2D algebraic model for shoulder abduction

This document is a complementary material to the article "Effect of supraspinatus deficiency on humerus translation and glenohumeral contact force during abduction", by Terrier A. et al. In this 2D model, shoulder abduction is considered in the plane of the scapula: forces and motion are restricted to that plane (figure 1). The ball-socket assumption is used for the glenohumeral joint and muscles are represented by cables. The muscle indeterminate is represented by 5 unknown ratios:

$$r^{MD} = \frac{F^{MD}}{F^{MD}} \equiv 1, r^{AD} = \frac{F^{AD}}{F^{MD}}, r^{PD} = \frac{F^{PD}}{F^{MD}}, r^{SS} = \frac{F^{SS}}{F^{MD}}, r^{IS} = \frac{F^{IS}}{F^{MD}}, r^{SS} = \frac{F^{SC}}{F^{MD}} \qquad (1)$$

relating the force amplitude of each muscle to the one of *MD*. Only *MD* and *SS* can wrap on the humeral head. For these two contacts, the bony side is represented by two rigid circles, with radius R^{MD} and R^{SS}. As in rope-pulley problems, continuous contact forces are replaced by a single force. The abduction angle α is decomposed into glenohumeral α_{gh} and scapulothoracic α_{st} angle according to the scapulohumeral rhythm:

$$\alpha = \alpha_{gh} + \alpha_{st} \ , \ \frac{\alpha_{gh}}{\alpha_{st}} = 2 \qquad (2)$$

At equilibrium,

$$\sum_K \vec{F}^K = 0 \qquad (3)$$

$$\sum_K \vec{X}^K \times \vec{F}^K = 0 \qquad (4)$$

The 8 forces $\vec{F}^K$ (6 muscles, glenoid reaction, and arm weight) are described in a local coordinate system Oxy fixed to the scapula (origin *O* on the humeral head centre and *y* parallel to the glenoid):

$$\vec{F}^K = \begin{pmatrix} F_x^K \\ F_y^K \end{pmatrix} = F^K \begin{pmatrix} \cos \alpha_{F^K} \\ \sin \alpha_{F^K} \end{pmatrix} \tag{5}$$

$$\vec{X}^K = \begin{pmatrix} X_x^K \\ X_y^K \end{pmatrix} = X^K \begin{pmatrix} \cos \alpha_{X^K} \\ \sin \alpha_{X^K} \end{pmatrix} \tag{6}$$

For muscles, the amplitude F^K is unknown, but X^K, α_{X^K} and α_{F^K} are determined geometrically. When the muscle is not in contact with the humeral head, $\vec{X}^K = \vec{I}^K$, where

$$\vec{I}^K = \begin{pmatrix} I_x^K \\ I_y^K \end{pmatrix} = \begin{pmatrix} I_{x0}^K \cos(\alpha_{gh}) - I_{y0}^K \sin(\alpha_{gh}) \\ I_{y0}^K \cos(\alpha_{gh}) + I_{x0}^K \sin(\alpha_{gh}) \end{pmatrix} \tag{7}$$

is the position of each muscle insertion, and $\left(I_{x0}^K, I_{x0}^K\right)$ are $\vec{I}^K$ coordinates in the initial configuration ($\alpha = 0$). Angles α_{X^K} and α_{F^K} are given by

$$\alpha_{X^K} = \begin{cases} \sin^{-1}\left(\dfrac{X_y^K}{X^K}\right), & X_x^K \geq 0 \\ \pi - \sin^{-1}\left(\dfrac{X_y^K}{X^K}\right), & X_x^K < 0 \end{cases} \tag{8}$$

$$\alpha_{F^K} = \pi - \sin^{-1}\left(\frac{O_y^K - X_y^K}{\sqrt{\left(O_x^K - X_x^K\right)^2 + \left(X_y^K - X_y^K\right)^2}}\right) \tag{9}$$

where $\left(O_x^K, O_y^K\right)$ are muscles origin coordinates. When *MD* or *SS* are in contact expressions of X^K, α_{X^K} and α_{F^K} are different: $X^K = R^K$,

$$\alpha_{X^K} = \sin^{-1}\left(\frac{X^K}{O^K}\right) - \sin^{-1}\left(\frac{O_x^K}{O^K}\right) \tag{10}$$

$$\alpha_{F^K} = \alpha_{X^K} + \frac{\pi}{2} \tag{11}$$

with $O^K = \sqrt{\left(O_x^K\right)^2 + \left(O_y^K\right)^2}$. The contact state (contact/no contact) of *MD* and *SS* respectively switches when

$$\alpha_{gh} > \alpha_{gh}^{MD_c} = \sin^{-1}\left(\frac{R^{MD}}{O^{MD}}\right) - \sin^{-1}\left(\frac{O_x^{MD}}{O^{MD}}\right) + \sin^{-1}\left(\frac{R^{MD}}{I^{MD}}\right) - \sin^{-1}\left(\frac{I_X^{MD}}{I^{MD}}\right) \quad (12)$$

