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Note

New polynomial-time algorithms for Camion bases☆

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Abstract

Let M be a finite set of vectors in $\mathbf{R}^n$ of cardinality m and $\mathscr{H}(M)=\{\{x\in\mathbf{R}^n : c^{\mathrm{T}}x=0\} : c\in M\}$ the central hyperplane arrangement represented by M. An independent subset of M of cardinality n is called a *Camion basis*, if it determines a simplex region in the arrangement $\mathscr{H}(M)$. In this paper, we first present a new characterization of Camion bases, in the case where M is the column set of the node-edge incidence matrix (without one row) of a given connected digraph. Then, a general characterization of Camion bases as well as a recognition procedure which runs in $\mathrm{O}(n^2m)$ are given. Finally, an algorithm which finds a Camion basis is presented. For certain classes of matrices, including totally unimodular matrices, it is proven to run in polynomial time and faster than the algorithm due to Fonlupt and Raco.

Keywords: Camion basis; Graph theory; Hyperplane arrangement; Pivot algorithm

1. Introduction

In what follows, we will consider a matrix as a set of column vectors. Let $M\in\mathbf{R}^{n\times m}$ be a matrix of rank n, $\mathscr{H}(M)=\{\{x\in\mathbf{R}^n : c^{\mathrm{T}}x=0\} : c\in M\}$ and B a basis of M. $\mathscr{H}(M)$ splits up $\mathbf{R}^n$ into a set S of full dimensional cones (regions). A *simplex region* is one which has exactly n facets. B is called a *Camion basis* if the corresponding hyperplanes determine the facets of a simplex region in S. It is known that there always exists a Camion basis. After some column permutations, we may write $M=(BN)$. Define a *signing* of a vector as multiplying it by -1. If A is a matrix, we write $A\geqslant 0$ if each entry of A is nonnegative. It is possible to show that B is a Camion basis if and only if there exists a signing of some columns of M so that $B^{-1}N\geqslant 0$. Geometrically, $B^{-1}N\geqslant 0$ means that the column vectors of N are contained in the cone generated by B. We define $A=B^{-1}N$ and denote by $m'=m-n$ the number of columns of A.

Camion bases were first investigated by Camion [1], who proved that one always exists. The geometric counterpart was studied by Shannon [6], who gave the best lower bound of the number of simplex regions, namely twice the number of distinct hyperplanes in $\mathscr{H}(M)$. Note that this lower bound does not translate immediately to a lower bound for the Camion bases, because one Camion basis might be associated with many (and at least two) simplex regions.

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There is no known polynomial-time algorithm to find a Camion basis in general. Fonlupt and Raco [2] described a finite procedure to find one based on the results of Camion. They also gave an algorithm which runs in time $O(n^3m^2)$ for totally unimodular matrices.

In this paper, we present a new characterization of Camion bases, in the case where M is the column set of the node-edge incidence matrix (without one row) of a given connected digraph. Then, a general characterization of Camion bases as well as a recognition procedure which runs in $O(n^2m)$ are given. Finally, an algorithm which finds a Camion basis is presented. For totally unimodular matrices, it is proven to run in time $O((nm)^2)$.

2. Camion bases of digraphs

Let $G = (V, E)$ be a connected digraph and M the $V \times E$-incidence matrix of G. Let $\tilde{M}$ be any submatrix of M obtained by deleting one row. It is well-known that a basis of $\tilde{M}$ corresponds to a spanning tree of G. Let B be a basis of $\tilde{M}$, $T = (V, E_0)$ the corresponding spanning tree and suppose $\tilde{M} = (BN)$. We call T a *Camion tree* if B is a Camion basis. Let $A = B^{-1}N$. Such a matrix is called a *network matrix* in the literature. It is known that for $e \in E_0$ and an edge $f = (u, v) \in E$:

$$a_{e,f} := \begin{cases} 1 & \text{if the unique } u\text{-}v\text{-path in } T \text{ passes through } e \text{ forwardly,} \\ -1 & \text{if the unique } u\text{-}v\text{-path in } T \text{ passes through } e \text{ backwardly,} \\ 0 & \text{if the unique } u\text{-}v\text{-path in } T \text{ does not pass through } e. \end{cases}$$

