/2}} \\
\leqslant C(t_\varepsilon - t)^{3/2+2\lambda} \frac{1}{\left((x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^{17/4}}.
\end{aligned} \tag{IV.2.132}
$$

Now, integrating (IV.2.132) with respect to x_2 yields

$$
\int_{x_2} |\psi_\varepsilon^2(x_2)||(vii)_2| \leqslant \frac{C}{(t_\varepsilon - t)^{7/4-3\lambda}}, \tag{IV.2.133}
$$

which is again of a smaller order.

Now, we have established the lower bound for $(iv)_1$ in (IV.2.70). The last step is to establish an upper bound for $(iv)_2$, $(iv)_3$ and $(iv)_4$ to conclude the proof. The case of $(iv)_2$ and $(iv)_3$ are similar. We start with $(iv)_2$.

We decompose as follows

$$(iv)_2 \leqslant C_\varepsilon \sum_{i+j+k=2} \int\int_\iota \frac{(x_1)^i |x_2 - v_\varepsilon^2|^j (t_\varepsilon - t)^k}{|\phi(t,x_1,x_2) - \phi(t,y,x_2)|^{1/2-2\lambda}} \cdot \frac{|y| + |x_2 - v_\varepsilon^2| + (t_\varepsilon - t)}{\phi_{x_1}(t,x_1,x_2)^2 \phi_{x_1}(t,y,x_2)^2}$$
$$= \alpha_1 + \alpha_2 + \alpha_3. \quad \text{(IV.2.134)}$$

We obtain

$$\alpha_1 \leqslant C_\varepsilon \sum_{i+j+k=2} \int\int_\iota \frac{(x_1)^i |x_2 - v_\varepsilon^2|^j (t_\varepsilon - t)^k}{|\phi(t,x_1,x_2) - \phi(t,y,x_2)|^{1/2-2\lambda}} \cdot \frac{|y|}{\phi_{x_1}(t,x_1,x_2)^2 \phi_{x_1}(t,y,x_2)^2}$$
$$\leqslant (t_\varepsilon - t)^{k-1/2+2\lambda} |x_2 - v_\varepsilon^2|^j$$
$$\cdot \int\int_\iota \frac{(x_1)^i \cdot |y|}{|x_1 - y|^{1/2-2\lambda} \cdot \phi_{x_1}(t,x_1,x_2)^2 \phi_{x_1}(t,y,x_2)^2} \quad \text{(IV.2.135)}$$

Now, we use Hölder in (IV.2.135),

$$\alpha_1 \leqslant \sum_{i+j+k=2} C_\varepsilon (t_\varepsilon - t)^{k-1/2+2\lambda} |x_2 - v_\varepsilon^2|^j \left(\int\int_\iota \frac{1}{|x_1 - y|^{3/4-3\lambda}} \right)^{2/3}$$
$$\cdot \left(\int\int_\iota \frac{(x_1)^{3i}}{\left(x_1^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^6} \right)^{1/3} \cdot \left(\int\int_\iota \frac{(y)^3}{\left(x_1^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)\right)^6} \right)^{1/3}$$
$$\leqslant \sum_{i+j+k=2} C_\varepsilon \frac{(t_\varepsilon - t)^{k-1/2+2\lambda} |x_2 - v_\varepsilon|^j}{\left((x_2 - v_\varepsilon)^2 + (t_\varepsilon - t)\right)^{19/6-i/2}}. \quad \text{(IV.2.136)}$$

To do this, we studied the two last integrals. In particular, we give the following estimates.

$$\begin{aligned}\left(\int_0^\infty \frac{1}{(x^2+a)^6}\right)^{1/3} &\leqslant \frac{C_\varepsilon}{a^{11/6}},\\ \left(\int_0^\infty \frac{x^3}{(x^2+a)^6}\right)^{1/3} &\leqslant \frac{C_\varepsilon}{a^{4/3}},\\ \left(\int_0^\infty \frac{x^6}{(x^2+a)^6}\right)^{1/3} &\leqslant \frac{C_\varepsilon}{a^{5/6}}.\end{aligned} \tag{IV.2.137}$$

Integrating (IV.2.136) with respect to x_2 yields

$$\int_{x_2} \alpha_1 \leqslant \sum_{i+j+k=2} \frac{1}{(t_\varepsilon - t)^{32/12-i/2-j/2-k+1/2-2\lambda}}. \tag{IV.2.138}$$

Given the fact that $i+j+k=2$, the highest order possible is $26/12$, which is indeed smaller than $9/4 = 27/12$. We will now go through α_2 and α_3. Because they are somehow similar, we will skip a few details.

For α_2, we obtain

$$\begin{aligned}\alpha_2 \leqslant \sum_{i+j+k=2} C_\varepsilon (t_\varepsilon - t)^{k-1/2+2\lambda} |x_2 - v_\varepsilon^2|^{j+1} \left(\int\int_\iota \frac{1}{|x_1 - y|^{3/4-3\lambda}}\right)^{2/3}\\ \cdot \left(\int\int_\iota \frac{(x_1)^{3i}}{(x_1^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t))^6}\right)^{1/3} \cdot \left(\int\int_\iota \frac{1}{(x_1^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t))^6}\right)^{1/3}\\ \leqslant \sum_{i+j+k=2} C_\varepsilon \frac{(t_\varepsilon - t)^{k-1/2+2\lambda} |x_2 - v_\varepsilon|^{j+1}}{((x_2 - v_\varepsilon)^2 + (t_\varepsilon - t))^{22/6-i/2}}.\end{aligned} \tag{IV.2.139}$$

Integrating (IV.2.139) with respect to x_2 yields

$$\int_{x_2} \alpha_2 \leqslant \sum_{i+j+k=2} \frac{1}{(t_\varepsilon - t)^{19/6-i/2-(j+1)/2-k+1/2-2\lambda}}. \tag{IV.2.140}$$

Given the fact that $i+j+k=2$, the highest order possible is $26/12$, which is indeed smaller than $9/4 = 27/12$. We will now go through α_2 and α_3. Because they are somehow similar, we will skip a few details.

For α_3, we obtain

$$\alpha_2 \leqslant \sum_{i+j+k=2} C_\varepsilon (t_\varepsilon - t)^{k+1-1/2+2\lambda} |x_2 - v_\varepsilon^2|^j \left(\iint_\iota \frac{1}{|x_1 - y|^{3/4-3\lambda}} \right)^{2/3}$$
$$\cdot \left(\iint_\iota \frac{(x_1)^{3i}}{(x_1^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t))^6} \right)^{1/3} \cdot \left(\iint_\iota \frac{1}{(x_1^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t))^6} \right)^{1/3}$$
$$\leqslant \sum_{i+j+k=2} C_\varepsilon \frac{(t_\varepsilon - t)^{k+1-1/2+2\lambda} |x_2 - v_\varepsilon|^j}{((x_2 - v_\varepsilon)^2 + (t_\varepsilon - t))^{22/6-i/2}}. \quad \text{(IV.2.141)}$$

Integrating (IV.2.141) with respect to x_2 yields

$$\int_{x_2} \alpha_2 \leqslant \sum_{i+j+k=2} \frac{1}{(t_\varepsilon - t)^{19/6-i/2-j/2-k-1+1/2-2\lambda}}. \quad \text{(IV.2.142)}$$

Given the fact that $i+j+k=2$, the highest order possible is $20/12$, which is indeed smaller than $9/4 = 27/12$. We will now go through α_2 and α_3. Because they are somehow similar, we will skip a few details.

