



# $K3$ surfaces, cyclotomic polynomials and orthogonal groups

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*In memory of Nikolai Vavilov*

Received: 11 February 2024 / Revised: 7 April 2024 / Accepted: 12 April 2024  
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## Abstract

Let  $X$  be a complex projective  $K3$  surface and let  $T_X$  be its transcendental lattice; the characteristic polynomials of isometries of  $T_X$  induced by automorphisms of  $X$  are powers of cyclotomic polynomials. Which powers of cyclotomic polynomials occur? The aim of this note is to answer this question, as well as related ones, and give an alternative approach to some results of Kondō, Machida, Oguiso, Vorontsov, Xiao and Zhang; this leads to questions and results concerning orthogonal groups of lattices.

**Keywords**  $K3$  surfaces · Orthogonal groups

**Mathematics Subject Classification** 14J28 · 20G30

## 1 Introduction

If  $X$  is a projective  $K3$  surface over the complex numbers; we denote by  $S_X$  its Picard lattice and by  $T_X$  its transcendental lattice; if  $a: X \rightarrow X$  is an automorphism, then  $a$  induces an isometry  $a^*$  of the lattice  $H^2(X, \mathbf{Z})$ , and the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is a power of a cyclotomic polynomial (see Proposition 2.1).

Let  $m, r$  be integers with  $m \geq 3$  and  $r \geq 1$ , and let  $C = \Phi_m^r$  (where  $\Phi_m$  is the  $m$ -th cyclotomic polynomial).

**Proposition 1.1** *Assume that  $\deg(C) \leq 20$ . Then there exists an automorphism  $a: X \rightarrow X$  of a projective  $K3$  surface  $X$  such that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$ .*

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We denote by  $\text{Aut}(X)$  the group of automorphisms of the  $K3$  surface  $X$ , and by  $\text{Aut}_s(X)$  the subgroup of  $\text{Aut}(X)$  acting trivially on  $T_X$ . We have the exact sequence

$$1 \rightarrow \text{Aut}_s(X) \rightarrow \text{Aut}(X) \rightarrow M_X \rightarrow 1,$$

where  $M_X$  is a finite cyclic group (see Nikulin [18, Theorem 10.1.2]); we denote by  $m_X$  the order of  $M_X$ .

**Corollary 1.2** *Let  $m \geq 4$  be an even integer such that  $\varphi(m) \leq 20$ . Then there exists a projective  $K3$  surface  $X$  with  $m_X = m$ .*

It is well known that there exist  $K3$  surfaces  $X$  with  $m_X = 1, 2$  (see for instance [11, Corollary 15.2.12]), but as far as we know, the following question is open.

**Question 1.3** Let  $m > 1$  be an odd integer. Does there exist a projective  $K3$  surface  $X$  with  $m_X = m$ ?

In [15], Machida and Oguiso obtain several results on related topics; see Remark 11.5, Proposition 12.1 and Remark 13.2 for details.

Following Vorontsov [23] and Kondō [12], we consider automorphisms that act trivially on the Picard lattice. Let  $N_X$  be the kernel of  $\text{Aut}(X) \rightarrow \text{O}(S_X)$ ; this is a finite cyclic group that can be identified with a subgroup of  $M_X$ ; we denote by  $n_X$  the order of  $N_X$ . The following proposition is proved in Sect. 5.

**Proposition 1.4** *There exists an automorphism  $a: X \rightarrow X$  of a projective  $K3$  surface  $X$  such that  $a^*$  is the identity on  $S_X$  and that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$  if and only if the following conditions hold:*

- (i)  $C(-1)$  is a square.
- (ii) If  $C(1) = 1$ , then  $\deg(C) \equiv 4 \pmod{8}$ .

The possible values of  $n_X$  can be deduced from Proposition 1.4, extending previous results of Vorontsov [23], Kondō [12], and Oguiso–Zhang [20]; see Sect. 13. Note that  $n_X$  divides  $m_X$ , since  $N_X$  can be identified with a subgroup of  $M_X$ . This suggests the following question.

**Question 1.5** What are the possible values of the pairs  $(m_X, n_X)$ ?

The proofs of the above propositions use some arithmetic results (see below), as well as the surjectivity of the period map, the strong Torelli theorem, and some results of McMullen [17].

The arithmetic results are valid in a greater generality than the one needed for the applications to  $K3$  surfaces. For instance, in order to prove Proposition 1.4, we introduce the following property, called property (P2):

(P2) Let  $R, S \geq 0$  be integers such that  $R \equiv S \pmod{8}$ , and set  $N = R + S$ ; suppose that  $\deg(C) < N$ .

Let  $c \geq 0$  be an even integer with  $c \leq \deg(C)$  such that  $c \leq R$  and  $\deg(C) - c \leq S$ .

**Definition 1.6** We say that *property (P2) holds* if there exist an even, unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t: L \rightarrow L$  such that

- The characteristic polynomial of  $t$  is  $C(X)(X - 1)^{N - \deg(C)}$ .
- The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ .

In application to *K3* surfaces, we have  $R = 3, S = 19$  and  $c = 2$ , and Proposition 1.4 is a consequence of the following result.

**Theorem 1.7** *Property (P2) holds if and only if the following conditions are satisfied:*

- (i)  $C(-1)$  is a square.
- (ii) If  $C(1) = 1$ , then  $\deg(C) \equiv 2c \pmod{8}$ .

In particular, Property (P2) always holds if  $C(-1)$  is a square and  $C(1) > 1$ .

If  $m$  is an odd prime number, this can be deduced from a result of Brandhorst and Cattaneo [6, Theorem 1.1]. Note that Theorem 1.7 gives a partial answer to a question of this paper (see [6, Outlook]).

With the same notation, we introduce property (P1).

**Definition 1.8** We say that *property (P1) holds* if there exist an even, unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t: L \rightarrow L$  such that

- (i) The characteristic polynomial of  $t$  is divisible by  $C$ .
- (ii) The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ .

**Theorem 1.9** *If  $C(1)C(-1) > 1$ , then property (P1) holds.*

If  $C(1) = C(-1) = 1$  and  $R = 0$  or  $S = 0$ , then property (P1) does not always hold, but the *indefinite* case seems to be open.

**Question 1.10** Does property (P1) always hold when  $R > 0$  and  $S > 0$ ?

A modified version of this property is used to prove Proposition 1.1. A more tractable question is to ask for isometries of *finite order*; this leads to the following definition.

**Definition 1.11** We say that *property (P1') holds* if there exist an even, unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t: L \rightarrow L$  of finite order such that

- The characteristic polynomial of  $t$  is divisible by  $C$ .
- The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ .

Note that properties (P1) and (P1') are equivalent if  $R = 0$  or  $S = 0$ , since then all isometries are of finite order. If  $R > 0$  and  $S > 0$ , it is possible that property (P1) always holds — however, this is not the case for property (P1'), as shown by the following example.

**Example 1.12** Let  $C = \Phi_{60}$ , and set  $R = 3, S = 19$  and  $c = 2$ . Then

- Property (P1) holds (see Example 9.4).
- Property (P1') does not hold (see Proposition 12.2).

The fact that property (P1') does not hold implies the well-known result that there does not exist a projective  $K3$  surface  $X$  having an automorphism  $a \in \text{Aut}(X)$  of finite order such that  $a^*$  induces multiplication by a primitive 60-th root of unity on  $T_X$ ; see Machida and Oguiso [15], Xiao [25], and Zhang [26]; see also Proposition 12.1.

It can be useful to replace “of finite order” by “of order  $m$ ”. This point of view is taken by several authors; see the paper of Brandhorst [5] and the references therein; see also Sects. 6, 7 and 8.

The paper is organized as follows. The first two sections are mainly preliminary, recalling notions and results on  $K3$  surfaces and isometries of lattices; Sects. 3 and 9 also recall some results of [2, 3], and give some examples that are used in the paper. Theorem 1.7 is proved in Sect. 4 (see Theorem 4.6), and Proposition 1.4 in Sect. 5 (see Proposition 5.1). A stronger form of Theorem 1.9 is in Sect. 6, see Theorem 6.1. Proposition 1.1 is proved in Sects. 10 and 11. The last section concerns the possible values of  $m_X$  and  $n_X$ , a discussion of some results of Kondō and Vorontsov, as well as a generalization of these results, and some open questions.

The proofs use results of [2, 3], of McMullen, [17], and of Takada [22].

## 2 $K3$ surfaces

We recall some notation and basic facts on  $K3$  surfaces and their automorphisms; see [11, 13] for details.

A  $K3$  surface  $X$  is a simply-connected compact complex surface with trivial canonical bundle. We have the Hodge decomposition

$$H^2(X, \mathbf{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

with  $\dim H^{2,0} = \dim H^{0,2} = 1$  and  $\dim H^{1,1} = 20$ . The Picard lattice of  $X$  is by definition

$$S_X = H^2(X, \mathbf{Z}) \cap H^{1,1}(X).$$

The intersection form  $H^2(X, \mathbf{Z}) \times H^2(X, \mathbf{Z}) \rightarrow \mathbf{Z}$  of  $X$  is an even unimodular lattice of signature  $(3, 19)$ . Such a form is unique up to isomorphism (see for instance [21, Chapter V, Theorem 5]). The transcendental lattice  $T_X$  is by definition the primitive sublattice of  $H^2(X, \mathbf{Z})$  of minimal rank such that  $T_X \otimes_{\mathbf{Z}} \mathbf{C}$  contains  $H^{2,0}(X) \oplus H^{0,2}(X)$ . Assume that  $X$  is projective; then lattices  $S_X$  and  $T_X$  are orthogonal to each other, and the orthogonal sum  $S_X \oplus T_X$  is of finite index in  $H^2(X, \mathbf{C})$ , the signature of  $S_X$  is  $(1, \rho_X - 1)$ , and the signature of  $T_X$  is  $(2, 20 - \rho_X)$ , where  $\rho_X$  is the rank of  $S_X$ .

If  $a: X \rightarrow X$  is an automorphism, then  $a^*: H^2(X, \mathbf{C}) \rightarrow H^2(X, \mathbf{C})$  respects the Hodge decomposition and is an isometry of the intersection form; hence  $a^*$  is also an isometry of the lattices  $S_X$  and  $T_X$ .

The following is a result of Oguiso [19, Theorem 2.4].

**Proposition 2.1** *Let  $a: X \rightarrow X$  be an automorphism of a projective K3 surface. Then the minimal polynomial of the restriction of  $a^*$  to  $T_X$  is a cyclotomic polynomial.*

**Proof** The minimal polynomial of the restriction of  $a^*$  to  $T_X$  is irreducible (see Oguiso [19, Theorem 2.4(1)]). Since  $X$  is projective,  $a^*|_{T_X}$  is of finite order (cf. Nikulin [18, Theorem 10.1.2], [19, Theorem 2.4(4)], or [11, Corollary 3.3.4 or Corollary 15.1.10]), hence its minimal polynomial is a cyclotomic polynomial.  $\square$

### 3 Isometries of lattices

In this section we summarize some notions and results from [2, 3, 10] in the special cases needed in this paper.

A lattice is a pair  $(L, q)$ , where  $L$  is a free  $\mathbf{Z}$ -module of finite rank, and  $q: L \times L \rightarrow \mathbf{Z}$  is a symmetric bilinear form; it is *unimodular* if  $\det(q) = \pm 1$ , and *even* if  $q(x, x)$  is an even integer for all  $x \in L$ . The following lemma is well known.

**Lemma 3.1** (See e.g., [21, Chapter V, Corollary 1]) *Let  $L$  be an even unimodular lattice of signature  $(r, s)$ . Then  $r \equiv s \pmod{8}$ .*

Let  $n \geq 1$  be an integer and let  $F \in \mathbf{Z}[x]$  be a monic polynomial of degree  $2n$  such that  $F(x) = x^{2n}F(x^{-1})$ . We say that  $F$  satisfies condition (C1) if  $|F(1)|, |F(-1)|$  and  $(-1)^n F(1)F(-1)$  are squares.

**Lemma 3.2** (Gross–McMullen, [10, Theorem 6.1] or [3, Corollary 2.3]) *Let  $L$  be an even unimodular lattice and let  $t: L \rightarrow L$  be an isometry with characteristic polynomial  $F$ . Then  $F$  satisfies condition (C1).*

Assume that  $F$  is a product of cyclotomic polynomials, and that  $F(1) \neq 0$  or  $F(-1) \neq 0$ . Let us write  $F = F_1 F_0$ , where  $F_1 \in \mathbf{Z}[x]$  is a monic polynomial such that  $F_1(1)F_1(-1) \neq 0$ , and  $F_0(x) = (x - 1)^{n_+}$  or  $F_0(x) = (x + 1)^{n_-}$ , where  $n_+, n_- \geq 0$  are integers. Set  $D_0 = (-1)^n F_1(1)F_1(-1)$ . Let  $I$  be the set of irreducible factors of  $F$  over  $\mathbf{Q}$ .

Following [3], we associate to  $F$  a finite group  $G_F$ ; we start by defining a set  $\Pi_{f,g}$  for all  $f, g \in I$ , as follows. We say that a monic polynomial  $h$  is  $(\pm)$ -symmetric if  $h(x) = \pm x^{\deg(h)} h(x^{-1})$ . We also use the terminology *symmetric* for  $(+)$ -symmetric.

If  $f, g \in I$  are such that  $\deg(f) \geq 2, \deg(g) \geq 2$ , then  $\Pi_{f,g}$  is the set of prime numbers  $p$  such that  $f \pmod{p}$  and  $g \pmod{p}$  have a common irreducible  $(\pm)$ -symmetric factor in  $\mathbf{F}_p[x]$ .

If  $f \in I$  is such that  $\deg(f) \geq 2$ , then  $\Pi_{f,x-1}$  is the set of prime numbers  $p$  such that  $f \pmod{p}$  is divisible by  $x - 1$  in  $\mathbf{F}_p[x]$ , and that if  $n^+ = 2$ , then  $D_0 \neq -1$  in  $\mathbf{Q}_p^\times / \mathbf{Q}_p^{\times 2}$ .

If  $f \in I$  is such that  $\deg(f) \geq 2$ , then  $\Pi_{f,x+1}$  is the set of prime numbers  $p$  such that  $f \pmod{p}$  is divisible by  $x + 1$  in  $\mathbf{F}_p[x]$ , and that if  $n^- = 2$ , then  $D_0 \neq -1$  in  $\mathbf{Q}_p^\times / \mathbf{Q}_p^{\times 2}$ .

