Index Assignment for N Balanced Multiple Description Scalar Quantization

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Abstract

In this paper, we address the design of any *number of balanced multiple descriptions using the multiple description scalar quantization(MDSQ) technique. The proposed scheme has the advantages of low complexity, the possibility of being extended easily to any number of descriptions and the possibility to trade off between the side, partial and central distortions. Unlike existing schemes, it can produce balanced descriptions at low rates, at the price however of a slightly higher distortion. The behavior of the proposed index assignment at high rate is in the same time similar to state-of-the-art schemes. The proposed scheme offers the possibility to adapt to loss probability, and rate constraints, in playing with both the number of descriptions, and the rate of each of them, to minimize the average distortion. The comparison with the systematic FEC* (N, k) *scheme shows that the FEC scheme in general gives smaller average distortion, but that our scheme seems to be more robust to sudden changes in network conditions and that receiving all the descriptions in general gives smaller distortions.*

I. Introduction

The multiple description (MD) coding was invented at Bell Laboratories, by Miller [1], in the late seventies, in connection with improving the reliability of the telephone network without the standby links. The idea of Jayant was to split and send the speech samples over two channels, one carrying odd and another even samples. Normally, both halves are received and the signal is reconstructed with the highest quality. However, receiving only half of the signal still guarantees the continuation of communication, though at reduced quality.

In parallel to this work, Witsenhausen realized that the channel splitting makes a very interesting information theoretic problem. At the information theory workshop in 1979, he posed the following problem: *"given the rates of the two descriptions, what are the qualities of these descriptions taken separately and jointly?"*. Since then, this problem has been known as the multiple description (MD) problem which, in the first decade, was considered only as an information theoretic problem. In that context, the research was focused on finding the set of simultaneously achievable rates and distortions for a particular source, known as the MD region. The first result on this topic was given by El Gamal and Cover [2] and it was followed by [3], [4] and [5] and many others.

With the very fast development of the lossy packet networks, such as Internet, the MD coding found a nice practical application. Namely, the goal of Multiple Description (MD) coding is the generation of N descriptions of a source, where each of them sent over the lossy network and possibly over a different path. All the descriptions are independently decodable, therefore the signal can be reconstructed even if only one of them is received. In addition, the more of them arrive at the decoder, the better the reconstructed signal will be. These properties of MD coding make it superior to existing techniques, like hierarchical or scalable coding, where losing a part of the information can cause dramatic degradations. Thus, it is of great practical importance to have a scheme that provides the generation and the reconstruction of arbitrary number of descriptions and under various loss scenarios.

However, the practical application of MD codes is usually limited to only two descriptions. The most common solution for producing $N > 2$ descriptions, the unequal error protection (UEP) based method, is optimized for exactly $N - k$ packet losses, $1 \leq k < N$. If fewer packets are lost, there is no gain in the quality, while if more packets are lost, there is a sharp degradation of performance. Finally, k and N are chosen based on the network state, that is highly varying, which makes the implementation of this scheme more difficult.

Multiple description scalar quantization (MDSQ) has given the first practical solution for generating two descriptions, [6]. Though quite simple, this scheme has remarkable asymptotic properties. Moreover, comparing to other techniques, it is less difficult to extend it to arbitrary number of descriptions. Still, most of the research has been focused on only two descriptions.

In this paper, we propose a simple method for producing any number N of descriptions, based on the multiple description scalar quantization (MDSQ). To our best knowledge, this is the only solution proposed in the literature that produces N descriptions with this technique. The proposed design easily extends to any number of descriptions, while keeping a very low complexity. We show that, in the case of uniformly distributed sources, we can achieve balanced descriptions even at low rates, which is not possible with state-of-the-art MDSQ schemes. Moreover, our scheme can easily trade off the distortions,

giving priority to some of them, depending on the lossy scenario. Finally, the fact that we can generate any number of descriptions is advantageously used to optimize the average distortion for given loss probabilities and rate constraints. It provides the flexibility to play with both the coding rates, and the number of descriptions, without being penalized by the cliff-effect observed in UEP-based solutions.
The paper is organized as follows. Bla bla nja nja

II. Overview of the MD coding techniques

A. Preliminaries

Suppose we want to generate N descriptions of a stationary ergodic source X, with a probability density function $p(x)$, which takes values in a finite alphabet $\mathcal{X}$. Description $n \in \{1, 2, ...N\}$, taking values in the alphabet $\mathcal{X}_n$, is sent over the channel n at the rate of R_n bits/sample. Here R_n corresponds to the average codeword length at the output of variable length encoder (in the entropy-constrained case), or to the rate at the output of the fixed-length encoder (which corresponds to the level-constrained case). Therefore, in the entropy-constrained case, the rate R_n is given by:

$$R_n = H_n = -\sum_{i=1}^{|\mathcal{X}_n|} p_i log_2 p_i \quad (1)$$

where p_i denotes the probability of $i \in \mathcal{X}_n$ and $|\mathcal{X}_n|$ denotes the cardinality of $\mathcal{X}_n$. In the level constrained case, the rate is given by:

$$R_n = log_2 |\mathcal{X}_n| \quad (2)$$

During the transmission some of the channels may fail, in which case the descriptions sent over these channels are completely lost. Suppose a set $\mathcal{A} \subseteq \{1, 2, ..., N\}$ of descriptions is received and let $Y_{\mathcal{A}}$, taking the values in alphabet $\mathcal{Y}_{\mathcal{A}}$, be the reconstructed value in that case.
Let $d(x, y) = f(|x - y|) \leq d_{max}$ be the bounded single-letter distortion measure, where $f(\cdot)$ is a nonnegative convex function with the only null point in 0. Receiving a set $\mathcal{A}$ of descriptions causes the distortion $D_{\mathcal{A}} = E[d(X, Y_{\mathcal{A}})]$. When $|\mathcal{A}| = 1$, we call the distortions $D_{\mathcal{A}} = D_n$ the *side* distortions. Receiving all the descriptions, i.e. $|\mathcal{A}| = N$, causes the *central* distortion D, while all the other distortions are called the *partial* distortions.
In general, MD systems that produce N descriptions involve $2^N + N - 1$ parameters: N rates and $2^N - 1$ possible sets of received descriptions. The performance analysis and the optimization of such scheme are not trivial. However, this problem can be reduced to only $N + 1$ dimensions under the assumption that all the rates are equal, $R_n = R$, and that any k received descriptions out of N always cause equal

distortions. This is the so called ***balanced*** case, that will be considered through the rest of the paper. The balanced case is somewhat motivated by the fact that the descriptions with the same rates will be equally treated from the network. This case is also interesting because the distortions depend only on the number of received descriptions, and not on which specific set is received.

The complete characterization of the achievable rates and distortions is given only for the case of two descriptions and for continuous sources with mean squared error distortion. For the special case when descriptions are balanced, Zamir [7] showed that the following expressions hold:

$$D_1 \geq P_x 2^{-2R} \tag{3}$$

$$D_{12} \geq \frac{P_x 2^{-4R}}{1-(\sqrt{\pi}-\sqrt{\Delta})^2} \tag{4}$$

where $P_x = \frac{1}{2\pi e}2^{2h(p)}$, $\pi = (1-\frac{D_1}{P_x})^2$ and $\Delta = \frac{D_1^2}{P_x^2} - 2^{-4R}$. Here $h(p)$ denotes the differential entropy of the source, defined as $h(p) = -\int_S p(x)logp(x)dx$, where $p(x)$ is a pdf of the source and S is the support set of the random variable. This product is tight when $R \rightarrow \infty$. It shows one very interesting relation between the side and the central distortion. Namely, one cannot decrease the central distortion if the side distortion is not increased and vice versa. Thus the tradeoff between the distortions.