We now look at $(iv)_4$, the expression being given by (IV.2.70). We establish an upper bound.

$$(iv)_4 \leqslant C_\varepsilon \sum_{\substack{i_1+j_1+k_1=2\\ i_2+j_2+k_2=2}} (t_\varepsilon - t)^{k_1+k_2} \iint_\iota \frac{|x_1|^{i_1} |y|^{i_2} |x_2 - v_\varepsilon^2|^{j_1+j_2}}{|\phi(t, v_\varepsilon^1 + x_1, x_2) - \phi(t, v_\varepsilon^1 + y, x_2)|^{1/2-2\lambda} \phi_{x_1}^2(x_1) \phi_{x_1}^2(y)}$$
$$\leqslant C_\varepsilon \sum_{\substack{i_1+j_1+k_1=2\\ i_2+j_2+k_2=2}} \frac{(t_\varepsilon - t)^{k_1+k_2}}{(t_\varepsilon - t)^{1/2-2\lambda}} \iint_\iota \frac{|x_1|^{i_1} |y|^{i_2} |x_2 - v_\varepsilon^2|^{j_1+j_2}}{|x_1 - y|^{1/2-2\lambda} \phi_{x_1}^2(x_1) \phi_{x_1}^2(y)} \quad \text{(IV.2.143)}$$

Now, we use Hölder's inequality in (IV.2.143) and simplify,

$$
\begin{aligned}
(iv)_4 \leqslant C_\varepsilon & \sum_{\substack{i_1+j_1+k_1=2\\ i_2+j_2+k_2=2}} (t_\varepsilon - t)^{k_1+k_1-1/2+2\lambda} |x_2 - \nu_\varepsilon^2|^{j_1+j_2} \int\int_\iota \frac{|x_1|^{i_1}|y|^{i_2}}{|x_1-y|^{1/2-2\lambda}\phi_{x_1}^2(x_1)\phi_{x_1}^2(y)} \\
\leqslant & \sum_{\substack{i_1+j_1+k_1=2\\ i_2+j_2+k_2=2}} C_\varepsilon (t_\varepsilon - t)^{k_1+k_1-1/2+2\lambda} |x_2 - \nu_\varepsilon^2|^{j_1+j_2} \left(\int\int_\iota \frac{1}{|x_1-y|^{3/4-3\lambda}} \right)^{2/3} \\
& \cdot \left(\int_0^\infty \frac{x_1^{3i_1}}{(x_1^2 + (x_2-\nu_\varepsilon^2)^2 + (t_\varepsilon - t))^6} \right)^{1/3} \cdot \left(\int_0^\infty \frac{y^{3i_2}}{(y^2 + (x_2-\nu_\varepsilon^2)^2 + (t_\varepsilon - t))^6} \right)^{1/3}
\end{aligned}
\tag{IV.2.144}
$$

From (IV.2.144) and (IV.2.137), we have

$$
\begin{aligned}
(iv)_4 \leqslant & \sum_{\substack{i_1+j_1+k_1=2\\ i_2+j_2+k_2=2}} \frac{C_\varepsilon (t_\varepsilon - t)^{k_1+k_1-1/2+2\lambda} |x_2 - \nu_\varepsilon^2|^{j_1+j_2}}{((t_\varepsilon - t) + (\nu_\varepsilon^2 - x_2)^2)^{\frac{11}{6}-\frac{3i_1}{6}} ((t_\varepsilon - t) + (\nu_\varepsilon^2 - x_2)^2)^{\frac{11}{6}-\frac{3i_2}{6}}} \\
& \leqslant \sum_{\substack{i_1+j_1+k_1=2\\ i_2+j_2+k_2=2}} \frac{C_\varepsilon (t_\varepsilon - t)^{k_1+k_1-1/2+2\lambda} |x_2 - \nu_\varepsilon^2|^{j_1+j_2}}{((t_\varepsilon - t) + (\nu_\varepsilon^2 - x_2)^2)^{\frac{22}{6}-3\frac{i_1+i_2}{6}}} \\
& \qquad \leqslant \sum_{\substack{i_1+j_1+k_1=2\\ i_2+j_2+k_2=2}} \frac{C_\varepsilon (t_\varepsilon - t)^{k_1+k_2-1/2+2\lambda}}{((t_\varepsilon - t) + (\nu_\varepsilon^2 - x_2)^2)^{\frac{22}{6}-\frac{i_1+j_1+i_2+j_2}{2}}}
\end{aligned}
\tag{IV.2.145}
$$

We will distinguish the different cases. First, if $k_1 + k_2 = 0$, then $i_1 + i_2 + j_1 + j_2 = 4$ and we have

$$
\frac{C_\varepsilon (t_\varepsilon - t)^{k_1+k_2-1/2+2\lambda}}{((t_\varepsilon - t) + (\nu_\varepsilon^2 - x_2)^2)^{\frac{22}{6}-\frac{i_1+j_1+i_2+j_2}{2}}} = \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2-2\lambda}} \frac{1}{((t_\varepsilon - t) + (\nu_\varepsilon^2 - x_2)^2)^{10/6}},
\tag{IV.2.146}
$$

hence, integrating (IV.2.146) yields

$$\left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2) \frac{C_\varepsilon}{(t_\varepsilon - t)^{1/2 - 2\lambda}} \frac{1}{((t_\varepsilon - t) + (v_\varepsilon^2 - x_2)^2)^{10/6}} \right| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{7/6 + 1/2 - 2\lambda}} << \frac{1}{(t_\varepsilon - t)^{9/4 - 3\lambda}}. \quad \text{(IV.2.147)}$$

If $k_1 + k_2 = 1$, then $i_1 + i_2 + j_1 + j_2 = 3$ and we have

$$\frac{C_\varepsilon (t_\varepsilon - t)^{k_1 + k_2 - 1/2 + 2\lambda}}{((t_\varepsilon - t) + (v_\varepsilon^2 - x_2)^2)^{\frac{22}{6} - \frac{i_1 + j_1 + i_2 + j_2}{2}}} = \frac{C_\varepsilon (t_\varepsilon - t)^{1/2 + 2\lambda}}{((t_\varepsilon - t) + (v_\varepsilon^2 - x_2)^2)^{13/6}}, \quad \text{(IV.2.148)}$$

hence, integrating (IV.2.148) yields

$$\left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2) \frac{C_\varepsilon (t_\varepsilon - t)^{1/2 + 2\lambda}}{((t_\varepsilon - t) + (v_\varepsilon^2 - x_2)^2)^{13/6}} \right| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{7/6 - 2\lambda}} << \frac{1}{(t_\varepsilon - t)^{9/4 - 3\lambda}}. \quad \text{(IV.2.149)}$$

If $k_1 + k_2 = 2$, then $i_1 + i_2 + j_1 + j_2 = 2$ and we have

$$\frac{C_\varepsilon (t_\varepsilon - t)^{k_1 + k_2 - 1/2 + 2\lambda}}{((t_\varepsilon - t) + (v_\varepsilon^2 - x_2)^2)^{\frac{22}{6} - \frac{i_1 + j_1 + i_2 + j_2}{2}}} = \frac{C_\varepsilon (t_\varepsilon - t)^{3/2 + 2\lambda}}{((t_\varepsilon - t) + (v_\varepsilon^2 - x_2)^2)^{16/6}}, \quad \text{(IV.2.150)}$$

hence, integrating (IV.2.150) yields

$$\left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2) \frac{C_\varepsilon (t_\varepsilon - t)^{3/2 + 2\lambda}}{((t_\varepsilon - t) + (v_\varepsilon^2 - x_2)^2)^{16/6}} \right| \leqslant \frac{C_\varepsilon}{(t_\varepsilon - t)^{4/6 - 2\lambda}} << \frac{1}{(t_\varepsilon - t)^{9/4 - 3\lambda}}. \quad \text{(IV.2.151)}$$

Now, we deduce from the same computations that the term corresponding to $k_1 + k_2 =$

3 is smaller than $\frac{C_\varepsilon}{(t_\varepsilon - t)^{1/6}}$ and that the term corresponding to $k_1 + k_2 = 4$ is bounded.