Let  $C(I)$  be the set of maps  $c: I \rightarrow \mathbf{Z}/2\mathbf{Z}$ , let  $C_0(I)$  the set of  $c \in C(I)$  such that  $c(f) = c(g)$  if  $\Pi_{f,g} \neq \emptyset$ ; note that  $C_0(I)$  is an abelian group. Let  $G_F$  be the quotient of  $C_0(I)$  by the subgroup of constant maps.

**Example 3.3** Let  $F(x) = \Phi_{15}^2(x)\Phi_3(x)(x - 1)^2$ . We have  $5 \in \Pi_{\Phi_{15}, \Phi_3}$  and  $3 \in \Pi_{\Phi_3(x), x-1}$ , hence  $G_F = 0$ .

When  $F$  has no linear factors, then  $G_F$  is already defined in [2], and several examples are given in [2, Section 25]. Here is another example that will be used in the proof of Proposition 10.5.

**Example 3.4** Let  $F = \Phi_{60}\Phi_{12}$ . The resultant of  $\Phi_{60}$  and  $\Phi_{12}$  is  $5^4$ , and the polynomials  $\Phi_{60}$  and  $\Phi_{12} \pmod{5}$  have the common irreducible factors  $x^2 + 2x + 4$  and  $x^2 + 3x + 4$  in  $\mathbf{F}_5[x]$ . These polynomials are not  $(\pm)$ -symmetric, hence  $\Pi_{\Phi_{60}, \Phi_{12}} = \emptyset$ , and  $G_F = \mathbf{Z}/2\mathbf{Z}$ .

The following is proved in [3, Corollary 12.4].

**Theorem 3.5** *Let  $r, s \geq 0$  be integers such that  $r \equiv s \pmod{8}$  and that  $r + s = \deg(F)$ . If  $G_F = 0$ , then there exists an even unimodular lattice of signature  $(r, s)$  having an isometry with characteristic polynomial  $F$ .*

We need a more precise result: it is not enough to fix the signature of the lattice, we also need information about the signature map of the isometry. We recall this notion from [3, Sections 3 and 4].

**Definition 3.6** Let  $V$  be a finite-dimensional vector space over  $\mathbf{R}$  and let  $q: V \times V \rightarrow \mathbf{R}$  be a non-degenerate quadratic form. Let  $t: V \rightarrow V$  be an isometry of  $q$ . If  $f \in \mathbf{R}[X]$ , set  $V_f = \text{Ker}(f(t))$  and let  $q_f$  be the restriction of  $q$  to  $V_f$ . Let

$$\text{sign}_t: \mathbf{R}[X] \rightarrow \mathbf{N} \times \mathbf{N}$$

be the map sending  $f \in \mathbf{R}[X]$  to the signature of  $(V_f, q_f)$ , where  $\mathbf{N}$  is the set of all nonnegative integers; it is called the signature map of the isometry  $t$ . The signature of  $q$  is called the maximum of the signature map, and the characteristic polynomial of  $t$  the polynomial associated to the signature map.

**Example 3.7** Let  $a: X \rightarrow X$  be an automorphism of a projective  $K3$  surface, and suppose that  $a^*|_{S_X}$  is the identity and that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is  $C$ . Let  $\rho_X$  be the rank of  $S_X$  and let  $\tau$  be the signature map of  $a^*$ . Then we have  $\tau(x - 1) = (1, \rho_X - 1)$  and  $\tau(C) = (2, \deg(C) - 2)$ . Note that  $\deg(C) = 22 - \rho_X$ , hence we also have  $\tau(C) = (2, 20 - \rho_X)$ .

**Theorem 3.8** ([3, Corollary 12.3]) *Let  $r, s \geq 0$  be integers such that  $r \equiv s \pmod{8}$  and that  $r + s = \deg(F)$ . Let  $\tau$  be a signature map of maximum  $(r, s)$  and associated polynomial  $F$ . If  $G_F = 0$ , then there exists an even unimodular lattice having an isometry with signature map  $\tau$ .*

## 4 Cyclotomic polynomials and property (P2)

The aim of this section is to prove Theorem 1.7 from the introduction. We start by recalling some basic properties of cyclotomic polynomials. Recall that  $\Phi_m$  denotes the  $m$ -th cyclotomic polynomial and that  $\deg(\Phi_m) = \varphi(m)$ .

**Lemma 4.1** *Let  $m$  be an integer with  $m \geq 3$ . We have*

- (i) *If  $m$  is a power of 2, then  $\Phi_m(1) = \Phi_m(-1) = 2$ .*
- (ii) *Let  $p$  be a prime number with  $p \neq 2$  and let  $k \geq 1$  be an integer. If  $m = p^k$ , then  $\Phi_m(1) = p$  and  $\Phi_m(-1) = 1$ ; if  $m = 2p^k$ , then  $\Phi_m(1) = 1$  and  $\Phi_m(-1) = p$ .*
- (iii) *In all other cases,  $\Phi_m(1) = \Phi_m(-1) = 1$ .*
- (iv) *If  $\Phi_m(1) = \Phi_m(-1) = 1$ , then  $\deg(\Phi_m) \equiv 0 \pmod{4}$ .*

**Proof** (i) Let  $m = 2^k$  for some integer  $k \geq 2$ . Then  $\Phi_m(x) = x^{2^{k-1}} + 1$ , hence  $\Phi_m(1) = \Phi_m(-1) = 2$ .

(ii) We have  $\Phi_p(x) = x^{p-1} + \dots + x + 1$ , and  $\Phi_{p^k}(x) = \sum_{i=0, \dots, p-1} x^{ip^{k-1}}$  for all integers  $k \geq 1$  (see for instance [14, Chapter IV, Section 1] or [9, Chapter VI, Section 1]). Hence  $\Phi_{p^k}(1) = p$  and  $\Phi_{p^k}(-1) = 1$  for all  $r \geq 1$ . We have  $\Phi_{2p^k}(x) = \Phi_{p^k}(-x)$ , hence  $\Phi_{2p^k}(1) = 1$  and  $\Phi_{2p^k}(-1) = p$  for all  $k \geq 1$ .

(iii) Suppose that  $m$  is divisible by at least two distinct prime numbers. Then  $\Phi_m(1) = 1$  by [24, Proposition 2.8]. We have  $\Phi_m(-1) = \Phi_{2m}(1)$ , therefore the same result implies that  $\Phi_m(-1) = 1$ .

(iv) If  $\Phi_m(1) = \Phi_m(-1) = 1$ , then by (i) and (ii) the integer  $m$  is not of the form  $p^k$  or  $2p^k$  for some prime number  $p$ . Therefore  $m$  is divisible by 4 and by an odd prime number, or by two distinct odd prime numbers. This implies that  $\varphi(m)$  is divisible by 4, hence  $\deg(\Phi_m) \equiv 0 \pmod{4}$ . □

**Lemma 4.2** *Let  $C = \Phi_m^r$  where  $m, r$  are integers with  $m \geq 3$  and  $r \geq 1$ . If  $C(1)$  and  $C(-1)$  are both squares, then  $\deg(C)$  is divisible by 4.*

**Proof** If  $C(1) = C(-1) = 1$ , then Lemma 4.1 (iv) implies that  $\deg(C)$  is divisible by 4. Suppose that  $C(1)C(-1) \neq 1$ . Then by Lemma 4.1 we have  $m = p^k$  or  $m = 2p^k$  for some prime number  $p$ , and hence  $\Phi_m(1) = p$  or  $\Phi_m(-1) = p$  (see Lemma 4.1 (i) and (ii)). This implies that  $r$  is even, and since  $\deg(\Phi_m)$  is even, the degree of  $C$  is divisible by 4. □

We now recall some notation from the introduction:

- $C = \Phi_m^r$  where  $m, r$  are integers with  $m \geq 3$  and  $r \geq 1$ ,
- $R, S \geq 0$  are integers such that  $R \equiv S \pmod{8}$  and  $\deg(C) < R + S$ . Set  $N = R + S$ .

Theorem 1.7 is a consequence of Lemmas 4.3 and 4.5.

**Lemma 4.3** *Let  $L$  be an even unimodular lattice of signature  $(R, S)$ , and let  $t: L \rightarrow L$  be an isometry with characteristic polynomial  $C(x)(x - 1)^{N - \deg(C)}$ . Let  $(c, \deg(C) - c)$  be the signature of the sublattice  $\text{Ker}(C(t))$ . Then*

- $C(-1)$  is a square.
- If  $C(1) = 1$ , then  $\deg(C) \equiv 2c \pmod{8}$ .

Moreover, if  $C(1) = 1$ , then the sublattice  $\text{Ker}(C(t))$  is unimodular.

**Proof** Set  $F(x) = C(x)(x - 1)^{N-\deg(C)}$ . Then by Lemma 3.2 the polynomial  $F$  satisfies condition (C1); this implies that  $|F(-1)|$  is a square. Note that  $N - \deg(C)$  is even, hence  $|C(-1)|$  is a square. Since  $C$  is a power of a cyclotomic polynomial, we have  $C(-1) \geq 0$  (cf. Lemma 4.1), hence  $C(-1)$  is a square, and hence (i) holds.

Set  $L_1 = \text{Ker}(C(t))$  and let  $L_2$  be the orthogonal complement of  $L_1$  in  $L$ . If  $C(1) = 1$ , then the polynomials  $x - 1$  and  $C$  are relatively prime over  $\mathbf{Z}$  (i.e. the resultant of  $x - 1$  and  $C$  is equal to 1); this implies that  $L = L_1 \oplus L_2$ , and hence the lattices  $L_1$  and  $L_2$  are both even and unimodular. Therefore  $\deg(C) - c \equiv c \pmod{8}$ , hence  $\deg(C) \equiv 2 \pmod{8}$ , and therefore (ii) holds.  $\square$

**Lemma 4.4** (i) *If  $C(1)$  and  $C(-1)$  are both squares, then  $\deg(C) \equiv 0 \pmod{4}$ .*

(ii) *If  $C(1)$  and  $C(-1)$  are both squares, then condition (C1) holds for  $C$ .*

*Let  $c \geq 0$  be an even integer such that  $c \leq \deg(C)$ ,  $c \leq R$  and  $\deg(C) - c \leq S$ . We have*

(iii) *If  $N = \deg(C) + 2$  and  $\deg(C) \equiv 0 \pmod{4}$ , then  $\deg(C) \equiv 2c \pmod{8}$ .*

**Proof** If  $C(1)$  and  $C(-1)$  are both squares, then Lemma 4.2 implies that  $\deg(C)$  is divisible by 4, hence (i) holds. Set  $\deg(C) = 2n$ ; then  $n$  is even, hence  $C(1)$ ,  $C(-1)$  and  $(-1)^n C(1)$  and  $C(-1)$  are all squares, therefore condition (C1) holds for  $C$  and this implies (ii).

Let us prove (iii). Since  $\deg(C)$  is divisible by 4 and  $\deg(C) = N - 2$ , we have  $N \equiv 2 \pmod{4}$ , and therefore  $R$  and  $S$  are both odd integers. Since  $\deg(C) = N - 2$ , the inequalities  $c \leq R$  and  $\deg(C) - c \leq S$  imply that  $R - 2 \leq c \leq R$ ; since  $R$  and  $S$  are odd, this implies that  $c = R - 1$  and  $\deg(C) - c = S - 1$ . We have  $R \equiv S \pmod{8}$  by hypothesis, hence  $\deg(C) - c \equiv c \pmod{8}$ , as claimed.  $\square$

**Lemma 4.5** *Let  $c \geq 0$  be an even integer such that  $c \leq \deg(C)$ ,  $c \leq R$  and  $\deg(C) - c \leq S$ . Suppose that the following conditions hold:*

- (i)  *$C(-1)$  is a square.*
- (ii) *If  $C(1) = 1$ , then  $\deg(C) \equiv 2c \pmod{8}$ .*

*Then there exist an even unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t: L \rightarrow L$  such that*

- *The characteristic polynomial of  $t$  is  $C(x)(x - 1)^{N-\deg(C)}$ .*
- *The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ .*

**Proof** Set  $F(x) = C(x)(x - 1)^{N-\deg(C)}$ . By (i), the polynomial  $F$  satisfies condition (C1).

Suppose that  $C(1) > 1$ , and note that by Lemma 4.1 this implies that  $m = p^k$  for some prime number  $p$ . If  $C(1) > 1$  and  $\deg(C) < N - 2$ , then with the notation of Sect. 3 we have  $n^+ \neq 2$ , hence  $\Pi_{\Phi_m(x), x-1} = \{p\}$ . Suppose now that  $C(1) > 1$ , that  $\deg(C) = N - 2$ , and that  $C(1)$  is not a square. Then we have  $D_0 \neq -1$  in  $\mathbf{Q}_p^\times / \mathbf{Q}_p^{\times 2}$ , hence  $\Pi_{\Phi_m(x), x-1} = \{p\}$  in this case as well. Therefore in both cases we have  $G_F = 0$ . By Theorem 3.8, there exist an even, unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t: L \rightarrow L$  with characteristic polynomial  $F$  and signature map  $\tau$  satisfying  $\tau(C) = (c, \deg(C) - c)$ . Let  $L_1 = \text{Ker}(C(t))$  and let  $L_2$  be the sublattice of  $L$  of fixed points by  $t$ .



It remains to consider the cases where  $C(1)$  is a square, and either  $C(1) = 1$  or  $\deg(C) = N - 2$ .

If  $C(1)$  is a square, then by Lemma 4.4 condition (C1) holds for  $C$ . Moreover, we have  $\deg(C) - c \equiv c \pmod{8}$ . If  $C(1) = 1$ , this follows from (ii), and if  $\deg(C) = N - 2$ , from Lemma 4.4 (i) and (iii).

The polynomial  $C$  is a power of an irreducible polynomial, hence the group  $G_C$  is trivial, and therefore in both cases we can apply Theorem 3.5, and conclude that there exist an even unimodular lattice  $L_1$  of signature  $(c, \deg(C) - c)$  and an isometry  $t_1: L_1 \rightarrow L_1$  of characteristic polynomial  $C$  (note that this also follows from [4, Theorem A]). Let  $L_2$  be an even unimodular lattice of signature  $(R - c, S - \deg(C) - c)$ , and let  $t_2: L_2 \rightarrow L_2$  be the identity. Set  $L = L_1 \oplus L_2$  and  $t = (t_1, t_2)$ .

The lattice  $L$  is even unimodular and of signature  $(R, S)$ , and  $t$  is an isometry of  $L$  with the required properties. □

The following is a reformulation of Theorem 1.7 from the introduction.

