Effect of Spatio-Temporal Channel Correlation on the Performance of Space-Time Codes

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Abstract **–The determinant and rank criteria used for space-time code design apply at high SNR [1]. Code design metrics developed for low SNR assume a channel autocorrelation matrix with equal eigenvalues [2], which does not hold in many practical scenarios. This paper shows that a space-time code designed to ensure full diversity at high SNR can suffer significant degradation when implemented at low-to-medium SNR because of the channel autocorrelation profile.**

We examine the effect of the channel autocorrelation matrix on a space-time code's performance and discuss how knowledge of this matrix can be used for code design, particularly from the aspect of space-time trellis code minimum memory requirements. Our discussion applies to both flat-fading and frequency-selective channels that are treated in a unified manner.

I. INTRODUCTION

Current space-time code designs minimize the largest average pairwise error probability (PEP). For large signal-to-noise ratio (SNR) and Rayleigh fading this is achieved by maximizing the diversity and coding gains [1]. It has been observed [3], [2] that these criteria give a good performance indication but do not fully quantify the performance of space-time codes and do not always lead to consistent code designs. The reason for the performance discrepancy is that only the minimum distance error event is examined. Transfer function bounds that take multiple error events into account (e.g. [3]) allow a better assessment of a space-time code's performance but are too complicated for code design.

On the other hand, the diversity and coding gain criteria are simple enough to be used in computer searches and have helped identify several good codes but do not always lead to consistent code design, especially when the channel taps are correlated in space and/or time.

For flat fading, it is reasonable to assume that the channels (between each transmit-receive antenna pair) have the same power; however, they are typically correlated. This correlation affects the eigenvalues of the spatio-temporal correlation matrix. We denote by CCME the channel correlation matrix eigenvalues. Even low channel correlation may result in significantly different CCME. For frequency-selective fading, unequal CCME is almost always the case.

Diversity order is the slope of the PEP in a log-log scale for large SNR. If the channel auto-correlation matrix has full rank, the code can achieve full diversity at high SNR. As observed in [2], we are interested in the code's performance at a finite SNR. Two codes that have the same asymptotic diversity may converge to their theoretical limit with different rates in the finite SNR range of interest. The rate of convergence depends on CCME and is not represented in the traditional rank and determinant design criteria.

Space-time codes designed for uncorrelated flat-fading channels demonstrate a certain robustness to spatial correlation [4], i.e. unequal CCME, in the sense that the space-time code's performance does not significantly degrade. However this does not necessarily imply that they achieve the best possible performance over the correlated channels.

Another interesting issue is the trade-off between decoder complexity and code performance. For space-time trellis codes, the number of memory elements in the encoder is an upper limit on the asymptotic diversity the code may achieve and determines the decoder complexity. For a practical system that operates in a finite SNR range and has complexity constraints, it is preferable to use the minimum number of memory elements required to achieve the highest possible slope in this SNR range. This is not necessarily the number of memory elements required to achieve the maximum diversity available at infinite SNR.

In this paper, we examine some effects of the channel autocorrelation matrix on a space-time code's performance, and discuss how knowledge of this matrix may be used for code design, particularly from the aspect of space-time trellis code minimum memory requirements. Our discussion applies to both flat-fading and frequency-selective channels that are treated in a unified manner.

The paper is organized as follows. Section II reviews basic background material and presents the system's model. Section III examines the minimum value

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achieved by the pairwise average probability bound for a specific channel autocorrelation matrix and PEP. The analysis offers some insight on the performance of different space-time codes over correlated channels. Section IV defines memory in terms of decoder complexity for space-time trellis codes and suggests minimum memory requirements for a code at a specific SNR value. Section V presents simulation results and Section VI concludes the paper.

II. System model

To simplify the discussion, we consider a system that employs M_T-transmit and one-receive antennas. The analysis can be generalized to multiple receive antennas.