Now, we use (IV.2.145) to conclude for the term corresponding to $(iv)_4$ in (IV.2.71).

$$\left| \int_{x_2 = v_\varepsilon^2 - 2\delta_\varepsilon}^{v_\varepsilon^2 + 2\delta_\varepsilon} \psi_\varepsilon^2(x_2)(iv)_4 \right| << \frac{C_\varepsilon}{(t_\varepsilon - t)^{9/4 - 3\lambda}}. \tag{IV.2.152}$$

This concludes this proof.

□

V Introducing a perturbation of the initial data

In this chapter, we study the stability of the instantaneous blow up exhibited in chapter I. In chapter II, we introduced a source term and showed that the pathological behaviour was preserved. Now, we are going to add a perturbation with respect to the initial data and show that the corresponding Cauchy problem is again ill-posed. More precisely, we consider the Cauchy problem

$$\begin{cases} \Box u = DuD^2 u + f(t, x_1, x_2, u), \\ u_{|t=0} = \tilde{u}_0, \ \dfrac{\partial u}{\partial t}_{|t=0} = -\chi(x_1) + \tilde{u}_1 \end{cases}, \quad (MQLW2) \tag{V.0.1}$$

and the corresponding regularized Cauchy problem

$$\begin{cases} \Box u = DuD^2 u + f(t, x_1, x_2, u), \\ u_{|t=0} = \tilde{u}_0, \ \dfrac{\partial u}{\partial t}_{|t=0} = -\chi_\varepsilon(x_1) + \tilde{u}_1, \end{cases} \quad (MQLW2)_\varepsilon \tag{V.0.2}$$

where χ and χ_ε are defined as in I.1.6 and (III.1.2) respectively. We will denote $D\tilde{u}(x_1, x_2) = \tilde{v}(x_1, x_2) = \partial_{x_1}\tilde{u}_0 - \tilde{u}_1$. We also assume the two following conditions

$$\begin{aligned} &(i)\ \forall \alpha, \left|\frac{\partial^\alpha f}{\partial_x^\alpha}\right| \leqslant C, \\ &(ii)\ (\tilde{u}_0, \tilde{u}_1) \in H^4(\mathbb{R}^2) \times H^3(\mathbb{R}^2). \end{aligned} \tag{V.0.3}$$

In this chapter, we do not go through all the previous computations as they will still hold. Instead, we are going to prove an equivalent of lemma IV.1.3 and the result will follow.

Now, rewritting (V.0.2) with $Du = v$ yields

$$\begin{cases} \left(\partial_t + \dfrac{1+v}{1-v}\partial_{x_1}\right) v = \dfrac{f(t,x_1,x_2,v)}{1-v} = g(t,x_1,x_2,v). \\ \dfrac{\partial u}{\partial t}_{|t=0} = -\chi + \tilde{u}_1, \quad u_{|t=0} = \tilde{u}_0, \end{cases} \tag{V.0.4}$$

where g also satisfies (i) of (V.0.3) as $v < 0$. We now state the main theorem of this chapter.

Theorem V.0.1. *Let u_ε be the solution of problem (V.0.2). There exists a time t_ε such that*

$$\begin{cases} ||u_\varepsilon(0,\cdot)||_{H^{11/4}(\mathbb{R}^2)} < \infty \\ ||u_\varepsilon(t,\cdot)||_{H^{11/4}(\mathbb{R}^2)} \to \infty \quad \text{as } t \to t_\varepsilon. \end{cases} \tag{V.0.5}$$

Also, we have that

$$t_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0. \tag{V.0.6}$$

We first do some preliminary work, and later state and prove an equivalent of lemma IV.1.3 for this case.

Preliminary work

As in (I.1.3), we define ϕ (implicitly depending on ε) as

$$\begin{cases} \phi(0,x_1,x_2) = x_1 \\ \partial_t\phi(t,x_1,x_2) = \dfrac{1+v(t,\phi(t,x_1,x_2),x_2)}{1-v(t,\phi(t,x_1,x_2),x_2)}. \end{cases} \tag{V.0.7}$$

From (V.0.4) and (IV.1.8), we get

$$\frac{\partial}{\partial_t}\left(v(t,\phi(t,x_1,x_2),x_2)\right) = g(t,\phi(t,x_1,x_2),x_2,v(t,\phi(t,x_1,x_2),x_2)). \tag{V.0.8}$$

From now on, we will not specify every variable, but only if the function is applied to x_1 of $\phi(t,x_1,x_2)$. Integrating (V.0.8) yields

$$v(\phi) = \chi(x_1) + \tilde{v}(x_1,x_2) + \int_{\tau=0}^{t} g(\tau,\phi)d\tau. \tag{V.0.9}$$

Differentiating (V.0.9) leads to the following expressions for the derivatives of v.

$$\begin{aligned}\partial_{x_1}\phi(x_1)\partial_{x_1}v(\phi) = \chi'(x_1)+\partial_{x_1}\tilde{v}+\int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_1 g(\tau,x_1)+\int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_{x_1}v(\tau,x_1)\partial_3 g(\tau,x_1)\\ \Rightarrow \partial_{x_1}v(\phi) = \frac{1}{\partial_{x_1}\phi(\tau,x_1)}\Bigg[\chi'(x_1)+\partial_{x_1}\tilde{v}\\ +\int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_1 g(\tau,\phi)+\int_\tau \partial_{x_1}\phi(x_1)\partial_{x_1}v(\tau,x_1)\partial_3 g(\tau,x_1)\Bigg]. \end{aligned}\tag{V.0.10}$$

$$\begin{aligned}\partial_{x_1}^2 v(\phi) = \frac{1}{(\partial_{x_1}\phi(x_1))^2}\Bigg[\chi''(x_1)+\partial_{x_1}^2\tilde{v}+\int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\partial_1^2 g(\tau,\phi)+\int_\tau \partial_{x_1}^2\phi(\tau,x_1)\partial_1 g(\tau,\phi)\\ +\int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\partial_{x_1}v(\tau,\phi)\partial_3\partial_1 g(\tau,\phi)+\int_\tau \partial_{x_1}^2\phi(\tau,x_1)\partial_3 g(\tau,\phi)\partial_{x_1}v(\tau,\phi)\\ +\int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\partial_1\partial_3 g(\tau,\phi)\partial_{x_1}v(\tau,\phi)+\int_\tau (\partial_{x_1}\phi(\tau,x_1))^2(\partial_{x_1}v(\tau,\phi))^2\partial_3^2 g(\tau,\phi)\\ +\int_\tau (\partial_{x_1}\phi(\tau,x_1))^2\partial_{x_1}^2 v(\tau,\phi)\partial_3 g(\tau,\phi)\Bigg]\\ -\frac{\partial_{x_1}^2\phi(x_1)}{(\partial_{x_1}\phi(x_1))^3}\Bigg[\chi'(x_1)+\partial_{x_1}\tilde{v}+\int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_1 g(\tau,\phi)+\int_\tau \partial_{x_1}\phi(\tau,x_1)\partial_3 g(\tau,\phi)\partial_{x_1}v(\tau,\phi)\Bigg]\\ = A-B. \end{aligned}\tag{V.0.11}$$

Again, we have

$$\phi_{tx_1}(x_1) = \frac{2v_{x_1}(\phi)}{(1-v(\phi))^2}. \tag{V.0.12}$$

$$\phi_{tx_1x_1}(x_1) = \frac{2v_{x_1x_1}(\phi)(1-v(\phi)) + 2v_{x_1}^2(\phi)}{(1-v(\phi))^3}. \tag{V.0.13}$$

