RELATIVE PROJECTIVITY AND RELATIVE ENDOTRIVIAL MODULES

CAROLINE LASSUEUR

Abstract. In this paper we use projectivity relative to kG-modules to define groups of relatively endotrivial modules, which are obtained by replacing the notion of projectivity with that of relative projectivity in the definition of ordinary endotrivial modules. To achieve this goal we develop the theory of projectivity relative to modules with respect to standard group operations such as induction, restriction and inflation. As a particular example, we show how these groups can generalise the Dade group. Finally, for finite groups having a cyclic Sylow p-subgroup, we determine all the different subcategories of relatively projective modules and, using the structure of the group $T(G)$ of endotrivial modules described in [MT07], the structure of all the different groups of relatively endotrivial modules.

1. Introduction

Let G be a finite group and k be an algebraically closed field of characteristic $p > 0$. The main purpose of this paper is to build a relative notion of endotrivial modules using the notion of projectivity relative to modules introduced by T. Okuyama in [Oku91] and then developed by J. Carlson and his coauthors in [Car96], [CP96] and [CPW98]. If V is a given finitely generated kG-module, in [CPW98] it is shown that one can define an analogue of the stable module category by setting that all the V-projective modules are isomorphic to zero. This construction results in the so-called *V-stable category*, denoted $\mathsf{stmod}_V(kG)$, which is triangulated. In particular, the notion of projectivity relative to a module encompasses the well-known notion of projectivity relative to a subgroup or to a family of subgroups.

Classically a finitely generated kG-module is endotrivial if its k-endomorphism ring is isomorphic to a trivial module in the stable module category $\mathsf{stmod}(kG)$. Therefore, for a quick description, a finitely generated kG-module shall be termed endotrivial relatively to the module V, required to be absolutely p-divisible, if it is invertible in the relative stable module category $\mathsf{stmod}_V(kG)$. This construction gives rise to a group structure $T_V(G)$ on the collection of relatively V-endotrivial modules endowed with the ordinary tensor product $\otimes_k$ over k. In particular, for $V = kG$ the group $T_V(G)$ is the group $T(G)$ of ordinary endotrivial modules.

One main reason of interest for relative endotrivial modules is that they provide a way to define a group structure on collections of representations of an arbitrary finite group G. This gives a generalization for the Dade Group $D(P)$ of a finite p-group P. Endo-permutation modules are defined only for p-groups, but not for finite groups in general. One way to obtain a similar notion for arbitrary groups is to consider endo-p-permutation modules as described in [Urf06]. However, the main drawback of this approach resides in the fact that there is not a unique indecomposable representative, up to isomorphism, in the resulting group structure. Now, whereas in the theory of endo-permutation modules, ordinary endotrivial module are seen as special cases, we shall take the problem the other way around and show that any endo-permutation module can be seen as a special case of a relative endotrivial module. Indeed, a good choice of a module $\widetilde{V}$ leads to a natural embedding of $D(P)$ in the group $T_{\widetilde{V}}(P)$, in which the equivalence classes do have a unique indecomposable representative, up to isomorphism.

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In section 2, we start by recalling general results concerned with projectivity relative to a module and establish a few facts describing the behaviour of this kind of relative projectivity with respect to the standard group operations such as induction, restriction and inflation. We also study relative versions of the ordinary syzygy modules as they are important building pieces for the theory of relative endotrivial modules.

Next, we shall consider groups having a normal Sylow 2-subgroup isomorphic to the Klein group, in which case we show that whatever the choice of the module V, the groups $T_V(G)$ are all isomorphic to the group $T(G)$ of ordinary endotrivial modules. Nonetheless it is worth keeping this case in mind because it is an easy-to-handle source of examples and counter-examples for the general theory.

Finally we treat groups having a cyclic Sylow p-subgroup $P \cong C_{p^n}$, in which case we give a description by generators and relations of all the different groups of relative endotrivial modules. This description is obtained through an induction argument from the structure of the group of ordinary endotrivial modules which was recently determined in [MT07]. In order to achieve, first we need to determine all the different subcategories of relative projective modules. It turns out that in this case projectivity relative to modules is reduced to the projectivity relative to the p-subgroups of P.

We point out that throughout this paper we use notation and develop a general theory concerned with projectivity relative to kG-modules in all generality, whereas in the Klein case, cyclic case and the aforementioned generalisation of the Dade group, it would be enough to work with projectivity relative to subgroups. Nonetheless, the groups $T_V(G)$ for a general module V are interesting for themselves as they can naturally be defined in the general theory of projectivity relative to kG-modules. Moreover, the multiplication being induced by the tensor product, the knowledge of the groups $T_V(G)$ can give some information on the multiplicative structure of the Green ring of the group G, about which very little is known. Also note that the Klein case in which there is, up to isomorphism, only one group of relatively endotrivial modules is an oddity, however in this case there are infinitely many different subcategories of V-projective modules which do not correspond to projectivity relative to a subgroup. Furthermore, the Dade group involves projectivity relative to a family of subgroups which is not reduced to a single element, however, it can be considered as projectivity relative to a single module which is much less cumbersome to work with.

2. Relative projectivity with respect to modules

Unless otherwise mentioned, throughout this text k shall denote an algebraically closed field of prime characteristic p, G a finite group whose order is divisible by p, all the modules shall be finitely generated, $\mathsf{mod}(kG)$ shall denote the category of finitely generated left kG-modules and $\mathsf{stmod}(kG)$ the corresponding stable category. Moreover, $\otimes$ shall denote the ordinary tensor product over k, $M^* = Hom_k(M, k)$ and $\Omega(M)$ the k-dual and the kernel of a projective cover of the kG-module M, respectively.

To begin with, the main purpose of this section is to recall and develop techniques and results concerned with projectivity relative to a module. This is a generalisation of the more classic projectivity relative to a subgroup widely used in the theory of vertices and sources. Its definition is just a special case of the relative homological algebra defined for a projective class of epimorphisms or a pair of adjoint exact functors in [HS71, Chap. 10]. Projectivity relative to a kG-module was first introduced in an unpublished manuscript by T. Okuyama [Oku91], then further developed and used by J. Carlson and several coauthors in [Car96], [CP96] and [CPW98].

Definition 2.0.1 ([Car96]). (a) A module $M \in \mathsf{mod}(kG)$ is termed *projective relative to* V or V*-projective* if there exists a kG-module N such that M is isomorphic to a direct summand of $V \otimes_k N$.

(b) A short exact sequence $E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ in $\mathsf{mod}(kG)$ is termed V*-split* if the tensored sequence $V \otimes E : 0 \longrightarrow V \otimes A \xrightarrow{V\otimes\alpha} V \otimes B \xrightarrow{V\otimes\beta} V \otimes C \longrightarrow 0$ splits.

The notion of a V-injective kG-module can be defined dually. However, the symmetry of the group algebra implies that the class of V-injective modules is equal to the class of V-projective modules. To settle notation, we shall denote by $Proj(V)$ the subcategory of all V-projective modules in $\mathsf{mod}(kG)$ and by $IProj(V)$ the collection of V-projective modules that are indecomposable. A module U is said to be a generator for $Proj(V)$ if and only if $Proj(U) = Proj(V)$. Moreover, in computations we shall often denote by $(V - proj)$ a module in $Proj(V)$ which does not need to be specified, and simply $(proj)$ for a projective module. Also note that $Proj(V)$ is a subcategory of $\mathsf{mod}(kG)$ and the properties summed up below show that it is functorially finite.

Although the notion of projectivity relative to a kG-module was first introduced by Okuyama, as he points out, the real point of origin of this notion is the following crucial observation due to Auslander and Carlson in [AC86]: if V is a kG-module, then the trace map $Tr : V^* \otimes V \longrightarrow k : f \otimes v \mapsto f(v)$ is always V-split. Moreover, it splits when $\dim_k V$ is coprime to p.

The following omnibus proposition sums up elementary properties of relative projectivity, that we shall use extensively in the sequel of this text.

Proposition 2.0.2 (Omnibus properties)**.** *Let M, N, U, V be kG-modules.*

(a) *Any direct summand of a V-projective module is V-projective.*
(b) *If $U \in Proj(V)$, then $Proj(U) \subseteq Proj(V)$.*
(c) *If $p \nmid \dim_k(V)$ then $Proj(V) = \mathsf{mod}(kG)$.*
(d) *$Proj(V) = Proj(V^*) = Proj(V^* \otimes V) = Proj(\Omega^n(V))$ for all $n \in \mathbb{Z}$.*
(e) *$Proj(U \oplus V) = Proj(U) \oplus Proj(V)$.*
(f) *$Proj(U) \cap Proj(V) = Proj(U \otimes V) \supseteq Proj(U) \otimes Proj(V)$.*
(g) *$Proj(\bigoplus_{j=1}^n V) = Proj(V) = Proj(\bigotimes_{j=1}^m V)$ for all $m, n \in \mathbb{N} - \{0\}$.*
(h) *$M \oplus N$ is V-projective if and only if both M and N are V-projective.*
(i) *$M \in Proj(V)$ if and only if $\operatorname{End}_k(M) \cong M^* \otimes M \in Proj(V)$.*
(j) *$M \in Proj(V)$ if and only if $M \mid V^* \otimes V \otimes M$.*
(k) *Let $g \in \tilde{G} \trianglerighteq G$, then ${}^gProj(V) = Proj({}^gV)$.*
(l) *$Proj(kG) \subseteq Proj(V)$ for all kG-modules V. Moreover, $Proj(kG) = Proj(P)$ for any non-zero projective kG-module P.*

Apart from property (k), all these properties appear either in [Car96, Sec. 8] or [CP96, Sec. 3.3], to which we refer for proofs. Moreover we give a proof for statement (f) which was mistyped (and not proven in [CP96, Lem. 3.3(iii)]) as $Proj(U)\otimes Proj(V) = Proj(U\otimes V)$ instead of $Proj(U)\cap Proj(V) = Proj(U\otimes V)$. We note that in general, $Proj(U)\otimes Proj(V) \neq Proj(U\otimes V)$. For instance, take $G := C_9$ the cyclic group of order 9, $U := k\uparrow_{C_3}^{C_9}$ and $V := kG$. Then, $Proj(V) = Proj(U \otimes V)$, the set of projective modules, whereas it will be easy to compute from the results we obtain in Section 7 for cyclic p-groups that $Proj(U) \otimes Proj(V) = \{kG^{\oplus 3n} \mid n \in \mathbb{N}\}$.

Proof. (g) $Proj(U \otimes V) \subseteq Proj(U) \cap Proj(V)$ by the very definition of $U \otimes V$-projectivity. If $M \in Proj(U) \cap Proj(V)$, by definition there are kG-modules N and L such that $M \mid U \otimes N$ and $M \mid V \otimes L$. Since the trace map $Tr : M^* \otimes M \longrightarrow k$ is M-split, $M \mid M \otimes M^* \otimes M$. These three relations brought together yield

$$M \mid M \otimes M^* \otimes M \mid U \otimes N \otimes M^* \otimes V \otimes L \cong U \otimes V \otimes N \otimes M^* \otimes L\,.$$

Hence $Proj(U) \cap Proj(V) = Proj(U \otimes V)$. In addition, if $M \in Proj(U) \otimes Proj(V)$, that is $M \cong M_U \otimes M_V$ with $M_U \in Proj(V)$ and $M_V \in Proj(V)$, then there are two modules $N_U, N_V \in \mathsf{mod}(kG)$ such that $M_U \mid U \otimes N_U$ and $M_V \mid V \otimes N_V$. It yields

$$M \cong M_U \otimes M_V \mid U \otimes N_U \otimes V \otimes N_V \cong U \otimes V \otimes N_U \otimes N_V\,.$$

Hence the inclusion $Proj(U \otimes V) \supseteq Proj(U) \otimes Proj(V)$.

(k) $M \in Proj(V)$ if and only if $M \mid V \otimes N$ for some $N \in \mathsf{mod}(kG)$ if and only if ${}^gM \mid {}^g(V \otimes N) \cong {}^gM \otimes {}^gN$ if and only if ${}^gM \in Proj({}^gV)$.

□

The notion of projectivity relative to a module encompasses the well-known notion of projectivity relative to a subgroup, as well as the notion of projectivity relative to a family of subgroups, with the advantage that it becomes somewhat less cumbersome when we look at it as projectivity relative to a single module. Using Frobenius reciprocity, it is easy to show that projectivity relative to the subgroup H of G is equivalent to projectivity relative to the kG-module $k\uparrow_H^G$ and also that a short exact sequence $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathsf{mod}(kG)$ is H-split if and only if it is $k\uparrow_H^G$-split. Similarly, one can show that projectivity relative to the family $\mathcal{H}$ of subgroups of G is equivalent to projectivity relative to the kG-module $\bigoplus_{H\in\mathcal{H}} k\uparrow_H^G$ and the sequence E is $\mathcal{H}$-split if and only if it is $\bigoplus_{H\in\mathcal{H}} k\uparrow_H^G$-split.

2.1. **Relative projectivity and operations on groups.** We now establish some notation and basic facts concerning projectivity relative to modules with respect to standard operations on groups.

Lemma 2.1.1. *Let H be a subgroup of G, N a normal subgroup of G.*

(a) ***Restriction****: Let Z be a V-projective kG-module, then $Z\downarrow_H^G$ is a $V\downarrow_H^G$-projective kH-module. We shall use the following short notation: $Proj(V)\downarrow_H^G \subseteq Proj(V\downarrow_H^G)$.*

(b) ***Induction****: Let Z be a V-projective kH-module, then $Z\uparrow_H^G$ is a $V\uparrow_H^G$-projective kG-module. We shall use the following short notation: $Proj(V)\uparrow_H^G \subseteq Proj(V\uparrow_H^G)$.*

(c) ***Inflation****: let Z be a V-projective $k[G/N]$-module, then $\mathrm{Inf}_{G/N}^G(Z)$ is an $\mathrm{Inf}_{G/N}^G(V)$-projective kG-module. We shall use the following short notation: $\mathrm{Inf}_{G/N}^G(Proj(V)) \subseteq Proj(\mathrm{Inf}_{G/N}^G(V))$.*

(d) ***Isomorphism****: let $\varphi : G \longrightarrow \widetilde{G}$ be a group homomorphism and Z be a V-projective kG-module, then $\mathrm{Iso}_G^{\widetilde{G}}(Z)$ is an $\mathrm{Iso}_G^{\widetilde{G}}(V)$-projective $k\widetilde{G}$-module.*

Proof. (a) Z is V-projective if and only if $Z \mid V \otimes L$ for some kG-module L, thus

$$Z\downarrow_H^G \mid (V \otimes L)\downarrow_H^G \cong V\downarrow_H^G \otimes L\downarrow_H^G,$$

i.e. Z is $V\downarrow_H^G$-projective.

(c) and (d) can be proven in like manner.

(b) Z is V-projective if and only if $Z \mid V \otimes L$ for some kH-module L, hence

$$Z\uparrow_H^G \mid (V \otimes L)\uparrow_H^G \mid V\uparrow_H^G \otimes L\uparrow_H^G$$

since $V\uparrow_H^G \otimes L\uparrow_H^G \cong \bigoplus_{[HgH]} [{}^gV\downarrow_{H\cap {}^gH}^H \otimes L\downarrow_{H\cap {}^gH}^H]\uparrow_H^G$. Thus $Z\uparrow_H^G \in Proj(V\uparrow_H^G)$.

□

As we shall use restriction extensively , note that the reverse inclusion for (a) does not hold in general. For instance, if $G = C_3 \times C_3$, let h be one of its generators, $H := < h >$ and $V := k\uparrow_H^{C_3\times C_3}$, then $V\downarrow_H^G \cong k \oplus k \oplus k$. It follows that $Proj(V\downarrow_H^G) = Proj(k^{\oplus 3}) = Proj(k) = \mathsf{mod}(kH)$, whereas using Green's indecomposability theorem it is easy to compute that

$$Proj(V)\downarrow_H^G = \{M \in \mathsf{mod}(kH) \mid M \cong a_1k \oplus a_2\Omega(k) \oplus a_3kH, a_1, a_2, a_3 \in 3\mathbb{Z}\}.$$

Next we focus on the behaviour of relatively projective modules with respect to restrictions and inductions.

Lemma 2.1.2. *Let G be a finite group and $X \leqslant H \leqslant G$ be subgroups. Let U, V be kG-modules and W, Z be kX-modules. The following inclusions and equalities hold:*

(a) *$Proj(V\downarrow_H^G)\downarrow_X^H = Proj(V\downarrow_X^G)$;*

(b) *$Proj(W\uparrow_X^H)\uparrow_H^G = Proj(W\uparrow_X^G)$;*

(c) $Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V\downarrow_H^G\uparrow_H^G) \subseteq Proj(V)$.
If, moreover, V is H-projective, then $Proj(V\downarrow_H^G\uparrow_H^G) = Proj(V)$;

(d) *If* $Proj(V) = Proj(U)$, *then* $Proj(V\downarrow_H^G) = Proj(U\downarrow_H^G)$.