However, rather than considering separately how distortions D_1 and D_{12} depend on the rate, in most of the cases the *distortion product* is used to qualify the "goodness" of the scheme. It can be shown that the following expression holds at high rate for the distortion product:

$$D_1 \cdot D_{12} \geq \frac{P_x^2}{4} 2^{-4R} \tag{5}$$

In the next section, some results on the existing distortion products will be given.

The excess rate, or redundancy, is also used to qualify the MDC schemes. It is defined as:

$$\rho = R_{tot} - R^* \tag{6}$$

where R_{tot} is the total rate budget used by MD coder and the R^* represents the rate required by a single description coder to achieve the same distortion comparing to the multiple description case. This measure tells us how robust to losses one scheme is.

B. *Multiple Description Scalar Quantization*

The design of an MDSQ system generally follows two steps: a scalar quantization and an index assignment method (the case of three descriptions can be represented as in Figure 1). The scalar quantizer, like in the single description case, maps the continuous random variable to the discrete set of quantized values. The index assignment is introduced to produce multiple descriptions of such a quantized value, by mapping it to a N-tuple of quantization indices $(i_1, i_2, ..., i_N)$. Equivalently, we can think of the MDSQ as having N scalar quantizers, each of them used to produce one of the descriptions. Each indice (or equivalently, each quantized version of the source) is then sent over a different channel and if all of them are correctly received, the signal will be reconstructed with the highest quality. If any channel fails, the decoder still reconstructs a version of the signal, though with a lower quality.

If a set $\mathcal{A}$ of descriptions is received, after the inverse index assignment, they are decoded with the inverse quantizer $IQ_{\mathcal{A}}$. The inverse quantizer simply takes the centroid of the maximum of all received quantization low levels and the minimum of all upper levels. Similarly to previous notation, if $|\mathcal{A}| = 1$, we call $IQ_{\mathcal{A}}$ the *side* inverse quantizer, while if $|\mathcal{A}| = N$ we call it the *central* inverse quantizer. All the other inverse quantizers are called the *partial* inverse quantizers.

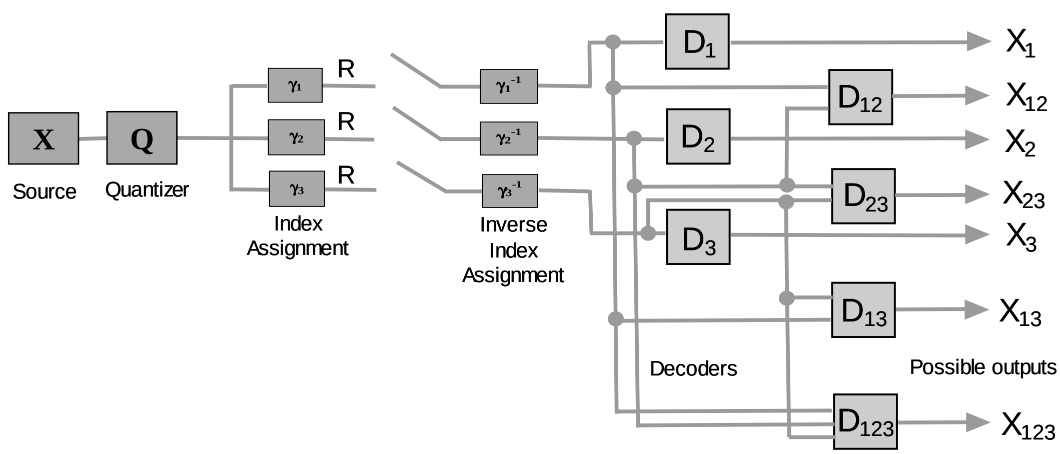


Fig. 1. Three Description Coding Scheme.

In the case of entropy constrained MDSQ, the rate of the description can be calculated in the following way:

$$R_n = H = -\sum_i p_i log_2 p_i \tag{7}$$

where p_i denotes the probability of indice i. On the other hand, in the case of the *level constrained* quantization, the rate is given by:

$$R_n = log_2 K \tag{8}$$

where K is the number of levels in quantizer Q_n.

Let the bin size that corresponds to indice i after receiving a set $\mathcal{A}$ of descriptions have the size $\Delta_{i_\mathcal{A}}$ and let its lower and upper bounds be $L_{\mathcal{A}_i}$ and $U_{\mathcal{A}_i}$ respectively. Then the distortion $D_\mathcal{A}$ can be written in the following way:

$$D = \sum_{\Delta_i} \int_{L_{\mathcal{A}_i}}^{U_{\mathcal{A}_i}} p(x)(x - \widehat{y_{\mathcal{A}_i}})^2 dx \tag{9}$$

where $\widehat{y_{\mathcal{A}_i}}$ is the centroid of the bin corresponding to the indice i.

The first idea for producing two balanced descriptions was very simple: use two quantizers that are offset to each other half of the quantization step size, see Figure 2(a). Such scheme gives completely balanced descriptions for symmetric pdf's both in terms of rates and distortions since both quantizers have the same set of bins. Receiving both descriptions results in approximately four times smaller distortion than in the case when only one description is received. This scheme is called the *staggered* index assignment scheme. It can be extended to the case of three descriptions (Figure 2(b)); however, it fails to give balanced descriptions at low rates since the quantizer Q_1 has different bins than quantizers Q_2 and Q_3 and thus different rates and distortions. Extending it to four and more descriptions gives even bigger disbalance both for rates and distortions and therefore is not used.

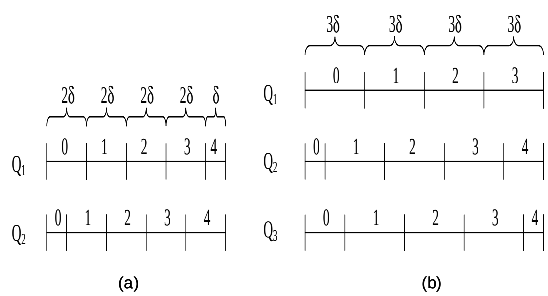


Fig. 2. Staggered index assignment for: (a) $N = 2$ and (b) $N = 3$.

Though very simple, the idea of staggered bins suffers from the high level of redundancy, thus being suitable when the losses in the network are high. In order to reduce the redundancy, the authors in [6], [8] proposed using noncontiguous quantization bins. They build the index assignment matrices, where the row corresponds to one of the descriptions and the column to another one. Making such matrices always starts with filling the main diagonal and it is subsequently followed by adding a certain number of diagonals below and above the main one. By varying the number of diagonals in such matrices, one can control the tradeoff between the side and central distortions. The authors showed later in [9] that the distortion product for their family of quantizers and in case of the level constrained quantizer satisfies:

$$D_1 \cdot D_{12} \geq \frac{3\pi^2 P_x^2}{16} 2^{-4R} \tag{10}$$

while in the case of entropy constrained quantizer it satisfies:

$$D_1 \cdot D_{12} \geq (\frac{2\pi e}{12})^2 \frac{P_x^2}{4} 2^{-4R} \tag{11}$$

Therefore, there is a 8.69 dB gap between the proposed level constrained quantizer and the rate distortion bound, and a gap of 3.07 dB for the level constrained quantizer.

In [10], the authors proposed index assignment matrices similar to the ones proposed in [6], but instead of using only the main diagonal in the first step, they propose using also the one diagonal above. For such a scheme, they showed that the distortion product improved for 0.4 dB compared to [6]. Later, in [11], the same authors proposed an additional refinement stage for the case when both descriptions are received and they show that the distortion product is equal to one obtained in [9].

Although the obtained results were remarkable, the design of more than two descriptions was not considered. Extending this method to the design of N descriptions would require the search for the solution in the hypercube of dimension N, which is not a trivial problem. In [12], the author proposed the encoding procedure in a multistage fashion, where each stage doubles the number of descriptions using the method proposed in [6]. However, his scheme allows only the number of descriptions that is a power of two.

