Gluing torsion endo-permutation modules

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Abstract : Let k be a field of characteristic p, and P be a finite p-group, where p is an odd prime. In this paper, we consider the problem of gluing compatible families of endo-permutation modules : being given a torsion element M_Q in the Dade group $D(N_P(Q)/Q)$, for each non-trivial subgroup Q of P, subject to obvious compatibility conditions, we show that it is always possible to find an element M in the Dade group of P such that $\mathrm{Defres}^P_{N_P(Q)/Q} M = M_Q$ for all Q, but that M need not be a torsion element of $D(P)$. The obstruction to this is controlled by an element in the zero-th cohomology group over $\mathbb{F}_2$ of the poset of elementary abelian subgroups of P of rank at least 2. We also give an example of a similar situation, when M_Q is only given for centric subgroups Q of P. Moreover, general results about biset functors and the Dade functor are given in two appendices.

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1. Introduction

Endo-permutation modules for finite p-groups play an important role in the representation theory of finite groups. Recently, the combined efforts of several authors led to the complete classification of all endo-permutation modules (see in particular [Al], [CaTh2], [CaTh3], [BoMa], [Bo4] and see [Th2] for a survey). Now new problems appear and the purpose of the present paper is to consider one of them, namely the question of gluing compatible families of endo-permutation modules.

More precisely, given a finite p-group P and a field k of characteristic p, let M_Q be a capped endo-permutation $k[N_P(Q)/Q]$-module, for every subgroup Q of P belonging to a suitable class $\mathcal{X}$ of subgroups of P, with some straightforward compatibility conditions between the M_Q's. We wish to find a capped endo-permutation kP-module M which 'glues' those data, in the sense that

$$\mathrm{Defres}^P_{N_P(Q)/Q}([M]) = [M_Q] \ ,$$

where $[M]$ denotes the class of M in the Dade group $D(P)$ of P, and Defres is the deflation–restriction map. We note that we are dealing in fact always with classes in the Dade group rather than endo-permutation modules.

This problem is important in block theory, particularly when $\mathcal{X}$ is either the class of centric subgroups of P or the class of all non-trivial subgroups of P. When P is abelian, the problem was completely solved by Puig [Pu2] and he used the result to construct suitable stable equivalences between blocks.

Here we address this question for an arbitrary p-group P with p odd and for the class $\mathcal{X}$ of all non-trivial subgroups of P (the class of centric subgroups is briefly treated in Section 6). Also, we consider only the subgroup $D_t(P)$ consisting of torsion elements of the Dade group, because on the one hand $D_t(P)$ requires special treatment, and on the other hand all endo-permutation modules arising from block theory are either known or expected to be torsion elements of the Dade group. The problem of gluing non-torsion endo-permutation modules will be treated in another paper [Bo5].

Our main result asserts that, if p is odd, it is almost always possible to glue torsion endo-permutation modules. The only exception occurs for groups having maximal elementary abelian subgroups of rank 2 and in this case we control entirely the obstruction, which lies in the zero-th cohomology group of the poset $\mathcal{A}_{\geq 2}(P)$ of elementary abelian subgroups of P of rank at least 2. More precisely, our main result takes the following form:

1.1. Theorem. *If p is an odd prime number and P is a non cyclic p-group, then there is an exact sequence of abelian groups*

$$0 \longrightarrow D_t(P) \xrightarrow{r_P} \lim_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D_t\big(N_P(Q)/Q\big) \xrightarrow{\widetilde{d}_P} \widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P \longrightarrow 0 \ .$$

In order to explain the connection with the problem described above, note that a compatible family is an element of the middle term of the sequence and that it can be 'glued' precisely when it is in the image of the map r_P.

The problem of gluing arbitrary classes of endo-permutation modules (not necessarily torsion) is solved in [Bo5] and it is interesting to notice that it is the first cohomology group of the same poset $\mathcal{A}_{\geq 2}(P)$ which plays a role. This shows the relevance of this poset, which is studied in [BoTh3].

The most important ingredients for the proof of our main result are the following. First we need the fact that the torsion subgroup of the group of endo-trivial modules is trivial when p is odd and P is not cyclic, a result proved in [CaTh2]. We also use a consequence of this result which asserts that the torsion subgroup $D_t(P)$ is annihilated by 2 (i.e. a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$). Note that we write the abelian group $D(P)$ additively (that is, $[M]+[N]=[M\otimes_k N]$ for all capped endo-permutation kP-modules M and N).

Finally, another ingredient is concerned with the detection map to all elementary abelian sections

$$\mathrm{Defres} \ : \ D(P) \quad \longrightarrow \quad \prod_{\substack{S\trianglelefteq T\leq P \\ T/S \ \text{elementary abelian}}} D(T/S) \ ,$$

which is known to be injective when p is odd [CaTh2]. By adapting a method used in [Bo2], we prove that, given any compatible family on the right hand side, it is possible to construct an element of $D(P)$ whose image is equal to $|P|$ times the family we started with. For the torsion subgroup $D_t(P)$ when p is odd, multiplication by $|P|$ is the identity and we thus obtain an isomorphism between $D_t(P)$ and the image of the above detection map restricted to the torsion subgroup.

The construction which we just mentioned works in a much more general context, namely for biset functors and we present the general proof in an appendix about biset functors (Section 7). For the Dade group, we state the result in Section 4 and we postpone the proof to the end of a second appendix (Section 8). In this appendix, we provide general results about the Dade functor, which follow from [BoTh1] but were not explicitly stated there. In particular, we prove a generalized Mackey formula and we explain why the Dade functor is actually a biset functor when p is odd.

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2. Limits of Dade groups

We first recall some basic facts concerning the Dade group and we fix notation. More details about the Dade group can be found in [BoTh1] and also in an appendix of the present paper (Section 8). Throughout this paper, we fix a prime number p, a finite p-group P, and a field k of characteristic p. We denote by $D(P)$ the Dade group of P, i.e. the group of equivalence classes of endo-permutation kP-modules which are capped, that is, which have at least one indecomposable direct summand with vertex P. Recall that we write the abelian group $D(P)$ additively :

$$[M] + [N] = [M \otimes_k N]$$

for all capped endo-permutation kP-modules M and N. We denote by $T(P)$ the subgroup of $D(P)$ consisting of equivalence classes of endo-trivial modules. We denote by $D_t(P)$ the torsion subgroup of $D(P)$ and by $T_t(P)$ the torsion subgroup of $T(P)$.

Recall that if Q is a subgroup of P, we have a restriction map $\mathrm{Res}^P_Q : D(P) \to D(Q)$ and a tensor induction map $\mathrm{Ten}^P_Q : D(Q) \to D(P)$. If Q is a normal subgroup of P, then there is an inflation map $\mathrm{Inf}^P_{P/Q} : D(P/Q) \to D(P)$ and a deflation map $\mathrm{Def}^P_{P/Q} : D(P) \to D(P/Q)$. All of those maps are group homomorphisms. Moreover, any group isomorphism $P \to R$ induces an isomorphism $D(P) \to D(R)$; in particular, if $x \in P$ and Q is a subgroup of P, then conjugation by x induces an isomorphism $\mathrm{Conj}_x : D(Q) \to D({}^xQ)$ (where ${}^xQ = xQx^{-1}$) and we write simply $\mathrm{Conj}_x(u) = {}^xu$ for any $u \in D(Q)$.

A *section* of a group P is by definition a pair (Q, R) of subgroups of P such that $R \trianglelefteq Q$. If (Q, R) is a section of P, we write $\mathrm{Defres}^P_{Q/R} = \mathrm{Def}^Q_{Q/R} \mathrm{Res}^P_Q$ and similarly $\mathrm{Teninf}^P_{Q/R} = \mathrm{Ten}^P_Q \mathrm{Inf}^Q_{Q/R}$.

The above maps satisfy transitivity properties which we shall use freely. First we have

$$\mathrm{Res}^P_P = \mathrm{Ten}^P_P = \mathrm{Inf}^P_{P/\mathbf{1}} = \mathrm{Def}^P_{P/\mathbf{1}} = \mathrm{id}_{D(P)} ,$$

for any p-group P (with the obvious identification $P/\mathbf{1} = P$). Next we have

$$\mathrm{Defres}^{Q/R}_{S/T} \mathrm{Defres}^P_{Q/R} = \mathrm{Defres}^P_{S/T} \qquad \text{and} \qquad \mathrm{Teninf}^P_{Q/R} \mathrm{Teninf}^{Q/R}_{S/T} = \mathrm{Teninf}^P_{S/T}$$

for any p-group P and any sections (Q, R), (S, T) of P satisfying $R \leq T \leq S \leq Q$ (with the obvious identification $(S/R)/(T/R) = S/T$, that we will use freely throughout this paper). As special cases of this, there are the familiar transitivity properties of restriction and tensor induction, as well as the transitivity of deflation and inflation.

A compatible family of elements of $D\big(N_P(Q)/Q\big)$, where Q runs over the non-trivial subgroups of P, is actually an element of a limit as follows. We denote by

$$\lim_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D_t\big(N_P(Q)/Q\big)$$

the set of sequences $(u_Q)_{\mathbf{1}<Q\leq P}$ indexed by non-trivial subgroups Q of P, where $u_Q \in D_t\big(N_P(Q)/Q\big)$, fulfilling the following two conditions :

1. if $Q \trianglelefteq R$, then
$$\mathrm{Defres}^{N_P(Q)/Q}_{N_P(Q,R)/R}(u_Q) = \mathrm{Res}^{N_P(R)/R}_{N_P(Q,R)/R}(u_R) ,$$
where $N_P(Q, R) = N_P(Q) \cap N_P(R)$,

2. if $Q \le P$ and $x \in P$, then ${}^x u_Q = u_{{}^x Q}$.

We denote by r_P the natural map induced by deflation–restriction

$$r_P \ : D_t(P) \longrightarrow \lim_{\substack{\longleftarrow \\ \mathbf{1}<Q\le P}} D_t\big(N_P(Q)/Q\big)$$

mapping $v \in D_t(P)$ to the sequence (v_Q) defined by $v_Q = \mathrm{Defres}^P_{N_P(Q)/Q}(v)$. Our gluing problem can be restated as the question of the surjectivity of the map r_P. However, we first settle the question of the injectivity of r_P.

2.1. Lemma. *(a) The kernel of the map r_P is equal to $T_t(P)$.*

(b) If p is odd and P is not cyclic, then r_P is injective.

Proof. (a) Recall that $T(P)$ is equal to the intersection of all the kernels of the deflation–restriction maps

$$\mathrm{Defres}^P_{N_P(Q)/Q} : \ D(P) \longrightarrow D\big(N_P(Q)/Q\big)$$

where Q runs over all non-trivial subgroups of P (see 2.1.2 in [Pu1]). It follows immediately that the kernel of r_P is equal to the torsion subgroup of $T(P)$.

(b) If p is odd and P is not cyclic, then $T(P)$ is torsion-free by [CaTh2]. □

In our next lemma, we consider one case where it is easy to prove that r_P is an isomorphism. It is a special case of the result of Puig mentioned in the introduction (see Proposition 3.6 in [Pu2]).

2.2. Lemma. *Let E be an elementary abelian p-group of rank at least 2. Then*

$$r_E \ : D_t(E) \longrightarrow \lim_{\substack{\longleftarrow \\ \mathbf{1}<F\le E}} D_t(E/F)$$

is an isomorphism.