(e) *If* $Proj(W) = Proj(Z)$, *then* $Proj(W\uparrow_X^G) = Proj(Z\uparrow_X^G)$.

Proof. (a)/(b) In both cases the inclusion $\subseteq$ was stated in Lemma 2.1.1. The reverse inclusion is a straightforward consequence of the transitivity of restrictions and inductions. E.g. $V\downarrow_X^G = (V\downarrow_H^G)\downarrow_X^H$ so that $V\downarrow_X^G \in Proj(V\downarrow_H^G)\downarrow_X^H$ and by the omnibus properties of relative projectivity $Proj(V\downarrow_X^G) \subseteq Proj(V\downarrow_H^G)\downarrow_X^H$. A similar argument can be carried through for induction.

(c) The inclusion $Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V\downarrow_H^G\uparrow_H^G)$ is a special case of Lemma 2.1.1, part (a). In addition, Frobenius reciprocity yields $V\downarrow_H^G\uparrow_H^G \cong V\otimes k\uparrow_H^G$, thus by 2.0.2,

$$Proj(V\downarrow_H^G\uparrow_H^G) = Proj(V)\cap Proj(k\uparrow_H^G) \subseteq Proj(V)\,.$$

Moreover, if V is H-projective, then $Proj(V) \subseteq Proj(k\uparrow_H^G)$ by 2.0.2 (b) again. Consequently, the argument of (a) implies that:

$$Proj(V\downarrow_H^G\uparrow_H^G) = Proj(V)\cap Proj(k\uparrow_H^G) = Proj(V)$$

(d)/(e) If $Proj(V) \subseteq Proj(U)$, then, in particular, $V \in Proj(U)$ so that $V\downarrow_H^G \in Proj(U\downarrow_H^G)$ by 2.1.1 (a), hence $Proj(V\downarrow_H^G) \subseteq Proj(U\downarrow_H^G)$ by 2.0.2. Swap the roles of V and U for the reverse inclusion. Property (e) is obtained likewise. ☐

The following lemma partly restates (a) and (b) of the two preceding ones, respectively, but focuses on a particular module rather than on a whole subcategory of relatively projective modules.

Lemma 2.1.3. *Let G be a finite group and H be a subgroup of G. Let M be an H-projective module. Then, the following conditions are equivalent:*

(a) *M is V-projective;*

(b) *$M\downarrow_H^G$ is $V\downarrow_H^G$-projective;*

(c) *$M\downarrow_H^G\uparrow_H^G$ is V-projective.*

Proof. (a)$\Rightarrow$ (b): is given by 2.1.1 (a).
(b)$\Rightarrow$ (c): Again by 2.1.1, $M\downarrow_H^G \in Proj(V\downarrow_H^G)$ implies that $M\downarrow_H^G\uparrow_H^G \in Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V)$.
(c)$\Rightarrow$ (a): Finally, by H-projectivity, $M \mid M\downarrow_H^G\uparrow_H^G \in Proj(V)$, therefore $M \in Proj(V)$ as well. ☐

As a consequence, one sees the following:

Corollary 2.1.4. *Let G be a finite group and P a Sylow p-subgroup of G. Let V and W be two kG-modules. Then $Proj(V) = Proj(W)$ if and only if $Proj(V\downarrow_P^G) = Proj(W\downarrow_P^G)$.*

Proof. The necessary condition was established in 2.1.2. For the sufficient condition, assume that $Proj(V\downarrow_P^G) = Proj(W\downarrow_P^G)$. Applying 2.1.3 twice yields the following equivalences: $M \in Proj(V)$ if and only if $M\downarrow_P^G \in Proj(V\downarrow_P^G) = Proj(W\downarrow_P^G)$ if and only if $M \in Proj(W)$. Hence $Proj(V) = Proj(W)$. ☐

Finally we establish links between V-projectivity, vertices, sources and the Green correspondence. In order to set up notation for the following sections, we recall that an admissible triple $(G,H;D)$ for the Green correspondence consists of a finite group G, a p-subgroup D and a subgroup H containing $N_G(D)$. Using notation of [CR90, Thm. 20.6], for each such triple, define $\mathcal{X} := \{{}^xD\cap D \mid x\in G\backslash H\}$, $\mathcal{Y} := \{{}^xD\cap H \mid x\in G\backslash H\}$ and $\mathcal{A} := \{D^* \leqslant D \mid D^* \nleqslant_G \mathcal{X}\}$. Then the Green correspondence is a bijection, that we shall denote by $\Gamma : [M] \rightleftarrows [N] : Gr$, from the set of isomorphism classes of indecomposable kG-modules M with vertex in $\mathcal{A}$ to the set of isomorphism classes of indecomposable kH-modules N with the same vertex in $\mathcal{A}$. Furthermore, an indecomposable kG-module M with vertex in $\mathcal{A}$ corresponds to an indecomposable kH-module N with the same vertex if and only if $M\downarrow_H^G \cong N\oplus(\mathcal{Y}-proj)$ or equivalently $N\uparrow_H^G \cong M\oplus(\mathcal{X}-proj)$.

Lemma 2.1.5. (a) *Let M be an indecomposable kG-module and (D,S) a vertex-source pair for M. Let V be a kG-module and W be a kD-module. If $S \in Proj(W)$, then $M \in Proj(W\uparrow_D^G)$, and if $M \in Proj(V)$, then $S \in Proj(V\downarrow_D^G)$.*

(b) *Let $(G,H;Q)$ be an admissible triple for the Green correspondence and V be any kG-module. Let U be an indecomposable kG-module with vertex Q and $Gr(U)$ be its kH-Green correspondent. Then $U \in Proj(V)$ if and only if $Gr(U) \in Proj(V\downarrow_H^G)$.*

Proof. (a) By 2.1.1 it is clear that if $S \in Proj(W)$, then $M \mid S\uparrow_D^G \in Proj(W\uparrow_D^G)$. If $M \in Proj(V)$, then $S \mid M\downarrow_D^G \in Proj(V\downarrow_D^G)$.

(b) If $U \in Proj(V)$, then by 2.1.1 $U\downarrow_H^G \in Proj(V\downarrow_H^G)$, therefore so does $Gr(U)$ as a direct summand of $U\downarrow_H^G$. Conversely, if $Gr(U) \in Proj(V\downarrow_H^G)$, then $Gr(U)\uparrow_H^G \in Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V)$ by 2.1.2. Hence $U \in Proj(V)$, as a direct summand of $Gr(U)\uparrow_H^G$. □

2.2. **Dimensional considerations and absolute p-divisibility.** Many arguments shall use the next result by D. Benson and J. Carlson [BC86, Thm. 2.1]:

Theorem 2.2.1. *Let k be an algebraically closed field of characteristic p (possibly $p = 0$). Let M, N be finite-dimensional indecomposable kG-modules, then*

$$k \mid M \otimes N \text{ if and only if } \begin{cases} (1)\ M \cong N^*\,; \\ (2)\ p \nmid \dim_k(N). \end{cases}$$

Moreover, if k is a direct summand of $N^* \otimes N$ then it has multiplicity one, i.e. $k \oplus k$ is not a summand.

In general, if M and N are finite-dimensional decomposable modules, write $M \cong \bigoplus_{i\in I} M_i$ and $N \cong \bigoplus_{j\in J} M_j$ as direct sums of indecomposable modules, then,

$$k \mid M \otimes N \text{ if and only if } \exists i \in I, j \in J \text{ such that } M_i \cong N_j^* \text{ and } p \nmid \dim_k(N_j).$$

In particular, if p divides the k-dimension of all direct summands of N then k is not a summand of $N^* \otimes N = \mathrm{End}_k(N)$. However, it is worth keeping in mind that the implication $(p \nmid \dim_k(N) \Rightarrow k \mid N^* \otimes N)$ is always true, that is even if N is decomposable, since in this case the trace map splits. Futhermore, the theorem enables us to characterize those kG-modules V relatively to which the trivial module is projective, which shall be essential later on to define the groups of relative endotrivial modules.

Proposition 2.2.2. *Let $V \in \mathsf{mod}(kG)$. Then, the following are equivalent:*

(a) *The trivial kG-module k is relatively V-projective;*
(b) *$p = \mathrm{char}(k)$ does not divide the k-dimension of at least one of the indecomposable direct summands of V;*
(c) *the subcategory $Proj(V)$ is equal to the whole category of finite-dimensional kG-modules $\mathsf{mod}(kG)$.*

Proof. (a)⇒(b): By 2.0.2, $k \in Proj(V)$ if and only if $k \mid V^* \otimes V$. Thus, V has an indecomposable direct summand whose k-dimension is not divisible by p.

(b)⇒(c): Since V is finitely generated, write $V = \bigoplus_{j\in J} V_j$ as a direct sum of indecomposable modules. Then by 2.0.2,

$$Proj(V) = \bigoplus_{j\in J} Proj(V_j)\,.$$

By assumption, there exists $j_0 \in J$ such that p does not divide $\dim_k(V_{j_0})$ so that by proposition 2.0.2, $Proj(V_{j_0}) = \mathsf{mod}(kG)$. Therefore $Proj(V) = \mathsf{mod}(kG)$ as well.

(c)⇒(a): is trivial. □

In other words, the proposition shows that relative projectivity to a module V is interesting essentially if the k-dimensions of all the indecomposable direct summands of V are divisible by $p = \mathrm{char}\, k$, that is when $Proj(V)$ is not equal to the whole category of finite-dimensional kG-modules $\mathsf{mod}(kG)$. To use the terminology introduced in [BC86], in the sequel, such a kG-module V shall be called *absolutely p-divisible.*

As another consequence of Theorem 2.2.1 we can rephrase [Ben98, Prop. 5.8.1] to get the following characterisation for dimensions of V-projective kG-modules.

Lemma 2.2.3. *Let V be an absolutely p-divisible kG-module and $U \in Proj(V)$. Then p divides $\dim_k U$.*

The behaviour of absolute p-divisibility with respect to restrictions shall turn out to be a key argument for the forthcoming study of relative endotrivial modules.

Lemma 2.2.4. *Let P be a Sylow p-subgroup of G. Let V be a kG-module with vertex $Q \leqslant P$.*

(a) *Then for every subgroup $H \geqslant P$, the module V is absolutely p-divisible if and only if $V\downarrow^G_H$ is absolutely p-divisible.*

(b) *Furthermore, if $Q \lneq P$, then for every subgroup R of P such that $P \geqslant R \gneq Q$, the module V is absolutely p-divisible if and only if $V\downarrow^G_R$ is absolutely p-divisible.*

Proof. It is straightforward to see that if V is a kG-module whose restriction $V\downarrow^G_H$ to some subgroup $H \leqslant G$ is absolutely p-divisible, then V is absolutely p-divisible itself. Thus in both cases we are left with the necessary condition to prove.

(a) Let $P \leqslant H \leqslant G$ be a subgroup and assume that $V\downarrow^G_H$ is not absolutely p-divisible. Then, by Lemma 2.2.2, $Proj(V\downarrow^G_H) = \mathsf{mod}(kH)$. Besides Lemma 2.1.2 yields:

$$\mathsf{mod}(kH)\uparrow^G_H = Proj(V\downarrow^G_H)\uparrow^G_H \subseteq Proj(V\downarrow^G_H\uparrow^G_H) = Proj(V)$$

We deduce, in particualr, that $k\uparrow^G_H \in Proj(V)$. Finally since $p \nmid \dim_k(k\uparrow^G_H) = |G:H|$, it follows from Lemma 2.2.3 that V is not absolutely p-divisible.

(b) Let R be a subgroup of P. By assumption, $V \in Proj(k\uparrow^G_Q)$, so that $V\downarrow^G_R \in Proj(k\uparrow^G_Q\downarrow^G_R)$ and the Mackey formula yields:

$$k\uparrow^G_Q\downarrow^G_R \cong \bigoplus_{g\in[R\backslash G/Q]} ({}^g k)\downarrow^{{}^gQ}_{{}^gQ\cap R}\uparrow^R_{{}^gQ\cap R} = \bigoplus_{g\in[R\backslash G/Q]} k\uparrow^R_{{}^gQ\cap R}$$

Therefore,

$$V\downarrow^G_R \in \bigoplus_{g\in[R\backslash G/Q]} Proj(k\uparrow^R_{{}^gQ\cap R})$$

and so do all its direct summands. Now, the assumption that $Q \lneq R$ implies that ${}^gQ \cap R \lneq R$ for every $g \in [R\backslash G/Q]$. Thus any direct summand of $V\downarrow^G_R$ has a vertex strictly smaller than R so that, by Lemma 3.4.1, p divides its k-dimension. Hence the result.

□

2.3. **V-projective resolutions and relative syzygy modules.** To end this section on properties of relative projectivity to modules, we recall some basic results linked to the corresponding relative homological algebra. The following definition is due to [Car96, Sec. 8].

Definition 2.3.1. A *V-projective resolution* of a module $M \in \mathsf{mod}(kG)$ is a nonnegative complex P_* of V-projective modules together with a surjective kG-homomorphism $P_0 \xrightarrow{\varepsilon} M$ such that the sequence

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is *totally V-split*, that is such that for all $i \leqslant 1$, the short exact sequences $0 \longrightarrow \ker(\partial_i) \longrightarrow P_i \xrightarrow{\partial_i} \mathrm{Im}(\partial_i) \longrightarrow 0$, as well as $0 \longrightarrow \ker(\varepsilon) \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ are V-split. The latter sequence is called a *V-projective presentation* of M.

Similarly, there is a notion of *V-injective resolution* $M \xrightarrow{\imath} I_*$. Noticing that $V^* \otimes V \otimes M \longrightarrow M$ is a V-projective presentation of M and iterating this constructions yields the existence of V-projective resolutions.

There is also a relative comparison theorem: if $P_* \xrightarrow{\varepsilon} M$ and $Q_* \xrightarrow{\theta} M$ are two V-projective resolutions of the module $M \in \mathsf{mod}(kG)$, then there is a chain map $\mu_* : (P_* \xrightarrow{\varepsilon} M) \longrightarrow (Q_* \xrightarrow{\theta} M)$ which lifts the identity on M. In particular, the comparison theorem applied to two V-projective presentations is the relative version of Schanuel's Lemma. Then using arguments similar to those proving the existence of minimal projective resolutions and minimal injective resolutions, it is possible to show the existence of minimal V-projective and V-injective resolutions. A corollary to this existence property provides a canonical generator for $Proj(V)$, that is, the V-projective cover V_k of the trivial module: $Proj(V) = Proj(V_k)$. It also leads to the following definition of the modules called the *relative syzygy modules* or the *relative Heller translates* of a kG-module M.

Definition 2.3.2. Let M be a kG-module, $P_* \xrightarrow{\varepsilon} M$ be a minimal V-projective resolution of M and $M \xrightarrow{\iota} I_*$ a minimal V-injective resolution of M. Define for all $n \geqslant 1$: $\Omega_V^n(M) := \ker \partial_n$, $\Omega_V^{-n}(M) := \mathrm{Coker}(\partial^{n-1})$. Finally, define Ω_V^0 to be the V-projective free part of M.

We sum the properties of the relative syzygy modules in the following omnibus proposition. They can all be found in [Car96, Sec. 8] or are more general versions of [Car96, Prop. 4.4], in which case the proofs are similar and straightforwardly obtained by replacing projectivity with relative projectivity.

Proposition 2.3.3 (Omnibus properties)**.** *Let $M, N, V \in \mathsf{mod}(kG)$ and $m, n \in \mathbb{Z}$.*

(a) *If $Proj(V) = Proj(W)$, then $\Omega_V^n(M) \cong \Omega_W^n(M)$.*
(b) *$\Omega_V^n(M) \cong \Omega_V^{-n}(M^*)^*$.*
(c) *$\Omega_V^n(M) = 0$ if and only if $M \in Proj(V)$.*
(d) *$\Omega_V^n(M)$ is V-projective-free.*
(e) *If $\widetilde{\Omega}_V(M)$ denotes the kernel of a V-projective presentation of M, then $\widetilde{\Omega}_V(M) \cong \Omega_V(M) \oplus (V - proj)$.*
(f) *$\Omega_V^n(M \oplus N) \cong \Omega_V^n(M) \oplus \Omega_V^n(N)$.*
(g) *$\Omega_V^n(\Omega_V^m(M)) \cong \Omega_V^{n+m}(M)$.*
(h) *$\Omega_V^m(M) \otimes N \cong \Omega_V^m(M \otimes N) \oplus (V - proj)$.*
(i) *$\Omega_V^m(M) \otimes \Omega_V^n(N) \cong \Omega_V^{m+n}(M \otimes N) \oplus (V - proj)$.*

We shall be particularly interested in the behaviour of the relative syzygy modules with respect to restriction and inflation.