*C. **Other techniques***

In parallel to scalar quantization, other ideas were proposed for the generation of two and more balanced descriptions. Most of them were based on unequal error protection (UEP) principles. The staggered index assignment in combination with UEP was proposed for the design of three balanced descriptions in [13] and as the first stage of the design in [14]. In [15], the authors use Reed-Solomon (n,k) codes to make equally important descriptions from the output of a progressive coder. In [16], they propose adding the controlled amount of redundancy to the same progressive coder. This is achieved by spreading the information about each wavelet tree in many descriptions, which guarantees recovery of the most important information. Schemes based on the UEP assume that at least k out of N descriptions are received and they might not be able to reconstruct the signal if fewer than k descriptions are received. Moreover, receiving more than k descriptions might not bring any improvement in the reconstructed quality. Besides, the protection level of these schemes ($\frac{k}{N}$) depends on the state of the channel, which may change very rapidly during the transmission.

In order to soften the cliff effect, the authors in [13], [17] propose using (N,k) source-channel erasure codes for the generation of N descriptions. They derive the complete rate region for their scheme and they show that when exactly k descriptions are received, the achievable distortion exactly matches the optimal distortion rate performance that would be achieved by a source rate of kR bits. However, their study is still based on the assumption that at least k descriptions are received.

The concepts of MDSQ apply also to vectors. Although the performance of the vector quantizer is better because the correlation between the components is also taken into account, it is not easy to extend it to more than two descriptions. The first reason is that the code vectors cannot be naturally ordered. The second is that the design complexity grows exponentially as a number of descriptions, N. In order to overcome these difficulties, the multiple description lattice vector quantization was proposed, [18], [19]. Recently, the authors in [20] proposed generating of arbitrary number of descriptions with this method. In the paper they show the necessity for having the flexibility to produce more than two descriptions. For example, when the packet loss ratio is 0.2, the decrease in distortion for 4 descriptions is 10.6 dB comparing to the case of only two descriptions.

Correlating transforms [21], [22], [23] can also be applied for the generation of the multiple descriptions. A correlating transform introduces a known and a controlled amount of redundancy between initially uncorrelated coefficients. Thus the statistical estimation of the lost coefficients based on the received ones. The method is successfully applied for two descriptions. Already for three descriptions there is no analytical solution for the transform, but the approximate solutions can be obtained if probabilities of channel breakdowns are small. For more than three descriptions, the cascade structure was proposed, but this solution is far from optimal.

Redundant frame expansions such as frames have very nice properties like resilience to additive noise, resilience to quantization, numerical stability of reconstruction and greater freedom to capture significant signal characteristics, [24], [25]. The redundancy of a frame mitigates the effect of losses in packet networks. Recently, in [26], a family of frames maximally robust to errors appeared, after which frames will hopefully be more widely used.

The general conclusion for the methods proposed for multiple descriptions is that they are mostly limited to only two descriptions, mostly due to the complexity and the lack of mathematical results that can tell how to expand $N = 2$ to arbitrary N. Except for the UEP based systems that are not very suitable because of already explained reasons, the MDSQ seems to be one of the most serious candidates to consider. They have excellent asymptotic properties, but it was also shown that they perform very good even at low rates. Moreover, it can have a very nice practical application since almost all video and image coding schemes use a quantization as a part of the standard. In the next section, we will show how we extended the MDSQ principles to the generation of N descriptions.

III. BALANCED INDEX ASSIGNMENT FOR N-DESCRIPTIONS

A. Balanced index assignment for N descriptions: uniform distribution

In this section, we consider the problem of index assignment, for the generation of N balanced descriptions, based on entropy constrained scalar quantization. We assume a source uniformly distributed on the interval $[0, 1]$. Each of the N descriptions has a rate $R_n = R$, and the total rate R_{tot} therefore becomes $R_{tot} = NR$. In addition, the proposed scheme balances the side and partial distortions, which are all measured by the mean squared error, MSE, relative to the input signal.

Since it is known that the uniform quantization minimizes the distortion for uniform sources, a proper index assignment should result in a uniform quantization of the source when all N descriptions are combined together. Let the quantization step size of such uniform quantizer be δ. Several index assignments strategies could fulfill this requirement. However, we are mostly interested in the particular ones that guarantee balanced descriptions in terms of rates and distortions.

To achieve both balanced rate and distortion, the index assignment strategy has to rely on N side quantizers with the same set of bins. Having the same set of bins guarantees balanced rates, even for low rate, since for the uniform source it is given by $R_n = -\sum_i \Delta_i log_2 \Delta_i$, where Δ_i is the size of bin i. Moreover, all the side distortions will also be equal since they are given by $D_n = \int_0^1 p(x)(x - \widehat{y})^2 dx = \sum_i \frac{\Delta_i^3}{12}$. In addition, to ensure balanced partial distortions, the combination of any k descriptions in a partial decoder should also provide the same set of bins, for all the partial decoders.

A trivial solution is to make all the side quantizers equal and uniform, with step size δ. Thus, the same information is sent N times. This solution is, however, expensive in terms of redundancy, since for R bits of useful information we spend NR bits. Moreover, the combination of several descriptions does not refine the quantization, and the distortion does not decrease as more descriptions is received. In order to reduce the redundancy and to make distortion decreasing as a function of the number of received descriptions, some of the bins in side quantizers can be made coarser, by merging several bins δ together. The remaining bins keep their initial size. Let p consecutive bins be merged into the coarse bin $p\delta$, where p is an integer smaller than the number of bins in the central quantizer ($p = 1$ means we did not do any merging). Recall that, to keep the rates and the side distortions still balanced, all the side quantizers need to have the same number of such coarse bins.

At the decoder side, we assume that the bins in partial inverse quantizers can be either $p\delta$ or δ. Moreover, we assume that each next received description refines one of the remaining coarse bins to bins δ. Finally, upon the reception of all descriptions, all the bins are refined and equal to δ.

The following lemma gives the minimal necessary number of bins per side quantizer.

Lemma 1: The minimal required number of coarse bins $p\delta$, $p > 1$, per each of the N side quantizers, that satisfies the following:

1) each next received description refines one of the coarse bins
2) receiving all the descriptions corresponds to the uniform central inverse quantizer, with the step size δ

is $N - 1$.

Proof: Receiving all the descriptions leaves no more coarse bins. Therefore, receiving any $N-1$ out of N descriptions should leave one coarse bin, the combination of any $N-2$ out of N leaves two of them etc. Following this logic, any $N-k$ descriptions leave k coarse bin. Thus, one description ($k = N-1$) has $N-1$ coarse bins. $\square$

If all the coarse bins have size $p\delta$, the minimal number of bins δ in each of the quantizers needs to be p. These bins are necessary to refine the quality of coarse bins and finally to ensure the uniform inverse quantization with step size δ when all descriptions are received. We will assume that the number of bins δ is some q, $q \geq p$. However, we still can achieve balanced rates and distortions, as will be shown later, if we allow for the increase of one of the bins $p\delta$. Let the size of the new bin be $(p+a)\delta$, $a \geq 0$, a is integer. The first reason for increasing the size of one bin is that, by varying a, we can increase the set of achievable rates for our descriptions. The second reason, as we will show later, is to add more flexibility to our scheme in terms of trading off the side, partial and central distortions. Allowing one of the bins to be bigger does not change the condition that the bins in partial inverse quantizers are either $p\delta$ or δ. However, now the number of bins δ needs to be bigger, so that also a bigger bin, $(p+a)\delta$, can be refined.

