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Why $H\mathbb{Z}$ -algebra Spectra are Differential Graded Algebras?

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Master Thesis

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List of notations

\mathcal{C}_*	Category of pointed objects of a category \mathcal{C}
$s\mathcal{C}$	Category of simplicial objects in a category \mathcal{C}
$\text{Ho } \mathcal{C}$	The homotopy category of a category \mathcal{C}
$\text{Mon}_{\mathcal{C}}$	Category of monoids in a category \mathcal{C}
$R\text{-Mod}_{\mathcal{C}}$	Category of R -modules in a category \mathcal{C}
$\mathcal{C}^{\mathbb{N}}$	Category of non-symmetric sequences in \mathcal{C}
\mathcal{C}^{Σ}	Category of symmetric sequences in \mathcal{C}
$Sp^{\mathbb{N}}(\mathcal{C})$	Category of non-symmetric spectra over \mathcal{C}
$Sp^{\Sigma}(\mathcal{C})$	Category of symmetric spectra over \mathcal{C}
$T\text{-alg}_{\mathcal{C}}$	Category of algebras over a monad T in \mathcal{C}
Set	Category of sets
Top	Category of topological spaces
Ab	Category of abelian groups
Ch_+	Category of non-negatively graded chain complexes
DGR-Mod	Category of differential graded R -modules
DG-Alg_R	Category of differential graded algebras over R
$(\mathcal{C}, \square, \mathbb{I})$	A monoidal category \mathcal{C} with product \square and unit \mathbb{I}
$(\mathcal{C}, \text{WE}, \text{Fib}, \text{Cof})$	A model category \mathcal{C} with distinguished classes of weak equivalences WE , fibrations Fib , cofibrations Cof
$\xrightarrow{\sim}$	A weak equivalence
\rightarrow	A fibration
\twoheadrightarrow	A cofibration
\mathcal{I}	Class of generating cofibrations
\mathcal{J}	Class of generating acyclic cofibrations
$\mathcal{C}(A, B)$	The set of maps from A to B in a category \mathcal{C}
$\text{Hom}(A, B)$	The object in \mathcal{C} of maps from A to B
$\text{Map}(A, B)$	The simplicial set of maps from A to B in a category \mathcal{C}
$[f]$	Homotopy class of the map f
S^n	Sphere of dimension n

Introduction

In homological algebra, to understand commutative rings R , one studies R -modules, chain complexes of R -modules and their monoids, the differential graded R -algebras. The category of R -modules has a rich structure, but too rigid to efficiently work with homological invariants and homotopy invariant properties. It appears more appropriate to operate in the derived category $\mathcal{D}(R)$, which is the homotopy category of differential graded R -modules.

Algebra of *symmetric* spectra offers a generalization of homological algebra. In this frame, spectra are objects that take the place of abelian groups; in particular, the analogue of the initial ring \mathbb{Z} is the sphere spectrum \mathbb{S} . Tensoring over \mathbb{S} endows the category of spectra with a symmetric monoidal smash product, analogous to the tensor product of abelian groups. Thus, spectra are \mathbb{S} -modules, and ring spectra, which extend the notion of rings, are the \mathbb{S} -algebras. To any discrete ring R , one can associate the Eilenberg-Mac Lane ring spectrum HR , which is commutative if R is.

It took time to elaborate a well-behaved smash product on spectra. Meanwhile, precisely this smash product gave birth to spectral algebra, since it made possible clear categorical definitions of ring, module and algebra spectra. Before, the notions of spectra and ring spectra already existed, but all algebraic structures had complex, up to homotopy, properties.

In this older context, Alan Robinson established a connection between rings and ring spectra. In [Rob], he defined a notion of A_∞ -modules over the ring spectrum HR , and showed that, up to a suitable notion of homotopy, the category of A_∞ -modules is equivalent to $\mathcal{D}(R)$. Nevertheless, it was difficult to obtain a similar result for algebras because of involved definitions.

This became achievable in the modern setting of algebra of spectra, where the homotopy theory is encoded in Quillen model structures. Given a commutative ring spectrum A , one defines an A -algebra spectrum to be simply a monoid in the category of A -modules. After strengthening the result of Robinson by showing that the category of HR -module spectra is Quillen equivalent to $R\text{-Mod}$, Brooke Shipley extended it in [Shi], and showed that the HR -algebra spectra capture the same “up to homotopy” information as differential graded R -algebras.

The aim of this Master thesis was to acquire a sufficient knowledge of algebra of spectra and of model category theory in order to understand the result of Shipley regarding HR -algebra spectra and differential graded algebras, and to explain the essential arguments used in the proof.

Guideline

Our paper is organized in six chapters, leading the reader step-by-step to the goal. The first four chapters gradually introduce the language, tools and notions, necessary to understand the main statement and its proof. They also contain fundamental theorems that will be applied to show the statement. The functorial zig-zag construction, at the heart of the proof, is studied in detail in Chapter 5. In the sixth chapter we discuss essential arguments used in the proof of the central result.

Chapters are tied up by Guidelines. They keep the reader up to date on our progression towards the objective, and give explanations on how the material from each chapter is related to the whole.

A two-part Appendix contains complementary information, which may help to have a better insight into cohomology theories and into the importance of the symmetric monoidal smash product on spectra.

Let us now get into the subject. The theorem to understand and to explain is the following.

Theorem [Shi, Theorem 1.1] *For any discrete commutative ring R , the model categories of unbounded differential graded R -algebras and HR -algebra spectra are Quillen equivalent. The associated composite derived functors are denoted*

$$\mathbb{H} : DG\text{-Alg}_R \longrightarrow HR\text{-Alg}_{Sp^\Sigma} \quad \text{and} \quad \Theta : HR\text{-Alg}_{Sp^\Sigma} \longrightarrow DG\text{-Alg}_R.$$

For simplicity, we will concentrate on $R = \mathbb{Z}$, as Shipley does in the article. However, in every step of the proof \mathbb{Z} could be replaced by any discrete commutative ring.

Chapter 1

Categories involved

1.1 Monoidal categories

1.1.1 A short reminder on simplicial structures

Recall that Δ is the category with objects the ordered sets $[n] := \{0, 1, \dots, n\}$, $n \geq 0$, and maps their order-preserving functions $\xi: [n] \rightarrow [m]$; i.e., such that $i \leq j$ implies $\xi(i) \leq \xi(j)$. It is easy to check that the maps in Δ are generated by the following special maps

$$d^i: [n-1] \rightarrow [n] \quad 0 \leq i \leq n \quad (n \geq 1) \quad (\text{coface maps})$$

$$s^j: [n+1] \rightarrow [n] \quad 0 \leq j \leq n \quad (n \geq 0) \quad (\text{codegeneracy maps}),$$

such that d^i is the order preserving injection which does not take the value $i \in [n]$, and s^j is the order preserving surjection which takes twice the value $j \in [n]$. One also checks that these maps satisfy the following *cosimplicial identities*

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, & i < j, \\ s^j s^i &= s^i s^{j+1}, & i \leq j, \\ s^j d^i &= \begin{cases} d^i s^{j-1}, & i < j, \\ \text{Id}, & i = j, \quad i = j + 1 \\ d^{i-1} s^j, & i > j + 1 \end{cases} \end{aligned}$$

and that all relations between composites of these special maps are consequences of these identities (see [DH], [GZ, II.2], [GJ, I.1]). Hence, the maps d^i, s^j together with the simplicial identities above can be thought of as a set of generators and relations for the category Δ .

If \mathcal{C} is a category and \mathcal{D} is a small category, recall that $\mathcal{C}^{\mathcal{D}}$ denotes the *functor category*, with objects the functors $X: \mathcal{D} \rightarrow \mathcal{C}$ and morphisms their natural transformations $\tau: X \rightarrow Y$.

Definition 1.1.1. Let \mathcal{C} be a category. A **simplicial object** in \mathcal{C} is a functor $X: \Delta^{\text{op}} \rightarrow \mathcal{C}$. Denote by $s\mathcal{C} := \mathcal{C}^{\Delta^{\text{op}}}$ the **category of simplicial objects in \mathcal{C}** . Limits and colimits in $s\mathcal{C}$ are calculated level-wise.

The cosimplicial identities above lead to another description of simplicial objects. If X is a simplicial object in \mathcal{C} , we usually use the notation

$$X_n := X([n]), \quad d_i := X(d^i), \quad \text{and} \quad s_j := X(s^j).$$

Hence, in order to define a simplicial object X in \mathcal{C} , it is sufficient to give a collection of objects X_n in \mathcal{C} ($n \geq 0$) together with maps

$$\begin{aligned} d_i: X_n &\longrightarrow X_{n-1} & 0 \leq i \leq n & \quad (n \geq 1) & \quad (\text{face maps}) \\ s_j: X_n &\longrightarrow X_{n+1} & 0 \leq j \leq n & \quad (n \geq 0) & \quad (\text{degeneracy maps}) \end{aligned}$$

in \mathcal{C} which satisfy the following identities, called the *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j, \\ s_i s_j &= s_{j+1} s_i, & i \leq j, \\ d_i s_j &= \begin{cases} s_{j-1} d_i, & i < j, \\ \text{Id}, & i = j, \quad i = j + 1 \\ s_j d_{i-1}, & i > j + 1 \end{cases} \end{aligned}$$

The simplicial categories that we will mainly come across in this paper are the category of pointed simplicial sets, $sSet_*$, and the category of simplicial abelian groups, sAb .

Every simplicial set has a natural decomposition, as illustrated in the following lemma, which is a consequence of [GZ, II.3.1].

Lemma 1.1.2. *Let X be a simplicial set. For each $k \geq 0$, denote by $(\mathcal{N}X)_k$ the set of non-degenerate k -simplices of X . There is an isomorphism between X and a simplicial set of the form*

$$(\mathcal{N}X)_0 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} (\mathcal{N}X)_0 \amalg (\mathcal{N}X)_1 \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} (\mathcal{N}X)_0 \amalg (\mathcal{N}X)_1 \amalg (\mathcal{N}X)_1 \amalg (\mathcal{N}X)_2 \dots,$$

which is given objectwise by isomorphisms

$$\coprod_{\substack{[n] \rightarrow [k] \\ \text{in } \Delta}} (\mathcal{N}X)_k \xrightarrow{\cong} X_n.$$

Here, the coproduct is indexed over the set of all surjections in Δ from $[n]$ to $[k]$.

Reminder 1.1.3. Given $X, Y \in sSet_*$, the **smash product** of X and Y , denoted $X \wedge Y \in sSet_*$, is

$$X \wedge Y := X \times Y / X \vee Y,$$

where $X \vee Y := X \times * \cup * \times Y$ is the wedge product of X and Y .

1.1.2 Monoidal categories

Definition 1.1.4. A **monoidal category** $(\mathcal{C}, \square, \mathbb{I})$ is a category \mathcal{C} , together with a bifunctor $-\square- : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called the **monoidal product** and an object $\mathbb{I} \in \mathcal{C}$, called the **unit**. Moreover, for every triple $A, B, C \in \mathcal{C}$, an isomorphism

$$\alpha_{A,B,C} : (A \square B) \square C \rightarrow A \square (B \square C)$$

is required, and for all $A \in \mathcal{C}$, two isomorphisms

$$l_A : \mathbb{I} \square A \rightarrow A \text{ and } r_A : A \square \mathbb{I} \rightarrow A$$

are required, such that they satisfy the following axioms.

(M1) The morphisms $\alpha_{A,B,C}$ are natural in A, B, C .

(M2) The morphisms l_A and r_A are natural in A .

(M3) *Associativity coherence:*

For all $A, B, C, D \in \mathcal{C}$ the diagram

$$\begin{array}{ccc} ((A \square B) \square C) \square D & \xrightarrow{\alpha_{A \square B, C, D}} & (A \square B) \square (C \square D) \\ \alpha_{A, B, C} \square \text{Id} \downarrow & & \downarrow \alpha_{A, B, C \square D} \\ (A \square (B \square C)) \square D & & \\ \alpha_{A, B \square C, D} \downarrow & & \\ A \square ((B \square C) \square D) & \xrightarrow{\text{Id} \square \alpha_{B, C, D}} & A \square (B \square (C \square D)) \end{array}$$

commutes.

(M4) *Unit coherence:*

For all $A, B \in \mathcal{C}$ the diagram

$$\begin{array}{ccc} (A \square \mathbb{I}) \square B & \xrightarrow{\alpha_{A, \mathbb{I}, B}} & A \square (\mathbb{I} \square B) \\ \searrow r_A \square \text{Id} & & \swarrow \text{Id} \square l_B \\ & A \square B & \end{array}$$

commutes.

(M5) It is also required that $l_{\mathbb{I}} = r_{\mathbb{I}} : \mathbb{I} \square \mathbb{I} \rightarrow \mathbb{I}$.

Definition 1.1.5. A monoidal category $(\mathcal{C}, \square, \mathbb{I})$ is **symmetric** if for all $A, B \in \mathcal{C}$ there exists an isomorphism

$$\tau_{A,B} : A \square B \rightarrow B \square A,$$

natural in A and B , and such that for all $A, B, C \in \mathcal{C}$ the following diagrams commute.

1. *Associativity coherence:*

$$\begin{array}{ccc}
(A \square B) \square C & \xrightarrow{\tau_{A,B} \square \text{Id}} & (B \square A) \square C \\
\alpha_{A,B,C} \downarrow & & \downarrow \alpha_{B,A,C} \\
A \square (B \square C) & & B \square (A \square C) \\
\tau_{A,B} \square C \downarrow & & \downarrow \text{Id} \square \tau_{A,C} \\
(B \square C) \square A & \xrightarrow{\alpha_{B,C,A}} & B \square (C \square A);
\end{array}$$

2. *Unit coherence:*

$$\begin{array}{ccc}
A \square \mathbb{I} & \xrightarrow{\tau_{A,\mathbb{I}}} & \mathbb{I} \square A; \\
r_A \searrow & & \swarrow l_A \\
& A &
\end{array}$$

3. *Symmetry axiom:*

$$\begin{array}{ccc}
A \square B & \xrightarrow{\tau_{A,B}} & B \square A. \\
& \searrow & \swarrow \tau_{B,A} \\
& A \square B &
\end{array}$$

The notion of a monoidal category $(\mathcal{C}, \square, \mathbb{I}_{\mathcal{C}})$ allows one to have a meaningful notion of a *monoid object* in \mathcal{C} . Suppose that for X in \mathcal{C} we are given a multiplication map $m : X \square X \rightarrow X$, satisfying the 3-fold associativity and coherent with a unit map $\eta : \mathbb{I}_{\mathcal{C}} \rightarrow X$.

A natural question that arises is the following: under what conditions does this 3-fold associativity imply n -fold associativity for the map m ? If one writes down all necessary conditions in terms of diagrams, he will discover that they result in asking all the diagrams in the definition of a monoidal category to commute, and that these diagrams are in fact sufficient.

Similarly, the notion of a *symmetric* monoidal category \mathcal{C} permits to have a meaningful notion of a *commutative* monoid object in \mathcal{C} .

We delay the exact definitions of (commutative) monoids in $(\mathcal{C}, \square, \mathbb{I}_{\mathcal{C}})$ until the next section, devoted to algebraic structures occurring on a monoidal category. For the time being, let us provide some simple and commonly known examples of symmetric monoidal categories. In each case, we give the definition of the monoidal product and describe the corresponding unit.

Examples 1.1.6. First, let us mention three “basic” examples.

$(\text{Set}, \times, *)$: is the category of sets and their maps, with monoidal product the cartesian product of sets. The unit is $*$, a one-element set in Set .

(Set_*, \wedge, S^0) : is the category of pointed sets and their basepoint preserving maps, with monoidal product the smash product of sets. Here, the unit $S^0 := * \amalg *$ denotes the 0-sphere.

$(Ab, \otimes, \mathbb{Z})$: is the category of abelian groups and their homomorphisms, with monoidal product the tensor product of abelian groups over \mathbb{Z} , and unit the group \mathbb{Z} .

Now follow examples of categories that we will actually work with.

$(sSet, \times, *)$: is the category of simplicial sets and their maps, with monoidal product the product of simplicial sets. Here, the unit $*$ is the terminal object in $sSet$, given by the simplicial set $\Delta[0] := \Delta(-, [0])$, which has a single simplex in each degree.

$(sSet_*, \wedge, S^0)$: is the category of pointed simplicial sets and their basepoint-preserving maps. The monoidal product is the smash product of pointed simplicial sets and the unit is the 0-sphere $S^0 := \Delta[0]_+ := \Delta[0] \amalg \Delta[0]$, obtained by adding a disjoint basepoint to the simplicial set $\Delta[0]$.

$(sAb, \otimes, \mathbb{Z})$: is the category of simplicial abelian groups and their homomorphisms. For $A, B \in sAb$, the monoidal product is given by the level-wise tensor product of simplicial abelian groups, i.e.,

$$(A \otimes B)_n := A_n \otimes B_n.$$

Here, the unit \mathbb{Z} denotes the constant simplicial abelian group with value \mathbb{Z} .

$(DG\mathbb{Z}\text{-Mod}, \otimes, \mathbb{Z}[0])$: is the category of differential graded \mathbb{Z} -modules (which are the same as \mathbb{Z} -graded unbounded chain complexes of abelian groups). The monoidal product for $C, C' \in DG\mathbb{Z}\text{-Mod}$ is defined at level n by

$$(C \otimes C')_n := \bigoplus_{p+q=n} C_p \otimes C'_q, \quad n \in \mathbb{Z}.$$

The unit for the product will be denoted $\mathbb{Z}[0]$; it is the chain complex concentrated at degree 0 with value \mathbb{Z} .

Notation 1.1.7. More generally, $\mathbb{Z}[n]$ will denote the chain complex concentrated at degree n with value \mathbb{Z} .

Remark 1.1.8. In general, one defines the category $DGR\text{-Mod}$ of differential graded R -modules (or, equivalently, \mathbb{Z} -graded chain complexes of R -modules), replacing \mathbb{Z} by an arbitrary ring R in the previous example.

$(Ch_+, \otimes, \mathbb{Z}[0])$: is the category of non-negatively graded chain complexes of abelian groups and their morphisms. Since it is a subcategory of $(DG\mathbb{Z}\text{-Mod}, \otimes, \mathbb{Z}[0])$, its monoidal structure is obtained by restriction of the monoidal structure of $(DG\mathbb{Z}\text{-Mod}, \otimes, \mathbb{Z}[0])$.

$(\mathcal{C}^{\circ}, \circ, \text{Id})$: denotes the category of endofunctors on \mathcal{C} . For $F, G \in \mathcal{C}^{\circ}$, the monoidal product is the usual composition of functors $F \circ G = FG$, defined objectwise by

$$(F \circ G)(B) := FG(B) := F(G(B)).$$

If $\tau : F \rightarrow F'$ and $\sigma : G \rightarrow G'$ are maps in \mathcal{C}° , then the following diagram

$$\begin{array}{ccc} (F \circ G)B & \xrightarrow{(\text{Id} \circ \sigma)B} & (F \circ G')B \\ (\tau \circ \text{Id})B \downarrow & & \downarrow (\tau \circ \text{Id})B \\ (F' \circ G)B & \xrightarrow{(\text{Id} \circ \sigma)B} & (F' \circ G')B \end{array} \quad (1.1)$$

in \mathcal{C} commutes for every object $B \in \mathcal{C}$. The map $\tau \circ \sigma : FG \rightarrow F'G'$ is defined objectwise by the composition of maps $(F \circ G)B \rightarrow (F' \circ G')B$ in the diagram above. The unit is the identity functor $\text{Id} : \mathcal{C} \rightarrow \mathcal{C}$. See [McL, II.5] for further details. It is easy to verify that composition \circ is not symmetric.

Definition 1.1.9. A symmetric monoidal category $(\mathcal{C}, \square, \mathbb{I})$ is **closed** if for all $B \in \mathcal{C}$, the functor $-\square B : \mathcal{C} \rightarrow \mathcal{C}$ has a right adjoint $\text{Hom}(B, -) : \mathcal{C} \rightarrow \mathcal{C}$, the *internal* hom functor, natural in B ; i.e., such that there are isomorphisms

$$\mathcal{C}(A \square B, C) \cong \mathcal{C}(A, \text{Hom}(B, C))$$

natural in A, B, C .

Remark 1.1.10. The categories $(\text{Set}, \times, \{*\})$, $(\text{Set}_*, \wedge, S^0)$, $(\text{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ are closed ([McL, VII.7]).

1.1.3 Algebraic structures defined from a monoidal category

Monoidal categories can be used to define the concepts of a *monoid object* and of an associated *action* on the objects of the category.

Recall that a ring R is precisely a monoid object in the category $(\text{Ab}, \otimes, \mathbb{Z})$; i.e., it is an abelian group R together with two maps $m : R \otimes R \rightarrow R$ and $\eta : \mathbb{Z} \rightarrow R$ which satisfy the usual associativity and two-sided unit diagrams. Similarly, a left R -module is an abelian group M together with a map $\nu : R \otimes M \rightarrow M$ satisfying the usual associativity and unit conditions.

The structure of a (symmetric) monoidal category permits one to make sense of (commutative) rings and their modules in other contexts besides the category $(Ab, \otimes, \mathbb{Z})$ of abelian groups.

Definition 1.1.11. Let $(\mathcal{C}, \square, \mathbb{I})$ be a symmetric monoidal category. A **monoid** in \mathcal{C} is an object $R \in \mathcal{C}$, equipped with two morphisms $\mu : R \square R \rightarrow R$, called the “multiplication”, and $\eta : \mathbb{I} \rightarrow R$, called the “unit”, such that the following diagrams commute.

1. *Associativity coherence:*

$$\begin{array}{ccc} R \square R \square R & \xrightarrow{\text{Id} \square \mu} & R \square R \\ \mu \square \text{Id} \downarrow & & \downarrow \mu \\ R \square R & \xrightarrow{\mu} & R \end{array}$$

2. *Right and left unit coherence:*

$$\begin{array}{ccccc} R \square \mathbb{I} & \xrightarrow{\text{Id} \square \eta} & R \square R & \xleftarrow{\eta \square \text{Id}} & \mathbb{I} \square R \\ & \searrow r_R & \downarrow \mu & \swarrow l_R & \\ & & R & & \end{array}$$

A monoid R in $(\mathcal{C}, \square, \mathbb{I})$ is **commutative** if the diagram

$$\begin{array}{ccc} R \square R & \xrightarrow{\tau_{R,R}} & R \square R \\ \mu \downarrow & & \downarrow \mu \\ R & \xlongequal{\quad} & R \end{array}$$

commutes.

Definition 1.1.12. Let $(\mathcal{C}, \square, \mathbb{I})$ be a symmetric monoidal category and let (R, μ, η) a monoid in \mathcal{C} . A **left R -module** in \mathcal{C} is an object $M \in \mathcal{C}$ together with a morphism $\nu : R \square M \rightarrow M$, called “left action”, such that the following diagrams are commutative.

1. *Action coherence:*

$$\begin{array}{ccc} R \square R \square M & \xrightarrow{\text{Id} \square \nu} & R \square M \\ \mu \square \text{Id} \downarrow & & \downarrow \nu \\ R \square M & \xrightarrow{\nu} & M \end{array}$$

2. *Left unit coherence:*

$$\begin{array}{ccc} \mathbb{I} \square M & \xrightarrow{\eta \square \text{Id}} & R \square M \\ \downarrow l_M & \swarrow \nu & \\ M & & \end{array}$$

Definition 1.1.13. Let (M, ν) and (M', ν') be two left R -modules in $(\mathcal{C}, \square, \mathbb{I})$. A **morphism** of left modules is a morphism $f \in \mathcal{C}(M, M')$ such that the square

$$\begin{array}{ccc} R \square M & \xrightarrow{\text{Id} \square f} & R \square M' \\ \nu \downarrow & & \downarrow \nu' \\ M & \xrightarrow{f} & M' \end{array}$$

commutes.

One can easily show the following.

Lemma 1.1.14. *Let $(\mathcal{C}, \square, \mathbb{I})$ be a symmetric monoidal category and R a commutative monoid in \mathcal{C} . Then left R -modules in \mathcal{C} and morphisms of left R -modules form a category, denoted $R\text{-Mod}_{\mathcal{C}}$.*

Note that *right* modules and their morphisms in $(\mathcal{C}, \square, \mathbb{I})$ are defined similarly, using right actions, and that an analogue of the previous lemma holds.

Lemma 1.1.15. *Every left R -module M has a compatible right R -module structure ν given by the composition*

$$M \square R \xrightarrow{\tau} R \square M \xrightarrow{\nu} M.$$

Question: Suppose that $(\mathcal{C}, \square, \mathbb{I})$ is symmetric monoidal and R is a commutative monoid in \mathcal{C} . Can the category of R -modules on \mathcal{C} be given a symmetric monoidal structure?

The answer is yes, assuming some extra conditions.

Proposition 1.1.16. [HSS, Lemma 2.2.2] *Let $(\mathcal{C}, \square, \mathbb{I})$ be a symmetric monoidal category that is cocomplete and let R be a commutative monoid in \mathcal{C} , such that the functor $R \square - : \mathcal{C} \rightarrow \mathcal{C}$ preserves coequalizers. Then there is a symmetric monoidal product \square_R on the category $R\text{-Mod}_{\mathcal{C}}$, with R as the unit. Explicitly, for M a right R -module and N a left R -module in \mathcal{C} we define*

$$M \square_R N := \text{colim} \left(M \square R \square N \begin{array}{c} \xrightarrow{\nu_M \square \text{Id}} \\ \xrightarrow{\text{Id} \square \nu_N} \end{array} \rightrightarrows M \square N \right),$$

and the category $(R\text{-Mod}_{\mathcal{C}}, \square_R, R)$ is symmetric monoidal.

Proof. A detailed proof can be found in [IS, Proposition 1.5.8]. We only emphasize a few points. First, requiring R to be *commutative* guarantees that the categories $R\text{-Mod}_{\mathcal{C}}$ and $\text{Mod}_{\mathcal{C}}\text{-}R$ are isomorphic, so we can simply speak of R -modules. We will adopt the notation $R\text{-Mod}_{\mathcal{C}}$ for the category

of R -modules. Secondly, since R is a *commutative* monoid, $M \square_R N$ inherits the structure of an R -module, because the diagram

$$\begin{array}{ccccc}
R \square M \square R \square N & \xrightarrow{\text{Id} \square \nu_M \square \text{Id}} & R \square M \square N & & \\
\downarrow \nu_M \square \text{Id} \square \text{Id} & \searrow \text{Id} \square p & \swarrow \text{Id} \square q & & \downarrow \nu_M \square \text{Id} \\
& & R \square (M \square_R N) & & \\
& & \uparrow \text{Id} \square \nu_N & & \\
M \square R \square N & \xrightarrow{\nu_M \square \text{Id}} & M \square N & & \\
& \searrow p & \swarrow q & & \\
& & M \square_R N & &
\end{array}$$

commutes, inducing a morphism $\nu : R \square (M \square_R N) \longrightarrow M \square_R N$. This does not work if R is not commutative. \square

The next proposition discusses the existence of the internal Hom functor in the category $\mathbf{R}\text{-Mod}_{\mathcal{C}}$.

Proposition 1.1.17. [HSS, Lemma 2.2.8] *Let $(\mathcal{C}, \square, \mathbb{I})$ be a closed symmetric monoidal category, which is bicomplete, and let R be a commutative monoid in \mathcal{C} . Then there is a functor*

$$\text{Hom}_R(-, -) : \mathbf{R}\text{-Mod}_{\mathcal{C}}^{\text{op}} \times \mathbf{R}\text{-Mod}_{\mathcal{C}} \longrightarrow \mathbf{R}\text{-Mod}_{\mathcal{C}}$$

such that $\text{Hom}_R(M, N)$ is natural in M and N , and such that $- \square_R M$ is left adjoint of the functor $\text{Hom}_R(M, -)$, for all M, N in $\mathbf{R}\text{-Mod}_{\mathcal{C}}$.

For M, N in $\mathbf{R}\text{-Mod}_{\mathcal{C}}$, the R -module $\text{Hom}_R(M, N)$ is defined by

$$\text{Hom}_R(M, N) := \lim \left(\text{Hom}(M, N) \begin{array}{c} \xrightarrow{m^*} \\ \xrightarrow{m_*} \end{array} \text{Hom}(M \square R, N) \right),$$

where m^* is the pullback along the multiplication $m : M \square R \longrightarrow M$, and m_* is the composition

$$\text{Hom}(M, N) \xrightarrow{- \square_R} \text{Hom}(M \square R, N \square R) \xrightarrow{m_*} \text{Hom}(M \square R, N).$$

Corollary 1.1.18. *Let $(\mathcal{C}, \square, \mathbb{I})$ be a closed symmetric monoidal category, which is bicomplete, and let R be a commutative monoid in \mathcal{C} . Then the category $\mathbf{R}\text{-Mod}_{\mathcal{C}}$ is a closed symmetric monoidal category.*

Finally, one more algebraic structure can be concocted from monoids and R -modules in a monoidal category.

Definition 1.1.19. Let R be a commutative monoid in $(\mathcal{C}, \square, \mathbb{I})$. An R -**algebra** is a monoid in $(\mathbf{R}\text{-Mod}_{\mathcal{C}}, \square_R, R)$.

1.1.4 Functors between monoidal categories

Suppose we have a monoid object X in a monoidal category $(\mathcal{C}, \otimes, \mathbb{I}_{\mathcal{C}})$. Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between \mathcal{C} and another monoidal category $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$, it is natural to look for conditions on F such that $F(X)$ always has a naturally occurring monoid structure in $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$. These conditions shape the correct definition of a functor between monoidal categories.

Definition 1.1.20. Given two monoidal categories $(\mathcal{C}, \otimes, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$, a **strong monoidal functor** from \mathcal{C} to \mathcal{D} is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, together with an isomorphism $\varphi_{X,Y} : F(X) \wedge F(Y) \xrightarrow{\cong} F(X \otimes Y)$, natural in X, Y , and an isomorphism $\nu : \mathbb{I}_{\mathcal{D}} \xrightarrow{\cong} F(\mathbb{I}_{\mathcal{C}})$ such that the following coherence diagrams are commutative for all $X, Y, Z \in \mathcal{C}$.

1. *Associativity coherence:*

$$\begin{array}{ccc}
 (FX \wedge FY) \wedge FZ & \xrightarrow{\alpha_{F(X), F(Y), F(Z)}} & F(X) \wedge (F(Y) \wedge F(Z)) \\
 \varphi_{X,Y} \wedge \text{Id} \downarrow & & \text{Id} \wedge \varphi_{Y,Z} \downarrow \\
 F(X \otimes Y) \wedge F(Z) & & F(X) \wedge F(Y \otimes Z) \\
 \varphi_{X \otimes Y, Z} \downarrow & & \downarrow \varphi_{X, Y \otimes Z} \\
 F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha_{X,Y,Z})} & F(X \otimes (Y \otimes Z)).
 \end{array}$$

2. *Right and left unit coherence:*

$$\begin{array}{ccc}
 F(\mathbb{I}_{\mathcal{C}}) \wedge F(X) & \xrightarrow{\varphi_{\mathbb{I}_{\mathcal{C}}, X}} & F(\mathbb{I}_{\mathcal{C}} \otimes X) & F(X) \wedge F(\mathbb{I}_{\mathcal{C}}) & \xrightarrow{\varphi_{X, \mathbb{I}_{\mathcal{C}}}} & F(X \otimes \mathbb{I}_{\mathcal{C}}) \\
 \nu \wedge \text{Id} \uparrow & & \downarrow F(l_X) & \text{Id} \wedge \nu \uparrow & & \downarrow F(r_X) \\
 \mathbb{I}_{\mathcal{D}} \wedge F(X) & \xrightarrow{l_{F(X)}} & F(X), & F(X) \wedge \mathbb{I}_{\mathcal{D}} & \xrightarrow{r_{F(X)}} & F(X).
 \end{array}$$

Remark 1.1.21. In Chapter 4, we introduce the notion of a *lax* monoidal functor. The difference with the definition above will be that the hypotheses on the morphisms φ and ν will be weakened. Examples of strong and lax monoidal functors will come in Chapters 5 and 6.

Definition 1.1.22. Let $(\mathcal{C}, \otimes, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$ be two symmetric monoidal categories. A **symmetric strong monoidal functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ between \mathcal{C} and \mathcal{D} is a strong monoidal functor, such that the diagram

$$\begin{array}{ccc}
 F(X) \wedge F(Y) & \xrightarrow{\varphi_{X,Y}} & F(X \otimes Y) \\
 \tau_{\mathcal{D}} \downarrow & & \downarrow F(\tau_{\mathcal{C}}) \\
 F(Y) \wedge F(X) & \xrightarrow{\varphi_{Y,X}} & F(Y \otimes X)
 \end{array}$$

commutes for all $X, Y \in \mathcal{C}$.