The input sequence is encoded by a space-time (ST) code to produce the signals $\mathbf{s}_i$ that are simultaneously transmitted over the M_T channels $\mathbf{h}_i$. Each channel is modeled as an FIR filter with L taps, i.e. $\mathbf{h}_i = [h_i(0) \ \ldots \ h_i(L-1)]$ for $i = 1 \ldots M_T$. For $L = 1$, we have flat fading, while for $L > 1$, we have frequency-selective fading. We assume that the channels $\mathbf{h}_i$ remain constant over the transmission of a block of N symbols and vary independently from block to block (quasi-static fading assumption).

The observed output can be expressed as

$$\mathbf{y} = \mathbf{hS} + \mathbf{z}, \tag{1}$$

where $\mathbf{h} = [\mathbf{h_1} \ \ldots \ \mathbf{h_{M_T}}]$, $\mathbf{S}' = [\mathbf{S'}_1 \ \ldots \ \mathbf{S'}_{M_T}]$, $\mathbf{y}$ and $\mathbf{z}$ are N-vectors, and $\mathbf{S}_i$, $i = 1, \ldots, M_T$ are Toeplitz matrices of dimension $L \times N$

$$\mathbf{S}_i = \begin{bmatrix} s_i(L-1) & \ldots & s_i(N+L-1) \\ s_i(L-2) & \ldots & s_i(N+L-2) \\ \vdots & & \vdots \\ s_i(0) & \ldots & s_i(N-1) \end{bmatrix}. \tag{2}$$

We assume that $\mathbf{z}$ is Additive White Gaussian Noise (AWGN) with auto-correlation matrix $R_z = E\mathbf{z}^H\mathbf{z} = 2\sigma^2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

The PEP between codewords $\mathbf{S}^i$ and $\mathbf{S}^j$ can be upper bounded by the Chernoff bound

$$Prob(\mathbf{S}^i \rightarrow \mathbf{S}^j|\mathbf{h})) \leq e^{-\mathbf{h}\mathbf{E}_{ij}\mathbf{h}^H\gamma}, \tag{3}$$

where $\mathbf{E}_{ij} = [\mathbf{S}^i - \mathbf{S}^j][\mathbf{S}^i - \mathbf{S}^j]^H$, $\gamma = E_s/4N_o$ is proportional to the SNR, and $(.)^H$ denotes the conjugate-transpose. Averaging over all channel realizations $\mathbf{h}$ leads to the well-known upper bound [1]

$$PEP \leq E_{\mathbf{h}} e^{-\mathbf{h}\mathbf{E}_{ij}\mathbf{h}^H\gamma} = det(\mathbf{I} + \gamma\mathbf{E}_{ij}\mathbf{R}_h)^{-1}, \tag{4}$$

where $E_{\mathbf{h}}$ denotes the averaging, $det(.)$ denotes the determinant and $\mathbf{R}_h = E\mathbf{h}^H\mathbf{h}$ is the channel auto-correlation matrix. Typically space-time code design tries to minimize the bound in (4) for the maximum average pairwise probability. For large γ and $\mathbf{R}_h = p\mathbf{I}$, i.e. multiple of identity for any $p > 0$ (corresponding to uncorrelated equal-power channels), this can be achieved by maximizing the *diversity* (or equivalently the rank of matrix $\mathbf{E}_{ij}$) and the *coding gain* (or equivalently the product of the nonzero eigenvalues of $\mathbf{E}_{ij}$).

III. Performance bound

In this section we investigate the minimum value the bound in (4) may take for a specific channel autocorrelation matrix and PEP. Consider the eigen-decomposition of the positive semi-definite matrices $\mathbf{R}_h = \mathbf{U}_h\mathbf{\Sigma}_h\mathbf{U}_h^H$ and $\mathbf{E}_{ij} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^H$. Diagonal elements of $\mathbf{\Sigma}_h$ and $\mathbf{\Sigma}$, σ_i^h and σ_i, are ordered : $\sigma_i^h \geq \sigma_{i+1}^h$ and $\sigma_i \geq \sigma_{i+1}$. If we assume that $\mathbf{R}_h$ and $\mathbf{E}_{ij}$ have full rank and $\mathbf{U}^H\mathbf{U}_h = \mathbf{I}$, then, using theorem 3.3.14 of [5], we obtain from (4)

$$log(PEP) \leq -\sum_i log(1 + \gamma\sigma_i\sigma_i^h), \tag{5}$$

If the condition $\mathbf{U}^H\mathbf{U}_h = \mathbf{I}$ is satisfied, we say that the codeword eigenvectors are matched to the channel eigenvectors.