**Theorem 4.6** *There exist an even unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t: L \rightarrow L$  such that*

- *The characteristic polynomial of  $t$  is  $C(x)(x - 1)^{N - \deg(C)}$ .*
- *The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$  if and only if the following conditions are satisfied:*
  - (i)  *$C(-1)$  is a square.*
  - (ii) *If  $C(1) = 1$ , then  $\deg(C) \equiv 2c \pmod{8}$ .*

**Proof** This is an immediate consequence of Lemmas 4.3 and 4.5. □

**Notation** If  $L$  is a lattice, we denote by  $L^\#$  its dual lattice, and set  $\Delta(L) = L^\# / L$ .

**Definition 4.7** Let  $p$  be a prime number. We say that a lattice  $L$  is  $p$ -elementary if  $p\Delta(L) = 0$ .

**Proposition 4.8** *Let  $L$  be a unimodular lattice of rank  $N$  and let  $t: L \rightarrow L$  be an isometry with characteristic polynomial  $C(x)(x - 1)^{N - \deg(C)}$ . Set  $L_C = \text{Ker}(C(t))$  and let  $L_0$  be the orthogonal complement of  $L_C$  in  $L$ . Then we have*

- (i) *If  $C(1) = 1$ , then  $L_C$  and  $L_0$  are both unimodular.*
- (ii) *If  $C(1) > 1$ , then  $L_C$  and  $L_0$  are both  $p$ -elementary, where  $p$  is such that  $\Phi_m(1) = p$ .*

**Proof** (i) Lemma 4.3 implies that  $L_C$  is unimodular, hence  $L_0$  is also unimodular.  
 (ii) The action of  $t$  endows  $L$  with a structure of  $\mathbf{Z}[\Gamma]$ -module with  $\Gamma$  infinite cyclic; this action stabilizes  $L_C$  and  $L_0$ , hence also  $\Delta(L_C)$  and  $\Delta(L_0)$ . Since  $L$  is unimodular, the  $\mathbf{Z}[\Gamma]$ -modules  $\Delta(L_C)$  and  $\Delta(L_0)$  are isomorphic.

Lemma 4.1 implies that  $C = \Phi_m^r$  with  $m = p^k$  for some integer  $k$ . Therefore  $t$  acts on  $L_C$  by  $t(x) = \alpha \cdot x$  where  $\alpha$  is the image of  $x$  in  $\mathbf{Z}[x]/(\Phi_{p^k})$ . On the other hand,  $t$  acts as the identity on  $L_0$ . Since the  $\mathbf{Z}[\Gamma]$ -modules  $\Delta(L_C)$  and  $\Delta(L_0)$  are isomorphic, this implies that  $p\Delta(L_C) = 0$  and  $p\Delta(L_0) = 0$ . □

### 5 Automorphisms acting trivially on Picard lattices

We now prove Proposition 1.4 from the introduction. Let  $m, r$  be integers with  $m \geq 3$  and  $r \geq 1$  and let  $C = \Phi_m^r$ . Assume that  $\deg(C) \leq 20$ .

**Proposition 5.1** *There exists an automorphism  $a: X \rightarrow X$  of a projective K3 surface  $X$  such that  $a^*$  is the identity on  $S_X$  and that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$  if and only if the following conditions hold:*

- (i)  $C(-1)$  is a square.
- (ii) If  $C(1) = 1$ , then  $\deg(C) \equiv 4 \pmod{8}$ .

*If  $C(1) = 1$ , then the lattice  $T_X$  is unimodular. Moreover, the K3 surface is unique up to isomorphism if and only if  $C$  is a cyclotomic polynomial (i.e.  $r = 1$ ).*

**Proof** Let  $a: X \rightarrow X$  be an automorphism of a projective K3 surface such that  $a^*|_{S_X}$  is the identity, and that the characteristic polynomial of  $a^*|_{T_X}$  is equal to  $C$ . Applying Lemma 4.3 with  $L = H^2(X, \mathbf{Z}), t = a^*, N = 22, R = 3, S = 19$ , and  $c = 2$  shows that (i) and (ii) hold.

Conversely, assume that (i) and (ii) hold, and set

$$F(x) = C(x)(x - 1)^{22 - \deg(C)}.$$

By Lemma 4.5 with  $N = 22, R = 3, S = 19$  and  $c = 2$  there exist an even unimodular lattice  $L$  of signature  $(3, 19)$  and an isometry  $t: L \rightarrow L$  with characteristic polynomial  $F$  such that the signature of the sublattice  $L_1 = \text{Ker}(C(t))$  is  $(2, \deg(C) - 2)$ . Let  $L_1 = \text{Ker}(C(t))$  and let  $L_2$  be the sublattice of  $L$  of fixed points by  $t$ .

Let  $V \subset L_1 \otimes_{\mathbf{Z}} \mathbf{R}$  be a 2-dimensional subspace of signature  $(2, 0)$  and stable by  $t$ ; for a generic choice of  $V$ , the intersection of  $L$  with the orthogonal of  $V$  is equal to  $L_2$ . The restriction of  $t$  to  $V$  has determinant 1. Since  $t_2$  is the identity, it is positive in the sense of McMullen [17, Section 2]. By [17, Theorem 6.1], there exist a projective K3 surface  $X$  and an automorphism  $a: X \rightarrow X$  such that  $a^* = t$ , that  $S_X = L_2$  and  $T_X = L_1$ . Therefore  $a^*$  is the identity on  $S_X$  and the restriction of  $a^*$  to  $T_X$  has characteristic polynomial  $C$ . If moreover  $C(1) = 1$ , then by Lemma 4.3 the lattice  $T_X$  is unimodular.

If  $r = 1$ , then the uniqueness of the K3 surface up to isomorphism follows from a result of Brandhorst [5, Theorem 1.2]. If  $r > 1$ , then varying the choice of the subspace  $V$  gives rise to an infinite family of K3 surfaces. □

**Corollary 5.2** *Let  $a: X \rightarrow X$  be an automorphism of a projective K3 surface  $X$  such that  $a^*$  is the identity on  $S_X$  and that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$ . Then one of the following holds:*

- (i)  $T_X$  and  $S_X$  are unimodular.
- (ii)  $T_X$  and  $S_X$  are  $p$ -elementary, where  $p$  is a prime number such that  $\Phi_m(1) = p$ .

**Proof** This is an immediate consequence of Proposition 4.8. □

Note that this implies that  $p \leq 19$ , hence we have the following

**Corollary 5.3** *If a projective  $K3$  surface  $X$  has a non-trivial automorphism that induces the identity on  $S_X$ , then the lattices  $T_X$  and  $S_X$  are either both unimodular or both  $p$ -elementary with  $p \leq 19$ .*

### 6 Cyclotomic polynomials and isometries of finite order

We keep the notation of Sect. 4; in particular,  $C = \Phi_m^r$  where  $m, r$  are integers with  $m \geq 3$  and  $r \geq 1$ . The following result implies Theorem 1.9 from the introduction; it is actually a *strengthening* of Theorem 1.9, since it implies the existence of an isometry is of order  $m$ .

**Theorem 6.1** *Suppose that  $C(1)C(-1) > 1$ . Then there exist an even, unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t : L \rightarrow L$  of order  $m$  such that*

- *The characteristic polynomial of  $t$  is divisible by  $C$ .*
- *The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \text{deg}(C) - c)$ .*

**Proof** If  $C(1) > 1$  and  $C(-1)$  is a square, then this follows from Theorem 4.6.

Suppose that  $C(-1) > 1$  and that  $C(1)$  is a square. In this case, set  $F(X) = C(X)(X + 1)^{N - \text{deg}(C)}$ . Since  $C(1)$  is a square, the polynomial  $F$  satisfies condition (C1).

If  $\text{deg}(C) < N - 2$ , then with the notation of Sect. 3 we have  $n^- \neq 2$ , hence  $\Pi_{\Phi_m(x), x-1} \neq \emptyset$ ; this implies that  $G_F = 0$ . Suppose now that  $\text{deg}(C) = N - 2$  and that  $C(-1)$  is not a square. Note that by Lemma 4.1 this implies that  $m = 2p^k$  for some odd prime number  $p$ ; with the notation of Sect. 3, we have  $D_0 \neq -1$  in  $\mathbf{Q}_p^\times / \mathbf{Q}_p^{\times 2}$ , hence  $\Pi_{\Phi_m(x), x-1} = \{p\}$ . This implies that  $G_F = 0$  in this case as well. By Theorem 3.8, there exist an even, unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t : L \rightarrow L$  with characteristic polynomial  $F$  and signature map  $\tau$  satisfying  $\tau(C) = (c, \text{deg}(C) - c)$ .

Suppose now that  $C(-1)$  is a square and  $\text{deg}(C) = N - 2$ ; then Lemma 4.4 implies that condition (C1) holds for  $C$  and that  $\text{deg}(C) - c \equiv c \pmod{8}$ . The polynomial  $C$  is a power of an irreducible polynomial, hence the group  $G_C$  is trivial, and therefore we can apply Theorem 3.5, and conclude that there exist an even unimodular lattice  $L_1$  of signature  $(c, \text{deg}(C) - c)$  and an isometry  $t_1 : L_1 \rightarrow L_1$  of characteristic polynomial  $C$  (note that this also follows from [4, Theorem A]). Let  $L_2$  be an even unimodular lattice of signature  $(R - c, S - \text{deg}(C) - c)$  and let  $t_2 : L_2 \rightarrow L_2$  be the identity. Set  $L = L_1 \oplus L_2$  and  $t = (t_1, t_2)$ ; then  $t : L \rightarrow L$  has the required properties.

Finally, suppose that  $C(1)$  and  $C(-1)$  are both non-squares. In this case,  $C(1) = C(-1) = 2$ , and  $C = \Phi_{2^k}$  for some integer  $k$  (see Lemma 4.1). Suppose first that  $N > \text{deg}(C) + 2$ . Set  $F(x) = C(x)(x + 1)^2(x - 1)^{N - \text{deg}(C) - 2}$ . We have  $\Pi_{C, x-1} = \Pi_{x-1, x+1} = \{2\}$  (see [3, Sections 7 and 12]), hence  $G_F = 0$ . Therefore by [3, Corollary 12.3], there exist exists an even, unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t : L \rightarrow L$  with characteristic polynomial  $F$  and signature map  $\tau$  satisfying  $\tau(C) = (c, \text{deg}(C) - c)$ . Assume now that  $N = \text{deg}(C) + 2$  and set  $F(x) = C(x)(x + 1)(x - 1)$ . By Takada [22, Theorem 6.11], there exist a lattice  $L$  and an isometry  $t : L \rightarrow L$  with the required properties. □

It remains to treat the case where  $C(1) = C(-1) = 1$ ; if  $\deg(C) \equiv 2c \pmod{8}$ , then Theorem 4.6 implies the following

**Corollary 6.2** *Suppose that  $C(1) = 1$  and that  $\deg(C) \equiv 2c \pmod{8}$ . Then there exist an even unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t : L \rightarrow L$  of order  $m$  such that*

- *The characteristic polynomial of  $t$  is divisible by  $C$ .*
- *The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ .*

The condition  $C(1) = C(-1) = 1$  implies that  $\deg(C) \equiv 0 \pmod{4}$  (see Lemma 4.1 (i)), and  $c$  is an even integer; hence we have either  $\deg(C) \equiv 2c \pmod{8}$  or  $\deg(C) \equiv c \pmod{8}$ . The first case is covered by Corollary 6.2, therefore we have the following two cases to consider

- (a)  $c \equiv 0 \pmod{4}$  and  $\deg(C) \equiv 4 \pmod{8}$ ,
- (b)  $c \equiv 2 \pmod{4}$  and  $\deg(C) \equiv 0 \pmod{8}$ .

We treat these cases in the next sections; the following results will be useful.

**Lemma 6.3** *Let  $t : L \rightarrow L$  be an isometry of a lattice  $L$  such that*

- *The characteristic polynomial of  $t$  is divisible by  $C$ .*
- *The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ .*

*If  $\deg(C) \equiv 0 \pmod{4}$  and  $c \equiv 2 \pmod{4}$ , then  $L$  is indefinite.*

**Proof** Indeed, the sublattice  $\text{Ker}(C(t))$  is indefinite: since  $\deg(C) \equiv 0 \pmod{4}$  and  $c \equiv 2 \pmod{4}$ , we have  $c \neq \deg(C)$  and  $c \neq 0$ . □

**Proposition 6.4** *Let  $m \geq 3$  be an integer and let  $p$  be a prime number that does not divide  $m$ . The following are equivalent:*

- (a) *The polynomial  $\Phi_m$  has a symmetric irreducible factor mod  $p$ .*
- (b) *The prime ideals of  $\mathbf{Q}(\zeta_m + \zeta_m^{-1})$  above  $p$  are inert in  $\mathbf{Q}(\zeta_m)$ .*
- (c) *The subgroup of  $(\mathbf{Z}/m\mathbf{Z})^\times$  generated by  $p$  contains  $-1$ .*

**Proof** The equivalence of (a) and (b) follows from [24, Proposition 2.14]. Let us prove that (b) and (c) are equivalent. Let  $G$  be the Galois group of  $\mathbf{Q}(\zeta_m)/\mathbf{Q}$  and let  $P$  be a prime ideal of  $\mathbf{Q}(\zeta_m)$  above  $p$ . The decomposition group  $G_P$  is by definition  $\{g \in G \mid g(P) = P\}$ . Since  $G$  is abelian, this group only depends on the prime number  $p$ ; set  $G_P = G_p$ . Condition (b) holds if and only if the element of  $G$  induced by  $\zeta_m \rightarrow \zeta_m^{-1}$  is contained in  $G_p$ . Let  $f : G \rightarrow (\mathbf{Z}/m\mathbf{Z})^\times$  be an isomorphism; then  $f(G_p)$  is the subgroup of  $(\mathbf{Z}/m\mathbf{Z})^\times$  generated by  $p$ . This implies the equivalence of (b) and (c). □

**Corollary 6.5** *Let  $m, p$  be distinct prime numbers. If  $p$  is not a square modulo  $m$  then the polynomial  $\Phi_m$  has a symmetric irreducible factor mod  $p$ .*

**Proof** We have  $p^{(m-1)/2} = \pm 1$ . If  $p^{(m-1)/2} = 1$ , then  $p$  is a square modulo  $m$ , hence  $p^{(m-1)/2} = -1$ . This implies that the subgroup of  $(\mathbf{Z}/m\mathbf{Z})^\times$  generated by  $p$  contains  $-1$ , and hence by Proposition 6.4 the polynomial  $\Phi_m$  has a symmetric irreducible factor mod  $p$ . □

We start by noting that if  $N$  is sufficiently large, then property (P1') holds.