So far, we have learned what a monoidal category is and have seen a number of relevant examples. Other examples of monoidal categories that we will work with are more involved. To understand them, we first need to become familiar with the notion of spectra.

1.2 Categories of spectra

Our main references to learn about symmetric sequences and symmetric spectra were [HSS] and [Sch]. The first one dealt exclusively with symmetric spectra over pointed simplicial sets, the second one considered symmetric spectra over topological spaces as well. It appeared quickly that one can define symmetric sequences and spectra over other categories, such as the category of simplicial abelian groups or the category of positive chain complexes.

Some material on spectra over an arbitrary symmetric monoidal (model) category \mathcal{C} , was offered by [Hov01]. Definitions and results in this enlarged context were based on [HSS].

This section provides an overview on symmetric sequences and symmetric spectra over an arbitrary symmetric monoidal category \mathcal{C} . It combines elements of [HSS] and [Hov01], and focuses on examples where $\mathcal{C} = sSet_*$, sAb or Ch_+ .

1.2.1 Symmetric sequences

Reminder 1.2.1. Denote by $\Sigma = \cup_{n \geq 0} \Sigma_n$ the category with objects finite sets $\bar{n} = \{1, \dots, n\}$ for $n \geq 0$ ($\bar{0} := \emptyset$), and the morphisms automorphisms of sets. Notice that Σ_n can be viewed as the group of permutations of n elements.

Definition 1.2.2. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{I}_{\mathcal{C}})$ be a monoidal category. A **symmetric sequence** in \mathcal{C} is a functor $X : \Sigma \rightarrow \mathcal{C}$. The functor category \mathcal{C}^{Σ} is called the **category of symmetric sequences in \mathcal{C}** .

Explicitly, an object of \mathcal{C}^{Σ} is a sequence $X = \{X_0, X_1, \dots\}$ with $X_n \in \mathcal{C}$, together with an action of Σ_n on X_n , for all $n \geq 0$. Given X, Y in \mathcal{C}^{Σ} , a morphism $f : X \rightarrow Y$ is a collection of morphisms $f_n : X_n \rightarrow Y_n$ in \mathcal{C} , compatible with the Σ_n -action, for all $n \geq 0$.

As a functor category, \mathcal{C}^{Σ} is bicomplete if \mathcal{C} is so; the limits and the colimits are calculated objectwise. If \mathcal{C} is closed, then so is \mathcal{C}^{Σ} .

Definition 1.2.3. Given $X, Y \in \mathcal{C}^{\Sigma}$, their **tensor product** $X \otimes Y$ is a symmetric sequence given at level n by

$$(X \otimes Y)_n := \coprod_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} (X_p \otimes_{\mathcal{C}} Y_q),$$

for all $n \geq 0$. Here \coprod denotes the coproduct in \mathcal{C} .

Let us explain the notations. On the one hand, the group $\Sigma_p \times \Sigma_q$ acts on Σ_n on the right

$$\begin{aligned} (\Sigma_p \times \Sigma_q) \times \Sigma_n &\longrightarrow \Sigma_n \\ ((\rho, \xi), \sigma) &\mapsto \sigma \circ (\rho^{-1}, \xi^{-1}). \end{aligned}$$

On the other hand, there is an action of Σ_p and Σ_q respectively on X_p and Y_q , by definition of a symmetric sequence.

The notation $\Sigma_n \times_{\Sigma_p \times \Sigma_q} X_p \otimes_{\mathcal{C}} Y_q$ denotes the quotient of $\Sigma_n \times X_p \otimes_{\mathcal{C}} Y_q$ by the diagonal action of $\Sigma_p \times \Sigma_q$ given by

$$(\sigma, X_p \otimes_{\mathcal{C}} Y_q) \mapsto (\sigma \circ (\rho^{-1}, \xi^{-1}), \sigma X_p \otimes_{\mathcal{C}} \sigma Y_q).$$

Remark 1.2.4. There is another equivalent way of describing the tensor product $X \otimes Y$. Denote by \mathcal{F} the category of all finite sets and their automorphisms. It can be shown that the inclusion functor $J : \Sigma \longrightarrow \mathcal{F}$ is fully faithful and essentially surjective, which implies by [McL, IV.4.1] that J is an *equivalence* of categories. Denote the inverse functor by

$$Res : \mathcal{F} \longrightarrow \Sigma$$

such that for all $n \geq 0$ and finite sets C

$$\begin{aligned} C \text{ s.t. } |C| = n &\mapsto \bar{n} \\ \alpha \in \text{Aut}(C) &\mapsto \sigma \in \text{Aut}(\bar{n}). \end{aligned}$$

In view of the equivalence, every symmetric sequence

$$X : \Sigma \longrightarrow \mathcal{C}$$

sending

$$\begin{aligned} \bar{n} &\mapsto X(\bar{n}) := X_n \\ \sigma \in \text{Aut}(\bar{n}) &\mapsto X\sigma : X_n \longrightarrow X_n, \end{aligned}$$

where $X\sigma$ is the action of Σ_n on X_n , has an extension $X : \mathcal{F} \longrightarrow \mathcal{C}$, unique up to isomorphism.

Let be two such extensions $X, Y : \mathcal{F} \longrightarrow \mathcal{C}$.

Definition 1.2.5. The tensor product of X and Y is the functor $X \otimes Y : \mathcal{F} \longrightarrow \mathcal{C}$, defined on a finite set C by

$$(X \otimes Y)(C) := \coprod_{A \cup B = C, A \cap B = \emptyset} X(A) \otimes Y(B).$$

To understand the relation between the two definitions of the tensor product, suppose that the finite set C in \mathcal{F} has n elements, and that is it partitioned into sets A and B having respectively p and q elements. With previous notations it is clear that the restriction functor sends $(X \otimes Y)(C)$ to $(X \otimes Y)_n$ and $X(A) \otimes Y(B)$ to $X_p \otimes_{\mathcal{C}} Y_q$. Now, given a set of n elements, there are $\binom{n}{p}$ possibilities to choose the p elements which constitute the subset A . The quotient of Σ by $\Sigma_p \times \Sigma_q$ in Definition 1.2.3 means that we want to identify all these different choices, and evaluate the functor X only once at a set A containing p elements; similarly for B .

Definition 1.2.6. The tensor product $f \otimes g : X \otimes Y \longrightarrow X' \otimes Y'$ of two morphisms f, g in \mathcal{C}^Σ is given by

$$(f \otimes g)(\alpha, X_p \otimes Y_q) := (\alpha, f_p(X_p) \otimes_{\mathcal{C}} g_q(Y_q))$$

for $\alpha : \bar{n} \longrightarrow \bar{n}$ in Σ_n , $n = p + q$. The restriction of α on the set \bar{p} of the first p elements induces the action of Σ_p on X_p and $f_p(X_p)$, the restriction on the last \bar{q} elements induces the action of Σ_q on Y_q and $g_q(Y_q)$.

Next proposition generalizes Proposition 2.1.4 in [HSS].

Proposition 1.2.7. [Hov01, Section 6] *Let $X, Y, Z \in \mathcal{C}^\Sigma$ be symmetric sequences. There is a natural isomorphism of sets*

$$\mathcal{C}^\Sigma(X \otimes Y, Z) \cong \prod_{p,q} \mathcal{C}^{\Sigma_p \times \Sigma_q}(X_p \otimes_{\mathcal{C}} Y_q, Z_{p+q}).$$

The twist isomorphism $\tau : X \otimes Y \longrightarrow Y \otimes X$ for $X, Y \in \mathcal{C}^\Sigma$ is the natural map given by

$$\tau(\alpha, X_p, Y_q) = (\alpha \circ \rho_{p,q}, Y_q, X_p)$$

for $\alpha \in \Sigma_n$. Here $\rho_{p,q} \in \Sigma_{p+q}$ denotes the p, q -**shuffle** permutation

$$\rho_{p,q} = \begin{pmatrix} 1 & 2 & \dots & p & p+1 & \dots & n \\ p+1 & p+2 & \dots & p+q & 1 & \dots & q \end{pmatrix}.$$

Intuitively, one describes the p, q -shuffle as follows: split an ordered set with n elements in two ordered subsets, the subset consisting of the first p elements, and the subset consisting of the last $q = n - p$ elements. The p, q shuffle changes the relative position of the two sets, but does not affect the internal order of their elements. Comparing the expression of τ to Definition 1.2.6 helps to realize why we need to precompose α with a p, q -shuffle here.

The following lemma is an extension of Lemma 2.1.6 in [HSS].

Lemma 1.2.8. [Hov01, Section 6] *The tensor product \otimes endows the category \mathcal{C}^Σ of symmetric sequences with a symmetric monoidal structure $(\mathcal{C}^\Sigma, \otimes, \mathbb{I})$.*

Examples 1.2.9. The following three examples of categories of symmetric sequences will be relevant to us.

- The category $(sSet_*^\Sigma, \otimes, \mathbb{I})$ of symmetric sequences of pointed simplicial sets. For $X, Y \in sSet_*^\Sigma$ the monoidal product is given by

$$(X \otimes Y)_m := \bigvee_{r+s=m} (\Sigma_m)_+ \wedge_{\Sigma_r \times \Sigma_s} (X_r \wedge Y_s).$$

The unit is the symmetric sequence $\mathbb{I} := (S^0, *, *, \dots)$.

- The category $(sAb^\Sigma, \otimes, \mathbb{I})$ of symmetric sequences of simplicial abelian groups. For $A, B \in sAb^\Sigma$ we have

$$(A \otimes B)_m := \bigoplus_{r+s=m} \Sigma_m \times_{\Sigma_r \times \Sigma_s} (A_r \otimes B_s),$$

and the unit is the symmetric sequence $\mathbb{I} := (\mathbb{Z}, 0, 0, \dots)$, where 0 stands for the constant trivial simplicial abelian group.

- The category $(Ch_+^\Sigma, \otimes, \mathbb{I})$ of symmetric sequences of non-negatively graded chain complexes of abelian groups. For $C, C' \in Ch_+^\Sigma$ the monoidal product is defined by

$$(C \otimes C')_m := \bigoplus_{r+s=m} \Sigma_m \times_{\Sigma_r \times \Sigma_s} (C_r \otimes C'_s),$$

and the unit is the symmetric sequence $\mathbb{I} := (\mathbb{Z}[0], 0, 0, \dots)$, where 0 denotes the trivial chain complex.

Definition 1.2.10. Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a symmetric monoidal category. For an object $K \in \mathcal{C}$, define $\text{Sym}(K) \in \mathcal{C}^\Sigma$, the **free commutative monoid on K** , to be the symmetric sequence

$$\text{Sym}(K) := (\mathbb{I}, K, K \otimes K, \dots, K^{\otimes n}, \dots).$$

- The *group action* of Σ_n on $\text{Sym}(K)_n = K^{\otimes n}$, permutes the tensored components.
- The *multiplication* on $\text{Sym}(K)$, $\mu : \text{Sym}(K) \otimes \text{Sym}(K) \longrightarrow \text{Sym}(K)$, is given by concatenation, via $\mu_{p,q} : K^{\otimes p} \otimes K^{\otimes q} \longrightarrow K^{\otimes p+q}$, $p + q = n \geq 0$.
- The *unit* $\eta : \mathbb{I} \longrightarrow \text{Sym}(K)$ is identity at level 0, and for $k > 0$ $\eta_k : \mathbb{I}_k \longrightarrow \text{Sym}(K)_k$ is the composition

$$\Phi \xrightarrow{\cong} \Phi^{\otimes k} \xrightarrow{\otimes_k \iota} K^{\otimes k},$$

where $\iota : \Phi \longrightarrow K$ is the unique map from the initial object Φ to K .

- The monoid $\text{Sym}(K)$ is *commutative*. Indeed, the twist map on $\text{Sym}(K)$ arises from maps

$$\tau_{p,q} : K^{\otimes p} \otimes K^{\otimes q} \longrightarrow K^{\otimes q} \otimes K^{\otimes p},$$

for $p+q = n \geq 0$, together with a permutation $\alpha \in \Sigma_{p+q}$. For $\text{Sym}(K)$ to be commutative, the diagram

$$\begin{array}{ccc} (\text{Sym}(K) \otimes \text{Sym}(K))_n & \xrightarrow{\tau_n} & (\text{Sym}(K) \otimes \text{Sym}(K))_n \\ m_n \downarrow & & \downarrow m_n \\ \text{Sym}(K)_n & \xlongequal{\quad\quad\quad} & \text{Sym}(K)_n \end{array}$$

must commute. It is equivalent to

$$\begin{array}{ccc} K^{\otimes p} \otimes K^{\otimes q} & \xrightarrow{\tau_{p,q}} & K^{\otimes q} \otimes K^{\otimes p} \\ m_{p,q} \downarrow & & \downarrow m_{q,p} \\ K^{\otimes p+q} & \xlongequal{\quad\quad\quad} & K^{\otimes q+p}, \end{array}$$

which commutes, because the monoidal product on \mathcal{C} is assumed symmetric and associative and at each level $n = p + q$ there is an action, given by α , of Σ_n on $K^{\otimes n}$.

Part A of the Appendix gives a brief overview of *non-symmetric* sequences and spectra and explains one of the problems that occurs if one does not take into account the symmetric group actions.

1.2.2 Symmetric spectra on a category \mathcal{C}

Symmetric spectra

Definition 1.2.11. Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a monoidal category and K an object in \mathcal{C} . The category of **symmetric spectra over \mathcal{C} with respect to K** is the category of modules over the commutative monoid $\text{Sym}(K)$ in $(\mathcal{C}^\Sigma, \otimes, \mathbb{I})$.

Explicitly, a **symmetric spectrum** X consists of

1. a sequence of objects X_0, X_1, \dots , such that X_n is in \mathcal{C}^{Σ_n} for $n \geq 0$,
2. a sequence of Σ_n -equivariant maps $\sigma_n : K \otimes X_n \longrightarrow X_{n+1}$ for $n \geq 0$, called the **structure maps**, such that the composite

$$K^{\otimes p} \otimes X_n \xrightarrow{\text{Id} \otimes \sigma_n} K^{\otimes p-1} \otimes X_{n+1} \xrightarrow{\text{Id} \otimes \sigma_{n+1}} \dots \xrightarrow{\text{Id} \otimes \sigma_{n+p-2}} K \otimes X_{n+p-1} \xrightarrow{\sigma_{n+p-1}} X_{n+p},$$

$\xrightarrow{\sigma^p}$

is $\Sigma_p \times \Sigma_n$ -equivariant for $p, n \geq 0$.

Symmetric spectra over \mathcal{C} form a category, which we denote $Sp^\Sigma(\mathcal{C})$.

Remark 1.2.12. In [Hov01], Hovey gives a more extensive definition of symmetric spectra. Given $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$, a bicomplete closed symmetric monoidal category and an object K in \mathcal{D} , consider a category $(\mathcal{C}, \otimes, \mathbb{I}_{\mathcal{C}})$, which is bicomplete, *enriched, tensored* and *cotensored over* \mathcal{D} (see for example [Kel05] for definitions). The category of symmetric spectra $Sp^{\Sigma}(\mathcal{C})$ is then defined to be the category of modules in \mathcal{C}^{Σ} over the commutative monoid $\text{Sym}(K)$ in \mathcal{D}^{Σ} .

However, for the purposes of the target article of Shipley, and hence for this report, reducing to the case $\mathcal{D} := \text{Set}$ was apparently sufficient (see Definition 2.8 in [Shi]). This reduction makes \mathcal{C} into an “ordinary category”, and simplifies the definition of symmetric spectra for us, so that Definition 1.2.11 is adequate here.

Definition 1.2.13. Let X, Y in $Sp^{\Sigma}(\mathcal{C})$. A **map of symmetric spectra** $f : X \rightarrow Y$ is a collection of maps $f_n : X_n \rightarrow Y_n$ for $n \geq 0$, such that f_n is Σ_n -equivariant, and compatible with the structure maps, i.e., the diagram

$$\begin{array}{ccc} K \otimes X_n & \xrightarrow{\sigma} & X_{n+1} \\ \text{Id} \otimes f_n \downarrow & & \downarrow f_{n+1} \\ K \otimes Y_n & \xrightarrow{\sigma} & Y_{n+1}. \end{array}$$

commutes for all $n \geq 0$.

Recall from the beginning of Chapter 1 the category Δ . One defines a functor $\Delta[-] : \Delta \rightarrow s\text{Set}$ on objects by

$$[n] \mapsto \Delta[n] := \Delta(-, [n]) : \Delta^{\text{op}} \rightarrow \text{Set}.$$

Definition 1.2.14. Assume that \mathcal{C} is tensored over $s\text{Set}$. For X, Y in $Sp^{\Sigma}(\mathcal{C})$, the **simplicial set of morphisms from X to Y** is $\text{Map}_{Sp^{\Sigma}(\mathcal{C})}(X, Y) := Sp^{\Sigma}(\mathcal{C})(X \otimes \Delta[-], Y)$, given at level n by

$$\text{Map}_{Sp^{\Sigma}(\mathcal{C})}(X, Y)_n := Sp^{\Sigma}(\mathcal{C})(X \otimes \Delta[n], Y).$$

Here, the tensor product between X in $Sp^{\Sigma}(\mathcal{C})$ and $\Delta[n]$ in \mathcal{C}^{Σ} is defined by prolongation of the monoidal product in \mathcal{C}^{Σ} . For $k \geq 0$ one sets

$$(X \otimes \Delta[n])_k := X_k \otimes \Delta[n].$$

Applying Corollary 1.1.18 to the category $(\mathcal{C}^{\Sigma}, \otimes, \mathbb{I})$ and to the commutative monoid $\text{Sym}(K)$ for any K in \mathcal{C} , yields the following theorem.

Theorem 1.2.15. *Let \mathcal{C} be a closed symmetric monoidal category, and $(\mathcal{C}^{\Sigma}, \otimes, \mathbb{I})$ the associated category of symmetric sequences. Then the category of symmetric spectra over \mathcal{C} , $(Sp^{\Sigma}(\mathcal{C}), \otimes_{\text{Sym}(K)}, \text{Sym}(K))$, is a closed symmetric monoidal category.*

Notation 1.2.16. The symmetric monoidal product $\otimes_{\text{Sym}(K)}$ in $Sp^{\Sigma}(\mathcal{C})$ is commonly known as the **smash product of spectra**, usually denoted by \wedge . We adopt this notation for all categories of spectra here. It follows from Theorem 1.2.15 that for $X, Y \in Sp^{\Sigma}(\mathcal{C})$, $X \wedge Y$ is defined by

$$X \wedge Y := \text{colim} \left(X \otimes \text{Sym}(K) \otimes Y \begin{array}{c} \xrightarrow{\nu \otimes \text{Id}} \\ \xrightarrow{\text{Id} \otimes \nu} \end{array} X \otimes Y \right).$$

Definition 1.2.17. A **symmetric ring spectrum** R is a monoid in the category $Sp^{\Sigma}(\mathcal{C})$ of symmetric spectra. In other words, R is a symmetric spectrum equipped with two morphisms of symmetric spectra $m : R \wedge R \rightarrow R$ and $\eta : \mathbb{S} \rightarrow R$, which are suitably associative and unital.

Examples of symmetric spectra on various categories

A. Simplicial set-valued spectra

To obtain the category of **symmetric spectra over pointed simplicial sets**, $(Sp^{\Sigma}(sSet_*), \wedge, \mathbb{S})$, take $K = S^1$ in Corollary 1.2.15. The simplicial model for the circle is $S^1 := \Delta[1]/\partial\Delta[1]$.

In order to understand the simplicial decomposition of S^1 , use Lemma 1.1.2. The first three levels of S^1 are then given by

$$S_0^1 = \{*\}, S_1^1 = \{*, (01)\}, S_2^1 = \{*, (01), (01)\}, \dots \quad (\star)$$

The commutative monoid

$$\mathbb{S} := \text{Sym}(S^1) = (S^0, S^1, S^2, \dots)$$

is called the **symmetric sphere spectrum**, where $S^n := (S^1)^{\wedge n}$. The structure maps of this spectrum are natural isomorphisms $S^1 \wedge S^n \xrightarrow{\cong} S^{n+1}$, and the Σ_n -action on S^n permutes the smashed components:

$$\begin{aligned} \Sigma_n \times (S^1)^{\wedge n} &\longrightarrow (S^1)^{\wedge n} \\ (\sigma, s_1 \wedge s_2 \wedge \dots \wedge s_n) &\mapsto s_{\sigma(1)} \wedge s_{\sigma(2)} \wedge \dots \wedge s_{\sigma(n)}. \end{aligned}$$

Notation 1.2.18. Later on, when working with the category $Sp^{\Sigma}(sSet_*)$, we will omit $sSet_*$ and simply write Sp^{Σ} . In cases where the underlying category is not $sSet_*$, we will always specify it explicitly.

B. Spectra on simplicial abelian groups

Let $\tilde{\mathbb{Z}}$ denote the reduced free abelian group functor

$$\tilde{\mathbb{Z}} : sSet_* \longrightarrow sAb : K \mapsto \tilde{\mathbb{Z}}(K),$$

which associates to every non-basepoint simplex $x \in K_n$ the free abelian group generated on x , for all $n \geq 0$. Take $K = \tilde{\mathbb{Z}}S^1$ in Corollary 1.2.15 to obtain the category of **symmetric spectra over simplicial abelian groups** $(Sp^{\mathcal{S}}(sAb), \wedge, \tilde{\mathbb{Z}}\mathbb{S})$.

It follows from (★) that the first terms of $\tilde{\mathbb{Z}}(S^1)$ are

$$\tilde{\mathbb{Z}}(S_0^1) = 0, \quad \tilde{\mathbb{Z}}(S_1^1) = \mathbb{Z} \cdot (01), \quad \tilde{\mathbb{Z}}(S_2^1) = \mathbb{Z} \cdot (01) \oplus \mathbb{Z} \cdot (01), \dots,$$

hence the commutative monoid $\tilde{\mathbb{Z}}\mathbb{S} := \text{Sym}(\tilde{\mathbb{Z}}S^1) = (\mathbb{Z}, \tilde{\mathbb{Z}}S^1, \tilde{\mathbb{Z}}S^2, \dots)$ looks like

$$\mathbb{Z}\mathbb{S} := \left(\begin{array}{cccccc} \mathbb{Z} & 0 & 0 & \dots & 0 & \dots \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \dots & \mathbb{Z} & \dots \\ \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}) & \dots & (\mathbb{Z} \oplus \mathbb{Z})^{\oplus n} & \dots \\ \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) & \dots & (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})^{\oplus n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ \mathbb{Z} & \mathbb{Z}^{\oplus k} & \mathbb{Z}^{\oplus k} \oplus \mathbb{Z}^{\oplus k} & \dots & (\mathbb{Z}^{\oplus k})^{\oplus n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{array} \right) \quad (\#).$$

level
 k

The simplicial maps are not represented for typographical reasons; however, each column in this matrix is a *simplicial* abelian group.

The Σ_n -action on $\tilde{\mathbb{Z}}S^n$ is trivial in levels $k = 0, 1$ and permutes the n -fold direct sums of k copies of \mathbb{Z} for $k \geq 2$, without changing anything within each k -block of \mathbb{Z} 's.

C. Spectra on non-negatively graded chain complexes

The corresponding category of spectra is $(Sp^{\mathcal{S}}(Ch_+), \wedge, \text{Sym}(\mathbb{Z}[1]))$. The unit $\text{Sym}(\mathbb{Z}[1]) = (\mathbb{Z}[0], \mathbb{Z}[1], \mathbb{Z}[2], \dots)$ is

$$\text{Sym}(\mathbb{Z}[1]) := \left(\begin{array}{cccccc} \mathbb{Z} & 0 & 0 & \dots & 0 & \dots \\ 0 & \mathbb{Z} & 0 & \dots & 0 & \dots \\ 0 & 0 & \mathbb{Z} \otimes \mathbb{Z} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & \dots & \mathbb{Z}^{\otimes n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{array} \right).$$

level
 n

Once again, the differentials do not figure in the above matrix; yet each column represents a positively graded chain complex.

The Σ_n -action on $\mathbb{Z}[n]$ is trivial in levels $k \neq n$ and permutes n tensored copies of \mathbb{Z} in level n . (All tensored expressions reduce obviously to \mathbb{Z} , but we keep $\mathbb{Z}^{\otimes n}$ to make the action explicit.)

D. Spectra on topological spaces

To build the category $(Sp^\Sigma(Top_*, \wedge, \mathbb{S}))$ of **topological symmetric spectra**, take K to be the circle S^1 . Natural homeomorphisms $S^1 \wedge S^n \cong S^{n+1}$, for all $n \geq 0$ imply that the n -sphere is given by $S^n := (S^1)^{\wedge n}$, the n -fold smash product in Top_* . By analogy with the simplicial case, we denote by $\mathbb{S} := (S^0, S^1, S^2, \dots)$ the topological sphere spectrum. The Σ_n -action on S^n is still given by permutation of the smashed copies of S^1 .

1.2.3 $H\mathbb{Z}$ -module spectra on Sp^Σ

In Section 1.1.3 we saw that algebraic structures, such as monoids and modules, emerged naturally on a monoidal category. Let us apply these constructions to $(Sp^\Sigma, \wedge, \mathbb{S})$.

The reduced free abelian group functor fits into an adjunction

$$\tilde{\mathbb{Z}} : sSet_* \rightleftarrows sAb : U,$$

where U is the forgetful functor. It simply forgets the abelian group structure of a simplicial abelian group X , returning its underlying pointed simplicial set. This adjunction lifts on corresponding categories of symmetric sequences

$$\tilde{\mathbb{Z}} : sSet_*^\Sigma \rightleftarrows sAb^\Sigma : U$$

(denoted using the same letters). It is given by functorial composites shown in the diagram

$$\begin{array}{ccccc} sAb & \xrightleftharpoons[U_1]{\Sigma_n \times -} & sAb^{\Sigma_n} & \xrightleftharpoons[ev_n]{(-)_n} & sAb^\Sigma \\ U \uparrow & & & & \uparrow U \\ & & & & \tilde{\mathbb{Z}} \\ & & & & \downarrow \\ sSet_* & \xrightleftharpoons[U_1]{\Sigma_n \wedge -} & sSet_*^{\Sigma_n} & \xrightleftharpoons[ev_n]{(-)_n} & sSet_*^\Sigma \end{array}$$

where left adjoints are displayed on top and on the right.

In the diagram above, for $\mathcal{C} = sSet_*$ or sAb , $U_1 : \mathcal{C}^{\Sigma_n} \rightarrow \mathcal{C}$ is the forgetful functor, which forgets the Σ_n -action. The functor $ev_n : \mathcal{C}^\Sigma \rightarrow \mathcal{C}^{\Sigma_n} : X \mapsto X_n$ is the evaluation functor at level n ; its left adjoint $(-)_n : \mathcal{C}^{\Sigma_n} \rightarrow \mathcal{C}^\Sigma$ is the inclusion in level n . Finally, $\Sigma_n \times - : \mathcal{C} \rightarrow \mathcal{C}^{\Sigma_n}$ induces the Σ_n -action on objects of \mathcal{C} .

Definition 1.2.19. The **Eilenberg-Mac Lane spectrum**, based on pointed simplicial sets, is defined by

$$H\mathbb{Z} := U \tilde{\mathbb{Z}} \mathbb{S} \in Sp^\Sigma.$$

In other words, at level n , $H\mathbb{Z}_n = U \tilde{\mathbb{Z}} S^n$ is the underlying simplicial set of the reduced free simplicial abelian group generated by the n -sphere.

Remark 1.2.20. This definition can be generalized for an arbitrary $A \in Ab$, setting $HA_n := U(A \otimes \tilde{\mathbb{Z}} S^n)$ (see [Sch, Example 2.7]). In a similar way, if R is a (commutative) ring, one defines the spectrum HR , which is a (commutative) ring spectrum. The functor H can be made into a lax symmetric monoidal functor (4.1.1) from the category Ab to the category Sp^Σ , which explains why it takes rings to ring spectra (see [Sch, Example 3.11]).

Lemma 1.2.21. *The spectrum $H\mathbb{Z}$ is a commutative ring spectrum, i.e., a commutative monoid in $(Sp^\Sigma, \wedge, \mathbb{S})$.*

Proof. The Σ_n -group action on $H\mathbb{Z}$ permutes the smash factors of S^n . The multiplication maps are determined by the collection

$$m_{p,q} : H\mathbb{Z}_p \wedge H\mathbb{Z}_q = U\tilde{\mathbb{Z}}S^p \wedge U\tilde{\mathbb{Z}}S^q \longrightarrow U\tilde{\mathbb{Z}}S^{p+q} = H\mathbb{Z}_n$$

$$\left(\sum_i a_i x_i\right) \wedge \left(\sum_i b_i y_i\right) \mapsto \sum_{i,j} (a_i b_j)(x_i \wedge y_j)$$

for $a_i, b_j \in \mathbb{Z}$, $x_i \in S^p$, $y_j \in S^q$. Here \sum denotes a formal finite sum of elements.

The unit map

$$\eta_0 : \mathbb{S}^0 \longrightarrow U\tilde{\mathbb{Z}}S_0$$

in degree 0 is the inclusion of generators. For $k \geq 1$, η_k is induced by the structure maps of the spectrum. Indeed, $\eta_1 : \mathbb{S}^1 \longrightarrow H\mathbb{Z}_1$ is the map making the diagram

$$\begin{array}{ccc} S^1 \wedge S^0 & \xrightarrow{S^1 \wedge \eta_0} & S^1 \wedge H\mathbb{Z}_0 \\ \sigma_S \downarrow & & \downarrow \sigma_{H\mathbb{Z}} \\ S^1 & \xrightarrow{\eta_1} & H\mathbb{Z}_1 \end{array}$$

commute. The argument continues similarly for $k \geq 2$.

Since \mathbb{Z} is a commutative ring, $H\mathbb{Z}$ is commutative, too. The twist maps are

$$\tau_{p,q} : H\mathbb{Z}_p \wedge H\mathbb{Z}_q \longrightarrow H\mathbb{Z}_q \wedge H\mathbb{Z}_p,$$

for $p+q = n \geq 0$, where at simplicial level k we have

$$\tau_{p,q}^k : (\mathbb{Z}^{\wedge k})^{\wedge p} \wedge (\mathbb{Z}^{\wedge k})^{\wedge q} \longrightarrow (\mathbb{Z}^{\wedge k})^{\wedge q} \wedge (\mathbb{Z}^{\wedge k})^{\wedge p}$$

$$(\alpha, (z_1 \wedge \dots \wedge z_k)^{\wedge p} \wedge (z'_1 \wedge \dots \wedge z'_k)^{\wedge q}) \mapsto (\alpha \circ p_{p,q}, (z'_1 \wedge \dots \wedge z'_k)^{\wedge q}) \wedge (z_1 \wedge \dots \wedge z_k)^{\wedge p}.$$

Here $\alpha \in \Sigma_n$ and $p_{p,q} \in \Sigma_{p+q}$ is the p, q -shuffle permutation. For the proof to be complete, one has to check that the suitable coherence diagrams commute, which we do not do here. \square

Combining the previous lemma and Lemma 1.1.14, it makes sense to define the category of $H\mathbb{Z}$ -modules in $(Sp^\Sigma, \wedge, \mathbb{S})$. We denote it by $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$.

Recall that by Theorem 1.2.15 the category $(Sp^\Sigma, \wedge, \mathbb{S})$ is symmetric monoidal. Since colimits exist in $sSet_*^\Sigma$, they also exist in $Sp^\Sigma(sSet_*)$.

Moreover, there is an adjunction

$$H\mathbb{Z} \wedge - : Sp^\Sigma \rightleftarrows Sp^\Sigma : U.$$

Being a left adjoint, $H\mathbb{Z} \wedge -$ preserves coequalizers.

Therefore, Proposition 1.1.16 applied to $(Sp^\Sigma, \wedge, \mathbb{S})$ and $H\mathbb{Z}$, endows the category $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$ with a symmetric monoidal product $\wedge_{H\mathbb{Z}}$, with unit $H\mathbb{Z}$. We summarize this writing $(H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}, \wedge_{H\mathbb{Z}}, H\mathbb{Z})$.