Moreover the inverse isomorphism maps $(u_F) \in \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<F\le E}} D(E/F)$ to

$$v = -\sum_{\mathbf{1}<F\le E} \mu(\mathbf{1},F)\, \mathrm{Inf}^E_{E/F}(u_F) \ ,$$

where $\mu(\mathbf{1},F)$ is the Möbius function of the poset of subgroups of F.

Proof. By the previous lemma, r_E is injective. For the surjectivity of r_E, we do not need to restrict to the torsion subgroup. Given $u = (u_F) \in \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<F\le E}} D(E/F)$, define $v \in D(E)$ as in the statement. We need the property that, for any subgroups A, F of E

$$\mathrm{Def}^E_{E/A}\, \mathrm{Inf}^E_{E/F} = \mathrm{Inf}^{E/A}_{E/AF}\, \mathrm{Def}^{E/F}_{E/AF}$$

which is proved in the second appendix (see Corollary 8.7) as a consequence of a much more general formula. Alternatively, a direct proof using the very definition of the deflation maps is not hard. Then for any non trivial subgroup A of E, we have

$$\begin{aligned}
\mathrm{Def}^E_{E/A}(v) &= \sum_{\mathbf{1}<F\leq E} -\mu(\mathbf{1},F)\,\mathrm{Def}^E_{E/A}\,\mathrm{Inf}^E_{E/F}(u_F)\\
&= \sum_{\mathbf{1}<F\leq E} -\mu(\mathbf{1},F)\,\mathrm{Inf}^{E/A}_{E/AF}\,\mathrm{Def}^{E/F}_{E/AF}(u_F)\\
&= \sum_{A\leq X\leq E}\ \sum_{\substack{\mathbf{1}<F\leq X\\ AF=X}} -\mu(\mathbf{1},F)\,\mathrm{Inf}^{E/A}_{E/X}\,\mathrm{Def}^{E/F}_{E/X}(u_F)\\
&= \sum_{A\leq X\leq E} \mathrm{Inf}^{E/A}_{E/X} \sum_{\substack{\mathbf{1}<F\leq X\\ AF=X}} -\mu(\mathbf{1},F)\,u_X\ .
\end{aligned}$$

Now the Möbius function satisfies the well-known property (see Cor. 3.9.3 in [St])):

$$\sum_{\substack{\mathbf{1}<F\leq X\\ AF=X}} -\mu(\mathbf{1},F) = \begin{cases} 0 & \text{if } A<X\ ,\\ 1 & \text{if } A=X\ .\end{cases}$$

It follows that $\mathrm{Def}^E_{E/A}(v)=u_A$. In other words $r_E(v)=u$. □

Another way of proving the surjectivity of r_E (for the torsion subgroup) is to notice that, by Dade's theorem for the abelian case, any torsion element of $D(E)$ is a sum of elements inflated from quotients E/M where M runs over the maximal subgroups of E. Therefore, ignoring inflation maps for simplicity, we have

$$D_t(E) = \prod_{M\text{ maximal}} D_t(E/M) = \varprojlim_{\mathbf{1}<F\leq E} D_t(E/F)\ ,$$

by direct inspection.

Note that Lemma 2.2 holds if $p=2$ but yields an uninteresting result because $D_t(E)=0$ when E is an elementary abelian 2-group.

If H is any subgroup of P, we denote by

$$\mathrm{Res}^P_H : \varprojlim_{\mathbf{1}<Q\leq P} D_t\big(N_P(Q)/Q\big) \longrightarrow \varprojlim_{\mathbf{1}<Q\leq H} D_t\big(N_H(Q)/Q\big)$$

the map defined by

$$\big(\mathrm{Res}^P_H(u)\big)_Q = \mathrm{Res}^{N_P(Q)/Q}_{N_H(Q)/Q}(u_Q)\ ,$$

for any $(u_Q)\in \varprojlim_{\mathbf{1}<Q\leq P} D_t\big(N_P(Q)/Q\big)$. Similarly, if $x\in P$, there is a conjugation map

$$u\in \varprojlim_{\mathbf{1}<Q\leq H} D_t\big(N_H(Q)/Q\big) \mapsto {}^xu\in \varprojlim_{\mathbf{1}<Q\leq {}^xH} D_t\big(N_{{}^xH}(Q)/Q\big)$$

defined by $({}^xu)_Q = {}^x(u_{Q^x})$.

These operations of restriction and conjugation have the usual properties of transitivity and commutation. For example, if $K\leq H\leq P$, then $\mathrm{Res}^H_K\circ\mathrm{Res}^P_H=\mathrm{Res}^P_K$. Moreover :

2.3. Lemma. *Let H be a subgroup of P. The diagram*

$$\begin{array}{ccc} D_t(P) & \xrightarrow{\;r_P\;} & \varprojlim\limits_{\mathbf{1}<Q\le P} D_t\big(N_P(Q)/Q\big) \\ {\scriptstyle \mathrm{Res}^P_H}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Res}^P_H} \\ D_t(H) & \xrightarrow[\;r_H\;]{} & \varprojlim\limits_{\mathbf{1}<Q\le H} D_t\big(N_H(Q)/Q\big) \end{array}$$

is commutative.

Proof. This is straightforward. □

3. The poset of elementary abelian subgroups

We denote by $\mathcal{A}_{\geq 2}(P)$ the poset of elementary abelian subgroups of P of rank at least 2, ordered by inclusion of subgroups. The cohomology group $H^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)$ of $\mathcal{A}_{\geq 2}(P)$, with values in the field $\mathbb{F}_2$, is the set of functions $f : \mathcal{A}_{\geq 2}(P) \to \mathbb{F}_2$ such that $f(F) = f(E)$ whenever F and E are elements of $\mathcal{A}_{\geq 2}(P)$ such that $F \leq E$. The reduced cohomology group $\widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)$ is the quotient of $H^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)$ by the one-dimensional subspace of all constant functions on $\mathcal{A}_{\geq 2}(P)$.

The group P acts on $\mathcal{A}_{\geq 2}(P)$ by conjugation, hence on $\widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)$, and we denote by $\widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P$ the set of fixed points under this action. When p is odd, we have $H^1(P, \mathbb{F}_2) = \{0\}$, and $\widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P$ is equal to the set of functions as above which are moreover invariant by P-conjugation, modulo constant functions again.

The dimension of the cohomology group $H^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P$ is the number of conjugacy classes of connected components of $\mathcal{A}_{\geq 2}(P)$. The case where $\mathcal{A}_{\geq 2}(P)$ is not connected is well understood and we now recall the result. We write $m(P)$ for the rank of P, that is, the maximum of the ranks of the elementary abelian subgroups of P. Clearly the isolated vertices of $\mathcal{A}_{\geq 2}(P)$ are precisely the maximal elementary abelian subgroups of rank 2. In particular, if $m(P) = 2$, then $\mathcal{A}_{\geq 2}(P)$ consists of isolated vertices.

Assume that $\mathcal{A}_{\geq 2}(P)$ is disconnected. We define a connected component $\mathcal{B}(P)$ of $\mathcal{A}_{\geq 2}(P)$ as follows. If P has a normal elementary abelian subgroup of rank 2, then we choose such a subgroup E_0, and define $\mathcal{B}(P)$ as the connected component of E_0 in $\mathcal{A}_{\geq 2}(P)$ (so in particular $\mathcal{B}(P) = \{E_0\}$ if $m(P) = 2$). If P has no normal elementary abelian subgroup of rank 2, then we choose some elementary abelian subgroup E_0 of rank 2 and we let $\mathcal{B}(P) = \{E_0\}$. It is well known that this second case occurs only if $p = 2$ and P is either dihedral or semi-dihedral, so $m(P) = 2$ and $\{E_0\}$ is indeed a connected component. This case will not play a role in this paper because we will assume that p is odd for our main result.

The following group-theoretic result is well-known and not very hard to prove (see Lemma 10.21 in [GLS] and Lemma 2.2 in [CaTh3]).

3.1. Lemma. *Assume that $\mathcal{A}_{\geq 2}(P)$ is disconnected.*

(a) P has a unique central subgroup Z of order p. Moreover Z is contained in every maximal elementary abelian subgroup of P.

(b) Any connected component of $\mathcal{A}_{\geq 2}(P)$ different from $\mathcal{B}(P)$ consists of a single subgroup of rank 2 (an isolated vertex).

(c) If E is a subgroup in $\mathcal{A}_{\geq 2}(P) - \mathcal{B}(P)$ and if S is a subgroup of E such that $E = Z \times S$, then $N_P(S)/S$ is cyclic or generalized quaternion.

(d) If E is a subgroup in $\mathcal{A}_{\geq 2}(P) - \mathcal{B}(P)$, then all the subgroups S of E of order p distinct from Z are conjugate in P.

Assume p is odd and P is non trivial, and choose some central subgroup Z of order p in P. In the case where $\mathcal{A}_{\geq 2}(P)$ is disconnected (which will turn out to be the only important case for the following construction), the subgroup Z is unique by the above lemma. In the other cases, we simply choose Z arbitrarily. Recall that $D(Z) = D_t(Z) \cong \mathbb{F}_2$ because p is odd. If $u = (u_Q) \in \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D_t\big(N_P(Q)/Q\big)$, let

$$d_P(u) : \mathcal{A}_{\geq 2}(P) \to \mathbb{F}_2$$

be the function defined by

$$d_P(u)(E) = \gamma_Z\left(\mathrm{Res}^{EZ}_Z\, r^{-1}_{EZ}\, \mathrm{Res}^P_{EZ}(u)\right) \ , \qquad (\mathbf{3.2})$$

where $\gamma_Z : D_t(Z) \to \mathbb{F}_2$ is the canonical isomorphism and where r^{-1}_{EZ} is the inverse of the isomorphism of Lemma 2.2 (note that EZ is elementary abelian). We let $\widetilde{d}_P(u)$ be the class of $d_P(u)$ in the quotient modulo the constant functions.

3.3. Lemma. *The assignement $u \mapsto \widetilde{d}_P(u)$ defines a group homomorphism*

$$\widetilde{d}_P : \lim_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D_t\big(N_P(Q)/Q\big) \longrightarrow \widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P \ .$$

Proof. Let $u \in \lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D_t\big(N_P(Q)/Q\big)$. In order to check that

$$\widetilde{d}_P(u) \in \widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P$$

it suffices to check that $d_P(u)$ lies in $H^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P$. If $F \leq E$ are elements of $\mathcal{A}_{\geq 2}(P)$, then

$$d_P(u)(E) = \gamma_Z\left(\mathrm{Res}^{EZ}_Z\, r^{-1}_{EZ}\, \mathrm{Res}^P_{EZ}(u)\right) \ ,$$

whereas

$$d_P(u)(F) = \gamma_Z\left(\mathrm{Res}^{FZ}_Z\, r^{-1}_{FZ}\, \mathrm{Res}^P_{FZ}(u)\right) \ .$$

By Lemma 2.3, we have

$$\begin{aligned} \mathrm{Res}^{FZ}_Z r^{-1}_{FZ}\, \mathrm{Res}^P_{FZ}(u) &= \mathrm{Res}^{FZ}_Z r^{-1}_{FZ}\, \mathrm{Res}^{EZ}_{FZ}\, \mathrm{Res}^P_{EZ}(u) \\ &= \mathrm{Res}^{FZ}_Z\, \mathrm{Res}^{EZ}_{FZ}\, r^{-1}_{EZ}\, \mathrm{Res}^P_{EZ}(u) \\ &= \mathrm{Res}^{EZ}_Z\, r^{-1}_{EZ}\, \mathrm{Res}^P_{EZ}(u) \ , \end{aligned}$$

so $d_P(u)(F) = d_P(u)(E)$. Now if $x \in P$, we have

$$\mathrm{Res}^{{}^xEZ}_Z\, r^{-1}_{{}^xEZ}\, \mathrm{Res}^P_{{}^xEZ}(u) = {}^x\big(\mathrm{Res}^{EZ}_Z\, r^{-1}_{EZ}\, \mathrm{Res}^P_{EZ}(u)\big) = \mathrm{Res}^{EZ}_Z\, r^{-1}_{EZ}\, \mathrm{Res}^P_{EZ}(u) \ ,$$

so $d_P(u)({}^xE) = d_P(u)(E)$.