The unique closed path going from u to v through f and then from v to u in T is called a *fundamental cycle*. Define an auxiliary graph H associated with T as follows. The set of vertices is E_0 and for $e, f \in E_0$, e and f are *adjacent* if and only if they share a common end-point in G and are in a same fundamental cycle. Then we have:

Proposition 1. *A spanning tree T is a Camion tree if and only if the auxiliary graph H is bipartite.*

Proof. A spanning tree T is a camion tree if and only if there exists an orientation of the edges of G so that for each non-basic edge $g = uv$ of G, all basic edges of the unique u-v-path in T are forward edges. Such an orientation will be called a *proper* orientation.

Suppose that there is a minimal odd cycle $\mathscr{C}$ in H. By minimality of $\mathscr{C}$, the subgraph of H induced by $\mathscr{C}$ has no chord. Using the definition of H, we deduce that the vertices of $\mathscr{C}$ determine a star in T. Denote the central vertex of the star by v_0. Since $\mathscr{C}$ is an odd cycle, for each orientation of G, there are two adjacent vertices in $\mathscr{C}$ corresponding to edges in G that are both either entering v_0 or leaving it. Thus, there is no proper orientation of the edges of T.

Now suppose that H is bipartite. We may suppose H is connected (otherwise what follows can be applied to each connected component of H). Here is a little procedure that finds a proper orientation of G.

> *Procedure Orientation_Propagation*. Choose any element $e_0 \in E_0$ and orient it in an arbitrary way in G. Let $F = \{e_0\}$. Then successively choose an element $e \in E_0 \backslash F$ adjacent in H to some $f \in F$; put e in F and orient it so that $\{e, f\}$ determines a directed path in G.

Denote by $T(F)$ the subgraph of G whose edge set is F. Let us prove by induction on the cardinality of F that during the above procedure, F always satisfies properties (a) and (b) below:

(a) $T(F)$ is a tree.
(b) For all $f_1, f_2 \in F$ such that (f_1, f_2) is an edge in H, the reoriented edges f_1 and f_2 determine a directed path in G.

Let $k = |F|$. If $k = 1$, then clearly (a) and (b) are true. Now suppose $k \geqslant 1$ and let $e \in E_0 \backslash F$, $f \in F$ such that (e, f) is an edge of H. Using the definition of H and the induction hypothesis, we have that $T(F \cup \{e\})$ is a tree in G. Let $N_F(e) = \{g \in F : (e, g) \in H\}$. Since e is a hanging edge of the tree $T(F \cup \{e\})$, from the definition of H we deduce that $N_F(e) \cup \{e\}$ determines a star in G with a central node, say v_0.

Let $f_1, f_2 \in N_F(e)$. As H is bipartite and the subgraph of H induced by F is connected by construction, there is a minimal path of even length in H between f_1 and f_2. Since the path linking f_1 to f_2 is of minimal length, all its vertices correspond to edges of the star. So f_1 and f_2 are both entering into v_0 or leaving this node. It follows that the elements of $N_F(e)$ are all either entering into v_0 or leaving it. Thus we can orient e in such a way that $F \cup \{e\}$ satisfies (b). The orientation of F given by the above procedure induces a proper orientation of G. □

Hoffman and Kruskal [4] and Heller and Hoffman [3] gave a characterization of positive network matrices. A directed graph is called *alternating* if in each circuit the edges are oriented alternately forwards and backwards. In Schrijver [5, pp. 278–279], it is shown that a {0, 1}-matrix is a network matrix if and only if its columns are the incidence vectors of some directed paths in an alternating digraph. To prove the necessary part of the condition, the alternating digraph $G' = (E_0, F')$ is defined, where for edges $e, e' \in E_0$, (e, e') is in F' if and only if the head of e is the same as the tail of e'. In Proposition 1, H is simply a subgraph of G' considered as an unoriented graph.