Definition 1.2.22. An $H\mathbb{Z}$ -algebra spectrum is a monoid in the category $(H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}, \wedge_{H\mathbb{Z}}, H\mathbb{Z})$.

We adopt the notation $H\mathbb{Z}\text{-Alg}_{Sp^\Sigma}$ for the category of $H\mathbb{Z}$ -algebra spectra.

1.2.4 A short note on the homotopy groups of a spectrum

Reminder 1.2.23. Let X be a pointed connected topological space (we do not write the base point). The n -th homotopy group of X , $\pi_n(X)$ is defined by

$$\pi_n(X) := [S^n, X].$$

This set of homotopy classes of basepoint-preserving maps has a group structure for $n \geq 1$, which is abelian for $n \geq 2$.

For a *topological* symmetric spectrum, the homotopy groups are defined as follows.

Definition 1.2.24. Let $X \in Sp^\Sigma(Top_*)$. The k -th homotopy group of X is defined to be the colimit

$$\pi_k(X) := \operatorname{colim}_n \pi_{k+n} X_n,$$

taken over the maps

$$\pi_{k+n} X_n \xrightarrow{- \wedge S^1} \pi_{k+n+1}(X_n \wedge S^1) \xrightarrow{\pi(\sigma_n)} \pi_{k+n+1} X_{n+1}.$$

To calculate the homotopy groups of a *simplicial-set valued* spectrum, use the adjunction

$$|\cdot| : sSet \rightleftarrows Top : \operatorname{Sing}(-)$$

between the geometric realization functor $|\cdot|$ and the singular complex functor $\operatorname{Sing}(-)$ (see [May, §16] for details; among other things, we might need to

request momentarily that Top be the category of compactly generated topological spaces). Note that this adjunction can be extended to the associated pointed categories $sSet_*$ and Top_* .

The functor $|\cdot|$ is a strong monoidal symmetric functor with respect to smash, with associated natural homeomorphisms

$$\varphi_{A,B} : |A| \wedge |B| \longrightarrow |A \wedge B|$$

for $A, B \in sSet_*$. Its adjoint $\text{Sing}(-)$ is only a lax symmetric monoidal functor (4.1.1) equipped with natural morphisms

$$\tilde{\varphi}_{X,Y} : \text{Sing}(X) \wedge \text{Sing}(Y) \longrightarrow \text{Sing}(X \wedge Y)$$

for $X, Y \in Top_*$.

The adjunction between $|\cdot|$ and $\text{Sing}(-)$ lifts on the associated categories of symmetric spectra. If $Y \in Sp^\Sigma$ we define a symmetric spectrum Y of topological spaces by $|Y|_n = |Y_n|$ with structure maps

$$|Y_n| \wedge S^1 \xrightarrow{\text{Id} \wedge h} |Y_n| \wedge |S^1| \xrightarrow{\varphi_{Y_n, S^1}} |Y_n \wedge S^1| \xrightarrow{|\sigma_n|} |Y_{n+1}|,$$

where $h : S^1 \longrightarrow |S^1|$ is a fixed choice of a homeomorphism.

Reciprocally, if $X \in Sp^\Sigma(Top_*)$, then we obtain a symmetric spectrum $\text{Sing}(X)$ of simplicial sets such that $\text{Sing}(X)_n = \text{Sing}(X_n)$, with structure maps

$$\text{Sing}(X_n) \wedge S^1 \xrightarrow{\text{Id} \wedge \hat{h}} \text{Sing}(X_n) \wedge \text{Sing}(S^1) \xrightarrow{\tilde{\varphi}_{X_n, S^1}} \text{Sing}(X_n \wedge S^1) \xrightarrow{\text{Sing}(\sigma_n)} \text{Sing}(X_{n+1}).$$

Here $h : S^1 \longrightarrow \text{Sing}(S^1)$ is the morphism of pointed simplicial sets, adjoint to the inverse homeomorphism $h^{-1} : |S^1| \longrightarrow S^1$. See [Sch, 1.8] for details.

Definition 1.2.25. Let $X \in Sp^\Sigma$. The k -th homotopy group of X is given by $\pi_k(|X|)$, the k -th homotopy group of the geometric realization of X .

Guideline

As the reader remembers from the Introduction, our goal is to understand the relationship (the Quillen equivalence) between the categories of $H\mathbb{Z}$ -algebra spectra and differential graded algebras. To do this, the strategy will be first to study first the relation between $H\mathbb{Z}$ -module spectra and differential graded modules, and then use the fact that $H\mathbb{Z}\text{-Alg}_{Sp^\Sigma}$ and $DG\text{-Alg}_{\mathbb{Z}}$ are the categories of monoids in $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$ and $DG\mathbb{Z}\text{-Mod}$ respectively.

To make a connection between $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$ and $DG\mathbb{Z}\text{-Mod}$, both of which are monoidal, we will need to consider two intermediate monoidal categories, the category $Sp^\Sigma(sAb)$ of symmetric spectra over simplicial abelian groups and the category $Sp^\Sigma(Ch_+)$ of symmetric spectra over non-negative chain complexes.

For the moment, the picture

$$H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \leftarrow \cdots \leftarrow Sp^\Sigma(sAb) \leftarrow \cdots \leftarrow Sp^\Sigma(Ch_+) \leftarrow \cdots \leftarrow DG\mathbb{Z}\text{-Mod}$$

looks quite mysterious. We need more tools to understand the nature of the relations involved.

Chapter 2

Model structures

Our presentation of model categories is based on three different sources. A clear introduction to model categories can be found in [H02]; for a good explanation of further details and for proofs the reader is referred to [DS] and [Hov99], Chapters 1 and 2.

2.1 Model categories

Model categories are categories that provide a natural setting for homotopy theory. The concept was originally introduced by Daniel G. Quillen in 1967, as a generalization of the homotopy theory of topological spaces.

2.1.1 Basic definitions

Definition 2.1.1.

- Let \mathcal{C} be a category and $\mathcal{I} \subseteq \text{Mor } \mathcal{C}$. A morphism $f \in \mathcal{C}(A, B)$ satisfies the **left lifting property** with respect to \mathcal{I} , denoted $f \in \text{LLP}(\mathcal{I})$, if for every commutative square diagram (*LLP*) of morphisms in \mathcal{C} with $g \in \mathcal{I}$, there exists a morphism $\hat{k} : B \rightarrow C$ such that $g\hat{k} = k$ and $\hat{k}f = h$, i.e., the two triangles commute. We call \hat{k} a **lift** in the original square diagram.

$$\begin{array}{ccc}
 \text{(LLP)} & \begin{array}{ccc} A & \xrightarrow{h} & C \\ f \downarrow & \nearrow \hat{k} & \downarrow g \\ B & \xrightarrow{k} & D \end{array} & \begin{array}{ccc} C & \xrightarrow{h} & A \\ g \downarrow & \nearrow \hat{k} & \downarrow f \\ D & \xrightarrow{k} & B \end{array} & \text{(RLP)}
 \end{array}$$

- Dually, we say that f satisfies the **right lifting property** with respect to \mathcal{I} , denoted $f \in \text{RLP}(\mathcal{I})$, if for every commutative square diagram (*RLP*) of morphisms in \mathcal{C} with $g \in \mathcal{I}$, there exists a morphism $\hat{k} : D \rightarrow A$ such that $f\hat{k} = k$ and $\hat{k}g = h$.

Definition 2.1.2. A morphism f in a category \mathcal{C} is a **retract** of a morphism g if there is a commutative diagram of morphisms in \mathcal{C}

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{i} & \bullet & \xrightarrow{r} & \bullet \\
 f \downarrow & & g \downarrow & & f \downarrow \\
 \bullet & \xrightarrow{j} & \bullet & \xrightarrow{s} & \bullet,
 \end{array}$$

such that $r \circ i$ and $s \circ j$ are identity morphisms.

Definition 2.1.3. A **model category** consists of a category \mathcal{C} , together with three distinguished classes of morphisms $WE, Fib, Cof \subseteq Mor(\mathcal{C})$ that are closed under composition and contain all identities, such that the following axioms are satisfied.

- (MC1) All finite limits and colimits exist.
- (MC2) Let $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$. If two of $f, g, g \circ f$ are in WE , then so is the third.
- (MC3) If f is a retract of g , and g belongs to WE (respectively Fib , respectively Cof), then f also belongs to WE (respectively Fib , respectively Cof).
- (MC4) (a) $Cof \subseteq LLP(Fib \cap WE)$
(b) $Fib \subseteq RLP(Cof \cap WE)$
- (MC5) If $f \in Mor(\mathcal{C})$, then there exist
 - (a) $i \in Cof$ and $p \in Fib \cap WE$ such that $f = pi$;
 - (b) $j \in Cof \cap WE$ and $q \in Fib$ such that $f = qj$.

We will write $(\mathcal{C}, WE, Fib, Cof)$ for a model category further on.

The morphisms belonging to the classes WE, Fib and Cof are called **weak equivalences**, **fibrations** and **cofibrations** respectively, and are symbolized by $\xrightarrow{\sim}, \twoheadrightarrow$ and \rightarrow . The elements of classes $Fib \cap WE$ and $Cof \cap WE$ are referred to as **acyclic fibrations** and **acyclic cofibrations** respectively. Since WE, Fib and Cof are closed under composition and contain all isomorphisms, we can view them as subcategories of \mathcal{C} , rather than simply as classes of morphisms.

Axiom (MC1) implies in particular that an initial object Φ , and a terminal object $*$ exist in \mathcal{C} (recall that they are defined to be respectively the colimit and the limit of an empty diagram $\emptyset \longrightarrow \mathcal{C}$).

Definition 2.1.4. An object A in \mathcal{C} is called **fibrant**, if the unique morphism $A \longrightarrow *$ is a fibration; it is called **cofibrant**, if the unique morphism $\Phi \longrightarrow A$ is a cofibration.

Given a model category \mathcal{C} , the category \mathcal{C}_* is the category under the terminal object $*$ of \mathcal{C} . Note that \mathcal{C}_* has arbitrary limits and colimits, since \mathcal{C} does. There is an adjunction

$$(-)_+ : \mathcal{C} \rightleftarrows \mathcal{C}_* : U,$$

where the left adjoint adds a disjoint basepoint to $X \in \mathcal{C}$, and the right adjoint is the forgetful functor. This adjunction plays an important role in defining the model structure on the category \mathcal{C}_* , if \mathcal{C} already has a model structure.

Proposition 2.1.5. [Hov99, Proposition 1.1.8] *Suppose \mathcal{C} is a model category. Define a map f in \mathcal{C}_* to be a cofibration (fibration, weak equivalence) if and only if $U(f)$ is a cofibration (fibration, weak equivalence) in \mathcal{C} . These definitions make \mathcal{C}_* in a model category.*

Proposition 2.1.6. [H02, Proposition 2.1.1] *Let $(\mathcal{C}, WE, Fib, Cof)$ be a model category, then we have*

1. $Cof = LLP(Fib \cap WE)$ and $Fib = RLP(Cof \cap WE)$;
2. the classes Cof and $Cof \cap WE$ are preserved under pushout.
3. the classes Fib and $Fib \cap WE$ are preserved under pullback.

Proofs can be found in [DS], Propositions 3.13 and 3.14; they are direct consequences of the axioms for a model category.

We end up this section with a helpful lemma. Note that there exists a dual version for the fibrant case, see [Hov99, Lemma 1.1.12].

Lemma 2.1.7. *Ken Brown's Lemma* [DS, 9.9] *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between model categories. If F carries acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

This property is useful in many circumstances; we will turn to it in Remark 3.1.2.

2.1.2 The homotopy category $Ho \mathcal{C}$ of a model category \mathcal{C}

Given a model category \mathcal{C} , we explain the essentials of two alternative constructions of the homotopy category of \mathcal{C} . They give rise to *equivalent* categories.

The homotopy category as a localization of \mathcal{C}

Definition 2.1.8. A **localization** of a category \mathcal{C} with respect to a class of morphisms $\mathcal{W} \subseteq \text{Mor } \mathcal{C}$ is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, such that

1. $\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{D})$, and F is the identity on objects;
2. For all $w \in \mathcal{W}$, $F(w)$ is an isomorphism in \mathcal{D} ;
3. *Universal property of the localization:*

If $G : \mathcal{C} \rightarrow \mathcal{E}$ is any other functor such that $G(w)$ is an isomorphism for all $w \in \mathcal{W}$, then there is a unique functor $\hat{G} : \mathcal{D} \rightarrow \mathcal{E}$, such that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ G \downarrow & \swarrow \hat{G} & \\ \mathcal{E} & & \end{array}$$

commutes.

Hence, two different localizations of \mathcal{C} with respect to \mathcal{W} are canonically isomorphic.

A commonly occurring situation is the following. One is given an arbitrary category \mathcal{C} , with a distinguished class of not necessarily invertible morphisms \mathcal{W} . One would like the elements of \mathcal{W} to be considered as isomorphisms in \mathcal{C} , i.e., to be invertible. Next definition explains how to construct a new category from \mathcal{C} , to which formal inverses of morphisms in \mathcal{W} will be added.

Definition 2.1.9. Let \mathcal{C} be a category with a subcategory \mathcal{W} . For each morphism w of \mathcal{W} , denote its formal inverse by w^{-1} . The **category** $\mathcal{C}[\mathcal{W}^{-1}]$ is constructed as follows.

1. First, form the category $\mathcal{F}(\mathcal{C}, \mathcal{W}^{-1})$. Its objects are the same as the objects of \mathcal{C} , and a morphism in $\mathcal{F}(\mathcal{C}, \mathcal{W}^{-1})$ is a finite string of composable morphisms $f = \{f_1, f_2, \dots, f_n\}$, where either $f_i \in \text{Mor}(\mathcal{C})$, or $f_i = w^{-1}$ for a $w \in \mathcal{W}$.

The composition in $\mathcal{F}(\mathcal{C}, \mathcal{W}^{-1})$ is defined by concatenation of strings, the identity for the composition is the empty string. At a particular object $A \in \mathcal{C}$, the empty string is the identity map Id_A .

2. Now, define $\mathcal{C}[\mathcal{W}^{-1}] := \mathcal{F}(\mathcal{C}, \mathcal{W}^{-1}) / \sim$, the quotient of $\mathcal{F}(\mathcal{C}, \mathcal{W}^{-1})$ by the following equivalence relation. Two finite strings $f = \{f_1, \dots, f_n\}$ and $g = \{g_1, \dots, g_k\}$ in $\mathcal{F}(\mathcal{C}, \mathcal{W}^{-1})$ are **equivalent**, denoted $f \sim g$, if we can pass from one to another using a chain of elementary equivalences of the following types:

- (i) in a chain of form $\{\dots, h, h', \dots\}$, any two composable morphisms h, h' of \mathcal{C} can be replaced by their composition $h \circ h'$;
- (ii) chains of form $\{w, w^{-1}\}$ and $\{w^{-1}, w\}$ can be replaced respectively by $\text{Id}_{\text{dom } w}$ and $\text{Id}_{\text{codom } w}$ for $w \in \mathcal{W}$.

The class of a finite string f under the equivalence relation \sim is written $[f]$.

Applying the previous construction to a model category \mathcal{C} and the class $\mathcal{W} := WE_{\mathcal{C}}$ gives the **homotopy category of \mathcal{C}** , which we denote $\text{Ho } \mathcal{C}$, instead of $\mathcal{C}[WE_{\mathcal{C}}^{-1}]$. There is a functor

$$\gamma : \mathcal{C} \longrightarrow \text{Ho } \mathcal{C}$$

which is identity on objects and sends a map f in \mathcal{C} to its class $[f]$ in $\text{Ho } \mathcal{C}$. In particular, for $w \in WE_{\mathcal{C}}$, the class $[w] = \gamma(w)$ is an isomorphism in $\text{Ho } \mathcal{C}$, with inverse given by the class $[w^{-1}]$ (in view of (ii) above).

Lemma 1.2.2 in [Hov99] shows that γ satisfies the universal property of Definition 2.1.8, and this fact characterizes the category $\text{Ho } \mathcal{C}$. Namely, if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor that sends weak equivalences to isomorphisms, then there is a unique functor $\text{Ho } F : \text{Ho } \mathcal{C} \longrightarrow \mathcal{D}$ such that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\gamma} & \text{Ho } \mathcal{C} \\ F \downarrow & \swarrow \text{Ho } F & \\ \mathcal{D} & & \end{array}$$

commutes. The functor $\text{Ho } F$ is defined to be the identity on objects and morphisms of \mathcal{C} , and $\text{Ho } F(w^{-1}) := (Fw)^{-1}$ for $w \in WE_{\mathcal{C}}$.

In other words, we have the following important result.

Theorem 2.1.10. *Let \mathcal{C} be a model category. The functor $\gamma : \mathcal{C} \longrightarrow \text{Ho } \mathcal{C}$ is a localization of \mathcal{C} with respect to the class $WE_{\mathcal{C}}$.*

Remark 2.1.11. Let us emphasize one important fact. If \mathcal{C} is an arbitrary category, not equipped with a model structure, we do not know if $\text{Ho } \mathcal{C}$ will be really a category. The issue is that $\text{Ho } \mathcal{C}(A, B)$ may not be a *set* in general. However, having a Quillen model structure on \mathcal{C} allows one to avoid this problem. In the next section, for \mathcal{C} model we will define the category \mathcal{C}_{cf} / \sim , and Theorem 2.1.23 will ensure us that there exists an equivalence of categories between $\text{Ho } \mathcal{C}$ and \mathcal{C}_{cf} / \sim . This will guarantee that $\text{Ho } \mathcal{C}$ is indeed a category in the usual sense when \mathcal{C} is a model category.

The homotopy category as a quotient of \mathcal{C}_{cf}

For a model category \mathcal{C} , we denote respectively by \mathcal{C}_c , \mathcal{C}_f and \mathcal{C}_{cf} the full subcategories of cofibrant, fibrant, cofibrant and fibrant objects of \mathcal{C} .

These subcategories come with associated inclusion functors $i_c : \mathcal{C}_c \rightarrow \mathcal{C}$, $i_f : \mathcal{C}_f \rightarrow \mathcal{C}$ and $i_{cf} : \mathcal{C}_{cf} \rightarrow \mathcal{C}$. There exist functors $Q : \mathcal{C} \rightarrow \mathcal{C}_c$ and $R : \mathcal{C} \rightarrow \mathcal{C}_f$ which go in the opposite directions; we explain their construction later. All together they will induce equivalences of categories on homotopy categories $\text{Ho } \mathcal{C}$, $\text{Ho } \mathcal{C}_c$, $\text{Ho } \mathcal{C}_f$ and $\text{Ho } \mathcal{C}_{cf}$, see Proposition 2.1.22.

Let us now prepare the groundwork for the second definition of $\text{Ho } \mathcal{C}$.

There are two different ways to define a homotopy of morphisms in an arbitrary model category $(\mathcal{C}, WE, Fib, Cof)$. Unless A and X satisfy certain criteria, these two definitions are not necessarily equivalent, and do not determine an equivalence relation on the hom-set $\mathcal{C}(A, X)$.

Definition 2.1.12. Given $A \in \mathcal{C}$, consider the pushout of $\Phi \rightarrow A$ with itself; see diagram (L1), where $\nabla : A \sqcup A \rightarrow A$ is the folding map.

A **cylinder** on A , denoted $\text{Cyl}(A)$, consists of a factorization of ∇ into a cofibration, followed by a weak equivalence; see diagram (L2).

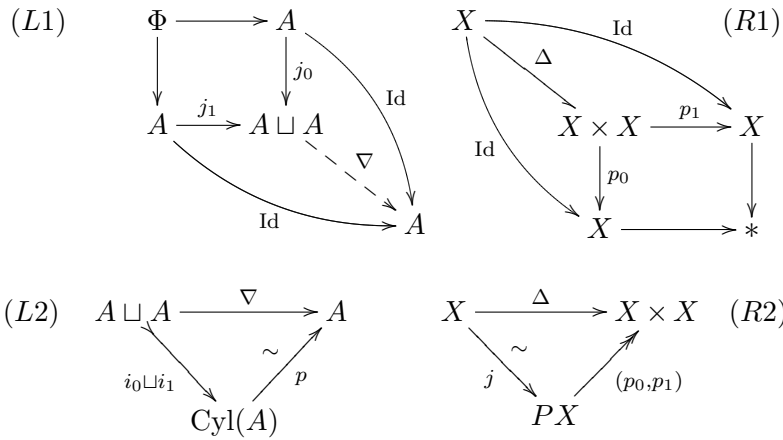
Let $f, g : A \rightarrow X$ be morphisms in \mathcal{C} . A **left homotopy** from f to g is a morphism $H : \text{Cyl}(A) \rightarrow X$ such that the diagram (L3) commutes. We say that f and g are **left homotopic** and write $f \sim_l g$.

Dually, we have the following definitions.

Definition 2.1.13. Given $X \in \mathcal{C}$, consider the pullback of $X \rightarrow *$ with itself; see diagram (R1), where $\Delta : X \rightarrow X \times X$ is the diagonal map.

A **path object** on X , denoted PX , consists of a factorization of Δ into a weak equivalence, followed by a fibration; see diagram (R2).

Let $f, g : A \rightarrow X$ be morphisms in \mathcal{C} . A **right homotopy** from f to g is a morphism $K : A \rightarrow PX$ such that the diagram (R3) commutes. We then say that f and g are **right homotopic** and write $f \sim_r g$.



$$(L3) \quad \begin{array}{ccc} A & \xrightarrow{i \circ j_0} & \text{Cyl}(A) & \xleftarrow{i \circ j_1} & A \\ & \searrow f & \downarrow H & \swarrow g & \\ & & X & & \end{array} \quad (R3) \quad \begin{array}{ccccc} X & \xleftarrow{p_1 \circ q} & PX & \xrightarrow{p_0 \circ q} & X \\ & \searrow f & \uparrow K & \swarrow g & \\ & & A & & \end{array}$$

Left and right homotopy satisfy numerous dual properties. The reader is referred to [Hov99] Proposition 1.2.5 or [DS] Section 4 for a complete list of these properties and their proofs. We mention only a few key statements here.

Proposition 2.1.14. [Hov99, Proposition 1.2.5.]

- If A is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(A, X)$. Dually, if X is fibrant, then right homotopy is an equivalence relation on $\mathcal{C}(A, X)$.
- If A is cofibrant, then $f \sim_l g$ implies $f \sim_r g$. Dually, if X is fibrant, then $f \sim_r g$ implies $f \sim_l g$.

Corollary 2.1.15. Suppose that A is a cofibrant object and X is a fibrant object in a model category \mathcal{C} . Then the left homotopy and the right homotopy coincide and determine an equivalence relation on $\mathcal{C}(A, X)$, denoted \sim . If $f \sim g$, we say that f and g are **homotopic**.

We have the following important corollary for the full subcategory \mathcal{C}_{cf} of cofibrant and fibrant objects in \mathcal{C} .

Corollary 2.1.16. [Hov99, Corollary 1.2.7.] *The homotopy relation on the morphisms of \mathcal{C}_{cf} is an equivalence relation and it is compatible with composition. Hence the category \mathcal{C}_{cf} / \sim exists.*

Definition 2.1.17. In a model category \mathcal{C} , a map $f : A \rightarrow X$ is a **homotopy equivalence**, if there exists a map $h : X \rightarrow A$ such that $f \circ h \sim \text{Id}_X$, and $h \circ f \sim \text{Id}_A$.

The quotient functor $\delta : \mathcal{C}_{cf} \rightarrow \mathcal{C}_{cf} / \sim$, given by identity on objects and sending a map in \mathcal{C}_{cf} to its homotopy class, inverts the homotopy equivalences in \mathcal{C}_{cf} . In fact, it inverts also the weak equivalences, as follows from the next proposition. One could think of this result as of the “generalized Whitehead Theorem”, because applied to Top with the model structure given later in Definition 2.2.12, it yields the Whitehead Theorem, which states that a weak equivalence between CW -complexes is a homotopy equivalence.

Proposition 2.1.18. [Hov99, Proposition 1.2.8] *Suppose \mathcal{C} is a model category. A morphism in \mathcal{C}_{cf} is a weak equivalence if and only if it is a homotopy equivalence.*

Similarly, apply (MC5)(ii) to the map $X \rightarrow *$ to obtain a **fibrant model** $X \xrightarrow{\sim} RX \rightarrow *$, with RX fibrant. If X is fibrant, set $RX = X$.

In the commutative diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & X & \xrightarrow{j_X} & RX \\
 \downarrow j_A \sim & & & \nearrow Rf & \downarrow \\
 RA & \xrightarrow{\quad} & & & *
 \end{array}$$

there exists an extension $Rf : RA \rightarrow RX$ of $j_X f$ over j_A by (MC4). Dual properties of right homotopy from Proposition 2.1.14(ii) imply that all such lifts are unique up to homotopy, because RX is fibrant.

Definition 2.1.21. Let \mathcal{C} be a model category such that all (MC5)-type factorizations are functorial. Define the **fibrant replacement functor** to be

$$\begin{aligned}
 R : \mathcal{C} &\longrightarrow \mathcal{C}_f \\
 A &\mapsto RA \\
 f : A &\longrightarrow X \mapsto Rf : RA \longrightarrow RX.
 \end{aligned}$$

Observe that one can compose the replacement functors in either way to obtain functors $RQ : \mathcal{C} \rightarrow \mathcal{C}_{cf}$ and $QR : \mathcal{C} \rightarrow \mathcal{C}_{cf}$.

We can finally state the following result.

Proposition 2.1.22. [Hov99, Proposition 1.2.3.] *Suppose \mathcal{C} is a model category. The inclusion functors*

$$\begin{array}{ccccc}
 & & \mathcal{C}_c & & \\
 & \nearrow i_{cf} & & \searrow i_c & \\
 \mathcal{C}_{cf} & & & & \mathcal{C} \\
 & \searrow i_{cf} & & \nearrow i_f & \\
 & & \mathcal{C}_f & &
 \end{array}$$

induce equivalences of categories

$$\mathrm{Ho} \mathcal{C}_{cf} \xrightleftharpoons[\mathrm{Ho}(R)]{\mathrm{Ho}(i_{cf})} \mathrm{Ho} \mathcal{C}_c \xrightleftharpoons[\mathrm{Ho}(Q)]{\mathrm{Ho}(i_c)} \mathrm{Ho} \mathcal{C} \quad \text{and} \quad \mathrm{Ho} \mathcal{C}_{cf} \xrightleftharpoons[\mathrm{Ho}(Q)]{\mathrm{Ho}(i_{cf})} \mathrm{Ho} \mathcal{C}_f \xrightleftharpoons[\mathrm{Ho}(R)]{\mathrm{Ho}(i_f)} \mathrm{Ho} \mathcal{C}.$$

The next theorem crowns the discussion about the two alternative constructions of the homotopy category for a model category \mathcal{C} .

Theorem 2.1.23. [Hov99, Theorem 1.2.10] *Suppose $(\mathcal{C}, WE, Fib, Cof)$ is a model category. Let $\gamma : \mathcal{C} \rightarrow \mathrm{Ho} \mathcal{C}$ denote the canonical functor, Q denote the cofibrant replacement functor of \mathcal{C} and R denote the fibrant replacement functor of \mathcal{C} .*

1. *There is an equivalence of categories*

$$\mathcal{C}_{cf} / \sim \xrightarrow[\cong]{j} \mathrm{Ho} \mathcal{C}_{cf} \xrightarrow[\cong]{\mathrm{Ho}(i_{cf})} \mathrm{Ho} \mathcal{C}.$$

2. *There are natural isomorphisms*

$$\mathcal{C}(QRX, QRY) / \sim \xrightarrow[\cong]{} \mathrm{Ho} \mathcal{C}(X, Y) \xrightarrow[\cong]{} \mathcal{C}(RQX, RQY) / \sim$$

Notation 2.1.24. From now on, we will abbreviate $\mathrm{Ho} \mathcal{C}(X, Y)$ by $[X, Y]$, and write X^c (respectively X^f) if X is a cofibrant (respectively fibrant) object in \mathcal{C} .

2.1.3 Quillen functors and derived functors

Given two model categories, we would like to define a good notion of a “morphism of model categories”. It appears sufficient to require that such a morphism (i.e., a functor) preserve half of the model structure, either the classes Cof and $Cof \cap WE$ or the classes Fib and $Fib \cap WE$. These functors are called *Quillen functors*.

Definition 2.1.25. Let $(\mathcal{C}, WE, Fib, Cof)$ and $(\mathcal{D}, WE, Fib, Cof)$ be two model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ an adjunction.

The left adjoint F is a **left Quillen functor** if $F(Cof_{\mathcal{C}}) \subseteq Cof_{\mathcal{D}}$ and $F((Cof \cap WE)_{\mathcal{C}}) \subseteq (Cof \cap WE)_{\mathcal{D}}$.

The right adjoint G is a **right Quillen functor** if $G(Fib_{\mathcal{D}}) \subseteq Fib_{\mathcal{C}}$ and $G((Fib \cap WE)_{\mathcal{D}}) \subseteq (Fib \cap WE)_{\mathcal{C}}$.

It is easy to check the following lemma, by combining the properties of an adjunction and the definitions of fibrations and cofibrations by RLP and LLP.

Lemma 2.1.26. [DS, Remark 9.8] *Let \mathcal{C} and \mathcal{D} be two model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ an adjunction. The following conditions are equivalent:*

1. $F(Cof_{\mathcal{C}}) \subseteq Cof_{\mathcal{D}}$ and $F((Cof \cap WE)_{\mathcal{C}}) \subseteq (Cof \cap WE)_{\mathcal{D}}$;
2. $G(Fib_{\mathcal{D}}) \subseteq Fib_{\mathcal{C}}$ and $G((Fib \cap WE)_{\mathcal{D}}) \subseteq (Fib \cap WE)_{\mathcal{C}}$;
3. $F(Cof_{\mathcal{C}}) \subseteq Cof_{\mathcal{D}}$ and $G(Fib_{\mathcal{D}}) \subseteq Fib_{\mathcal{C}}$.

Definition 2.1.27. In either of the three cases stated in Lemma 2.1.26, we say that the adjoint pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ forms a **Quillen pair**, we will write then (F, G) .

Composition of Quillen pairs is a Quillen pair, and a Quillen pair between two model categories gives rise to a Quillen pair between corresponding pointed categories.

Quillen functors between two model categories induce functors on the corresponding homotopy categories as follows.

Definition 2.1.28. Let \mathcal{C} and \mathcal{D} be two model categories.

- Given $F : \mathcal{C} \rightarrow \mathcal{D}$ a left Quillen functor, define its **total left derived functor** $\mathbb{L}F : \text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$ to be the composite

$$\text{Ho } \mathcal{C} \xrightarrow[\simeq]{\text{Ho}(Q)} \text{Ho } \mathcal{C}_c \xrightarrow{\text{Ho}(F)} \text{Ho } \mathcal{D}.$$

If $\tau : F \rightarrow F'$ is a natural transformation between left Quillen functors, i.e., if for all $f : A \rightarrow B$ in \mathcal{C} the square

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \tau(A) \downarrow & & \downarrow \tau(B) \\ F'(A) & \xrightarrow{F'(f)} & F'(B) \end{array}$$

commutes, define the **total derived natural transformation** $\mathbb{L}\tau : \mathbb{L}F \rightarrow \mathbb{L}F'$ to be

$$\mathbb{L}\tau(A) := \tau_{QA},$$

for all $A \in \mathcal{C}$, so that the square

$$\begin{array}{ccc} \text{Ho}(F) \circ \text{Ho}(Q)(A) & \xrightarrow{\mathbb{L}F(f)} & \text{Ho}(F) \circ \text{Ho}(Q)(B) \\ \mathbb{L}\tau(A) \downarrow & & \downarrow \mathbb{L}\tau(B) \\ \text{Ho}(F') \circ \text{Ho}(Q)(A) & \xrightarrow{\mathbb{L}F'(f)} & \text{Ho}(F') \circ \text{Ho}(Q)(B) \end{array}$$

commutes.

- Similarly, given $G : \mathcal{D} \rightarrow \mathcal{C}$ a right Quillen functor, define its **total right derived functor** $\mathbb{R}G : \text{Ho } \mathcal{D} \rightarrow \text{Ho } \mathcal{C}$ to be the composite

$$\text{Ho } \mathcal{D} \xrightarrow[\simeq]{\text{Ho}(R)} \text{Ho } \mathcal{D}_f \xrightarrow{\text{Ho}(G)} \text{Ho } \mathcal{C}.$$

If $\tau : G \rightarrow G'$ is a natural transformation between right Quillen functors, define the **total derived natural transformation** $\mathbb{R}\tau : \mathbb{R}G \rightarrow \mathbb{R}G'$ to be

$$\mathbb{R}\tau(A) := \tau_{RA}$$

for all $A \in \mathcal{D}$.