**Proposition 6.6** *Suppose that  $C(1) = C(-1) = 1$ . Let  $p$  be a prime number such that  $\Pi_{\Phi_{mp}, \Phi_m} = \{p\}$ . If  $N > \deg(C) + \varphi(mp)$ , then there exist an even unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t : L \rightarrow L$  of order  $mp$  such that*

- *The characteristic polynomial of  $t$  is divisible by  $C$ .*
- *The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ .*

**Proof** Set  $F(x) = C(x)\Phi_{mp}(x)(x - 1)^k$  with  $k = N - \deg(C) - \varphi(mp)$ . The polynomial  $F$  satisfies condition (C1). Since  $\Pi_{\Phi_{mp}, \Phi_m} = \{p\}$ , we have  $G_F = 0$ . Therefore by Theorem 3.8 there exist a lattice  $L$  and an isometry  $t$  with the required properties.

Note that Proposition 6.4 implies that there exist infinitely many prime numbers  $p$  such that  $\Pi_{\Phi_{mp}, \Phi_m} = \{p\}$ . In the following sections, we give conditions on  $N$  for the existence of an isometry of order  $m$ . □

### 7 $C(1) = C(-1) = 1$ and $c \equiv 0 \pmod{4}$

We keep the notation of the previous sections and assume in addition that  $C(1) = C(-1) = 1$ ; this implies that  $\deg(C) \equiv 0 \pmod{4}$  (see Lemma 4.1 (i)). Suppose that  $c \equiv 0 \pmod{4}$ .

Corollary 6.2 implies that if  $\deg(C) \equiv 0 \pmod{8}$ , then there exists an even, unimodular lattice  $L$  of signature  $(R, S)$  having an isometry  $t : L \rightarrow L$  of order  $m$  such that the signature of  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ .

Suppose that  $\deg(C) \equiv 4 \pmod{8}$ ; then  $r$  is odd, and Lemma 4.1 implies that  $m$  is of one of the following forms:

- $m = 4p^k$  where  $p$  is a prime number with  $p \equiv 3 \pmod{4}$  and  $k \geq 1$  is an integer;
- $m = p^k q^s$  where  $p$  and  $q$  are distinct prime numbers with  $\equiv 3 \pmod{4}$  and  $k, s \geq 1$  are integers;
- $m = 2p^k q^s$  where  $p$  and  $q$  are distinct prime numbers with  $\equiv 3 \pmod{4}$  and  $k, s \geq 1$  are integers.

**Lemma 7.1**  $N \geq \deg(C) + 4$ .

**Proof** Let us show that  $N \neq \deg(C) + 2$ . Set  $c' = \deg(C) - c$ . We have  $c \leq R$ ,  $c' \leq S$  and  $N = R + S$ ,  $\deg(C) = c + c'$ ; moreover,  $c$  and  $c'$  are even. Therefore if  $N = \deg(C) + 2$ , then  $R = c + 1$  and  $S = c' + 1$ . We have  $R \equiv S \pmod{8}$ , hence this implies that  $c \equiv c' \pmod{8}$ ; but  $\deg(C) = c + c'$  is congruent to  $4 \pmod{8}$ , so this is impossible. Since  $N$  and  $\deg(C)$  are both even, this implies that  $N \geq \deg(C) + 4$ , as claimed. □

**Proposition 7.2** *Suppose that  $m = 4p^k$  where  $p$  is a prime number with  $p \equiv 3 \pmod{4}$  and  $k \geq 1$  is an integer. Then there exist an even unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t : L \rightarrow L$  of order  $m$  such that*

- *The characteristic polynomial of  $t$  is divisible by  $C$ .*

- The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \text{deg}(C) - c)$ .

**Proof** Set  $F = C\Phi_4^2(x-1)^{N-\text{deg}(C)-4}$ ; note that Lemma 7.1 implies that  $N - \text{deg}(C) - 4 \geq 0$ , and that  $F$  satisfies condition (C1). Since  $p \equiv 3 \pmod{4}$ ,  $\Phi_4$  is irreducible mod  $p$ , and therefore  $\Pi_{\Phi_m, \Phi_4} = \{p\}$ . This implies that  $G_F = 0$ , and hence by Theorem 3.8 there exist a lattice  $L$  and an isometry  $t$  with the required properties.

If  $p$  and  $q$  are distinct prime numbers  $\equiv 3 \pmod{4}$ , then by quadratic reciprocity either  $p$  is a square modulo  $q$  or  $q$  is a square modulo  $p$ , and these cases are mutually exclusive. Therefore we may assume that  $p$  is a square modulo  $q$ . □

**Proposition 7.3** *Suppose that  $m = p^kq^s$  or  $2p^kq^s$  where  $p$  and  $q$  are distinct prime numbers with  $p, q \equiv 3 \pmod{4}$  and  $k, s \geq 1$  are integers, and assume that  $p$  is a square modulo  $q$ . Suppose that  $N \geq \text{deg}(C) + (p - 1)p^{k-1} + 2$ . Then there exist an even unimodular lattice  $L$  of signature  $(R, S)$  and an isometry  $t : L \rightarrow L$  of order  $m$  such that*

- The characteristic polynomial of  $t$  is divisible by  $C$ .
- The signature of the sublattice  $\text{Ker}(C(t))$  is  $(c, \text{deg}(C) - c)$ .

**Proof** Set  $N' = N - \text{deg}(C) - (p - 1)p^{k-1} - 2$ ; set  $F(x) = C(x)\Phi_{p^k}(x)(x - 1)^{N'}$  if  $m = p^kq^s$ , and  $F(x) = C(x)\Phi_{2p^k}(x)(x + 1)^{N'}$  if  $m = 2p^kq^s$ . The polynomial  $F$  satisfies condition (C1), and Lemma 4.1 implies that  $G_F = 0$ . Theorem 3.8 implies that there exist a lattice  $L$  and an isometry  $t$  with the required properties. □

### 8 $C(1) = C(-1) = 1$ and $c \equiv 2 \pmod{4}$

We keep the notation of Sect. 4; in particular,  $C = \Phi_m^r$  where  $m, r$  are integers with  $m \geq 3$  and  $r \geq 1$ . The case where  $C(1)C(-1) > 1$  is covered by Theorem 6.1.

Assume now that  $C(1) = C(-1) = 1$  and that  $c \equiv 2 \pmod{4}$ . Since  $C(1) = C(-1) = 1$ , by Lemma 4.1 we have  $\text{deg}(C) \equiv 0 \pmod{4}$ ; hence Lemma 6.3 implies that if  $t : L \rightarrow L$  is an isometry of a lattice such that the signature of  $\text{Ker}(C(t))$  is  $(c, \text{deg}(C) - c)$ , then  $L$  is indefinite. This implies that  $R > 0$  and  $S > 0$ ; recall that since  $R \equiv S \pmod{8}$ , there exists up to isomorphism a unique even, unimodular lattice of signature  $(R, S)$  (see for instance [21, Chapter V]); we denote it by  $\Lambda_{R,S}$ .

In the applications to  $K3$  surfaces, we have  $R = 3, S = 19$ , and  $c = 2$ .

Note that the case where  $\text{deg}(C) \equiv 4 \pmod{8}$  was already handled in Corollary 6.2.

**Proposition 8.1** *If  $\text{deg}(C) \equiv 4 \pmod{8}$  then the lattice  $\Lambda_{R,S}$  has an isometry  $t$  of order  $m$  such that the signature of  $\text{Ker}(C(t))$  is  $(c, \text{deg}(C) - c)$ .*

**Proof** Indeed, we are assuming that  $c \equiv 2 \pmod{4}$ , hence the hypothesis  $\text{deg}(C) \equiv 4 \pmod{8}$  implies that  $\text{deg}(C) \equiv 2c \pmod{8}$ . Therefore by Corollary 6.2 there exist an even, unimodular lattice  $L$  and an isometry  $t : L \rightarrow L$  of order  $m$  such that the signature of  $\text{Ker}(C(t))$  is  $(c, \text{deg}(C) - c)$ . By Lemma 6.3, such a lattice is indefinite, hence  $L$  is isomorphic to  $\Lambda_{R,S}$ . □

Suppose that  $\deg(C) \equiv 0 \pmod{8}$ . Recall that  $N = R + S$ . Using the results of [3] (in particular, Theorem 3.8) and Proposition 6.4 it is possible to determine the values of  $N$  for which  $\Lambda_{R,S}$  has an isometry  $t$  of order  $m$  such that the signature of  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$ ; since this would be quite long, we only give some partial results that will be useful for the for the proof of Proposition 1.1.

**Proposition 8.2** *Let  $m = 2^n p$  with  $n \geq 2$  and  $p$  be a prime number such that  $p \equiv 3, 5 \pmod{8}$ , or  $m = pq$  with  $p$  and  $q$  distinct prime numbers such that  $p$  is not a square modulo  $q$ . Suppose that  $N \geq \deg(C) + p + 1$ ; set  $M = N - \deg(C) - p + 1$  and  $F(x) = C(x)\Phi_p(x)(x - 1)^M$ . Then  $\Lambda_{R,S}$  has an isometry  $t$  with characteristic polynomial  $F$  such that the signature of  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$  and that the signature of  $\text{Ker}((t - 1)^M)$  is  $(1, M - 1)$ .*

**Proof** Since  $C(1) = C(-1) = 1$ , the polynomial  $F$  satisfies condition (C1). We have  $G_F = 0$ ; indeed, by Corollary 6.5 we have  $\Pi_{\Phi_m, \Phi_p} = \{2\}$  if  $m = 2^n p$  and  $\Pi_{\Phi_m, \Phi_p} = \{q\}$  if  $m = pq$ ; moreover,  $\Pi_{\Phi_p(x), x-1} = \{p\}$ . Therefore by Theorem 3.8 there exists an isometry with the required properties.  $\square$

**Proposition 8.3** *Let  $m = 2pq$  with  $p$  and  $q$  distinct prime numbers such that  $p$  is not a square modulo  $q$ . Suppose that  $N \geq \deg(C) + p + 1$ ; set  $M = N - \deg(C) - p - 1$  and  $F(x) = C(x)\Phi_{2p}(x)(x+1)^2(x-1)^M$ . Then  $\Lambda_{R,S}$  has an isometry  $t$  with characteristic polynomial  $F$  such that the signature of  $\text{Ker}(C(t))$  is  $(c, \deg(C) - c)$  and that the signature of  $\text{Ker}((t - 1)^M)$  is  $(1, M - 1)$ .*

**Proof** We have  $\Pi_{\Phi_m, \Phi_{2p}} = \{q\}$  by Corollary 6.5,  $\Pi_{\Phi_{2p}(x), x+1} = \{p\}$ ,  $\Pi_{x+1, x-1} = \{2\}$ , hence  $G_F = 0$ . The polynomial  $F$  satisfies condition (C1). Therefore by Theorem 3.8 there exists an isometry with the required properties.  $\square$

### 9 Salem polynomials and isometries of lattices

A Salem polynomial is a monic irreducible polynomial  $S \in \mathbf{Z}[X]$  such that  $S(X) = X^{\deg(S)}S(X^{-1})$  and that  $S$  has exactly two roots outside the unit circle, both positive real numbers.

**Example 9.1** Let  $n$  be an integer  $\geq 0$  and set

$$S_n(X) = X^6 - nX^5 - X^4 + (2n - 1)X^3 - X^2 - nX + 1.$$

The polynomials  $S_n$  are Salem polynomials (see [16, Section 4] or [10, Section 7, Example 1]); we have  $S_n(1) = -1$  and  $S_n(-1) = 1$ .

If  $a: X \rightarrow X$  is an automorphism of a projective K3 surface, then the characteristic polynomial of  $a^*: H^2(X, \mathbf{C}) \rightarrow H^2(X, \mathbf{C})$  is either a product of cyclotomic polynomials or it is of the form  $SC$ , where  $S$  is a Salem polynomial and  $C$  a product of cyclotomic polynomials (see [16, Theorem 3.2 and Corollary 3.3]).

We recall some notions and results from [3, Sections 7 and 12].

**Notation 1** Let  $S$  be a Salem polynomial such that  $S(1) = -1$  and  $S(-1) = 1$ , and let  $C$  be a cyclotomic polynomial. Let  $\Pi_{S,C}$  be the set of prime numbers  $p$  such that  $S \pmod{p}$  and  $C \pmod{p}$  have a common irreducible symmetric factor in  $\mathbb{F}_p[x]$ .

**Example 9.2** Let  $S_2(x) = x^6 - 2x^5 - x^4 + 3x^3 - x^2 - 2x + 1$  (cf. Example 9.1) and let  $C = \Phi_{60}$ . The polynomials  $S_2 \pmod{359}$  and  $C \pmod{359}$  have the common irreducible factor  $x^2 - 15x + 1$  in  $\mathbb{F}_{359}[x]$ ; this polynomial is symmetric, hence  $359 \in \Pi_{S,C}$ .

**Notation** Let  $F = SC$  for  $S$  and  $C$  as in Notation 1. We define a group  $G_F$  as in [3, Section 7] (see also Sect. 3); we have  $G_F = 0$  if  $\Pi_{S,C} \neq \emptyset$ , and  $G_F = \mathbb{Z}/2\mathbb{Z}$  if  $\Pi_{S,C} = \emptyset$ .

**Proposition 9.3** Let  $S$  be a Salem polynomial such that  $S(1) = -1$  and  $S(-1) = 1$ , and let  $C$  be a cyclotomic polynomial; set  $F = SC$ . Suppose that  $\deg(F) = 22$ , that condition (C1) holds for  $F$ , and that  $G_F = 0$ . Then the lattice  $\Lambda_{3,19}$  has an isometry  $t$  of signature map  $\tau$  satisfying  $\tau(C) = (2, \deg(C) - 2)$ .

**Proof** This is a consequence of [3, Corollary 12.3]. □

**Example 9.4** Let  $S(x) = S_2(x) = x^6 - 2x^5 - x^4 + 3x^3 - x^2 - 2x + 1$ , and  $C = \Phi_{60}$ ; set  $F = SC$ . We have  $359 \in \Pi_{S,C}$  (cf. Example 9.2), therefore  $G_F = 0$ . Condition (C1) holds for  $F$ , hence by Proposition 9.3 the lattice  $\Lambda_{3,19}$  has an isometry  $t$  with characteristic polynomial  $F$  such that the signature of  $\text{Ker}(C(t))$  is  $(2, 14)$ .