Lemma 2.3.4. (a) *Let H be a subgroup of G and M, V be kG-modules, then:*

$$\Omega_V(M)\downarrow_H^G \cong \Omega_{V\downarrow_H^G}(M\downarrow_H^G) \oplus (V\downarrow_H^G -proj)$$

(b) *Let N be a normal subgroup of G and M be a $k[G/N]$-module, then :*

$$\Omega_{k\uparrow_N^G}(\mathrm{Inf}_{G/N}^G(M)) \cong \mathrm{Inf}_{G/N}^G(\Omega(M))$$

Proof. (a) The restriction of a minimal V-projective resolution is a $V\downarrow_H^G$-projective resolution of $M\downarrow_H^G$, it is not necessarily minimal though. Then, the formula follows from the comparison theorem.

(b) This formula is a version for projectivity relative to modules of a formula given in [Bou00, Cor. 4.1.2] for relative syzygies of P-sets, with P a p-group. The proof is identical.

□

2.4. **Relative syzygies, vertices, sources and Green correspondence.** To start with, the following characterisation of the vertices of relative syzygy modules was established in [Oku91, Cor. 9.9].

Lemma 2.4.1. *Let V and W be kG-modules.*

(a) *Let $M \in Proj(W)$. Then $\Omega_V^n(M) \in Proj(W)$ for all $n \in \mathbb{Z}$.*
(b) *Let M be an indecomposable non-V-projective kG-module. Then, for all $n \in \mathbb{Z}$, M and $\Omega_V^n(M)$ have the same vertices.*

In consequence, relative Heller operators commute with the Green correspondence and we can also compute their sources.

Corollary 2.4.2. *Let V be a kG-module.*

(a) *Let $(G,H;Q)$ be an admissible triple for the Green correspondence. Let U be a non V-projective indecomposable kG-module with vertex Q. If T is the kH-Green correspondent of U, then $\Omega_{V\downarrow_H^G}(T)$ is the kH-Green correspondent of $\Omega_V(U)$.*

(b) *Let M be an indecomposable non-V-projective kG-module and (D,S) a vertex-source pair for M. Then $\Omega_{V\downarrow_D^G}(S)$ is a source for $\Omega_V(M)$.*

Proof. (a) First, the assumption that U is non V-projective ensures that neither $\Omega_V(U)$, nor $\Omega_{V\downarrow_H^G}(T)$ is zero. Indeed, by 2.1.5 $U \notin Proj(V)$ if and only if $Gr(U) \notin Proj(V\downarrow_H^G)$. Then, by assumption, both the modules U and T have vertex Q, thus, by the lemma, so do the modules $\Omega_V(U)$ and $\Omega_{V\downarrow_H^G}(T)$. Therefore, it suffices to prove that $\Omega_{V\downarrow_H^G}(T)$ is a direct summand of $\Omega_V(U)\downarrow_H^G$. Indeed, as seen before $\Omega_{V\downarrow_H^G}(U\downarrow_H^G) \,|\, \Omega_V(U)\downarrow_H^G$. In addition, by the Green correspondence, $T \,|\, U\downarrow_H^G$, so that, by the properties of relative syzygies, $\Omega_{V\downarrow_H^G}(T) \,|\, \Omega_{V\downarrow_H^G}(U\downarrow_H^G)$.

(b) Let $\Omega_{V\downarrow_D^G}(S) \hookrightarrow P_{V\downarrow_D^G}(S) \twoheadrightarrow S$ be a minimal $V\downarrow_D^G$-projective presentation of S. Then $\Omega_{V\downarrow_D^G}(S)\uparrow_G^D \hookrightarrow P_{V\downarrow_D^G}(S)\uparrow_G^D \twoheadrightarrow S\uparrow_G^D$ is a V-projective presentation of $S\uparrow_G^D$, but it is not necessarily minimal though. Nonetheless, the relative version of Shanuel's lemma yields:

$$\Omega_{V\downarrow_D^G}(S)\uparrow_G^D \cong \Omega_V(S\uparrow_G^D) \oplus (V-proj)\,.$$

By assumption, S is a source of M, thus M is a direct summand of $S\uparrow_G^D$ and so $\Omega_V(M)$ is a direct summand of $\Omega_V(S\uparrow_G^D)$, which is, as seen above, in turn a direct summand of $\Omega_{V\downarrow_D^G}(S)\uparrow_G^D$. Furthermore, according to the previous lemma, M and $\Omega_V(M)$ have a common vertex. It follows that $\Omega_{V\downarrow_D^G}(S)$ is a source for $\Omega_V(M)$.

□

3. The group of relative endotrivial modules

Recall that a module $M \in \mathsf{mod}(kG)$ is called *endotrivial* if its endomorphism ring is of the form $\mathrm{End}_k(M) \cong k \oplus (proj)$. In this section we generalize this family of modules to weaker versions by replacing ordinary projectivity with projectivity relative to modules. From now on, unless otherwise stated, V a fixed absolutely p-divisible kG-module so that the subcategory $Proj(V)$ is not the whole category $\mathsf{mod}(kG)$ of kG-modules, which, as we have pointed out in the previous section, is equivalent to requiring that the trivial module k is not projective relative to V.

3.1. Relative endotrivial modules.

Definition 3.1.1. A kG-module M is termed *endotrivial relative to the kG-module V* or *relatively V-endotrivial* or simply *V-endotrivial* if its k-endomorphism ring is the direct sum of a trivial module and a V-projective module. That is, M is endotrivial relative to V if and only if

$$\mathrm{End}_k(M) \cong M^* \otimes M \cong k \oplus (V-proj)\,.$$

It is equivalent to requiring that $\mathrm{End}_k(M)$ is isomorphic to a trivial module in the relative stable category $\mathsf{stmod}_V(kG)$.

To begin with, here is a rudimentary but extremely useful dimensional characterisation.

Lemma 3.1.2. *Let V be an absolutely p-divisible kG-module and M be a V-endotrivial module. Then:*

(a) $\dim_k(M)^2 \equiv 1 \mod p$.

(b) *In case $V = k\uparrow_Q^G$, that is if we consider projectivity relative to the p-subgroup Q of G, then* $\dim_k(M)^2 \equiv 1 \mod |P:Q|$ *where P is a Sylow p-subgroup of G containing Q.*

Proof. (a) By 2.2.3 the k-dimension of any V-projective module is divisible by p, hence

$$\dim_k(M)^2 = \dim_k(\mathrm{End}_k(M)) = \dim_k(k \oplus (V - proj)) \equiv 1 \mod p\,.$$

(b) As a consequence of Green's indecomposability theorem, the k-dimension of a module is divisible by the index of one of its vertices in the corresponding Sylow p-subgroup. (See [CR90].) □

3.2. **Constructions and stable operations for relative endotrivial modules.** As the projective modules belong to any subcategory of relatively projective modules $Proj(V)$, it is clear that ordinary endotrivial modules are also endotrivial relatively to any kG-module V. In particular, so is any one-dimensional kG-module. The other class of obvious examples of V-endotrivial modules is given by the kernels and cokernels of V-projective resolutions of the trivial module and in particular, the relative syzygies $\Omega_V(k)$. More generally we have the following:

Lemma 3.2.1. (a) *Let $P \in Proj(V)$ and $0 \longrightarrow L \longrightarrow P \longrightarrow N \longrightarrow 0$ be a V-split short exact sequence. Then N is V-endotrivial if and only if L is.*

(b) *Let M be a V-endotrivial kG-module. Then the kG-modules $\Omega_W^n(M)$ are V-endotrivial modules for every kG-module $W \in Proj(V)$ and for every $n \in \mathbb{Z}$.*

Proof. (a) follows from (b). Indeed, $L = \widetilde{\Omega}_V(N) \cong \Omega_V(N) \oplus (V - proj)$ and $N = \widetilde{\Omega}_V^{-1}(L) \cong \Omega_V^{-1}(L) \oplus (V - proj)$. However this proof can be done directly using the methods of the previous section and in particular the relative version of Shanuel's lemma.

(b) Using the arithmetic of the relative syzygies that we developed in the previous section, we compute:

$$\begin{aligned}\mathrm{End}_k(\Omega_W^n(M)) &\cong \Omega_W^n(M)^* \otimes \Omega_W^n(M) \cong \Omega_W^0(M^* \otimes M) \oplus (V - proj)\\ &\cong \Omega_W^0(k \oplus (V - proj)) \oplus (V - proj)\\ &\cong k \oplus (V - proj) \oplus (V - proj) = k \oplus (V - proj)\end{aligned}$$

For, $k \notin Proj(W) \subseteq Proj(V) \neq \mathsf{mod}(kG)$ and the W-projective-free part of a V-projective module is clearly V-projective. □

Lemma 3.2.2. *If M, N are V-endotrivial kG-modules, then so are the modules M^*, $M \otimes N$ and $\mathrm{Hom}_k(M,N)$.*

Proof. First, $\mathrm{End}_k(M^*) \cong \mathrm{End}_k(M)$. Then, using the properties of V-projectivity, compute

$$\mathrm{End}_k(M \otimes N) \cong \mathrm{End}_k(M) \otimes \mathrm{End}_k(N) \cong (k \oplus (V - proj)) \otimes (k \oplus (V - proj)) \cong k \oplus (V - proj)\,.$$

Finally, $\mathrm{Hom}_k(M,N) \cong M^* \otimes N$ which is V-endotrivial by the two preceding constructions. □

Next, we investigate the behaviour of relative endotrivial modules with respect to standard group operations.

Lemma 3.2.3. (a) *If H is a subgroup of G and M a V-endotrivial kG-module, then $M{\downarrow}_H^G$ is a $V{\downarrow}_H^G$-endotrivial module.*

(b) *If N is a normal subgroup of G and M a V-endotrivial $k[G/N]$-module, then $\mathrm{Inf}_{G/N}^G(M)$ is an $\mathrm{Inf}_{G/N}^G(V)$-endotrivial module.*

(c) *Let $\varphi : G_1 \longrightarrow G_2$ be a group isomorphism and M a kG_1-module. Then M can be seen as a kG_2-module, denoted by $\mathrm{Iso}_{G_1}^{G_2}(M)$, the action of G_2 on M being given via φ^{-1}. Furthermore, if V is a kG_1-module and M is a V-endotrivial kG_1-module then $\mathrm{Iso}_{G_1}^{G_2}(M)$ becomes an $\mathrm{Iso}_{G_1}^{G_2}(V)$-endotrivial kG_2-module.*

Proof.

$$\mathrm{End}_k(M\downarrow_H^G) \cong \mathrm{End}_k(M)\downarrow_H^G \cong (k \oplus (V-proj))\downarrow_H^G \cong k\downarrow_H^G \oplus (V-proj)\downarrow_H^G \cong k \oplus (V\downarrow_H^G - proj)$$

where the last isomorphism is obtained by 2.1.1, part (a). This proves (a). The proofs for (b) and (c) are similar. □

Induction. Relative endotrivial modules are, in general, not stable under induction. This is easily seen by considering the group $G := C_3 \times C_3$ and its index 3 subgroup $H := C_3 \times 1$. Then, the trivial kH-module k is endotrivial, but the induced module $k\uparrow_H^G$ can't be endotrivial relatively to any kG-module since it is indecomposable and thus by the Benson-Carlson Theorem 2.2.1, the module $(k\uparrow_H^G)^* \otimes k\uparrow_H^G$ does not have the trivial module as a direct summand. (This example extends to any indecomposable relative endotrivial kH-module M and any $G \geqslant H$ satisfying the hypothesis of Green's indecomposability criterion, since then $dim_k(M\uparrow_H^G) = |G : H| \dim_k(M)$ is divisible by p.)

3.3. **Direct sum decomposition structure of relative endotrivial modules.** The first step towards the construction of an abelian group structure on the class of relative endotrivial modules is the following:

Lemma 3.3.1. *Let M be a V-endotrivial kG-module and assume there is a direct sum decomposition $M \cong M_0 \oplus M_1$, then one of M_0 or M_1 is V-endotrivial and the other is V-projective. In consequence, M is V-endotrivial if and only if its V-projective-free part is indecomposable and V-endotrivial.*

Proof. By assumption, we have

$$k \oplus (V-proj) \cong \mathrm{End}_k(M) \cong \mathrm{End}_k(M_0) \oplus \mathrm{Hom}_k(M_0, M_1) \oplus \mathrm{Hom}_k(M_1, M_0) \oplus \mathrm{End}_k(M_1)\,.$$

As a result, the Krull-Schmidt theorem forces the trivial module k to be a direct summand of either $\mathrm{End}_k(M_0)$, or $\mathrm{End}_k(M_1)$. Indeed, if it were not the case, k would be a direct summand of $\mathrm{Hom}_k(M_0, M_1)$ or $\mathrm{Hom}_k(M_1, M_0)$. But the two latter modules being dual to each other, $k \oplus k$ would be a direct summand of $\mathrm{End}_k(M)$, which is not possible because, by the assumption that V is absolutely p-divisible, $k \notin Proj(V)$ (Proposition 2.2.2). Thus we may assume that $\mathrm{End}_k(M_0) \cong k \oplus (V-proj)$ and $\mathrm{End}_k(M_1) \in Proj(V)$. But, by 2.0.2 $M_1 \in Proj(V)$ if and only if $M_1 \otimes M_1^* \in Proj(V)$. In conclusion, M_0 is V-endotrivial and $M_1 \in Proj(V)$, as required. □

3.4. **Vertices and sources.**

Lemma 3.4.1. *Let V be an absolutely p-divisible kG-module. Let M be any indecomposable V-endotrivial kG-module.*

(a) *The vertices of M are the Sylow p-subgroups of G.*
(b) *If (P, S) is a vertex-source pair for M, then S is a $V\downarrow_P^G$-endotrivial module.*
(c) *Assume moreover that $M\downarrow_P^G \cong k \oplus (V\downarrow_P^G - proj)$, then the trivial kP-module is a source for M.*

Proof. (a) It is well-known that the vertices of an indecomposable kG-module whose dimension is coprime to the characteristic p are the Sylow p-subgroups. Now, the dimensional characterisation of Lemma 3.1.2 yields $\dim_k(M) \not\equiv 0 \mod p$, hence the result.

(b) By assumption $S \mid M\downarrow_P^G$, so that $S^* \mid M^*\downarrow_P^G$ and

$$S \otimes S^* \mid M\downarrow_P^G \otimes S^* \mid M\downarrow_P^G \otimes M^*\downarrow_P^G \cong (M \otimes M^*)\downarrow_P^G \cong k \oplus (V\downarrow_P^G - proj)\,.$$

Thus it remains to show that $k \mid S \otimes S^*$. Assume ab absurdo that it is not the case, then $S \otimes S^*$ has to be $V\downarrow_P^G$-projective by the above and therefore, so is S by 2.0.2. In consequence,

$$M \mid S\uparrow_P^G \in (Proj(V\downarrow_P^G))\uparrow_P^G \subseteq Proj(V)$$

by Lemma 2.1.2, which contradicts the fact that for an absolutely p-divisible module V, an indecomposable V-endotrivial module is V-projective-free.

(c) Since P is a Sylow p-subgroup, M is P-projective so that

$$M \mid M\downarrow_P^G\uparrow_P^G \cong (k \oplus (V\downarrow_P^G - proj))\uparrow_P^G \cong k\uparrow_P^G \oplus (V\downarrow_P^G - proj))\uparrow_P^G = k\uparrow_P^G \oplus (V - proj),$$

by Lemma 2.1.2 (c). Moreover, M is V-projective-free by assumption, thus the Krull-Schmidt theorem yields that $M \mid k\uparrow_P^G$. In consequence, P being a vertex of M, k is a source of M.

□

3.5. **Group structure.** We can now copy the group structure on the ordinary endotrivial modules. Let $V \in \mathsf{mod}(kG)$ be an absolutely p-divisible module and set an equivalence relation $\sim_V$ on the class of V-endotrivial kG-modules as follows: for M and N two V-endotrivial modules let

$$M \sim_V N \text{ if and only if } M_0 \cong N_0\,,$$

where M_0 and N_0 are the unique V-endotrivial indecomposable summands of M and N, respectively, given by 3.3.1. This amounts to requiring that M and N are isomorphic in $\mathsf{stmod}_V(kG)$. Then let $T_V(G)$ denote the resulting set of equivalence classes. In particular, any equivalence class in $T_V(G)$ consists of an indecomposable V-endotrivial module M_0 and all the modules of the form $M_0 \oplus (V - proj)$.

Proposition 3.5.1. *The ordinary tensor product $\otimes_k$ induces an abelian group structure on the set $T_V(G)$ defined as follows:*

$$[M] + [N] := [M \otimes_k N]$$

The zero element is $[k]$ and the opposite of a class $[M]$ is the class $[M^*]$.