We now state the lemma. It is in fact identical to lemma IV.1.3

Lemma V.0.2. *There exists a time t_ε such that the following properties are verified.*

$$t_\varepsilon \to 0 \quad as\, \varepsilon \to 0. \tag{V.0.14}$$

$$\begin{aligned} &\phi_{x_1} > 0 \ \ \forall t < t_\varepsilon \\ &\exists (v_\varepsilon^1, v_\varepsilon^2) \ \ s.t. \ \ \phi_{x_1}(t_\varepsilon, v_\varepsilon^1, v_\varepsilon^2) = 0 \end{aligned} \tag{V.0.15}$$

$$\begin{aligned} &0 < \phi_{x_1} < 1,\ \phi_{t,x_1} < 0 \ \ ,\forall t < t_\varepsilon, \\ &\exists C_\varepsilon,\ \ |\phi_{t,x_1,x_1}| \leqslant C_\varepsilon |\ln(t_\varepsilon - t)|. \\ &v_{x_1} < 0,\ \ v < 0 \\ &\exists C_\varepsilon,\ \ |v_{x_1}(\phi)| \leqslant \frac{C_\varepsilon(\chi'(x_1) + C_\varepsilon)}{|\phi_{x_1}|}. \end{aligned} \tag{V.0.16}$$

If x_1, x_2 and t are sufficiently close to $(v_\varepsilon^1, v_\varepsilon^2, t_\varepsilon)$, we have the following estimates.

$$\begin{aligned} \exists C_\varepsilon^1, C_\varepsilon^2 > 0,\ \ &\frac{C_\varepsilon^1}{(x_1 - v_\varepsilon^1)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)} \leqslant \frac{1}{\phi_{x_1}(x_1)} \\ &\leqslant \frac{C_\varepsilon^2}{(x_1 - v_\varepsilon^1)^2 + (x_2 - v_\varepsilon^2)^2 + (t_\varepsilon - t)} \\ \exists C_\varepsilon^1, C_\varepsilon^2, C_\varepsilon^3,\ \ &\phi_{x_1x_1}(x_1) = C_\varepsilon^1(x_1 - v_\varepsilon^1) + C_\varepsilon^2(x_2 - v_\varepsilon^2) + C_\varepsilon^3(t_\varepsilon - t) \\ &+ \sum_{i+j+k=2} (x_1 - v_\varepsilon^1)^i (x_2 - v_\varepsilon^2)^j (t_\varepsilon - t)^k f_{i,j,k}(t, x_1, x_2), \end{aligned} \tag{V.0.17}$$

where all the involved f functions are bounded near $(t, v_\varepsilon^1, v_\varepsilon^2)$. We also have

$$|A| \leqslant \frac{C_\varepsilon |\ln(t_\varepsilon - t)|}{|\phi_{x_1}|^2}, \tag{V.0.18}$$

where A is the A involved in (V.0.11).

Lastly, for ε small enough, we have

$$\begin{aligned} \frac{C_\varepsilon \phi_{x_1 x_1}(x_1)}{(\phi_{x_1}(x_1))^3} \frac{\chi'}{2} \leqslant B \leqslant \frac{C_\varepsilon \phi_{x_1 x_1}(x_1)}{(\phi_{x_1}(x_1))^3} 2\chi', \\ \frac{C_\varepsilon \phi_{x_1 x_1}(x_1)}{(\phi_{x_1}(x_1))^3} 2\chi' \leqslant B \leqslant \frac{C_\varepsilon \phi_{x_1 x_1}(x_1)}{(\phi_{x_1}(x_1))^3} \frac{\chi'}{2}. \end{aligned} \tag{V.0.19}$$

The first inequality being verified when $\phi_{x_1 x_1} \leqslant 0$, and the second being verified when $\phi_{x_1 x_1} \geqslant 0$.

We stop here for this case. The rest of the proof should be identical to what have been done when we added the source term $f(t, x_1, x_2, \nabla u)$.

A APPENDIX

A.1 Initial condition in the classical Sobolev space

First, we will consider (I.1.6) and prove the following theorem. In the next section, we will use this theorem to find an uniform bound on the sobolev norm of an extension of (I.1.7).

Theorem A.1.1. *With* $g : (x_1, x_2) \in \Omega \mapsto -\int_{y=0}^{x_1} |\ln(y)|^{\alpha} dy$*, there exists* $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\tilde{g}_{|\Omega_0}(x_1, x_2) = g(x_1, 0)$ *and* $||\tilde{g}||_{H^{7/4}(\mathbb{R}^2)} < \infty$.

First, we show the following technical lemma. We will use it for $t = 0$ in this subsection and for $t > 0$ in the next section, when we will consider extensions of our functions defined respectively on Ω_0 and Ω_t.

Lemma A.1.2. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *and* $\omega \subseteq \mathbb{R}^2$ *such that* $f = 0$ *outside of* ω.

Then for $t > 0$,

$$\begin{aligned} ||f||_{\dot{H}^{7/4}_{x_1}(\mathbb{R}^2)} = C \int\int_{(x_1,x_2)\in\omega} \left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1, x_2) \\ \cdot \int_{y|\ (y,x_2)\in\omega} |x_1 - y|^{-1/2} \left(\frac{\partial^2 f}{\partial x_1^2}\right)(y, x_2) dy dx_2 dx_1 \end{aligned} \tag{A.1.1}$$

Proof.

$$
\begin{aligned}
||f||_{\dot{H}^{7/4}_{x_1}(\mathbb{R}^2)} &= ||\frac{\partial^2 f}{\partial x_1^2}||_{\dot{H}^{-1/4}} \\
&= \int\int_{(x_1,x_2)\in\mathbb{R}^2} |\nabla_{x_1}^{-1/4}| \left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)\cdot|\nabla_{x_1}^{-1/4}|\left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)dx \\
&= \int\int_{(x_1,x_2)\in\mathbb{R}^2} |\nabla_{x_1}^{-1/2}| \left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)\cdot\left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)dx \\
&=_{(*)} C\int\int_{(x_1,x_2)\in\mathbb{R}^2} \left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)\cdot\int_{y\in\mathbb{R}} \frac{\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2)}{\sqrt{|x_1-y|}}dy \\
&= C\int\int_{(x_1,x_2)\in\omega} \left(\frac{\partial^2 f}{\partial x_1^2}\right)(x_1,x_2)\cdot\int_{y|\ (y,x_2)\in\omega} |x_1-y|^{-1/2}\left(\frac{\partial^2 f}{\partial x_1^2}\right)(y,x_2)
\end{aligned}
\tag{A.1.2}
$$

For $(*)$, we used that

$$(-\Delta)^{s/2}(f)(x) = \left((2\pi|\xi|)^s\widehat{f}(\xi)\right)^{\check{}}(x), \tag{A.1.3}$$

and that for $s>0$,

$$(2\pi)^{-s}\left(|\xi|^{-s}\right)^{\check{}}(x) = (2\pi)^{-s}\frac{\pi^{\frac{s}{2}}\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n-s}{2}}\Gamma\left(\frac{s}{2}\right)}|x|^{s-n}. \tag{A.1.4}$$

In our case, because we integrate only with respect to x_1, we obtain from (A.1.3) and (A.1.4)

$$|\nabla_{x_1}|^{-1/2}f(x) = C\int_{y\in\mathbb{R}}\frac{f(y)}{\sqrt{|x_1-y|}}. \tag{A.1.5}$$

□

We will also use the following technical lemma. It gives the finiteness of an expression that will appear when we will show theorem A.1.1.

Lemma A.1.3. *The expression*

$$I_{\alpha,\delta} = \int_{x_1=0}^{\frac{1}{2}} \frac{|\ln(x_1)|^{\alpha-1}}{x_1} \int_{0\leqslant x_2 \leqslant \frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} \int_{y, x_2 \leqslant \frac{\sqrt{y}}{|\ln(y)|^{\delta}}}^{\frac{1}{2}} \frac{|\ln(y)|^{\alpha-1}}{y\sqrt{|x_1-y|}}, \tag{A.1.6}$$

is finite for $\alpha < \frac{1}{6}$ *and* $\delta > \frac{1}{3}$.