Note that if for any constant a, the minimum-distance codeword difference is equal to $\mathbf{E}_{ij} = a\mathbf{I}$, then the code is matched in terms of eigenvectors to all channel autocorrelation matrices, since we can always write $\mathbf{I} = \mathbf{U}_h\mathbf{U}_h^H$ for any unitary $\mathbf{U}_h$. For example, in the case of one-memory-element delay-diversity code with two-transmit and one-receive antennas over a flat fading channel [1] $\mathbf{E}_{ij} = d_{min}^2\mathbf{I}$ (where d_{min}^2 is the smallest constellation distance). Alamouti's block code [6] over flat-fading also has smallest distance codeword differences that lead to $\mathbf{E}_{ij}$ equal to a multiple of identity.

Similarly, if $\mathbf{R}_h = p\mathbf{I}$ for any p, which is a typical assumption for space-time code design, then again there is no need to match the codeword eigenvectors to the channel eigenvectors.

Finally, assume that $\mathbf{E}_{ij} \neq a\mathbf{I}$, $\mathbf{R}_h \neq p\mathbf{I}$, and that the channel autocorrelation matrix $\mathbf{R}_h$ is known at the transmitter[1]. Then, it is straightforward to match $\mathbf{E}_{ij}$ (for the smallest distance codeword difference $\mathbf{E}_{ij}$) to the channel eigenvectors by transmitting the signal $\mathbf{S}' = \mathbf{U}_h\mathbf{U}^H\mathbf{S}$

[1] Note that $\mathbf{R}_h$ knowledge implies that only the statistics of the channel (not instantaneous values) are known at the transmitter, which is reasonable to assume.

instead of $\mathbf{S}$. Indeed we have

$$\mathbf{E}_{ij}^{'}\mathbf{E}_{ij}^{'H} = \mathbf{U}_h\mathbf{U}^H\mathbf{E}_{ij}\mathbf{U}\mathbf{U}_h^H = \mathbf{U}_h\mathbf{\Sigma}\mathbf{U}_h^H.$$

We conclude that mismatch in the eigenvectors can easily be taken into account. Thus for the rest of the paper we will assume that the code satisfies the bound in (5) and focus on the effect of unequal CCME on performance.

Equation (5) leads to some interesting observations. First, assume that we have flat fading and the capability to transmit unequal power levels γ_i over the different channels i, i.e.

$$log(PEP) \leq -\sum_i log(1+\gamma_i\sigma_i\sigma_i^h), \qquad (6)$$

subject to the total transmit power constraint $\sum_i \gamma_i = \gamma M_T$. Then, the optimal power allocation follows from water-filling on $\sigma_i\sigma_i^h$ [7]. An interesting question that motivates the development in Section IV is: Given that we are going to perform water-filling for each γ_i, and assuming that $\sum_i \sigma_i^h = const$, what are the most desirable CCME σ_i^h in terms of PEP? The answer is that it depends on γ: the PEP curves that correspond to different σ_i^h actually intersect. For very small γ, it is optimal to concentrate all the transmit power at only one channel, while for large γ it is optimal to equally divide the power among the available channels. This implies that for finite SNR, having unequal CCME does not necessarily cause a performance degradation. In fact, for finite SNR, unequal CCME may result in a better PEP performance.

A second observation is that given specific CCME and the power constraint constant γ, the eigenvalues of the matrices $\mathbf{E}_{ij}\mathbf{E}_{ij}$ should be matched to the channel's eigenvalues, as opposed to being equal.