3. A polynomial-time recognition algorithm

We are going to see a characterization of Camion bases and a procedure that recognizes them in polynomial time. Let us remark that when one of m or n is fixed, there is a trivial polynomial algorithm. We will present an algorithm to check for a given basis B whether B is a Camion basis in time $O(n^2m)$.

Now suppose that B (a basis), N and $A = B^{-1}N$ are given. Consider the bipartite digraph $G(A)$, whose vertex set is the index set of the rows and columns of A and $e = (i, j)$ is an edge of $G(A)$ if and only if $a_{ij} \neq 0$. Set a weight function w on the set of edges:

$$w((i, j)) = \begin{cases} 2 & \text{if } \operatorname{sign}(a_{ij}) = +, \\ 0 & \text{if } \operatorname{sign}(a_{ij}) = -. \end{cases}$$

If C is a cycle of $G(A)$, set $w(C) = \sum_{e \in C} w(e)$. If A' is obtained from A by multiplying a row or a column of A by -1 and keeping the rest of A unchanged, we will say that *A' is obtained from A by a basic signing operation*. The weight function on the edges of the graph $G(A')$ will be denoted by w'. Then, we have the following characterization of Camion bases.

Proposition 2. *A basis B is a Camion basis if and only if each non-oriented cycle of $G(A)$ has a total weight equal to* $0 \bmod (4)$.

Lemma 1. *Assume that A' is obtained from A by successive applications of basic signing operations. If C is a cycle of* $G(A)$, $w'(C) \equiv w(C) \bmod (4)$.

Proof. We can assume that A' is obtained from A by application of a basic signing operation. Note that $G(A) = G(A')$. We can assume that C is an elementary cycle of $G(A) = G(A')$. But clearly $w'(C) = w(C) \pm 4$ or $w'(C) = w(C)$ and the result follows. □

Proof of Proposition 2. First we prove the necessity. Let B be a Camion basis. There exists a matrix $A' \geqslant 0$ which can be obtained from A by successive applications of basic signing operations. If C is a cycle of $G(A') = G(A)$, $w'(C) = 2|C|$ and $w'(C) \equiv 0 \bmod (4)$ since C is even. The result follows from the previous lemma.

To prove the sufficiency part, we can assume that $G(A)$ is connected and we use the procedure below. In particular, the procedure returns a cycle with a total weight equal to $2 \bmod (4)$ if B is not a Camion basis. For a given spanning tree T of $G(A)$ and an edge $e \in G(A) - T$, let C_e denote the unique cycle in $T \cup \{e\}$.

Procedure IS_CAMION

Input: A matrix $A \in \mathbf{R}^{n \times m'}$, where $A = B^{-1}N$ such that $G(A)$ is connected.

Output: Yes (if B is a Camion basis) with a matrix $A' \geqslant 0$ or No (otherwise) with a certificate, that is a cycle in $G(A')$ of total weight equal to $2 \bmod (4)$, where A' is A after some basic signing operations.

(1) Let T be a subset of edges of $G(A)$ which induces a spanning tree of $G(A)$.
Assign to an initial node v of $G(A)$ the label $l(v) = +1$, the other nodes are unlabelled.
(2) **while** $\exists (i, j) \in T$ with i (resp. j) labelled and j (resp. i) unlabelled **do**
label j (resp. i) with the label $+1$ or -1 in such a way that $a_{ij} \cdot l(i) \cdot l(j) > 0$.
(3) Let A' be the matrix obtained by multiplying each row i of A by its label $l(i)$ and each column j of A by its label $l(j)$. If $A' \geqslant 0$, output Yes and A'. Otherwise, there exists $e = (i, j) \notin T$ such that $a'_{ij} < 0$. Output No and C_e.

All the nodes of $G(A)$ are labelled by $+1$ or -1 at the end of this procedure. Let A' be the matrix constructed (at step 3). Note that for all $(i, j) \in T$, $a'_{ij} = a_{ij} \cdot l(i) \cdot l(j) > 0$.