Proof. (of the technical lemma)

We first establish the following estimation.

$$\begin{aligned}\int_{y=x/2}^{\frac{1}{2}} \frac{1}{\sqrt{|y-x|}}\frac{1}{y}dy &\leqslant \int_{y=x/2}^{x} \frac{1}{\sqrt{x-y}}\frac{1}{y}dy + \int_{y=x}^{\frac{1}{2}} \frac{1}{\sqrt{y-x}}\frac{1}{y}dy \\ &\leqslant \frac{2}{x}\int_{y=x/2}^{x}\frac{1}{\sqrt{x-y}} + \left[2\sqrt{y-x}\frac{1}{y}\right]_x^{\frac{1}{2}} + \int_{y=x}^{\frac{1}{2}} 2\sqrt{y-x}\frac{1}{y^2}dy \\ &\leqslant C\frac{1}{\sqrt{x}} - \left[2\sqrt{0}/x - 4\sqrt{1/2}\right] + \int_{y=x}^{\frac{1}{2}}\frac{2\sqrt{y-x}}{y^2}dy \\ &\leqslant C\frac{1}{\sqrt{x}} + C\int_{y=x}^{\frac{1}{2}}\frac{\sqrt{y}}{y^2}dy \leqslant \frac{C}{\sqrt{x}}\end{aligned} \tag{A.1.7}$$

Also, we remark that for $m < 1$,

$$\int_0^b |\ln(x)|^m \leqslant \left(\int_0^b |\ln(x)|\right)^m (b)^{m^*} \leqslant Cb^{m+m^*}|\ln(b)|^m = Cb|\ln(b)|^m, \tag{A.1.8}$$

where $m^* = 1-m$.

Now, for $m \in [1,2[$,

$$\int_0^b |\ln(x)|^m \leqslant b|\ln(x)|^m + \int_0^b |\ln(x)|^{m-1} \leqslant_{(A.1.8)} Cb|\ln(b)|^m. \tag{A.1.9}$$

Now, we split the domain of integration of I as in (A.1.6) in y into two parts, corresponding to the smallness respectively of y and of $y-x$.

$$\begin{aligned} I^+_{\alpha,\delta} \leqslant & \int_{x_1=0}^{\frac{1}{2}} \frac{|\ln(x_1)|^{\alpha-1}}{|x_1|} \int_{x_2 \leqslant \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \int_{y=x_2^2}^{x_1/2} \frac{|\ln(y)|^{\alpha-1}}{|y|\sqrt{|x_1-y|}} \\ & + \int_{x_1=0}^{\frac{1}{2}} \frac{|\ln(x_1)|^{\alpha-1}}{|x_1|} \int_{x_2 \leqslant \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \int_{y=x_1/2}^{\frac{1}{2}} \frac{|\ln(y)|^{\alpha-1}}{|y|\sqrt{|x_1-y|}} = (*)^1 + (*)^2. \end{aligned} \tag{A.1.10}$$

Now, we study $(*)^1$ of (A.1.10), and use (A.1.8).

$$\begin{aligned} (*)^1 \leqslant & \int_{x_1=0}^{\frac{1}{2}} \frac{|\ln(x_1)|^{\alpha-1}}{x_1\sqrt{x_1}} \int_{x_2 \leqslant \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \int_{y=x_2^2}^{x_1/2} \frac{|\ln(y)|^{\alpha-1}}{|y|} \\ \leqslant & \int_{x_1=0}^{\frac{1}{2}} \frac{|\ln(x_1)|^{\alpha-1}}{x_1\sqrt{x_1}} \int_{x_2 \leqslant \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} |\ln(x_2)|^\alpha \leqslant \int_{x_1=0}^{\frac{1}{2}} \frac{|\ln(x_1)|^{\alpha-1}}{x_1\sqrt{x_1}} \frac{|\ln(x_2)|^\alpha\sqrt{x_1}}{|\ln(x_1)|^\delta} \\ & \leqslant \int_{x_1=0}^{\frac{1}{2}} \frac{|\ln(x_1)|^{2\alpha-1-\delta}}{x_1}. \end{aligned} \tag{A.1.11}$$

This integral converges when $\alpha < \frac{1}{6}$ and $\delta > \frac{1}{3}$.

Now, for $(*)^2$, we have

$$\begin{aligned} (*)^2 \leqslant & \int_{x_1=0}^{3} \frac{|\ln(x_1)|^\alpha}{x_1} \int_{x_2 \leqslant \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} |\ln(x_1)|^{\alpha-1} \int_{x_2 \leqslant \frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \int_{y=x/2}^{3} \frac{1}{y\sqrt{|x-y|}} \\ \leqslant & \int_{x_1=0}^{3} \frac{|\ln(x_1)|^{2\alpha-1-\delta}\sqrt{x}}{x} \int_{y=x_1/2}^{3} \frac{1}{y\sqrt{|x-y|}} \leqslant \int_{x_1=0}^{3} \frac{|\ln(x_1)|^{2\alpha-1-\delta}}{x}. \end{aligned} \tag{A.1.12}$$

The last inequality holds because of (A.1.7). This integral converges when $\alpha < \frac{1}{6}$ and $\delta > \frac{1}{3}$.

□

We now prove theorem A.1.1. To do so, we find an extension of the considered function that is defined globally and has finite Sobolev norm.

Proof. of theorem A.1.1. Let

$$f:(x_1,x_2)\mapsto\begin{cases}-\int_{y=0}^{x_1}|\ln(y)|^\alpha, & \text{for } x_1\geqslant 0,\\ 0 \quad \text{for } x_1<0.\end{cases} \tag{A.1.13}$$

Let $\phi:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $\phi(x)=1$ for $|x|\leqslant\frac{1}{4}$, $\phi(x)=0$ for $|x|\geqslant\frac{1}{2}$, and $0\leqslant\phi(x)\leqslant 1$ for all x in $\mathbb{R}$. Let $\psi:\mathbb{R}\to\mathbb{R}$ denote a smooth function such that $\psi(x)=1$ for $|x|\leqslant\frac{1}{4}$, $\psi(x)=0$ for $|x|\geqslant\frac{1}{2}$, and $0\leqslant\psi(x)\leqslant 1$ for all $x\in\mathbb{R}$.

Define

$$h(x)=f_{x_1}(x)\cdot\phi\left(\frac{\ln^\delta(x_1)x_2}{\sqrt{x_1}}\right)\cdot\psi(x_1). \tag{A.1.14}$$

We multiply f_x by a cutoff function in x_1 that almost respect the geometry of Ω, i.e. $\phi\left(\frac{\ln^\delta(x_1)x_2}{\sqrt{x_1}}\right)=0$ when $x_2\geqslant\frac{1}{2}\sqrt{x_1}\ln(x_1)^{-\delta}$; and we multiply f_x by a simple cutoff function ψ in x_2.

Now,

$$\frac{\partial}{\partial x_1}\left(\phi\left(\frac{\ln^\delta(x_1)x_2}{\sqrt{x_1}}\right)\right)=x_2\frac{2\delta\ln^{\delta-1}(x_1)-\ln^\delta(x_1)}{2x_1\sqrt{x_1}}\phi'\left(\frac{\ln^\delta(x_1)x_2}{\sqrt{x_1}}\right), \tag{A.1.15}$$

and

$$\frac{\partial}{\partial x_2}\left(\phi\left(\frac{\ln^\delta(x_1)x_2}{\sqrt{x_1}}\right)\right)=\frac{\ln^\delta(x_1)}{\sqrt{x_1}}\phi'\left(\frac{\ln^\delta(x_1)x_2}{\sqrt{x_1}}\right). \tag{A.1.16}$$

Let $\kappa:(x_1,x_2)\mapsto\phi\left(\frac{\ln^\delta(x_1)x_2}{\sqrt{x_1}}\right)\psi(x_2)$.