There is some freedom in placing bins $p\delta$, $(p+a)\delta$ and δ in the side quantizers structure. For example, Figure 3 shows two preliminary quantizer structures for $N = 4$ balanced descriptions, with parameters $p = 3$, $a = 1$ and $q = 10$. Both structures satisfy all the requirements for having both rates and distortions balanced. However, the second one places bins more uniformly in the quantizer structures and might be preferred also for other symmetric distributions different from uniform. On the other hand, the first one has a structure that could be better applied to nonsymmetric distributions. For example, for the exponential source on the interval $[0, \infty]$ with $f(x) = e^{-x}$, the inverted first structure will perform better than the second one, since it provides finer quantization in the area when the samples of the signal are more probable. We will explain this more in detail in the following sections.

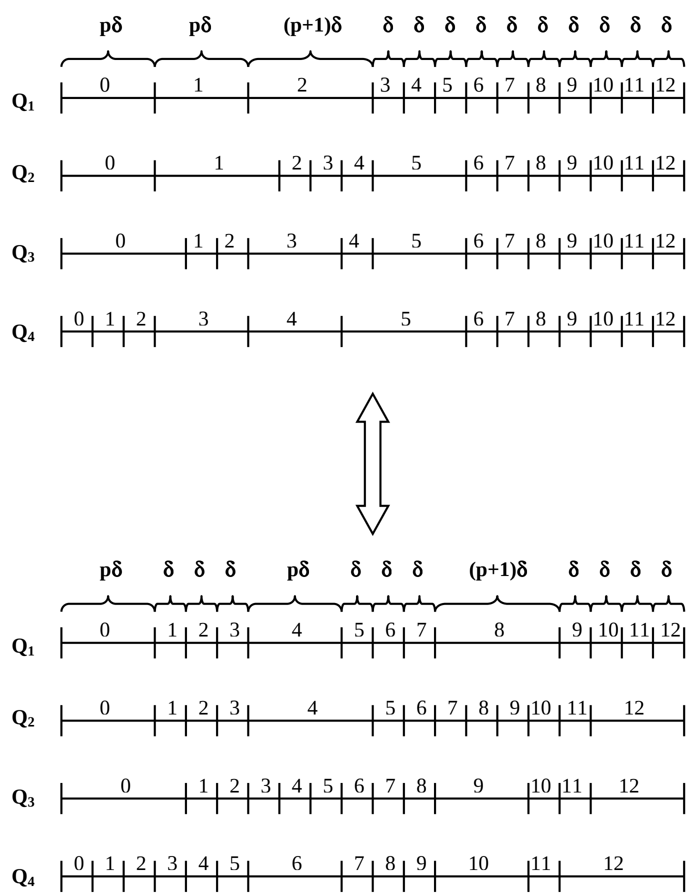


Fig. 3. Two equivalent quantizer structures that give us balanced descriptions for $N = 4$, $p = 3$, $a = 1$ and $q = 10$.

So far, we described what our quantizers should fulfill to achieve balanced descriptions and we gave two possible and equivalent constructions for one set of parameters. However, we would like to give a generic rule for their design, that holds for arbitrary choice of N, p, q and a. The solution we propose is to make the quantizer Q_i, $i = 2, 3, ...N$, have a structure of Q_{i-1} cyclically permuted for a certain number of bins. We stress that this solution is not unique; other equivalent solutions can be found. However, we chose this one because it is generic and easily extendable. First, what we would like to have, is that all the coarse bins $p\delta$ and $(p + a)\delta$ start on the same positions in all quantizers. This fact ensures that no matter which and how many descriptions are received, the intersection bins can be either $p\delta$ or δ. Next, to make bins δ appear more uniformly in the quantizer structures, we propose putting some number of them after each coarse bin.

What we need to do, is to define the structure of the quantizer Q_1; all the other structures will be obtained by the simple cyclic permutation. Without loss of generality, we propose placing first the coarse bins in the quantizer Q_1, and placing $a-1$ bins δ between each pair of coarse bins. If $a \leq 1$, we do not insert these bins between the coarse bins. Also, without loss of generality, we first place $N-2$ bins $p\delta$ and then a bin $(p+a)\delta$. The remaining bins δ we place in the end of the structure of Q_1. Since we would like that all the coarse bins start at the same positions in all the quantizers, we propose cyclic permutation for a bin $p\delta$ and $a-1$ bins δ (in case bins δ are inserted between the coarse bins). The cyclic permutation of quantizer structures moves bins δ along the interval $[0,1]$, that each time provides refinement of some of the coarse bins that starts on that position in some other quantizer.

More formally, we now show the proposed structure of quantizer Q_1, depicted in Figure 4. The basic building block is the structure that contains a bin $p\delta$, followed by $a-1$ bins δ. This basic building block is then repeated $N-2$ times, The structure is subsequently followed by a block that contains a bin $(p+a)\delta$ and $q_{min}-(N-2)\cdot(a-1)$ bins δ. The minimal number of bins δ in each of the quantizers for the permutation scheme that we propose is: $q_{min}=(N-2)\cdot(a-1)+p+a-1$; $a-1$ bins following each bin $p\delta$ and $p+a-1$ bins following the bin $(p+a)\delta$. Finally, a remaining part of $q-q_{min}$ bins δ is added to the end of the structure. Further on, we make quantizer Q_i, $2 \leq i < N$, have the structure of Q_{i-1} cyclically permuted for the basic building block (bin $p\delta$ followed by $a-1$ bins δ) or for $(p+a)\delta$ if $i=N$. We do not permute the last part of the structure, that contains $q-q_{min}$ bins δ, it is always at the end of all quantizers. In addition, one bin δ needs to be added to the permuting structure of all the quantizers, to compensate for edge effects that results from shifting of the quantizer Q_{N-1} for the bin $(p+a)\delta$ and thus leaving one bin δ in the beginning of the last quantizer structure.

Some examples of quantizer structures are given on Figures 5, 6 and 7, where the influence of parameter a

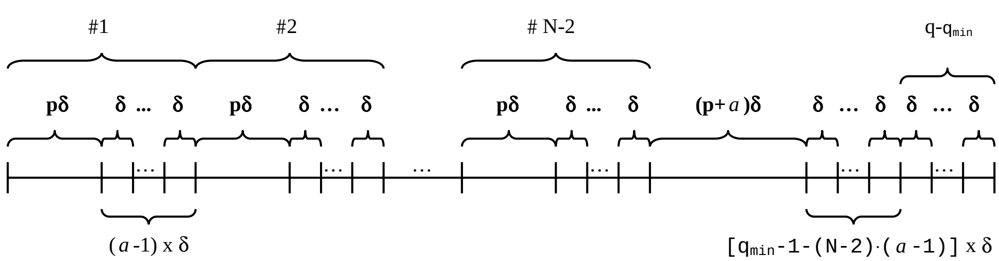


Fig. 4. The structure of the quantizer Q_1.

on the quantizer structure can be seen. The case when $a = 0$, as in Figure 5, corresponds to the simplest way of obtaining balanced descriptions, where we start from a uniform quantizer with step sizes $p\delta$. Subsequently, we divide the last bin of that quantizer into p bins δ. Finally, we get the structure of remaining $N-1$ quantizers by pure shifting by bin $p\delta$. The case when $a \neq 0$, depicted in Figures 6 and 7, is somewhat different due to the edge effect explained above.

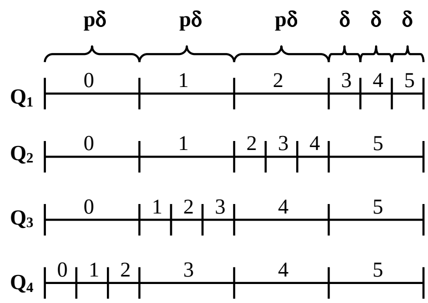


Fig. 5. Balanced descriptions when $N = 4$, $p = 3$, $a = 0$ and $q = 3$.