The total derived functors preserve adjunctions.

Lemma 2.1.29. [Hov99, Lemma 1.3.10] *Suppose that (F, G) is a Quillen pair between model categories \mathcal{C} and \mathcal{D} . Then there is an adjunction on the associated homotopy categories*

$$\mathbb{L}F : \mathrm{Ho} \mathcal{C} \rightleftarrows \mathrm{Ho} \mathcal{D} : \mathbb{R}G ,$$

called the *derived adjunction*.

Definition 2.1.30. A Quillen pair (F, G) between model categories \mathcal{C} and \mathcal{D} is called a **Quillen equivalence**, if for all cofibrant A in \mathcal{C} and fibrant B in \mathcal{D}

$$f : FA \longrightarrow B \in WE_{\mathcal{D}} \iff f^{\sharp} : A \longrightarrow GB \in WE_{\mathcal{C}},$$

where f^{\sharp} is the adjoint of f .

Proposition 2.1.31. [Hov99, Proposition 1.3.13] *If (F, G) is a Quillen equivalence, then the derived adjunction $\mathbb{L}F : \mathrm{Ho} \mathcal{C} \rightleftarrows \mathrm{Ho} \mathcal{D} : \mathbb{R}G$ is an equivalence on corresponding homotopy categories.*

2.2 Cofibrantly generated model categories

In many model categories it is possible to identify reasonably small sets of cofibrations and acyclic cofibrations that generate the model category structure in a natural way. In this case, proving results and detecting properties of maps, as for instance, checking if a given map is an (acyclic) fibration, is easier, because then the lifting conditions have to be checked only on these relatively small generating sets.

2.2.1 The notion of a cofibrantly generated category

We begin with some technical but necessary definitions.

Definitions 2.2.1.

- A **cardinal** is an isomorphism class of sets; an **ordinal** is an isomorphism class of well-ordered sets.
- We call λ a **limit ordinal** if it can not be reached by the successor operation, i.e., if $\lambda > 0$ and for any $\beta < \lambda$, there exists γ such that $\beta < \gamma < \lambda$.
- A set of ordinals S' is **cofinal** in a set of ordinals S , if for all $\lambda \in S$ there is $\lambda' \in S'$ such that $\lambda' \geq \lambda$.

Example 2.2.2. Consider two following well-ordered sets:

$$\begin{aligned} \mathbb{N} &:= 0\ 1\ 2\ 3\dots && \text{(natural numbers)} \\ \mathbb{N}_1 \cup \mathbb{N}_2 &:= 0\ 1\ 2\ 3\dots 0\ 1\ 2\ 3\dots && \text{(two copies of natural numbers)}. \end{aligned}$$

One shows that \mathbb{N} and $\mathbb{N}_1 \cup \mathbb{N}_2$ are *equinumerous* sets (have the same cardinal, denoted $\mathbb{N} \approx \mathbb{N}_1 \cup \mathbb{N}_2$), but they do not have the same ordinal. Indeed, one can always find $\gamma \in \mathbb{N}_1 \cup \mathbb{N}_2$ such that for all $\beta \in \mathbb{N}_1 \cup \mathbb{N}_2$, $\beta < \gamma$ (take γ in the second copy \mathbb{N}_2 , and β in the first). This is not true in \mathbb{N} .

Definition 2.2.3.

1. Let λ be an ordinal. A λ -**sequence** in \mathcal{C} is a functor $X : \lambda \rightarrow \mathcal{C}$, i.e., a diagram of the form

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_\beta \rightarrow \dots \rightarrow X_\gamma \rightarrow X_{\gamma+1} \rightarrow \dots \quad (\beta < \lambda),$$

such that the induced morphism $\text{colim}_{\beta < \gamma} X_\beta \xrightarrow{\cong} X_\gamma$ is an isomorphism for every limit ordinal γ .

Drawing a parallel between this definition and the previous example, we can imagine, for example, that X_γ represents the image by X of the 0 of the second copy \mathbb{N}_2 .

2. The **composition** of a λ -sequence is the morphism $X_0 \rightarrow \text{colim}_{\beta < \lambda} X_\beta$.
3. Let \mathcal{D} be a subcategory of \mathcal{C} . A **transfinite composition** of \mathcal{D} -morphisms is the composition in \mathcal{C} of a λ -sequence, such that the map $X_\beta \rightarrow X_{\beta+1}$ is a morphism in \mathcal{D} for all $\beta < \lambda$.
4. Let \mathcal{D} be a subcategory of \mathcal{C} . An object A is **small with respect to** \mathcal{D} if there is a cofinal set S of ordinals, such that for all $\lambda \in S$ and for all λ -sequences $X : \lambda \rightarrow \mathcal{C}$, the induced map in *Set*

$$\text{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \rightarrow \mathcal{C}(A, \text{colim}_{\beta < \lambda} X_\beta) \quad (S)$$

is an isomorphism (thus, a bijection).

The fourth point means that for a small object A , a morphism from A into a sufficiently long composition will factor through some stage of the composition.

Remark 2.2.4. To understand better the isomorphism (S), consider the commutative diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{j_0} & X_1 & \xrightarrow{j_1} & X_2 & \longrightarrow \dots & \longrightarrow X_\beta & \longrightarrow \dots \\ & \searrow i_0 & \downarrow i_1 & \swarrow i_2 & \swarrow i_\beta & & \swarrow i_\beta & \\ & & \text{colim}_{\beta < \lambda} X_\beta & & & & & \end{array} \quad \text{in } \mathcal{C},$$

and consider the hom-set of maps $\mathcal{C}(A, \operatorname{colim}_{\beta < \lambda} X_\beta)$ for $A \in \mathcal{C}$ and λ in a cofinal set of ordinals.

Applying the covariant hom-set functor $\mathcal{C}(A, -)$ to (S) gives the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{C}(A, X_0) & \xrightarrow{j_{0*}} & \mathcal{C}(A, X_1) & \xrightarrow{j_{1*}} & \mathcal{C}(A, X_2) & \longrightarrow & \dots \longrightarrow \mathcal{C}(A, X_\beta) \longrightarrow \dots \\
& \searrow^{i_{0*}} & \downarrow^{i_{1*}} & \swarrow^{i_{2*}} & & \nearrow^{i_{\beta*}} & \\
& & \operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) & & & & \text{in } \mathit{Set}.
\end{array}$$

By the universal property of colimits, there exists a unique map up to isomorphism in Set

$$\varphi : \operatorname{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \longrightarrow \mathcal{C}(A, \operatorname{colim}_{\beta < \lambda} X_\beta).$$

Condition (4) of Definition 2.2.3 actually requests that this map have an inverse.

Definition 2.2.5. Let \mathcal{C} be a model category, and $\mathcal{I} \subseteq \operatorname{Mor} \mathcal{C}$. The class of morphisms \mathcal{I} gives rise to the following three other classes in $\operatorname{Mor} \mathcal{C}$:

1. $\mathcal{I}\text{-inj} := \operatorname{RLP}(\mathcal{I})$,
2. $\mathcal{I}\text{-cof} := \operatorname{LLP}(\mathcal{I}\text{-inj})$,
3. $\mathcal{I}\text{-cell}$ is the class of morphisms $f : A \longrightarrow B$ in \mathcal{C} , for which there exist an ordinal λ and a λ -sequence $X : \lambda \longrightarrow \mathcal{C}$, such that
 - $X_0 = A$;
 - each $j_\beta : X_\beta \longrightarrow X_{\beta+1}$ is a pushout of a morphism in \mathcal{I} ; and
 - the composition $i_0 : X_0 \longrightarrow \operatorname{colim}_{\beta < \lambda} X_\beta$ of the λ -sequence is isomorphic to f .

The Small Object Argument

The Small Object Argument is originally due to Quillen [Qui67, Lemma II.3.3]. It is a method allowing to produce functorial factorizations of maps, needed to establish model category structures.

Theorem 2.2.6. (The Small Object Argument). *Let \mathcal{C} be a cocomplete model category. Suppose that $\mathcal{I} \subseteq \mathcal{C}$ is a set of maps such that the source of every morphism in \mathcal{I} is small with respect to $\mathcal{I}\text{-cell}$. Then there is a functor*

$$(i, p) : \operatorname{Mor} \mathcal{C} \longrightarrow \mathcal{I}\text{-cell} \times \mathcal{I}\text{-inj}$$

$$f \mapsto (i(f), p(f)),$$

such that $f = p(f) \circ i(f)$, for all $f \in \operatorname{Mor} \mathcal{C}$.

The SOA permits to factorize maps into factors that have appropriate lifting properties, and it can be applied when the domains of a set of maps are small with respect to the class of pushouts of those maps.

Definition 2.2.7. A model category $\mathcal{C}(WE, Fib, Cof)$ is **cofibrantly generated** if \mathcal{C} is bicomplete and if there exist two classes of morphisms $\mathcal{I}, \mathcal{J} \subseteq Mor \mathcal{C}$, such that

1. the source of morphisms in \mathcal{I} is small with respect to \mathcal{I} -cell;
2. the source of morphisms in \mathcal{J} is small with respect to \mathcal{J} -cell;
3. $Fib := RLP(\mathcal{J}) = \mathcal{J} -inj$ and $Fib \cap WE := RLP(\mathcal{I}) = \mathcal{I} -inj$.

The class \mathcal{I} is then referred to as the class of **generating cofibrations** in \mathcal{C} , and the class \mathcal{J} as the class of **generating acyclic cofibrations** in \mathcal{C} .

Remark 2.2.8. Note that part (3) of the previous definition implies that $Cof = \mathcal{I} -cof$ and $Cof \cap WE = \mathcal{J} -cof$.

Cofibrantly generated structures are “well-behaved” in pointed model categories.

Proposition 2.2.9. [Hov99, Lemma 2.1.21] *Suppose \mathcal{C} is a cofibrantly generated model category. Then the model category \mathcal{C}_* is cofibrantly generated.*

We are now aware that in a cofibrantly generated model category (acyclic) fibrations can be detected by checking the RLP against the *set* \mathcal{J} (respectively \mathcal{I}). On the contrary, in a general model category, lifting properties have to be checked against the *whole* classes of (acyclic) cofibrations.

Another advantage of cofibrantly generated model structures will become evident in the sequel (see, for instance, Lemma 3.1.10). At that time, to check required properties for a model (monoidal) category \mathcal{C} , it will be once again sufficient to investigate them only for the generating classes, rather than for all morphisms.

2.2.2 Relevant examples of cofibrantly generated model categories

At first, let us recall some basic concepts from topology. Here I denotes the closed interval $[0, 1]$

Reminder 2.2.10. Let $f : X \longrightarrow Y$ be a map of topological spaces.

- f is called a **weak homotopy equivalence** if for each choice of the basepoint $x \in X$, the induced map $f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is a bijection of pointed sets for $n = 0$, and an isomorphism of groups for $n \geq 1$.

- f is called a **Serre fibration** if for each CW -complex A , it has the RLP with respect to the inclusions $A \times \{0\} \hookrightarrow A \times I$.
- f is called a **Hurewicz fibration** if it has the homotopy lifting property (HLP), i.e., given any commutative square diagram of continuous maps

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow i_0 & \nearrow \hat{H} & \downarrow f \\ A \times I & \xrightarrow{H} & Y, \end{array}$$

there exists a continuous map $\hat{H} : A \times I \rightarrow X$, such that $\hat{H}i_0 = h$ and $p\hat{H} = H$.

- Suppose now that $f : X \hookrightarrow Y$ is an inclusion of a closed subspace. It is called a **Hurewicz closed cofibration** if it has the homotopy extension property (HEP), i.e., given any commutative square diagram of continuous maps

$$\begin{array}{ccc} X & \xrightarrow{K} & W^I \\ \downarrow f & \nearrow \hat{K} & \downarrow p_0 \\ Y & \xrightarrow{k} & W, \end{array}$$

there exists a continuous map $\hat{K} : Y \rightarrow W^I$ such that $p_0\hat{K} = k$ and $\hat{K}f = K$.

Model structures on topological spaces

There exist two model category structures on the category Top ; both of them define the same homotopy theory of topological spaces. The first one was established by Strøm [Str], and is defined as follows.

Definition 2.2.11. Let f be a map in Top , then

- (i) f is a *weak equivalence* if f is a homotopy equivalence;
- (ii) f is a *fibration* if f is a Hurewicz fibration;
- (iii) f is a *cofibration* if f is a closed Hurewicz cofibration.

The second model structure on Top is due to Quillen [Qui67, II.3].

Definition 2.2.12. Let f be a map in Top , then

- (i) f is a *weak equivalence* if f is a weak homotopy equivalence;
- (ii) f is a *fibration* if f is a Serre fibration;

(iii) f is a *cofibration* if f is a retract of an inclusion $A \hookrightarrow X$, such that X is obtained from A by attaching cells.

With respect to this structure, all objects in Top are fibrant, and the cofibrant objects are the generalized CW -complexes (retracts of spaces build up from attaching cells, without the requirement that the cells are attached in order by dimension).

The next proposition involves the second model structure on Top .

Proposition 2.2.13. [Hov99, Theorem 2.4.19] *The category Top is a cofibrantly generated model category, where $\mathcal{I} := \{S^{n-1} \hookrightarrow D^n\}_{n \geq 0}$ is the set of generating cofibrations and $\mathcal{J} := \{D^n \hookrightarrow D^n \times I\}_{n \geq 0}$ is the set of generating acyclic cofibrations. By convention, $S^{-1} := \emptyset$.*

Model structure on simplicial sets

The model category structure on $sSet$ is established in [Qui67, II.3].

Definition 2.2.14. For $f : X \rightarrow Y$ a map in $sSet$, we have that

- (i) f is a *weak equivalence* if its geometric realization $|f| : |X| \rightarrow |Y|$ is a weak homotopy equivalence (i.e., f is a weak equivalence in Top with respect to the second model structure);
- (ii) f is a *fibration* if f is a **Kan fibration**. By definition, this means that f has the RLP with respect to maps $\Lambda^i[n] \rightarrow \Delta[n]$ for $n \geq 0$ and $0 \leq i \leq n$, where $\Lambda^i[n]$ is the i -horn of $\Delta[n]$;
- (iii) f is a *cofibration* if $f \in \text{LLP}(Fib \cap WE)$, namely if f is a monomorphism.

Proposition 2.2.15. [Hov99, Theorem 3.6.5] *The category $sSet$ is a cofibrantly generated model category, by letting $\mathcal{I} := \{\partial\Delta[n] \rightarrow \Delta[n]\}$ to be the set of generating cofibrations, and $\mathcal{J} := \{\Lambda^i[n] \rightarrow \Delta[n]\}$ to be the set of generating acyclic cofibrations, for $n \geq 0$, $0 \leq i \leq n$.*

Every object in $sSet$ is cofibrant. Indeed, the initial object in $sSet$ is $*$, and a map from a point in each level to any simplicial set is always a monomorphism, hence a cofibration.

Model structure on pointed simplicial sets

The model structure on $sSet_*$ is induced from that on $sSet$, according to Propositions 2.1.5 and 2.2.9, and given in the following corollary.

Corollary 2.2.16. *The category $sSet_*$ is a cofibrantly generated model category, where the generating set of cofibrations is $\mathcal{I} := \{\partial\Delta[n]_+ \rightarrow \Delta[n]_+\}$ and the generating set of acyclic cofibrations is $\mathcal{J} := \{\Lambda^i[n]_+ \rightarrow \Delta[n]_+\}$ for $n \geq 0$, $0 \leq i \leq n$.*

Model structures on differential graded R -modules

Let us first recall one of many equivalent characterizations of a projective R -module. We say that an R -module P is **projective** if it is a direct summand of a free R -module.

A. The projective model structure on $DGR\text{-}Mod$

The **projective model structure** on $DGR\text{-}Mod$ is the following.

Definition 2.2.17. Define f a map in $DGR\text{-}Mod$ to be

- (i) a *projective weak equivalence* if f is a quasi-isomorphism, i.e., if the induced map in homology $H_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$;
- (ii) a *projective fibration* if for each $n \in \mathbb{Z}$, f_n is an epimorphism;
- (iii) a *projective cofibration* if for each $n \in \mathbb{Z}$, f_n is a monomorphism with a projective R -module as its cokernel.

This structure is cofibrantly generated, as we will see in a moment.

Definition 2.2.18. Let R be a ring and M an R -module. For $n \in \mathbb{Z}$ define the chain complexes $S^n(M)$ and $D^n(M)$ (the analogues of an n -sphere and an n disk in $DGR\text{-}Mod$) by

$$S^n(M)_k = \begin{cases} M, & \text{if } k = n \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad D^n(M)_k = \begin{cases} M, & \text{if } k = n, n-1 \\ 0, & \text{otherwise} \end{cases}$$

Note that the differential in $S^n(M)$ is necessarily trivial in all degrees, and the differential $d : D^n(M)_k \rightarrow D^n(M)_{k-1}$ is identity if $k = n$, and trivial otherwise. There are obvious injections $S^{n-1}(M) \hookrightarrow D^n(M)$ for all $n \in \mathbb{Z}$.

Proposition 2.2.19. [Hov99, Theorem 2.3.11] *The category $DGR\text{-}Mod$ is a cofibrantly generated model category, where the projective model structure is determined by the set of generating acyclic cofibrations $\mathcal{J} := \{0 \hookrightarrow D^n(M)\}_{n \in \mathbb{Z}}$, and the set of generating cofibrations $\mathcal{I} := \{S^{n-1}(M) \hookrightarrow D^n(M)\}_{n \in \mathbb{Z}}$. Here 0 denotes the trivial chain complex.*

Proposition 2.2.19 applies obviously to the particular case $R = \mathbb{Z}$, hence to the category $DG\mathbb{Z}\text{-}Mod$, as well as to its sub-category Ch_+ of non-negatively graded chain complexes of abelian groups.

B. The injective model structure on $DGR\text{-}Mod$

One can define another model structure on the category $DGR\text{-}Mod$, which we use in Chapter 6. It is called the **injective model structure** and is settled in the following way.

Definition 2.2.20. A map f in $DGR\text{-}Mod$ is

- (i) an *injective weak equivalence* if f is a projective weak equivalence;
- (ii) an *injective cofibration* if f_n is a injection (monomorphism) for all $n \in \mathbb{Z}$;
- (iii) an *injective fibration* if f has the RLP with respect to maps which are both weak equivalences and injections. They are given by surjections with fibrant kernel.

Theorem 2.2.21. [Hov99, Theorem 2.3.13] *The injections, injective fibrations, and weak equivalences are part of a cofibrantly generated injective model structure on $DGR\text{-}Mod$.*

Model structure on simplicial abelian groups

Definition 2.2.22. We say that a simplicial abelian group homomorphism f is

- (i) a *weak equivalence* if f is a weak equivalence of underlying simplicial sets (see Definition 2.2.14 (i));
- (ii) a *fibration* if f is a fibration of underlying simplicial sets (see Definition 2.2.14 (ii));
- (iii) a *cofibration* if f has the LLP with respect to all maps which are weak equivalences and fibrations. They are given by level-wise monomorphisms.

Theorem 2.2.23. [GJ, III Theorem 2.8] *With the above definitions, the category sAb of simplicial abelian groups is a closed model category.*

It can be shown [GJ, III.2] that the model structure on the category sAb is cofibrantly generated. The sets of generating cofibrations and acyclic cofibrations are given by images of the corresponding generating sets in $sSet$ under the free abelian group functor $\mathbb{Z} : sSet \rightarrow sAb$. Explicitly, $\mathcal{I} := \{\mathbb{Z}(\partial\Delta[n]) \rightarrow \mathbb{Z}(\Delta[n])\}$, and $\mathcal{J} := \{\mathbb{Z}(\Lambda^i[n]) \rightarrow \mathbb{Z}(\Delta[n])\}$ for $n \geq 0$, $0 \leq i \leq n$.

Remark 2.2.24. The model structure on sAb is induced from $sSet$ via the Quillen pair $\mathbb{Z} : sSet \rightleftarrows sAb : U$. This Quillen pair factors as follows:

$$\begin{array}{ccccc}
 & & \mathbb{Z} & & \\
 & \curvearrowright & & \curvearrowleft & \\
 sSet & \xrightarrow{(\)_+} & sSet_* & \xrightarrow{\tilde{\mathbb{Z}}} & sAb \\
 & \xleftarrow{U} & & \xleftarrow{U} & \\
 & & U & &
 \end{array}$$

2.3 Model structures on the category Sp^Σ

In this section we present different model structures that can be chosen on the category Sp^Σ of simplicial set-valued symmetric spectra, following [HSS]. We discuss the model structures on the categories $Sp^\Sigma(sAb)$ and $Sp^\Sigma(Ch_+)$ in Section 2.4.

Just one step before spectra there are symmetric sequences. They can be endowed with a *level model structure*.

2.3.1 Level model structure on $sSet_*^\Sigma$

There exists a chain of Quillen pairs between the category $sSet$ of simplicial sets and the category Sp^Σ , as displayed below (left adjoints are on top).

$$\begin{array}{ccccccc}
 & & & & F_n & & \\
 & & & & \curvearrowright & & \\
 & & & G_n & \curvearrowright & & \\
 & & & \curvearrowright & & & \\
 sSet & \xleftrightarrow{(-)_+} & sSet_* & \xleftrightarrow{\Sigma_{n+} \wedge -} & sSet_*^{\Sigma_n} & \xleftrightarrow{(-)_n} & sSet_*^\Sigma & \xleftrightarrow{\mathbb{S} \otimes -} & Sp^\Sigma & \text{leaf} \\
 & \xleftarrow{U} & & \xleftarrow{U_1} & & \xleftarrow{ev_n} & & \xleftarrow{U} & & \\
 & & & & \curvearrowleft & & & & & \\
 & & & & Ev_n & & & & &
 \end{array}$$

The first adjunction is already familiar to us (see passage above Proposition 2.1.5). The two middle adjoint pairs were described in the section devoted to $H\mathbb{Z}$ -module spectra on Sp^Σ in Chapter 1.

For each $n \geq 0$, the *evaluation functor* $Ev_n : sSet_*^\Sigma \rightarrow sSet_*$ is given by $Ev_n(X) = X_n$ and $Ev_f = f_n$. Its left adjoint, the *free functor* $G_n : sSet_* \rightarrow sSet_*^\Sigma$, is such that for a pointed simplicial set K , we have $(G_n K)_n = (\Sigma_n)_+ \wedge K$, and $(G_n K)_k = *$ for $k \neq n$. In particular, note that $G_0 K = (K, *, *, \dots)$.

Finally, the functor $\mathbb{S} \otimes - : sSet_*^\Sigma \rightarrow Sp^\Sigma$ gives the *free \mathbb{S} -module* $\mathbb{S} \otimes X$ generated by the symmetric sequence X . Its right adjoint is the forgetful functor $U : Sp^\Sigma \rightarrow sSet_*^\Sigma$, which forgets the \mathbb{S} -action. The composite functor $F_n = \mathbb{S} \otimes G_n$ is called the *free functor*.

Next proposition suggests the definition of a level model structure on the category $sSet_*^\Sigma$.

Proposition 2.3.1. [HSS, Proposition 2.1.9] *Let g be a map of pointed simplicial sets.*

- *If g is a monomorphism, then $G_n(g)$ is a monomorphism for $n \geq 0$;*

- If g is a weak equivalence, then $G_n(g)$ is a weak equivalence for $n \geq 0$.

Definition 2.3.2. A map $f : X \rightarrow Y$ in $sSet_*^\Sigma$ is defined to be

- (i) a *weak equivalence* if $f_n : X_n \rightarrow Y_n$ is a weak equivalence of underlying simplicial sets for each $n \geq 0$;
- (ii) a *fibration* if $f_n : X_n \rightarrow Y_n$ is a fibration of underlying simplicial sets for each $n \geq 0$;
- (iii) a *cofibration* if f has the LLP with respect to all maps which are weak equivalences and fibrations. This amounts for f to being a level-wise monomorphism.

Weak equivalences and fibrations in $sSet_*^\Sigma$ are called respectively **level equivalences** and **level fibrations**, since they are defined level-wise.

Remark 2.3.3. In fact, given an arbitrary model category \mathcal{C} , the category \mathcal{C}^Σ of symmetric sequences over \mathcal{C} can be given a **level model structure** in a similar way. The classes of weak equivalences and fibrations in \mathcal{C}^Σ are defined to be the level-wise classes of underlying weak equivalences and fibrations in \mathcal{C} . The definition of cofibrations follows naturally by the LLP. Moreover, the category \mathcal{C}^Σ is cofibrantly generated if \mathcal{C} is, and has all small objects if \mathcal{C} does.

One also has the following proposition:

Proposition 2.3.4. [HSS, Proposition 2.2.6] *Let f be a map of pointed simplicial sets.*

- *The functor $F_n : sSet_* \rightarrow Sp^\Sigma$ preserves colimits.*
- *If f is a monomorphism, then $F_n(f)$ is a monomorphism.*
- *If f is a weak equivalence, then $F_n(f)$ is a level equivalence.*

2.3.2 The projective level structure on Sp^Σ

The **projective level structure** on Sp^Σ is given by the classes of maps defined below.

Definition 2.3.5. Let $f : X \rightarrow Y$ be a map of symmetric spectra.

- (i) The map f is a *level equivalence* if each map $f_n : X_n \rightarrow Y_n$ is a weak equivalence of simplicial sets.
- (ii) The map f is a *level fibration* if each map $f_n : X_n \rightarrow Y_n$ is a fibration of simplicial sets.
- (iii) The map f is a *level cofibration* if f has the LLP with respect to every level acyclic fibration.

Theorem 2.3.6. [HSS Theorem 5.1.2] *The projective level structure defines a model structure on the category of symmetric spectra Sp^Σ .*

2.3.3 The projective stable model structure on Sp^Σ

One of the subtlety of the theory of *symmetric* spectra is that one cannot define a map f to be a stable equivalence if it induces isomorphisms $\pi_* f$ on the homotopy groups of the spectrum, i.e., on stable homotopy groups of its spaces. Look at [HSS, Example 3.1.10] to see why it does not work. Instead, the class of stable equivalences on the category Sp^Σ is defined using *cohomology theories*. Part B of the Appendix offers an outline of cohomology theories, giving the main definitions and results on their representability.

Some prior notions are required before we can understand the definition of the stable equivalences in Sp^Σ .

Definitions 2.3.7.

- A simplicial set X is a **Kan complex** if every map $\alpha : \Lambda^i[n] \rightarrow X$ may be extended to a map $\hat{\alpha}$ defined on $\Delta[n]$, so that the diagram below commutes.

$$\begin{array}{ccc} \Lambda^i[n] & \xrightarrow{\alpha} & X \\ \downarrow i & \nearrow \hat{\alpha} & \\ \Delta[n] & & \end{array}$$

- An **Ω -spectrum** is a spectrum $X \in Sp^\Sigma$ such that for each $n \geq 0$ the underlying simplicial set X_n is a Kan complex, and the adjoint of the structure map $S^1 \wedge X_n \rightarrow X_{n+1}$ is a weak equivalence of simplicial sets.
- A symmetric spectrum E is **injective** if for every diagram in Sp^Σ

$$\begin{array}{ccc} X & \xrightarrow{g} & E \\ \downarrow f & \nearrow h & \\ Y & & \end{array}$$

where f is a monomorphism and a level equivalence, there is a map $h : Y \rightarrow E$ making this diagram commute.

- For $E \in Sp^\Sigma$, define E^0 to be the contravariant functor given by

$$E^0 := \pi_0(\text{Map}_{Sp^\Sigma}(-, E)) : Sp^\Sigma \rightarrow \text{Set},$$

where the simplicial set $\text{Map}_{Sp^\Sigma}(-, -)$ was defined in 1.2.14.

Here comes the definition of the **stable model structure** on Sp^Σ .

Definition 2.3.8. Let $f : X \rightarrow Y$ be a map of symmetric spectra.

- (i) The map f is a *stable equivalence* if the induced map in cohomology $E^0(f) : E^0(Y) \longrightarrow E^0(X)$ is an isomorphism for every injective Ω -spectrum E .
- (ii) The map f is a *stable cofibration* if it has the LLP with respect to every level acyclic fibration.
- (iii) The map f is a *stable fibration* if it has the RLP with respect to every map that is a stable acyclic cofibration.

Theorem 2.3.9. [HSS, Theorem 3.4.4] *The category of symmetric spectra Sp^Σ with the class of stable equivalences, the class of stable cofibrations, and the class of stable fibrations is a model category.*

The stable model structure is **cofibrantly generated**. It is shown in [HSS, Sections 3.3, 3.4] that the sets of generating (acyclic) cofibrations in Sp^Σ are obtained via the chain of adjunctions \mathcal{L} , applying functors F_n for $n \geq 0$ to the sets of generating (acyclic) cofibrations in $sSet$.

In particular, let the set $I_\partial := \{\partial\Delta[r]_+ \longrightarrow \Delta[r]_+\}_{r \geq 0}$. The set of **generating cofibrations in Sp^Σ** is by definition

$$\mathcal{I} := FI_\partial = \cup_{n \geq 0} F_n(I_\partial).$$

Let also $I_\Lambda := \{\Lambda^k[r]_+ \longrightarrow \Delta[r]_+\}_{r > 0, 0 \leq k \leq r}$, and define $FI_\Lambda := \cup_{n \geq 0} F_n(I_\Lambda)$. The set of **generating acyclic cofibrations in Sp^Σ** is then given by

$$\mathcal{J} := FI_\Lambda \cup K.$$

We preferred to not give the definition of maps in K here, for the sake of simplicity, and since we will never need to use their explicit characterization within this report. One finds the description of the set K in Definition 3.4.9 of [HSS].

Remark 2.3.10. The condition (i) in Definition 2.3.8 is in fact equivalent to the following:

- (i') The map f is a *stable equivalence* if the induced map in cohomology $E^*(f) : E^*(Y) \longrightarrow E^*(X)$ is an isomorphism for every generalized cohomology theory E , assuming that the Ω -spectrum E is injective.

The implication (i') \implies (i) is obvious. To see the other direction, let E^* be a cohomology theory, represented by an injective Ω -spectrum $E \in Sp^\Sigma$ with structure maps $\sigma_n : \Sigma E_n \longrightarrow E_{n+1}$. The collection E' of symmetric sequences defined by $E'_n := \Sigma E_n$ for all $n \geq 0$ is also an injective Ω -spectrum (by restriction of E) with structure maps $\Sigma(\sigma_n) : \Sigma E'_n \longrightarrow E'_{n+1}$, $n \geq 0$. By hypothesis a map of spectra $f : X \longrightarrow Y$ is a stable equivalence if the map $E^0(f) : E^0(Y) \longrightarrow E^0(X)$ is an isomorphism for every injective Ω -spectrum E . In particular, this holds for the spectrum $E' = \Sigma E$, hence the map

$E'_0(f) := E^1(f) : E^1(Y) \longrightarrow E^1(X)$ is an isomorphism. The same argument works for all $n \geq 0$.

One finds a further discussion about the functor $E^0(-)$ in Appendix B.

The following property is very useful, as we will see in a moment.