Finally $\widetilde{d}_P$ is a group homomorphism because it is constructed by composing group homomorphisms. □

4. Inverting the detection map

We denote by $\mathcal{E}(P)$ the set of sections (T,S) of P such that T/S is elementary abelian. When p is odd, by Theorem 13.1 in [CaTh2], we can detect elements in $D(P)$ by means of the following injective map

$$\text{Defres} \;:\; D(P) \longrightarrow \prod_{(T,S)\in\mathcal{E}(P)} D(T/S)\ ,$$

which we call the detection map. Clearly the image of the detection map consists of compatible families, so that we actually obtain an injective homomorphism

$$\text{Defres} \;:\; D(P) \longrightarrow \varprojlim_{(T,S)\in\mathcal{E}(P)} D(T/S)\ .$$

Here the limit is defined to be the set of all families

$$u = (u_{T,S})_{(T,S)\in\mathcal{E}(P)} \;\in \prod_{(T,S)\in\mathcal{E}(P)} D(T/S)$$

satisfying the following two conditions :

(4.1)
$$\begin{cases} 1.\ \text{if } (T,S),(T',S') \in \mathcal{E}(P) \text{ with } S \le S' \le T' \le T, \text{ then } \mathrm{Defres}^{T/S}_{T'/S'}(u_{T,S}) = u_{T',S'}\ , \\ 2.\ \text{if } x\in P \text{ and } (T,S)\in\mathcal{E}(P), \text{ then } {}^{x}u_{T,S} = u_{{}^{x}T,{}^{x}S}\ . \end{cases}$$

We want to show that, on restriction to the torsion subgroup, the map Defres above is an isomorphism. This is a consequence of the following key result, which is concerned with the limit over elementary abelian sections of a fixed p-group P.

4.2. Theorem. *Let P be a p-group. Let $u = (u_{T,S})_{(T,S)\in\mathcal{E}(P)} \in \varprojlim_{(T,S)\in\mathcal{E}(P)} D(T/S)$. Define $v\in D(P)$ by the formula*

$$v = \sum_{(T,S)\in\mathcal{E}(P)} |S|\,\mu(S,T)\,\mathrm{Teninf}^{P}_{S/\Phi(T)}(u_{S,\Phi(T)})\ ,$$

where $\Phi(T)$ denotes the Frattini subgroup of T and where μ denotes the Möbius function of the poset of subgroups of P. Then v satisfies

$$\mathrm{Defres}^{P}_{T/S}(v) = |P|\cdot u_{T,S}$$

for any $(T,S)\in\mathcal{E}(P)$.

In fact, this is a general theorem for arbitrary biset functors which is proved in Section 7 (Theorem 7.2), while the above form for the Dade group is proved at the end of Section 8. The result also plays a crucial role in another paper [BoTh2].

4.3. Corollary. *If p is odd, then the detection map*

$$\text{Defres} \;:\; D_t(P) \longrightarrow \varprojlim_{(T,S)\in\mathcal{E}(P)} D_t(T/S)$$

is an isomorphism.

Proof. Let $(T,S) \in \mathcal{E}(P)$. When p is odd, any element $u_{T,S} \in D_t(T/S)$ has order 2 (see Theorem 1.5 in [CaTh2]) and hence multiplication by $|P|$ is the identity on $D_t(T/S)$. Therefore, with the same notation as in Theorem 4.2, there exists an element $v \in D_t(P)$ such that

$$\mathrm{Defres}^P_{T/S}(v) = |P| \cdot u_{T,S} = u_{T,S}$$

for any $(T,S) \in \mathcal{E}(P)$. Note that it is clear from its definition that v is a torsion element whenever every $u_{T,S}$ is a torsion element. This shows that the detection map is surjective. ∎

5. The main theorem

Using the map r_P of Section 2 and the map $\widetilde{d}_P$ of Section 3, we can now state the main result.

5.1. Theorem. *Let p be an odd prime number and let P be a non cyclic p-group. Then the sequence*

$$0 \longrightarrow D_t(P) \xrightarrow{r_P} \lim_{\substack{\longleftarrow \\ 1<Q\leq P}} D_t\big(N_P(Q)/Q\big) \xrightarrow{\widetilde{d}_P} \widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P \longrightarrow 0$$

is exact.

In particular r_P is an isomorphism if every maximal elementary abelian subgroup of P has rank ≥ 3.

Proof. Recall that r_P is injective by Lemma 2.1. Recall also that $\mathcal{A}_{\geq 2}(P)$ is connected if every maximal elementary abelian subgroup of P has rank ≥ 3 (by Lemma 3.1), in which case $\widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big) = 0$. Thus the additional statement follows immediately from the exactness of the sequence.

We first check that $\widetilde{d}_P \circ r_P = 0$. Let $v \in D_t(P)$. Then for $Q \leq P$, we have

$$r_P(v)_Q = \mathrm{Defres}^P_{N_P(Q)/Q}(v) \ .$$

Recall that $\widetilde{d}_P$ is induced by the map d_P defined by (3.2). Thus if $E \in \mathcal{A}_{\geq 2}(P)$, by Lemma 2.3

$$\begin{aligned} d_P\, r_P(v)(E) &= \gamma_Z\left(\mathrm{Res}^{EZ}_Z\, r_{EZ}^{-1}\, \mathrm{Res}^P_{EZ}\, r_P(v)\right) \\ &= \gamma_Z\left(\mathrm{Res}^{EZ}_Z\, r_{EZ}^{-1} r_{EZ}\, \mathrm{Res}^P_{EZ}(v)\right) \\ &= \gamma_Z\left(\mathrm{Res}^{EZ}_Z\, \mathrm{Res}^P_{EZ}(v)\right) \\ &= \gamma_Z\left(\mathrm{Res}^P_Z(v)\right) \ , \end{aligned}$$

and this is independent of E. Therefore $d_P\, r_P(v)$ is a constant function and it follows that $\widetilde{d}_P \circ r_P = 0$.

We next prove that $\widetilde{d}_P$ is surjective. There is nothing to do if $\mathcal{A}_{\geq 2}(P)$ is connected, for $\widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big) = 0$ in that case. If $\mathcal{A}_{\geq 2}(P)$ is not connected, then by Lemma 3.1, the group $\widetilde{H}^0\big(\mathcal{A}_{\geq 2}(P), \mathbb{F}_2\big)^P$ is generated by the classes of the characteristic functions χ_A where A runs over the set $\mathcal{A}_{\geq 2}(P) - \mathcal{B}(P)$ up to conjugacy. Explicitly χ_A denotes the function having value 1 on A and its conjugates and zero elsewhere. We now fix such an isolated point A and we want to prove that χ_A is in the image of d_P. By

Lemma 3.1 again, P has a unique central subgroup Z of order p, we can write $A = SZ$ where S is a non-central subgroup of order p, and the group $N_P(S)/S$ is cyclic. If Q is any non trivial subgroup of P, we define

$$a_Q = \begin{cases} \Omega_{N_P(Q)/Q} & \text{if } Q \text{ is conjugate to } S \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

where for any p-group X, we write Ω_X for the class of the Heller translate $\Omega^1_X(k)$ in the Dade group $D(X)$. Note that a_Q only depends on A and not on the choice of the subgroup S of A (by Lemma 3.1). Recall that Ω_X is endo-trivial, hence satisfies $\mathrm{Defres}^X_{U/V}(\Omega_X) = 0$ whenever $V \trianglelefteq U \leq X$ and $V \neq \mathbf{1}$. Our next lemma shows that the map $\widetilde{d}_P$ is surjective.

5.2. Lemma. *Let A be a subgroup in $\mathcal{A}_{\geq 2}(P) - \mathcal{B}(P)$, let S be a subgroup of A of order p such that $A = SZ$, and let $a = (a_Q)_{\mathbf{1}<Q\leq P}$ be the sequence defined above.*

(a) The sequence $a = (a_Q)_{\mathbf{1}<Q\leq P}$ is an element of $\varprojlim_{\mathbf{1}<Q\leq P} D_t\big(N_P(Q)/Q\big)$.

(b) $\widetilde{d}_P(a) = \chi_A$.

Proof. (a) By construction, it is clear that ${}^x a_Q = a_{{}^xQ}$ for any $Q \leq P$ and any $x \in P$. Now if $\mathbf{1} < Q \trianglelefteq R$, we have to check that

$$\mathrm{Defres}^{N_P(Q)/Q}_{N_P(Q,R)/R}(a_Q) = \mathrm{Res}^{N_P(R)/R}_{N_P(Q,R)/R}(a_R) \ .$$

If $Q = R$, there is nothing to check. If $Q < R$, the only case where both sides of the above equality are not identically 0 is when Q is conjugate to S. In that case, the right hand side is 0 and the left hand side is

$$\mathrm{Defres}^{N_P(Q)/Q}_{N_P(Q,R)/R}(a_Q) = \mathrm{Defres}^{N_P(Q)/Q}_{N_P(Q,R)/R}(\Omega_{N_P(Q)/Q}) = 0$$

by the above remark.

(b) We now compute $d_P(a)$, and check that if $E \in \mathcal{A}_{\geq 2}(P)$, then

$$d_P(a)(E) = \begin{cases} 1 & \text{if } E \text{ is conjugate to } A \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

By definition

$$d_P(a)(E) = \gamma_Z\left(\mathrm{Res}^{EZ}_Z \, r^{-1}_{EZ} \, \mathrm{Res}^P_{EZ}(a)\right) \ .$$

If $\mathbf{1} < Q \leq EZ$, then

$$\mathrm{Res}^P_{EZ}(a)_Q = \mathrm{Res}^{N_P(Q)/Q}_{EZ/Q}(a_Q) \ .$$

This is zero unless $Q = {}^xS$, for some $x \in P$. In this case EZ is an elementary abelian subgroup containing $QZ = {}^x(SZ) = {}^xA$, and therefore $EZ = {}^xA$ by maximality of A (since A is an isolated point of $\mathcal{A}_{\geq 2}(P)$). Since E has rank ≥ 2, this forces the equality $E = {}^xA$. Thus $d_P(a)(E) = 0$ unless E is conjugate to A.

Now we can assume $E = A$ and in this case $AZ = A$. By Lemma 2.2

$$r^{-1}_A \, \mathrm{Res}^P_A(a) = -\sum_{\mathbf{1}<F\leq A} \mu(\mathbf{1}, F) \, \mathrm{Inf}^A_{A/F} \, \mathrm{Res}^{N_P(F)/F}_{A/F}(a_F) \ .$$

The non-zero terms in the sum are obtained for the subgroups F of A which are conjugate to S. These are all the subgroups of order p of A, different from Z (by Lemma 3.1).