If B is not a Camion basis, there exists $e = (i, j) \notin T$ such that $a'_{ij} < 0$ and $w'(e) = 0$. We have $w'(C_e) \equiv 2 \bmod (4)$ since $w'(C_e) = 2(|C_e| - 1)$ and C_e is even. But by the previous lemma we have also $w(C_e) \equiv 2 \bmod (4)$.

If B is Camion, then for each edge $e = (i, j) \in G(A) - T$, $w'(C_e) \equiv 0 \bmod (4)$ as proved above and since $w'(C_e) = 2(|C_e| - 1) + w'(e)$, it follows that $w'(e) = 2$ and $a'_{ij} > 0$. □

Corollary 1. *There is a polynomial-time algorithm to check whether a basis B is Camion.*

Proof. To check whether a given basis B is a Camion basis, we simply apply the procedure IS_CAMION to the matrix $A = B^{-1}N$. Since the complexity of calculating $B^{-1}N$ is $O(n^2 m)$, computing A dominates the procedure IS_CAMION. Thus, the complexity of checking a given basis B is $O(n^2 m)$. □

4. A new algorithm to find a Camion basis

Suppose that B, N (and $A = B^{-1}N$) are given. If $A = \begin{bmatrix} \alpha & \mathbf{c} \\ \mathbf{b} & D \end{bmatrix}$, where α is a non-zero entry, $\mathbf{b}$ is a column vector, $\mathbf{c}$ is a row vector, and D is a submatrix of A, then the transforsmation

$$A \longrightarrow A' = \begin{bmatrix} \dfrac{1}{\alpha} & \dfrac{\mathbf{c}}{\alpha} \\ -\dfrac{\mathbf{b}}{\alpha} & D - \dfrac{\mathbf{bc}}{\alpha} \end{bmatrix}$$

is called a *pivoting operation* on the element α. Such an operation corresponds to the replacement of a basic vector by a nonbasic one. Remark that a pivoting operation is described in general on the standard matrix $M = [I, A]$. For our purpose, it is convenient to use the "condensed" matrix by ignoring the identity matrix part. Finding a Camion basis is equivalent to transforming the matrix A into a nonnegative one by application of the following two operations:

- pivoting operations,
- basic signing operations.

Signing a row of A is equivalent to signing an associated basic vector. A matrix A' obtained from A after some signing and pivoting operations will be called *equivalent* to A. Denote by $\varepsilon(A)$ the set of matrices that are equivalent to A. Since the number of bases of A and possibilities of signing some columns of A is finite, the cardinality of $\varepsilon(A)$ is finite.

Let us remark that for an implementation of the algorithm that is described below, it is important to maintain the correspondence between the rows of A and the basic vectors and between the columns of A and the nonbasic ones.

We will see an algorithm, called Simp, that runs in $O(\Delta^3 (nm')^2)$ if M is an integral matrix, where Δ is the greatest determinant (in absolute value) of a basis. So, for the particular case of totally unimodular matrices (where $\Delta = 1$), Simp is faster than the algorithm of Fonlupt and Raco [2]. Moreover, the procedure Simp applied to real matrices is also finite.

For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m'\}$, let us denote by A_i the ith row of A and $A_{\bullet j}$ the jth column of A. Moreover, $A^{(i,j)}$ denotes the matrix obtained from A by pivoting on a_{ij}. For any vector or matrix V, the function $\mathrm{sum}(V)$ evaluates the sum of all components of V.

Let us present the algorithm SIMP.

Procedure: SIMP

Input: A real matrix A.

Output: A nonnegative matrix A' (obtained from A by a sequence of pivots and signings).

(1) **for** $i = 1, \ldots, n$ **do**
  if $sum(A_i) < 0$, **then** multiply the ith row by -1.
(2) **for** $j = 1, \ldots, m'$ **do**
  if $A_{\bullet j} \leqslant 0$, **then** multiply the jth column by -1.