Proposition 2.3.11. [HSS], Lemma 3.4.5 *A map f of symmetric spectra is a stable acyclic fibration if and only if it is a level acyclic fibration.*

Corollary 2.3.12. *The functor $F_n : sSet_* \longrightarrow Sp^\Sigma$ preserves cofibrations for all $n \geq 0$.*

Proof. Consider for all $n \geq 0$ the following adjunction, coming from \mathcal{B} ,

$$F_n : sSet_* \rightleftarrows Sp^\Sigma : Ev_n,$$

and let $f : A \twoheadrightarrow B$ be a cofibration in $sSet_*$. For F_n to preserve cofibrations, we need to show that there exists a lift in any commutative diagram in Sp^Σ of the form

$$\begin{array}{ccc} F_n(A) & \xrightarrow{h} & X \\ F_n(f) \downarrow & & \sim \downarrow g \\ F_n(B) & \xrightarrow{k} & Y, \end{array}$$

where g is an acyclic fibration of spectra. To do this, use the adjoint diagram in $sSet_*$

$$\begin{array}{ccc} A & \xrightarrow{h^\sharp} & Ev_n(X) \\ f \downarrow & \nearrow \xi^\sharp & \sim \downarrow Ev_n(g) \\ B & \xrightarrow{k^\sharp} & Ev_n(Y). \end{array}$$

Proposition 2.3.11 implies that for all $n \geq 0$ the map $Ev_n(g) : X_n \longrightarrow Y_n$ is an acyclic fibration of pointed simplicial sets. Since f is a cofibration, there exists a lift ξ^\sharp that gives the desired lift ξ in the original diagram. \square

Corollary 2.3.13. *The sphere spectrum \mathbb{S} is cofibrant in Sp^Σ .*

Proof. Since $G_0(S^0) = (S^0, *, *, \dots)$ and $F_0(S^0) := \mathbb{S} \otimes G_0(S^0)$, one calculates that $F_0(S^0) = \mathbb{S}$. Furthermore, S^0 is cofibrant in $sSet_*$, i.e., the map $* \longrightarrow S^0$ is a cofibration in $sSet_*$. Since F_0 preserves colimits, it preserves the initial objects, too, and $F_0(*)$ is the initial spectrum, given by $*$ at each level. By Corollary 2.3.12, the map $F_0(*) \longrightarrow \mathbb{S}$ is then a cofibration in Sp^Σ . \square

2.4 Comments on the model structures on $Sp^\Sigma(sAb)$ and $Sp^\Sigma(Ch_+)$

The categories of spectra $Sp^\Sigma(sAb)$ and $Sp^\Sigma(Ch_+)$ can be equipped with projective level and projective stable model structures, as well. To find out how, we invite the reader to look in [Hov01], Sections 3 and 7, where the method is explained in general for $Sp^\Sigma(\mathcal{C})$. Here the category \mathcal{C} has to be model with specific properties, which are verified by sAb and Ch_+ , according to [Shi].

In [Hov01], the projective level structure on $Sp^\Sigma(\mathcal{C})$ is defined exactly in the same way as for Sp^Σ . On the other hand, the projective *stable* structure on $Sp^\Sigma(\mathcal{C})$ is constructed as the *left Bousfield localization* of the corresponding projective level structure. One can learn a lot about Bousfield localization in [Hir], or read a good summary in [Hov01, Section 3]. Hovey points out that his construction gives a different construction of the stable model structure on Sp^Σ than the one in [HSS].

Guideline

We have just learned that certain categories can be equipped with a model structure, which allows to define a homotopy theory on them. In fact, all of the four related (monoidal) categories in

$$H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \leftarrow \cdots \leftarrow Sp^\Sigma(sAb) \leftarrow \cdots \leftarrow Sp^\Sigma(Ch_+) \leftarrow \cdots \leftarrow DG\mathbb{Z}\text{-Mod}$$

carry model structures, as we will see in Chapter 6.

These categories will be put in correspondence via a zig-zag of adjunctions. For this zig-zag to be coherent, two types of conditions must be satisfied. First, there exist some particular constraints on the categories themselves, and secondly there are requirements on the nature of the adjoint functor pairs. The next chapter explains the conditions on the categories, which require compatibility between the model and the monoidal structure.

Chapter 3

Monoidal model categories

So far, we have seen some examples of categories endowed with two different structures. They carried a closed symmetric monoidal product, encoding an algebraic structure, and were also given a model structure that made possible the definition of a homotopy theory on these categories. If one wants to take full advantage of both structures, they must be compatible in a certain way. This interaction leads to the definition of a monoidal model category.

3.1 What is a monoidal model category?

3.1.1 Definitions and axioms

Definition 3.1.1. A model category $(\mathcal{C}, WE, Fib, Cof)$ is called a **monoidal model category** if it is endowed with a closed symmetric monoidal structure $(\mathcal{C}, \otimes, \mathbb{I})$ and satisfies the following two axioms.

Push-out product axiom (PP): Let $f : A \rightarrow B$ and $g : K \rightarrow L$ be cofibrations in \mathcal{C} . Then the map

$$f \square g : A \otimes L \coprod_{A \otimes K} B \otimes K \longrightarrow B \otimes L$$

induced by the pushout diagram

$$\begin{array}{ccc}
 A \otimes K & \xrightarrow{f \otimes \text{Id}} & B \otimes K \\
 \text{Id} \otimes g \downarrow & & \downarrow j_0 \\
 A \otimes L & \xrightarrow{j_1} & A \otimes L \sqcup_{A \otimes K} B \otimes K \\
 & & \text{---} f \square g \text{---} \\
 & & \downarrow \\
 & & B \otimes L
 \end{array}$$

$f \otimes \text{Id}$ (curved arrow from $A \otimes L$ to $B \otimes L$)
 $\text{Id} \otimes g$ (curved arrow from $B \otimes K$ to $B \otimes L$)

is also a cofibration. If in addition one of the maps f or g is a weak equivalence, then so is the map $f \square g$.

Unit axiom (U): Let $Q\mathbb{I} \xrightarrow[\simeq]{q} \mathbb{I}$ be a cofibrant replacement for the unit.

Then the natural map $Q\mathbb{I} \otimes X \xrightarrow{q \otimes \text{Id}} \mathbb{I} \otimes X$ is a weak equivalence for all cofibrant X in \mathcal{C} .

Axioms (PP) and (U) encode the desired compatibility between the model and the monoidal structures on \mathcal{C} .

Remarks 3.1.2.

- Observe that if the unit of the monoidal product in \mathcal{C} is cofibrant, the unit axiom holds automatically. In Propositions 6.1.2, 6.1.7, 6.1.6 and 6.1.4 we show that the four categories we are interested in ($H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$, $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ and $DG\mathbb{Z}\text{-Mod}$) all have cofibrant units. It follows that the model-monoidal compatibility conditions for these categories reduce to checking the (PP) axiom.
- The (PP) axiom is important, because it guarantees that for cofibrant objects in \mathcal{C} the monoidal product is an invariant of the weak equivalence type. To see this, suppose that A is cofibrant and $f : X \xrightarrow{\simeq} Y$ is an acyclic cofibration between cofibrant objects. In the pushout diagram

$$\begin{array}{ccc}
 \Phi \cong \Phi \otimes X & \xrightarrow{\cong} & \Phi \otimes Y \cong \Phi \\
 \downarrow & & \downarrow \\
 A \otimes X & \xrightarrow{\cong} & A \otimes X \\
 & \searrow \text{Id} \otimes f & \nearrow \text{dashed} \\
 & & A \otimes Y
 \end{array}$$

the map $\text{Id} \otimes f : A \otimes X \xrightarrow{\simeq} A \otimes Y$ is then an acyclic cofibration by (PP). Hence the functor $F := A \otimes - : \mathcal{C} \rightarrow \mathcal{C}$ satisfies the hypotheses of Ken Brown's Lemma 2.1.7, and therefore preserves all weak equivalences between cofibrant objects. Thus for A cofibrant and a weak equivalence $f : X \xrightarrow{\simeq} Y$, where X, Y are cofibrant, the map $\text{Id} \otimes f : A \otimes X \xrightarrow{\simeq} A \otimes Y$ is a weak equivalence.

This property comes into play when one wants to construct a well-defined (i.e., independent of the homotopy type) monoidal product on the homotopy category $\text{Ho } \mathcal{C}$, induced by the monoidal product on \mathcal{C} .

Proposition 3.1.3. [Hov99, Proposition 4.2.9] *Suppose \mathcal{C} is a monoidal model category, whose unit is the terminal object $*$, and that $*$ is cofibrant. Then \mathcal{C}_* is also a monoidal model category, which is symmetric if \mathcal{C} is.*

Finally, the next theorem states that the homotopy category of a monoidal model category is itself a closed monoidal category.

Theorem 3.1.4. [Hov01, Theorem 4.3.2] *Let \mathcal{C} be a (symmetric) monoidal model category. The homotopy category $\mathrm{Ho}\mathcal{C}$ can be given a natural closed (symmetric) monoidal structure, induced by the monoidal structure of \mathcal{C} .*

3.1.2 Examples of monoidal model categories

The following propositions give us examples of monoidal model categories.

Proposition 3.1.5. [Hov99, Proposition 4.2.8] *The model category $s\mathrm{Set}$ of simplicial sets, with the model structure given in Definition 2.2.14, forms a symmetric monoidal model category. Its monoidal structure was given in Examples 1.1.6.*

Applying Proposition 3.1.3 to $s\mathrm{Set}$ implies a similar statement for pointed simplicial sets.

Proposition 3.1.6. [Hov99, Corollary 4.2.10] *The model category $s\mathrm{Set}_*$ of pointed simplicial sets, with the model structure of Corollary 2.2.16, forms a symmetric monoidal model category. See Examples 1.1.6 for the description of the monoidal structure.*

Proposition 3.1.7. [Hov99, Proposition 4.2.13] *Let R be a commutative ring. Then the category $DGR\text{-}Mod$ of unbounded chain complexes of R -modules, given the projective model structure of Definition 2.2.17, is a symmetric monoidal model category. The monoidal structure on $DGR\text{-}Mod$ was discussed in Examples 1.1.6.*

According to Hovey, Shipley and Smith, both the projective and stable model structures on the category Sp^Σ of pointed simplicial set-valued symmetric spectra are compatible with the symmetric monoidal structure described in Section 1.2.2. This compatibility is established in [HSS] as Corollary 5.3.8.

3.1.3 The monoid axiom

The monoid axiom is a crucial ingredient for the purposes of Section 3.2, where we will focus on the conditions required to extend a model structure on a model monoidal category \mathcal{C} to model structures on the associated categories of monoids and modules.

Definition 3.1.8. Given \mathcal{C} , a monoidal model category, we say that \mathcal{C} satisfies the **monoid axiom (MA)** if every map obtained by pushouts and (possibly transfinite) compositions of maps of the form

$$f \otimes \mathrm{Id} : A \otimes Z \longrightarrow B \otimes Z,$$

where $f : A \xrightarrow{\sim} B$ is an acyclic cofibration and Z is any object of \mathcal{C} , is a weak equivalence.

Notation 3.1.9. For a class of maps \mathcal{K} in \mathcal{C} , use the notation

$$\mathcal{K} \otimes Z := \{f \otimes \text{Id} \mid f \in \mathcal{K}, Z \in \mathcal{C}\}.$$

Now, take \mathcal{K} to be the class of acyclic cofibrations in \mathcal{C} , and recall the notation \mathcal{K} -cell from Definition 2.2.5. The monoid axiom requests the maps in $(\mathcal{K} \otimes Z)$ -cell to be weak equivalences.

We have already mentioned some advantages of having a *cofibrantly generated* model category in Section 2.2.1. At present, we can appreciate another of its benefits. For \mathcal{C} cofibrantly generated, if (PP) and (MA) axioms hold only for *a set* of generating (acyclic) cofibrations, this implies that they hold for *all* maps in \mathcal{C} , as the following lemma states.

Lemma 3.1.10. [SS00, Lemma 3.5] *Let \mathcal{C} be a cofibrantly generated model category endowed with a closed symmetric monoidal structure.*

1. *If (PP) holds for a set of generating cofibrations and a set of generating acyclic cofibrations, then it holds in general.*
2. *Let \mathcal{J} be a set of generating acyclic cofibrations. If every map in $(\mathcal{J} \otimes Z)$ -cell is a weak equivalence, then (MA) holds.*

3.2 Induced model structures on modules and algebras

Question: Given \mathcal{C} , a cofibrantly generated model monoidal category, under what conditions do there exist induced model structures on the associated categories of monoids, modules and algebras, and how are they defined?

Schwede and Shipley in [SS00] establish the necessary conditions to be satisfied for \mathcal{C} in order to define the induced model structures on the categories $Mon_{\mathcal{C}}$, $R\text{-Mod}_{\mathcal{C}}$ and $R\text{-Alg}_{\mathcal{C}}$ of monoids, R -modules and R -algebras on \mathcal{C} , for a (commutative) monoid R . Their approach is in fact more general, since they show how to induce the model structure on the category $T\text{-alg}_{\mathcal{C}}$ of algebras over a monad T in \mathcal{C} (Proposition 3.2.6). The general procedure is then applied to construct the model structure for modules and algebras (Theorem 3.2.7). We present the results and explain the strategy used.

3.2.1 Preliminary notions and results

Throughout this section \mathcal{C} denotes a model monoidal category. Let us first give the definition of a monad in \mathcal{C} . Recall the monoidal category $(\mathcal{C}^{\mathcal{C}}, \circ, \text{Id})$ from Examples 1.1.6.

Definition 3.2.1. A **monad** T in \mathcal{C} is a monoid in $(\mathcal{C}^{\mathcal{C}}, \circ, \text{Id})$, i.e., a triple (T, μ, η) where $T : \mathcal{C} \rightarrow \mathcal{C}$ is a functor and $\mu : T \circ T \rightarrow T$, $\eta : \text{Id} \rightarrow T$ are natural transformations, making the following diagrams commute:

1. *Associativity coherence:*

$$\begin{array}{ccc} T \circ T \circ T & \xrightarrow{\mu \circ \text{Id}} & T \circ T \\ \text{Id} \circ \mu \downarrow & & \downarrow \mu \\ T \circ T & \xrightarrow{\mu} & T \end{array}$$

2. *Left and right unit coherence:*

$$\begin{array}{ccccc} T \circ \text{Id} & \xrightarrow{\text{Id} \circ \eta} & T \circ T & \xleftarrow{\eta \circ \text{Id}} & \text{Id} \circ T \\ & \searrow & \downarrow \mu & \swarrow & \\ & & T & & \end{array}$$

Lemma 3.2.2. [McL, VI.1] *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction, and denote*

$$\Phi(A, B) : \mathcal{D}(FA, B) \cong \mathcal{C}(A, GB)$$

the associated isomorphism, natural in A, B . The functor $GF : \mathcal{C} \rightarrow \mathcal{C}$ defines a monad on \mathcal{C} , where $\eta : \text{Id} \rightarrow GF$ is given by the unit of the adjunction, defined on objects by

$$\eta_A = \Phi(A, FA)(\text{Id}_{FA}) : A \rightarrow GF(A).$$

The multiplication μ is given on objects by

$$\mu_A := G\varepsilon F(A) : GF \circ GF(A) \rightarrow GF(A),$$

where $\varepsilon : FG \rightarrow \text{Id}$ is the counit, given by $\Phi(GB, B)^{-1}(\text{Id}_{GB}) : FG(B) \rightarrow B$ for all $B \in \mathcal{D}$.

Examples 3.2.3.

1. Let R be a monoid in a monoidal category \mathcal{C} and consider the adjunction

$$R \otimes - : \mathcal{C} \rightleftarrows \mathbf{R}\text{-Mod}_{\mathcal{C}} : U.$$

The associated monad is $T_R := U(R \otimes -) : \mathcal{C} \rightarrow \mathcal{C}$ sending

$$M \mapsto U(R \otimes M)$$

$$(f : M \rightarrow N) \mapsto U(\text{Id}_R \otimes f) : U(R \otimes M) \rightarrow U(R \otimes N).$$

In other words, for an object M , T_R returns $R \otimes M \in \mathcal{C}$ forgetting its R -module structure. For all M in \mathcal{C} , the multiplication is induced by the monoid structure on R :

$$\begin{aligned} \mu_{T_R} : U(R \otimes U(R \otimes M)) &\longrightarrow U(R \otimes M) \\ r' \otimes (r \otimes m) &\mapsto [(r' \otimes r) \otimes m \xrightarrow{m \otimes \text{Id}} r'' \otimes m], \end{aligned}$$

and the unit is

$$\eta_{T_R} : M \longrightarrow U(R \otimes M) : m \mapsto 1_R \otimes m.$$

2. The second example is the free-forgetful adjunction

$$T : \mathcal{C} \rightleftarrows \text{Mon}_{\mathcal{C}} : U,$$

where for all $X \in \mathcal{C}$, $T(X)$ gives the **free-algebra over X** , defined by

$$T(X) := \coprod \coprod X \coprod (X \otimes X) \coprod \dots \coprod X^{\otimes n} \coprod \dots,$$

with \coprod the coproduct in \mathcal{C} . Using the monoidal product \otimes on \mathcal{C} , define $T(X) \otimes T(X) \in \text{Ob } \mathcal{C}$ by

$$(T(X) \otimes T(X))_n := \coprod_{p+q=n} T_p(X) \otimes T_q(X),$$

where $T_p(X) = \coprod \coprod X \coprod (X \otimes X) \coprod \dots \coprod X^{\otimes p}$ for all $p \geq 0$. Notice that $T(X)$ can be equipped with a **monoid structure** such that the unit $\eta : \mathbb{1} \longrightarrow T(X)$ is the inclusion, and the multiplication $\mu : T(X) \otimes T(X) \longrightarrow T(X)$ is given by the collection of maps for $p + q = n$

$$\begin{aligned} \mu_{p,q} : T(X)_p \otimes T(X)_q &\longrightarrow T(X)_{p+q} \\ (x_1 \otimes \dots \otimes x_p) \otimes (x'_1 \otimes \dots \otimes x'_q) &\mapsto x_1 \otimes \dots \otimes x_p \otimes x'_1 \otimes \dots \otimes x'_q, \end{aligned}$$

where we used concatenation of terms. This adjunction defines the **free monoid monad**

$$U(T) : \mathcal{C} \longrightarrow \mathcal{C} : K \mapsto U(T(K)).$$

Definition 3.2.4. Let (T, μ, η) be a monad on a category \mathcal{C} . A **T -algebra** X is a pair (X, ξ) , where $X \in \mathcal{C}$ and $\xi : T(X) \longrightarrow X \in \text{Mor}(\mathcal{C})$ such that the two diagrams

$$\begin{array}{ccc} X & \xrightarrow{\epsilon_X} & UT(X) \\ & \searrow & \downarrow U(\xi) \\ & & U(X) \cong X \end{array} \qquad \begin{array}{ccc} T \circ T(X) & \xrightarrow{\mu_X} & T(X) \\ T(\xi) \downarrow & & \downarrow \xi \\ T(X) & \xrightarrow{\xi} & X \end{array}$$

commute.

If (Y, ζ) is another T -algebra, a **morphism** $f : (X, \xi) \longrightarrow (Y, \zeta)$ of T -algebras is a morphism $f_T : X \longrightarrow Y$ in \mathcal{C} , such that the diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{T(f)} & T(Y) \\ \xi \downarrow & & \downarrow \zeta \\ X & \xrightarrow{f} & Y. \end{array}$$

commutes. Composition of morphisms and identities follow from those in \mathcal{C} , and the category of T -algebras on \mathcal{C} , will be denoted $T\text{-alg}_{\mathcal{C}}$.

If \mathcal{C} admits a model structure, it is possible to define a **model structure** on the category $T\text{-alg}_{\mathcal{C}}$.

Definition 3.2.5. Let T be a monad in \mathcal{C} . A morphism f of T -algebras is

- (i) a *weak equivalence* if f is a weak equivalence in \mathcal{C} ;
- (ii) a *fibration* if f is a fibration in \mathcal{C} ;
- (iii) a *cofibration* if $f \in \text{LLP}(\text{Cof} \cap \text{WE})_{T\text{-alg}_{\mathcal{C}}}$.

However, perhaps not all model axioms are satisfied by $T\text{-alg}_{\mathcal{C}}$ in general then.

Now, choose a monad T in \mathcal{C} and consider the adjunction

$$F : \mathcal{C} \rightleftarrows T\text{-alg}_{\mathcal{C}} : U, \quad (\diamond)$$

where F sends

$$\begin{aligned} X &\mapsto F(X) := (T(X), \mu_X), \\ (f : X \longrightarrow Y) &\mapsto F(f) := (T(X), \mu_X) \longrightarrow (T(Y), \mu_Y), \end{aligned}$$

and U is the forgetful functor given by $(X, \xi) \mapsto X$; $f_T \mapsto f$.

The following key proposition gives two different situations in which one can lift a model category structure from \mathcal{C} to the one on $T\text{-alg}_{\mathcal{C}}$.

Proposition 3.2.6. [SS00, Lemma 2.3] *Let \mathcal{C} be a cofibrantly generated model category, and let $T \in \mathcal{C}^{\mathcal{C}}$ be a monad in \mathcal{C} . Consider the adjunction $F : \mathcal{C} \rightleftarrows T\text{-alg}_{\mathcal{C}} : U$ as in (\diamond) . Let \mathcal{I} be a set of generating cofibrations and \mathcal{J} be the set of generating acyclic cofibrations for \mathcal{C} . Denote by \mathcal{I}_T and \mathcal{J}_T the images of \mathcal{I} (respectively \mathcal{J}) by F . Assume that:*

1. *The functor T commutes with filtered direct limits;*
2. *The domains of maps in \mathcal{I}_T and \mathcal{J}_T are small with respect to \mathcal{I}_T -cell, respectively \mathcal{J}_T -cell;*

3. (1) Every map f in \mathcal{J}_T -cell is a weak equivalence; or
- (2) Every object of \mathcal{C} is fibrant and every T -algebra has a path object.

Then the category $T\text{-alg}_{\mathcal{C}}$ is a cofibrantly generated model category with \mathcal{I}_T a generating set of cofibrations and \mathcal{J}_T a generating set of acyclic cofibrations.

Let us highlight some key facts in the proof. The assumption on T commuting with filtered colimits is necessary to ensure the existence of colimits in $T\text{-alg}_{\mathcal{C}}$ (see [Bor, II 4.3.6]). The model axioms (MC1), (MC2), (MC3) for $T\text{-alg}_{\mathcal{C}}$ follow more or less directly from assumptions; the axiom (MC4)(a) holds by the definition of cofibrations in $T\text{-alg}_{\mathcal{C}}$. The definition of the model structure on $T\text{-alg}_{\mathcal{C}}$ (3.2.5) is used to prove the remaining axioms, as well as the small object argument, which applies because of the assumption on the smallness of domains of maps in \mathcal{I}_T and \mathcal{J}_T . Notice that assumptions (1) or (2) are necessary for (MC5)(a) to hold.

3.2.2 The main theorem for modules and algebras

Theorem 3.2.7. [SS00, Theorem 4.1] *Let \mathcal{C} be a cofibrantly generated, monoidal model category. Assume that every object in \mathcal{C} is small relative to the whole category \mathcal{C} , and that \mathcal{C} satisfies the monoid axiom.*

- (I) *Let R be a monoid in \mathcal{C} . Then the category of (left) R -modules $R\text{-Mod}_{\mathcal{C}}$ is a cofibrantly generated model category.*
- (II) *Let R be a monoid in \mathcal{C} and suppose R commutative. Then the category $R\text{-Mod}_{\mathcal{C}}$ is a cofibrantly generated, monoidal model category, satisfying the monoid axiom.*
- (III) *Let R be a monoid in \mathcal{C} and suppose R commutative. Then the category $R\text{-Alg}_{\mathcal{C}}$ is a cofibrantly generated model category. Every cofibration of R -algebras whose source is cofibrant as an R -module is also a cofibration of R -modules. In particular, if the unit of the monoidal product $\mathbb{1}_{\mathcal{C}}$ is cofibrant in \mathcal{C} , then every cofibrant R -algebra is also cofibrant as an R -module.*

This theorem essentially says that monoids, modules and algebras in a cofibrantly generated monoidal model category \mathcal{C} again form a cofibrantly generated model category, if the monoid axiom holds. Next definition describes these model structures.

Definition 3.2.8. Let \mathcal{C} be a model monoidal category and fix a monoid R in \mathcal{C} . In the associated categories of monoids $Mon_{\mathcal{C}}$, (left) R -modules $R\text{-Mod}_{\mathcal{C}}$, and (when R is commutative) R -algebras $R\text{-Alg}_{\mathcal{C}}$, the **model structure** is defined as follows.

- (i) A map f in $Mon_{\mathcal{C}}, R\text{-Mod}_{\mathcal{C}}, R\text{-Alg}_{\mathcal{C}}$ is a weak equivalence if f is a weak equivalence in \mathcal{C} ;

- (ii) A map f in $Mon_{\mathcal{C}}, R\text{-Mod}_{\mathcal{C}}, R\text{-Alg}_{\mathcal{C}}$ is a *fibration* if f is a fibration in \mathcal{C} ;
- (iii) A map f in $Mon_{\mathcal{C}}, R\text{-Mod}_{\mathcal{C}}, R\text{-Alg}_{\mathcal{C}}$ is a *cofibration* if f has the LLP with respect to all acyclic fibrations in the corresponding category.

Remark 3.2.9. If we take R to be the unit \mathbb{I} of the monoidal product in \mathcal{C} in part (III) of the Theorem, we conclude that the category of monoids in \mathcal{C} forms a model category.

General strategy of the proof: Given a model category \mathcal{C} satisfying assumptions of Lemma 3.2.6, we obtain a model structure on the category of T -algebras over a monad T in \mathcal{C} . The particular case of the monad $T := T_R$ (Example 3.2.3(1)) gives the result (I) for R -modules. Showing part (II) completes the proof that the category of R -modules is itself a cofibrantly generated monoidal model category, satisfying the monoid axiom. Since R -algebras are monoids in $R\text{-Mod}$ under the free-algebra monad $T := UT$ (Example 3.2.3(2)), Lemma 3.2.6 can be applied to $\mathcal{D} := R\text{-Mod}_{\mathcal{C}}$ to obtain the result (III) for R -algebras.

Comments: As we said, part (I) is a direct application of Lemma 3.2.6. Indeed, the category of R -modules in \mathcal{C} is the category of T -algebras over the monad $T := T_R$. Since the monoidal product \otimes is assumed to be closed symmetric, this means by definition that the functor $R \otimes -$ is a left adjoint of $\text{Hom}(R, -)$, hence it commutes with all colimits, and in particular with filtered directed limits (i.e., colimits (!) of diagrams indexed by filtered directed sets).

Domains of maps of generating sets \mathcal{I}_T and \mathcal{J}_T are small with respect to all the category \mathcal{C} by hypothesis, hence in particular with respect to $\mathcal{I}_T\text{-cell}$ and $\mathcal{J}_T\text{-cell}$.

If \mathcal{J} is the set of generating acyclic cofibrations in \mathcal{C} , the set \mathcal{J}_T of generating acyclic cofibrations in $R\text{-Mod}$ consists of maps of \mathcal{J} , tensored with R , i.e., $\mathcal{J}_T \otimes_R R\text{-Mod} = \mathcal{J} \otimes \mathcal{C}$. The monoid axiom on \mathcal{C} implies that maps in $\{(\mathcal{J} \otimes Z)\text{-cell}\}$ are weak equivalences in \mathcal{C} for all $Z \in \mathcal{C}$. In particular, it follows that maps in $\{(\mathcal{J}_T \otimes_R R\text{-Mod})\text{-cell}\}$ are weak equivalences in \mathcal{C} , and hence, by Definition 3.2.8, in $R\text{-Mod}$. This allows us to apply Lemma 3.2.6 to the monad T_R and gives (I).

The model category part of (II) is (I), so one only needs to verify the (PP) and (MA) axioms, which are sufficient to be checked for the sets of generating (acyclic) cofibrations (Lemma 3.1.10). See [SS00] for the details of these proofs.

By (II), the category of R -modules satisfies the required hypotheses of Lemma 3.2.6. On the other hand, R -algebras are monoids in $R\text{-Mod}$. In

order to apply the Lemma, we recognize monoids in $\mathbf{R}\text{-Mod}$ as being algebras over the free-algebra monad $T := UT$. The main issue here is to check the condition (1) on regular cofibrations, required by Lemma 3.2.6. This condition is taken care of by Lemma 6.2 in [SS00], where a filtration of a certain pushout in the category of monoids is used to reduce the problem to arguments based on Quillen's Small Object Argument, and to finally show (III). \square

Guideline

The desired Quillen equivalence between $H\mathbb{Z}$ -algebra spectra and differential graded algebras will partially follow from applying Theorem 3.2.7 to the four categories $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$, $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ and $DG\mathbb{Z}\text{-Mod}$. Chapter 6 will provide an overview of arguments showing that these categories satisfy indeed the hypotheses of Theorem 3.2.7. This will imply that the corresponding categories of monoids, which are precisely $H\mathbb{Z}$ -algebra spectra for $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$, and differential graded algebras in the case of $DG\mathbb{Z}\text{-Mod}$, are equipped with induced model structures.

We mentioned in the previous Guideline that, in addition to conditions on categories, a functorial coherence was required for the zig-zag of functors. In general, one needs to be sure that monoidal-model category structures are appropriately transferred via a chain of adjunctions. For this to happen, the involved functors must respect this structure. We call such functors monoidal Quillen functors.

Chapter 4

Monoidal Quillen functors

Question: When does a Quillen equivalence between two monoidal model categories induce Quillen equivalences on the associated categories of monoids, modules and algebras?

The answer is given by Schwede and Shipley in [SS03b], which is a sequel to [SS00]. They give sufficient conditions for extending Quillen equivalences on the associated categories of monoids, modules and algebras. For the purposes of this report we will only need to focus on the induced equivalences of the categories of monoids and modules.

Throughout this chapter let $(\mathcal{C}, \otimes, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$ denote two monoidal model categories. Recall the notions of a Quillen pair (2.1.27) and a Quillen equivalence (2.1.30) from Chapter 2.

4.1 Monoidal Quillen functors

The reader is already familiar with the notion of a strong monoidal functor between monoidal categories from Definition 1.1.20. One can consider monoidal functors with weakened assumptions on the maps φ and ν , no more required to be *isomorphisms*.

Definition 4.1.1. Given monoidal categories \mathcal{C} and \mathcal{D} , a **lax monoidal functor** from \mathcal{C} to \mathcal{D} is a functor $R : \mathcal{C} \longrightarrow \mathcal{D}$, together with a *morphism*

$$\varphi_{X,Y} : R(X) \wedge R(Y) \longrightarrow R(X \otimes Y)$$

natural in X and Y , and a *morphism*

$$\nu : \mathbb{I}_{\mathcal{D}} \longrightarrow R(\mathbb{I}_{\mathcal{C}})$$

such that the coherence diagrams of 1.1.20 commute for all $X, Y, Z \in \mathcal{C}$.

Suppose now to have a lax monoidal functor $R : \mathcal{C} \longrightarrow \mathcal{D}$, equipped with a left adjoint $\lambda : \mathcal{D} \longrightarrow \mathcal{C}$, and denote by ε and η the counit and the unit of the adjunction. In this case, the adjoint map of ν ,

$$\tilde{\nu} : \lambda(\mathbb{I}_{\mathcal{D}}) \longrightarrow \mathbb{I}_{\mathcal{C}},$$

and the natural map

$$\tilde{\varphi} : \lambda(X \wedge Y) \longrightarrow \lambda X \otimes \lambda Y,$$

defined by composite

$$\lambda(X \wedge Y) \xrightarrow{\lambda(\eta_X \wedge \eta_Y)} \lambda(R\lambda X \wedge R\lambda Y) \xrightarrow{\lambda(\varphi_{\lambda X, \lambda Y})} \lambda R(\lambda X \otimes \lambda Y) \xrightarrow{\varepsilon_{\lambda X \otimes \lambda Y}} \lambda X \otimes \lambda Y$$

make the left adjoint λ into a **comonoidal functor**. The map $\tilde{\varphi}$ need not be an isomorphism in general.

Now we define adjoint functors that take care of both the monoidal and model structure on categories.

Definitions 4.1.2.

- A **weak monoidal Quillen pair** between monoidal model categories \mathcal{C} and \mathcal{D} consists of a Quillen pair $\lambda : \mathcal{D} \rightleftarrows \mathcal{C} : R$, with a lax monoidal structure on the right adjoint

$$\varphi_{X,Y} : R(X) \wedge R(Y) \longrightarrow R(X \otimes Y), \quad \nu : \mathbb{I}_{\mathcal{D}} \longrightarrow R(\mathbb{I}_{\mathcal{C}})$$

such that the following two conditions hold:

1. for all cofibrant objects X and Y in \mathcal{D} , the comonoidal map $\tilde{\varphi} : \lambda(X \wedge Y) \longrightarrow \lambda X \otimes \lambda Y$ is a weak equivalence in \mathcal{C} ;
2. for any cofibrant replacement $Q \mathbb{I}_{\mathcal{D}} \xrightarrow{q} \mathbb{I}_{\mathcal{D}}$ of the unit object in \mathcal{D} , the composite map $\lambda(Q \mathbb{I}_{\mathcal{D}}) \longrightarrow \lambda(\mathbb{I}_{\mathcal{D}}) \xrightarrow{\tilde{\nu}} \mathbb{I}_{\mathcal{C}}$ is a weak equivalence in \mathcal{C} .

- A **strong monoidal Quillen pair** is a weak monoidal Quillen pair for which the comonoidal maps $\tilde{\varphi}$ and $\tilde{\nu}$ are isomorphisms.
- A weak (resp. strong) monoidal Quillen pair is a **weak (resp. strong) monoidal Quillen equivalence** if the underlying Quillen pair is a Quillen equivalence.