The proof of this result is straightforward and analogous to that of the endotrivial case. We shall call $T_V(G)$ the *group of V-endotrivial modules*. Also note that we use an additive notation, which is consistent with the choice made in [BT00] and related articles treating endo-permutation and endotrivial modules.

Corollary 3.5.2. *If M is a self-dual, V-endotrivial kG-module, then $[M]$ has order two in $T_V(G)$.*

Proof. By assumption $M \otimes M \cong M^* \otimes M \cong k \oplus (V - proj)$ so that $2[M] = [k]$. □

To give a first example, this simple observation can be applied at once to the concrete case of a cyclic p-group C_{p^n}, $n \geqslant 1$. Indeed, all the indecomposable kC_{p^n}-modules are self-dual. Therefore, whatever the choice of the absolutely p-divisible module V, we can conclude that the group $T_V(C_{p^n})$ is an elementary abelian 2-group. We shall give a complete description of all the different groups of relative endotrivial modules for cyclic p-groups in section 7.

The following lemma points out relations of inclusion between groups of relative endotrivial modules.

Lemma 3.5.3. *Let $U, V \in \mathsf{mod}(kG)$ be absolutely p-divisible modules. Assume moreover that $Proj(V) \subseteq Proj(U)$, then:*

(a) *every V-endotrivial kG-module is U-endotrivial;*
(b) *if M and N are V-endotrivial modules such that $M \sim_V N$, then $M \sim_U N$ as well. In consequence, $T_V(G)$ can be identified with a subgroup of $T_U(G)$ via the injective group homomorphism $\imath : T_V(G) \longrightarrow T_U(G) \,:\, [M]_V \mapsto [M]_U$.*

Proof. (a) Let M be a V-endotrivial module, then $\mathrm{End}_k(M) \cong k \oplus (V - proj) = k \oplus (U - proj)$, i.e. M is U-endotrivial.

(b) There exists two indecomposable V-endotrivial modules M_0 and N_0 such that $M \cong M_0 \oplus (V - proj)$ and $N \cong N_0 \oplus (V - proj)$ and $M \sim_V N$ implies that $M_0 \cong N_0$. By (i), M_0 and N_0 are U-endotrivial, so that $M \sim_U N$. In consequence $\imath$ is a well-defined group homomorphism. The injectivity follows from the uniqueness of the summand M_0.

□

The study of relative endotrivial modules for the Klein Group $C_2 \times C_2$ will show that it is possible to have a strict inclusion $Proj(V) \subsetneq Proj(U)$ but an isomorphism $T_V(G) \cong T_U(G)$. Nevertheless, a strict inclusion $Proj(V) \subsetneq Proj(U)$ implies that the class of V-endotrivial modules is strictly contained in the class of U-endotrivial modules. Indeed, let $M \in Proj(U) \backslash Proj(V)$. Then, on the one hand, $L := k \oplus M$ is U-endotrivial, since

$$\mathrm{End}_k(M) \cong k \oplus M \oplus M^* \oplus (M \otimes M^*) = k \oplus (U - proj),$$

but on the other hand it is not V-endotrivial, otherwise M would be V-projective. Besides, this argument also shows that there are more modules belonging to the class $[k]$ in $T_U(G)$ than in $T_V(G)$.

3.6. **Standard homomorphisms between groups of relative endotrivial modules.** In order to make further links between different groups of relative endotrivial modules, we are now going to define group homomorphisms and actions which are induced by group operations.

Restriction. Let M be a kG-module, H a subgroup of G and V an absolutely p-divisible kG-module such that $V\downarrow_H^G$ is also absolutely p-divisible. Then the groups $T_V(G)$ and $T_{V\downarrow_H^G}(H)$ are well-defined and the module $M\downarrow_H^G$ is a $V\downarrow_H^G$-endotrivial kH-module. Therefore, restriction to a subgroup induces a well-defined group homomorphism:

$$\begin{array}{rccc} \mathrm{Res}_H^G\colon & T_V(G) & \longrightarrow & T_{V\downarrow_H^G}(H) \\ & [M] & \longmapsto & [M\downarrow_H^G] \end{array}$$

Indeed, Res_H^G is a group homomorphism since restriction and $\otimes_k$ commute. Furthermore, by Lemma 2.1.2, the map Res_H^G is independent of the choice of the generator V for $Proj(V)$.

Inflation. Let N be a normal subgroup of a group G, M be a $k[G/N]$-module. Then M can be seen as a kG-module, denoted $\mathrm{Inf}_{G/N}^G(M)$, by letting N act by identity. Furthermore, if V is an absolutely p-divisible $k[G/N]$-module, then the groups $T_V(G/N)$ and $T_{\mathrm{Inf}_{G/N}^G(V)}(G)$ are well-defined. In addition, if M is V-endotrivial, then $\mathrm{Inf}_{G/N}^G(M)$ is $\mathrm{Inf}_{G/N}^G(V)$-endotrivial, therefore, inflation induces an injective group homomorphism:

$$\begin{array}{rccc} \mathrm{Inf}_{G/N}^G\colon & T_V(G/N) & \hookrightarrow & T_{\mathrm{Inf}_{G/N}^G(V)}(G) \\ & [M] & \longmapsto & [\mathrm{Inf}_{G/N}^G(M)] \end{array}$$

Isomorphism. Let $\varphi : G_1 \longrightarrow G_2$ be a group isomorphism and M a kG_1-module. Then M can be seen as a kG_2-module, denoted by $\mathrm{Iso}_{G_1}^{G_2}(M)$, the action of G_2 on M being given via φ^{-1}. Furthermore, if V is an absolutely p-divisible kG_1-module, then $\mathrm{Iso}_{G_1}^{G_2}(V)$ is an absolutely p-divisible kG_2-module and if M is a V-endotrivial kG_1-module then M becomes a $\mathrm{Iso}_{G_1}^{G_2}(V)$-endotrivial kG_2-module. This operation induces a group isomorphism:

$$\begin{array}{rccc} \mathrm{Iso}_{G_1}^{G_2}\colon & T_V(G_1) & \longrightarrow & T_{\mathrm{Iso}_{G_1}^{G_2}(V)}(G_2) \\ & [M] & \longmapsto & [\mathrm{Iso}_{G_1}^{G_2}(M)] \end{array}$$

A concrete example of such an isomorphism between groups of relative endotrivial modules is provided below by conjugation.

Remark 3.6.1. It should be noted that the three cases of restriction, inflation, and isomorphism can be unified in the single case of *restriction along* a group homomorphism $G_1 \longrightarrow G_2$. Nonetheless, we do not do it in these terms because the case of restriction, in which we need to require that the module $V\downarrow_H^G$ is absolutely p-divisible, shows that an arbitrary group homomorphism, and in particular an inclusion of subgroups, would not necessarily induce a well-defined group homomorphism between the corresponding groups of relative endotrivial modules.

Conjugation. Let $H \trianglelefteq G$ be a normal subgroup and V be an absolutely p-divisible G-invariant kH-module. Then, for all $g \in G$, ${}^{g}Proj(V) = Proj(V)$ and ${}^{g}H = H$. Therefore, conjugation induces a well-defined action of G (or rather G/H), on the group $T_V(H)$ given by:

$$\begin{array}{ccc} G \times T_V(H) & \longrightarrow & T_V(H) \\ (g, [M]) & \longmapsto & [{}^{g}M] \end{array}$$

In case the subgroup H and the module V are not assumed to be normal nor G-invariant, then the above assignment does not yield a group action, nevertheless, for any element $g \in G$, the conjugation isomorphism $\gamma_g : H \longrightarrow {}^{g}H$ induces a group isomorphism

$$\begin{array}{cccc} \gamma_g\colon & T_V(H) & \longrightarrow & T_{{}^{g}V}({}^{g}H) \\ & [M] & \longmapsto & [{}^{g}M] \end{array}$$

In particular, if $H \trianglelefteq G$, then $T_V(H) \cong T_{{}^{g}V}(H)$. Also, if V is G-invariant then $T_V(H) \cong T_V({}^{g}H)$.

4. Properties of restriction maps

The purpose of this section is to relate groups of relative endotrivial modules for a group G to those for a Sylow p-subgroup P of G or a subgroup H containing P. In particular, links between endotrivial modules for G and the normalizer $N_G(P)$ of the Sylow subgroup can be obtained by Green correspondence. Most of the result presented in this section are generalisations of results concerning ordinary endotrivial modules which can be found in [MT07], [CMN06] and [Maz07].

4.1. **Restrictions to Sylow p-subgroups.** To begin with, we describe restrictions to a Sylow p-subgroup. The following easy properties generalise [CMN06, Prop. 2.6].

Lemma 4.1.1. *Let P be a Sylow p-subgroup of G and H a subgroup of G containing P.*

(a) *Let M be a V-endotrivial kG-module. Then M is a direct summand of a $V\downarrow^G_H$-endotrivial module induced from H to G, namely the module $M\downarrow^G_H\uparrow^G_H$.*
(b) *Assume $V\downarrow^G_H$ is absolutely p-divisible and let M be a kG-module such that $M\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial, then M is V-endotrivial.*

Proof. (a) Since $H \geqslant P$, by H-projectivity $M \mid M\downarrow^G_H\uparrow^G_H$ where $M\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial by 3.2.3.
(b) As $M\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial and $V\downarrow^G_H$ absolutely p-divisible,

$$(\dim_k M)^2 = (\dim_k M\downarrow^G_H)^2 \equiv 1 \mod p\,.$$

In consequence both the trace map and its restriction to H split, so that $M^*\otimes M \cong k\oplus\ker(Tr)$ and $(M\downarrow^G_H)^*\otimes M\downarrow^G_H \cong k\oplus\ker(Tr)\downarrow^G_H$, where $\ker(Tr)\downarrow^G_H$ has to be $V\downarrow^G_H$-projective by the assumption that $M\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial. Besides, by H-projectivity, $\ker(Tr) \mid \ker(Tr)\downarrow^G_H\uparrow^G_H \in Proj(V)$ by 2.1.3. Therefore $\ker(Tr)$ is a V-projective module as well and $M^*\otimes M \cong k\oplus(V-proj)$ as required.

□

We now treat the special case of a normal Sylow p-subgroup. The next proposition and its corollary partly generalise [CMN06, Prop. 2.6, (d)] and [Maz07, Cor. 2.7].

Proposition 4.1.2. *Let P be a normal Sylow p-subgroup of G. Let V be an absolutely p-divisible kG-module. Then, an indecomposable kG-module M is V-endotrivial if and only if its restriction to P is an indecomposable $V\downarrow^G_P$-endotrivial module.*

Proof. Assume that M is an indecomposable V-endotrivial module. Let $M\downarrow^G_P \cong N_1\oplus\cdots\oplus N_s$, $s\in\mathbb{N}$, be a decompostion into indecomposable summands. Since P is a vertex of M (see 3.4.1), one may assume, without loss of generality, that N_1 is a source for M, so that $M \mid N_1\uparrow^G_P$ as well. Thus, given that P is normal in G, the Mackey formula yields

$$M\downarrow^G_P \mid N_1\uparrow^G_P\downarrow^G_P \cong \bigoplus_{g\in[G/P]} {}^{g}N_1\,.$$

Now, on the one hand $M\downarrow^G_P$ is $V\downarrow^G_P$-endotrivial, which is more accurately the direct sum of an indecomposable $V\downarrow^G_P$-endotrivial module, whose k-dimension is coprime to p, and a $V\downarrow^G_P$-projective module, all of whose indecomposable summands have k-dimension divisible by p. On the other hand the G-conjugates ${}^gN_1$ of N_1 are all indecomposable with k-dimension equal to that of N_1. Therefore, this forces $M\downarrow^G_P$ to be indecomposable ($V\downarrow^G_P$-endotrivial). Conversely, let M be such that $M\downarrow^G_P$ is an indecomposable $V\downarrow^G_P$-endotrivial module. Firstly the fact that $M\downarrow^G_P$ is indecomposable forces M to be indecomposable as well, and secondly it follows from part (c) of Lemma 4.1.1 that M is V-endotrivial. □

As a consequence, when the Sylow p-subgroup P is normal in the group G, then the V-endotrivial modules are detected upon restriction to P. Since the restriction of a V-endotrivial module is G-invariant, at the level of groups of relatively endotrivial modules, there is an inclusion

$$\mathrm{Im}(\mathrm{Res}^G_P) \leqslant T_{V\downarrow^G_P}(P)^{N_G(P)/P}\,.$$

A natural question is to ask when this inclusion is indeed an equality, that is when the restriction map is actually surjective onto the $N_G(P)/P$-fixed points of $T_{V\downarrow^G_P}(P)$. We shall see further in the last section that, for instance, it is always the case for groups with cyclic Sylow p-subgroups.

Corollary 4.1.3. *Let P be a normal Sylow p-subgroup of G. Let V be an absolutely p-divisible kG-module, M be an indecomposable V-endotrivial module and (X_*,∂_*) be a V-projective resolution of M. Then:*

- (a) *(X_*,∂_*) is minimal if and only if $(X_*\downarrow^G_P,\partial_*\downarrow^G_P)$ is a minimal $V\downarrow^G_P$-projective resolution of $M\downarrow^G_P$;*
- (b) *in particular, $\Omega^n_V(M)\downarrow^G_P\cong\Omega^n_{V\downarrow^G_P}(M\downarrow^G_P)$ for all integer n.*

Proof. Given that (X_*,∂_*) is a minimal V-projective resolution, for each integer $n\geqslant 0$ there is a V-split short exact sequence

$$0\longrightarrow \Omega^{n+1}_V(M)\longrightarrow X_n\xrightarrow{\partial_n}\Omega^n_V(M)\longrightarrow 0\ .$$

Restricting it from G to P yields a $V\downarrow^G_P$-projective presentation of $\Omega^n_{V\downarrow^G_P}(M)\downarrow^G_P$:

$$0\longrightarrow \Omega^{n+1}_V(M)\downarrow^G_P\longrightarrow X_n\downarrow^G_P\xrightarrow{\partial_n\downarrow^G_P}\Omega^n_V(M)\downarrow^G_P\longrightarrow 0$$

although, it is not necessarily minimal. However, by 2.3.3,

$$\Omega^{n+1}_V(M)\downarrow^G_P\cong\Omega^{n+1}_V(M\downarrow^G_P)\oplus(V\downarrow^G_P-proj)\ \text{and}\ \Omega^n_V(M)\downarrow^G_P\cong\Omega^n_V(M\downarrow^G_P)\oplus(V\downarrow^G_P-proj)\,.$$

Besides, by the proposition, both these modules are indecomposable so that the $V\downarrow^G_P$-projective factors are zero. Therefore the above short exact sequence is indeed

$$0\longrightarrow \Omega^{n+1}_V(M\downarrow^G_P)\longrightarrow X_n\downarrow^G_P\xrightarrow{\partial_n\downarrow^G_P}\Omega^n_V(M\downarrow^G_P)\longrightarrow 0\ .$$

Hence the minimality of $(X_*\downarrow^G_P,\partial_*\downarrow^G_P)$. The converse is trivial. □

4.2. **Restriction to the normalizer of a Sylow p-subgroup.** The goal is now to figure out the behaviour of restriction maps from a group G to the normalizer of a Sylow p-subgroup P or a subgroup H containing $N_G(P)$. It follows from Sections 2.2 and 3.6 that for every absolutely p-divisible kG-module V there is a well-defined restriction map

$$\mathrm{Res}^G_H: T_V(G)\longrightarrow T_{V\downarrow^G_H}(H)\,.$$

The following statement generalises [CMN06, Prop. 2.6.(a)].