**Example 9.5** Let  $C = \Phi_{30}^2$  and  $S_1(x) = x^6 - x^5 - x^4 + x^3 - x^2 - x + 1$ , as in Example 9.1. We have  $3 \in \Pi_{S_1, \Phi_{30}}$ , hence  $G_{S_1C} = 0$ . Proposition 9.3 implies that  $\Lambda_{3,19}$  has an isometry  $t$  such that the signature of  $\text{Ker}(C(t))$  is  $(2, 14)$ .

**Example 9.6** Let  $C = \Phi_{10}^4$ . We have  $S_0(x) = x^6 - x^4 - x^3 - x^2 + 1$  (cf. Example 9.1) and  $3 \in \Pi_{S_0, \Phi_{10}}$ , hence  $G_{S_0C} = 0$ . Proposition 9.3 implies that  $\Lambda_{3,19}$  has an isometry  $t$  such that the signature of  $\text{Ker}(C(t))$  is  $(2, 14)$ .

**Remark 9.7** The sets  $\Pi_{S,C}$  of the above examples were computed by PARI GT.

**Notation** Let  $C$  be a cyclotomic polynomial and let  $S_n$  be as in Example 9.1. Let  $N(C)$  be the set of integers  $n \geq 0$  such that  $\Pi_{C, S_n} \neq \emptyset$ .

**Example 9.8** Let  $C = \Phi_{60}$ . We have  $0, 2, 5, 6, 7, \dots \in N(C)$ .

**Question 9.9** Let  $C$  be a cyclotomic polynomial. Is the set  $N(C)$  infinite?

### 10 Proof of Proposition 1.1—first part

In this section and the next one, we prove Proposition 1.1 from the introduction. Let  $m, r$  be integers with  $m \geq 3$  and  $r \geq 1$ , and let  $C = \Phi_m^r$ . Assume that  $\deg(C) \leq 20$ .

**Proposition 10.1** *There exists an automorphism  $a : X \rightarrow X$  of a projective K3 surface  $X$  such that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$ .*



The proof of the proposition is divided into several parts, according to the value of  $m$ . Note first that if  $m = p^k$  for some prime number  $p \neq 2$ , then Proposition 10.1 follows from Proposition 5.1.

**Proposition 10.2** *Suppose that  $m = p^k$  where  $p$  is a prime number,  $p \neq 2$ , and  $k \geq 1$  is an integer. Then there exists an automorphism  $a : X \rightarrow X$  of a projective K3 surface  $X$  such that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$ .*

**Proof** We have  $C(1) = p^r$  and  $C(-1) = 1$ , hence Proposition 5.1 implies the existence of an automorphism  $a : X \rightarrow X$  of a projective K3 surface  $X$  with the required properties.  $\square$

**Proposition 10.3** *Suppose that  $m = 2p^k$  where  $p$  is a prime number, and  $k \geq 1$  is an integer. Suppose that  $r$  is even and that  $\deg(C) \equiv 4 \pmod{8}$ . Then there exists an automorphism  $a : X \rightarrow X$  of a projective K3 surface  $X$  such that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$ .*

**Proof** We have  $C(1) = 1$  and  $C(-1) = p^r$ ; since  $r$  is even,  $C(-1)$  is a square, hence the conditions of Proposition 5.1 are satisfied; therefore this implies the existence of an automorphism  $a : X \rightarrow X$  of a projective K3 surface  $X$  with the required properties.  $\square$

In the remaining cases, the proofs use modified versions of the results of the previous sections. The following lemma is based on results of McMullen in [17], and will be used in the proof of Proposition 10.1. Recall from [17, Section 2] that an isometry of a hyperbolic lattice is said to be *positive* if it stabilizes a chamber.

**Lemma 10.4** *Let  $(L, q)$  be an even unimodular lattice of signature  $(3, 19)$  and let  $t : L \rightarrow L$  be an isometry of  $L$ . Let  $L_1$  and  $L_2$  be mutually orthogonal sublattices of  $L$  such that  $L_1 \oplus L_2$  is of finite index in  $L$ , that  $t(L_1) = L_1$ ,  $t(L_2) = L_2$ , that the signature of  $L_1$  is  $(2, \text{rank}(L_1) - 2)$  and the signature of  $L_2$  is  $(1, \text{rank}(L_2) - 1)$ . Suppose moreover that the restriction of  $t$  to  $L_2$  preserves a connected component of  $\{x \in L_2 \otimes_{\mathbb{Z}} \mathbb{R} \mid q(x, x) > 0\}$ . Then we have*

- (i) *The lattice  $L$  has an isometry  $t' : L \rightarrow L$  such that the restriction of  $t'$  to  $L_2$  is positive, and that  $t'$  and  $t$  coincide on  $L_1$ .*
- (ii) *Let  $V \subset L_1 \otimes_{\mathbb{Z}} \mathbb{R}$  be a 2-dimensional subspace of signature  $(2, 0)$  and stable by  $t$  such that the intersection of  $L$  with the orthogonal of  $V$  is equal to  $L_2$  and that the restriction of  $t$  to  $V$  is in  $\text{SO}(V)$ . Then there exist a projective K3 surface  $X$  and an automorphism  $a : X \rightarrow X$  such that  $T_X \simeq L_1$ ,  $S_X \simeq L_2$ , and  $a^*|_{T_X} = t$ .*

**Proof** (i) Set  $t_1 = t|_{L_1}$  and  $t_2 = t|_{L_2}$ . For  $i = 1, 2$ , set  $\bar{L}_i = L_i^\# / L_i$ , and let  $\bar{q}_i$  and  $\bar{t}_i$  be the induced symmetric bilinear forms and isometries; since  $L$  is unimodular, we have  $(\bar{L}_1, \bar{q}_1, \bar{t}_1) \simeq (\bar{L}_2, -\bar{q}_2, \bar{t}_2)$ . If  $L_2$  has no roots, then  $t_2$  is a positive isometry in the sense of McMullen [17, Section 2]; otherwise, let  $\rho$  be an element of the Weyl group of  $L_2$  such that  $\rho \circ t_2$  is positive. Set  $t'_2 = t_2$  in the first case, and  $t'_2 = \rho \circ t_2$  in the second one. The elements of the Weyl group of  $L_2$  induce the identity on  $\bar{L}_2$ , hence we have  $(\bar{L}_1, \bar{q}_1, \bar{t}_1) \simeq (\bar{L}_2, -\bar{q}_2, \bar{t}'_2)$ . This implies that there exists an isometry  $t' : L \rightarrow L$  such that  $t'|_{L_1} = t_1$  and  $t'|_{L_2} = t'_2$ ; the isometry  $t'_2$  is positive, and this implies (i).

(ii) Applying [17, Theorem 6.1] to the isometry  $t' : L \rightarrow L$  constructed in part (i), we conclude that there exist a projective K3 surface  $X$  with  $T_X \simeq L_1$ ,  $S_X \simeq L_2$ , and an automorphism  $a : X \rightarrow X$  such that  $a^* = t'$ . By construction, we have  $t'|L_1 = t|L_1$ , hence the restriction of  $a^*$  to  $T_X$  is equal to  $t|L_1$ .  $\square$

**Proposition 10.5** *Suppose that  $C(1) = C(-1) = 1$ . Then there exists an automorphism  $a : X \rightarrow X$  of a projective K3 surface  $X$  such that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$ .*

**Proof** If  $\deg(C) \equiv 4 \pmod{8}$ , then this follows from Proposition 5.1. Suppose that  $\deg(C) \equiv 0 \pmod{8}$ , and that  $m \neq 30, 60$ . Then we have  $m = 15, 20, 24$  and  $r = 1$  or  $2$ , or  $m = 40, 48$  and  $r = 1$ . By Proposition 8.2 the lattice  $\Lambda_{3,19}$  has an isometry  $t$  such that the characteristic polynomial of  $t$  is divisible by  $C$  and by  $(x - 1)^4$ , and that the signature of  $\text{Ker}(C(t))$  is  $(2, \deg(C) - 2)$ . The same property holds for  $m = 30$  and  $r = 1$  by Proposition 8.3. If  $m = 60$ , then by Example 9.4, the lattice  $\Lambda_{3,19}$  has an isometry  $t$  such that the characteristic polynomial of  $t$  is  $CS_2$ ; if  $m = 30$  and  $r = 2$ , then this holds for  $CS_1$  by Example 9.5.

Set  $L_1 = \text{Ker}(C(t))$  and let  $L_2$  be the orthogonal complement of  $L_1$  in  $L$ . The hypotheses of Lemma 10.4 are fulfilled; hence by Lemma 10.4 there exist a projective K3 surface  $X$  with  $T_X = L_1$  and an automorphism  $a : X \rightarrow X$  such that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$ .  $\square$

**Proposition 10.6** *Let  $p$  be a prime number, let  $r, k \geq 0$  be integers, and let  $C = \Phi_{2p^k}^r$ . Suppose that  $\deg(C) \leq 16$ , or  $r$  is even and  $\deg(C) \leq 20$ . Then there exists an automorphism  $a : X \rightarrow X$  of a projective K3 surface  $X$  such that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is equal to  $C$ .*

**Proof** The hypothesis implies that  $C(1) = 1$  (if  $p \neq 2$ ) or  $C(1) = 2^r$  (if  $p = 2$ ) and  $C(-1) = p^r$ . If  $r$  is odd, set  $C'(x) = (x + 1)^2(x - 1)^{20 - \deg(C)}$ . If  $C = \Phi_5^4$ , set  $C = S_0$  (cf. Example 9.1). Then  $CC'$  satisfies condition (C1) and  $G_{CC'} = 0$ , hence by Theorem 3.8 the lattice  $\Lambda_{3,19}$  has an isometry with characteristic polynomial  $CC'$  such that the signature of  $\text{Ker}(C(t))$  is  $(2, \deg(C) - 2)$ . If  $r$  is even and  $p = 2$  or  $\deg(C) \equiv 4 \pmod{8}$ , then the existence of such an isometry (with  $C'$  a power of  $x - 1$ ) follows from Proposition 5.1. We conclude as in the proof of Proposition 10.5.  $\square$

### 11 Proof of Proposition 1.1—continued

The aim of this section is to prove Proposition 1.1 (that is, Proposition 10.1) in the remaining cases; the results are stated in a more general setting than needed.

**Notation** Let  $q$  be a prime number. If  $V = (V, b)$  is a quadratic form over  $\mathbf{Q}_q$ , we denote by  $d(V) \in \mathbf{Q}_q^\times / \mathbf{Q}_q^{\times 2}$  its determinant and by  $w(V) \in \text{Br}_2(\mathbf{Q}_q)$  its Hasse–Witt invariant.

**Lemma 11.1** *Let  $V$  be a quadratic form over  $\mathbf{Q}_2$ . Then  $V$  contains an even, unimodular  $\mathbf{Z}_2$ -lattice if and only if*

- $\dim(V) \equiv 2 \pmod{4}$ ,  $d(V) = -1$  and  $w(V) = 0$  or  $d(V) = 3$  and  $d(V) = 1$ ;
- $\dim(V) \equiv 0 \pmod{4}$ ,  $d(V) = 1$  and  $w(V) = 1$  or  $d(V) = 5$  and  $d(V) = 0$ .

**Proof** Let  $H = \langle 1, -1 \rangle$  and  $N = \langle 2, 6 \rangle$ . By [8, Proposition 5.2], we see that  $V$  contains an even, unimodular  $\mathbf{Z}_2$ -lattice if and only if  $V$  is an orthogonal sum of copies of  $H$  and  $N$ ; the lemma follows by computing the invariants of these orthogonal sums.  $\square$

**Notation** Let  $K$  be a field, and let  $E$  be an étale  $K$ -algebra with a  $K$ -linear involution  $\sigma : E \rightarrow E$ ; set  $E_0 = \{x \in E \mid \sigma(x) = x\}$ . Let  $\lambda \in E_0^\times$ . We denote by  $b_\lambda$  the quadratic form  $b_\lambda : E \times E \rightarrow K$  given by  $b_\lambda(x, y) = \text{Tr}_{E/K}(\lambda x \sigma y)$ .

**Proposition 11.2** *Let  $p$  be a prime number,  $p \neq 2$ , let  $r, k \geq 0$  be integers, and let  $C = \Phi_{2p^k}^r$ . Suppose that if  $r$  is even, then  $\deg(C) \equiv 4 \pmod{8}$ . The lattice  $(L, q) = \Lambda_{3,19}$  has an isometry  $t$  with characteristic polynomial  $CC'$ , where  $C'(x) = (x^2 - 1)(x - 1)^{20 - \deg(C)}$ , such that the sublattice  $\text{Ker}(C(t))$  has signature  $(2, 18)$  and that the restriction of  $t$  to  $\text{Ker}(C'(t))$  stabilizes one of the connected components of  $\{x \in \text{Ker}(C'(t)) \otimes_{\mathbf{Z}} \mathbf{R} \mid q(x, x) > 0\}$ .*

**Proof** Set  $2n = 22 - \deg(C)$ , and let  $U$  be the  $\mathbf{Q}$ -vector space with basis  $e_1, \dots, e_n, f_1, \dots, f_n$ ; let  $Q : U \times U \rightarrow \mathbf{Q}$  be the orthogonal sum of the quadratic form equal to  $\langle 2, -2p^r \rangle$  on the subspace generated by  $e_1$  and  $f_1$ , and of the diagonal form  $\langle -2, \dots, -2 \rangle$  on the subspace generated by  $e_i, f_j$  for  $i, j \neq 1$ . Let  $T : U \rightarrow U$  be the isometry of  $Q$  given by  $T(f_1) = -f_1$  and by  $T(e_i) = e_i$  for all  $i$ ,  $T(f_i) = f_i$  if  $i \neq 1$ .

Suppose first that  $r$  is even. Since  $C(1) = 1, C(-1) = p^r$  and  $\deg(C) \equiv 0 \pmod{4}$ , the polynomial satisfies condition (C1). Moreover,  $G_F = 0$ , since  $C$  is a power of an irreducible polynomial. We are assuming that  $\deg(C) \equiv 4 \pmod{8}$ , hence  $\deg(C) - 2 \equiv 2 \pmod{8}$ . Therefore Theorem 3.8 implies that  $\Lambda_{2, \deg(C) - 2}$  has an isometry  $T'$  with characteristic polynomial  $C$ . Note that  $(U, Q)$  contains a lattice isomorphic to  $\Lambda_{1, 2n-1}$  stable by  $T$ , hence  $\Lambda_{3,19}$  has an isometry with the required properties.