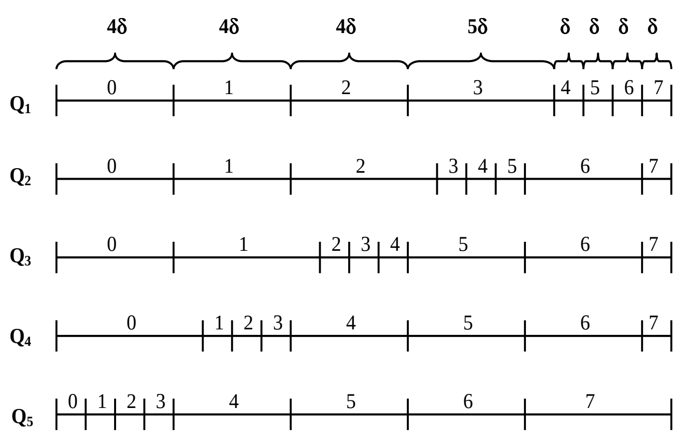


Fig. 6. Balanced descriptions when $N = 5$, $p = 4$, $a = 1$ and $q = 4$.

Obviously, the combination of *any* k out of N descriptions results in $p(k-1)+q+a$ bins of size δ and $(N-k)$ bins of size $p\delta$. This guarantees balanced side and partial distortions, as explained above. Therefore, our descriptions remain completely balanced for any choice of parameters N, p, q and a.

The minimal rate, for the given N, that the proposed scheme can give, can be obtained by putting $p = 2$, $a = 0$ and $q = 2$: $R_{min}(N) = log_2(2N) - \frac{N-1}{N}$. This scheme corresponds to the *step split*

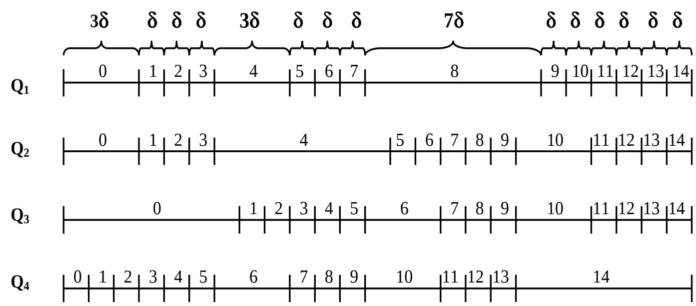


Fig. 7. Balanced descriptions when $N = 4$, $p = 3$, $a = 4$ and $q = 12$.

scheme that we proposed in [27]. It consists of $N-1$ bins 2δ and two bins δ. Each quantizer is obtained from the previous one by a simple shifting by 2δ. Putting $p = 2$ and $a = 1$ leads us to the *merge and split* scheme also proposed in [27], that consists of $N-2$ bins 2δ, one bin 3δ and two bins δ.

One can change the rate by playing with p, q and a. However, there is one more way to increase the rate. It consists of the simple repetition of the structure explained above m times. For such a general case, we have $\delta = \frac{1}{m(p[N-1]+q+a)}$ and we write the following expressions for rates and distortions:

$$R_n = -log_2\delta - \frac{(p+a)log_2(p+a) + (N-2)plog_2(p)}{p(N-1)+q+a} \tag{12}$$

$$D_n = \frac{(N-2)p^3 + (p+a)^3 + q}{12} m\delta^3 = \frac{(N-2)p^3 + (p+a)^3 + q}{((N-1)p+q+a) \cdot 2^{\frac{2(N-2)plog_2p + 2(p+a)log_2(p+a)}{(N-1)p+q+a}}} \cdot \frac{2^{-2R_n}}{12} \tag{13}$$

$$D_{12\ldots k} = \frac{(N-k)p^3 + (k-1)p + q + a}{12} m\delta^3 = \frac{(N-k)p^3 + (k-1)p + q + a}{((N-1)p+q+a) \cdot 2^{\frac{2(N-2)plog_2p + 2(p+a)log_2(p+a)}{(N-1)p+q+a}}} \cdot \frac{2^{-2R_n}}{12}, k \geq 2 \tag{14}$$

It can be seen that there is an explicit relation between the rate and distortion for our scheme, which holds even at low rates. Besides, we see that all the distortions have the same decay rate (2^{-2R_n}) and that they decay linearly as a function of the number of received descriptions, *k*.

All the distortions obviously depend on parameters p, q, a and m. The partial derivation over the parameter a shows that the distortion D_1 increases with the increase of a, while on the other side all the other distortions decrease. Moreover, further calculation shows that all the distortions are decreasing functions of m and q. This is because increasing these parameters decreases δ for a fixed interval $[0, 1]$, and all the distortions are proportional to the third power of δ. The influence of parameter p is especially interesting. While it will make some partial distortions monotonically decreasing or increasing, some of the distortions will not be monotonic at all! Figure 8 shows how all the distortions depend on the parameter p when $N = 5$, $a = 5$, $q = 10$ and $m = 1$. It can be seen, for example. that the distortions D_1 and D_{1234} are not monotonic and have the minimal value when $p = 3$. Distortions D_{12} and D_{123} are monotonically increasing with p, while the central distortion D_{12345} is a monotonically decreasing function of p. Therefore, the parameter p plays the central role in trading off the side, partial and central distortions.

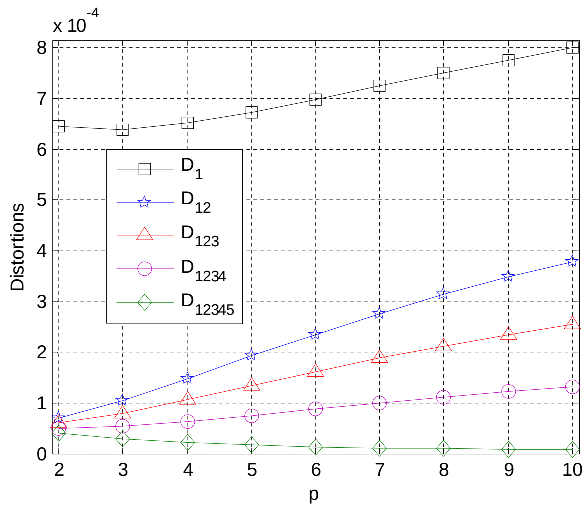


Fig. 8. Distortions as a function of p when $N = 5$, $a = 5$, $q = 10$ and $m = 1$.

It is also interesting to analyze the excess rate or redundancy, that in our case turns to be $\rho = NR + log_2 \delta$. We show that ρ decreases as we increase a or p, while it increases with the increase of q. This is very intuitive result. Less redundant schemes should have higher a and p and they should keep q low. Such schemes are suitable for very low description loss probabilities, where more attention is put to minimizing the central distortion, rather than the partial ones. On the other side, more redundant schemes keep parameters a and p low, while they repeat a lot of bins δ. This is suitable for scenarios where we expect more losses and when we penalize the central distortion in order to make partial distortions lower.

B. Balanced index assignment for N descriptions: arbitrary source distribution

The proposed method for the uniform source distribution can be further extended to arbitrary sources, when all the quantizers decision levels are adapted to the different distribution. This can be done by using compander functions.

For example, we show how the coding of Gaussian sources is derived from the method described above. Denote by $Q_{U,N}$ the set of N quantizers used for the uniform source and by $Q_{G,N}$ the same set for the Gaussian source. The lower and upper decision levels of each bin in $Q_{U,N}$ is transformed to a new set of lower and upper decision levels in $Q_{G,N}$, that will adapt to the new signal. If the Gaussian source has zero mean and unit variance, the transform is given simply by:

$$g = \frac{1 + erf(\frac{u}{\sqrt{2}})}{2}, \tag{15}$$

where u corresponds to levels in $Q_{U,N}$, and g corresponds to new levels in $Q_{G,N}$. Note that such transformation guarantees that the rates of the descriptions stay balanced. However, the side and partial distortions are not balanced any more at low rate. The reason is that trying to keep the linear metrics balanced (i.e, rates) for the Gaussian distribution, does not allow for keeping the square metrics (i.e., MSE distortion) balanced, at the same time. However, the descriptions will be asymptotically balanced in distortions when the rate increases, as we will show later. When choosing between the balanced rates or balanced distortions, we give preference to balanced rates since the packets formed for each of the descriptions will have the same size and thus be equally treated from the network.