Lemma 4.2.1. *Let P be a Sylow p-subgroup of G, H a subgroup of G containing $N_G(P)$ and V an absolutely p-divisible kG-module. Then the restriction map $\mathrm{Res}^G_H: T_V(G)\longrightarrow T_{V\downarrow^G_H}(H)$ is injective.*

Proof. Let M be an indecomposable V-endotrivial kG-module. By Lemma 3.4.1 P is a vertex of M. Then, on the one hand, the Green correspondence for the triple $(G, H; P)$ yields:

$$M\downarrow^G_H \cong Gr(M) \oplus X$$

where $Gr(M)$ is an indecomposable kH-module with vertex P and $X \in Proj(\mathcal{Y})$ with $\mathcal{Y} = \{{}^xP \cap H \mid x \in G \backslash H\}$. But ${}^xP \cap H \lneq {}^xP$ for all $x \in G\backslash H$, otherwise ${}^xP$ would be a Sylow p-subgroup of H which is not possible, since then there would be $h \in H$ such that ${}^{hx}P = P$, that is $hx \in N_G(P) \subseteq H$ and $x \in H$. Therefore all the direct summands of X have a vertex strictly smaller than P. On the other hand, $M\downarrow^G_H$ is a $V\downarrow^G_H$-endotrivial module, that is:

$$M\downarrow^G_H \cong M_0 \oplus (V\downarrow^G_H -proj)$$

with M_0 an indecomposable $V\downarrow^G_H$-endotrivial module, thus with vertex P by 3.4.1. In consequence, the Krull-Schmidt theorem implies that $M_0 \cong Gr(M)$, the kH-Green correspondent of M, whose uniqueness yields the injectivity of Res^G_H. □

4.3. **Cases in which restriction maps are isomorphisms.** Knowing that the restriction map $\mathrm{Res}^G_H : T_V(G) \longrightarrow T_{V\downarrow^G_H}(H)$ is injective for every subgroup H containing the normalizer $N_G(P)$ of a Sylow p-subgroup P, the next question that arises is to understand when this map is an isomorphism. The last section on groups with cyclic Sylow p-subgroup shall provide us with examples in which the answer depends on the module V to which relative projectivity is considered. Notwithstanding, one can show that in case the subgroup H is strongly p-embedded in G, then Res^G_H is always an isomorphism, however the choice of the module V. This result generalises the similar result for ordinary endotrivial modules that can be found, for instance, in [MT07, Lem. 2.7]. Furthermore, the proof of this result provides us with the following more general sufficient condition on the module V for the restriction map to be an isomorphism.

Lemma 4.3.1. *Let P be a Sylow p-subgroup of G and $H \leqslant G$ a subgroup containing the normalizer $N_G(P)$ of P. Let V be an absolutely p-divisible kG-module. If $Proj(V\downarrow^G_H) \supseteq Proj(\mathcal{Y})$, where $\mathcal{Y}$ is the family of subgroups $\{{}^gP \cap H \mid g \in G\backslash H\}$ involved in the Green correspondence, then the restriction map $\mathrm{Res}^G_H : T_V(G) \longrightarrow T_{V\downarrow^G_H}(H)$ is an isomorphism. Furthermore, the inverse map is induced by induction, so that*

$$T_V(G) = \{[M\uparrow^G_H] \mid [M] \in T_{V\downarrow^G_H}(H)\} \cong T_{V\downarrow^G_H}(H)$$

More accurately, on indecomposable $V\downarrow^G_H$-endotrivial modules, the inverse map is induced by the Green correspondence, that is, if $\Gamma(M)$ denotes the Green correspondent of an indecomposable kH-module M, then

$$T_V(G) = \{[\Gamma(M)] \mid M \text{ is an indecomposable } V\downarrow^G_H \text{-endotrivial } kH\text{-module}\}\,.$$

Proof. By 4.2.1, the map Res^G_H is one-to-one, therefore it only remains to show that it is onto as well. Let L be an indecomposable $V\downarrow^G_H$-endotrivial module. The Mackey formula yields the isomorphism.

$$L\uparrow^G_H\downarrow^G_H \cong L \oplus \bigoplus_{\substack{g\in[H\backslash G/H]\\ g\in G\backslash H}} ({}^gL)\downarrow^{{}^gH}_{{}^gH\cap H}\uparrow^H_{{}^gH\cap H} =: L \oplus L'\,,$$

where, by the proof of the Green correspondence, $L' \in Proj(\mathcal{Y})$, so that

$$L\uparrow^G_H\downarrow^G_H \cong L \oplus (\mathcal{Y} - proj) = L \oplus (V\downarrow^G_H -proj)$$

by assumption. In other words, $L\uparrow^G_H\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial and consequently $L\uparrow^G_H$ is V-endotrivial by 4.1.1. Therefore, $\mathrm{Res}^G_H([L\uparrow^G_H]) = [L\uparrow^G_H\downarrow^G_H] = [L]$. Hence the surjectivity of Res^G_H. Moreover, the proof of the injectivity of Res^G_H shows that the unique indecomposable V-endotrivial summand of $L\uparrow^G_H$ has to be isomorphic to the kG-Green correspondent of L.
It follows from the proof of the injectivity (Lemma 4.2.1) that the inverse map is induced by Green correspondence on the indecomposable modules. To see that, alternatively, it is induced by induction,

let $[M] \in T_{V\downarrow^G_H}(H)$ an write $M \cong M_0 \oplus (V\downarrow^G_H -proj)$ with M_0 an indecomposable $V\downarrow^G_H$-endotrivial module. Then,

$$M\uparrow^G_H \cong M_0\uparrow^G_H \oplus (V\downarrow^G_H)\uparrow^G_H \cong \Gamma(M_0) \oplus (\mathcal{X} - proj) \oplus (V - proj)$$

where $\mathcal{X}$ is the familly of subgroups involved in the Green correspondence, as described in section 2, and $Proj(V\downarrow^G_H)\uparrow^G_H \subseteq Proj(V)$ by Lemma 2.1.2. As just mentioned above, $\Gamma(M_0)$ is V-endotrivial, therefore it remains to check that $Proj(\mathcal{X}) \subseteq Proj(V)$. But this is a consequence of the hypothesis that $Proj(V\downarrow^G_H) \supseteq Proj(\mathcal{Y})$. Indeed, at the level of kH-modules, $Proj(\mathcal{Y}) \supseteq Proj(\mathcal{X})$ by definition of the families $\mathcal{X}$ and $\mathcal{Y}$, thus $Proj(V\downarrow^G_H) \supseteq Proj(\mathcal{X})$. Inducing to G yields in $\mathsf{mod}(kG)$ the required inclusions

$$Proj(V) \supseteq Proj(V\downarrow^G_H)\uparrow^G_H \supseteq Proj(\mathcal{X})\uparrow^G_H = Proj(\mathcal{X})\,.$$

□

Corollary 4.3.2. *If the subgroup H is strongly p-embedded in G, then* $\mathrm{Res}^G_H : T_V(G) \longrightarrow T_{V\downarrow^G_H}(H)$ *is an isomorphism.*

Proof. If H is strongly p-embedded in G, then for any $g \in G\backslash H$ the subgroup ${}^gH \cap H$ has order coprime to p, thus $\mathcal{Y} = \{\{1\}\}$. Therefore Res^G_H is an isomorphism, regardless of the module V, since then $Proj(V\downarrow^G_H) \supseteq Proj(\mathcal{Y}) = Proj$ for any kG-module V.

□

For instance, if the Sylow p-subgroup P is a trivial intersection subgroup (TI), then $N_G(P)$ is strongly p-embedded in G. Moreover, any strongly p-embedded subgroup contains the normalizer of some Sylow p-subgroup of G.

Besides, the first explicit example that springs to mind for a module satisfying the hypotheses of the lemma is the absolutely p-divisible module

$$V := \bigoplus_{Q\in\mathcal{F}} k\uparrow^G_Q$$

where $\mathcal{F} := \{Q \lneq P\}$ is the family of all proper p-subgroups of the Sylow p-subgroup P. Indeed, for any p-subgroup $Q \lneq P \leqslant G$ it results from the Mackey formula that $k\uparrow^H_Q \mid k\uparrow^G_Q\downarrow^G_H$, thus

$$Proj(V\downarrow^G_H) \supseteq Proj(\mathcal{F}) = Proj(\overline{\mathcal{F}}) \supseteq Proj(\mathcal{Y})\,.$$

Finally, it is also worth emphasizing that in general the kG-Green correspondent $\Gamma(L)$ of an indecomposable $V\downarrow^G_H$-endotrivial module L might or might not be a V-endotrivial module. Again, the section on groups with cyclic Sylow p-subgroups shall provide us with a handful of examples illustrating this phenomenon. Nonetheless, whenever the lemma applies $\Gamma(L)$ is always V-endotrivial.

4.4. **On the kernels of the restriction maps.** If G is a finite group, we shall follow the notation of [MT07] and denote by $X(G)$ the abelian group of all isomorphism classes of one-dimensional kG-modules endowed with the group law induced by $\otimes_k$, which can also be identified with the group $Hom(G, k^\times)$ of k-linear characters of G. It is a p'-group, isomorphic to the p'-part of the abelianization $G/[G,G]$ of G.

Any one-dimensional module χ is endotrivial ($\chi^* \otimes_k \chi \cong k$ since in this case the trace map splits). Therefore, for all absolutely p-divisible kG-module V, there is an embedding

$$\begin{array}{ccc} X(G) & \longrightarrow & T_V(G) \\ \chi & \longmapsto & [\chi]\ . \end{array}$$

mapping a one-dimensional module to its class in $T_V(G)$. Formalism would require to denote by $X_V(G)$ the image of $X(G)$ in $T_V(G)$, where the law is written additively, nonetheless, in order to keep light notation, when no confusion is to be made we shall simply use $X(G)$ instead of $X_V(G)$.

The next lemma gives conditions on the module V under which the kernel of the restriction map $\mathrm{Res}^G_Q : T_V(G) \longrightarrow T_{V\downarrow^G_Q}(P)$ is exactly $X(G)$. This generalises [MT07, Lem. 2.6]. The proof is the same, it is only analysed more deeply in order to state the results in terms of V-projectivity, which is less restricting than ordinary projectivity. This criterion shall be especially useful for the forthcoming case of groups having a cyclic Sylow p-subgroup.

Lemma 4.4.1. *Let G be a finite group and P be a Sylow p-subgroup of G. Let V be an absolutely p-divisible kG-module.*

(a) *Let Q be any p-subgroup of G such that the restriction map $\mathrm{Res}^G_Q : T_V(G) \longrightarrow T_{V\downarrow^G_Q}(Q)$ is well-defined, that is such that $V\downarrow^G_Q$ is absolutely p-divisible. Then $X(G) \leqslant \ker(\mathrm{Res}^G_Q)$.*

(b) *If all the direct summands of $V\downarrow^G_P$ have a vertex strictly included in ${}^xP \cap P$, up to conjugation, for all $x \in G\backslash N_G(P)$, then $X(G) = \ker(\mathrm{Res}^G_P)$. In particular, if P is normal in G, then $X(G) = \ker(\mathrm{Res}^G_P)$.*

Proof. (a) This is clear since the only one-dimensional kQ-module is the trivial module.

(b) It remains to show the reverse inclusion. So let M be an indecomposable V-endotrivial kG-module such that $[M] \in \ker(\mathrm{Res}^G_P)$, i.e. $M\downarrow^G_P \cong k \oplus (V\downarrow^G_P - proj)$. Thus, by P-projectivity, we have:

$$M \mid M\downarrow^G_P\uparrow^G_P \cong k\uparrow^G_P \oplus (V\downarrow^G_P - proj)\uparrow^G_P = k\uparrow^G_P \oplus (V - proj)$$

where by 2.1.1 and 2.1.2 $Proj(V\downarrow^G_P)\uparrow^G_P \subseteq Proj(V\downarrow^G_P\uparrow^G_P) = Proj(V)$. Now, since by assumption M is indecomposable and V-endotrivial, that is V-projective-free, M must be a direct summand of $k\uparrow^G_P$, therefore restricting to P and applying the Mackey formula yields:

$$M\downarrow^G_P \mid k\uparrow^G_P\downarrow^G_P \cong k^{\oplus|N_G(P):P|} \oplus \bigoplus_{\substack{x\in[P\backslash G/P] \\ x\notin N_G(P)}} k\uparrow^P_{{}^xP\cap P}$$

Each summand $k\uparrow^P_{{}^xP\cap P}$ has a vertex equal to ${}^xP \cap P$. Write $V\downarrow^G_P \cong \bigoplus_{i=1}^m V_i$, $m \in \mathbb{N}$, as a sum of indecomposable modules and for all $i = \overline{1,n}$ and let Q_i be a vertex of V_i, then

$$Proj(V\downarrow^G_P) = \bigoplus_{i=1}^m Proj(V_i) \subseteq \bigoplus_{i=1}^m Proj(k\uparrow^G_{Q_i}) .$$

Assume then that $k\uparrow^P_{{}^xP\cap P} \in Proj(V\downarrow^G_P)$, thus $k\uparrow^P_{{}^xP\cap P} \in Proj(k\uparrow^G_{Q_i})$ for some $1 \leqslant i \leqslant m$. However, $Q_i \lneq_G {}^xP \cap P$ by assumption, contradicting the fact that ${}^xP \cap P$ is a vertex. Therefore, none of the summands $k\uparrow^P_{{}^xP\cap P}$ belongs to $Proj(V\downarrow^G_P)$, which forces $M\downarrow^G_P$ to be a direct summand of $k^{\oplus|N_G(P):P|}$. Using once more that $M\downarrow^G_P$ is $V\downarrow^G_P$-endotrivial allows us to deduce that $M\downarrow^G_P \cong k$, for $V\downarrow^G_P$ being absolutely p-divisible, $k \notin Proj(V\downarrow^G_P)$. Hence $[M] \in X(G)$.

□

Notice that in case $V = kG$, that is if we consider ordinary endotrivial modules, then condition (b) is equivalent to requiring that ${}^xP \cap P$ is non trivial for all $x \in G$, as is stated in [MT07, Lem. 2.6].

5. Relative endotrivial modules as a generalisation for the Dade group

We come in this section to a chief reason of interest in relative endotrivial modules: it provides a way to define a group structure on collections of representations of an arbitrary finite group G which gives a generalization for the Dade group $D(P)$ of a finite p-group P.

Endo-permutation modules are defined only for p-groups, but not for finite groups in general. One way to obtain a similar notion for arbitrary groups is to consider endo-p-permutation modules as described in [Urf06]. If P is a p-subgroup of a group G, this notion induces a group structure, denoted by $D_P(G)$, on a set of equivalence classes of indecomposable endo-p-permutation kG-modules with vertex P. However, the main drawback of this approach resides in the fact that there is not a unique indecomposable representative, up to isomorphism, for the classes in $D_P(G)$.

So let us see in which way our notion of relative endotrivial modules can generalize the Dade group. Let us fix P a finite p-group. The first observation to make is that an indecomposable capped endo-permutation kP-module M (i.e. with vertex P) is in fact a relative endotrivial module as well, that is relatively to some intrinsincally defined kP-module V_M. Indeed, it is an elementary fact about capped endo-permutation modules that the trivial kP-module k has to be a direct summand of $End_k(M)$ ([Dad78a, Prop. 3.7]), while by 2.2.1 it is clear that the multiplicity of k is exactly one. It yields the characterization:

$$\mathrm{End}_k(M) = (\text{permutation module}) \cong k \oplus k\uparrow_{Q_1}^P \oplus \cdots \oplus k\uparrow_{Q_s}^P$$

for some subgroups $Q_1, \ldots, Q_s \lneq P$, $s \in \mathbb{N}$. Therefore one can set $V_M := \bigoplus_{i=1}^s k\uparrow_{Q_i}^P$ (clearly absolutely p-divisible!) so that, by very definition, M becomes a V_M-endotrivial module. Besides, adding to V_M direct summands which are permutation modules, with no trivial summand, allows to build modules $W := V_M \oplus (perm)$ such that M is also W-endotrivial. Such a construction always results in an absolutely p-divisible module since a non trivial indecomposable permutation kP-module is of the form $k\uparrow_Q^P$ for some $Q \lneq P$. Therefore, one can easily define a universal $\widetilde{V}$, relatively to which any indecomposable capped endo-permutation module is endotrivial, namely

$$\widetilde{V} := \bigoplus_{Q \lneq P} k\uparrow_Q^P \ .$$

This construction leads to the following natural embedding of $D(P)$ in $T_{\widetilde{V}}(P)$, in which the equivalence classes do have a unique indecomposable representative, up to isomorphism.

Theorem 5.0.2. *The Dade group $D(P)$ can be identified with a subgroup of $T_{\widetilde{V}}(P)$ via the canonical injective homomorphism*

$$\begin{array}{ccc} D(P) & \longrightarrow & T_{\widetilde{V}}(P) \\ [M] & \longmapsto & [Cap(M)] \end{array} .$$

Proof. According to the above construction, any indecomposable capped endo-permutation module M is a $\widetilde{V}$-endotrivial module. Since both in $D(P)$ and in $T_{\widetilde{V}}(P)$ there is a unique indecomposable representative for the classes, the map $D(P) \longrightarrow T_{\widetilde{V}}(P) : [M] \longmapsto [Cap(M)]$ is a well-defined, injective group morphism. □

Open Problem 5.0.3. The question that arises naturally at this stage is the question of wether or not $D(P)$ is actually isomorphic to $T_{\widetilde{V}}(P)$.
We shall see in the next sections that if $P = C_{p^n}$, a cyclic p-group, or if $p = 2$ and $P = C_2 \times C_2$, then the answer is positive.