Assume now that  $r$  is odd. Set  $F = \mathbf{Q}[x]/(\Phi_{2p^k})$  and let  $\alpha \in F$  be the image of  $x$ . Let  $\sigma_F : F \rightarrow F$  be the involution induced by  $\alpha \mapsto \alpha^{-1}$  and let  $F_0$  be the fixed field of this involution. Let  $E_0$  be an extension of  $F_0$  of degree  $r$  that is linearly independent of  $F$  over  $F_0$ , and set  $E = E_0 \otimes_{F_0} F$ . Then  $E$  is a field, and the characteristic polynomial of  $\alpha \in E$  is equal to  $C = \Phi_{2p^k}^r$ . Let  $\sigma$  be the extension of  $\sigma_F$  to  $E$ .

If  $q$  is a prime number, set  $E_q = E \otimes_{\mathbf{Q}} \mathbf{Q}_q$  and  $(E_q)_0 = E_0 \otimes_{\mathbf{Q}} \mathbf{Q}_q$ . With the notation of [4], let  $\lambda_p \in (E_p)_0^\times$  be such that  $\partial(E_p, b_{\lambda_p}, \alpha) = -\partial(U, Q, T)$ ; if  $q \neq p$ , let  $\lambda_q \in (E_q)_0^\times$  be such that  $\partial(E_q, b_{\lambda_q}, \alpha) = 0$  and that  $(E_q, b_{\lambda_q})$  contains an even, unimodular  $\mathbf{Z}_q$ -lattice stabilized by  $\alpha$ ; this is possible by [4, Propositions 7.1 and 9.1] and the fact that  $\det(E_q, b_{\lambda_q}) = p$  and  $\deg(C) \equiv p - 1 \pmod{4}$ . Let  $\lambda_\infty \in \mathbf{R}^\times$  be such that the signature of  $(E \otimes_{\mathbf{Q}} \mathbf{R}, b_{\lambda_\infty})$  is equal to  $(2, \deg(C) - 2)$ .

For all prime numbers  $q$ , set  $U_q = (U, Q) \otimes_{\mathbf{Q}} \mathbf{Q}_q$  and  $W_q = (E_q, b_{\lambda_q})$ ; we have  $d(U_q) = p$  and  $d(W_q) = -p$ . Note that this implies that  $w(W_q \oplus U_q) = w(W_q) + w(U_q)$ . If  $q \neq 2, p$ , we have  $w(W_q) = w(U_q) = 0$ .

Set  $W_\infty = (E \otimes_{\mathbf{Q}} \mathbf{R}, b_{\lambda_\infty})$  and set  $w(W_\infty) = w_2(W_\infty)$  in  $\text{Br}_2(\mathbf{R})$ . We have  $w(W_\infty) = 0 \iff p \equiv 3 \pmod{4} \iff n \equiv 0 \pmod{2}$  and  $w(W_\infty) = 1 \iff p \equiv 1 \pmod{4} \iff n \equiv 1 \pmod{2}$ .

By construction,  $W_p \oplus U_p$  contains a unimodular lattice, hence  $w(W_p) = w(U_p)$ , and we have  $w(U_p) = 0 \iff p \equiv \pm 1 \pmod{8}$ .

Together with Lemma 11.1, this allows us to compute  $w(W_q)$  for all  $q$ , as follows. Assume first that  $p \equiv 3 \pmod{4}$ . Then  $w(W_\infty) = 0$  and

$$w(W_p) = 0 \iff p \equiv 7 \pmod{8}.$$

By Lemma 11.1, we have

$$w(W_2) = 0 \iff p \equiv 7 \pmod{8}.$$

Since  $w(W_q) = 0$  if  $q \neq 2, p$ , the sum of the invariants  $w(W_q)$  (for  $q$  a prime number) and  $w(W_\infty)$  is 0.

Suppose that  $p \equiv 1 \pmod{4}$ . Then  $w(W_\infty) = 1$  and

$$w(W_p) = 0 \iff p \equiv 1 \pmod{8}.$$

By Lemma 11.1, we have

$$w(W_2) = 0 \iff p \equiv 5 \pmod{8}.$$

Again, since  $w(W_q) = 0$  if  $q \neq 2, p$ , the sum of the invariants  $w(W_q)$  (for  $q$  a prime number) and  $w(W_\infty)$  is 0.

We have  $w_2(E_q, b_{\lambda_q}) = w(b_1) + \text{cor}_{E_q/\mathbf{Q}_q}(\lambda_q, d)$  for all prime numbers  $q$  (see [3, Proposition 5.4]). Let  $\mathcal{V}$  be the set of all places of  $\mathbf{Q}$ ; since  $b_1$  is a global form, the above argument shows that  $\sum_{v \in \mathcal{V}} \text{cor}_{E_v/\mathbf{Q}_v}(\lambda_p, d) = 0$ . By [3, Theorem 9.6], this implies that there exists  $\lambda \in E_0^\times$  such that  $(E, b_\lambda) \otimes_{\mathbf{Q}} \mathbf{Q}_q \simeq (E_p, b_{\lambda_p})$  for all  $q$ .

Let  $(V, B, t)$  be the orthogonal sum of  $(E, b_\lambda, \alpha)$  and  $(U, Q, T)$ . Set  $V_2 = (V, B) \otimes_{\mathbf{Q}} \mathbf{Q}_2$ . We have  $d(V) = -1$  and  $w(V_2) = w(W_2) + w(U_2)$ . Recall that  $\text{deg}(C) \equiv p - 1 \pmod{4}$ , hence  $\text{dim}(W_2) \equiv p - 1 \pmod{4}$ ; we have  $\text{dim}(U) = 22 - \text{deg}(C)$ , hence  $\text{dim}(U) \equiv p + 1 \pmod{4}$ . This implies that

If  $p \equiv 1 \pmod{4}$ , then

$$w(U_2) = 0 \iff p \equiv 1 \pmod{8} \quad \text{and} \quad w(W_2) = 0 \iff p \equiv 5 \pmod{8}.$$

If  $p \equiv 3 \pmod{4}$ , then

$$w(U_2) = 0 \iff p \equiv 3 \pmod{8} \quad \text{and} \quad w(W_2) = 0 \iff p \equiv 7 \pmod{8}.$$

In both cases, we have  $w(W_2) + w(U_2) = 1$ , hence  $w(V_2) = 1$ .

The quadratic form  $V$  has determinant  $-1$ , signature  $(3, 19)$ ,  $w(V_2) = 1$  and all the other Hasse–Witt invariants of  $V$  are trivial. This implies that  $V$  is isomorphic to  $\Lambda_{3,19} \otimes_{\mathbf{Z}} \mathbf{Q}$ . The characteristic polynomial of  $t$  is  $CC'$ . The quadratic form  $V$  contains

an even unimodular lattice stabilized by  $t$  everywhere locally; this is clear by construction at all prime numbers  $q \neq 2$ , and for  $q = 2$  it follows from Takada [22, Theorem 4.2]. The intersection of these lattices is an even unimodular lattice stabilized by the isometry  $t$ ; this lattice is isomorphic to  $\Lambda_{3,19}$ . By construction,  $t$  has the required properties.  $\square$

**Proposition 11.3** *Let  $C = \Phi_{2k}^r$  with  $k = 2$  and  $r = 9$  or  $k = 3$  and  $r = 5$ . Then the lattice  $(L, q) = \Lambda_{3,19}$  has an isometry  $t$  with characteristic polynomial  $CC'$ , where  $C'(x) = (x^2 - 1)(x - 1)^{20 - \deg(C)}$ , such that the sublattice  $\text{Ker}(C(t))$  has signature  $(2, 18)$  and that the restriction of  $t$  to  $\text{Ker}(C'(t))$  stabilizes one of the connected components of  $\{x \in \text{Ker}(C'(t)) \otimes_{\mathbf{Z}} \mathbf{R} \mid q(x, x) > 0\}$ .*

**Proof** Set  $E = \mathbf{Q}[x]/(\Phi_{2k})$  with  $k = 2$  or  $k = 4$ . We denote by  $x \mapsto \bar{x}$  the complex conjugation and let  $E_0$  be the fixed subfield of  $E$ : we have  $E_0 = \mathbf{Q}$  if  $r = 2$  and  $E_0 = \mathbf{Q}(\sqrt{2})$  if  $k = 4$ .

Suppose first that  $k = 2$ , and let  $X = (X, q_X, t_X)$  be defined by  $X = E, q_X(x, y) = \text{Tr}_{E/\mathbf{Q}}(\frac{1}{2}x\bar{y})$ ; let  $t_X$  be induced by multiplication by  $i = \zeta_4$ ; note that  $t_X$  is an isometry of  $q_X$  with characteristic polynomial  $\Phi_4$ . Let  $W_2 = (W_2, q_2, t_2)$  be the orthogonal sum of a copy of  $X$  with 9 copies of  $-X$ . We have  $\dim(W_2) = 18, d(W_2) = 1,$  and  $w(W_2) = 0$ . The signature of  $W_2$  is  $(2, 16)$ , and the characteristic polynomial of  $t_2$  is  $\Phi_4^9$ .

Let  $U_2$  be the  $\mathbf{Q}$ -vector space of basis  $e_1, e_2, f_1, f_2$  and  $q_2: U_2 \times U_2 \rightarrow \mathbf{Q}$  be the quadratic form such that  $q_2(e_1, e_1) = 1$  and  $q_2(f_1, f_1) = q_2(e_2, e_2) = q_2(f_2, f_2) = -1$ . Let  $t_2: U_2 \rightarrow U_2$  be the isometry given by  $t_2(f_2) = -f_2$  and  $t_2(e_i) = e_i$  for  $i = 1, 2, t_2(f_2) = f_2$ . We have  $\dim(U_2) = 4, d(U_2) = -1, w(U_2) = 1$  at 2 and  $\infty,$  and 0 elsewhere.

If  $k = 2$ , we set  $(V, q, t) = (W_2, q_2, t_2) \oplus (U_2, q_2, t_2)$ . The signature of  $V$  is  $(3, 19)$ , and  $d(V) = -1, w(V) = 1$  at 2 and  $\infty,$  and 0 elsewhere. The characteristic polynomial of  $t$  is  $\Phi_4^9(x)(x^2 - 1)(x - 1)^2$ .

Assume now that  $k = 4$ . Let  $X = (X, q_X, t_X)$  be defined by  $X = E, q_X(x, y) = \text{Tr}_{E/\mathbf{Q}}(\frac{\sqrt{2}}{4}x\bar{y})$ ; let  $t_X$  be induced by multiplication by  $\zeta_8$ ; note that  $t_X$  is an isometry of  $q_X$  with characteristic polynomial  $\Phi_8$ . Let  $Y = (Y, q_Y, t_Y)$  be defined by  $Y = E, q_Y(x, y) = \text{Tr}_{E/\mathbf{Q}}(\frac{1}{4}x\bar{y})$ ; let  $t_Y$  be induced by multiplication by  $\zeta_8$ . Let  $W_4 = (W_4, q_4, t_4)$  be the orthogonal sum of  $X$  with 4 copies of  $-Y$ . We have  $\dim(W_4) = 20, d(W_2) = 1,$  and  $w(W_2) = 1$  at 2 and at  $\infty,$  and 0 elsewhere. The signature of  $W_2$  is  $(2, 18)$ , and the characteristic polynomial of  $t_2$  is  $\Phi_8^5$ .

Let  $U_2$  be the  $\mathbf{Q}$ -vector space of basis  $e_1, f_1$  and  $q_2: U_2 \times U_2 \rightarrow \mathbf{Q}$  be the quadratic form such that  $q_2(e_1, e_1) = 1$  and  $q_2(f_1, f_1) = -1$ . Let  $t_2: U_2 \rightarrow U_2$  be the isometry given by  $t_2(f_2) = -f_2$  and  $t_2(e_1) = e_1$ . We have  $\dim(U_2) = 2, d(U_2) = -1, w(U_2) = 0$ .

If  $k = 2$ , we set  $(V, q, t) = (W_2, q_2, t_2) \oplus (U_2, q_2, t_2)$ . The signature of  $V$  is  $(3, 19)$ , and  $d(V) = -1, w(V) = 1$  at 2 and  $\infty,$  and 0 elsewhere. The characteristic polynomial of  $t$  is  $\Phi_4^9(x)(x^2 - 1)(x - 1)^2$ .

If  $k = 4$ , we set  $(V, q, t) = (W_4, q_4, t_4) \oplus (U_4, q_4, t_4)$ . The signature of  $V$  is  $(3, 19)$ , and  $d(V) = -1, w(V) = 1$  at 2 and  $\infty,$  and 0 elsewhere. The characteristic polynomial of  $t$  is  $\Phi_8^5(x)(x^2 - 1)$ .

In both cases,  $(V, q, t)$  contains an even unimodular lattice stabilized by  $t$  everywhere locally; at the prime 2, this follows from Takada [22, Theorem 4.2]. Let  $L$  be the intersection of these lattices;  $L$  is stabilized by  $t$ , and we have  $L \simeq \Lambda_{3,19}$ .  $\square$

**Proof of Proposition 10.1** If  $m = p^k$  where  $p$  is a prime number with  $p \neq 2$ , then the proposition follows from Proposition 10.2; if  $C(1) = C(-1)$ , then from Proposition 10.5. Suppose that  $C = \Phi_{2^r}^k$ . If  $r$  is even or if  $\deg(C) \leq 16$ , it is a consequence of Proposition 10.6. Suppose that  $r$  is odd and that  $\deg(C) = 18$  or  $20$ ; apply Proposition 11.2 if  $p \neq 2$ , and Proposition 11.3 if  $p = 2$ .  $\square$

Proposition 10.1 implies the following result.

**Corollary 11.4** *Let  $m$  be an integer such that  $m \geq 1$  and that  $\varphi(m) \leq 20$ . Then there exists an automorphism  $a: X \rightarrow X$  of a projective K3 surface  $X$  inducing multiplication by a primitive  $m$ -th root of unity on  $T_X$ .*

**Proof** For  $m = 1$ , we can take the identity, and there are many examples of automorphisms of projective K3 surfaces  $X$  inducing  $-\text{id}$  on  $T_X$  (see for instance [11, Corollary 15.2.12]). Suppose that  $m \geq 3$ , and let  $C = \Phi_m$ ; Proposition 10.1 implies that there exists an automorphism  $a: X \rightarrow X$  of a projective K3 surface such that the characteristic polynomial of  $a^*|_{T_X}$  is equal to  $C$ ; hence  $a^*|_{T_X}$  acts by multiplication by a primitive  $m$ -th root of unity.  $\square$

**Remark 11.5** Corollary 11.4 follows from results of Machida–Oguiso [15], Xiao [25], and Zhang [26] when  $m \neq 60$ ; more precisely, they prove the existence of an automorphism  $a: X \rightarrow X$  of finite order inducing multiplication by a primitive  $m$ -th root of unity on  $T_X$ . They also show that this is not the case for  $m = 60$ ; in the next section we give another proof of this result.