Also,

$$\begin{aligned}||\kappa(x_1,x_2)||^2_{\dot{H}^{3/4}(\mathbb{R}^2)} &=< |\xi|^{3/4}\hat{\kappa}, |\xi|^{3/4}\hat{\kappa} >_{L_2(\mathbb{R}^2)} \\ &=< |\xi|^{-1/2}(|\xi|\hat{\kappa}), |\xi|\hat{\kappa} >_{L_2(\mathbb{R}^2)} \\ &= \int_{\mathbb{R}^2} \Big(\nabla^{-1/2}(\nabla\kappa)\Big)\cdot(\nabla\kappa)\end{aligned} \tag{A.1.17}$$

Using this formula, we get that

$$\begin{aligned}&||\kappa(x_1,x_2)||^2_{\dot{H}^{3/4}(\mathbb{R}^2)} \\ &\leqslant C\int_{x\in\mathbb{R}^2}\left|\partial_{x_1}^{-1/2}\frac{\partial}{\partial x_1}\phi\left(\frac{|\ln|^\delta(x_1)x_2}{\sqrt{x_1}}\right)\right|\cdot|\nabla\kappa(x_1,x_2)|\,dx \\ &+ C\int_{x\in\mathbb{R}^2}\left|\partial_{x_2}^{-1/2}\frac{\partial}{\partial x_2}\phi\left(\frac{|\ln|^\delta(x_1)x_2}{\sqrt{x_1}}\right)\right|\cdot|\nabla\kappa(x_1,x_2)|\,dx,\end{aligned} \tag{A.1.18}$$

where C is a constant depending only on ψ and its derivatives. The non-integrability of this integral may only occur near $x_1 = 0$. Also, when $|x_2| > \frac{1}{2}\frac{\sqrt{|x_1|}}{|\ln|^\delta(x_1)}$, $\nabla\kappa(x_1,x_2) = 0$.

Thus, there exists a constant $K > 0$ such that

$$\left|\frac{\partial}{\partial x_1}\kappa(x_1,x_2)\right| \leqslant \text{ constant } \cdot \frac{|2\delta\ln^{\delta-1}(x_1)| + |\ln^\delta(x_1)|}{2|x_1|\cdot|\ln^\delta(x_1)|} \leqslant K\frac{1}{|2x_1|}, \tag{A.1.19}$$

and

$$\left|\frac{\partial}{\partial x_2}\kappa(x_1,x_2)\right| \leqslant K\frac{1}{2|x_1|}. \tag{A.1.20}$$

Hence,

$$
\begin{aligned}
&||\kappa(x_1,x_2)||^2_{H^{3/4}(\mathbb{R}^2)}\\
&\leqslant (K+1)\int_{x_1\in[0,\frac{1}{2}]}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{|x_1|}}{|\ln|^{\delta}(x_1)}}\left|\partial_{x_1}^{-1/2}\frac{\partial}{\partial x_1}\kappa(x_1,x_2)\right|\cdot\frac{1}{2x_1}dx_2dx_1\\
&+(K+1)\int_{x_1\in[0,\frac{1}{2}]}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{|x_1|}}{|\ln|^{\delta}(x_1)}}\left|\partial_{x_2}^{-1/2}\frac{\partial}{\partial x_2}\kappa(x_1,x_2)\right|\cdot\frac{1}{2x_1}dx_2dx_1\\
&=: (i)+(ii)
\end{aligned}
\tag{A.1.21}
$$

Let us now find an upper bound for these two quantities. First, we divide (i) into two terms. We use the formula given by lemma I.2.4. We consider $x, y, x_2 > 0$ by symmetry.

$$
\begin{aligned}
(i) &\leqslant C\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln|^{\delta}(x_1)}}\int_{y,x_2\leqslant\frac{\sqrt{y}}{|\ln(y)|^{\delta}}}\frac{1}{x_1y\sqrt{|y-x_1|}}\\
&\leqslant C\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln|^{\delta}(x_1)}}\int_{y=(x_2)^2}^{\frac{x_1}{2}}\frac{1}{x_1y\sqrt{|y-x_1|}}\\
&\quad + C\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln|^{\delta}(x_1)}}\int_{y=\frac{x_1}{2}}^{c}\frac{1}{x_1y\sqrt{|y-x_1|}}
\end{aligned}
\tag{A.1.22}
$$

For the first term, we get

$$
\begin{aligned}
&\int_{x_1=0}^{3}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}}\int_{y=(x_2)^2}^{x_1/2}\frac{1}{x_1y\sqrt{|x-y|}}\\
&\leqslant C\int_{x_1=0}^{\frac{1}{2}}\frac{1}{x_1\sqrt{x_1}}\int_{x_2<\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}}|\ln(x_2)| \leqslant C\int_{x_1=0}^{\frac{1}{2}}\frac{1}{\sqrt{x_1}x_1}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}|\ln(x_1)|\\
&\leqslant \int_{x_1=0}^{\frac{1}{2}}\frac{1}{x_1|\ln(x_1)|^{\delta-1}},
\end{aligned}
\tag{A.1.23}
$$

which converges when $\delta > 2$.

Now, for the second term, we obtain

$$
\begin{aligned}
& C\int_0^{\frac{1}{2}}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{|x_1|}}{|\ln|^{\delta}(x_1)}}\int_{y=x_1/2}^{\frac{1}{2}}\frac{1}{\sqrt{|y-x_1|}}\frac{1}{2y}dy\frac{1}{2|x_1|}dx_1 \\
& \leqslant C\int_0^{\frac{1}{2}}\frac{\frac{1}{2}\sqrt{x_1}}{|\ln^{\delta}(x_1)|}\int_{y=x_1/2}^{\frac{1}{2}}\frac{1}{\sqrt{|y-x_1|}}\frac{1}{2y}dy\frac{1}{2|x_1|}dx_1 \\
& \leqslant C\int_0^{\frac{1}{2}}\frac{\sqrt{x_1}}{\sqrt{x_1}}\frac{1}{2|x_1|\cdot|\ln^{\delta}(x_1)|}dx_1 \text{ by (A.1.7)} \\
& \leqslant C\int_0^{\frac{1}{2}}\frac{1}{x_1\cdot|\ln^{\delta}(x_1)|}dx_1
\end{aligned}
\tag{A.1.24}
$$

This integral converges whenever $\delta > 1$.

We continue with (ii).

$$
\begin{aligned}
(ii) & \leqslant K''\int_{x_1=0}^{\frac{1}{2}}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{|x_1|}}{|\ln|^{\delta}(x_1)}}\int_{y=(x_2)^2}^{\frac{1}{2}}\frac{1}{\sqrt{|y-x_2|}}\frac{1}{\sqrt{x_1}}dy\frac{1}{2x_1}dx_2dx_1 \\
& \textit{(where } K'' \textit{ is a constant depending only on } \phi\textit{, its derivatives, } \psi \textit{ and its derivatives)} \\
& \leqslant C\int_{x_1=0}^{\frac{1}{2}}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{|x_1|}}{|\ln|^{\delta}(x_1)}}C\frac{1}{\sqrt{x_1}}\frac{1}{2x_1}dx_2dx_1 \\
& \leqslant C\int_{x_1=0}^{\frac{1}{2}}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{|x_1|}}{|\ln|^{\delta}(x_1)}}\frac{1}{\sqrt{x_1}}\frac{1}{2x_1}dx_2dx_1 \\
& \leqslant C\int_{x_1=0}^{\frac{1}{2}}\frac{\sqrt{|x_1|}}{|\ln|^{\delta}(x_1)}\frac{1}{\sqrt{x_1}}\frac{1}{2x_1}dx_1 \\
& \leqslant K''\int_{x_1=0}^{\frac{1}{2}}\frac{1}{2x_1\cdot|\ln^(x_1)|\delta}dx_1,
\end{aligned}
\tag{A.1.25}
$$

which converges whenever $\delta > 1$.