Finally we note that the embedding of $D(P)$ in $T_{\widetilde{V}}(P)$ shall provide us with the right setting to define an analog of the Dade group for an arbitrary finite group G with $char(k) \,|\, |G|$. If P is a Sylow p-subgroup of G, let $\widetilde{V}_G := \bigoplus_{Q \lneq P} k\uparrow_Q^G$. The idea is then to find a subgroup of $T_{\widetilde{V}_G}(G)$, that we shall denote by $D(G)$ and that has many properties in common with the Dade group of a p-group. In particular, the indecomposable representatives of the classes shall be endo-p-permutation modules and if G is a p-group then $D(G)$ coincides with the Dade group. This new group is studied in the Ph.D. Thesis of the author [Las12].

6. $C_2 \times C_2$: THE NORMAL CASE

For a first example, we consider groups G having a normal Sylow 2-subgroup isomorphic to the Klein group $C_2 \times C_2$, which we shall rather denote by V_4 for ease of notation. Furthermore, k shall denote an algebraically closed field of characteristic 2. In this case, we show that relative endotrivial modules are not of much interest, since any group of relative endotrivial modules turns out to be isomorphic to the group $T(G)$ of ordinary endotrivial modules, whose structure is made explicit in [Maz07]. Nonetheless this case is still worth considering because it is a nice source of examples and counter-examples for general behaviours of the groups of relative endotrivial modules.

Theorem 6.0.4. *Let G be a finite group with a normal Sylow 2-subgroup isomorphic to the Klein group V_4. Let V be any absolutely 2-divisible kG-module.*
Then there is a group isomorphism $\varphi : T_V(G) \longrightarrow T(G) : [M]_V \mapsto [M_0]$, where $M \cong M_0 \oplus (V-proj)$ with M_0 indecomposable and V-endotrivial. In particular, if $G = V_4$, then $T_V(G) = <[\Omega(k)]_V> \cong \mathbb{Z}$.

Proof. To begin with, consider the case $G = V_4$ itself. The Klein group is a 2-group, therefore the indecomposable modules that bear chances to be V-endotrivial must have odd k-dimension. By the classification of indecomposable kV_4-modules, the odd-dimensional indecomposable modules are precisely the modules $\Omega^n(k)$, $n \in \mathbb{Z}$, which are all endotrivial modules in the usual sense. In consequence, on the one hand, $<[\Omega(k)]_V> \cong <[\Omega(k)]> = T(G) \cong \mathbb{Z}$ and on the other hand, $T_V(G) \cong T(G)$ via φ. Hence

$$T_V(G) = <[\Omega(k)]_V> \cong \mathbb{Z}$$

although the classes in $T_V(G)$ may contain more modules than the classes in $T(G)$. Now, let G be an arbitrary group with a normal Sylow 2-subgroup isomorphic to V_4. By 4.1.2, a kG-module M is indecomposable V-endotrivial if and only if its restriction $M\downarrow^G_{V_4}$ is indecomposable and $V\downarrow^G_{V_4}$-endotrivial. But we have just shown that any such kV_4-module is in fact an ordinary endotrivial module hence, by the same criterion, M is actually endotrivial. In consequence, φ is a well-defined group homomorphism. The uniqueness of the summand M_0 then yields the bijection. □

Remark 6.0.5. For $G = V_4$, $D(G) \cong T(G)$ (see [Dad78b]), hence the positive answer to Problem 5.0.3 in this case. Also note in the general case that the structure of $T(G)$ is described more accurately in [Maz07, Thm. 2.6] as follows:

$$T(G) = X(G) \oplus <\Omega(k)> \cong X(G) \oplus \mathbb{Z}$$

By Corollary 4.1.3, the indecomposable endotrivial kG-modules consist of all the extensions to G of the kV_4-modules $\Omega^n(k)$, $n \in \mathbb{Z}$, which are given by the family of modules $\Omega^n(k) \otimes k_\omega$ such that $n \in \mathbb{Z}$ and k_ω is a one-dimensional kG-module.

Although there is, up to isomorphism, only one group of relatively endotrivial kG-modules, there are infinitely many different subcategories of V-projective modules which, in particular, do not correspond to projectivity relative to a subgroup.

Lemma 6.0.6. *Let $\lambda \in \mathbb{P}^1(k)$ and $n \geqslant 1$ be an integer. Let $M_{2n}(\lambda)$ be the unique $2n$-dimensional indecomposable kV_4-module with projective variety $\{\lambda\}$. Then the indecomposable modules projective relative to $M_{2n}(\lambda)$ are:*

(a) $IProj(M_{2n}(\lambda)) = \{M_{2m}(\lambda) \mid 1 \leqslant m \leqslant n\} \cup \{kV_4\}$ *if* $\lambda = 0, 1, \infty$;
(b) $IProj(M_2(\lambda)) = \{M_2(\lambda), M_4(\lambda), kV_4\}$ *if* $\lambda \neq 0, 1, \infty$;
(c) $IProj(M_{2n}(\lambda)) = \{M_{2m}(\lambda) \mid 1 \leqslant m \leqslant n\} \cup \{kV_4\}$ *if* $\lambda \neq 0, 1, \infty$ *and* $n \geqslant 2$.

Proof. This is a consequence of the Green ring structure on kV_4 computed by Conlon [Con65]. □

7. The cyclic case: relative endotrivial modules for cyclic p-groups.

Let $G := C_{p^n}$ be a cyclic p-group of order p^n, $n \geqslant 1$, generated by g and k an algebraically closed field of characteristic p. Then $kC_{p^n} \cong k[X]/(X-1)^{p^n}$ as k-algebras and $M_i := k[X]/(X-1)^i$ is the unique indecomposable kC_{p^n}-module of dimension i, up to isomorphism. Moreover, for $1 \leqslant i \leqslant p^n$ this provides a complete list of indecomposable kC_{p^n}-modules, up to isomorphism. In particular, $M_1 = k$, the trivial module, and $M_{p^n} = kC_{p^n}$ is the indecomposable projective module. (See [Thé95, Exercises

5.4, 17.2 and 28.3] for details.) Besides, for all $1 \leqslant i \leqslant p^n$, a simple comparison of dimensions yields $\Omega(M_i) \cong M_{p^n-i}$. Also note that the indecomposable absolutely p-divisible modules are the M_i's with p dividing their dimension i. Finally, according to notation used in [MT07], for all integers $0 \leqslant r \leqslant n$, we shall denote by Z_r the unique cyclic subgroup of P of order p^r, with $Z_0 = 1$, $Z_1 =: Z$ and $Z_n = P$. Thus there are isomorphisms $M_{p^r} \cong k\uparrow_{Z_{n-r}}^{G}$. In this section we shall classify the relative endotrivial modules relatively to any absolutely p-divisible kG-module V.

7.1. **Determination of all types of relative projectivities to modules.** The aim is first to find out all the absolutely p-divisible modules V for which the subcategories $Proj(V)$ are strictly different and secondly to describe explicitly all the modules they contain.

Lemma 7.1.1. *For every integer $1 \leqslant r \leqslant n$, $IProj(M_{p^r}) = \{M_{\alpha p^r} \mid \alpha \in \mathbb{N}, 1 \leqslant \alpha \leqslant p^{n-r}\}$.*

Proof. Let $1 \leqslant r \leqslant n$ and $1 \leqslant \alpha \leqslant p^{n-r}$ be integers. Consider the subgroup $Z_{n-r} \leqslant C_{p^n}$ of index p^r. By 2.0.2 (c), $\mathsf{mod}(kZ_{n-r}) = Proj(k)$. In particular, the kZ_{n-r}-module $M_\alpha \in Proj(k)$ thus, by Lemma 2.1.1, we get

$$M_\alpha \uparrow_{Z_{n-r}}^{C_{p^n}} \in Proj(k\uparrow_{Z_{n-r}}^{C_{p^n}})\,.$$

In addition, by Green's indecomposability theorem, both $M_\alpha \uparrow_{Z_{n-r}}^{C_{p^n}}$ and $k\uparrow_{Z_{n-r}}^{C_{p^n}}$ are indecomposable. Because for every $1 \leqslant i \leqslant p^n$, there is a unique indecomposble kC_{p^n}-module with k-dimension i, it is necessary that $M_\alpha \uparrow_{Z_{n-r}}^{C_{p^n}} \cong M_{\alpha p^r}(C_{p^n})$ and $k\uparrow_{Z_{n-r}}^{C_{p^n}} \cong M_{p^r}$. In other words $M_{\alpha p^r} \in Proj(M_{p^r})$. This yields the inclusion

$$\{M_{\alpha p^r} \mid 1 \leqslant \alpha \leqslant p^{n-r}\} \subseteq IProj(M_{p^r})\,.$$

On the other hand, projectivity relative to the module M_{p^r} is exactly the same thing as projectivity relative to the subgroup Z_{n-r} of C_{p^n}. Therefore, if M is projective relatively to Z_{n-r}, then by 3.1.2 the index $p^r = |C_{p^n} : Z_{n-r}|$ divides $\dim_k(M)$, which proves the second inclusion. □

Corollary 7.1.2. *For every integer $1 \leqslant r \leqslant n$, the collection of kC_{p^n}-modules projective relatively to the kC_{p^n}-module M_{p^r} is given as follows:*

$$Proj(M_{p^r}) = \{\bigoplus_{I\,finite} M_{\alpha_i p^r} \mid \alpha_i \in \mathbb{N} \text{ and } 1 \leqslant \alpha_i \leqslant p^{n-r}\ \forall i \in I\}$$

Lemma 7.1.3. *Let M_i be an indecomposable kC_{p^n}-module such that p^r, with $1 \leqslant r \leqslant n-1$, is the largest power of p dividing $\dim_k(M_i) = i$. Write $i := \alpha_i p^r$ with $1 \leqslant \alpha_i \leqslant p-1$ an integer. Then $Proj(M_i) = Proj(M_{p^r})$.*

Proof. By Lemma 7.1.1, $M_i = M_{\alpha_i p^r} \in IProj(M_{p^r})$. In consequence, $Proj(M_i) \subseteq Proj(M_{p^r})$. In order to show the reverse inclusion, consider again the subgroup Z_{n-r}. Since $p \nmid \alpha_i$, by 2.0.2 (c), $Proj(M_{\alpha_i}) = \mathsf{mod}(kZ_{n-r})$. In particular, the trivial kZ_{n-r}-module $k \in Proj(M_{\alpha_i})$, hence

$$M_{p^r} = k\uparrow_{Z_{n-r}}^{C_{p^n}} \in Proj(M_{\alpha_i}\uparrow_{Z_{n-r}}^{C_{p^n}}) = Proj(M_{\alpha_i p^r})$$

by Green's indecomposability theorem again. Thus $Proj(M_i) \supseteq Proj(M_{p^r})$. □

We shall now show that in $\mathsf{mod}(kC_{p^n})$ projectivity relative to modules is indeed reduced to projectivity relative to subgroups. In other words:

Proposition 7.1.4. *Let V be an absolutely p-divisible kC_{p^n}-module. Then $Proj(V) = Proj(M_{p^r}) = Proj(k\uparrow_{Z_{n-r}}^{C_{p^n}})$ for some subgroup Z_{n-r} of C_{p^n} with $r \geqslant 1$.*

Proof. If V is indecomposable then $V \cong M_i$ for some i divisible by p, and the result has been shown in the preceding lemma. If V is decomposable, write $V := \bigoplus_{i=1}^{s} M_i$, $s \in \mathbb{N}$. Factor every $1 \leqslant i \leqslant s$, as $i := \alpha_i p^{r_i}$ with $1 \leqslant \alpha_i \leqslant p-1$ and $1 \leqslant r_i \leqslant n-1$. Let $m := \min_i\{r_i\}$. Then, using 2.0.2, compute:

$$Proj(V) = Proj(\bigoplus_{i=1}^{s} M_i) = \bigoplus_{i=1}^{s} Proj(M_i) = \bigoplus_{i=1}^{s} Proj(M_{p^{r_i}}) = \bigoplus_{i=1}^{s} Proj(M_{p^m}) = Proj(M_{p^m})$$

where clearly $Proj(M_{p^{r_i}}) \subseteq Proj(M_{p^m})$ either by a classical argument on projectivity relative to subgroups or by Lemma 7.1.1. $\square$

In particular, note that for $G = C_p$ a cyclic group of prime order, there is no relative projectivity to modules other than ordinary projectivity. More generally, we note that there is a unique chain of strict inclusions of subcategories of relatively projective kC_{p^n}-modules given as follows:

$$Proj = Proj(M_{p^n}) \subsetneq Proj(M_{p^{n-1}}) \subsetneq \cdots \subsetneq Proj(M_{p^2}) \subsetneq Proj(M_p)\,.$$

7.2. **Structure of the groups of relatively endotrivial modules.** There are exactly n different proper subcategories of relatively projective modules in $\mathsf{mod}(kC_{p^n})$, given by $Proj(M_{p^r})$ for $0 \leqslant r \leqslant n$ and therefore also n different groups of relatively endotrivial modules: $T_{M_{p^r}}(C_{p^n})$ for $0 \leqslant r \leqslant n$. Besides, since there is a unique indecomposable kC_{p^n}-module for each k-dimension between 1 and p^n, it is clear that every such module is self-dual. Therefore, by Corollary 3.5.2, any group $T_{M_{p^r}}(C_{p^n})$ is an elementary abelian 2-group (or trivial). It remains to figure out their respective ranks. First, we need the following technical result.

Lemma 7.2.1. *Let C_{p^n} be a cyclic p-group with $n \geqslant 2$ and let $1 \leqslant r \leqslant n$ be an integer. Then, there is no indecomposable kC_{p^n}-module, whose k-dimension lies between p^{n-1} and $p^n - p^{n-1}$, which is M_{p^r}-endotrivial.*

The main idea of the proof is based on the following restriction formula (see [Thé95, Exercise 28.3 (a)]):

$$M_i \downarrow_Z^{C_{p^n}} \cong sM_{a+1} \oplus (p^{n-1} - s)M_a \tag{1}$$

with $i = ap^{n-1} + s$, $0 \leqslant s < p^{n-1}$ and $0 \leqslant a < p$, for all $1 \leqslant i \leqslant p^n$.

Proof. The case $p = 2$ is trivial since $2^{n-1} = 2^n - 2^{n-1}$, therefore, we may assume that p is odd. Furthermore, an M_{p^r}-endotrivial module is necessarily M_p-endotrivial since $Proj(M_{p^r}) \subseteq Proj(M_p)$, hence we may also asume that $r = 1$. The indecomposable modules, candidates to be M_p-endotrivial are the indecomposable modules of the form $M_{\beta p \pm 1}$ for some $1 \leqslant \beta \leqslant p^{n-1}$. We claim that, if $p^n < \beta p \pm 1 < p^n - p^{n-1}$, then $M_{\beta p \pm 1}$ is not M_p-endotrivial.
First note that the symmetry given by the Heller operator Ω allows us to consider only the case $\beta p + 1$. The proof proceeds ab absurdo: we assume that $M_{\beta p+1}$ is M_p-endotrivial and compute $\mathrm{End}_k(M_{\beta p+1})\downarrow_Z^{C_p^n}$. Since $p^n < \beta p + 1 < p^n - p^{n-1}$, we have $p^{n-2} \leqslant \beta < p^{n-1}$ and we can write $\beta := \gamma p^{n-2} + \sigma$ with γ and σ integers such that $1 \leqslant \gamma < p - 1$ and $0 \leqslant \sigma < p^{n-2}$. So that

$$\beta p + 1 = (\gamma p^{n-2} + \sigma)p + 1 = \gamma p^{n-1} + \sigma p + 1.$$

Now, $M_{\beta p+1}$ is M_p-endotrivial, thus $\mathrm{End}_k(M_{\beta p+1}) \cong k \oplus (M_p - proj)$ and

$$\mathrm{End}_k(M_{\beta p+1}) \downarrow_Z^{C_{p^n}} \cong k \oplus (M_p \downarrow_Z^{C_{p^n}} -proj)\,.$$

Let us count the number of trivial summands on both sides of this isomorphism. On the right-hand side, there is one modulo p by formula (1) (this easily follows from the fact that M_p-projective modules have dimension divisible by p by 2.2.3). On the left-hand side we get by formula (1):

$$\begin{aligned}\mathrm{End}_k(M_{\beta p+1}) \downarrow_Z^{C_{p^n}} &\cong (M_{\beta p+1} \downarrow_Z^{C_{p^n}}) \otimes (M_{\beta p+1} \downarrow_Z^{C_{p^n}}) \\ &\cong ((\sigma p+1)M_{\gamma+1} \oplus (p^{n-1} - \sigma p - 1)M_\gamma)^{\otimes 2} \\ &\cong (\sigma p+1)^2 (M_{\gamma+1})^{\otimes 2} \oplus 2(p^{n-1} - \sigma p - 1)(\sigma p+1)(M_{\gamma+1} \otimes M_\gamma) \\ &\quad \oplus (p^{n-1} - \sigma p - 1)^2 (M_\gamma)^{\otimes 2}\end{aligned}$$

Since $1 \leqslant \gamma < p - 1$, $p \nmid \dim_k M_\gamma$ and $p \nmid \dim_k M_{\gamma+1}$, but by 2.2.1 there is exactly one trivial summand k in $M_\gamma \otimes M_\gamma$ as well as in $M_{\gamma+1} \otimes M_{\gamma+1}$ and, moreover, k is not a direct summand of $M_{\gamma+1} \otimes M_\gamma$. Therefore, altogether there are $(\sigma p+1)^2 + (p^{n-1} - \sigma p - 1)^2 \equiv 2 \bmod p$ trivial summands in $\mathrm{End}_k(M_{\beta p+1}) \downarrow_Z^{C_{p^n}}$, which is a contradiction. Hence the result. $\square$

For simplicity of notation, we shall, from now on, denote by $\Omega_{M_{p^s}}$ the class of the relative syzygy module $\Omega_{M_{p^s}}(k)$ in $T_{M_{p^r}}(C_{p^n})$ and simply use $\Omega := \Omega_{M_{p^n}}$. The classification theorem is the following.