## 12 The primitive 60-th roots of unity

The following result was proved by Machida–Oguiso [15], Xiao [25], and Zhang [26].

**Proposition 12.1** *There does not exist any automorphism of finite order of a projective K3 surface inducing multiplication by a primitive 60-th root of unity on its transcendental lattice.*

The aim of this section is to give another proof of Proposition 12.1. Set  $C = \Phi_{60}$ .

**Proposition 12.2** *Let  $L$  be an even unimodular lattice of signature  $(3, 19)$ . The lattice  $L$  does not have any isometry  $t: L \rightarrow L$  having the following properties:*

- (i) *The characteristic polynomial of  $t$  is  $CC'$ , where  $C'$  is a product of cyclotomic polynomials.*
- (ii) *The signature of the sublattice  $L_C = \text{Ker}(C(t))$  of  $L$  is  $(2, 14)$ .*

The main ingredient of the proof of Proposition 12.2 is the following lemma.

**Lemma 12.3** *Set  $C' = \Phi_{12}$  and let  $M$  be an even unimodular lattice of signature  $(2, 18)$ . The lattice  $M$  does not have any isometry  $t : M \rightarrow M$  having the following properties:*

- (i) *The characteristic polynomial of  $t$  is  $CC'$ .*
- (ii) *The signature of the sublattice  $M_C = \text{Ker}(C(t))$  of  $M$  is  $(2, 14)$ .*

We give two proofs of this lemma; the first one is based on some results of [3], the second one is a direct proof.

**First proof of Lemma 12.3** Set  $F = CC'$  and note that  $F$  satisfies condition (C1). By Example 3.4, we have  $G_F = \mathbf{Z}/2\mathbf{Z}$ .

Set  $I = \{C, C'\}$ . Since  $G_F = \mathbf{Z}/2\mathbf{Z}$ , we have  $C_0(I) = C(I)$ . Let  $c : I \rightarrow \mathbf{Z}/2\mathbf{Z}$  be such that  $c(C) = 1$  and  $c(C') = 0$ , and let  $c' : I \rightarrow \mathbf{Z}/2\mathbf{Z}$  be such that  $c'(C) = 0$  and  $c'(C') = 1$ .

As in [3, Sections 9 and 12], we define a homomorphism  $\epsilon_F^{\text{finite}} : C(I) \rightarrow \mathbf{Z}/2\mathbf{Z}$ .

Let  $\tau$  be a signature map with characteristic polynomial  $F$  and maximum  $(2, 18)$  such that  $\tau(C) = (2, 14)$  and  $\tau(C') = (0, 4)$ , and let  $\epsilon_\tau^\infty : C(I) \rightarrow \mathbf{Z}/2\mathbf{Z}$  be the associated homomorphism (see [3, Sections 9 and 12]). We obtain a homomorphism  $\epsilon_\tau : G_F \rightarrow \mathbf{Z}/2\mathbf{Z}$  by setting  $\epsilon_\tau = \epsilon_F^{\text{finite}} + \epsilon_\tau^\infty$ . By [3, Theorem 12.1], there exists an even unimodular lattice  $M$  having an isometry  $t : M \rightarrow M$  with properties (i) and (ii) if and only if  $\epsilon_\tau = 0$ .

With the notation of [3, Section 9], we have  $a_\tau^\infty(C) = 1$  and  $a_\tau^\infty(C') = 0$ . This implies that  $\epsilon_\tau^\infty(c) = 1$  and  $\epsilon_\tau^\infty(c') = 0$ .

Similarly, let  $\tau'$  be a signature map with characteristic polynomial  $F$  and maximum  $(2, 18)$  such that  $\tau'(C) = (0, 16)$  and  $\tau'(C') = (2, 2)$ , and let  $\epsilon_{\tau'}^\infty : C(I) \rightarrow \mathbf{Z}/2\mathbf{Z}$  be the associated homomorphism. We have  $a_{\tau'}^\infty(C) = 0$  and  $a_{\tau'}^\infty(C') = 1$ , hence  $\epsilon_{\tau'}^\infty(c) = 0$  and  $\epsilon_{\tau'}^\infty(c') = 1$ . We obtain a homomorphism  $\epsilon_{\tau'} : G_F \rightarrow \mathbf{Z}/2\mathbf{Z}$  by setting  $\epsilon_{\tau'} = \epsilon_F^{\text{finite}} + \epsilon_{\tau'}^\infty$ .

Both  $C$  and  $C'$  satisfy condition (C 1), and since they are irreducible, we have  $G_C = G_{C'} = 0$ . Therefore by Theorem 3.5 there exists an even unimodular lattice  $N$  of signature  $(0, 16)$  having an isometry with characteristic polynomial  $C$ , and an even unimodular lattice  $N'$  of signature  $(2, 2)$  having an isometry with characteristic polynomial  $C'$ . The lattice  $N \oplus N'$  is even unimodular of signature  $(2, 18)$ , and has an isometry of characteristic polynomial  $F$  and of signature map  $\tau'$ . Applying [3, Theorem 12.1], this implies that  $\epsilon_{\tau'} = 0$ . Since  $\epsilon_{\tau'} = \epsilon_F^{\text{finite}} + \epsilon_{\tau'}^\infty$ , we have  $\epsilon_F^{\text{finite}}(c) = 0$  and  $\epsilon_F^{\text{finite}}(c') = 1$ .

On the other hand, we have  $\epsilon_\tau = \epsilon_F^{\text{finite}} + \epsilon_\tau^\infty$ , and this implies that  $\epsilon_\tau \neq 0$ ; therefore there does not exist any even unimodular lattice  $M$  having an isometry  $t : M \rightarrow M$  with properties (i) and (ii). □

The second proof of Lemma 12.3 uses the notion of Hasse–Witt invariant of a quadratic form.

**Notation** Let  $K$  be a field of characteristic  $\neq 2$ , let  $V$  be a finite-dimensional  $K$ -vector space, and let  $q : V \times V \rightarrow K$  be a non-degenerate quadratic form. The Hasse–Witt invariant of  $q$  is denoted by  $w_2(q)$ ; it is an element of  $\text{Br}_2(K)$ .

If  $K$  is a  $p$ -adic field or  $\mathbf{R}$ , then  $\text{Br}_2(K)$  is a group of order two, that we identify with  $\{0, 1\}$ ; see [21, Chapitre IV] for the properties of Hasse–Witt invariants of quadratic forms that are needed here.

**Second proof of Lemma 12.3** Set  $F = CC'$ . Suppose that the even unimodular lattice  $(M, q)$  of signature  $(2, 18)$  has an isometry  $t : M \rightarrow M$  with characteristic polynomial  $F$ , and let us show that (ii) does not hold.

Set  $M_1 = \text{Ker}(C(t))$  and let  $q_1$  be the restriction of  $q$  to  $M_1 \times M_1$ ; similarly, set  $M_2 = \text{Ker}(C'(t))$  and let  $q_2$  be the restriction of  $q$  to  $M_2 \times M_2$ . Set  $V = M \otimes_{\mathbf{Z}} \mathbf{Q}$ ,  $V_1 = M_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ , and  $V_2 = M_2 \otimes_{\mathbf{Z}} \mathbf{Q}$ . Since  $C$  and  $C'$  are distinct irreducible polynomials, the quadratic space  $(V, q)$  is the orthogonal sum of  $(V_1, q_1)$  and  $(V_2, q_2)$ .

The resultant of  $C$  and  $C'$  is  $5^4$ . This implies that if  $p$  is a prime number with  $p \neq 5$ , then  $(M, q) \otimes_{\mathbf{Z}} \mathbf{Z}_p$  is the orthogonal sum of  $(M_1, q_1) \otimes_{\mathbf{Z}} \mathbf{Z}_p$  and  $(M_2, q_2) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ ; therefore the  $\mathbf{Z}_p$ -lattices  $(M_1, q_1) \otimes_{\mathbf{Z}} \mathbf{Z}_p$  and  $(M_2, q_2) \otimes_{\mathbf{Z}} \mathbf{Z}_p$  are unimodular.

Let  $p$  be a prime number  $p \neq 2, 5$ . Since  $(V_1, q_1) \otimes_{\mathbf{Q}} \mathbf{Q}_p$  and  $(V_2, q_2) \otimes_{\mathbf{Q}} \mathbf{Q}_p$  contain unimodular  $\mathbf{Z}_p$ -lattices, we have  $w_2(q_1) = w_2(q_2) = 0$  in  $\text{Br}_2(\mathbf{Q}_p)$ .

Let  $U$  be the hyperbolic plane; if  $n$  is an integer with  $n \geq 1$ , we denote by  $U^n$  the orthogonal sum of  $n$  copies of  $U$ . Since  $(M_1, q_1) \otimes_{\mathbf{Z}} \mathbf{Z}_2$  and  $(M_2, q_2) \otimes_{\mathbf{Z}} \mathbf{Z}_2$  are unimodular, we have  $(M_1, q_1) \otimes_{\mathbf{Z}} \mathbf{Z}_2 \simeq U^8 \otimes_{\mathbf{Z}} \mathbf{Z}_2$  and  $(M_2, q_2) \otimes_{\mathbf{Z}} \mathbf{Z}_2 \simeq U^2 \otimes_{\mathbf{Z}} \mathbf{Z}_2$  (see for instance [8, Proposition 5.2]). This implies that  $w_2(q_1) = 0$  and  $w_2(q_2) = 1$  in  $\text{Br}_2(\mathbf{Q}_2)$ .

Let  $K$  be the cyclotomic field of the 12-th roots of unity and let  $K_0$  be its maximal real subfield. The prime 5 is inert in the extension  $K_0/\mathbf{Q}$ , and splits in  $K/K_0$ . This implies that there exists a degree 2 polynomial  $f \in \mathbf{Z}_5[x]$  such that  $C' = ff^*$  in  $\mathbf{Z}_5[x]$ , where  $f^*(x) = x^2 f(x^{-1})$ , and such that  $f \neq f^*$ . Let us denote by  $t_2$  the restriction of  $t$  to  $V_2$ , and note that  $\text{Ker}(f(t_2))$  is an isotropic subspace of dimension 2 of  $V_2 \otimes_{\mathbf{Q}} \mathbf{Q}_5$ ; this implies that  $(V_2, q_2) \otimes_{\mathbf{Q}} \mathbf{Q}_5 \simeq U^2 \otimes_{\mathbf{Q}} \mathbf{Q}_5$ , and therefore  $w_2(q_2) = 0$  in  $\text{Br}_2(\mathbf{Q}_5)$ .

Suppose that (ii) holds. Then the signature of  $(V_1, q_1)$  is  $(2, 14)$ , and the signature of  $(V_2, q_2)$  is  $(0, 4)$ ; hence  $w_2(q_2) = 0$  in  $\text{Br}_2(\mathbf{R})$ . This leads to a contradiction, since  $w_2(q_2) = 1$  in  $\text{Br}_2(\mathbf{Q}_2)$ , and  $w_2(q_2) = 0$  in  $\text{Br}_2(\mathbf{Q}_p)$  for all prime numbers  $p$  with  $p \neq 2$ . □

**Proof of Proposition 12.2** Let  $C'$  be a product of cyclotomic polynomials and let  $t : L \rightarrow L$  be an isometry with characteristic polynomial  $CC'$ , and set  $F = CC'$ . Suppose first that  $C'$  is not divisible by  $\Phi_{12}$ . Then  $C$  and  $C'$  are relatively prime over  $\mathbf{Z}$ . Indeed,  $\deg(C') = 6$ , and if  $\Phi$  is a cyclotomic polynomial of degree  $\leq 6$  such that  $\Phi$  is not relatively prime to  $C$  over  $\mathbf{Z}$ , then  $\Phi = \Phi_{12}$ ; this follows from the values of the resultants of cyclotomic polynomials, see for instance [1]. Since  $C$  and  $C'$  are relatively prime over  $\mathbf{Z}$ , the lattice  $L$  is the orthogonal sum of the even unimodular lattices  $L_C$  and  $\text{Ker}(C'(t))$ . If (ii) holds, then the signature of  $L_C$  is  $(2, 14)$ ; this contradicts the fact that  $L_C$  is an even unimodular lattice.

Assume now that  $\Phi_{12}$  divides  $C'$ . The polynomial  $F$  satisfies condition (C1); this implies that  $C' = \Phi_{12}C''$  such that the irreducible factors of  $C''$  are in  $\{x - 1, x + 1\}$ . Therefore  $C''$  is relatively prime over  $\mathbf{Z}$  to  $C\Phi_{12}$ . Set  $M = \text{Ker}(C\Phi_{12}(t))$  and  $M' = \text{Ker}(C''(t))$ . The lattice  $L$  is the orthogonal sum of  $M$  and  $M'$ , hence both these lattices are even and unimodular. This implies that the signature of  $M'$  is  $(1, 1)$ , and hence the signature of  $M$  is  $(2, 14)$ . By Lemma 12.3, this is impossible. □



**Proof of Proposition 12.1** Let  $a : X \rightarrow X$  be an automorphism of a projective  $K3$  surface such that  $a^*$  induces multiplication by a primitive 60-th root of unity on  $T_X$ . This implies that the characteristic polynomial of  $a^*|_{T_X}$  is equal to  $\Phi_{60}$ . Assume that  $a$  is of finite order. Then the characteristic polynomial of  $a^*|_{S_X}$  is a product of cyclotomic polynomials. Since  $T_X = \text{Ker}(\Phi_{60}(a^*))$ , the signature of the lattice  $\text{Ker}(\Phi_{60}(a^*))$  is  $(2, 14)$ . Proposition 12.2 implies that this is impossible, hence no such automorphism exists. □

### 13 Two cyclic groups

Recall from the introduction that if  $X$  is a projective  $K3$  surface, we have the exact sequences

$$1 \rightarrow \text{Aut}_s(X) \rightarrow \text{Aut}(X) \rightarrow M_X \rightarrow 1$$

and

$$1 \rightarrow N_X \rightarrow \text{Aut}(X) \rightarrow \text{O}(S_X),$$

where  $M_X$  and  $N_X$  are finite cyclic groups, of order  $m_X$ , respectively  $n_X$ . The group  $N_X$  can be identified to a subgroup of  $M_X$ , hence  $n_X$  divides  $m_X$ .