For such a subgroup F, we have $a_F = \Omega_{N_P(F)/F}$, so $\mathrm{Res}^{N_P(F)/F}_{A/F}(a_F) = \Omega_{A/F}$. Moreover $\mu(\mathbf{1}, F) = -1$ and so

$$r_A^{-1}\,\mathrm{Res}^P_A(a) = \sum_{\substack{|F|=p\\ Z\neq F<A}} \mathrm{Inf}^A_{A/F}(\Omega_{A/F})$$

Now for any subgroup F of order p of A, different from Z, we have

$$\mathrm{Res}^A_Z\left(\mathrm{Inf}^A_{A/F}(\Omega_{A/F})\right) = \Omega_Z$$

because $\mathrm{Inf}^A_{A/F}(\Omega_{A/F})$ is the class of the kA-module

$$\Omega^1_{A/F}(k) = \mathrm{Ker}\big(k[A/F] \to k\big)$$

and on restriction to Z, we obtain $\Omega^1_Z(k) = \mathrm{Ker}(kZ \to k)$, whose class in the Dade group is Ω_Z. Since there are p such subgroups F in A, it follows that

$$d_P(a)(A) = \gamma_Z\left(\mathrm{Res}^A_Z\, r_A^{-1}\,\mathrm{Res}^P_A(a)\right) = \gamma_Z(p\cdot\Omega_{Z/\mathbf{1}}) = p\cdot 1 = 1$$

because p is odd. Hence $\widetilde{d}_P(a)$ is the characteristic function χ_A. This proves the lemma. □

The only thing left to prove is that the kernel of $\widetilde{d}_P$ is equal to the image of r_P. Let $w = (w_Q)_{\mathbf{1}<Q\leq P} \in \mathrm{Ker}(\widetilde{d}_P)$. Our aim is to construct an element $u \in \varprojlim_{(T,S)\in\mathcal{E}(P)} D_t(T/S)$ and use the isomorphism of Corollary 4.3 to produce an element $v \in D_t(P)$ such that $r_P(v) = w$.

Let $(T,S) \in \mathcal{E}(P)$. If $S \neq \mathbf{1}$, set $u_{T,S} = \mathrm{Res}^{N_P(S)/S}_{T/S}(w_S)$. If $T \in \mathcal{A}_{\geq 2}(P)$, set $u_{T,\mathbf{1}} = r_T^{-1}\,\mathrm{Res}^P_T(w)$. If T is a subgroup of order p different from Z, set $u_{T,\mathbf{1}} = \mathrm{Res}^{TZ}_T(u_{TZ,\mathbf{1}})$, which makes sense since $u_{TZ,\mathbf{1}}$ is already defined for $TZ \in \mathcal{A}_{\geq 2}(P)$. Finally, if $T = Z$, choose any element $E \in \mathcal{A}_{\geq 2}(P)$ and set $u_{Z,\mathbf{1}} = \mathrm{Res}^{EZ}_Z(u_{EZ,\mathbf{1}})$. Since $w \in \mathrm{Ker}(\widetilde{d}_P)$, this does not depend on the choice of E. Indeed the function $d_P(w)$ is constant, and by definition

$$\gamma_Z\left(\mathrm{Res}^{EZ}_Z(u_{EZ,\mathbf{1}})\right) = \gamma_Z\left(\mathrm{Res}^{EZ}_Z\, r_{EZ}^{-1}\,\mathrm{Res}^P_{EZ}(w)\right) = d_P(w)(E) = \text{ constant }.$$

To be complete, we set $u_{\mathbf{1},\mathbf{1}} = 0$, so that the element $u_{T,S}$ is now defined for every $(T,S) \in \mathcal{E}(P)$.

We want to prove that the elements $u_{T,S}$ fulfill the conditions of the limit appearing before Theorem 4.2, namely conditions (4.1). The second condition on conjugation is obviously satisfied. For the first one, suppose that (T,S) and (T',S') are elements of $\mathcal{E}(P)$ such that $S \leq S' \leq T' \leq T$. We have to check that

$$\mathrm{Defres}^{T/S}_{T'/S'}(u_{T,S}) = u_{T',S'}\ .$$

This is trivially true if $T' = S'$, or if $(T,S) = (T',S')$. So we can assume $T' \neq S'$, and $S \neq S'$ or $T \neq T'$.

If $S \neq \mathbf{1}$, then

$$\begin{aligned}
\mathrm{Defres}^{T/S}_{T'/S'}(u_{T,S}) &= \mathrm{Defres}^{T/S}_{T'/S'}\,\mathrm{Res}^{N_P(S)/S}_{T/S}(w_S)\\
&= \mathrm{Defres}^{N_P(S)/S}_{T'/S'}(w_S)\\
&= \mathrm{Res}^{N_P(S,S')/S'}_{T'/S'}\,\mathrm{Defres}^{N_P(S)/S}_{N_P(S,S')/S'}(w_S)\\
&= \mathrm{Res}^{N_P(S,S')/S'}_{T'/S'}\,\mathrm{Res}^{N_P(S')/S'}_{N_P(S,S')/S'}(w_{S'})
\end{aligned}$$

because $w \in \varprojlim_{\mathbf{1}<Q\leq P} D_t\big(N_P(Q)/Q\big)$. Thus we are left with

$$\mathrm{Defres}^{T/S}_{T'/S'}(u_{T,S}) = \mathrm{Res}^{N_P(S')/S'}_{T'/S'}(w_{S'}) = u_{T',S'} \ .$$

Now if $S = \mathbf{1} \neq S'$, then T is elementary abelian, and since $S' < T' \leq T$, the group T has order at least p^2, so $T \in \mathcal{A}_{\geq 2}(P)$. Hence

$$\begin{aligned}
\mathrm{Defres}^{T}_{T'/S'}(u_{T,\mathbf{1}}) &= \mathrm{Defres}^{T}_{T'/S'}\, r_T^{-1}\, \mathrm{Res}^P_T(w) \\
&= \mathrm{Def}^{T'}_{T'/S'}\, \mathrm{Res}^T_{T'}\, r_T^{-1}\, \mathrm{Res}^P_T(w) \\
&= \mathrm{Def}^{T'}_{T'/S'}\, r_{T'}^{-1}\big(\mathrm{Res}^P_{T'}(w)\big) && \text{(by Lemma 2.3)} \\
&= \big(\mathrm{Res}^P_{T'}(w)\big)_{S'} && \text{(by definition of } r_{T'}\text{)} \\
&= \mathrm{Res}^{N_P(S')/S'}_{T'/S'}(w_{S'}) = u_{T',S'} \ .
\end{aligned}$$

Finally, if $S = S' = \mathbf{1}$, then $\mathbf{1} < T' < T$, and both T and T' are elementary abelian. Moreover $T \in \mathcal{A}_{\geq 2}(P)$. If $T' \in \mathcal{A}_{\geq 2}(P)$, then

$$\mathrm{Res}^T_{T'}(u_{T,\mathbf{1}}) = \mathrm{Res}^T_{T'}\, r_T^{-1}\, \mathrm{Res}^P_T(w) = r_{T'}^{-1}\, \mathrm{Res}^P_{T'}(w) = u_{T',\mathbf{1}} \ ,$$

by Lemma 2.3. If $T' = Z$, then we are done since by definition $u_{Z,\mathbf{1}} = \mathrm{Res}^T_Z(u_{T,\mathbf{1}})$. Now if T' has order p and $T' \neq Z$, then, using Lemma 2.3 again, we obtain

$$\begin{aligned}
\mathrm{Res}^T_{T'}(u_{T,\mathbf{1}}) &= \mathrm{Res}^T_{T'}\, r_T^{-1}\, \mathrm{Res}^P_T(w) \\
&= \mathrm{Res}^T_{T'}\, r_T^{-1}\, \mathrm{Res}^{TZ}_T\, \mathrm{Res}^P_{TZ}(w) \\
&= \mathrm{Res}^T_{T'}\, \mathrm{Res}^{TZ}_T\, r_{TZ}^{-1}\, \mathrm{Res}^P_{TZ}(w) \\
&= \mathrm{Res}^{T'Z}_{T'}\, \mathrm{Res}^{TZ}_{T'Z}\, r_{TZ}^{-1}\, \mathrm{Res}^P_{TZ}(w) \\
&= \mathrm{Res}^{T'Z}_{T'}\, r_{T'Z}^{-1}\, \mathrm{Res}^P_{T'Z}(w) \\
&= \mathrm{Res}^{T'Z}_{T'}(u_{T'Z,\mathbf{1}}) = u_{T',\mathbf{1}} \ .
\end{aligned}$$

All this shows that the family $u = (u_{T,S})$ belongs to $\varprojlim_{(T,S)\in\mathcal{E}(P)} D_t(T/S)$ and so we can apply Corollary 4.3. Hence there exists an element $v \in D_t(P)$ such that $\mathrm{Defres}^P_{T/S}(v) = u_{T,S}$ for every $(T,S) \in \mathcal{E}(P)$. We have now to prove that $r_P(v) = w$, that is, we have to show that $\mathrm{Defres}^P_{N_P(Q)/Q}(v) = w_Q$ for every non-trivial subgroup Q of P. Since elements of the Dade group of $N_P(Q)/Q$ are detected on deflation–restriction to all elementary abelian sections, it suffices to prove that we have equality after applying all the maps $\mathrm{Defres}^{N_P(Q)/Q}_{T/S}$, where $(T,S) \in \mathcal{E}(P)$ such that $Q \leq S \leq T \leq N_P(Q)$. This is proved as follows :

$$\begin{aligned}
\mathrm{Defres}^{N_P(Q)/Q}_{T/S}\, \mathrm{Defres}^P_{N_P(Q)/Q}(v) &= \mathrm{Defres}^P_{T/S}(v) \\
&= u_{T,S} \\
&= \mathrm{Res}^{N_P(S)/S}_{T/S}(w_S) \\
&= \mathrm{Res}^{N_P(Q,S)/S}_{T/S}\, \mathrm{Res}^{N_P(S)/S}_{N_P(Q,S)/S}(w_S) \\
&= \mathrm{Res}^{N_P(Q,S)/S}_{T/S}\, \mathrm{Defres}^{N_P(Q)/Q}_{N_P(Q,S)/S}(w_Q) \\
&= \mathrm{Defres}^{N_P(Q)/Q}_{T/S}(w_Q) \ ,
\end{aligned}$$

as was to be shown. □

5.3. Remark. In view of Lemma 2.1, one can include cyclic groups in the statement of Theorem 5.1 by adding one term in the exact sequence, as follows. Let p be an odd prime number and let P be a p-group. Then the sequence

$$0 \longrightarrow T_t(P) \longrightarrow D_t(P) \xrightarrow{r_P} \lim_{\substack{\longleftarrow \\ \mathbf{1}<Q\le P}} D_t\big(N_P(Q)/Q\big) \xrightarrow{\widetilde{d}_P} \widetilde{H}^0\big(\mathcal{A}_{\ge 2}(P),\mathbb{F}_2\big)^P \longrightarrow 0$$

is exact.

5.4. Remark. Theorem 5.1 does not hold when $p=2$ because there are exceptional torsion endo-trivial modules for quaternion groups and semi-dihedral groups. An easy modification would be either to exclude these cases or to try and state a result in the form of the previous remark. However, the method used for the proof when p is odd does not work when $p=2$ because of Section 4. For the detection map of Section 4, elementary abelian sections are not sufficient and one must include cyclic groups of order 4 and quaternion groups of order 8. Also, $D_t(P)$ is not annihilated by 2, but by 4. But the most important problem is that multiplication by $|P|$ is a power of 2, hence annihilates $D_t(P)$, whereas it is the identity when p is odd. Thus the final argument of Section 4 collapses completely.