Finally, a similar transform could also be applied to other source distributions.

C. *Loss analysis of our scheme*

Multiple description coding scheme is intended for lossy scenarios. It intentionally adds redundancy in order to recover a certain part of possibly lost information. Depending on flexibility of the scheme, one can chose which part and which percent of the information will be protected. Comparing to other state-of-the-art source coding techniques, like hierarchical coding schemes, it provides better error resilience at the price of, however, higher redundancy.

In this section, we show how our scheme can be applied to different lossy scenarios. Assume we can send N descriptions over a lossy network, each one over a different channel, like in Figure 9. Assume also that the probability that each channel will break down is equal to PLR, and that the breakdowns are independent. If all descriptions are received, which happens with the probability PLR^N, the signal will be reconstructed with the lowest, central distortion (D). If we receive any k descriptions out of N, which happens with the probability $\binom{N}{k} PLR^{N-k}(1 - PLR)^k$, the reconstructed signal will be with the distortion $(D_{12...k})$. The most severe case will correspond to the case when all descriptions are lost, which happens with the probability $(1 - PLR)^N$. In that case, the receiver can just guess what was sent by the sender and the distortion will be equal to the variance of the source. Since all the descriptions

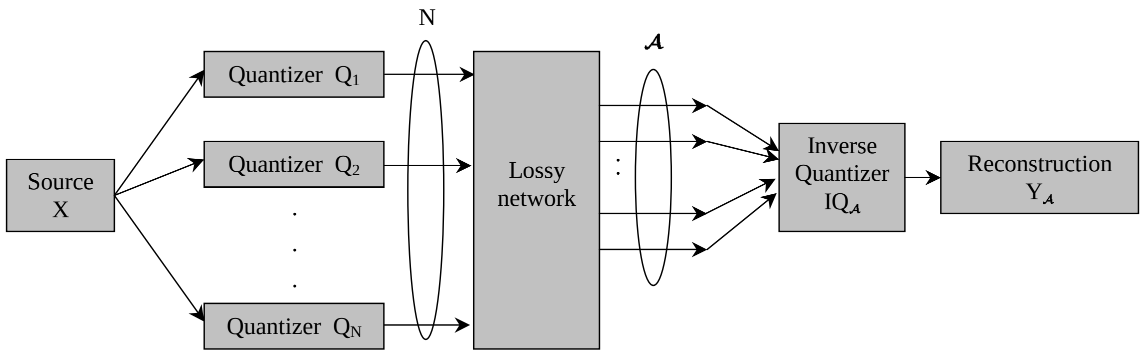


Fig. 9. Lossy scenario that we consider.

have the same rate and they are balanced, we can write the average distortion in the presence of losses in the following way:

$$D_{av} = \sum_{k=0}^{N-1} \binom{N}{k} PLR^k (1 - PLR)^{N-k} D_{12\ldots(N-k)} + PLR^N \sigma^2 \tag{16}$$

Therefore, the average distortion in the lossy scenario will depend on the number of descriptions, rates of descriptions and the probability of error.

We will show in the next section that, given the losses on a network and the total rate constraint, we can choose the parameters

IV. Experimental results

A. R-D performance

In this section, we consider first the generation of $N = 3$ balanced descriptions for the uniform source, and we compare our scheme with the existing staggered scheme. The comparison is done at higher rates since the staggered scheme does not allow for balanced descriptions at low rates. It is not difficult to derive the following expressions for the staggered scheme when $R \to \infty$:

$$D_n \approx 0.083 \cdot 2^{-2R} \tag{17}$$

$$D_{12} \approx 0.025 \cdot 2^{-2R} \tag{18}$$

$$D_{123} \approx 0.0093 \cdot 2^{-2R} \tag{19}$$

This scheme gives fixed expressions for rates and distortions. Moreover, it gives the constant relations between the distortions: each next received description reduces the distortion approximately three times.

Our scheme has only one parameter fixed: $N = 3$. All the other parameters can be chosen based on which distortion we might want to minimize, or how fast we want to decrease the distortion, or which rates and rate granularity we want to have. Basically, our objective will be to minimize the average distortion seen by the client, and this will be discussed in detail in the next section. Here we will just show what is the advantage of our scheme in terms of having the ability to arbitrarily choose the parameters: p, a and m.

We will show the performance of our scheme on few examples. First, let us for example set the following parameters in our scheme: $p = q = 6$ and $a = 1$. This set of parameters gives the following expressions for distortions:

$$D_n \approx 0.123 \cdot 2^{-2R} \tag{20}$$

$$D_{12} \approx 0.077 \cdot 2^{-2R} \tag{21}$$

$$D_{123} \approx 0.0064 \cdot 2^{-2R} \tag{22}$$

that hold both at low and high rates. Putting $p = 4$, $q = 30$ and $a = 14$ on the other hand gives us the following expressions:

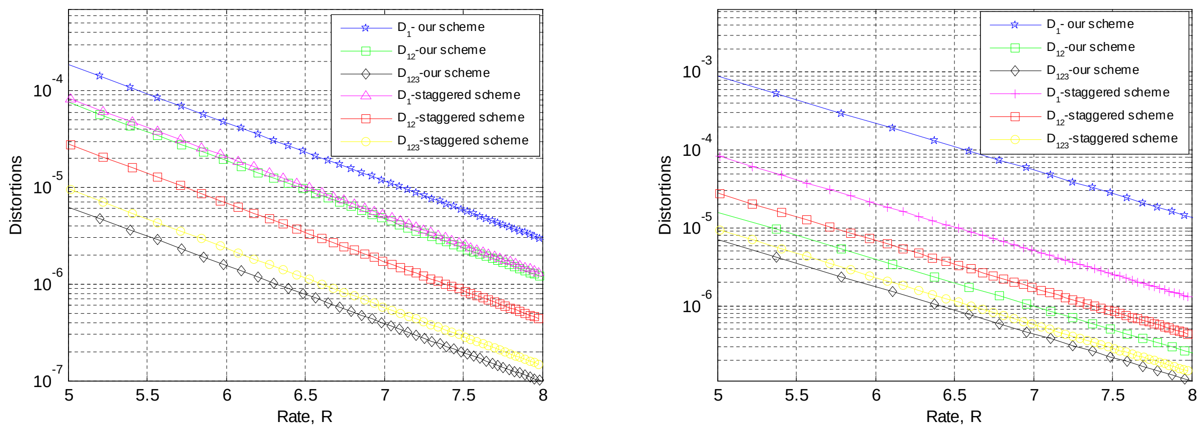
$$D_n \approx 1.03 \cdot 2^{-2R} \tag{23}$$

$$D_{12} \approx 0.0195 \cdot 2^{-2R} \tag{24}$$

$$D_{123} \approx 0.009 \cdot 2^{-2R} \tag{25}$$

Rate distortion functions for these two cases and for the staggered scheme are given on Figure 10. With the first se of parameters, the central distortion is made 1.4 smaller, at the expense of having 1.47 times higher side distortion and 3.1 times higher partial distortion. The second set of parameters gives 12 times higher side distortion. However, the partial distortion is made 1.66 times smaller and the central distortion is made 1.03 times smaller. Thus, by playing with the parameters p, q, a and m, one can change all the distortions and the relations between them. Different sets of parameters can clearly be chosen to favor different scenarios. For example, the first set of parameters might be chosen if the losses on the network are low and one expects receiving all three descriptions. On the other side, the second set of parameters tends to minimize the distortion D_{12}. Therefore, we can sacrifice the performance of the distortion D_1 if we expect receiving two or three descriptions.