(3) **if** $A \geqslant 0$, **return** A.
otherwise, let k, k' and l such that $a_{kl} > 0$ and $a_{k'l} < 0$. **do**
if $sum(A_{\bullet l}) < 1$, **then** pivot on a_{kl}.
otherwise pivot on $a_{k'l}$.
go to step 1.

Theorem 1. *For any real matrix A, the procedure Simp returns a nonnegative matrix after a finite number of steps. Furthermore if M is integral, then Simp runs in time* $O(\Delta^3(nm')^2)$.

Proof. Suppose that $\text{sum}(A_i) < 0$ at step 1 or $A_{\bullet j} \leqslant 0$ at step 2 for some i or some j. Let A' be the matrix obtained from A by multiplying its ith row by -1 (at step 1) or its jth column by -1 (at step 2). Clearly, we have $\text{sum}(A') > \text{sum}(A)$.

At the beginning of step 3, $\text{sum}(A_i) \geqslant 0$ for all i and there is no column $A_{\bullet j}$ such that $A_{\bullet j} \leqslant 0$. So, if there exists an entry $a_{k'l} < 0$, then there exists k such that $a_{kl} > 0$. Denote by $\mathbf{c}$ the ith row of A without the lth column and $\mathbf{b}$ the lth column of A without the ith row. We have that

$$A^{(i,l)} - A = \begin{pmatrix} \frac{1}{\alpha} - \alpha & \frac{\mathbf{c}}{\alpha} - \mathbf{c} \\ -\frac{\mathbf{b}}{\alpha} - \mathbf{b} & -\frac{\mathbf{bc}}{\alpha} \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} 1 - \alpha \\ -\mathbf{b} \end{pmatrix} \cdot (1 + \alpha \ \mathbf{c}).$$

(Note that in the middle of this equation, the matrix is written as if $i = l = 1$.) Thus,

$$\text{sum}(A^{(i,l)}) - \text{sum}(A) = \frac{1 - \text{sum}(A_{\bullet l})}{a_{il}} (\text{sum}(A_i) + 1) \quad \text{for } 1 \leqslant i \leqslant n.$$

So, if $\text{sum}(A_{\bullet l}) \neq 1$, $\text{sum}(A)$ will increase by pivoting either on a_{kl} or $a_{k'l}$. If $\text{sum}(A_{\bullet l}) = 1$, then $\text{sum}(A)$ neither increases nor decreases. However, since $\sum_{j \neq l} a_{k'j} \geqslant 0$ and $a_{k'l} < 0$, we deduce that $\text{sum}(A_{k'}^{(k',l)}) = 1/a_{k'l}(1 + \sum_{j \neq l} a_{k'j}) < 0$. Therefore, $\text{sum}(A)$ will increase at the next step 1.

As $|\varepsilon(A)| < \infty$, it follows that Simp generates a well-oriented matrix after a finite number of steps.

Now, suppose that M is integral. By Cramer's rule, for each matrix A' equivalent to A, we have $a'_{ij} = \pm \det(B'')/\det(B')$ $\forall i, j$, where B', B'' are two bases of M. Since $\det(B'') \leqslant \Delta$ and $|\det(B')| \geqslant 1$, we deduce that $a'_{ij} \leqslant \Delta \ \forall i, j$. Moreover, if $A' = (B')^{-1}N'$ where B' is a basis and N' a submatrix of M after some signing operations of columns of M, then we may write each entry of A' as a fraction of an integer over $\det(B')$. Thus $\text{sum}(A')$, respectively $\text{sum}(A)$, is a ratio of some integer to $\det(B')$, respectively $\det(B)$. So, if A' is obtained from A at some step and $\text{sum}(A') - \text{sum}(A) > 0$, then $\text{sum}(A') - \text{sum}(A) \geqslant 1/\Delta^2$. Since $\text{sum}(A)$ never decreases, but increases after at most two passages at step 1 and is between $-\Delta nm$ and Δnm, the number of passages at step 1 is $O(\Delta^3 nm)$.

Since the number of elementary operations at steps 1,2 and 3 is $O(nm)$, we conclude that the complexity of SIMP is $O(\Delta^3(nm)^2)$. □

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