We can be more explicit about the compatible families which do not glue to an element of $D_t(P)$. We now show that they actually glue to a non-torsion element in $D(P)$.

5.5. Proposition. *Let p be an odd prime and let P be a non-cyclic p-group. Let $a=(a_Q)_{\mathbf{1}<Q\le P}$ be an element of $\lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\le P}} D_t\big(N_P(Q)/Q\big)$ which is not in the image of $D_t(P)$ under the map r_P. Then there exists a (non-torsion) element $b\in D(P)$ such that*

$$\mathrm{Defres}^P_{N_P(Q)/Q}(b) = a_Q \quad \textit{for every } Q\neq 1\,.$$

Proof. Since a is not in the image of $D_t(P)$, it is not in the kernel of $\widetilde{d}_P$. This forces $\mathcal{A}_{\ge 2}(P)$ to be disconnected and $\widetilde{H}^0\big(\mathcal{A}_{\ge 2}(P),\mathbb{F}_2\big)^P \neq 0$. By linearity, we can assume that $\widetilde{d}_P(a)$ is a basis element χ_A of $\widetilde{H}^0\big(\mathcal{A}_{\ge 2}(P),\mathbb{F}_2\big)^P$, where A is in $\mathcal{A}_{\ge 2}(P)-\mathcal{B}(P)$ and χ_A is the corresponding characteristic function (as in Lemma 5.2). We write $A=SZ$ where S is a non-central subgroup of order p and Z is central of order p. By Lemma 5.2, we can assume that our family $a=(a_Q)_{\mathbf{1}<Q\le P}$ is defined by

$$a_Q = \begin{cases} \Omega_{N_P(Q)/Q} & \text{if } Q \text{ is conjugate to } S\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

because this family maps to the basis element χ_A under $\widetilde{d}_P$. Now we show that this family glues to an element of $D(P)$, namely $b=\Omega_{P/S}$, where $\Omega_{P/S}$ denotes the class in the Dade group of the relative syzygy $\Omega^1_{P/S}(k)=\mathrm{Ker}(k[P/S]\to k)$. Note that P/S is the P-set consisting of cosets xS for $x\in S$. Relative syzygies are endo-permutation modules, as shown by Alperin [Al], and their properties are studied in [Bo2].

We have to prove that

$$\mathrm{Defres}^P_{N_P(Q)/Q}(\Omega_{P/S}) = a_Q \quad \text{for every } Q\neq 1\,.$$

By Section 4 of [Bo2], we have $\mathrm{Defres}^P_{N_P(Q)/Q}(\Omega_{P/S}) = \Omega_{(P/S)^Q}$, where the set of Q-fixed points $(P/S)^Q$ is viewed as an $N_P(Q)/Q$-set in the obvious way. Since Q has no fixed point on P/S if Q is not conjugate to S, we obtain $\mathrm{Defres}^P_{N_P(Q)/Q}(\Omega_{P/S}) = 0$ in that case (Lemma 4.2.1 of [Bo2]). If now Q is conjugate to S, then $P/S \cong P/Q$ as P-sets, $(P/Q)^Q = N_P(Q)/Q$ and therefore $\Omega_{(P/Q)^Q} = \Omega_{N_P(Q)/Q}$. □

5.6. Remark. As mentioned in the introduction, the problem of gluing arbitrary endo-permutation modules (not necessarily torsion) will be solved in [Bo5] for p odd. The result is the existence of an exact sequence

$$0 \longrightarrow T(P) \longrightarrow D(P) \xrightarrow{r_P} \lim_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big) \xrightarrow{\widetilde{d}_P} H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^P .$$

5.7. Remark. Theorem 5.1 and the previous remark show the relevance of the poset $\mathcal{A}_{\geq 2}(P)$. It is proved in [BoTh3] that $\mathcal{A}_{\geq 2}(P)$ has the homotopy type of a bouquet of spheres (of possibly different dimensions). This implies that $H^1\big(\mathcal{A}_{\geq 2}(P), \mathbb{Z}\big)^P$ is free abelian and this shows again that any torsion compatible family can always be glued to a (possibly non-torsion) element in $D(P)$. Indeed the torsion part of $\lim\limits_{\substack{\longleftarrow \\ \mathbf{1}<Q\leq P}} D\big(N_P(Q)/Q\big)$ maps to zero under $\widetilde{d}_P$, hence is in the image of r_P.

6. Centric subgroups

A subgroup Q of a p-group P is called *centric* (or *self-centralizing*) if $C_P(Q) \leq Q$. Such subgroups play an important role in block theory (see Chapter 41 in [Th1]). In particular, the problem of gluing endo-permutation modules also appears when a compatible family of elements of $D\big(N_P(Q)/Q\big)$ is given, but Q runs only over the centric subgroups of P.

Let $\mathcal{C}$ be the family of centric subgroups of P. Here we address the question of gluing a compatible family of torsion elements, defined for subgroups $Q \in \mathcal{C}$ only. As before, the question is to know whether or not there is an element of $D_t(P)$ which glues the data. The purpose of this section is to show that the phenomenon described in Proposition 5.5 may appear again.

We construct an example of a compatible family $(a_Q)_{Q\in\mathcal{C}}$ which is not in the image of $D_t(P)$, but is the image of some non-torsion $b \in D(P)$, that is, $\mathrm{Defres}^P_{N_P(Q)/Q}(b) = a_Q$ for every $Q \in \mathcal{C}$.

Let p be an odd prime. Let $Q = E \rtimes \langle a_1 \rangle$ where $E = \langle v, x_1, x_2, z\rangle$ is an elementary abelian p-group of rank 4 and a_1 has order p, with respect to the action of a_1 on E defined as follows (in multiplicative notation, hence via commutators) :

$$[a_1, v] = x_1 \ , \ [a_1, x_1] = 1 \ , \ [a_1, x_2] = z \ , \ [a_1, z] = 1 \ .$$

Let $P = Q \rtimes \langle a_2 \rangle$, where a_2 has order p, with respect to the action of a_2 on Q defined as follows :

$$[a_2, v] = x_2 \ , \ [a_2, x_2] = 1 \ , \ [a_2, x_1] = z \ , \ [a_2, z] = 1 \ , \ [a_1, a_2] = v \ .$$

One can check that this is indeed an action because p is odd. The group P is a non-split extension

$$1 \longrightarrow E \longrightarrow P \longrightarrow \overline{A} \longrightarrow 1 ,$$

where $\overline{A} = \langle \bar{a}_1, \bar{a}_2 \rangle$ is elementary abelian of rank 2 (here $\bar{a}_i$ is the class of a_i). Note that $Z(P) = \langle z \rangle$.

We let $F = \langle x_1, x_2, z \rangle$ and we notice that P/F is extraspecial of order p^3 and exponent p, because $[a_1, a_2] = v$. It is not hard to see that the sequence

$$1 \longrightarrow F \longrightarrow P \longrightarrow P/F \longrightarrow 1$$

does not split. The whole point of the construction is that every subgroup Q containing F properly is a centric subgroup (as one can check easily), but F itself is not centric (because $C_P(F) = E$).

The group P/F is such that $\mathcal{A}_{\geq 2}(P/F)$ is disconnected and we can apply Proposition 5.5. Consider a family

$$(a_Q)_{F<Q\leq P} \in \varprojlim_{F<Q\leq P} D_t\big(N_P(Q)/Q\big)$$

which is not in the image of $D_t(P/F)$. By Proposition 5.5, there exists a non-torsion element $\bar{b} \in D(P/F)$ such that

$$\mathrm{Defres}^{P/F}_{N_P(Q)/Q}(\bar{b}) = a_Q \quad \text{whenever } F < Q \leq P\,.$$

We now define $b = \mathrm{Inf}^P_{P/F}(\bar{b})$ and we extend the family to all centric subgroups Q by setting $a_Q = \mathrm{Defres}^P_{N_P(Q)/Q}(b)$ for every $Q \in \mathcal{C}$. This extends the previously defined family because $\mathrm{Def}^P_{P/F}\,\mathrm{Inf}^P_{P/F} = \mathrm{id}_{D(P/F)}$.

6.1. Lemma. *For every $Q \in \mathcal{C}$, the element a_Q belongs to the torsion subgroup $D_t(N_P(Q)/Q)$.*

Assuming this lemma, we obtain a family in $\varprojlim_{Q\in\mathcal{C}} D_t\big(N_P(Q)/Q\big)$ which can be glued to an element $b \in D(P)$ by construction. However, this family cannot be glued to a torsion element. Otherwise the deflation of this element in the group $D_t(P/F)$ would glue the family $(a_Q)_{F<Q\leq P}$, but this is impossible by our choice of $(a_Q)_{F<Q\leq P}$.

This example shows that the phenomenon of Proposition 5.5 may happen again with the family of centric subgroups. We have a family of torsion elements (defined for centric subgroups) which cannot be glued to a torsion element of $D(P)$, but which can be glued to a non-torsion element of $D(P)$.

Proof of Lemma 6.1. By construction, we know that a_Q lies in $D_t\big(N_P(Q)/Q\big)$ whenever $F < Q$. Let now Q be a centric subgroup not containing F. Then FQ is centric (because every overgroup of a centric subgroup is centric) and therefore FQ contains F properly because F is not centric. By Proposition 8.6 (applied with $S = P$, $R = F$, $B = N_P(Q)$, and $A = Q$), we have

$$\mathrm{Defres}^P_{N_P(Q)/Q}\,\mathrm{Inf}^P_{P/F} = \Phi\,\mathrm{Defres}^{P/F}_{FN_P(Q)/FQ}$$

for some homomorphism $\Phi : D\big(FN_P(Q)/FQ\big) \to D\big(N_P(Q)/Q\big)$ (actually a composition of an isomorphism and an inflation). Moreover, since $FN_P(Q) \leq N_P(FQ)$, we clearly have

$$\mathrm{Defres}^{P/F}_{FN_P(Q)/FQ} = \mathrm{Res}^{N_P(FQ)/FQ}_{FN_P(Q)/FQ}\,\mathrm{Defres}^{P/F}_{N_P(FQ)/FQ}$$

Since $b = \mathrm{Inf}_{P/F}^{P}(\bar{b})$, it follows that

$$\begin{aligned}
a_Q &= \mathrm{Defres}_{N_P(Q)/Q}^{P}(b) \\
&= \mathrm{Defres}_{N_P(Q)/Q}^{P}\,\mathrm{Inf}_{P/F}^{P}(\bar{b}) \\
&= \Phi\,\mathrm{Defres}_{FN_P(Q)/FQ}^{P/F}(\bar{b}) \\
&= \Phi\,\mathrm{Res}_{FN_P(Q)/FQ}^{N_P(FQ)/FQ}\,\mathrm{Defres}_{N_P(FQ)/FQ}^{P/F}(\bar{b}) \\
&= \Phi\,\mathrm{Res}_{FN_P(Q)/FQ}^{N_P(FQ)/FQ}(a_{FQ})\,.
\end{aligned}$$

Since we know that a_{FQ} lies in $D_t\big(N_P(Q)/Q\big)$ because $F < FQ$, the lemma follows. □

Note that, since the group is small enough, it is possible to make a complete list of all centric subgroups, but the proof of the lemma shows that this is not necessary.