Theorem 7.2.2. *Let $G := C_{p^n}$ with $n \geqslant 1$ be a cyclic p-group and M_{p^r} with $1 \leqslant r \leqslant n$ be an absolutely p-divisible kC_{p^n}-module.*

(a) *If p is odd, or if $p = 2$ and $r \geqslant 2$, then*

$$T_{M_{p^r}}(C_{p^n}) = <\{\Omega_{M_{p^s}} \mid r \leqslant s \leqslant n\}> \cong \prod_{j=1}^{n-(r-1)} C_2 \,.$$

(b) *If $p = 2$ and $r = 1$, then*

$$T_{M_2}(C_{2^n}) = <\{\Omega_{M_{p^s}} \mid 1 < s \leqslant n\}> \cong \prod_{j=1}^{n-1} C_2 \,.$$

To begin with, the following lemma on the structure of $T_{M_{p^r}}(C_{p^n})$ shall enable us to prove the theorem by induction on the integer n.

Lemma 7.2.3. *Assume $G = C_{p^n}$ with $n \geqslant 2$ and write $C_{p^{n-1}} = C_{p^n}/Z$. Then for every integer $1 \leqslant r \leqslant n$,*

$$T_{M_{p^r}}(C_{p^n}) = \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(T_{M_{p^r}}(C_{p^{n-1}})) \times <\Omega> \cong T_{M_{p^r}}(C_{p^{n-1}}) \times C_2 \,.$$

Proof. Inflation induces an injective group homomorphism $\mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}} : T_{M_{p^r}}(C_{p^{n-1}}) \hookrightarrow T_{M_{p^r}}(C_{p^n})$. The indecomposable representatives for the classes in the image subgroup $\mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(T_{M_{p^r}}(C_{p^{n-1}}))$ are kC_{p^n}-modules whose k-dimension is less than or equal to p^{n-1}. Moreover, as inflation commutes with direct sums, it is clear that $M_i = \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(M_i)$ is M_{p^r}-endotrivial if and only if M_i, seen as a $kC_{p^{n-1}}$-module, is an M_{p^r}-endotrivial $kC_{p^{n-1}}$-module. As seen in Lemma 7.2.1 there is no indecomposable M_{p^r}-endotrivial module with k-dimension between p^{n-1} and $p^n - p^{n-1}$. Furthermore, for all $p^n - p^{n-1} \leqslant i \leqslant p^n$, $M_i \cong \Omega(M_{p^n-i})$ is M_{p^r}-endotrivial if and only if M_{p^n-i} is and for such a module in $T_{M_{p^r}}(C_{p^n})$ we have $[M_i] = [\Omega(M_{p^n-i})] = \Omega + [M_{p^n-i}]$. Whence the direct product

$$T_{M_{p^r}}(C_{p^n}) = \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(T_{M_{p^r}}(C_{p^{n-1}})) \times <\Omega> \;.$$

□

Proof of Theorem 7.2.2. (a) The proof proceeds by induction on n. First, the cyclic p-group of smallest order for which projectivity relative to an indecomposable module of dimension p^r can be considered is C_{p^r}, in which case $Proj(M_{p^r}) = Proj$ as $M_{p^r} \cong kC_{p^r}$. Therefore $T_{M_{p^r}}(C_{p^r}) = T(C_{p^r}) = <\Omega> \cong C_2$ by the classification made in [Dad78b]. Then by the lemma and the induction hypothesis we get:

$$\begin{aligned} T_{M_{p^r}}(C_{p^n}) &= \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(T_{M_{p^r}}(C_{p^{n-1}})) \times <\Omega> \\ &= \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(<\{\Omega_{M_{p^s}} \mid r \leqslant s \leqslant n-1\}>) \times <\Omega> \\ &= <\{\Omega_{M_{p^s}} \mid r \leqslant s \leqslant n-1\}> \times <\Omega> = <\{\Omega_{M_{p^s}} \mid r \leqslant s \leqslant n\}> \\ &\cong \prod_{j=1}^{n-(r-1)} C_2 \end{aligned}$$

since by Corollary 3.5.2 any element of $T_{M_{p^r}}(C_{p^n})$ has order 2.
(b) If $r = 1$, then $T_{M_2}(C_2) = T(C_2) = \{[k]\} \cong \{1\}$. Hence the missing factor C_2 in the product. Notice that $\Omega_{M_2}(k) \cong k$, which is the reason why Ω_{M_2} is not a generator for $T_{M_2}(C_{2^n})$. Nonetheless the set of generators is obtained in like manner as it was in part (a). □

Corollary 7.2.4. *Let C_{p^n} with $n \geqslant 1$ be a cyclic p-group. Then the Dade group $D(C_{p^n}) \cong T_{M_p}(C_{p^n})$.*

Proof. By the description of the Dade group for cyclic p-groups made in [Dad78b], any indecomposable M_p-endotrivial kC_{p^n}-module is an endo-permutation module, therefore the injective homomorphism $D(C_{p^n}) \longrightarrow T_{M_p}(C_{p^n}) : [M] \longmapsto [Cap(M)]$ of Theorem 5.0.2 is an isomorphism. □

Remark 7.2.5. Even though we showed that for cyclic p-groups projectivity relative to modules is reduced to projectivity relative to subgroups, we kept notation using modules rather than subgroups because it was more manageable firstly in the description of relatively projective modules, secondly in computations and thirdly in arguments involving inflation. Nevertheless in the next section, treating the case of groups having a cyclic Sylow p-subgroup, it will be easier to think in terms of subgroups. In this system of notation the groups of relatively endotrivial modules are generated as follows:

$T_{M_{p^r}}(C_{p^n}) = T_{k\uparrow_{Z_{n-r}}^{C_{p^n}}}(C_{p^n}) = <\{\Omega_{k\uparrow_{Z_s}^{C_{p^n}}} \mid 0 \leqslant s \leqslant n-r\}>$ if $M_{p^r} \neq M_2$ and
$T_{M_2}(C_{2^n}) = T_{k\uparrow_{Z_{n-1}}^{C_{p^n}}}(C_{2^n}) = <\{\Omega_{k\uparrow_{Z_s}^{C_{2^n}}} \mid 0 \leqslant s < n-1\}>$.

8. The cyclic case: groups with cyclic Sylow p-subgroups.

In this section G is a finite group having a non-trivial cyclic Sylow p-subgroup $P \cong C_{p^n}$, $n \geqslant 1$. Recall from the previous section that Z_r denotes the unique cyclic subgroup of P of order p^r with $0 \leqslant r \leqslant n$. Moreover, for $0 \leqslant r \leqslant n-1$, one has the following chain of inclusions of subgroups of G: $Z_r < Z_{r+1} \leqslant P \leqslant N_G(P) \leqslant N_G(Z_{r+1}) \leqslant N_G(Z_r) \leqslant G$.

8.1. Determination of the different types of V-projectivities.

To start with we show that the only types of relative projectivity that occur are again the projectivities relative to subgroups.

Proposition 8.1.1. *Let G be a finite group with a cyclic Sylow p-subgroup $P \cong C_{p^n}$ with $n \geqslant 1$.*

(a) *Let V be any kG-module. Then $Proj(V) = Proj(k\uparrow_Q^G)$ for some subgroup Q of P. In particular, V is absolutely p-divisible if and only if Q is a proper subgroup of P.*
(b) *There is a unique chain of proper inclusions of subcategories of relatively projective kG-modules:*

$$Proj \subsetneq Proj(k\uparrow_{Z_1}^G) \subsetneq Proj(k\uparrow_{Z_2}^G) \subsetneq \cdots \subsetneq Proj(k\uparrow_{Z_{n-1}}^G) \subsetneq Proj(k\uparrow_P^G) = \mathsf{mod}(kG)$$

Proof. (a) Recall that, by Lemma 2.1.4, subcategories of relatively projective modules are determined upon restriction to P in the sense that for all $U, W \in \mathsf{mod}(kG)$, $Proj(U\downarrow_P^G) = Proj(W\downarrow_P^G)$ if and only if $Proj(U) = Proj(W)$. First of all, by 7.1.4 there exists a subgroup Q of P such that $Proj(V\downarrow_P^G) = Proj(k\uparrow_Q^P)$. ($Q = P$ in case $V\downarrow_P^G$ is not absolutely p-divisible.) Therefore, by the above remark, in order to show that $Proj(V) = Proj(k\uparrow_Q^G)$, it is enough to check that $Proj(k\uparrow_Q^G\downarrow_P^G) = Proj(k\uparrow_Q^P)$. Indeed, applying the Mackey formula yields

$$k\uparrow_Q^G\downarrow_P^G \cong \bigoplus_{x\in[P\backslash G/Q]} k\uparrow_{{}^xQ\cap P}^P$$

where the subgroups ${}^xQ \cap P$ form a chain of subgroups of $Q = {}^1Q \cap P$, since P is cyclic. Hence $Proj(k\uparrow_{{}^xQ\cap P}^P) \subseteq Proj(k\uparrow_Q^P)$ for all $x \in [P\backslash G/Q]$ so that

$$Proj(k\uparrow_Q^G\downarrow_P^G) = \bigoplus_{x\in[P\backslash G/Q]} Proj(k\uparrow_{{}^xQ\cap P}^P) = Proj(k\uparrow_Q^P)\,.$$

Now, the module V is absolutely p-divisible if and only if $Proj(V) \neq \mathsf{mod}(kG) = Proj(k)$, if and only if $Proj(V\downarrow_P^G) = Proj(k\uparrow_Q^P) \neq Proj(k\downarrow_P^G) = Proj(k)$. Thus by the characterization given in 7.1.4, V is absolutely p-divisible if and only if Q is a proper subgroup of P.

(b) For $G = P$, we have shown in the previous section that there is a unique chain of inclusions of subcategories of relatively projective kP-modules given by

$$Proj \subsetneq Proj(k\uparrow_{Z_1}^P) \subsetneq Proj(k\uparrow_{Z_2}^P) \subsetneq \cdots \subsetneq Proj(k\uparrow_{Z_{n-1}}^P) \subsetneq Proj(k\uparrow_P^P) = \mathsf{mod}(kP)\,.$$

But we proved in (a) that $Proj(k\uparrow_Q^G\downarrow_P^G) = Proj(k\uparrow_Q^P)$ for all subgroup $Q \leqslant P$, therefore another application of Lemma 2.1.4 yields the result.

□

As a corollary of the proof, we obtain that for all $0 \leqslant r \leqslant n$, projectivity relatively to the p-subgroup Z_r of G restricted to a subgroup H of G such that either $P \leqslant H$ or $Z_r \lneqq H \leqslant P$ remains projectivity relative to Z_r, i.e. $Proj(k\uparrow_{Z_r}^G\downarrow_H^G) = Proj(k\uparrow_{Z_r}^H)$. For, we showed in the proof of the proposition that $Proj(k\uparrow_{Z_r}^G\downarrow_P^G) = Proj(k\uparrow_{Z_r}^P)$, but the argument remains true if P is replaced with a subgroup H as given above.

Groups of relatively endotrivial modules are defined only for absolutely p-divisible modules V, in consequence and in view of Proposition 8.1.1, we shall assume for the remainder of the section that $V = k\uparrow_{Z_r}^G$ for some proper subgroup Z_r of P. The remainder of the section is devoted to the determination of the structure of the groups $T_{k\uparrow_{Z_r}^G}(G)$ with $0 \leqslant r < n$.

Remark 8.1.2. Proposition 8.1.1 also gives full control of absolute p-divisibility with respect to restrictions: the restriction of an absolutely p-divisible kG-module V remains absolutely p-divisible whenever either $P \leqslant H$ or $Z_r \lneqq H \leqslant P$. Indeed, we have $Proj(V) = Proj(k\uparrow_{Z_r}^G)$ for some $Z_r \lneqq P$ then, by Lemma 2.1.2 and the above remarks

$$Proj(V\downarrow_H^G) = Proj(k\uparrow_{Z_r}^G\downarrow_H^G) = Proj(k\uparrow_{Z_r}^H) \neq \mathsf{mod}kH\,.$$

Hence $V\downarrow_H^G$ is an absolutely p-divisible kH-module. In consequence, for all subgroups H as above, the restriction maps Res_H^G from the group $T_{k\uparrow_{Z_r}^G}(G)$ are well-defined and all have the form

$$\mathrm{Res}_H^G : T_{k\uparrow_{Z_r}^G}(G) \longrightarrow T_{k\uparrow_{Z_r}^H}(H)\,.$$

8.2. **Properties of the restriction maps and the structure theorem.** To begin with we develop a few more properties of the restriction maps. We shall then use them to deduce the structure of the groups of relatively endotrivial modules $T_{k\uparrow_{Z_r}^G}(G)$ from our knowledge of the structure of $T_{k\uparrow_{Z_r}^P}(P)$.

In order to ease up notation we simply use the symbole Ω_V to denote the class $[\Omega_V(k)]$ in $T_V(G)$ and $k\uparrow_Q$ instead of $k\uparrow_Q^H$ in indices when it is clear to which subgroup $H \leqslant G$ induction goes. We avoid to use a simpler notation like Ω_Q because it has been widely used to denote the class of the ordinary syzygy $\Omega(k)$ in $\mathsf{mod}(kQ)$ in articles concerned with endotrivial and endo-permutation modules.

Lemma 8.2.1. *Let $H \leqslant G$ be a subgroup such that either $P \leqslant H$ or $Z_r \lneqq H \leqslant P$ and Res_H^G : $T_{k\uparrow_{Z_r}}(G) \longrightarrow T_{k\uparrow_{Z_r}}(H)$ be a restriction map. Then:*

(a) $\mathrm{Res}_H^G(\Omega_{k\uparrow_{Z_s}}) = \Omega_{k\uparrow_{Z_s}}$ *for all $Z_s \leqslant Z_r$ so that* $<\{\Omega_{k\uparrow_{Z_s}} \mid 0 \leqslant s \leqslant r\}> \leqslant \mathrm{Im}(\mathrm{Res}_H^G)$.
(b) *If $Z_r \lneqq H \leqslant P$, then Res_H^G is surjective.*

Proof. (a) By 2.3.4, $\Omega_{k\uparrow_{Z_s}}(k)\downarrow_H^G \cong \Omega_{k\uparrow_{Z_s}}(k) \oplus (k\uparrow_{Z_s}^H -proj)$. Hence $\mathrm{Res}_H^G(\Omega_{k\uparrow_{Z_s}}) = \Omega_{k\uparrow_{Z_s}}$.
(b) Follows from (a) since by 7.2.2 the group $T_{k\uparrow_{Z_r}}(H)$ is generated by the set of all relative syzygies $\Omega_{k\uparrow_{Z_s}}(k)$ such that $Z_s \leqslant Z_r$.

□

Corollary 8.2.2. *Let P be a cyclic p-group and Z_r a proper subgroup of P. Then, the restriction maps $\mathrm{Res}_H^P : T_{k\uparrow_{Z_r}}(P) \longrightarrow T_{k\uparrow_{Z_r}}(H)$ are isomorphisms for all $Z_r \lneqq H \leqslant P$.*

Proof. By the lemma Res_H^P is surjective and, by 7.2.2, $|T_{k\uparrow_{Z_r}}(P)| = |T_{k\uparrow_{Z_r}}(H)|$. □

Using the criterion described in Lemma 4.3.1, we can show that the group $T_{k\uparrow_{Z_r}}(G)$ is indeed entirely determined by restriction to $N_G(Z_{r+1})$.