IV. Space-time Trellis code memory and diversity.

In this section, we relate the decoder complexity of a space-time trellis code to the code's performance.

A. Effective memory

A convolutional code with m memory elements, k inputs, and n outputs can be described by the state-space equations

$$\mathbf{s}_{k+1} = \mathbf{A}\mathbf{s}_k + \mathbf{B}\mathbf{u}_k, \quad \mathbf{y}_k = \mathbf{C}\mathbf{s}_k + \mathbf{D}\mathbf{u}_k$$

where $\mathbf{s}_k$ is the $1 \times m$ state vector $\mathbf{u}_k$ is the $1 \times k$ input vector, $\mathbf{y}_k$ is the $1 \times n$ output vector at time k, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are binary matrices. This encoder has a trellis representation with 2^m states. For a feedforward encoder, it always holds that $\mathbf{A}^m = \mathbf{0}$. We define as *effective memory* of a feedforward encoder the smallest integer $m_o \leq m$ such that $\mathbf{A}^{m_o} = \mathbf{0}$. Effective memory of m_o means that the current output at time k depends on the $m_o + 1$ most recent inputs.

A joint trellis decoder/equalizer over channels with L taps and constellation size C requires C^{L+m_o} states, since the channel output at time k can always be expressed as a function of only $m_o + L + 1$ inputs. Thus the effective memory m_o rather than the number of encoder memory elements m determines the required decoder complexity.

As an example, consider the 8-PSK space-time trellis code with 8-states [1] for two transmit and one receive antennas. This code has $m = 3$ memory elements, but effective memory $m_o = 1$. Thus, a joint trellis decoder/equalizer requires C^{L+1} states. A general trellis codes with $m = 3$ memory elements could require up to C^{L+3} states.

B. Memory and diversity

Consider a system with M_T transmit antennas over channels with L taps each, and assume a full-rank channel auto-correlation matrix. The following propositions relate the effective memory of a space-time trellis code to the maximum diversity it can achieve.

Proposition 1 *To achieve diversity order of $M_T L$ (maximum possible) it is necessary to have effective memory $m_o \geq (M_T - 1)L$. Generally, with effective memory $m_o \leq M_T - 1$, we can achieve diversity order of at most $(m_o + 1)L$.*

Proposition 2 *For a trellis code with k inputs, to achieve effective memory of m_o, it is necessary to use at least m_o memory elements, and it is not necessary to use more than km_o memory elements.*

The proofs are straightforward and omitted. Note that to achieve effective code memory m_o with exactly m_o memory elements, these memory elements have to be connected in series. It is also worth emphasizing that, from a decoder's point of view, only the effective memory determines its complexity and not the number of encoder memory elements. Thus, we can increase the number of encoder memory elements to achieve a better coding gain without increasing the decoder complexity.

V. Simulation results

In this section, we support with simulation results the observation that, at a finite SNR, codes with different diversity levels can exhibit similar performance over channels with full-rank channel auto-correlation matrix but different CCME.

Such a channel is the EDGE [2] typical urban (TU) channel which is a frequency-selective channel with four taps and full-rank auto-correlation matrix. We consider two-transmit and one-receive antennas, where each of the two channels is an EDGE-TU channel. The maximum achievable diversity is equal to $LM_T = 8$.

The 8-state 8-PSK space-time trellis code [1] has effective memory of one. This specific code can achieve diversity of at most $L + 1 = 5$ over our example channel. On the other hand, the time-reversal space-time block code (TR-STBC) [8] may be thought of as having effective memory equal to half the block length, and can achieve full diversity (order 8) over the two EDGE channels. Our simulations show that the Frame Error Rate curve for these two codes over the EDGE channel has the same slope (approximately equal to 2), although they have different diversity levels.