Remark 4.1.3. Note that if $\mathbb{I}_{\mathcal{D}}$ is cofibrant and λ is strong monoidal, then R is lax monoidal and the Quillen pair (λ, R) is a strong monoidal Quillen pair. Indeed, if $\mathbb{I}_{\mathcal{D}}$ is cofibrant then $\mathbb{I}_{\mathcal{D}} \cong Q \mathbb{I}_{\mathcal{D}}$, and λ strong monoidal implies that $\tilde{\varphi}$ and $\tilde{\nu}$ are isomorphisms, hence in particular weak equivalences.

4.2 Associated lifts and left adjoints

4.2.1 The induced lifts

Let $\lambda : \mathcal{D} \rightleftarrows \mathcal{C} : R$ be a weak monoidal Quillen pair. As any lax monoidal functor, equipped with maps φ and ν , the right adjoint R induces a functor on the category of monoids as follows.

Let a monoid A in \mathcal{C} with multiplication $\mu : A \otimes A \longrightarrow A$ and unit $\eta : \mathbb{I}_{\mathcal{C}} \longrightarrow A$. There exists a monoid-valued lift of R

$$R^{mon} : Mon_{\mathcal{C}} \longrightarrow Mon_{\mathcal{D}}, \quad A \mapsto R(A),$$

such that the **monoid structure** on RA is given by the composite maps

$$RA \wedge RA \xrightarrow{\varphi_{A,A}} R(A \otimes A) \xrightarrow{R(\mu)} RA, \quad \mathbb{I}_{\mathcal{D}} \xrightarrow{\nu} R(\mathbb{I}_{\mathcal{C}}) \xrightarrow{R(\eta)} RA.$$

Similarly, for an A -module M with action $\alpha : A \otimes M \longrightarrow M$, there exists a module-valued lift

$$R^{mod} : A\text{-Mod}_{\mathcal{C}} \longrightarrow RA\text{-Mod}_{\mathcal{D}}, \quad M \mapsto R(M),$$

such that the **module structure** on RA is given via the composite morphism

$$RA \wedge RM \xrightarrow{\varphi_{A,M}} R(A \otimes M) \xrightarrow{R(\alpha)} RM.$$

4.2.2 The left adjoints

Regarding the induced left adjoints, two different situations can occur. If the Quillen pair (λ, R) is strong monoidal (which, by definition, means that $\tilde{\varphi}$ and $\tilde{\nu}$ are isomorphisms), the left adjoint λ becomes itself a **strong monoidal functor** via the inverses of natural maps

$$\tilde{\varphi}^{-1} : \lambda X \otimes \lambda Y \longrightarrow \lambda(X \wedge Y), \quad \tilde{\nu}^{-1} : \mathbb{I}_{\mathcal{C}} \longrightarrow \lambda(\mathbb{I}_{\mathcal{D}}),$$

and lifts to a functor on monoids. This lift is then adjoint to the corresponding lift on monoids induced by R .

In a situation where (λ, R) is only *weak* monoidal, the functor induced by R on monoids will still have a left adjoint. However, this induced adjoint will not usually coincide with a lift of λ .

Notation 4.2.1. We will abuse notation and always denote by R both the original lax monoidal right adjoint functor R and its structured versions on monoids and modules, introduced before as R^{mon} and R^{mod} . The same will apply to the left adjoints λ in a strong monoidal case.

In a weak monoidal case it is therefore necessary to use different symbols for the different induced functors, which we now describe.

The case of monoids

Recall the free-forgetful adjunction

$$T : \mathcal{C} \rightleftarrows \text{Mon}_{\mathcal{C}} : U$$

from Examples 3.2.3, where $T(X)$ is the free algebra over X . We assume that the forgetful functor U creates a model structure as in Definition 3.2.8. The monoid-valued lift of R , $R : \text{Mon}_{\mathcal{C}} \rightarrow \text{Mon}_{\mathcal{D}}$, has a left adjoint

$$L^{mon} : \text{Mon}_{\mathcal{D}} \rightarrow \text{Mon}_{\mathcal{C}}$$

defined for any monoid B in \mathcal{D} to be the coequalizer

$$L^{mon}(B) := \text{colim} \left(T_{\mathcal{C}} \circ \lambda \circ U \circ T_{\mathcal{D}} \circ U(B) \begin{array}{c} \xrightarrow{\xi} \\ \xrightarrow{\psi} \end{array} T_{\mathcal{C}} \circ \lambda \circ U(B) \right).$$

The two involved morphisms of \mathcal{C} -monoids are

$$\xi := (T_{\mathcal{C}} \circ \lambda \circ U)\varepsilon_B \quad \text{and} \quad \psi := T_{\mathcal{C}} \circ \hat{\psi},$$

where $\hat{\psi} : \mathcal{C} \rightarrow \mathcal{C}$ is given by

$$\coprod_{n \geq 0} \lambda(U(B)^{\wedge n}) \xrightarrow{\sqcup \hat{\psi}} \coprod_{n \geq 0} (\lambda U(B))^{\otimes n}.$$

Since R preserves the underlying objects, the monoid-valued left adjoint L^{mon} and the original left adjoint λ are related via a natural isomorphism

$$\tau : L^{mon} \circ T_{\mathcal{D}} \xrightarrow{\cong} T_{\mathcal{C}} \circ \lambda$$

of functors from \mathcal{D} to $\text{Mon}_{\mathcal{C}}$. More precisely, the commutative square of adjunctions (with left adjoints on top and on the right)

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \xrightarrow{T_{\mathcal{C}}} \\ \xleftarrow{U} \end{array} & \text{Mon}_{\mathcal{C}} \\ \begin{array}{c} \uparrow R \\ \downarrow \lambda \end{array} & & \begin{array}{c} \uparrow R^{mon} \\ \downarrow L^{mon} \end{array} \\ \mathcal{D} & \begin{array}{c} \xrightarrow{T_{\mathcal{D}}} \\ \xleftarrow{U} \end{array} & \text{Mon}_{\mathcal{D}} \end{array}$$

shows that the isomorphism $L^{mon} \circ T_{\mathcal{D}} \cong T_{\mathcal{C}} \circ \lambda$ holds if and only if the isomorphism $U \circ R^{mon} \cong R \circ U$ holds. The latter is true, because $R^{mon} = R$ precisely since R preserves the underlying objects, and because U denotes the forgetful functors “of the same type”, only in different categories.

The case of modules

We will not give all the details for this case; they can be worked out similarly, using [SS03b, 3.3].

Recall the other adjunction

$$A \otimes - : \mathcal{C} \rightleftarrows \mathbf{A}\text{-Mod}_{\mathcal{C}} : U.$$

from Examples 3.2.3, where A is any monoid in \mathcal{C} . We assume that the forgetful functor U creates a model structure as in Definition 3.2.8. The module-valued lift of R , $R : \mathbf{A}\text{-Mod}_{\mathcal{C}} \rightarrow \mathbf{R}\mathbf{A}\text{-Mod}_{\mathcal{D}}$, has a left adjoint

$$L^A : \mathbf{R}\mathbf{A}\text{-Mod}_{\mathcal{D}} \rightarrow \mathbf{A}\text{-Mod}_{\mathcal{C}}$$

defined for any $\mathbf{R}\mathbf{A}$ -module M in \mathcal{D} to be the coequalizer

$$L^A(M) := \operatorname{colim} \left((A \otimes -) \circ \lambda \circ U \circ (A \wedge -) \circ U(M) \rightrightarrows (A \otimes -) \circ \lambda \circ U(M) \right).$$

The module-valued left adjoint L^A and the original left adjoint λ are again related, via another natural isomorphism

$$\tau : L^A \circ (R\mathbf{A} \wedge -) \xrightarrow{\cong} (A \otimes -) \circ \lambda$$

of functors from \mathcal{D} to $\mathbf{A}\text{-Mod}_{\mathcal{C}}$, since R preserves the underlying objects.

4.3 The second main theorem: lifting a Quillen equivalence

Definition 4.3.1. Let $(\mathcal{C}, \otimes, \mathbb{I}_{\mathcal{C}})$ be a monoidal model category such that the forgetful functor creates model structures for modules over any monoid in \mathcal{C} . We say that **Quillen invariance holds in \mathcal{C}** if for every weak equivalence of monoids in \mathcal{C} $f : R \xrightarrow{\sim} S$, the adjoint functors of restriction and extension of scalars along f

$$S \otimes_R - : \mathbf{R}\text{-Mod}_{\mathcal{C}} \rightleftarrows \mathbf{S}\text{-Mod}_{\mathcal{C}} : \operatorname{Res}(-)$$

(see Definition 5.1.1 for further details) induce a Quillen equivalence between the respective module categories.

For a sufficient condition for Quillen invariance in \mathcal{C} , see for example [SS00, Theorem 4.3].

Theorem 4.3.2. [SS03b, Theorem 3.12] *Let \mathcal{C} and \mathcal{D} be two monoidal model categories and $R : \mathcal{C} \rightarrow \mathcal{D}$ be the right adjoint of a weak monoidal Quillen equivalence. Suppose that the unit objects in \mathcal{C} and \mathcal{D} are cofibrant.*

1. If the forgetful functor creates model structures for monoids in \mathcal{C} and \mathcal{D} , then the adjoint functor pair

$$L^{\text{mon}} : \text{Mon}_{\mathcal{D}} \rightleftarrows \text{Mon}_{\mathcal{C}} : R$$

is a Quillen equivalence between the respective model categories of monoids.

2. Suppose that Quillen invariance holds in \mathcal{C} and \mathcal{D} . Then for any fibrant monoid A in \mathcal{C} , such that the forgetful functors create model structures for modules over A and modules over RA , the adjoint functor pair

$$L^A : \text{RA-Mod}_{\mathcal{D}} \rightleftarrows \text{A-Mod}_{\mathcal{C}} : R$$

is a Quillen equivalence. If the right adjoint R preserves weak equivalences between monoids and the forgetful functors create model structures for modules over any monoid, then this holds for any monoid A in \mathcal{C} .

Guideline

The next chapter finally uncovers the definitions of the adjoint functor pairs involved in the zig-zag below:

$$H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \begin{array}{c} \xrightarrow{?} \\ \xleftarrow{?} \end{array} Sp^\Sigma(sAb) \begin{array}{c} \xleftarrow{?} \\ \xrightarrow{?} \end{array} Sp^\Sigma(Ch_+) \begin{array}{c} \xrightarrow{?} \\ \xleftarrow{?} \end{array} DG\mathbb{Z}\text{-Mod} .$$

However, the reader will have to wait until Chapter 6 to learn that each of the adjoint pairs is a monoidal Quillen equivalence, satisfying conditions of Theorem 4.3.2, and thus induces Quillen equivalences on the associated categories of monoids.

Chapter 5

The zig-zag of functors between $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$ and $DG\mathbb{Z}\text{-Mod}$

NB: Throughout this chapter, our purpose of labeling categories and functors using full symbols was to make the constructions easier to understand, rather than to burden the reader with cumbersome notations... All symbols were analyzed carefully before they saw the light.

5.1 General strategy for the construction of the zig-zag

The zig-zag of functors

$$H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \begin{array}{c} \xrightarrow{Z} \\ \xleftarrow{U} \end{array} Sp^\Sigma(sAb) \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{\varphi^*N} \end{array} Sp^\Sigma(Ch_+) \begin{array}{c} \xleftarrow{D} \\ \xrightarrow{R} \end{array} DG\mathbb{Z}\text{-Mod}$$

will be built using three more or less “classical” adjoint pairs, defined on the underlying categories

$$sSet_* \begin{array}{c} \xrightarrow{\tilde{Z}} \\ \xleftarrow{U} \end{array} sAb \begin{array}{c} \xleftarrow{N} \\ \xrightarrow{\Gamma} \end{array} Ch_+ \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{C_0} \end{array} DG\mathbb{Z}\text{-Mod} . \text{**}$$

We have already come across the free-forgetful adjunction \tilde{Z}, U on many occasions (see, for instance, Section 1.2.3). The pair N, Γ lies at the basis of the *Dold-Kan equivalence*, and is actually an *equivalence* of categories. One of the useful *a posteriori* consequences would be that N can also be seen as a *right* adjoint of Γ . Finally, to make a connection between non-negatively graded chain complexes Ch_+ and \mathbb{Z} -graded chain complexes $DG\mathbb{Z}\text{-Mod}$, we will use the inclusion functor i and the connective cover functor C_0 , which fit into an adjoint pair.

The zig-zag will emerge from lifting the adjunctions in \mathbb{K}^* to the categories of corresponding symmetric sequences, and then to spectra. However, simple liftings and functorial compositions will not systematically bring us to the destination. Another stratagem will be to use a certain type of adjunctions, naturally induced by maps between monoids, as described below.

Definition 5.1.1. Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a closed symmetric monoidal category, and R, S two monoids in \mathcal{C} with respective multiplications $\mu_R : R \otimes R \rightarrow R$ and $\mu_S : S \otimes S \rightarrow S$. Suppose there exist a morphism of monoids $\xi : R \rightarrow S$.

The map ξ makes S into a right R -module, via

$$\beta : S \otimes R \rightarrow S : s \otimes r \mapsto \mu_S(s \otimes \xi(r)),$$

and induces an adjunction on the categories of modules as follows:

$$\begin{aligned} S \otimes_R - : R\text{-Mod}_{\mathcal{C}} &\rightleftarrows S\text{-Mod}_{\mathcal{C}} : \text{Res}(-) \\ (M, \alpha : R \otimes M \rightarrow M) &\mapsto (S \otimes_R M, \varphi : S \otimes (S \otimes_R M) \rightarrow S \otimes_R M) \\ (N, \gamma : R \otimes N \xrightarrow{\xi \otimes \text{Id}_N} S \otimes N \xrightarrow{\theta} N) &\leftarrow (N, \theta : S \otimes N \rightarrow N). \end{aligned}$$

Here, $S \otimes_R M \in \mathcal{C}$ is by definition the coequalizer in \mathcal{C} of

$$S \otimes R \otimes M \begin{array}{c} \xrightarrow{\beta \otimes \text{Id}_M} \\ \xrightarrow{\text{Id}_S \otimes \alpha} \end{array} S \otimes M,$$

see Proposition 1.1.16. The S -action φ follows from 1.1.16 as well. Since $S \otimes -$ is a left adjoint (\mathcal{C} is assumed closed), it preserves colimits, hence the colimit of

$$S \otimes (S \otimes R \otimes M) \begin{array}{c} \xrightarrow{\text{Id}_S \otimes \beta \otimes \text{Id}_M} \\ \xrightarrow{\text{Id}_S \otimes \text{Id}_S \otimes \alpha} \end{array} S \otimes (S \otimes M),$$

is $S \otimes (S \otimes_R M)$. Thus the diagram

$$\begin{array}{ccccc} S \otimes S \otimes R \otimes M & \begin{array}{c} \xrightarrow{\text{Id}_S \otimes \beta \otimes \text{Id}_M} \\ \xrightarrow{\text{Id}_S \otimes \text{Id}_S \otimes \alpha} \end{array} & S \otimes S \otimes M & & \\ \downarrow \mu_S \otimes \text{Id}_R \otimes \text{Id}_M & \searrow & \downarrow \mu_S \otimes \text{Id}_M & \swarrow & \\ & S \otimes (S \otimes_R M) & & & \\ & \downarrow \beta \otimes \text{Id}_M & & & \\ S \otimes R \otimes M & \begin{array}{c} \xrightarrow{\text{Id}_S \otimes \alpha} \\ \xrightarrow{\text{Id}_S \otimes \alpha} \end{array} & S \otimes M & & \\ & \searrow & \downarrow & \swarrow & \\ & S \otimes_R M & & & \end{array}$$

commutes, inducing φ .

Notation 5.1.2. For convenience, we will denote the functor $S \otimes_R -$ by ξ_* , and refer to it as the **push-forward** of the monoid map ξ . Similarly, we adopt the notation ξ^* for $\text{Res}(-)$, and call it the **pullback** of ξ .

5.2 The pair (Z, U)

The first adjoint pair from \mathbb{K}^* induces the adjunction

$$\tilde{\mathbb{Z}} : sSet_*^\Sigma \rightleftarrows sAb^\Sigma : U$$

on symmetric sequences. The adjunction between corresponding spectra is constructed in several steps, as follows:

$$Sp^\Sigma \xrightleftharpoons[\iota^*]{H\mathbb{Z} \wedge -} H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \xrightleftharpoons[U]{\tilde{\mathbb{Z}}} \tilde{\mathbb{Z}}(H\mathbb{Z})\text{-Mod}_{sAb^\Sigma} \xrightleftharpoons[\mu^*]{\mu_*} Sp^\Sigma(sAb).$$

Exploit the definition of spectra to be modules over suitable monoids to rewrite the chain as

$$\mathbb{S}\text{-Mod}_{Sp^\Sigma} \xrightleftharpoons[\iota^*]{H\mathbb{Z} \wedge_{\mathbb{S}} -} H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \xrightleftharpoons[U]{\tilde{\mathbb{Z}}} \tilde{\mathbb{Z}}(H\mathbb{Z})\text{-Mod}_{sAb^\Sigma} \xrightleftharpoons[\mu^*]{\tilde{\mathbb{Z}}\mathbb{S} \wedge_{\tilde{\mathbb{Z}}(H\mathbb{Z})} -} \tilde{\mathbb{Z}}\mathbb{S}\text{-Mod}_{sAb^\Sigma}, \otimes$$

where we used the fact that $X \in Sp^\Sigma$ can also be seen as an \mathbb{S} -module with respect to \wedge , since $X \wedge \mathbb{S} := X \otimes_{\mathbb{S}} \mathbb{S} \cong X$.

The first adjunction is of type 5.1.1, with respect to the map of monoids in Sp^Σ , $\iota : \mathbb{S} \rightarrow H\mathbb{Z}$, which is the unit map for $H\mathbb{Z}$. The spectrum $H\mathbb{Z}$ was shown to be a monoid in Lemma 1.2.21.

Before we continue, let us investigate how $\tilde{\mathbb{Z}}$ behaves with respect to the smash products \wedge of spectra. For $X, Y \in Sp^\Sigma$, we have

$$X \wedge Y := \text{colim} \left(X \otimes \mathbb{S} \otimes Y \rightrightarrows X \otimes Y \right)$$

in $sSet_*^\Sigma$. Applying $\tilde{\mathbb{Z}}$, we get

$$\tilde{\mathbb{Z}}(X \wedge Y) := \text{colim} \left(\tilde{\mathbb{Z}}(X \otimes \mathbb{S} \otimes Y) \rightrightarrows \tilde{\mathbb{Z}}(X \otimes Y) \right)$$

in sAb^Σ , since $\tilde{\mathbb{Z}}$ is a left adjoint, hence commutes with colimits. Now,

$$(X \otimes Y)_m := \bigvee_{r+s=m} (\Sigma_m)_+ \wedge_{\Sigma_r \times \Sigma_s} (X_r \wedge Y_s)$$

and so

$$\tilde{\mathbb{Z}}(X \otimes Y)_m := \bigoplus_{r+s=m} \Sigma_m \times_{\Sigma_r \times \Sigma_s} \tilde{\mathbb{Z}}(X_r \wedge Y_s),$$

because $\tilde{\mathbb{Z}}$ commutes with coproducts and copower functors. Since the underlying free reduced functor $\tilde{\mathbb{Z}} : sSet_* \rightarrow sAb$ from \mathbb{K}^* is *strong symmetric monoidal*, i.e., there is an isomorphism $\tilde{\mathbb{Z}}(X_r \wedge Y_s) \cong \tilde{\mathbb{Z}}X_r \otimes \tilde{\mathbb{Z}}Y_s$, for all

$X_r, Y_s \in sSet_*$. Hence, for all m , there is an isomorphism $\tilde{\mathbb{Z}}(X \otimes Y)_m \cong (\tilde{\mathbb{Z}}X \otimes \tilde{\mathbb{Z}}Y)_m$ of symmetric sequences, and therefore

$$\tilde{\mathbb{Z}}(X \wedge Y) \cong \operatorname{colim}\left(\tilde{\mathbb{Z}}X \otimes \tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}Y \rightrightarrows \tilde{\mathbb{Z}}X \otimes \tilde{\mathbb{Z}}Y\right) =: \tilde{\mathbb{Z}}X \wedge \tilde{\mathbb{Z}}Y,$$

which shows that the smash product in $Sp^\Sigma(sAb)$ is defined by $\tilde{\mathbb{Z}}X \wedge \tilde{\mathbb{Z}}Y := \tilde{\mathbb{Z}}X \otimes_{\tilde{\mathbb{Z}}\mathbb{S}} \tilde{\mathbb{Z}}Y$. This is consistent with Examples of Section 1.2.2, where the equivalent of the sphere spectrum on $Sp^\Sigma(sAb)$ was claimed to be $\tilde{\mathbb{Z}}\mathbb{S}$, the image of the sphere spectrum \mathbb{S} by $\tilde{\mathbb{Z}}$.

Remark 5.2.1. Notice that to define a category of symmetric spectra over a monoidal category \mathcal{C} , one only needs to pin down the (equivalent of the) 1-sphere in $Sp^\Sigma(\mathcal{C})$; the equivalent of the sphere spectrum is then defined “automatically”. This is clear from Definition 1.2.11 and refines our previous comment: the important thing was that the 1-sphere in $Sp^\Sigma(sAb)$ was defined by applying the functor $\tilde{\mathbb{Z}}$ from \mathbb{S} to the 1-sphere in Sp^Σ .

We now know that $\tilde{\mathbb{Z}}$ respects the smash products between spectra. The second step in \mathfrak{B} results in applying $\tilde{\mathbb{Z}}$ to an $H\mathbb{Z}$ -module Y and gives a $\tilde{\mathbb{Z}}H\mathbb{Z}$ -module $\tilde{\mathbb{Z}}Y$. Recall from Definition 1.2.19 that $H\mathbb{Z} := U\tilde{\mathbb{Z}}\mathbb{S} \in Sp^\Sigma$, to see that the action $\tilde{\mathbb{Z}}H\mathbb{Z} \wedge \tilde{\mathbb{Z}}Y \longrightarrow \tilde{\mathbb{Z}}Y$ is by definition

$$\tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \otimes_{\tilde{\mathbb{Z}}\mathbb{S}} \tilde{\mathbb{Z}}Y \longrightarrow \tilde{\mathbb{Z}}Y,$$

arising from two maps

$$\begin{aligned} \tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}Y &\xrightarrow{\cong} \tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}(\mathbb{S} \otimes Y) \xrightarrow{\operatorname{Id} \otimes \tilde{\mathbb{Z}}(\sigma)} \tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}Y \\ &\xrightarrow{\cong} \tilde{\mathbb{Z}}(U\tilde{\mathbb{Z}}\mathbb{S} \otimes Y) \xrightarrow{\tilde{\mathbb{Z}}(\varphi)} \tilde{\mathbb{Z}}Y \end{aligned}$$

and

$$\tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}Y \xrightarrow{\cong} \tilde{\mathbb{Z}}(U\tilde{\mathbb{Z}}\mathbb{S} \otimes Y) \xrightarrow{\tilde{\mathbb{Z}}(\varphi)} \tilde{\mathbb{Z}}Y.$$

The last adjunction in \mathfrak{B} is again of type 5.1.1; it is induced by the map of monoids

$$\mu : \tilde{\mathbb{Z}}U(\tilde{\mathbb{Z}}\mathbb{S}) \longrightarrow \tilde{\mathbb{Z}}\mathbb{S}$$

in sAb^Σ , which sends a formal finite sum $\sum_i n_i x_i$ of elements of $\tilde{\mathbb{Z}}\mathbb{S}$ to their well-defined sum $\bigoplus_i n_i x_i$ in the abelian group $\tilde{\mathbb{Z}}\mathbb{S}$. The monoidal structure on $\tilde{\mathbb{Z}}\mathbb{S}$ is determined by the monoidal structure on \mathbb{S} and the fact that $\tilde{\mathbb{Z}}$ is strong monoidal.

The monoidal structure on $\tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \in sAb^\Sigma$ is induced in a more subtle way. Even if the forgetful functor U is only lax monoidal, this is sufficient to make $\tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S}$ into a monoid. Indeed, applying U to the universal property diagram of the tensor product in Ab gives a natural map

$$\psi : UA \wedge UB \longrightarrow U(A \otimes B)$$

for all $A, B \in Ab$, since U preserves the cartesian product and the smash product (it is a right adjoint). Thus the composition

$$U(\tilde{\mathbb{Z}}\mathbb{S}) \wedge U(\tilde{\mathbb{Z}}\mathbb{S}) \xrightarrow{\psi} U(\tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}\mathbb{S}) \xrightarrow{U(m)} U(\tilde{\mathbb{Z}}\mathbb{S}),$$

Ψ

makes sense. The multiplication map $\mathcal{M} : \tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \longrightarrow \tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S}$ is then defined to be

$$\tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}U\tilde{\mathbb{Z}}\mathbb{S} \xrightarrow{\cong} \tilde{\mathbb{Z}}(U\tilde{\mathbb{Z}}\mathbb{S} \wedge U\tilde{\mathbb{Z}}\mathbb{S}) \xrightarrow{\tilde{\mathbb{Z}}(\Psi)} \tilde{\mathbb{Z}}(U\tilde{\mathbb{Z}}\mathbb{S}).$$

Finally, the pair (Z, U) is shown below

$$Sp^\Sigma \xrightleftharpoons[t^*]{H\mathbb{Z} \wedge -} H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \xrightleftharpoons[U]{\tilde{\mathbb{Z}}} \tilde{\mathbb{Z}}(H\mathbb{Z})\text{-Mod}_{sAb^\Sigma} \xrightleftharpoons[\mu^*]{\mu_*} Sp^\Sigma(sAb).$$

Z
 U

5.3 The pair (L, φ^*N)

A short note on the Dold-Kan correspondence

Let us first recall how the adjoint pair

$$N : sAb \rightleftarrows Ch_+ : \Gamma$$

from \ast is defined. The left adjoint $N : sAb \longrightarrow Ch_+$ is called the **normalization functor**, it can be built in two steps. The chain complex functor

$$C : sAb \longrightarrow Ch_+$$

$$A \mapsto (CA, d)$$

is defined by $(CA)_n := A_n$ with differential $d_n : (CA)_n \longrightarrow (CA)_{n-1}$ the alternating sum of the face maps

$$d_n := \sum_{i=0}^n (-1)^i d_i,$$

for all $n \geq 0$. The chain complex CA has a natural subcomplex DA , the complex of **degenerate simplexes**. By definition, the subgroup $(DA)_n$ of A_n is generated by all degenerate simplexes. The differential on DA_n is given by restriction of d_n . It can be shown that the quotient of CA by DA is well-defined, and induces the **normalization functor**

$$N : sAb \longrightarrow Ch_+$$

$$A \mapsto NA := CA/DA.$$

A compact way of defining the right adjoint Γ of N is provided by [GJ, III.2], where one learns that

$$\Gamma : Ch_+ \longrightarrow sAb$$

is given in degree n by

$$\Gamma(C)_n := \bigoplus_{[n] \twoheadrightarrow [k]} C_k.$$

Here the direct sum is indexed over the set of all surjections from $[n]$ to $[k]$ in Δ . We refer the reader to [GJ, III.2] for the definition of the simplicial structure maps on $\Gamma(C)$, as well as for a detailed discussion about this adjoint pair.

The **Dold-Kan correspondence**, which states that N and Γ form an *equivalence of categories*, is established as Theorem 1.9 in [Do58]. A later reference, which might be easier to read, is Corollary III.2.3 in [GJ].

Remark 5.3.1. It can be shown that the *left* adjoint of an equivalence of categories is at the same time the *right* adjoint for this equivalence, and similarly for the other functor (see [McL, IV.4]).

The categories of simplicial abelian groups sAb and non-negative chain complexes Ch_+ figured among examples of monoidal categories given in Chapter 1. Both chosen monoidal products were symmetric, but different in an essential way, since the product in Ch_+ was graded, while the one in sAb was defined level-wise. The equivalence of categories given by the normalization functor and its inverse does not take one tensor product to another, yet they can be related via two lax monoidal transformations, the *shuffle map* and the *Alexander-Whitney map*.

For our purposes, we are mainly interested in the **shuffle map**

$$\nabla : CA \otimes CB \longrightarrow C(A \otimes B),$$

introduced by Eilenberg and Mac Lane. We allow ourselves to skip its technical definition, since there will be no explicit need for it in the report, and refer the reader to [May67, 29.7] for details. The shuffle map is appropriately associative and unital, the unit being the unique chain map $\eta_A : \mathbb{Z}[0] \longrightarrow CA$, for all $A \in sAb$, which is the unit map $\mathbb{Z} \longrightarrow A_0$ in degree 0 and the inclusion of the trivial group at upper levels.

It turns out that ∇ preserves the subcomplexes of degenerate simplexes, and hence factors over normalized chain complexes, inducing a map

$$\nabla : NA \otimes NB \longrightarrow N(A \otimes B).$$

This induced map is again lax monoidal, and the restricted unit map is now an isomorphism $\mathbb{Z}[0] \cong N(\mathbb{Z})$.

Corollary 5.3.2. *The normalization functor N is a lax symmetric monoidal functor, with structure map $\nabla : NA \otimes NB \longrightarrow N(A \otimes B)$ induced by the shuffle map, and unit the isomorphism $\mathbb{Z}[0] \xrightarrow{\cong} N(\mathbb{Z})$.*

Now that the pair N, Γ has lost its mystery, notice that it induces an adjunction

$$N : sAb^\Sigma \rightleftarrows Ch_+^\Sigma : \Gamma$$

on symmetric sequences. To establish the required adjunction on spectra, let us start with the chain of functors

$$\begin{array}{ccc} Sp^\Sigma(sAb) & & Sp^\Sigma(Ch_+) \\ \parallel & & \parallel \\ \tilde{\mathbb{Z}}\mathbb{S}\text{-Mod}_{sAb^\Sigma} & \xrightleftharpoons[\Gamma]{N} & N(\tilde{\mathbb{Z}}\mathbb{S})\text{-Mod}_{Ch_+^\Sigma} \xrightleftharpoons[\varphi^*]{N\tilde{\mathbb{Z}}\mathbb{S} \otimes_{\text{Sym}(\mathbb{Z}[1])} -} \text{Sym}(\mathbb{Z}[1])\text{-Mod}_{Ch_+^\Sigma}, \quad \clubsuit \end{array}$$

which will evolve gradually.

Note for the moment that, given a $\tilde{\mathbb{Z}}\mathbb{S}$ -module X together with an action $\alpha : \tilde{\mathbb{Z}}\mathbb{S} \otimes X \longrightarrow X$, the image NX is a $N(\tilde{\mathbb{Z}}\mathbb{S})$ -module. To see this, observe first that $N\tilde{\mathbb{Z}}\mathbb{S} = N(\text{Sym}(\tilde{\mathbb{Z}}S^1)) = (N\tilde{\mathbb{Z}}S^0, N\tilde{\mathbb{Z}}S^1, N\tilde{\mathbb{Z}}S^2, \dots)$ is indeed a commutative monoid, since the functor N is monoidal. It respects symmetries, too, and the multiplication is given by the composition

$$N\tilde{\mathbb{Z}}\mathbb{S} \otimes N\tilde{\mathbb{Z}}\mathbb{S} \xrightarrow{\nabla} N(\tilde{\mathbb{Z}}\mathbb{S} \otimes \tilde{\mathbb{Z}}\mathbb{S}) \xrightarrow{N(\mu)} N\tilde{\mathbb{Z}}\mathbb{S},$$

which is well-defined at each level m . One easily checks it, using the lax monoidal map ∇ and the fact that $\tilde{\mathbb{Z}}\mathbb{S}$ is already a commutative monoid. The action $N(\tilde{\mathbb{Z}}\mathbb{S}) \otimes NX \longrightarrow NX$ arises from the collection of maps for $p + q = n$

$$N\tilde{\mathbb{Z}}\mathbb{S}^p \otimes NX_q \xrightarrow{\nabla} N(\tilde{\mathbb{Z}}\mathbb{S}^p \otimes X_q) \xrightarrow{N(\alpha)} N(X_n).$$

The second adjunction in \clubsuit is again of type 5.1.1, induced by the monoid map in Ch_+^Σ

$$\varphi : \text{Sym}(\mathbb{Z}[1]) \longrightarrow N(\tilde{\mathbb{Z}}\mathbb{S}),$$

such that $\varphi_n : \text{Sym}(\mathbb{Z}[1])_n \longrightarrow N(\text{Sym}(\tilde{\mathbb{Z}}S^1))_n$ is induced by the monoidal structure on N , as follows:

$$(\mathbb{Z}[1])^{\otimes n} \cong (N\tilde{\mathbb{Z}}S^1)^{\otimes n} \xrightarrow{\nabla_n} N((\tilde{\mathbb{Z}}S^1)^{\otimes n}).$$

We use here that $N\tilde{\mathbb{Z}}S^1$ can be identified with $\mathbb{Z}[1]$, which is clear if one writes $N\tilde{\mathbb{Z}}S^1$ explicitly.