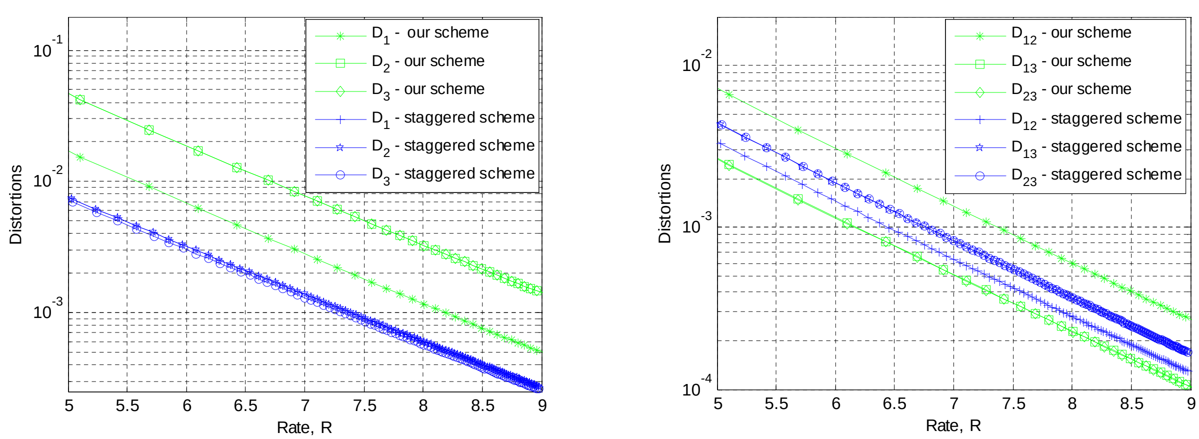
Thus, contrary to the staggered scheme, our scheme provides the possibility to make a trade off between the distortions and gives the possibility to choose parameters that can minimize only one of them. This, together with the fact that that our descriptions are completely balanced at both low and high rates, makes our scheme superior to the existing scheme.



(a) $p = q = 6, a = 1$.

(b) $p = 4, a = 14, q = 30$.

Fig. 10. Comparison of our scheme and staggered scheme for $N = 3$ and different setd of parameters p, q and a.



(a) Side distortions

(b) Partial distortions

Fig. 11. Comparison of our scheme for $p = 4, a = 14$ and $q = 30$ and the existing scheme for the Gaussian distribution and D_i and D_{ij}, $i, j = 1, 2, 3$.

Finally, we compare our scheme and the staggered scheme for the Gaussian distribution function. The relation between the distortions D_i, $i = 1, 2, 3$ for the two schemes and the choice of parameters $p = 4, a = 14$ and $q = 30$ is given on Figure 11(a), while the relation between the distortions D_{ij}, $i, j = 1, 2, 3$ is given on Figure 11(b). From these figures, the similar conclusion like in the case of the uniform source can be made. All the side distortions are better for the staggered scheme, while two of our partial distortions are better than all the partial distortions for the staggered scheme.

Similarly, with the different set of parameters, we can again tradeoff our distortions and choose parameters that will outperform the staggered scheme for the specific scenario. However, what is most interesting to notice here is that the distortions in the staggered scheme are more balanced, even though they are not balanced in terms of rate. On the other side, our descriptions are completely balanced in terms of rate, but tend to give less balanced distortions. This is mostly visible at low rates.

V. Lossy scenarios

A. Our scheme in lossy scenarios

In this section, we intend to investigate the behavior of our scheme in lossy scenarios, as well as to compare it to the FEC scheme under the same conditions. As we will show, our scheme tends to give higher average distortions in case of losses, but it is also more robust to sudden changes of network conditions. Now we can formulate the following problem.

Problem formulation*:* Given *the probability PLR that any of the channels will break down, and the total rate R_{tot} available for generation of balanced descriptions,* find *the parameters N, m, p, q and a that will minimize the average distortion, D_{av}, given by* (25).

The solution to this question is given in Figure 12, which shows the minimal achievable average distortion as a function of the PLR, and in Table 1 which shows the best parameters for the proposed scenario. It can be seen that the case when the rate budget is small is much more sensitive to losses of descriptions. This is due to the fact that we can produce fewer descriptions at lower rates, but also because these descriptions are necessarily less redundant. Losing one of them therefore causes higher increase in distortion than in the case of more redundant descriptions. We also see from the Table 1 that, for a given rate R_{tot}, with the increase of packet loss ratio, the optimal number of descriptions is increasing and the rate of each description decreasing. In addition, the redundancy is also increasing. This is not a surprising result, since as the losses tend to increase, it makes sense to make and send more descriptions and to make them more redundant, with the hope that at least one of the descriptions will be received and at least the most important information will be recovered. We can also see that the parameters p and a are decreasing as the PLR increases. This is because the decrease of these parameters increases the redundancy of the scheme. Not surprisingly, when the losses are very high, there is a tendency to increase the number of descriptions and to increase the redundancy of the scheme. An interesting effect appears when $R_{tot} = 40$ bits and $PLR = 0.9$: the multiple description scheme degenerates to the simple repetition of the same information in all descriptions. More specifically, all $N_{opt} = 11$ descriptions will contain the information of the source that is uniformly quantized with the quantizer with step size $\delta = \frac{1}{12}$.

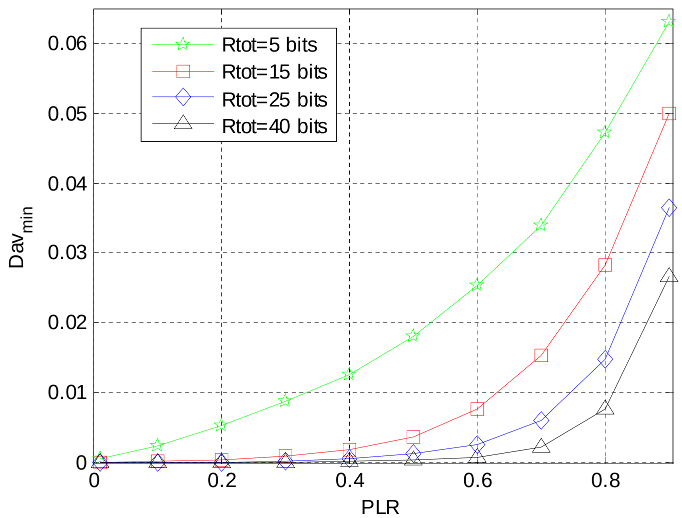


Fig. 12. Minimal achievable distortion, as a function of packet loss ratio.

B. *FEC scheme in lossy scenarios*

For the forward error correction (FEC) scheme, we assume the equal error protection (EEP) scheme, since we do not give any priority to some quantized values over the other ones.

Most of the FEC schemes are based on the Reed-Solomon codes or X-OR functions that can, in general, correct as many losses as the number of redundancy packets. In the systematic (N, k) FEC scheme (Figure 13), the set of k data packets is followed by $(N - k)$ redundancy packets. If at least k out of N packets are correctly received, all the data can be correctly decoded. Otherwise, none of the lost packets can be recovered by the receiver.