The EDGE channel auto-correlation matrix can be expressed as $R_h = \sum_{i=1}^{4} \lambda_i u_i u_i^H$, where λ_i are the CCME and u_i the eigenvectors. Figure 1 compares the performance of the TR-STBC over the full-rank EDGE channel and on a channel with the rank-1 auto-correlation matrix $\mathbf{R}_h^1 = \lambda_1 u_1 u_1^H$ where u_1 (λ_1) is the dominant eigenvector (eigenvalue) of $\mathbf{R}_h$. As expected intuitively, the two channels result in almost identical performance for SNR up to 20 dB (corresponding to $log(E_s/4N_o) = 2.53$).

The first point we try to make is that for code design, the minimum required effective memory for good performance can be based on the CCME. Transmit power water-filling according to the CCME leads to zero power for some channels (temporal or spatial), which in turn can serve as a guideline for the maximum achievable diversity (and thus effective memory) at the finite SNR value (or range) of interest. For our simulations over the EDGE-TU channel with two-transmit and one-receive antennas, power water-filling at 20dB SNR (assuming equal code eigenvalues) indeed suggests distributing transmit power only to the two largest overall CCME.

A second set of simulation results considers two-transmit antennas over flat-fading channels with channel auto-correlation matrix and corresponding eigenvalues equal to

$$\mathbf{R}_h = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}, \lambda_{1,2} = 1 \pm b \, .$$

Figures 2 and 3 depict the BER and log(Frame Error rate) performance of the Delay-Diversity code with one memory element, with and without power water-filling. The power water-filling is performed separately for each γ.

[2] EDGE stands for *Enhanced Data Rates for Global Evolution* and is a 3G TDMA cellular standard.

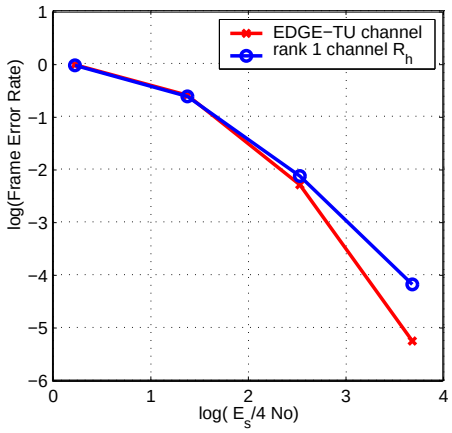

Fig. 1. Performance of the TR-STBC over the two EDGE channels each with normalized CCME [0.9981 0.0618 0.0007 0.0002] and two channel s each with CCME [1 0 0 0].

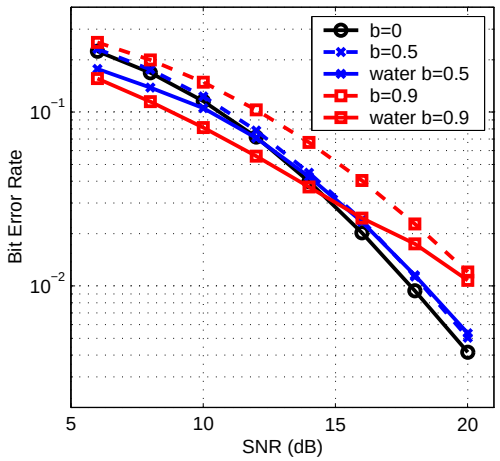

Fig. 2. BER performance of the Delay-Diversity code with 2 transmit antennas over correlated flat-fading channels with eigenvalues $\lambda_{1,2} = 1 \pm b$, with and without power water-filling.

Two remarks are in order. First, power water-filling offers a big performance improvement for low SNR and large CCME difference. Second, at low γ, unequal

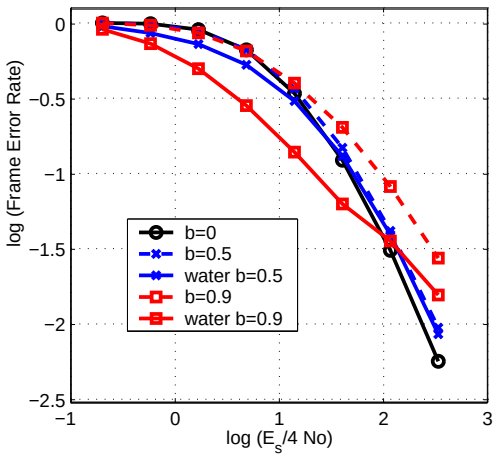

Fig. 3. PEP performance of the Delay-Diversity code with 2 transmit antennas over flat correlated channels with eigenvalues $\lambda_{1,2} = 1 \pm b$, with and without power water-filling.