Remark 5.3.3. The definition of the map φ incidentally suggests that the 1-sphere $\mathbb{Z}[1]$ in $Sp^\Sigma(Ch_+)$ is in fact given by the image of the 1-sphere $\tilde{\mathbb{Z}}S^1$ in $Sp^\Sigma(sAb)$ by N .

Remarks 5.2.1 and 5.3.3 illustrate that the “classical” adjunctions from \ast are indeed fundamental in constructing the zig-zag of functors on the associated categories of spectra.

Finally, consider the composition φ^*N in the diagram

$$\begin{array}{ccccc}
& & \xleftarrow{\quad L \quad} & & \\
\tilde{\mathbb{Z}}\mathbb{S}\text{-Mod}_{sAb^\Sigma} & \xrightarrow{N} & \mathbb{N}(\tilde{\mathbb{Z}}\mathbb{S})\text{-Mod}_{Ch_+^\Sigma} & \xrightarrow{\varphi^*} & \text{Sym}(\mathbb{Z}[1])\text{-Mod}_{Ch_+^\Sigma} \\
\parallel & & & & \parallel \\
Sp^\Sigma(sAb) & & & & Sp^\Sigma(Ch_+)
\end{array}$$

In view of Remark 5.3.1, N can be seen as a *right* adjoint, too. A left adjoint to φ^*N exists by [SS03b, 3.3]. In fact, it is given by the module-valued lift (see Section 4.2.2)

$$L^{\tilde{\mathbb{Z}}\mathbb{S}} : \text{Sym}(\mathbb{Z}[1])\text{-Mod}_{Ch_+^\Sigma} \longrightarrow \tilde{\mathbb{Z}}\mathbb{S}\text{-Mod}_{sAb^\Sigma},$$

simply denoted L above.

5.4 The pair (D, R)

The last adjoint pair in the zig-zag is

$$Sp^\Sigma(Ch_+) \xrightleftharpoons[R]{D} DG\mathbb{Z}\text{-Mod}.$$

To understand the construction of these functors, we need first some additional notions.

The underlying adjunction

Let $n > 0$. Observe that tensoring a complex $X \in DG\mathbb{Z}\text{-Mod}$ with $\mathbb{Z}[n]$ shifts its groups by n to the right, i.e., $(\mathbb{Z}[n] \otimes X)_k = X_{k-n}$, while tensoring X with $\mathbb{Z}[-n]$ shifts its groups by n to the left, i.e., $(\mathbb{Z}[-n] \otimes X)_k = X_{k+n}$.

Now, consider the following chain of adjunctions

$$Ch_+ \xrightleftharpoons[C_0]{i} DG\mathbb{Z}\text{-Mod} \xrightleftharpoons[\mathbb{Z}[-1] \otimes -]{\mathbb{Z}[1] \otimes -} DG\mathbb{Z}\text{-Mod} . \ast$$

Here i denotes simply the inclusion functor of non-negatively graded chain complexes into \mathbb{Z} -graded chain complexes and the functor C_0 is the **connective cover**.

Given X in $DG\mathbb{Z}\text{-Mod}$, one would like to associate to it a chain complex $C_0(X)$ in Ch_+ in a “good” way. The idea would be to make a *truncation*

at level 0 (see [Wei] 1.2.7), i.e., to consider a subcomplex of X , where all negative groups are set to 0. One could make a *brutal truncation*

$$X = (\dots \leftarrow X_{-2} \leftarrow X_{-1} \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots) \xrightarrow{\text{truncate}} (\dots \leftarrow 0 \leftarrow 0 \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots),$$

but doing this neither respects the homology information of the complex, nor gives an adjoint functor to the inclusion. Indeed, having an adjunction (i, C_0) supposes to have a natural bijection

$$DG \mathbb{Z}\text{-Mod}(i(Y), X) \cong Ch_+(Y, C_0(X)).$$

In particular, the diagram

$$(I) \quad \begin{array}{ccccccc} \dots & \longleftarrow & 0 & \longleftarrow & 0 & \xleftarrow{0} & Y_0 \xleftarrow{\partial_1} Y_1 \longleftarrow \dots \\ & & \downarrow & & \downarrow 0 & \Delta & f_0 \downarrow & & f_1 \downarrow \\ \dots & \longleftarrow & X_{-2} & \longleftarrow & X_{-1} & \xleftarrow{\partial_0} & X_0 \xleftarrow{\partial_1} & X_1 \longleftarrow \dots \end{array}$$

must commute if and only if the diagram

$$(II) \quad \begin{array}{ccc} Y_0 & \xleftarrow{\partial_1} & Y_1 \longleftarrow \dots \\ f_0 \downarrow & & f_1 \downarrow \\ X_0 & \xleftarrow{\partial_1} & X_1 \longleftarrow \dots \end{array}$$

commutes. The problem occurs in Δ when showing $(II) \implies (I)$, because no conditions on f_0 would guarantee that $\partial_0 \circ f_0 = 0$.

To solve this problem, it is worthwhile to make a *good truncation* as follows:

$$Y = (\dots \leftarrow Y_{-2} \leftarrow Y_{-1} \leftarrow Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \dots) \xrightarrow{\text{truncate}} C_0(Y) := (\dots \leftarrow 0 \leftarrow 0 \leftarrow \text{Ker}(\partial_0) \leftarrow Y_1 \leftarrow Y_2 \leftarrow \dots).$$

In this case, the square

$$\begin{array}{ccc} 0 & \xleftarrow{0} & Y_0 \\ 0 \downarrow & \Delta & \downarrow f_0 \\ \text{Ker}(\partial_0) & \xleftarrow{\partial_0} & X_0 \end{array}$$

commutes, and the pair (i, C_0) is adjoint.

The right adjoint R

The functor

$$R : DG \mathbb{Z}\text{-Mod} \longrightarrow \text{Sym}(\mathbb{Z}[1])\text{-Mod}_{Ch_+^\Sigma},$$

is given for all $Y \in DG \mathbb{Z}\text{-Mod}$ by

$$(RY)_m := C_0(Y \otimes \mathbb{Z}[m]).$$

Each RY_m is indeed an object of Ch_+ ; the structure of a $\text{Sym}(\mathbb{Z}[1])$ -module to RY is provided by the adjunction \ast . By calculation, one checks that there are isomorphisms

$$f_m : (RY)_m \xrightarrow{\cong} C_0(\mathbb{Z}[-1] \otimes (RY)_{m+1}),$$

for all m , adjoint to the maps

$$g_m : \mathbb{Z}[1] \otimes (RY)_m \longrightarrow (RY)_{m+1}.$$

At level n , the action $\alpha_n : (\text{Sym}(\mathbb{Z}[1]) \otimes RY)_n \longrightarrow RY_n$ is determined by a collection of maps for $n = p + q$

$$\alpha_{p,q} : \mathbb{Z}[p] \otimes (RY)_q \longrightarrow (RY)_n,$$

given by the composite

$$\mathbb{Z}[p] \otimes (RY)_q \xrightarrow{\text{Id} \otimes g_q} \mathbb{Z}[p-1] \otimes (RY)_{q+1} \xrightarrow{\text{Id} \otimes g_{q+1}} \dots \xrightarrow{g_{n-1}} (RY)_n.$$

The left adjoint D

The remaining functor from the zig-zag

$$D : Sp^\Sigma(Ch_+) \longrightarrow DG \mathbb{Z}\text{-Mod}$$

is defined in several steps.

Let I be the **skeleton** of the category of finite ordered sets and injections, denoted \mathcal{I} , with objects \mathbf{n} . This means that I is a full subcategory of \mathcal{I} , such that all isomorphic sets in \mathcal{I} are given by the same object in I . For $X \in Sp^\Sigma(Ch_+)$, let $\tilde{\sigma} : X_n \longrightarrow \mathbb{Z}[n-m] \otimes X_m$ be the adjoint of the structure map σ at level n .

Define a diagram of shape I in Ch_+ , $D_X : I \longrightarrow Ch_+$, as follows. For an object \mathbf{n} of I , set

$$D_X(\mathbf{n}) := \mathbb{Z}[-n] \otimes X_n.$$

Observe that any monomorphism $\mathbf{a} : \mathbf{n} \longrightarrow \mathbf{m}$ for $n \leq m$ can be seen as an inclusion, followed by a permutation. This gives on elements

$$(1, 2, \dots, n) \xrightarrow{i} (i(1), i(2), \dots, i(n), 1, \dots, m-n) \xrightarrow{i} (\rho \circ i(1), \rho \circ i(2), \dots, \rho \circ i(n), \rho(1), \dots, \rho(m-n)),$$

where $\rho \in \Sigma_m$. If $n = m$, then \mathbf{a} is just a permutation in Σ_n .

For a standard inclusion \mathbf{a} (case $n < m$), define $D_X(\mathbf{a})$ to be the composite

$$\mathbb{Z}[-n] \otimes X_n \xrightarrow{\mathbb{Z}[-n] \otimes \tilde{\sigma}} \mathbb{Z}[-n] \otimes \mathbb{Z}[n-m] \otimes X_m \xrightarrow{\cong} \mathbb{Z}[-m] \otimes X_m.$$

$\xrightarrow{D_X(\mathbf{a})}$

If $n = m$, $D_X(\mathbf{a})$ is an isomorphism. An action on $\mathbb{Z}[-n] \otimes X_n$ is given by the tensor product of the sign action with the action of Σ_n on X_n .

The functor $D : Sp^\Sigma(Ch_+) \rightarrow DG\mathbb{Z}\text{-Mod}$ is defined by

$$DX := \text{colim}_I D_X.$$

To better understand this definition, recall that in the category $DG\mathbb{Z}\text{-Mod}$ a colimit is given by the quotient of a coproduct (direct sum) by relations induced from the diagram category. In particular, here we have

$$DX := \bigoplus_{\mathbf{k}} D_X[\mathbf{k}] / \sim$$

where $(\mathbb{Z}[-n] \otimes X_n, \mathbf{n}) \xrightarrow{D_X(\mathbf{a})} (\mathbb{Z}[-m] \otimes X_m, \mathbf{m})$, for all $\mathbf{a} : \mathbf{n} \rightarrow \mathbf{m}$ in I .

Proposition 4.5 in [Shi] proves that the functors D and R are adjoint. The proof establishes the adjunction on free spectra, and this restriction appears to be sufficient to show the adjunction for all spectra, probably because further in the paper any spectrum Z is claimed to be the coequalizer of the two maps $FFZ \rightrightarrows FZ$, where $FZ = \bigoplus_n F_n(Z_n)$.

Unfortunately, we were not able to find a satisfying reference that would shed light on this decomposition, as well as on the reason why the restriction on free spectra is sufficient. At least, let us explore the following lemma.

Lemma 5.4.1. *The values of the functor D on free spectra are given by*

$$D(F_m K) = \mathbb{Z}[-m] \otimes K$$

for $K \in Ch_+$.

Proof. There is an adjunction in $Sp^\Sigma(Ch_+^\Sigma)$, similar to the adjunction \textcircled{e} from Section 2.3.1

$$\begin{array}{ccccc} & & F_n & & \\ & & \curvearrowright & & \\ & & G_n & & \\ & & \curvearrowleft & & \\ Ch_+ & \xleftrightarrow{\Sigma_n \times -} & Ch_+^{\Sigma_n} & \xleftrightarrow{(-)_n} & Ch_+^{\Sigma} \xleftrightarrow{\text{Sym}(\mathbb{Z}[1]) \otimes -} Sp^\Sigma(Ch_+) \\ & \xleftarrow{U_1} & & \xleftarrow{ev_n} & \xleftarrow{U} \\ & & Ev_n & & \end{array}$$

By definition, we have

$$\begin{aligned} F_m(K)_n &:= (G_m(K) \otimes \text{Sym}(\mathbb{Z}[1]))_n \\ &= \bigoplus_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} G_m(K)_p \otimes \mathbb{Z}[q] \end{aligned}$$

Recall now that $G_m(K)_m := \Sigma[m] \otimes K$ and $G_m(K)_n := 0$ for $n \neq m$. Thus for $n \geq m$

$$\begin{aligned} F_m(K)_n &\stackrel{p=m; q=n-m}{=} \Sigma_n \times_{\Sigma_m \times \Sigma_{m-n}} G_m(K)_m \otimes \mathbb{Z}[m-n] \\ &= \Sigma_n \times_{\Sigma_m \times \Sigma_{m-n}} \Sigma[m] \otimes K \otimes \mathbb{Z}[m-n] \\ &= \Sigma_n \times_{\Sigma_{m-n}} K \otimes \mathbb{Z}[m-n] \end{aligned}$$

and for $n < m$, $F_m(K)_n = 0$. Therefore, for all $k \geq 0$

$$\begin{aligned} D(F_m(K))_k &= D_{F_m(K)}[\mathbf{k}] \\ &\stackrel{k \geq m}{=} \mathbb{Z}[-k] \otimes F_m(K)_k \\ &= \mathbb{Z}[-k] \otimes \Sigma_k \times_{\Sigma_{k-m}} K \otimes \mathbb{Z}[k-m] \\ &= \Sigma_k \times_{\Sigma_{k-m}} K \otimes \mathbb{Z}[-m]. \end{aligned}$$

To see that $D(F_m K) = \mathbb{Z}[-m] \otimes K$ we will try to show that

$$\text{colim}_k D(F_m(K))_k = K \otimes \mathbb{Z}[-m],$$

i.e., that $K \otimes \mathbb{Z}[-m]$ satisfies the universal property of the colimit.

For any $l \in I$, the map $\varphi_l : D(F_m(K))_l \rightarrow K \otimes \mathbb{Z}[-m]$ of positively graded chain complexes is determined by a collection of maps for $q \geq 0$

$$\varphi_l^q : (\Sigma_l \times_{\Sigma_{l-m}} K \otimes \mathbb{Z}[-m])_q \rightarrow (K \otimes \mathbb{Z}[-m])_q,$$

compatible with the differentials, and such that

$$\begin{aligned} \varphi_l^q &: \Sigma_l \times_{\Sigma_{l-m}} K_{m+q} \rightarrow K_{m+q} \\ &(g, x) \mapsto (g \cdot x), \end{aligned}$$

where $x \in K_{m+q}$ and $g \in \Sigma_l \times_{\Sigma_{l-m}}$ is a permutation of a form

$$\begin{pmatrix} 1 & 2 & \dots & m & m+1 & \dots & l \\ id(1) & id(2) & \dots & id(m) & g(m+1) & \dots & g(l) \end{pmatrix}$$

(hence, in particular g is an element of Σ_l).

For an injection $\mathbf{a} : l \rightarrow p$ in I , the induced map

$$D_{F_m(K)}(\mathbf{a}) : \Sigma_l \times_{\Sigma_{l-m}} K \otimes \mathbb{Z}[-m] \longrightarrow \Sigma_p \times_{\Sigma_{p-m}} K \otimes \mathbb{Z}[-m]$$

is also determined by a collection of differential-compatible maps for $q \geq 0$

$$\begin{aligned} D(\mathbf{a})_q : \Sigma_l \times_{\Sigma_{l-m}} K_{m+q} &\longrightarrow \Sigma_p \times_{\Sigma_{p-m}} K_{m+q} \\ (g, x) &\mapsto (\hat{g}, x), \end{aligned}$$

where $g \in \Sigma_l$ is defined as before and $\hat{g} \in \Sigma_p$ is of a form

$$\begin{pmatrix} 1 & 2 & \dots & m & m+1 & \dots & l & l+1 & \dots & p \\ id(1) & id(2) & \dots & id(m) & g(m+1) & \dots & g(l) & id(l+1) & \dots & id(p) \end{pmatrix}.$$

In other words, we use the fact that an injection $\mathbf{a} : l \rightarrow p$ in I induces an injection $\Sigma_l \rightarrow \Sigma_p$, and the permutation \hat{g} is finally g , regarded in Σ_p .

For $(g, x) \in \Sigma_l \times_{\Sigma_{l-m}} K_{m+q}$, we have

$$\varphi_p^q \circ D(\mathbf{a})(g, x) = \varphi_p^q(\hat{g}, x) = \hat{g} \cdot x = g \cdot x = \varphi_l^q(g, x).$$

To have the universal property of diagrams

$$\begin{array}{ccc} \dots D(F_m(K))_l & \xrightarrow{D_{F_m(K)}(\mathbf{a})} & D(F_m(K))_p \dots \\ & \searrow \varphi_l & \swarrow \varphi_p \\ & K \otimes \mathbb{Z}[-m] & \\ & \searrow f_l & \swarrow f_p \\ & A & \end{array}$$

$\downarrow \Phi$

in Ch_+ , and for all $\mathbf{a} : l \rightarrow p$, it is sufficient to have it in all degrees $q \geq 0$. Hence we have to show that for all $q \geq 0$, for all $\mathbf{a} : l \rightarrow p$, given maps f_l^q and f_p^q as in the diagram

$$\begin{array}{ccc} \dots \Sigma_l \times_{\Sigma_{l-m}} K_{m+q} & \xrightarrow{D^q(\mathbf{a})} & \Sigma_p \times_{\Sigma_{p-m}} K_{m+q} \dots \\ & \searrow \varphi_l^q & \swarrow \varphi_p^q \\ & K_{m+q} & \\ & \searrow f_l^q & \swarrow f_p^q \\ & A_{m+q} & \end{array}$$

$\downarrow \Phi^q$

and which satisfy $f_l^q = f_p^q \circ D^q(\mathbf{a})$, there exists a unique map

$$\Phi^q : K_{m+q} \longrightarrow A_{m+q},$$

such that $f_l^q = \Phi^q \circ \varphi_l^q$ and $f_p^q = \Phi^q \circ \varphi_p^q$.

For $\mathbf{a} : l \rightarrow p$, an arbitrary map

$$f_l^q : \Sigma_l \times_{\Sigma_{l-m}} K_{m+q} \longrightarrow A_{m+q}$$

should be of form

$$(g, x) \mapsto f^{q+m}(g_l \cdot x)$$

(we only use that the complex K bears a certain action of Σ_l , and that we are given a map $f = \{f_{q+m} : K_{q+m} \rightarrow A_{q+m}\}_{q+m \geq 0}$ of chain complexes). Here, the subscript l means $g_l = g \in \Sigma_l$. Similarly, the map

$$f_p^q : \Sigma_p \times_{\Sigma_{p-m}} K_{m+q} \longrightarrow A_{m+q}$$

has the form

$$(g, x) \mapsto f^{q+m}(g_p \cdot x),$$

where the subscript p means $g_p \in \Sigma_p$.

Thus, given $(g, x) \in \Sigma_l \times_{\Sigma_{l-m}} K_{m+q}$, we have

$$f_p^q \circ D^q(\mathbf{a})(g, x) = f_p^q(\hat{g}, x) = f^{q+m}(\hat{g} \cdot x) = f^{q+m}(g \cdot x) = f_l^q(g, x).$$

Define the map $\Phi^q : K_{m+q} \rightarrow A_{m+q}$ for all $q \geq 0$, and for all $y \in K_{m+q}$ by

$$y \mapsto f^{q+m}(\text{Id} \cdot y) = f^{q+m}(y).$$

It follows that for $(g, x) \in \Sigma_l \times_{\Sigma_{l-m}} K_{m+q}$

$$\Phi^q \circ \varphi_l^q(g, x) = \Phi^q(g \cdot x) = f^{q+m}(g \cdot x) = f_l^q(g, x),$$

and for $(g, x) \in \Sigma_p \times_{\Sigma_{p-m}} K_{m+q}$

$$\Phi^q \circ \varphi_p^q(g, x) = \Phi^q(g_p \cdot x) = f^{q+m}(g_p \cdot x) = f_p^q(g, x),$$

which shows the universal property of colimit for $K \otimes \mathbb{Z}[-m]$. □

Guideline

The following two propositions summarize the properties of the categories and functors involved in the zig-zag.

Proposition 5.4.2. [Shi, Proposition 2.9] *The categories $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$, $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ and $DG\mathbb{Z}\text{-Mod}$ satisfy the hypotheses of Theorem 3.2.7. It follows that there are model structures on the categories of monoids and modules on these categories, with fibrations and weak equivalences defined in the underlying model category.*

Proposition 5.4.3. [Shi, Proposition 2.10]

1. *The pair $Z : H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \rightleftarrows Sp^\Sigma(sAb) : U$ is a strong monoidal Quillen equivalence.*
2. *The pair $L : Sp^\Sigma(Ch_+) \rightleftarrows Sp^\Sigma(sAb) : \varphi^*N$ is a weak monoidal Quillen equivalence.*
3. *The pair $D : Sp^\Sigma(Ch_+) \rightleftarrows DG\mathbb{Z}\text{-Mod} : R$ is a strong monoidal Quillen equivalence.*

Moreover, the right adjoints in each of these pairs preserve all weak equivalences.

Now let us recall the main theorem. It will follow once the two propositions above are proved, and once Theorem 4.3.2 is applied to the zig-zag.

Theorem [Shi, Theorem 1.1] *The model categories of unbounded differential graded \mathbb{Z} -algebras and $H\mathbb{Z}$ -algebra spectra are Quillen equivalent. The associated composite derived functors are denoted*

$$\mathbb{H} : DG\text{-Alg}_{\mathbb{Z}} \longrightarrow H\mathbb{Z}\text{-Alg}_{Sp^\Sigma} \quad \text{and} \quad \Theta : H\mathbb{Z}\text{-Alg}_{Sp^\Sigma} \longrightarrow DG\text{-Alg}_{\mathbb{Z}}.$$

Proof. Propositions 5.4.2 and 5.4.3 verify all of the hypotheses required to apply Theorem 4.3.2 to the zig-zag of functors

$$H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} \begin{array}{c} \xrightarrow{Z} \\ \xleftarrow{U} \end{array} Sp^\Sigma(sAb) \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{\varphi^*N} \end{array} Sp^\Sigma(Ch_+) \begin{array}{c} \xleftarrow{D} \\ \xrightarrow{R} \end{array} DG\mathbb{Z}\text{-Mod}$$

(see Chapter 6). By Theorem 4.3.2 these three adjoint pairs then induce Quillen equivalences on the categories of monoids as follows:

$$H\mathbb{Z}\text{-Alg}_{Sp^\Sigma} \xrightleftharpoons[U]{Z} \text{Mon}_{Sp^\Sigma(sAb)} \xrightleftharpoons[\varphi^*N]{L^{mon}} \text{Mon}_{Sp^\Sigma(Ch_+^\Sigma)} \xrightleftharpoons[R]{D} \text{DG-Alg}_{\mathbb{Z}}.$$

Since original pairs (Z, U) and (D, R) are both strong monoidal Quillen equivalences, the left adjoints induced on the categories of monoids are just the restrictions of the underlying functors. Thus they are denoted by the same letters (Notation 4.2.1). The pair (L, φ^*N) being only a weak monoidal Quillen equivalence, the induced left adjoint on monoids here is L^{mon} ; it does not agree with L on underlying objects (see Section 4.2.2).

The composite derived functors \mathbb{H} and Θ on the homotopy categories, mentioned in the main Theorem, are explicitly given by the compositions

$$\begin{array}{ccccc} \text{Ho DG-Alg}_{\mathbb{Z}} & \xrightarrow{\text{Ho } R} & \text{Ho Mon}_{Sp^\Sigma(Ch_+^\Sigma)} & \xrightarrow{\text{Ho } Q} & \text{Ho Mon}_{Sp^\Sigma(Ch_+^\Sigma)}^c \\ & & & & \downarrow \text{Ho } L^{mon} \\ & & & & \text{Ho Mon}_{Sp^\Sigma(sAb)} \\ & & & & \downarrow \text{Ho } U \\ & & & & \text{Ho } H\mathbb{Z}\text{-Alg}_{Sp^\Sigma} \end{array}$$

\mathbb{H}

and

$$\begin{array}{ccccc} \text{Ho } H\mathbb{Z}\text{-Alg}_{Sp^\Sigma} & \xrightarrow{\text{Ho } Q} & \text{Ho } H\mathbb{Z}\text{-Alg}_{Sp^\Sigma}^c & \xrightarrow{\text{Ho } Z} & \text{Ho Mon}_{Sp^\Sigma(sAb)} \\ & & & & \downarrow \text{Ho } \varphi^*N \\ & & & & \text{Ho Mon}_{Sp^\Sigma(Ch_+^\Sigma)} \\ & & & & \downarrow \text{Ho } Q \\ & & & & \text{Ho Mon}_{Sp^\Sigma(Ch_+^\Sigma)}^c \\ & & & & \downarrow \text{Ho } D \\ & & & & \text{Ho DG-Alg}_{\mathbb{Z}}. \end{array}$$

Θ

Here Q denotes the cofibrant replacement functor in the category of monoids, and the superscript c indicates the subcategory of cofibrant objects. Note that the fibrant replacement functors are not needed, because the right adjoints preserve all weak equivalences. \square

Chapter 6

A few comments on the proofs

This chapter presents some of the arguments that prove that the categories $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$, $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$, $DG\mathbb{Z}\text{-Mod}$ and the pairs of functors (Z, U) , (L, φ^*N) , (D, R) effectively satisfy the hypotheses of Theorems 3.2.7 and 4.3.2. This shows Propositions 5.4.2 and 5.4.3.

Due to the amount and the complexity of the material, this chapter is an overview. We concentrate on explaining facts that were accessible to us, and that gave the impression of being interesting or important to mention. We also sketch some “hidden” proofs and carry out a few verifications that did not appear explicitly in the target article.

6.1 Conditions on categories

Recall that the hypotheses of Theorem 3.2.7 require five conditions to be checked for each category:

- ① The category is model monoidal;
- ② It is cofibrantly generated;
- ③ The monoid axiom is satisfied;
- ④ The unit object for the monoidal product is cofibrant;
- ⑤ All objects in the category are small.

Remark 6.1.1. The smallness condition ⑤ is established for all categories in Propositions 3.7 and 3.8 in the target article of Shipley.

6.1.1 Category $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$

Conditions ① - ③ for the category $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$ of $H\mathbb{Z}$ -module spectra will follow from Theorem 3.2.7(2), when applied to Sp^Σ and to $H\mathbb{Z}$, which is shown to be a commutative ring spectrum in Lemma 1.2.21. Next lemma gives the condition ④ for $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$.

Lemma 6.1.2. *In the category $(H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}, \wedge_{H\mathbb{Z}}, H\mathbb{Z})$, the spectrum $H\mathbb{Z}$ is cofibrant.*

Proof. There is an adjunction

$$H\mathbb{Z} \wedge - : Sp^\Sigma \rightleftarrows H\mathbb{Z}\text{-Mod}_{Sp^\Sigma} : U$$

between the free $H\mathbb{Z}$ -module functor and the forgetful functor U . By Theorem 3.2.7, a morphism f in $H\mathbb{Z}\text{-Mod}_{Sp^\Sigma}$ is a weak equivalence (a fibration) if and only if Uf is a weak equivalence (a fibration) in Sp^Σ . Hence U preserves fibrations and acyclic fibrations, making $(H\mathbb{Z} \wedge -, U)$ into a Quillen pair. Lemma 2.1.26 implies that the left adjoint $H\mathbb{Z} \wedge -$ preserves cofibrations.

We already know that the sphere spectrum \mathbb{S} is cofibrant (Corollary 2.3.13). Applying $H\mathbb{Z} \wedge -$ to $* \rightarrow \mathbb{S}$ gives the map $* \rightarrow H\mathbb{Z}$, since $H\mathbb{Z} \wedge -$ preserves the initial objects, and since by definition $H\mathbb{Z} \wedge \mathbb{S} := H\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{S} \cong H\mathbb{Z}$. By the above argument this map is a cofibration. \square

Summary of ① - ④ for Sp^Σ

To apply Theorem 3.2.7(2) to Sp^Σ , one needs to check the five conditions listed above. They follow from results in Sections 2, 3, and 5 in [HSS], some of which were cited in previous chapters. Corollary 2.3.13 verifies ④.

6.1.2 Category $DG\mathbb{Z}\text{-Mod}$

Conditions ① and ② for $DG\mathbb{Z}\text{-Mod}$ follow from 2.3.11 and 4.3.13 in [Hov01]. We will consider the monoid axiom ③ for $DG\mathbb{Z}\text{-Mod}$ in the next section.

To check that the unit $\mathbb{Z}[0]$ is cofibrant in $(DG\mathbb{Z}\text{-Mod}, \otimes, \mathbb{Z}[0])$ we use the following lemma.

Lemma 6.1.3. [Hov99, Lemma 2.3.6] *Suppose R is a ring. Any bounded below complex of projective R -modules is cofibrant.*

Corollary 6.1.4. *The complex $\mathbb{Z}[0]$ is cofibrant in $DG\mathbb{Z}\text{-Mod}$.*

Proof. The non-negatively graded chain complex $\mathbb{Z}[0]$ is clearly bounded below. Since a projective R -module is a module that is a direct summand of a free R -module, $\mathbb{Z}[0]$ is clearly projective in each degree. Hence, it is cofibrant. \square

6.1.3 Categories $Sp^\Sigma(sAb)$ and $Sp^\Sigma(Ch_+)$

The fact that these two categories of spectra are model monoidal and cofibrantly generated ①, ② follows from Section 8 of the article [Hov01].

The monoid axiom

The strategy Shipley uses to show the monoid axiom ③ for the categories $DG\mathbb{Z}\text{-Mod}$, $Sp^\Sigma(sAb)$ and $Sp^\Sigma(Ch_+)$ is modeled on the approach for Sp^Σ in [HSS, Section 5.5], where two different stable model structures, referred to as **projective** and **injective**, are employed. We explain the central idea of the construction.

Recall the projective and the injective model structures on the category $DG\mathbb{Z}\text{-Mod}$ from Definitions 2.2.17 and 2.2.20. Note that in both of these structures the weak equivalences are the quasi-isomorphisms.

Stable model structures on $Sp^\Sigma(sAb)$ and $Sp^\Sigma(Ch_+)$, introduced in [Hov01] (see Section 2.4 for comments), are called *projective*. Within these structures, a map f is a projective weak equivalence if it is a stable equivalence (2.3.8), and a map f is a projective cofibration if it has the LLP with respect to level acyclic fibrations.

The aim is to establish the monoid axiom for $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ (and $DG\mathbb{Z}\text{-Mod}$) in the projective model context. To do this, Shipley creates **injective** stable model structures on the two categories of spectra ([Shi, Proposition 3.2]), by setting the class of injective weak equivalences to be the class of stable equivalences, and the class of injective cofibrations to be the level cofibrations, i.e., monomorphisms. Notice again that the class of weak equivalences in both stable structures on spectra is the same.

The following proposition is crucial in showing the monoid axiom holds on $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ and $DG\mathbb{Z}\text{-Mod}$. It claims that there is an action of the respective projective stable model categories on the injective stable model categories, and that this action satisfies an analogue of the pushout product axiom.

Proposition 6.1.5. [Shi, Proposition 3.3] *If $i : A \rightarrow B$ is an injective cofibration in $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ or $DG\mathbb{Z}\text{-Mod}$, and $i' : K \rightarrow L$ is a projective cofibration on the same underlying category, then the induced map*

$$i \square i' : A \otimes L \coprod_{A \otimes K} B \otimes K \longrightarrow B \otimes L$$

is an injective cofibration, which is a weak equivalence if either i or i' is.

The key argument in establishing the monoid axiom is the following ([Shi, Corollary 3.4]). Suppose that $f : A \xrightarrow{\sim} B$ is a projective acyclic cofibration in any of the three categories $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ or $DG\mathbb{Z}\text{-Mod}$. To establish (MA) in the projective case, one needs to show that any map

obtained by pushouts and (transfinite) compositions from maps of the form $A \wedge Z \longrightarrow B \wedge Z$, where Z is an object in the corresponding category, is a projective stable equivalence.