7. Appendix on biset functors

The ordinary Mackey formula describes the behaviour of induction (respectively tensor induction, whenever needed) followed by restriction. It holds in the general framework of Mackey functors. We need a generalization of this formula which occurs when inflation and deflation are also involved. It works in the general framework of biset functors, introduced in [Bo1] and studied in [BoTh1], and [Bo4]. In the same framework, we prove a result about elementary abelian sections of a p-group which is the key for the completion of the proof of Theorem 4.2 (but this completion appears only at the end of the next section).

We first recall the notion of biset functor. Let $\mathcal{C}_p$ be the category whose objects are the finite p-groups and where, for any finite p-groups P and Q, $\mathrm{Hom}_{\mathcal{C}_p}(P,Q)$ is the Grothendieck group of finite (Q,P)-bisets. Recall that a (Q,P)-biset is a finite set endowed with a left action of Q and a right action of P which commute; note that (Q,P)-bisets are called Q-sets-P in [BoTh1]. Addition in $\mathrm{Hom}_{\mathcal{C}_p}(P,Q)$ is induced by the disjoint union of bisets, written $\sqcup$.

If R is another finite p-group and if V is a (R,Q)-biset and U is a (Q,P)-biset, then $V \times_Q U$ is a (R,P)-biset defined as the quotient $V \times_Q U = (V \times Q)/\sim$ with respect to the equivalence relation $(vx,u) \sim (v,xu)$ for all $v \in V$, $u \in U$, and $x \in Q$. The composition of morphisms is induced by this product of bisets. A *biset functor* is an additive functor from $\mathcal{C}_p$ to the category of abelian groups. For important examples of biset functors, we refer to [Bo1], [BoTh1], and [Bo4].

If F is a biset functor and (S,R) is a section of P, then the following homomorphisms are defined:

- $\mathrm{Inf}_{S/R}^{S} : F(S/R) \to F(S)$, induced by the $(S, S/R)$-biset S/R (inflation);
- $\mathrm{Ind}_{S}^{P} : F(S) \to F(P)$, induced by the (P,S)-biset P (induction);
- $\mathrm{Def}_{S/R}^{S} : F(S) \to F(S/R)$, induced by the $(S/R, S)$-biset S/R (deflation);
- $\mathrm{Res}_{S}^{P} : F(P) \to F(S)$, induced by the (S,P)-biset P (restriction);
- Whenever $\alpha : P \to Q$ is an isomorphism, the corresponding isomorphism $\mathrm{Iso}_\alpha : F(P) \to F(Q)$ is induced by the (Q,P)-biset P with left action of Q via α^{-1}. In particular, $\mathrm{Conj}_x : F(S/R) \to F({}^{x}S/{}^{x}R)$ (where $x \in P$) is induced by the $({}^{x}S/{}^{x}R, S/R)$-biset S/R (conjugation by x).

The five types of bisets above will be called *basic bisets*. We also consider the composites :

• $\mathrm{Indinf}_{S/R}^P = \mathrm{Ind}_S^P\mathrm{Inf}_{S/R}^S$, induced by the $(P,S/R)$-biset $P\times_S(S/R)\cong P/R$ (where P/R is the set of all cosets xR for $x\in P$);

• $\mathrm{Defres}_{S/R}^P = \mathrm{Def}_{S/R}^S\mathrm{Res}_S^P$, induced by the $(S/R,P)$-biset $(S/R)\times_S P\cong R\backslash P$ (where $R\backslash P$ is the set of all cosets Rx for $x\in P$).

If V is (Q,P)-biset, we denote by $Q\backslash V/P$ the set of all (Q,P)-orbits QvP in V. We also write $[Q\backslash V/P]$ for an arbitrary set of representatives v of the orbits $QvP\in Q\backslash V/P$.

The following result is the generalized Mackey formula, which is best understood by drawing the so-called 'butterfly' diagram going back to Zassenhaus (see Chap. 4 in [La]).

7.1. Proposition. *Let F be a biset functor and let (S,R) and (B,A) be two sections of a p-group P. Then as homomorphisms from $F(S/R)$ to $F(B/A)$, we have*

$$\begin{aligned}&\mathrm{Defres}_{B/A}^P\,\mathrm{Indinf}_{S/R}^P\\ &=\sum_{x\in[B\backslash P/S]}\mathrm{Indinf}_{(B\cap{}^xS)A/(B\cap{}^xR)A}^{B/A}\,\mathrm{Iso}_{\alpha_x}\,\mathrm{Defres}_{(B\cap{}^xS)\,{}^xR/(A\cap{}^xS)\,{}^xR}^{{}^xS/\,{}^xR}\,\mathrm{Conj}_x\end{aligned}$$

where $\alpha_x : (B\cap{}^xS)\,{}^xR/(A\cap{}^xS)\,{}^xR \to (B\cap{}^xS)A/(B\cap{}^xR)A$ denotes the group isomorphism of the Zassenhaus lemma.

Proof. Actually the proof is just a property of the product of bisets and has nothing to do with the functor F. The map $\mathrm{Defres}_{B/A}^P\,\mathrm{Indinf}_{S/R}^P$ is induced by the $(B/A,S/R)$-biset

$$A\backslash P\times_P P/R\cong A\backslash P/R$$

and this decomposes into orbits as follows

$$A\backslash P/R=\bigsqcup_{x\in[B\backslash P/S]}A\backslash BxS/R\cong\bigsqcup_{x\in[B\backslash P/S]}(A\backslash B\cdot{}^xS/\,{}^xR)\ \times_{{}^xS/{}^xR}C_x\ ,$$

where C_x denotes the $({}^xS/\,{}^xR\,,S/R)$-biset S/R, which induces the conjugation map Conj_x. Taking $x=1$ for simplicity, we analyze further the term $A\backslash BS/R$, which is a transitive $(B/A,S/R)$-biset. The stabilizer Σ of its element AR is the set of pairs (bA,sR) in $(B/A)\times(S/R)$ such that $bs^{-1}\in AR$ and

$$A\backslash BS/R\cong\big((B/A)\times(S/R)\big)/\Sigma\ .$$

By Lemma 7.4 of [BoTh1] (or Lemma 3 in [Bo1]), this biset can be decomposed according to the projections and intersections of Σ. The projection of Σ on the group S/R is equal to $(B\cap S)R/R$ and the intersection of Σ with $\mathbf{1}\times(S/R)$ is equal to $\mathbf{1}\times((A\cap S)R/R)$. Similarly, the projection of Σ on B/A is equal to $(B\cap S)A/A$ and the intersection of Σ with $(B/A)\times\mathbf{1}$ is equal to $((B\cap R)A/A)\times\mathbf{1}$. It follows that $\big((B/A)\times(S/R)\big)/\Sigma$ is the product of the following sequence of bisets :

• the $\big(B/A,(B\cap S)A/A\big)$-biset B/A (which induces $\mathrm{Ind}_{(B\cap S)A/A}^{B/A}$),

• the $\big((B\cap S)A/A,(B\cap S)A/(B\cap R)A\big)$-biset $(B\cap S)A/(B\cap R)A$ (which induces $\mathrm{Inf}_{(B\cap S)A/(B\cap R)A}^{(B\cap S)A/A}$),

• the $\big((B\cap S)A/(B\cap R)A,(B\cap S)R/(A\cap S)R\big)$-biset $(B\cap S)R/(A\cap S)R$ (which induces the isomorphism α_1 of the Zassenhaus lemma),

• the $\big((B\cap S)R/(A\cap S)R,(B\cap S)R/R\big)$-biset $(B\cap S)R/(A\cap S)R$ (which induces $\mathrm{Def}_{(B\cap S)R/(A\cap S)R}^{(B\cap S)R/R}$),

- and the $\big((B\cap S)R/R, S/R\big)$-biset S/R (which induces $\mathrm{Res}^{S/R}_{(B\cap S)R/R}$).

The other terms of the orbit decomposition are treated similarly, replacing S by ${}^xS$, R by ${}^xR$, and α_1 by α_x, and the result follows. Note that each such term induces the following sequence of homomorphisms $F(S/R)\to F(B/A)$:

$$\mathrm{Indinf}^{B/A}_{(B\cap {}^xS)A/(B\cap {}^xR)A}\,\mathrm{Iso}_{\alpha_x}\,\mathrm{Defres}^{{}^xS/{}^xR}_{(B\cap {}^xS)\,{}^xR/(A\cap {}^xS)\,{}^xR}\,\mathrm{Conj}_x\ .$$

□

The next theorem is concerned with the family of elementary abelian groups. Its version for the Dade group is Theorem 4.2 and will be proved at the end of the next section. It also plays a crucial role in another paper [BoTh2].

Recall that $\mathcal{E}(P)$ denotes the set of all sections (T,S) of a p-group P such that T/S is elementary abelian and that $\varprojlim_{(T,S)\in\mathcal{E}(P)} F(T/S)$ denotes the limit defined by the conditions (4.1). The result is inspired by Section 7.4 of [Bo2], in particular by formula (7.4.7).

7.2. Theorem. *Let F be a biset functor and let P be a p-group. Let $u=(u_{T,S})_{(T,S)\in\mathcal{E}(P)}\in \varprojlim_{(T,S)\in\mathcal{E}(P)} F(T/S)$. Define $v\in F(P)$ by the formula*

$$v=\sum_{(T,S)\in\mathcal{E}(P)}|S|\,\mu(S,T)\,\mathrm{Indinf}^P_{S/\Phi(T)}(u_{S,\Phi(T)})\ ,$$

where $\Phi(T)$ denotes the Frattini subgroup of T and where μ denotes the Möbius function of the poset of subgroups of P. Then v satisfies

$$\mathrm{Defres}^P_{T/S}(v)=|P|\cdot u_{T,S}$$

for any $(T,S)\in\mathcal{E}(P)$.

Proof. Note that T/S is elementary abelian if and only if $\Phi(T)\le S\le T$. This explains why the formula for v is well-defined.

We have to compute $\mathrm{Defres}^P_{B/A}(v)$ for every $(B,A)\in\mathcal{E}(P)$ and so we apply the generalized Mackey formula of Proposition 7.1. Since u belongs to the given limit, the conjugation and deflation–restriction involved in the formula can be replaced by the following:

$$\begin{aligned}\mathrm{Defres}^{{}^xS/{}^x\Phi(T)}_{(B\cap {}^xS)\,{}^x\Phi(T)/(A\cap {}^xS)\,{}^x\Phi(T)}({}^xu_{S,\Phi(T)}) &= u_{(B\cap {}^xS)\,{}^x\Phi(T),(A\cap {}^xS)\,{}^x\Phi(T)}\\ &= u_{(B\cap {}^xS)\Phi({}^xT),(A\cap {}^xS)\Phi({}^xT)}=u_x\ ,\end{aligned}$$

where we define u_x in this way for simplicity of notation.