Now, we do an estimation of the Sobolev norm of $v_x\kappa$ to show that h is in $\dot{H}^{7/4}(\mathbb{R}^2)$.

$$||\kappa f_{x_1}||_{\dot{H}^{3/4}(\mathbb{R}^2)} = \int_{x_1\in[0,\frac{1}{2}]}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln^{\delta}(x_1)|}} \nabla^{-1/2}(\nabla h)(x_1,x_2)\cdot\nabla h(x_1,x_2)dx_2dx_1 = (*) \tag{A.1.26}$$

As we have previously seen, the biggest order terms near $x_1 = 0$ are obtained by differentiating with respect to x_1. Hence,

$$\begin{aligned}
(*) \leqslant\; & C\int_{x_1\in[0,\frac{1}{2}]}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} |\kappa_{x_1}f_{x_1}|\left(\int_{y=(x_2)^2}^{\frac{1}{2}}\frac{1}{\sqrt{|y-x_1|}}|\kappa_{x_1}(y)f_{x_1}(y)|\,dy\right)dx_2dx_1\\
&+C\int_{x_1\in[0,\frac{1}{2}]}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} |\kappa f_{x_1x_1}|\left(\int_{y=(x_2)^2}^{\frac{1}{2}}\frac{1}{\sqrt{|y-x_1|}}|\kappa(y)f_{x_1x_1}(y)|\,dy\right)dx_2dx_1\\
&+C\int_{x_1\in[0,\frac{1}{2}]}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} |\kappa f_{x_1x_1}|\left(\int_{y=(x_2)^2}^{\frac{1}{2}}\frac{1}{\sqrt{|y-x_1|}}|\kappa_{x_1}(y)f_{x_1}(y)|\,dy\right)dx_2dx_1\\
&+C\int_{x_1\in[0,\frac{1}{2}]}\int_{|x_2|\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} |\kappa_{x_1}f_{x_1}|\left(\int_{y=(x_2)^2}^{\frac{1}{2}}\frac{1}{\sqrt{|y-x_1|}}|\kappa(y)f_{x_1x_1}(y)|\,dy\right)dx_2dx_1\\
=:\; & C\left((i)+(ii)+(iii)+(iv)\right)
\end{aligned} \tag{A.1.27}$$

Let us majorate these quantities independently.

Now, we study (i) and split the domain in y into two parts as previously.

$$\begin{aligned}
(i) \leqslant & \int_{x_1=0}^{\frac{1}{2}}\int_{x_2\leqslant\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}}\int_{y=(x_2)^2}^{x_1/2}\frac{|\ln(x_1)|^{\alpha}}{x_1}\frac{1}{\sqrt{|x_1-y|}}\frac{|\ln(y)|^{\alpha}}{y}\\
&+\int_{x_1=0}^{\frac{1}{2}}\int_{x_2\leqslant\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}}\int_{y=x_1/2}^{1/2}\frac{|\ln(x_1)|^{\alpha}}{x_1}\frac{1}{\sqrt{|x_1-y|}}\frac{|\ln(y)|^{\alpha}}{y}
\end{aligned} \tag{A.1.28}$$

For the first part, we have that

$$\int_{x_1=0}^{\frac{1}{2}}\int_{x_2\leqslant\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}}\int_{y=(x_2)^2}^{x_1/2}\frac{|\ln(x_1)|^\alpha}{x_1}\frac{1}{\sqrt{|x_1-y|}}\frac{|\ln(y)|^\alpha}{y}$$
$$\leqslant\int_{x_1=0}^{\frac{1}{2}}\frac{|\ln(x_1)|^\alpha}{x_1\sqrt{x_1}}\int_{x_2\leqslant\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}}\int_{y=(x_2)^2}^{x_1/2}\frac{|\ln(y)|^\alpha}{y}$$
$$\leqslant\int_{x_1=0}^{\frac{1}{2}}\frac{|\ln(x_1)|^\alpha}{x_1\sqrt{x_1}}\int_{x_2\leqslant\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}}|\ln(x_2)|^{\alpha+1}\leqslant_{(A.1.9)}\int_{x_1=0}^{\frac{1}{2}}\frac{|\ln(x_1)|^{2\alpha+1-\delta}}{x_1}. \quad \text{(A.1.29)}$$

This integral converges when $\delta>\frac{7}{3}$ and $\alpha<\frac{1}{6}$. For the second part,

$$\int_{x_1=0}^{\frac{1}{2}}\int_{x_2\leqslant\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}}\int_{y=x_1/2}^{\frac{1}{2}}\frac{|\ln(x_1)|^\alpha}{x_1\sqrt{|x_1-y|}}|\ln(y)|^\alpha\leqslant C\int_0^{\frac{1}{2}}\frac{1}{x_1\cdot|\ln(x_1)|^\delta}|\ln(x_1)|^{2\alpha}dx_1 \quad \text{(A.1.30)}$$

Using the previous upper bounds. Since $2\alpha<1/3$, this integral converges for $\delta>\frac{4}{3}$.

$$(ii)\leqslant C||\kappa||^2_{L^\infty}\int_{x_1\in[0,\frac{1}{2}]}\frac{\sqrt{|x_1|}}{|\ln(x_1)|^\delta}f_{x_1x_1}(x_1)\int_{y=(x_2)^2}^{\frac{1}{2}}\frac{f_{x_1x_1}(y)}{\sqrt{y-x_1}}dydx_1$$
$$\leqslant C||\kappa||^2_{L^\infty}\int_{x_1\in[0,\frac{1}{2}]}\frac{\sqrt{|x_1|}}{|\ln(x_1)|^{\delta-\alpha}}\int_{y=(x_2)^2}^{\frac{1}{2}}\frac{|\ln(y)|^\alpha}{\sqrt{y-x_1}}dydx_1 \quad \text{(A.1.31)}$$

By the calculus made in lemma A.1.3, this integral converges provided that $\alpha<\frac{1}{6}$ and $\delta>\frac{7}{3}$.

Now, we look at (iii).

$$
\begin{aligned}
(iii) \leqslant\, & C\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} ||\kappa||_\infty \frac{\alpha|\ln(x_1)|^{\alpha-1}}{x_1}\left(\int_{y=x_2^2}^{x_1/2}\frac{1}{\sqrt{|y-x_1|}}\frac{1}{y}|\ln(y)|^{\alpha}dy\right)dx_2dx_1\\
&+C\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} ||\kappa||_\infty \frac{\alpha|\ln(x_1)|^{\alpha-1}}{x_1}\left(\int_{y=x_1/2}^{\frac{1}{2}}\frac{1}{\sqrt{|y-x_1|}}\frac{1}{y}|\ln(y)|^{\alpha}dy\right)dx_2dx_1\\
&=(*)^1+(*)^2.
\end{aligned}
\tag{A.1.32}
$$

For the first term, using (A.1.9), we obtain

$$
\begin{aligned}
(*)^1 &\leqslant C\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} ||\kappa||_\infty \frac{\alpha|\ln(x_1)|^{\alpha-1}}{x_1\sqrt{x_1}}\int_{y=x_2^2}^{x_1/2}\frac{1}{y}|\ln(y)|^{\alpha}\\
&\leqslant C\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} ||\kappa||_\infty \frac{\alpha|\ln(x_1)|^{\alpha-1}}{x_1\sqrt{x_1}}|\ln(x_2)|^{\alpha+1}\leqslant C\int_{x_1=0}^{\frac{1}{2}}\frac{|\ln(x_1)|^{2\alpha-\delta}}{x_1},
\end{aligned}
\tag{A.1.33}
$$

and for the second term, using (A.1.7),

$$
\begin{aligned}
(*)^2 &\leqslant C\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^{\delta}}} ||\kappa||_\infty \frac{\alpha|\ln(x_1)|^{2\alpha-1}}{x_1}\int_{y=x_1/2}^{\frac{1}{2}}\frac{1}{y\sqrt{|x-y|}}\\
&\qquad\qquad\leqslant C\int_{x_1=0}^{\frac{1}{2}}\frac{|\ln(x_1)|^{2\alpha-1-\delta}}{x_1}.
\end{aligned}
\tag{A.1.34}
$$

Both those expressions converge when $\alpha<\frac{1}{6}$ and $\delta>\frac{7}{3}$.