Proposition 8.2.3. *Let G be a finite group with a non-trivial cyclic Sylow p-subgroup P and Z_r be a proper subgroup of P. Then, the restriction map*

$$\mathrm{Res}^G_{N_G(Z_{r+1})} : T_{k\uparrow_{Z_r}}(G) \longrightarrow T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$$

is an isomorphism, with inverse map induced by Green correspondence or alternatively by induction:

$$\begin{aligned} T_{k\uparrow_{Z_r}}(G) &= \{[\Gamma(M)]\,|\, M \text{ is an indecomposable } k\uparrow_{Z_r}\text{-endotrivial } kN_G(Z_{r+1})\text{-module}\} \\ &= \{[M\uparrow^G_{N_G(Z_{r+1})}] \mid [M] \in T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))\} \end{aligned}$$

Proof. The isomorphism and both the descriptions of $T_{k\uparrow_{Z_r}}(G)$ using Green correspondence and induction follow from the criterion given in Lemma 4.3.1. Thus, it suffices to check that

$$Proj(k\uparrow^{N_G(Z_{r+1})}_{Z_r}) \supseteq Proj(\mathcal{Y}),$$

where $\mathcal{Y} = \{\, {}^gP \cap N_G(Z_{r+1}) \,|\, g \in G\backslash N_G(Z_{r+1})\}$. For all $g \in G\backslash N_G(Z_{r+1})$, the subgroup ${}^gP \cap N_G(Z_{r+1})$ is a p-subgroup of ${}^gP$, hence of the form ${}^gZ_l$ for some $l \in \mathbb{N}_n$ since $P \cong C_{p^n}$ is cyclic. Besides, ${}^gZ_l \leqslant N_G(Z_{r+1})$ as well, thus contained in some Sylow p-subgroup of $N_G(Z_{r+1})$, say ${}^hP$ with $h \in N_G(Z_{r+1})$, so that by uniqueness of the subgroup of order p^l in ${}^hP$, we have ${}^gZ_l = {}^hZ_l$. Hence $h^{-1}g$ normalizes Z_l and $g \in hN_G(Z_l) \subseteq N_G(Z_{r+1})N_G(Z_l) \gneqq N_G(Z_{r+1})$ since g does not normalize Z_{r+1}. This forces $N_G(Z_l)$ to contain strictly $N_G(Z_{r+1})$, because the subgroups $N_G(Z_i)$ are totally ordered by inclusion, hence $Z_l \leqslant Z_r$. As a consequence, $Proj({}^gP \cap N_G(Z_{r+1})) = Proj(Z_l) \subseteq Proj(k\uparrow^{N_G(Z_{r+1})}_{Z_r})$ and as required:

$$Proj(\mathcal{Y}) = \bigoplus_{g\in G\backslash N_G(Z_{r+1})} Proj({}^gP \cap N_G(Z_{r+1})) \subseteq Proj(k\uparrow^{N_G(Z_{r+1})}_{Z_r})$$

□

In view of the proposition we can restrict our attention to the groups $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$. First of all, computing the kernel of the restriction map $\mathrm{Res}^{N_G(Z_{r+1})}_P : T_{k\uparrow_{Z_r}}(N_G(Z_{r+1})) \longrightarrow T_{k\uparrow_{Z_r}}(P)$ provides us with a set of generators.

Lemma 8.2.4. (a) *There is an exact sequence*

$$0 \longrightarrow X(N_G(Z_{r+1})) \longrightarrow T_{k\uparrow_{Z_r}}(N_G(Z_{r+1})) \xrightarrow{\mathrm{Res}^{N_G(Z_{r+1})}_P} T_{k\uparrow_{Z_r}}(P) \longrightarrow 0\,.$$

(b) *The group $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$ is a finite abelian group generated by $X(N_G(Z_{r+1}))$ and the $r+1$ relative syzygy modules $\Omega = \Omega_{k\uparrow_1}, \Omega_{k\uparrow_Z}, \ldots, \Omega_{k\uparrow_{Z_r}}$.*

Proof. (a) The map $\mathrm{Res}^{N_G(Z_{r+1})}_P$ is surjective by 8.2.1. In addition, $V := k\uparrow^{N_G(Z_{r+1})}_{Z_r}$ fulfills the hypotheses of Lemma 4.4.1. Indeed, recall from the proof of Proposition 8.1.1 that

$$k\uparrow^{N_G(Z_{r+1})}_{Z_r}\downarrow^{N_G(Z_{r+1})}_P \cong \bigoplus_{g\in[P\backslash N_G(Z_{r+1})/Z_r]} k\uparrow^P_{{}^gZ_r\cap P}$$

where each indecomposable summand $k\uparrow^P_{{}^gZ_r\cap P}$ has a vertex equal to ${}^gZ_r \cap P \leqslant Z_r$, which is strictly contained in ${}^xP \cap P$ for all $x \in N_G(Z_{r+1})\backslash N_G(P)$. Indeed, any such x normalizes Z_{r+1}, thus $Z_r \lneqq Z_{r+1} \leqslant {}^xP \cap P$. Therefore 4.4.1 yields $\ker(\mathrm{Res}^{N_G(Z_{r+1})}_P) = X(N_G(Z_{r+1}))$.

(b) By Theorem 7.2.2, $T_{k\uparrow_{Z_r}}(P) =< \{\Omega_{k\uparrow_{Z_s}} \,|\, 0 \leqslant s \leqslant r\} >$. Now, by 8.2.1, $\Omega_{k\uparrow_{Z_s}}$ is a preimage by $\mathrm{Res}^{N_G(Z_{r+1})}_P$ for the generator $\Omega_{k\uparrow_{Z_s}}$ of $T_{k\uparrow_{Z_r}}(P)$ for all $0 \leqslant s \leqslant r$. Thus $X(N_G(Z_{r+1})) \cup \{\Omega_{k\uparrow_{Z_s}} \,|\, 0 \leqslant s \leqslant r\}$ is a set of generators for $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$. The finiteness of $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$ follows from both that of $T_{k\uparrow_{Z_r}}(N_G(P))$ and of $X(N_H(Z_{r+1}))$.

□

Knowing that the generators $\Omega, \Omega_{k\uparrow_{Z_1}}, \ldots, \Omega_{k\uparrow_{Z_r}}$ of $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$ are preimages for the generators $\Omega_{k\uparrow_{Z_s}}, 0 \leqslant s \leqslant r$, of $T_{k\uparrow_{Z_r}}(P)$ which all have order 2, it remains to identify $2\Omega_{k\uparrow_{Z_s}}$, for all $0 \leqslant s \leqslant r$, with an element of the kernel, that is, a one-dimensional representation of $N_G(Z_{r+1})$. These identifications will follow from an induction argument and use the structure of the group of endotrivial modules $T(G)$ described in [MT07, Thm. 3.2]. This result makes use of a distinguished element of $X(N_G(Z))$, which we need to describe and understand before use.

For Z the unique subgroup of P of order p, let $H := N_G(Z)$ be its normalizer in G. As H acts by conjugation on Z, the quotient $H/C_G(Z)$ embeds as a subgroup of $\mathrm{Aut}(Z) \cong (\mathbb{Z}/p\mathbb{Z})^\times$, thus given $c \in H$, for all $u \in Z$ we have

$$ {}^{c}u = u^{\nu(c)} \text{ for some } \nu(c) \in (\mathbb{Z}/p\mathbb{Z})^\times $$

where in addition $\nu(c)$ can be considered as an element of $k^\times$ via the canonical embedding $\mathbb{Z}/p\mathbb{Z} \hookrightarrow k$. In consequence, the composition $H \longrightarrow H/C_G(Z) \longrightarrow \mathrm{Aut}(Z) \cong \mathbb{Z}/p\mathbb{Z}^\times \longrightarrow k^\times$ defines a linear character of H. For simplicity, ν is then identified with a one-dimensional module in $X(H)$.
In fact, a similar construction can be applied to any subgroup of G which normalizes Z. Furthermore, a Frattini argument applied to H and its normal subgroup $C_G(Z)$ yields the decomposition $H = N_H(P)C_G(Z) = N_G(P)C_G(Z)$, therefore as $C_G(Z)$ acts trivially on Z, ν is entirely defined by its value on $N_G(P)$. In other words, ν can be viewed as a $kN_G(P)$-module which can be extended in a $k\widetilde{H}$-module for all subgroup $\widetilde{H}$ such that $H \geqslant \widetilde{H} \geqslant N_G(P)$, and for ease of notation, we also denote these modules by ν, so that:

$$ \mathrm{Res}^{\widetilde{H}_1}_{\widetilde{H}_2}(\nu) = \nu \quad \text{whenever } H \geqslant \widetilde{H}_1 \geqslant \widetilde{H}_2 \geqslant N_G(P)\,. $$

In particular, we are interested in the subgroup $H_{r+1} := N_G(Z_{r+1})$. Our aim will be to apply an induction argument to its quotient H_{r+1}/Z_r. In this respect, note that P/Z_r is a cyclic Sylow p-subgroup of H_{r+1}/Z_r, Z_{r+1}/Z_r its unique cyclic p-subgroup of order p, moreover $H_{r+1}/Z_r = N_{H_{r+1}/Z_r}(Z_{r+1}/Z_r)$ and $N_{H_{r+1}/Z_r}(P/Z_r) = N_G(P)/Z_r$. Moreover, a Frattini argument yields more precisely

$$ H_{r+1}/Z_r = N_G(P)/Z_r \cdot C_{H_{r+1}/Z_r}(Z_{r+1}/Z_r)\,. $$

Therefore, there is also a corresponding $kN_G(P)/Z_r$-module $\nu = \nu_{N_G(P)/Z_r}$ which extends to H_{r+1}/Z_r. Finally, the following technical result computes the inflation of $\nu_{N_G(P)/Z_r}$ to a $kN_G(P)$-module.

Lemma 8.2.5. *With the notation above we have* $\mathrm{Inf}^{N_G(P)}_{N_G(P)/Z_r}(\nu_{N_G(P)/Z_r}) = \nu_{N_G(P)}$*, that is, by abuse of notation,* $\mathrm{Inf}^{N_G(P)}_{N_G(P)/Z_r}(\nu) = \nu$.

Proof. Let $P := <u \mid u^{p^n} = 1>$. Then $Z = <u^{p^{n-1}}>$, $Z_r = <u^{p^{n-r}}>$ and $Z_{r+1}/Z_r = <u^{p^{n-r-1}}Z_r>$.
Let $d \in N_G(P)$, we have ${}^{d}u = u^j$ for some integer j such that $1 \leqslant j \leqslant p^n$. Then ${}^{d}(u^x) = (u^x)^j$ for all $1 \leqslant x \leqslant n$. Therefore ${}^{d}(u^{p^{n-1}}) = (u^{p^{n-1}})^j$ so that $\nu(d) \equiv j \mod p$. Likewise ${}^{dZ_r}(u^{p^{n-r-1}}Z_r) = (({}^{d}u)Z_r)^{p^{n-r-1}} = (u^jZ_r)^{p^{n-r-1}} = (u^{p^{n-r-1}}Z_r)^j$, hence $\mathrm{Inf}^{N_G(P)}_{N_G(P)/Z_r}(\nu)(d) \equiv j \mod p$. Hence the result. □

Theorem 8.2.6. *Let G be a finite group with a non trivial cyclic Sylow p-subgroup $P \cong C_{p^n}$. For all $0 \leqslant r \lneq n$, let Z_r be the unique proper p-subgroup of P of order p^r and H_{r+1} be its normalizer in G. Then*

$$ \begin{aligned} T_{k\uparrow_{Z_r}}(H_{r+1}) &= < X(H_{r+1}), \{\Omega_{k\uparrow_{Z_s}} \mid 0 \leqslant s \leqslant r\} > \\ &\cong \Big(X(H_{r+1}) \oplus < \{\Omega_{k\uparrow_{Z_s}} \mid 0 \leqslant s \leqslant r\} > \Big) \Big/ \Big([\nu] - 2\Omega_{k\uparrow_{Z_s}}, 0 \leqslant s \leqslant r\Big)\,. \end{aligned} $$

Proof. We need to identify each class $2\Omega_{k\uparrow_{Z_s}}$ with an element of $X(H_{r+1})$. We claim that $2\Omega_{k\uparrow_{Z_s}} = [\nu]$ for all $0 \leqslant s \leqslant r$. The proof proceeds by induction on r. The case $r = 0$ holds by [MT07, Thm 3.2], because projectivity relative to $Z_0 = \{1\}$ is ordinary projectivity, thus $r = 0$ is the ordinary endotrivial case. So we may assume that $r > 0$ and as $T_{k\uparrow_{Z_{r-1}}}(H_{r+1})$ can be seen as a subgroup of $T_{k\uparrow_{Z_r}}(H_{r+1})$, by induction hypothesis, we may assume that the relations $2\Omega_{k\Uparrow_{Z_s}} = [\nu]$ hold for all $0 \leqslant s \leqslant r-1$. Thus it remains to show that $2\Omega_{k\Uparrow_{Z_r}} = [\nu]$.

Factoring out $H_{r+1} = N_G(Z_{r+1})$ by its normal subgroup Z_r enables us to apply the induction hypothesis again to the group $T_{k\Uparrow_{Z_r/Z_r}}(H_{r+1}/Z_r) = T(H_{r+1}/Z_r)$, for which [MT07, Thm. 3.2] provides the relation

$$2\Omega = [\nu] \text{ in } T(H_{r+1}/Z_r), \text{ that is, } 2\Omega_{k\Uparrow_{Z_r/Z_r}} = [\nu] \text{ in } T_{k\Uparrow_{Z_r/Z_r}}(H_{r+1}/Z_r)\,.$$

The following commutative square yields the desired relation for $T_{k\Uparrow_{Z_r}}(H_{r+1})$:

$$\begin{array}{ccc}
T_{k\Uparrow_{Z_r}}(H_{r+1}) & \xhookrightarrow{\ \mathrm{Res}\ } & T_{k\Uparrow_{Z_r}}(N_G(P)) \\
\uparrow {\scriptstyle \mathrm{Inf}_{H_{r+1}/Z_r}^{H_{r+1}}} & \circlearrowleft & \uparrow {\scriptstyle \mathrm{Inf}_{N_G(P)/Z_r}^{N_G(P)}} \\
T_{k\Uparrow_{Z_r/Z_r}}(H_{r+1}/Z_r) & \xhookrightarrow[\ \mathrm{Res}\ ]{} & T_{k\Uparrow_{Z_r/Z_r}}(N_G(P)/Z_r)
\end{array}$$

By Lemma 2.3.4, $\mathrm{Inf}_{H_{r+1}/Z_r}^{H_{r+1}}(\Omega_{k\Uparrow_{Z_r/Z_r}}) = \Omega_{k\Uparrow_{Z_r}}$, therefore, inflationing our relation to $T_{k\Uparrow_{Z_r}}(H_{r+1})$ yields

$$2\Omega_{k\Uparrow_{Z_r}} = [\mathrm{Inf}_{H_{r+1}/Z_r}^{H_{r+1}}(\nu)] \text{ in } T_{k\Uparrow_{Z_r}}(H_{r+1})\,.$$

By the previous lemma $[\mathrm{Inf}_{N_G(P)/Z_r}^{N_G(P)}(\nu)] = [\nu]$, so that the result follows from the injectivity of $\mathrm{Res}_{N_G(P)}^{H_{r+1}}$ (Lemma 4.2.1). □

Remark 8.2.7. Since $2\Omega_{k\Uparrow_{Z_s}} = [\nu] = 2\Omega$ for all $0 \leqslant s \leqslant r$ the generators $\Omega_{k\Uparrow_{Z}}, \ldots, \Omega_{k\Uparrow_{Z_r}}$ can be replaced with the generators $\Omega - \Omega_{k\Uparrow_{Z}}, \ldots, \Omega - \Omega_{k\Uparrow_{Z_r}}$, all of which have order 2. Thus the abelian group $T_{k\Uparrow_{Z_r}}(N_G(Z_{r+1}))$ contains a direct sum of r copies of $\mathbb{Z}/2\mathbb{Z}$.

Finally, using the isomorphism of Proposition 8.2.3, the description by generators and relations of $T_{k\Uparrow_{Z_r}}(N_G(Z_{r+1}))$ extends to $T_{k\Uparrow_{Z_r}}(G)$ which is a finite abelian group generated by the relative syzygy modules $\Omega = \Omega_{k\Uparrow_1^G}, \Omega_{k\Uparrow_Z^G}, \ldots, \Omega_{k\Uparrow_{Z_r}^G}$ and an isomorphic copy of $X(N_G(Z_{r+1}))$, made up of all the classes of the Green correspondents of the one-dimensional $kN_G(Z_{r+1})$-modules, with the relations $2\Omega_{k\Uparrow_{Z_s}^G} = [\Gamma(\nu)]$ for all $0 \leqslant s \leqslant r$.

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EPFL/SB/CTG, Station 8, CH-1015 Lausanne, Switzerland
E-mail address: caroline.lassueur@epfl.ch