Thus, by Proposition 7.1 we obtain that $\mathrm{Defres}^P_{B/A}(v)$ is equal to

$$\begin{aligned}&\sum_{(T,S)\in\mathcal{E}(P)}|S|\,\mu(S,T)\sum_{x\in[B\backslash P/S]}\mathrm{Indinf}^{B/A}_{(B\cap {}^xS)A/(B\cap {}^x\Phi(T))A}\,\mathrm{Iso}_{\alpha_x}\,(u_x)\\
=&\sum_{(T,S)\in\mathcal{E}(P)}\mu(S,T)\sum_{x\in[B\backslash P/S]}|S|\,\mathrm{Indinf}^{B/A}_{(B\cap {}^xS)A/(B\cap {}^x\Phi(T))A}\,\mathrm{Iso}_{\alpha_x}\,(u_x)\\
=&\sum_{(T,S)\in\mathcal{E}(P)}\mu(S,T)\sum_{x\in[B\backslash P]}|B\cap {}^xS|\,\mathrm{Indinf}^{B/A}_{(B\cap {}^xS)A/(B\cap {}^x\Phi(T))A}\,\mathrm{Iso}_{\alpha_x}\,(u_x)\\
=&\sum_{x\in[B\backslash P]}\sum_{(T,S)\in\mathcal{E}(P)}\mu({}^xS,{}^xT)|B\cap {}^xS|\,\mathrm{Indinf}^{B/A}_{(B\cap {}^xS)A/(B\cap {}^x\Phi(T))A}\,\mathrm{Iso}_{\alpha_x}\,(u_x)\\
=&\ |P:B|\sum_{(T,S)\in\mathcal{E}(P)}\mu(S,T)|B\cap S|\,\mathrm{Indinf}^{B/A}_{(B\cap S)A/(B\cap\Phi(T))A}\,\mathrm{Iso}_{\alpha_1}\,(u_1)\end{aligned}$$

using the invariance of the Möbius function by conjugation and the fact that $({}^xT, {}^xS)$ runs over $\mathcal{E}(P)$ whenever (T,S) does. Here $u_1 = u_{(B\cap S)\Phi(T),(A\cap S)\Phi(T)}$.

Grouping together the terms for which T is given and $B\cap S$ is a given subgroup R of $B\cap T$, we get that $\mathrm{Defres}^P_{B/A}(v)$ is equal to

$$|P:B| \sum_{\substack{T\leq P\\ R\leq B\cap T}} s_{R,T}\,|R|\,\mathrm{Indinf}^{B/A}_{RA/(R\cap\Phi(T))A}\,\mathrm{Iso}_\alpha\left(u_{R\Phi(T),(A\cap R)\Phi(T)}\right) ,$$

where $s_{R,T} = \sum_{\substack{\Phi(T)\leq S\leq T\\ S\cap B=R}} \mu(S,T)$ and $\alpha : R\Phi(T)/(A\cap R)\Phi(T) \to RA/(R\cap\Phi(T))A$ is the isomorphism of the Zassenhaus lemma. Since $\mu(S,T)=0$ if S does not contain $\Phi(T)$ (see Lemma 4.6 in [BoTh1]), we can write

$$s_{R,T} = \sum_{\substack{S\leq T\\ S\cap(B\cap T)=R}} \mu(S,T) .$$

Now by a classical combinatorial lemma (see Cor. 3.9.3 in [St]), this is zero unless $T = B\cap T$, i.e. $T\leq B$. In this case $s_{R,T}=\mu(R,T)$ and this is zero unless $R\geq\Phi(T)$. Moreover $\Phi(T)\leq\Phi(B)$ since B is a p-group, and $\Phi(B)\leq A$ since B/A is elementary abelian. Thus $\Phi(T)\leq A$, and this gives

$$\mathrm{Defres}^P_{B/A}(v) = |P:B| \sum_{R\leq T\leq B} \mu(R,T)\,|R|\,\mathrm{Indinf}^{B/A}_{RA/A}\,\mathrm{Iso}_\alpha\left(u_{R,A\cap R}\right) ,$$

where $\alpha : R/A\cap R \to RA/A$ is the canonical isomorphism. Now for a given R, the sum $\sum_{R\leq T\leq B} \mu(R,T)$ is zero unless $R=B$, in which case $RA=BA=B$, $A\cap R = A\cap B = A$, and $\alpha : B/A \to B/A$ is the identity. Therefore

$$\mathrm{Defres}^P_{B/A}(v) = |P:B|\,|B|\,\mathrm{Indinf}^{B/A}_{B/A}\,\mathrm{Iso}_{\mathrm{id}}\left(u_{B,A}\right) = |P|\cdot u_{B,A} ,$$

as was to be shown. ∎

8. Appendix on the Dade functor

The purpose of this section is to show various results about the Dade group, viewed as a functor $D(-)$ defined on a suitable category of p-groups. Many of these results follow from our paper [BoTh1] but are not explicitly stated there. So this appendix could be viewed as a complement to [BoTh1]. First we state a formula for the composition of morphisms between Dade groups, and then derive a generalized Mackey formula for the Dade group, which resembles the formula of the previous section (Proposition 7.1) but is slightly more complicated. Next we prove that the Dade functor D is a biset functor when p is odd and we conclude with a proof of Theorem 4.2.

We start by reviewing some of the main constructions in [BoTh1]. Let P be a p-group and let k be a field of characteristic p. A *permutation P-algebra A* is a finite dimensional algebra A over k endowed with a left action of P by algebra automorphisms and having a P-invariant basis X. The *multiplication constants* $m(x, y, z) \in k$ for A and X are defined by

$$xy = \sum_{z \in X} m(x, y, z)z$$

for all $x, y \in X$. The P-invariant basis X is a left P-set and the multiplication constants are P-invariant, i.e. $m(gx, gy, gz) = m(x, y, z)$ for all $g \in P$ and $x, y, z \in X$. All P-invariant bases of A are isomorphic as P-sets, because P is a p-group and k has characteristic p. However, the multiplication constants can be different for different P-invariant bases.

If q is a power of p, the map $\lambda \mapsto \lambda^q$ is an endomorphism of k which is used to define a new permutation P-algebra $\Gamma_q(A)$, with the same basis X as A, the same action of P on X, and the new multiplication constants

$$\Gamma_q(m)(x, y, z) = m(x, y, z)^q$$

for all $x, y, z \in X$. The permutation P-algebra $\Gamma_q(A)$ (written $\gamma_q(A)$ in [BoTh1]) only depends on A up to isomorphism (see page 297 in [BoTh1]).

Let P, Q be finite p-groups and let U be a (Q, P)-biset (called a Q-set-P in [BoTh1]). For any P-set X, we define the Q-set $T_U(X) = \mathrm{Hom}_P(U^{op}, X)$ (see page 287 in [BoTh1]). Moreover, if X is a P-invariant basis for a permutation P-algebra A, then $T_U(X)$ is a Q-invariant basis for a permutation Q-algebra $T_U(A)$ with multiplication constants

$$T_U(m)(\theta, \phi, \psi) = \prod_{u \in [U/P]} m(\theta(u), \phi(u), \psi(u))$$

for all $\theta, \phi, \psi \in T_U(X)$, where $[U/P]$ denotes a set of representatives of the right P-orbits in U. The permutation Q-algebra $T_U(A)$ only depends on A and U up to isomorphism, and not on the choice of basis X (see pages 292–293 in [BoTh1]).

If U' is a finite (Q, P)-biset disjoint from U, then the disjoint union $U \sqcup U'$ is again a (Q, P)-biset and for any P-set X, there is a natural isomorphism of Q-sets $T_{U \sqcup U'}(X) \cong T_U(X) \times T_{U'}(X)$. It follows that if A is a permutation P-algebra, there is an isomorphism of Q-algebras

$$T_{U \sqcup U'}(A) \cong T_U(A) \otimes_k T_{U'}(A)$$

(see Proposition 2.10(d) in [BoTh1]).

Let U be a (Q, P)-biset and let V be a (R, Q)-biset where R is another finite p-group. Then, for any P-set X, there is a natural isomorphism of R-sets

$$(T_V \circ T_U)(X) \cong T_{V \times_Q U}(X) ,$$

where $V \times_Q U$ is defined as in the previous section (see Proposition 3.5(a) in [BoTh1]). Therefore, if A is a permutation P-algebra with P-invariant basis X, the R-algebras $(T_V \circ T_U)(A)$ and $T_{V \times_Q U}(X)$ have isomorphic R-sets as invariant bases. However, the algebras need not be isomorphic and the following result describes the difference.

8.1. Proposition. *With the notation above, consider the decomposition of the* (R, P)*-biset* $V \times_Q U$ *into orbits*

$$V \times_Q U = \bigsqcup_{(v,_Q u) \in [R \backslash (V \times_Q U)/P]} R(v,_Q u)P ,$$

where $(v,_Q u)$ *runs over a set of representatives of the* (R,P)*-orbits in* $V \times_Q U$ *(and where* $(v,_Q u)$ *denotes the class in* $V \times_Q U$ *of the pair* $(v,u) \in V \times U$*). Then there are isomorphisms of* R*-algebras*

$$T_{V\times_Q U}(A) \cong \bigotimes_{(v,_Q u)\in[R\backslash(V\times_Q U)/P]} T_{R(v,_Q u)P}(A) \ ,$$

and

$$(T_V \circ T_U)(A) \cong \bigotimes_{(v,_Q u)\in[R\backslash(V\times_Q U)/P]} \Gamma_{|Q_v:Q_{v,uP}|}\big(T_{R(v,_Q u)P}(A)\big) \ ,$$

where Q_v *is the stabilizer of* v *in* Q *and* $Q_{v,uP}$ *is the stabilizer of the orbit* uP *in* Q_v*.*

Proof. It should be noted that the index $|Q_v : Q_{v,uP}|$ only depends on the orbit $R(v,_Q u)P$ and not on the choice of $(v,_Q u)$ (see page 300 in [BoTh1]). The first isomorphism follows from a remark above about disjoint unions. The second isomorphism follows easily from Proposition 3.5(b) in [BoTh1]), together with the definition of the multiplication constants in the various algebras involved. □

Now we can discuss the Dade group, viewed as a functor on p-groups. If M is an endo-permutation kP-module, then $\mathrm{End}_k(M)$ is a permutation P-algebra. Moreover M is capped if and only if a P-invariant basis of $\mathrm{End}_k(M)$ has at least one P-fixed point. In that case, $\mathrm{End}_k(M)$ is a *Dade* P*-algebra*, that is, a permutation P-algebra which is split and simple as k-algebra and whose P-invariant basis contains at least one P-fixed point. The map $M \mapsto \mathrm{End}_k(M)$ induces a bijective correspondence from the set of equivalence classes of capped endo-permutation kP-modules (i.e. the Dade group $D(P)$) to the set of equivalence classes of Dade P-algebras (for an equivalence relation defined on page 280 of [BoTh1]). This allows us to view $D(P)$ as a group of equivalence classes of Dade P-algebras. The tensor product $A \otimes_k B$ of two Dade P-algebras A and B is a Dade P-algebra whose equivalence class corresponds to the sum in the additive group $D(P)$.

If U is a (Q,P)-biset, the correspondence T_U induces a well defined map

$$D(U) : D(P) \longrightarrow D(Q) \ ,$$

sending the class of a P-algebra A to the class of the Q-algebra $T_U(A)$. This map $D(U)$ is a group homomorphism (Corollary 2.13 in [BoTh1]). If U is one of the five basic bisets described in Section 7, then the corresponding homomorphism $D(U)$ is one of the familiar maps, namely inflation, tensor induction, deflation, restriction, or isomorphism. More details appear in Examples 2.5–2.9 in [BoTh1]. We emphasize that if S is a subgroup of P, the map induced by the (P,S)-biset P, called induction and written Ind_S^P in the case of biset functors (Section 7), is now tensor induction and is therefore written Ten_S^P, as in the rest of this paper.

The behaviour of the correspondence T_U with respect to disjoint unions immediately implies that if U and U' are disjoint (Q,P)-bisets, then

$$D_{U\sqcup U'} = D_U + D_{U'}$$

as homomorphisms from $D(P)$ to $D(Q)$. Also the correspondence Γ_q induces a well-defined group homomorphism, still called Γ_q, from $D(P)$ to itself (by Lemma 3.1(b) in [BoTh1]). Moreover, by Lemma 3.2(b) in [BoTh1], we have the following result.