In general, the average distortion in the presence of losses can be written in the following way:

$$D_{avFEC} = \overline{p} \cdot \sigma^2 + (1 - \overline{p}) \cdot D(R) \qquad (26)$$

	R_{tot}[bits]	$p = 10^{-3}$	$p = 10^{-2}$	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
N	5	2	2	2	3	3	3	3
m		1	1	1	1	1	1	1
p		1	1	1	1	1	1	1
q		9	9	5	1	1	1	1
a		9	9	2	1	1	0	0
R		2.499	2.499	2.41	1.5	1.5	1.585	1.585
ρ		0.751	0.751	1.81	2.5	2.5	3.17	3.17
N	15	2	3	3	4	5	5	5
m		19	5	7	2	1	1	1
p		20	10	4	3	1	1	1
q		24	10	4	5	4	4	4
a		5	1	1	1	1	0	0
R		7.49	4.98	4.99	3.74	2.95	3	3
ρ		5.12	7.65	8.49	10.05	11.57	12	12
N	25	3	3	4	6	8	8	8
m		32	48	13	1	1	1	1
p		39	12	5	3	2	2	2
q		39	12	5	18	2	2	2
a		1	1	1	4	1	1	0
R		8.33	8.33	6.25	4.14	3.101	3.101	3.125
ρ		13.11	14.2	16.9	19.78	20.73	20.73	21
N	40‘	3	4	5	7	9	11	11
m		176	100	28	6	2	1	1
p		40	19	4	5	3	4	1
q		250	25	8	5	4	4	2
a		106	3	2	1	1	1	0
R		13.33	10	7.99	5.71	4.43	3.99	3.59
ρ		23.76	26.95	30.43	32.23	34.05	35.11	35.85

TABLE I

OPTIMAL PARAMETERS N, m, p, q AND a THAT MINIMIZE THE AVERAGE DISTORTION IN THE PRESENCE OF LOSSES.

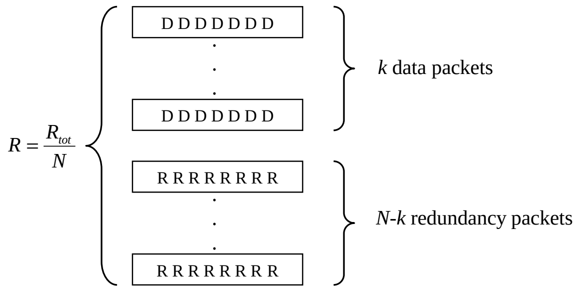


Fig. 13. (N,k) FEC scheme.

Here $\overline{p}$ denotes the average probability that the data will not be recovered after the FEC decoding, σ denotes the variance of the source, while $D(R)$ denotes the source distortion-rate function.

The average probability that the data will not be recovered after the FEC decoding can be written in the following way:

$$\overline{p} = \frac{\sum i \cdot p_i(N,k)}{k} \tag{27}$$

where $p_i(N,k)$ denotes the probability that i data packets will not be recovered after the FEC decoding. This corresponds to the case when i data packets are lost and at least $N-k-i+1$ FEC packets are also lost, which happens with the probability:

$$p_i(N,k) = \binom{k}{i} PLR^i (1-PLR)^{k-i} \sum_{j=\lfloor n-k+1-i \rfloor}^{n-k} \binom{n-k}{j} PLR^j (1-PLR)^{n-k-j} \tag{28}$$

The rate used for the source coding is $R = \frac{k}{N} R_{tot}$. For the purpose of further comparisons, we assume that this rate is used for the uniform quantization of the source. Therefore, for the uniform source, we can write:

$$D_{avFEC} = \overline{p} \cdot \frac{1}{12} + (1-\overline{p}) \frac{2^{\frac{-kR_{tot}}{N}}}{12} \tag{29}$$

C. Comparison of our and FEC scheme

To make a fair comparison of both schemes from the network point of view, we set the number of FEC packets to be equal to the number of descriptions in our scheme, for each R_{tot} and PLR.

Figure 14 shows the average distortions for the FEC and MDSQ scheme, for the total rates of 5 bits/symbol and 25 bits/symbol, when PLR changes from 10^{-4} up to 0.3. We see that when $R_{tot} = 5$ bits/symbol, the FEC scheme outperforms our scheme in the range of very low losses. However, when PLR exceeds $3 \cdot 10^{-3}$, our scheme tends to give smaller average distortion. The results favor even more our scheme for higher total rate constraints. For example, when $R_{tot} = 25$ bits/symbol, our scheme starts outperforming the FEC scheme even at very low losses, in this case $3 \cdot 10^{-4}$.

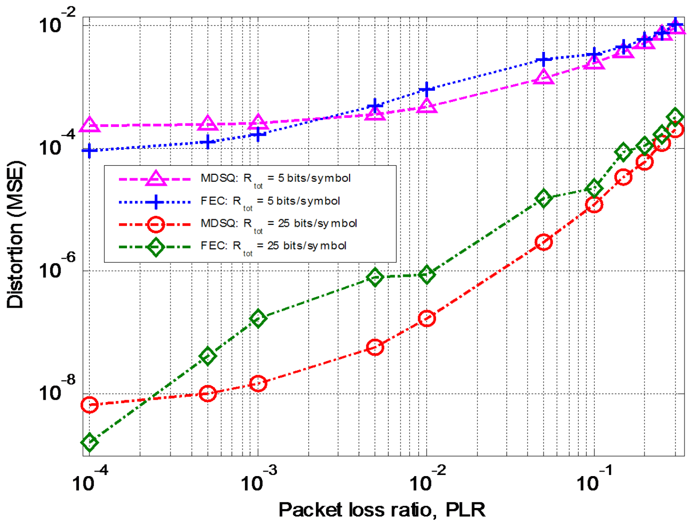


Fig. 14. Comparison of the FEC scheme and the MDSQ scheme for $R_{tot} = 5$ bits and $R_{tot} = 25$ bits.

Next, we examine how both schemes perform when the network conditions change. Namely, both schemes, optimized for some PLR and R_{tot}, continue to work in the conditions they are not optimized for. Figure 15 shows the comparison in two cases: one when the schemes are optimized for the $R_{tot} = 5$ bits/symbol and $PLR = 10^{-3}$ and the second when $R_{tot} = 25$ bits and $PLR = 0.05$. For the first case, when the losses in the network drop, the FEC scheme will tend to give smaller distortions. However, as the losses increase, our scheme will give smaller average distortion. When there is a big increase of PLR (due to congestion for example) the schemes will give similar distortions, but ours remains better. In the second case, we see that our scheme strongly outperforms the FEC scheme up to the point when $PLR = 0.2$, after which they give practically equal results.

From these results, we conclude that besides giving smaller average distortion, our scheme is also more robust to network changes.

VI. Conclusions

$\mathcal{A}$

In this paper bla bla bla...

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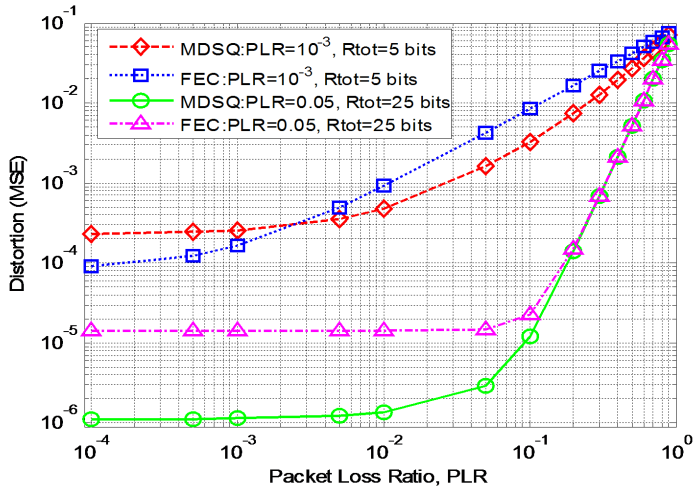

Fig. 15. Comparison of the FEC scheme and the MDSQ scheme for sudden network condition changes

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