Lastly, we do the same for (iv).

$$
\begin{aligned}
(iv) \leqslant C &\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \frac{|\ln(x_1)|^\alpha}{x_1}\int_{y=(x_2)^2}^{x_1/2}\frac{||\kappa||_\infty|\ln(y)|^{\alpha-1}}{\sqrt{|x-y|}y} \\
&+\int_{x_1\in[0,\frac{1}{2}]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \frac{|\ln(x_1)|^\alpha}{x_1}\int_{y=x_1/2}^{\frac{1}{2}}\frac{||\kappa||_\infty|\ln(y)|^{\alpha-1}}{\sqrt{|x-y|}y} \\
\leqslant &(*)^1+(*)^2.
\end{aligned}
\tag{A.1.35}
$$

For the first term, we obtain using (A.1.8)

$$
\begin{aligned}
(*)^1 \leqslant C &\int_{x_1\in[0,1/2]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \frac{|\ln(x_1)|^\alpha}{x_1\sqrt{x_1}}\int_{y=(x_2)^2}^{x_1/2}\frac{||\kappa||_\infty|\ln(y)|^{\alpha-1}}{y} \\
\leqslant C &\int_{x_1\in[0,1/2]}\int_{x_2\leqslant\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \frac{|\ln(x_1)|^\alpha}{x_1\sqrt{x_1}}|\ln(x_2)|^\alpha \leqslant C\int_{x_1\in[0,1/2]}\frac{|\ln(x_1)|^{2\alpha-\delta}}{x_1},
\end{aligned}
\tag{A.1.36}
$$

and for the second term, using (A.1.7),

$$
\begin{aligned}
(*)^2 \leqslant C &\int_{x_1\in[0,1/2]}\int_{x_2\leqslant\frac{1}{2}\frac{\sqrt{x_1}}{|\ln(x_1)|^\delta}} \frac{|\ln(x_1)|^{2\alpha-1}}{x_1}\int_{y=x_1/2}^{1/2}\frac{1}{y\sqrt{|x-y|}} \\
&\leqslant C\int_{x_1\in[0,1/2]}\frac{|\ln(x_1)|^{2\alpha-1-\delta}}{x_1}.
\end{aligned}
\tag{A.1.37}
$$

Again, both those expressions converge when $\delta \geqslant \frac{7}{3}$ and $\alpha \leqslant \frac{1}{6}$.

□

Definitions

Definition A.1.4. *We say that u is a proper solution, if it is a distributional solution, and if u is the weak limit of a sequence of smooth solutions u_ε with data $(\phi_\varepsilon \star f, \phi_\varepsilon \star g)$,* *where* $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon})\varepsilon^{-n}$ *for some function* ϕ *satisfying* $\phi \in C_0^\infty$ *and* $\int \phi = 1$.

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Curriculum Vitae

PERSONAL DATA

Place and date of birth: *Pontoise, France, 14th August 1992.*

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Education

2016-2021 **Ph.D at EPFL, Switzerland** under the supervision of Prof. Joachim Krieger. The thesis in entitled *A sharp counterexample to local well-posedness of quasilinear wave equations in dimension 2, and its stability.* The thesis uses Fourier analysis, fractional order differentiations, and characteristics method.

2015-2016 **MFOCS at Oxford University, United Kingdom**. It is a master in mathematics and theoretical computer science. The subjects I chose were algebra, topology, number theory, algorithmics, quantum computing and categories.

2013-2015 **Parisian master of research in computer science (MPRI), at Ecole normale superieure de Cachan**. The main subjects include algorithmics, logics, linear logics, quantum computing, bio-informatics, game theory and discrete mathematics.

2012-2013 **Bachelor at ENS Cachan in theoretical computer science**. The main subjects include algorithmics, discrete mathematics, logics, programming, network and systems.

2009-2012 **CPGE**, Classe preparatoire aux Grandes Ecoles, at Pierre-Corneille and Lycee Louis le Grand, Paris.

Multi-thread optimization for topological sort andvalues propagation on big graphs.

Experience

Apr.-Sep. 2015 **6 months Research internship at Oxford university in PDEs, United Kingdom**. Supervised By Prof. Robert Van Gorder. The title of the research project is "Classification of elliptic potentialsfor the 3x3 spectral problem". The subjects include the Kortweg de Vries equations, the AKNS hierarchy and the inverse scattering problem.

Mar.-Aug. 2015 **6 months Research internship at Sungkyunkwan University in combinatorics, South Korea**. Supervised By Prof. Jang Soo Kim, the project included Askey Wilson polynomials, Motzkin paths, reversing sign involution method and general discrete mathematics. Submitted preprint: clic here.

Apr.-Aug. 2014 **5 months Research internship at SAP in algorithmics, Walldorf, Germany**. The title of the project is "Multi-thread optimization for topological sort and values propagation on big graphs". My task was to the design and develop algorithms that work with very large graphs that we can for instance find in social networks. Languages involved: C++, C, LLVM, SQL (HANA). Master thesis: clic here.

Jun.-Aug. 2012 **3 months Research internship at CNRS, Grenoble** in partnership with Grenoble's hospital. Supervised by Marie-Christine Rousset, my job was to design and implement efficient web data reasoning algorithms. Languages involved: C++, Java, MySQL.

Teaching

Teaching assistants have the following duties: Participate to the design of exercise sheets, do the exercise sessions, supervise the tutors when there are tutors for the lecture, participate to the design of the exam, grade the exam, replace the teacher for the lecture when he can not do it (conferences, illness...).

Evolutional PDE (Teaching assistant), EPFL (Master course). The main subjects include: First order PDE, linear, semi-linear and quasilinear, method of characteristics, envelope , complete integral, characteristic variety, Cauchy-Kowalevski problem, parabolic equation, heat equation, regularity. Contraction semigroups, Hile-Yosida theorem.

Dynamics and bifurcations (Teaching assistant), EPFL (3^{rd} year Bachelor course). The main subjects include: one dimensional flows, equilibrium points and stability, bifurcations for flows and maps, phase protrait, Lyapunov stability and functions, ω limit sets and orbits, Poincaré-Bendixon's theorem. This lecture contains computational aspects.

Functional analysis II (Teaching assistant), EPFL. (Second year master course) The main subjects include: locally convex spaces, inductive limit topology, spectral families of commuting operators, associated commuting operator, Reid's inequality, integration with respect to a spectral family, differential calculus in Banach spaces.

Sobolev spaces and elliptic equations (Teaching assistant), EPFL. (3rd year Bachelor course) The main subjects include: Laplace equation and the study of harmonic functions, the notion of fundamental solution, Green functions, Lorentz and Sobolev space, complex analysis (holomorphic functions, Goursat theorem etc.), trace theory, extension operators, Lax-Milgram's theory.

Analysis I and Analysis II (Teaching assistant), EPFL. (1st year Bachelor course) General analysis, integration theory and vectorial calculus.

Languages

English: Fluent.

French: Mothertongue.

Spanish and German: Basic knowledge.