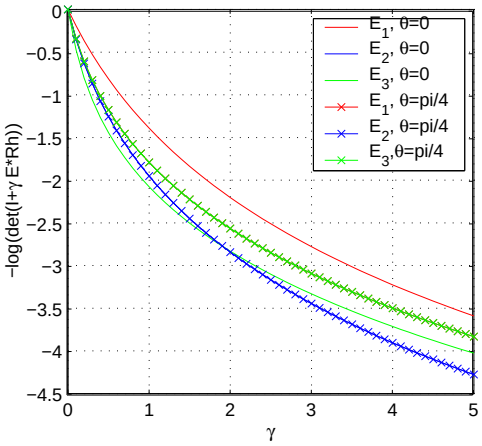

Fig. 4. PEP bounds corresponding to the three $\mathbf{E}_k$ events, for $\mathbf{U}_h(\theta = 0)$ and $\mathbf{U}_h(\theta = \pi/4)$ and CCME: $\sigma_1^h = 1.7$, $\sigma_2^h = 0.3$.

CCME may lead to better performance than equal CCME if transmit power water-filling is performed.

Finally, we investigate the effect of eigenvector matching for the 8-state 8-PSK trellis space-time code [1]. Let $d_1 = 0.5858$, $d_2 = 2$ and $d_3 = 3.4142$ be the three smallest 8-PSK constellation distances. The minimum distance events for this code are $\mathbf{E}_1 = diag[d_1, d_3]$, $\mathbf{E}_2 = diag[d_3, d_1]$, and $\mathbf{E}_3 = diag[d_2, d_2]$.

Consider a channel with unequal eigenvalues $\sigma_1 > \sigma_2$, and $\mathbf{\Sigma}_h = diag[\sigma_1, \sigma_2]$. Each of the $\mathbf{E}_k$, $k = 1, 2, 3$ leads to a different bound

$$log(PEP) \leq -log(det(\mathbf{I} + \gamma \mathbf{E}_k \mathbf{U}_h \mathbf{\Sigma}_h \mathbf{U}_h^H)), \quad (9)$$

where the 2×2 unitary matrix $\mathbf{U}_h$ can be expressed without loss of generality as a function of a rotation angle θ

$$\mathbf{U}_h = \mathbf{U}_h(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}, \quad \theta \in [0, \pi]. \quad (10)$$

The bound in (9) achieves its minimum value for $\mathbf{U}_h(\theta = \pi/2)$ if the error event corresponds to $\mathbf{E}_1$, and for $\mathbf{U}_h(\theta = 0)$ if the error event corresponds to $\mathbf{E}_2$. This intuitively corresponds to waterfiling: matching the largest codeword eigenvalue to the largest CCME. For $\mathbf{E}_3$ both choices lead to the same bound. Since both $\mathbf{E}_1$ and $\mathbf{E}_2$ occur with the same probability, their average is minimized for $\mathbf{U}_h(\theta = \pi/4)$, as Fig. 4 shows.

VI. Conclusions

This paper argued that for spatially or temporally correlated channels at low-to-moderate SNR, the exact CCME determine the achievable space-time code performance. Thus it may not always be necessary to design for maximum diversity over spatially or temporally correlated channels. Space-time code design should take into account the CCME, the operating SNR, and decoder complexity to achieve the best performance-complexity trade-off.

We introduced the notion of effective memory for the encoder and related it to the decoder complexity. The effective memory upper limits the diversity order achieved by a space-time code.

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