$$\alpha_{gh} > \alpha_{gh}^{SS_c} = \sin^{-1}\left(\frac{R^{SS}}{O^{SS}}\right) - \sin^{-1}\left(\frac{O_y^{SS}}{O^{SS}}\right) + \sin^{-1}\left(\frac{R^{SS}}{I^{SS}}\right) - \sin^{-1}\left(\frac{I_Y^{SS}}{I^{SS}}\right) \quad (13)$$

with $I^K = \sqrt{\left(I_X^K\right)^2 + \left(I_Y^K\right)^2}$. The amplitude F^{AW} of the arm weight is fixed, and its orientation is $\alpha_{F^{AW}} = 3/2\,\pi - \alpha_{st}$, according to equation (2). The distance X^{AW} of its application point from the origin is also fixed, and its direction is $\alpha_{X^{AW}} = 3/2\,\pi + \alpha - \alpha_{st}$. The amplitude F^{GH} and direction $\alpha_{P^{GH}}$ of the glenohumeral reaction are unknown, but the frictionless ball-socket joint gives

$$\alpha_{X^{GH}} = \pi - \alpha_{F^{GH}} \quad (14)$$

Finally, only three unknowns remain: F^{MD}, F^{GH} and $\alpha_{P^{GH}}$. The moment equation (4) gives:

$$F^{MD} = F^{AW} \frac{X^{AW}\sin(\alpha)}{\sum_K r^K X^K \sin(\Delta\alpha^K)} \quad (15)$$

with $K = AD, PD, SS, SC, IS$. Note that X^K and $\Delta\alpha^K = \alpha_{F^K} - \alpha_{X^K}$ depend on α through equations (5) to (13). The force amplitude of all other muscles is determined from the ratios r^K equations (1), while the amplitude F^{GH} and orientation $\alpha_{F^{GH}}$ of the glenoid reaction are obtained from the force equation (3).

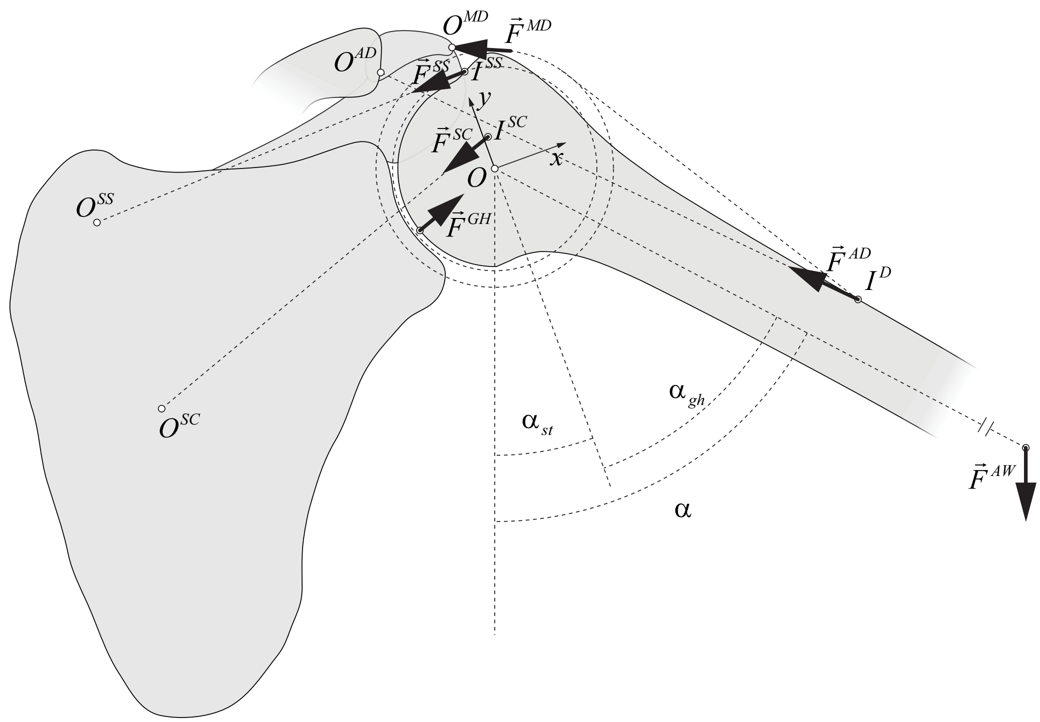

Figure 1. The shoulder model with the local reference frame Oxy, the abduction angles ($\alpha, \alpha_{st}, \alpha_{gh}$), the origin O^K, insertion I^K, muscle forces $\vec{F}^K$, glenoid reaction $\vec{F}^{GH}$, and arm weight $\vec{F}^{AW}$. The two contact circles are also represented.