The question of determining the possible values of  $m_X$  and  $n_X$  was raised by Huybrechts in [11, p. 336]; the characterization of the pairs  $(m_X, n_X)$  is also of interest.

The values of  $n_X$  were studied much earlier, by Vorontsov [23], Kondō [12], and Oguiso–Zhang [20]; assuming that  $\text{rank}(T_X) = \varphi(n_X)$ , they give a complete list of these values (see Corollary 13.6 below).

We start with the integers  $m_X$ . The following is Corollary 1.2 from the introduction.

**Corollary 13.1** *Let  $m \geq 2$  be an even integer such that  $\varphi(m) \leq 20$ . Then there exists a projective  $K3$  surface  $X$  with  $m_X = m$ .*

**Proof** There exist projective  $K3$  surfaces  $X$  with  $m_X = 2$ , see for instance [11, Corollary 15.2.12]. Suppose that  $m \geq 4$ . By Proposition 10.1 there exist a projective  $K3$  surface  $X$  and an automorphism  $a : X \rightarrow X$  such that the characteristic polynomial of the restriction of  $a^*$  to  $T_X$  is  $\Phi_m$ . Since  $m$  is even, we have  $m_X = m$ . □

**Remark 13.2** In [15], Machida and Oguiso consider a related problem; they are interested in the images of the *finite subgroups* of  $\text{Aut}(X)$  in  $M_X$ . Their results imply that there exist projective  $K3$  surfaces  $X$  with  $m_X = 28, 30, 32, 34, 38, 40, 42, 44, 48, 54$  and  $66$  (see [15, Proposition 4]).

The possible values of  $n_X$  can be deduced from Proposition 5.1. As we will see, it is enough to consider  $K3$  surfaces  $X$  such that  $\text{rank}(T_X) = \varphi(n_X)$  or  $\text{rank}(T_X) = 2\varphi(n_X)$ .

**Proposition 13.3** *Let  $m$  be an integer with  $m \geq 1$  and  $\varphi(m) \leq 20$ . The following are equivalent:*

- (i) *There exists a projective K3 surface  $X$  such that  $n_X = m$ .*
- (ii) *There exists a projective K3 surface  $X$  such that  $n_X = m$  and  $\text{rank}(T_X) = \varphi(m)$  or  $\text{rank}(T_X) = 2\varphi(m)$ .*

This will be proved at the end of the section.

We start with the case where  $C$  is a cyclotomic polynomial, i.e. when the rank of  $T_X$  is  $\varphi(n_X)$ .

**Corollary 13.4** *Let  $m$  be an integer such that  $m \geq 3$  and that  $\varphi(m) \leq 20$ ; set  $C = \Phi_m$ . There exists a projective K3 surface  $X$  such that  $n_X = m$  and the rank of  $T_X$  is  $\varphi(n_X)$  if and only if the following conditions hold:*

- (i)  $C(-1) = 1$ .
- (ii) *If  $C(1) = 1$ , then  $m \equiv 0 \pmod{2}$  and  $\deg(C) \equiv 4 \pmod{8}$ .*

*The K3 surface is unique up to isomorphism. Moreover, the lattice  $T_X$  is unimodular if and only if  $C(1) = 1$ , and  $p$ -elementary with  $p = C(1)$  if  $C(1) > 1$ .*

**Proof** Suppose that conditions (i) and (ii) hold. Then by Proposition 5.1, there exists an automorphism  $a: X \rightarrow X$  of a projective K3 surface such that the restriction of  $a^*$  is the identity and that the characteristic polynomial of  $a^*|T_X$  is equal to  $C$ ; this implies that  $m$  divides  $n_X$ . Since  $C = \Phi_m$ , the rank of  $T_X$  is equal to  $\varphi(m)$ ; hence  $\varphi(m) = \varphi(n_X)$ . Suppose that  $C(1) > 1$ ; then  $m = p^r$ , where  $p$  is an odd prime number (cf. Lemma 4.1). Then either  $n_X = m$  or  $n_X = 2m$ ; but  $\Phi_{2p^r}$  does not satisfy (i), hence  $n_X \neq 2p^r$ , and this implies that  $n_X = m$ . Assume now that  $C(1) = 1$ . By (ii), we have  $m \equiv 0 \pmod{2}$ ; since  $m$  divides  $n_X$  and  $\varphi(m) = \varphi(n_X)$ , this implies that  $n_X = m$ .

Conversely, suppose that there exists a projective K3 surface  $X$  such that  $n_X = m$  and  $\text{rank}(T_X) = \varphi(m)$ , and let  $a$  be a generator of the cyclic group  $N_X$ . Then the restriction of  $a^*$  to  $S_X$  is the identity and the characteristic polynomial of  $a^*|T_X$  is equal to  $C$ . Therefore Proposition 5.1 (i) implies that  $C(-1)$  is a square; since  $C$  is a cyclotomic polynomial, we have  $C(-1) = 1$ , hence (i) holds. Proposition 5.1 (ii) implies that if  $C(1) = 1$ , then  $\deg(C) \equiv 4 \pmod{8}$ . Moreover, the hypothesis  $C(1) = 1$  implies that  $T_X$  is unimodular; this implies that  $n_X \equiv 0 \pmod{2}$ , hence  $m \equiv 0 \pmod{2}$ , and therefore (ii) holds.

The uniqueness of the K3-surface follows from Proposition 5.1. If  $C(1) = 1$  then  $T_X$  is unimodular (cf. Proposition 5.1). Conversely, suppose that  $T_X$  is unimodular; then  $C$  satisfies condition (C1). Since  $C$  is a cyclotomic polynomial, this implies that  $C(1) = 1$ . If  $C(1) > 1$ , then by Proposition 5.2 the lattice  $T_X$  is  $p$ -elementary with  $p = C(1)$ . Conversely, if  $T_X$  is  $p$ -elementary, then by Proposition 5.1 we have  $C(1) > 1$ .  $\square$

Using Corollary 13.4, we recover the lists of Vorontsov [23] and Kondō [12], as follows. Set

$$A = \{12, 28, 36, 42, 44, 66\} \quad \text{and} \quad B = \{3, 9, 27, 5, 25, 7, 11, 13, 17, 19\}.$$

**Lemma 13.5** *Let  $m$  be an integer such that  $m \geq 3$  and that  $\varphi(m) \leq 20$ . Then we have*

$$\begin{aligned}
 m \in A &\iff \Phi_m(1) = \Phi_m(-1) = 1, \text{ } m \text{ is even, and } \deg(C) \equiv 4 \pmod{8}, \\
 m \in B &\iff \Phi_m(-1) = 1 \text{ and } \Phi_m(1) > 1.
 \end{aligned}$$

**Proof** This follows from Lemma 4.1. □

Combining Corollary 13.4 and Lemma 13.5, we obtain

**Corollary 13.6** (Kondō, Vorontsov) *Let  $m \in A \cup B$ . Then there exists a projective K3 surface  $X$  such that  $n_X = m$  and that the rank of  $T_X(X)$  is equal to  $\varphi(n_X)$ . Moreover,  $X$  is unique up to isomorphism, and*

$$\begin{aligned}
 n_X \in A &\iff \text{the lattice } T_X \text{ is unimodular,} \\
 n_X \in B &\iff \text{the lattice } T_X \text{ is } p\text{-elementary with } p = \Phi_{n_X}(1).
 \end{aligned}$$

**Proof** See [12, 23]; this also follows from Corollary 13.4 and Lemma 13.5. □

We now consider the case where  $\text{rank}(T_X) = 2\varphi(n_X)$ .

**Corollary 13.7** *Let  $m$  be an integer such that  $m \geq 3$ , that  $m \equiv 0 \pmod{2}$ , and that  $\varphi(m) \leq 10$ . Suppose that if  $\Phi_m(1) = 1$ , then  $\varphi(m) \equiv 2 \pmod{4}$ . Then there exist infinitely many projective K3 surfaces  $X$  such that  $n_X = m$ , and that  $\text{rank}(T_X) = 2\varphi(m)$ .*

**Proof** Set  $C = \Phi_m^2$  and note that  $C$  satisfies conditions (i) and (ii) of Proposition 5.1. Therefore there exist infinitely many projective K3 surfaces  $X$  having automorphisms  $a: X \rightarrow X$  such that  $a^*|S_X$  is the identity, and that the characteristic polynomial of  $a^*|T_X$  is  $C$ . If  $m$  is a power of 2, then  $m = n_X$ . Otherwise, we have  $m = 2m'$  where  $m' = 3, 7, 9$  or  $11$ ; therefore  $n_X = m$  or  $n_X = 2m = 4m'$ .

If  $Y$  is a K3 surface with  $n_Y = 4m'$  and  $\text{rank}(T_Y) = 2\varphi(m) = \varphi(4m')$ , then  $Y$  has an automorphism  $b: Y \rightarrow Y$  such that  $b^*|T_Y$  is the identity and the characteristic polynomial of  $b^*|T_Y$  is  $C' = \Phi_{4m'}$ ; by Corollary 13.4, the surface  $Y$  is unique up to isomorphism.

Therefore there exist infinitely many non-isomorphic projective K3 surfaces  $X$  such that  $n_X = m$ . □

Set  $C = \{4, 6, 8, 14, 16, 18, 22\}$ .

**Lemma 13.8** *Let  $m$  be an integer such that  $m \geq 3$  and that  $\varphi(m) \leq 10$ . We have*

$$m \in C \iff m \text{ is even, and if } \Phi_m(1) = 1, \text{ then } \varphi(m) \equiv 2 \pmod{4}.$$

**Proof** This follows from Lemma 4.1. □

Corollary 13.7 and Lemma 13.8 imply the following

**Corollary 13.9** *If  $m \in C$ , then there exist infinitely many projective K3 surfaces  $X$  such that  $n_X = m$ , and that  $\text{rank}(T_X) = 2\varphi(n_X)$ .*

Finally, we show that Corollaries 13.4 and 13.7 characterize the possible values of  $n_X$ . We start with a lemma.

**Lemma 13.10** *Let  $m, r$  be integers with  $m \geq 3$  and  $r \geq 1$ , and let  $C = \Phi_m^r$ . Conditions (i) and (ii) of Proposition 5.1 hold for  $C$  if and only if*

- (i)  $r \equiv 1 \pmod{2}$ ,  $\Phi_m(-1) = 1$  and if  $\Phi_m(1) = 1$ , then  $\varphi(m) \equiv 4 \pmod{8}$ .
- (ii)  $r \equiv 2 \pmod{4}$ , and if  $\Phi_m(1) = 1$ , then  $\varphi(m) \equiv 2 \pmod{4}$ .
- (iii)  $r \equiv 0 \pmod{4}$ , and  $\Phi(1) > 1$ .

**Proof** This is a straightforward verification, using Lemma 4.1. □

**Corollary 13.11** *Let  $m$  be an integer such that  $m \geq 3$  and that  $\varphi(m) \leq 20$ . There exists a projective  $K3$  surface  $X$  with  $n_X = m$  and  $\text{rank}(T_X) = r\varphi(n_X)$  if and only if*

- (i)  $r = 1$  and  $m \in A \cup B$ .
- (ii)  $r = 2$  and  $m \in C \cup \{3, 5, 7, 11\}$ .
- (iii)  $r = 3$  and  $m \in \{3, 5, 7, 9, 12\}$ .
- (iv)  $r = 4$  and  $m \in \{3, 4, 5, 8\}$ .
- (v)  $r = 5$  and  $m \in \{3, 5\}$ .
- (vi)  $r = 6$  or  $10$  and  $m \in \{3, 4, 6\}$ .
- (vii)  $r = 8$  and  $m \in \{3, 4\}$ .
- (viii)  $r = 7$  or  $9$  and  $m = 3$ .

**Proof** This follows from Proposition 5.1 and Lemma 13.10. □

**Corollary 13.12** *Let  $m$  be an integer such that  $m \geq 3$  and that  $\varphi(m) \leq 20$ . There exists a projective  $K3$  surface  $X$  with  $n_X = m \iff m \in A \cup B \cup C$ .*

**Proof** This is an immediate consequence of Corollary 13.11. □

**Proof of Proposition 13.3** It is clear that (ii)  $\implies$  (i). Suppose (i). Then by Corollary 13.12, we have  $n_X \in A \cup B \cup C$ , and (i) follows from Corollaries 13.6 and 13.9. □

**Notation** If  $X$  is a  $K3$  surface, set  $r_X = \frac{\text{rank}(T_X)}{\varphi(n_X)}$  and  $t_X = \frac{\text{rank}(T_X)}{\varphi(m_X)}$ .

Corollary 13.11 characterizes the possible pairs  $(n_X, r_X)$ . This suggests the following question.

**Question 13.13** What are the possibilities for  $(n_X, m_X, r_X, t_X)$ ?

Huybrechts asked for explicit examples of  $K3$  surfaces with  $m_X > n_X$  (see [11, Remark 15.1.13]). The following construction is due to Brandhorst and Elkies [7].

**Example 13.14** Let  $X$  be the  $K3$  surface constructed by Brandhorst and Elkies in [7]. This surface has an automorphism  $a : X \rightarrow X$  such that the characteristic polynomial of  $a^*|T_X$  is  $\Phi_{14}$  (see [7, Section 3]), hence  $m_X = 14$ . Since  $14 \notin A \cup B$ , we have  $n_X \neq 14$ , therefore  $n_X < m_X$ .

This implies that  $n_X = 7, 2$  or  $1$ . The results of Vorontsov [23] and Oguiso–Zhang [20] imply that up to isomorphism, there exists a unique projective  $K3$  surface  $Y$  such that  $\text{rank}(T_Y) = 6$  and  $n_Y = 7$  (note that this also follows from Proposition 13.4); the discriminant of  $S_Y$  is equal to  $7$  (see [20, Lemma 1.3]). For the  $K3$  surface  $X$  constructed by Brandhorst and Elkies, the discriminant of  $S_X$  is  $7 \cdot 13^2$ , hence  $X$  and  $Y$  are not isomorphic; therefore  $n_X \neq 7$ . One can show that  $n_X = 1$ .

**Funding** Open access funding provided by EPFL Lausanne

**Data availability** No datasets were generated or analysed during the current study.

## Declarations

**Conflict of interest** The author declares no conflict of interest.

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