8.2. Lemma. *For any (Q,P)-biset U, we have $D(U)\circ\Gamma_q = \Gamma_q\circ D(U)$ as maps from $D(P)$ to $D(Q)$.*

Finally, Proposition 8.1 implies the following rule regarding the composition of the maps $D(U)$.

8.3. Proposition. *With the notation of Proposition 8.1, consider the decomposition of the (R,P)-biset $V\times_Q U$ into orbits*

$$V\times_Q U = \bigsqcup_{(v,_Q u)\in[R\backslash(V\times_Q U)/P]} R(v,_Q u)P\ ,$$

where $(v,_Q u)$ runs over a set of representatives of the (R,P)-orbits in $V\times_Q U$. Then

$$D(V\times_Q U) = \sum_{(v,_Q u)\in[R\backslash(V\times_Q U)/P]} D\big(R(v,_Q u)P\big)\ ,$$

while

$$D(V)\circ D(U) = \sum_{(v,_Q u)\in[R\backslash(V\times_Q U)/P]} \Gamma_{|Q_v:Q_{v,uP}|}\, D\big(R(v,_Q u)P\big)\ .$$

Proof. The sum in the additive group $D(R)$ is induced by the tensor product of Dade R-algebras. Therefore the result follows directly from Proposition 8.1. □

This proposition indicates why the Dade functor $D(-)$ need not be a biset functor. The presence of the maps Γ_q shows that $D(V)\circ D(U)$ need not be equal to $D(V\times_Q U)$ (although it turns out that Γ_q is often the identity, as we shall see). This is why the results of the previous section cannot be applied directly to Dade groups. However, it should be noted that it is possible to view $D(-)$ as a functor defined on a category whose morphisms are more complicated, namely $\mathrm{End}(k)$-graded bisets, including both bisets and maps Γ_q. This approach is quickly explained in Remark 7.2 of [BoTh1] and is developed in [Bo3].

We now derive a few consequences of Proposition 8.3. First we give a few cases where the map Γ_q does not appear, so that $D(V)\circ D(U)$ is equal to $D(V\times_Q U)$.

8.4. Proposition. *With the notation above, assume that either the right action of Q on V is free or the right action of P on U is transitive. Then $D(V)\circ D(U) = D(V\times_Q U)$, as maps $D(P)\to D(R)$.*

Proof. In both cases, the index $|Q_v : Q_{v,uP}|$ is equal to 1 (see Corollaries 3.7 and 3.9 in [BoTh1]). □

As mentioned after Corollary 3.7 in [BoTh1], this proposition applies whenever V corresponds to a restriction, a tensor induction, an inflation, or an isomorphism, or whenever U corresponds to a restriction, an inflation, a deflation, or an isomorphism. This implies in particular $D(-)$ is a Mackey functor defined on p-groups. In contrast, the map Γ_q (for a suitable q) appears in the formula for the composition of a deflation and a tensor induction (Proposition 3.10 in [BoTh1]).

8.5. Corollary. *Let (S,T) be a section of P and consider the $(T/S,P)$-biset $S\backslash P$ under left and right multiplication, and the $(P,T/S)$-biset P/S under left and right multiplication. Then $D(S\backslash P) = \mathrm{Defres}^P_{T/S}$ and $D(P/S) = \mathrm{Teninf}^P_{T/S}$.*

Proof. The (T,P)-biset P corresponds to restriction Res^P_T and the $(T/S,T)$-biset T/S corresponds to deflation $\mathrm{Def}^T_{T/S}$. The product $(T/S)\times_T P$ is a $(T/S,P)$-biset isomorphic to $S\backslash P$, and since Proposition 8.4 applies, we get

$$D(S\backslash P) = D\big((T/S)\times_T P\big) = D(T/S)\circ D(P) = \mathrm{Def}^T_{T/S}\circ \mathrm{Res}^P_T = \mathrm{Defres}^P_{T/S}\ .$$

The equality $D(P/S) = \mathrm{Teninf}^P_{T/S}$ is proved similarly. □

Proposition 8.4 can also be applied to show that the maps Defres satisfy the transitivity properties mentioned at the beginning of Section 2 (and similarly for Teninf).

The next consequence of Proposition 8.3 is the generalized Mackey formula, which is similar to the formula of the previous section, but with maps Γ_q involved.

8.6. Proposition. *Let (S,R) and (B,A) be two sections of a p-group P. Then as homomorphisms between the Dade groups $D(S/R)$ and $D(B/A)$, we have*

$$\begin{aligned}&\mathrm{Defres}^P_{B/A}\,\mathrm{Teninf}^P_{S/R}\\ &= \sum_{x\in[B\backslash P/S]} \Gamma_{|A:A\cap {}^xS|}\,\mathrm{Teninf}^{B/A}_{(B\cap {}^xS)A/(B\cap {}^xR)A}\,\mathrm{Iso}_{\alpha_x}\,\mathrm{Defres}^{{}^xS/{}^xR}_{(B\cap {}^xS)\,{}^xR/(A\cap {}^xS)\,{}^xR}\,\mathrm{Conj}_x\end{aligned}$$

where $\alpha_x : (B\cap {}^xS)\,{}^xR/(A\cap {}^xS)\,{}^xR \to (B\cap {}^xS)A/(B\cap {}^xR)A$ denotes the group isomorphism of the Zassenhaus lemma.

Proof. We wish to apply Proposition 8.3 to the $(B/A,P)$-biset $V = A\backslash P$ and the $(P,S/R)$-biset $U = P/R$. By Corollary 8.5, $D(V) = \mathrm{Defres}^P_{B/A}$ and $D(U) = \mathrm{Teninf}^P_{S/R}$. The product $V\times_P U$ decomposes into orbits (as in the proof of Proposition 7.1) :

$$V\times_P U = A\backslash P\times_P P/R \cong A\backslash P/R = \bigsqcup_{x\in[B\backslash P/S]} A\backslash BxS/R\ .$$

For every $x\in P$, the double coset AxR of the right hand side corresponds to the pair

$$(v\,,_P u) = (A{\cdot}1\,,_P xR)\in A\backslash P\times_P P/R$$

of the left hand side, where $v = A{\cdot}1$ and $u = xR$. The corresponding stabilizers are $P_v = A$ and

$$P_{v,u(S/R)} = A_{x(S/R)} = \{\, y\in A \mid yx(S/R) = x(S/R)\,\} = A\cap xSx^{-1}\ .$$

It follows now from Proposition 8.3 that

$$\mathrm{Defres}^P_{B/A}\circ\mathrm{Teninf}^P_{S/R} = D(V)\circ D(U) = \sum_{x\in[B\backslash P/S]} \Gamma_{|A:A\cap {}^xS|} D(A\backslash BxS/R)\ .$$

The rest of the proof consists of the decomposition of the transitive $(B/A,S/R)$-biset $A\backslash BxS/R$ as a product of basic bisets and this follows exactly as in the proof of Proposition 7.1. For every successive product of basic bisets in this decomposition, Proposition 8.4 applies and we obtain the composition of the corresponding basic maps between Dade groups, without any map Γ_q. Therefore, as in the proof of Proposition 7.1, we obtain that $D(A\backslash BxS/R)$ is equal to the following sequence of homomorphisms :

$$\mathrm{Teninf}^{B/A}_{(B\cap {}^xS)A/(B\cap {}^xR)A}\,\mathrm{Iso}_{\alpha_x}\,\mathrm{Defres}^{{}^xS/{}^xR}_{(B\cap {}^xS)\,{}^xR/(A\cap {}^xS)\,{}^xR}\,\mathrm{Conj}_x\ .$$

The result follows. □

Numerous special cases of the formula of Proposition 8.6 can be derived, including of course the ordinary Mackey formula. Here is one case which is used in this paper.

8.7. Corollary. *Let A and R be normal subgroups of a p-group P. Then*

$$\mathrm{Def}^P_{P/A}\,\mathrm{Inf}^P_{P/R} = \mathrm{Inf}^{P/A}_{P/AR}\,\mathrm{Def}^{P/R}_{P/AR}\,.$$

Proof. This is the special case of Proposition 8.6 when $S = P$ and $B = P$.

For a more direct way of proving this special case, one can notice the isomorphisms of $(P/A, P/R)$-bisets

$$(P/A) \times_P (P/R) \cong P/AR \cong (P/AR) \times_{P/AR} (P/AR)$$

and apply Proposition 8.4. □

As mentioned earlier, the homomorphism Γ_q is often the identity and we have already seen this in Proposition 8.4. But there are also the following important cases :

8.8. Proposition. *Let q be a power of p.*

(a) If A is an abelian p-group, then $\Gamma_q : D(A) \to D(A)$ is the identity.

(b) If p is an odd prime, then $\Gamma_q : D(P) \to D(P)$ is the identity for every p-group P. In particular, D is a biset functor if p is an odd prime.

(c) Let (S,R) and (B,A) be two sections of a p-group P. If either S/R or B/A is abelian, then each map $\Gamma_{|A:A\cap {}^xS|}$ is the identity in the generalized Mackey formula of Proposition 8.6.

Proof. (a) For any abelian p-group A, every element of $D(A)$ is defined over the base field $\mathbb{F}_p$ by the classification of Dade in the abelian case (see [Da]). Since the map $\lambda \mapsto \lambda^q$ is the identity on $\mathbb{F}_p$ for any power q of the prime p, the corresponding map $\Gamma_q : D(A) \to D(A)$ is the identity.

(b) If p is odd and P is any p-group, then by Theorem 13.1 in [CaTh2], the map

$$\mathrm{Defres}\;: D(P) \longrightarrow \prod_{(T,S)\in\mathcal{E}(P)} D(T/S)$$

is injective, where $\mathcal{E}(P)$ denotes the set of sections (T,S) of P such that T/S is elementary abelian. Now Γ_q is the identity on each $D(T/S)$ by (a) and Γ_q commutes with deflation and restriction by Lemma 8.2. Therefore Γ_q is the identity on $D(P)$.

It follows that the map Γ_q can be ignored in the formula for the composition of homomorphisms $D(V)\circ D(U)$ in Proposition 8.3. Therefore $D(V)\circ D(U) = D(V \times_Q U)$. This means that D is a biset functor.

(c) The map Γ_q commutes with every basic homomorphism between Dade groups, by Lemma 8.2. Therefore, in the formula of Proposition 8.6, $\Gamma_{|A:A\cap {}^xS|}$ can be applied first to $D(S/R)$, or alternatively to $D(B/A)$ at the end of the composition. If either S/R or B/A is abelian, it follows from (a) that this map can be suppressed in the composition. □

We can finally prove Theorem 4.2.

Proof of Theorem 4.2. If p is odd, then D is a biset functor by Proposition 8.8. Therefore Theorem 7.2 applies (replacing induction by tensor induction, hence Indinf by Teninf). This gives precisely the statement of Theorem 4.2.

If $p = 2$, then Theorem 7.2 cannot be applied directly. We must go back to its proof and notice that the main ingredient is the generalized Mackey formula of Proposition 7.1. The version of this formula for the Dade group appears in Proposition 8.6 and involves maps $\Gamma_{|A:A\cap {}^xS|}$. However, by part (c) of Proposition 8.8, every such map $\Gamma_{|A:A\cap {}^xS|}$ is the identity, because the group B/A involved in the proof is abelian. Therefore the whole proof of Theorem 7.2 actually goes through also when $p = 2$. □

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