By definition, in any of the three categories the injective cofibrations are monomorphisms, hence any object Z is injective cofibrant, i.e., the map from the initial object $\iota : \Phi \rightarrow Z$ is an injective cofibration. Apply Proposition 6.1.5 to maps $\iota : \Phi \rightarrow Z$ and $f : A \xrightarrow{\sim} B$ to obtain the pushout diagram

$$\begin{array}{ccc}
 \Phi \cong A \wedge \Phi & \longrightarrow & B \wedge \Phi \cong \Phi \\
 \downarrow & & \downarrow \\
 A \wedge Z & \xrightarrow{\cong} & A \wedge Z \\
 & \searrow f \wedge \text{Id} & \downarrow f \square \iota \\
 & & B \wedge Z.
 \end{array}$$

$\text{Id} \wedge \iota$

and to deduce that smashing with Z takes projective acyclic cofibrations in $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ or $DG\mathbb{Z}\text{-Mod}$ to injective acyclic cofibrations in these categories. Since pushouts and directed colimits of monomorphisms are again monomorphisms, the maps in $(f \wedge Z)$ -cell are injective acyclic cofibrations, hence, in particular, injective weak equivalences in $Sp^\Sigma(sAb)$, $Sp^\Sigma(Ch_+)$ or $DG\mathbb{Z}\text{-Mod}$. By definition of the involved model structures, the injective stable equivalences agree with the projective stable equivalences, and this establishes the monoid axiom for the projective structures.

The units in $Sp^\Sigma(sAb)$ and in $Sp^\Sigma(Ch_+)$ are cofibrant

We need to show that $\tilde{\mathbb{Z}}\mathbb{S}$ and $\text{Sym}(\mathbb{Z}[1])$ are cofibrant in their respective categories of spectra, which will give condition $\textcircled{4}$ for $Sp^\Sigma(sAb)$ and $Sp^\Sigma(Ch_+)$. The idea is to exploit the definition of the categories of spectra as categories of modules, and to apply Theorem 3.2.7 in both cases to obtain the result. We will give a detailed proof for $Sp^\Sigma(Ch_+)$. As the argument for the category $Sp^\Sigma(sAb)$ follows in a similar vein, we will only give a few comments on that case.

Note first that $\mathbb{Z}[1]$ is a cofibrant object in Ch_+ for the same reasons that $\mathbb{Z}[0]$ is, by Corollary 6.1.4.

Proposition 6.1.6. *The unit $\text{Sym}(\mathbb{Z}[1])$ in $(Sp^\Sigma(Ch_+), \wedge, \text{Sym}(\mathbb{Z}[1]))$ is cofibrant.*

Proof. Consider the adjunction

$$\text{Sym}(\mathbb{Z}[1]) \otimes - : Ch_+^\Sigma \rightleftarrows \text{Sym}(\mathbb{Z}[1])\text{-Mod}_{Ch_+^\Sigma} : U$$

between the free $\text{Sym}(\mathbb{Z}[1])$ -module functor and the forgetful functor U . The cofibrantly generated model structure from Ch_+ is transferred to the

category of symmetric sequences Ch_+^Σ , where it is defined level-wise. Recall that the weak equivalences in Ch_+ are the quasi-isomorphisms, and the cofibrations are the monomorphisms. Since colimits in Ch_+^Σ are taken level-wise, and every object is small in Ch_+ , all objects in Ch_+^Σ are small as well (see Remark 2.3.3).

From Lemma 1.2.8 we know that Ch_+^Σ is given a symmetric monoidal structure, which is closed. To show that it is compatible with the model structure, we have to check the pushout product (PP) axiom (Definition 3.1.1) in Ch_+^Σ . By Lemma 3.1.10, it appears sufficient to check (PP) on the generating sets \mathcal{I} and \mathcal{J} of cofibrations and acyclic cofibrations.

Since the weak equivalences and the fibrations in Ch_+^Σ are defined level-wise, a map f in Ch_+^Σ is a cofibration (i.e., has LLP with respect to $WE \cap Fib$) if and only if there exists a collection of lifts for all f_m , $m \geq 0$, i.e., cofibrations are also defined level-wise. In particular, the generating set \mathcal{I} in Ch_+^Σ consists of a family of generating sets $\mathcal{I}_m := \{S^{n-1}(M) \hookrightarrow D^n(M)\}_{n \in \mathbb{Z}}$, and similarly, \mathcal{J} is a family $\mathcal{J}_m := \{0 \hookrightarrow D^n(M)\}_{n \in \mathbb{Z}}$, for all $m \geq 0$, where M is a \mathbb{Z} -module (see Definition 2.2.18). To simplify the notation, we will write S^{n-1} and D^n in the rest of the proof.

For all $m \geq 0$, let $i_m : S^{n-1} \hookrightarrow D^n$ be a generating cofibration, and let $f : A \hookrightarrow B$ be a cofibration in Ch_+^Σ . The map

$$f_m \square i_m : A_m \otimes D^n \coprod_{A_m \otimes S^{n-1}} B_m \otimes S^{n-1} \longrightarrow B_m \otimes D^n$$

is induced by the pushout diagram

$$\begin{array}{ccc} A_m \otimes S^{n-1} & \xrightarrow{f_m \otimes \text{Id}} & B_m \otimes S^{n-1} \\ \text{Id} \otimes i_m \downarrow & & \downarrow \\ A_m \otimes D^n & \longrightarrow & A_m \otimes D^n \coprod_{A_m \otimes S^{n-1}} B_m \otimes S^{n-1}(M) \\ & \searrow f_m \otimes \text{Id} & \downarrow \text{Id} \otimes i_m \\ & & B_m \otimes D^n \end{array}$$

(A dashed arrow labeled $f_m \square i_m$ points from the middle object to the bottom object.)

is a cofibration, for all $m \geq 0$, because the (PP) axiom holds in Ch_+ . Hence $f \square i$ is a cofibration in Ch_+^Σ . If in addition the map $f : A \hookrightarrow B$ is a weak equivalence in Ch_+^Σ , i.e., $f_m \in WE_{Ch_+}$ for all $m \geq 0$, the (PP) in Ch_+ implies that $f \square i$ is a weak equivalence in Ch_+^Σ as well.

Now, for all $m \geq 0$ let $j_m : 0 \xrightarrow{\sim} D^n$ be a generating acyclic cofibration, and let $f : A \hookrightarrow B$ be a cofibration in Ch_+^Σ . A collection of pushout product diagrams for all m , similar to the one above induces maps $f_m \square j_m$ which are acyclic cofibrations, since the (PP) axiom holds in Ch_+ . We therefore conclude that (PP) holds in Ch_+^Σ .

By Lemma 3.1.10 again, it is sufficient to check the monoid axiom (MA) for the set \mathcal{J} of generating acyclic cofibrations in Ch_+^Σ . Let $j : 0 \xrightarrow{\sim} D^n$ in \mathcal{J}

and $Z \in Ch_+^\Sigma$. The map $j \otimes Z : 0 \longrightarrow D^n \otimes Z$ in Ch_+^Σ is given by a collection of Σ_n -equivariant maps $j_m \otimes Z : 0 \longrightarrow D^n \otimes Z$ for all $m \geq 0$ in Ch_+ . Since the (MA) is true for Ch_+ , each map $j_m \otimes Z$ is a weak equivalence. Hence, $j \otimes Z$ is a weak equivalence in Ch_+^Σ .

The unit $(\mathbb{Z}[0], 0, 0, \dots)$ of the symmetric monoidal product in Ch_+^Σ is cofibrant, since it is cofibrant level-wise (see Corollary 6.1.4). Also, $\text{Sym}(\mathbb{Z}[1])$ is a commutative monoid in Ch_+^Σ by Definition 1.2.10. The category Ch_+^Σ then satisfies all the hypotheses of Theorem 3.2.7, and a morphism f in $\text{Sym}(\mathbb{Z}[1])\text{-Mod}_{Ch_+^\Sigma}$ is a weak equivalence (a fibration) if and only if Uf is a weak equivalence (a fibration) in Ch_+^Σ . Hence U preserves fibrations and acyclic fibrations, making $(\text{Sym}(\mathbb{Z}[1]) \otimes -, U)$ into a Quillen pair. Lemma 2.1.26 implies that the left adjoint $\text{Sym}(\mathbb{Z}[1]) \otimes -$ preserves cofibrations. Hence, to prove that $\text{Sym}(\mathbb{Z}[1]) \cong \text{Sym}(\mathbb{Z}[1]) \otimes_{\mathbb{Z}[0]} \mathbb{Z}[0]$ is cofibrant, it is sufficient to show that $\mathbb{Z}[0]$ is cofibrant in Ch_+^Σ , which follows from Corollary 6.1.4. \square

Proposition 6.1.7. *The unit $\tilde{\mathbb{Z}}\mathbb{S}$ in $(Sp^\Sigma(sAb), \wedge, \tilde{\mathbb{Z}}\mathbb{S})$ is cofibrant.*

First, we need a lemma.

Lemma 6.1.8. *The simplicial abelian group $\mathbb{Z} \cong \tilde{\mathbb{Z}}(S^0)$ is cofibrant in sAb .*

Proof. From Remark 2.2.24 and the fact that S^0 is cofibrant in $sSet_*$, we conclude that $\tilde{\mathbb{Z}}(S^0)$ is cofibrant in sAb , since $\tilde{\mathbb{Z}}$ is a left Quillen functor and preserves initial objects. \square

Comments on the proof of 6.1.7: Here the adjunction to consider is

$$\tilde{\mathbb{Z}}\mathbb{S} \otimes - : sAb^\Sigma \rightleftarrows \tilde{\mathbb{Z}}\mathbb{S}\text{-Mod}_{sAb^\Sigma} : U,$$

where the left adjoint is the free $\tilde{\mathbb{Z}}\mathbb{S}$ -module functor.

To show that there exists a cofibrantly generated model monoidal structure on sAb^Σ and that the (MA) is satisfied, one proceeds with arguments similar to those given for Ch_+ and Ch_+^Σ . Recall that the generating sets for sAb^Σ are collections $\mathcal{I}_m := \{\mathbb{Z}(\partial\Delta[n]) \longrightarrow \mathbb{Z}(\Delta[n])\}$ and $\mathcal{J}_m := \{\mathbb{Z}(\Lambda^i[n]) \longrightarrow \mathbb{Z}(\Delta[n])\}$ for $n \geq 0$, $0 \leq i \leq n$ and for all levels $m \geq 0$.

Observe that the unit $(\mathbb{Z}, 0, 0, \dots)$ of the symmetric monoidal product in sAb^Σ is cofibrant (see Lemma 6.1.8), and, finally, note that $\tilde{\mathbb{Z}}\mathbb{S} \cong \text{Sym}(\tilde{\mathbb{Z}}S^1)$ is a commutative monoid in sAb^Σ by Definition 1.2.10. \square

6.2 Conditions on functors

The adjoint pairs (Z, U) and (D, R) are shown to be strong monoidal Quillen equivalences, and the pair (L, ϕ^*N) to be a weak monoidal Quillen equivalence respectively in Propositions 4.3, 4.7 and 4.4 of the target article of

Shipley. The proofs turn out to be technically involved; they use complex criteria and methods from [SS03a] and [SS03b]. We therefore decided to refer the reader to the article to find out the details, and only make a few general remarks here.

There are several things to check; to start with, the fact that each pair of functors is adjoint and that it is a Quillen pair.

In order to show that they are compatible with the monoidal structure, Shipley uses a criterion for Quillen pairs between monoidal stable model categories established in [SS03b], Proposition 3.17. For this criterion to be applied, each right adjoint $R : \mathcal{C} \rightarrow \mathcal{D}$ has to be lax monoidal, and the map $\tilde{\nu} : \lambda(\mathbb{I}_{\mathcal{D}}) \rightarrow \mathbb{I}_{\mathcal{C}}$ has to be a weak equivalence in \mathcal{C} . Furthermore, the unit $\mathbb{I}_{\mathcal{D}}$ has to *stably detect weak equivalences*, see Section 3.4 in [SS03b]. To understand this definition, one first has to become familiar with the notion of *cosimplicial objects* in a model category and with the specific model structure they admit, called the *Reedy model structure*, see [Hir, Chapter 15], [Hov99, Chapter 5] or [GJ, VII]. The notion of a *cosimplicial resolution* (or *frame*), introduced by Dwyer and Kan [DK, 4.3], then allows to define what it means to (stably) detect weak equivalences.

Once this criterion is verified, one concludes that all functor pairs in the zig-zag are weak monoidal Quillen pairs, and it remains to show that these adjunctions are Quillen *equivalences*. A sufficient condition for a Quillen pair to be an equivalence is offered by [HSS]:

Lemma 6.2.1. [HSS, Lemma 4.1.7] *Suppose $L : \mathcal{C} \rightarrow \mathcal{D}$ is a left Quillen functor with right adjoint R , and suppose K denotes a fibrant replacement functor on \mathcal{D} . Suppose R detects and preserves weak equivalences between fibrant objects, and the composition*

$$X^c \longrightarrow RLX^c \xrightarrow{Ri} RK LX^c$$

is a weak equivalence for all cofibrant objects X^c of \mathcal{C} . Then the pair (L, R) is a Quillen equivalence.

This is the condition Shipley will use for all of the three Quillen pairs.

Conclusion

In the stable homotopy theory of topological spaces one is concerned with all structures and phenomena that remain after sufficiently many applications of the suspension functor. In the modern treatment of stable homotopy, spaces are replaced with spectra. A stable homotopy category of spectra can then be created by inverting the stable homotopy equivalences. It is equipped with “nice” properties; however, operating with objects and carrying out calculations in this context can be difficult, especially in a purely topological case.

Things can be made easier when manipulating a simplicial combinatorial version of symmetric spectra, where the existence of a symmetric monoidal smash product enables one furthermore to use methods of homological algebra and facilitates the calculations within a rigid algebraic context.

Working on the result of Brooke Shipley allowed us to discover, study and analyze the existence of a Quillen equivalence between $H\mathbb{Z}$ -algebra spectra and differential graded \mathbb{Z} -algebras. To succeed in showing this correspondence, it was crucial for Shipley to show that the homotopy theory and the algebraic structures “in both worlds” can be combined in a good way. This required the building of a connecting zig-zag of functors, involving two intermediate categories. Using various complex techniques and a number of previously established results permitted her to prove that monoidal structures on all of the involved categories were sufficiently compatible with corresponding Quillen model structures, since the functors in the zig-zag respected the product structures. This fact was essential to formulate rigorously the statement of a Quillen equivalence between the two initial categories.

To be able to translate back and forth between the language of $H\mathbb{Z}$ -spectral algebras or, by restriction \mathbb{S} -algebras, and the language of differential graded algebras has appeared fruitful for the purposes of [BK], [BG], [BJP], [TB] and allows one to extend the results of [DS06] from ring spectra to algebra spectra.

Appendix

A When the sphere spectrum becomes a *commutative* monoid?

Given a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{I}_{\mathcal{C}})$, it is possible to construct the category of (non-symmetric) sequences in \mathcal{C} , where one “forgets” about the symmetric group actions. One can then describe *non-symmetric* spectra over \mathcal{C} as modules over (an analog of) the sphere spectrum in $\mathcal{C}^{\mathbb{N}}$.

The problem is that then the analog of the sphere spectrum is not a *commutative* monoid. This fact prevents one from defining a suitable closed symmetric monoidal smash product on non-symmetric spectra.

In this Appendix we briefly define non-symmetric sequences and spectra over \mathcal{C} and explain why the monoid $T(K) := (\mathbb{I}, K, K \otimes K, \dots, K^{\otimes n}, \dots)$, for K any object of \mathcal{C} , fails to be a commutative monoid in this context. Throughout this section $(\mathcal{C}, \otimes, \mathbb{I}_{\mathcal{C}})$ is a symmetric monoidal category.

Definition A.1.

- The category \mathbb{N} is the category with the non-negative integers as its objects and with identity maps of the objects as its only maps. A **sequence** in \mathcal{C} is a functor $X : \mathbb{N} \rightarrow \mathcal{C}$. The functor category $\mathcal{C}^{\mathbb{N}}$ is called the **category of sequences in \mathcal{C}** .
- Given $X, Y \in \mathcal{C}^{\mathbb{N}}$, their **graded tensor product** $X \otimes Y$ is the sequence given at level n by

$$(X \otimes Y)_n := \coprod_{p+q=n} (X_p \otimes Y_q),$$

for all $n \geq 0$. Here \coprod denotes the coproduct in \mathcal{C} .

The following lemma extends Lemma 2.3.3 in [HSS].

Lemma A.2. *The category of sequences is a bicomplete category, and the graded tensor product \otimes endows the category $\mathcal{C}^{\mathbb{N}}$ with a symmetric monoidal structure $(\mathcal{C}^{\mathbb{N}}, \otimes, \mathbb{I})$, with $\mathbb{I} := (\mathbb{I}_{\mathcal{C}}, \Phi, \Phi, \dots)$.*

Lemma A.3. *Given any K object of \mathcal{C} , the sequence*

$$T(K) := (\mathbb{1}, K, K \otimes K, \dots, K^{\otimes n}, \dots)$$

is a monoid in $\mathcal{C}^{\mathbb{N}}$.

Proof. The monoid structure on $T(K)$ is defined exactly as in Definition 1.2.10. The multiplication $m : T(K) \otimes T(K) \rightarrow T(K)$ is given by concatenation of terms, and the unit $\nu : \mathbb{1} \rightarrow T(K)$ is induced by the maps from the initial object Φ in \mathcal{C} . \square

Let us see why the monoid $T(K)$ is not commutative. The twist isomorphism is induced by $\tau_{p,q} : K^{\otimes p} \otimes K^{\otimes q} \rightarrow K^{\otimes q} \otimes K^{\otimes p}$. For $T(K)$ to be commutative the diagram

$$\begin{array}{ccc} (T(K) \otimes T(K))_n & \xrightarrow{\tau_n} & (T(K) \otimes T(K))_n \\ m_n \downarrow & & \downarrow m_n \\ T(K)_n & \xlongequal{\quad\quad\quad} & T(K)_n \end{array}$$

must commute. It is equivalent to

$$\begin{array}{ccc} K^{\otimes p} \otimes K^{\otimes q} & \xrightarrow{\tau_{p,q}} & K^{\otimes q} \otimes K^{\otimes p} \\ m_{p,q} \downarrow & & \downarrow m_{p,q} \\ K^{\otimes p+q} & \xrightarrow{\cong} & K^{\otimes q+p}. \end{array}$$

Since the monoidal product on \mathcal{C} is assumed symmetric and associative, the bottom arrow is only an *isomorphism*. There is no action on $K^{\otimes n}$ which would turn this isomorphism into identity.

Definition A.4. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category and K an object in \mathcal{C} . The category of **non-symmetric spectra over \mathcal{C}** , denoted $Sp^{\mathbb{N}}(\mathcal{C})$ is the category of modules over the monoid $T(K)$ in $(\mathcal{C}^{\mathbb{N}}, \otimes, \mathbb{1})$.

Because the monoid $T(K)$ is not commutative, the category of non-symmetric spectra $Sp^{\mathbb{N}}(\mathcal{C})$ cannot be equipped with a closed symmetric tensor product over $T(K)$ (compare to Corollary 1.1.18 and Theorem 1.2.15). In other words, there is no closed symmetric monoidal smash product on $Sp^{\mathbb{N}}(\mathcal{C})$.

As we already know, working with symmetric sequences and spectra solves this problem. Having a good symmetric monoidal category of spectra is important, since it allows one to perform algebraic constructions on spectra that were impossible without such a category. In addition, the smash product of spectra interacts well with the model structure, so it becomes possible to construct a symmetric monoidal homotopy category of ring spectra and algebra spectra.

B Spectra and generalized cohomology theories

This part of Appendix has the following goals. Motivated by our previous study [Ka1], we first give an overview of reduced generalized cohomology theories defined on the category Top_* of pointed topological spaces and underline that Ω -spectra in $Sp(Top_*)$ represent cohomology theories satisfying the wedge axiom (this is *Brown representability*, [Brw63]).

Secondly, using the definition of the groups E^n of a cohomology theory associated to a spectrum E from [Swi], we make an attempt to understand the relation between the group E^0 thus defined, and the definition of the functor E^0 (2.3.7) involved in characterizing stable equivalences in the model category Sp^Σ .

B.1 Cohomology theories on Top_*

Definition B.1. Let Top_* denote the category of non-degenerately based spaces. A **reduced cohomology theory** h^* consists of a collection of contravariant functors and natural isomorphisms

$$h^n : \text{Ho } Top_* \longrightarrow Ab, \quad \sigma^n : h^n \xrightarrow{\cong} h^{n+1} \circ \Sigma,$$

for $n \in \mathbb{Z}$, satisfying the following axioms.

Exactness: If $i : A \hookrightarrow X$ is a cofibration, then the induced sequence

$$h^n(X/A) \xrightarrow{p^*} h^n(X) \xrightarrow{i^*} h^n(A)$$

is exact.

Weak homotopy equivalence (WHE): If $f : X \longrightarrow Y$ is a weak homotopy equivalence, then

$$f^* : h^n(Y) \xrightarrow{\cong} h^n(X)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Wedge axiom (W): For every collection $\{X_\alpha\}_{\alpha \in A}$ of pointed spaces, the inclusions $i_\beta : X_\beta \hookrightarrow \bigvee_{\alpha \in A} X_\alpha$ induce an isomorphism

$$i^* : h^n\left(\bigvee_{\alpha \in A} X_\alpha\right) \xrightarrow{\cong} \prod_{\alpha \in A} h^n(X_\alpha), \quad n \in \mathbb{Z}.$$

We will mention some examples of cohomology theories later. For the moment, let us point out that there are cohomology theories for which the axiom (W) is not true, unless the set A is finite; see [Hi, 1.8] for an example. However, the isomorphism in (W), with A an arbitrary set, is a sufficient condition for a cohomology theory to be *realizable*, and this property is important for the theories we are interested in.

Definition B.2. [Brw] A cohomology theory h^* is said to be **realizable** if for each $n \in \mathbb{Z}$, there exists a space Y_n , such that the contravariant functors h^n and $[-, Y_n]$ are natural equivalent.

The important thing is that one can construct cohomology theories using spectra with values in Top_* . Here, spectra need not to be assumed symmetric, so we write $Sp(Top_*)$ without any Σ or \mathbb{N} .

Let E in $Sp(Top_*)$ with structure maps $\sigma_n : \Sigma E_n \rightarrow E_{n+1}$. Let also X in Top_* and $k \in \mathbb{N}$. Consider the diagram

$$\begin{array}{ccc} [X, E_k] & \xrightarrow{\pi_{(0)}} & [\Sigma X, E_{k+1}], \\ & \searrow \Sigma & \nearrow (\sigma_k)_* \\ & & [\Sigma X, \Sigma E_k] \end{array}$$

where $\pi_{(0)} = (\sigma_k)_* \circ \Sigma$. For each $n \geq 0$ we can similarly define a morphism

$$\pi_{(n)} : [\Sigma^n X, E_{k+n}] \rightarrow [\Sigma^{n+1} X, E_{k+n+1}],$$

which is a homomorphism of abelian groups for $n \geq 2$.

In this way, to each space X and integer k we associate an abelian group

$$h_E^k(X) := \operatorname{colim}_n [\Sigma^n X, E_{k+n}].$$

The natural transformations $\sigma^k : h_E^k \rightarrow h_E^{k+1} \circ \Sigma$ are defined by passing to the colimit. It may be shown that σ^k thus defined is actually a natural isomorphism, see [Whi].

We will see in a moment that restricting to Ω -spectra gives important results in representability of cohomology theories.

Definition B.3. A spectrum $E \in Sp(Top_*)$ is an **Ω -spectrum** if the adjoints $\tilde{\sigma}_n : E_n \rightarrow \Omega E_{n+1}$ of the structure maps are homotopy equivalences (compare with Definition 2.3.7).

The adjunction $\Sigma : Top_* \rightleftarrows Top_* : \Omega$ induces an adjunction on the homotopy category as well. The bijective natural correspondence

$$[\Sigma B, A] \cong [B, \Omega A]$$

of homotopy classes of maps is in fact an isomorphism of groups for all $A, B \in Top_*$, see [Sw, 2.14; 2.21].

Given a spectrum E and an integer k , take the sequence

$$[X, E_k] \xrightarrow{(\tilde{\sigma}_k)_*} [X, \Omega E_{k+1}] \xrightarrow{\Omega(\tilde{\sigma}_{k+1})_*} [X, \Omega^2 E_{k+2}] \rightarrow \dots$$

and define

$$h_E^k(X) := \operatorname{colim}_n [X, \Omega^n E_{k+n}],$$

which is equivalent to the first definition, by adjunction (Σ, Ω) .

Observe now that if E is an Ω -spectrum, we have $E_k \simeq \Omega^n E_{k+n}$, so the previous definition gives simply

$$h_E^k(X) := [X, E_k],$$

and we say that the cohomology theory h_E^* is **representable**, or **represented by the spectrum E** .

There are two important results on the representation of generalized reduced cohomology theories by Ω -spectra. The first one was implicitly suggested by the above construction and says that for every Ω -spectrum E there exists a reduced cohomology theory h_E^* associated to it.

Theorem B.4. [Sw, Theorem 8.42] *Let E be an Ω -spectrum. The collection of contravariant functors $h_E^n : \text{Ho Top}_* \rightarrow \text{Ab}$ given by*

$$h_E^n(X) := [X; E_n]$$

and of natural isomorphisms $\sigma^n : h_E^n \rightarrow h_E^{n+1} \circ \Sigma$ given by

$$[X, E_n] \xrightarrow[\cong]{\tilde{\sigma}_*^n} [X, \Omega E_{n+1}] \xrightarrow[\cong]{} [\Sigma X, E_{n+1}]$$

defines a generalized reduced cohomology theory h_E^* satisfying the wedge axiom (W), for all $n \in \mathbb{Z}$, $X \in \text{Top}_*$.

The converse exists and follows from the Brown Representability Theorem (see [Sw, Theorem 9.12]).

Theorem B.5. [Sw, Theorem 9.27] *Let k^* be a generalized reduced cohomology theory satisfying the wedge axiom (W). Then there exists an Ω -spectrum E , such that $k^n(X) \xrightarrow[\cong]{} h_E^n(X)$ is a natural isomorphism of cohomology theories for all $n \in \mathbb{Z}$, $X \in \text{Top}_*$.*

We will give only three examples among a very large variety of existing cohomology theories.

Examples B.6.

- The **singular cohomology** H^* with coefficients in an abelian group A , is an example of a general cohomology theory. It is associated to the Eilenberg-Mac Lane Ω -spectrum HA , which in degree n is given by $HA_n := K(A, n)$. Here $K(A, n)$ is the *Eilenberg-Mac Lane space*, i.e., a connected topological space with homotopy concentrated in degree n , all other homotopy groups of it are trivial. By Brown representability we have

$$H^n(X, A) := [X, K(A, n)].$$

Note that the singular cohomology satisfies in addition the *dimension axiom*, which asserts that for a 0-sphere S^0 , $H^n(S^0) = 0$ if $n \neq 0$.

- The **complex K-theory** K^* is represented by the Ω -spectrum KU . It has spaces defined by

$$\begin{cases} KU_{2i} := BU \times \mathbb{Z} \\ KU_{2i+1} := U, \end{cases}$$

for all $i \in \mathbb{Z}$. Here U is the unitary group and BU is the space defined by $BU := \text{colim}_{n,k} G_n(\mathbb{C}^k)$, where $G_n(\mathbb{C}^k)$ denotes the complex Grassmanian manifold. For every pointed CW -complex X , the K -groups of X are defined inductively by

$$K^n(X) \cong K^{n-2}(X).$$

Complex Bott periodicity implies that these groups have period 2.

- The **stable cohomotopy** $\pi_{\mathbb{S}}^*$ is the cohomology theory associated to the sphere spectrum \mathbb{S} . For a pointed topological space X , Its groups are

$$\pi_{\mathbb{S}}^n(X) := \text{colim}_k [\Sigma^k X, S^{n+k}],$$

the stable cohomotopy groups of X .

B.2 The group E^0 of cohomology of spectra

Let E be an Ω -spectrum, and denote by E^* a cohomology theory associated to it. According to Switzer, [Sw, 8.40], one can extend E^* from Top_* to $Sp(Top_*)$ by defining

$$E^n(F) := [F, \Sigma^n E]$$

for all topological spectra F in $Sp(Top_*)$. In particular, the 0^{th} cohomology group is then

$$E^0(F) := [F, E]$$

in $\text{Ho } Sp(Top_*)$. It can be given the structure of an abelian group for all spectra E, F ([Swi, 8.27]).

Remark B.7. The notion of homotopy on $Sp(Top_*)$ is defined in [Swi, 8.20]. One must be able to show that, since left (and right) homotopy are equivalence relations for all spaces in Top_* , it follows that the homotopy relation on $Sp(Top_*)$ is also an equivalence relation for all topological spectra F, E .

Now, let us come back to the simplicial context and to what we have learned before. Recall from Chapter 2 that a map $f : X \rightarrow Y$ is a stable equivalence in $Sp^{\Sigma}(sSet_*)$, if, by definition, the induced map $E^0(f) : E^0(Y) \rightarrow E^0(X)$ is an isomorphism for all injective Ω -spectra E , where

$$E^0 := \pi_0(\text{Map}_{Sp^{\Sigma}}(-, E)) : Sp^{\Sigma} \rightarrow \text{Set}.$$

We will try to investigate the relationship between these two definitions of E^0 . Namely, by transferring the definition of Switzer in the simplicial

context and unrolling the one given in [HSS], we would like to see them coincide.

First there are the following facts to remember.

If X is a Kan complex (Definition 2.3.7) in $sSet$, the 0^{th} simplicial homotopy group of X is given by

$$\pi_0(X) = X_0 / \partial_0(y) \sim \partial_1(y)$$

for each $y \in X_1$ (see [Wei, 8.3.2]). In other words,

$$\pi_0(X) = \operatorname{colim} \left(X_1 \begin{array}{c} \xrightarrow{\partial_1} \\ \xrightarrow{\partial_0} \end{array} X_0 \right).$$

One can show that $\operatorname{Map}_{Sp^\Sigma}(X, E)$ is a Kan complex for any injective spectrum E , and for all $X \in Sp^\Sigma$ (see [IS, Lemma 2.6.12]). Therefore,

$$\begin{aligned} E^0(X) &:= \pi_0(\operatorname{Map}_{Sp^\Sigma}(X, E)) \\ &= \operatorname{colim} \left(Sp^\Sigma(X \wedge \Delta[1]_+, E) \begin{array}{c} \xrightarrow{\partial_1} \\ \xrightarrow{\partial_0} \end{array} Sp^\Sigma(X \wedge \Delta[0]_+, E) \right). \end{aligned}$$

Writing things explicitly, it appears that the colimit is given by all maps $X \wedge \Delta[0]_+ \rightarrow E$, modulo homotopy relation. Hence,

$$\begin{aligned} E^0(X) &\cong Sp^\Sigma(X \wedge \Delta[0]_+, E) / \simeq \\ &\cong [X \wedge S^0, E] \\ &\cong [X, E] \end{aligned}$$

in $\operatorname{Ho} Sp^\Sigma$. The homotopy classes are well-defined, since E is injective, which is analogous to fibrant for a model structure where every spectrum is cofibrant, see [HSS, Section 5.1].

We realize that this tends to match what would be a “simplicial imitation” of the definition given in [Swi], by setting $E^0(X) := [X, E]$ in $\operatorname{Ho} Sp(sSet_*)$ with appropriate fibrancy/cofibrancy conditions on X and E , requested by the rules of a model category. However, lacking in any solid reference that would help us to polish this observation, we remain skeptical. One of the main difficulties here was that most sources on generalized cohomology theories we had at hand only treated the “traditional” case of theories, defined on spaces. In papers devoted to spectra, the cohomology theories were hardly addressed.

Coming to think of it, the following questions arose: would a notion of a certain “cohomology theory”, associated to a spectrum E in $Sp^\Sigma(\mathcal{C})$ for \mathcal{C} an arbitrary model monoidal category (at least such that the category $Sp^\Sigma(\mathcal{C})$ is meaningful) make any sense? Does it exist? Cohomology theories are relevant for the categories of topological spaces, CW-complexes and

pointed simplicial sets (i.e., for categories that have “space-like” objects), but are there any other such categories? Certainly, cohomology theories are part of topological invariants, created to study and obtain information on topological spaces. So, the question is whether it would be anyhow useful, if even possible, to consider similar invariants on other categories? It might be, providing they are reasonably operational, they give interesting information, and can be wisely related to other tools, available to solve topological problems.

Happy is the reader who has better answers than we do on these questions! We will continue our inquiry...

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