Asymptotic Capacity of Multi-Level Amplify-and-Forward Relay Networks

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***Abstract*— This paper analyzes the capacity of a wireless relay network composed of a large number of nodes that operate in an amplify-and-forward mode and that divide into a fixed number of levels. The capacity computation relies on the study of products of large random matrices, whose limiting eigenvalue distribution is computed via a set of recursive equations.**

I. Introduction

A typical relay network consists of three groups of nodes: the sources, the destinations and the relays. In one-direction transmission, the messages are sent from the sources to the destinations with assistance from the relays. A special case is when there is no direct link between the information sources and the destinations. Direct links are absent when the sources and the destinations are far from each other or when there are obstructions between the sources and the destinations. Therefore, the sources must first send the messages to the relays. Then the relays forward the received signals to the destinations. In an amplify-and-forward (AF) relay network, the relays simply scale the received signals according to their power constraint and forward the scaled signals to the destinations. This amplify-and-forward operation is a reasonable strategy when relays have a limited computation power and no centralized control nor feedback exists. The AF relay network is illustrated in Figure 1.

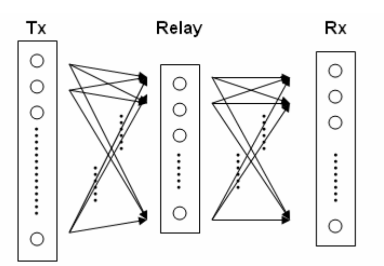

Fig. 1. AF relay network

When the sources and the destinations are too distant or numerous obstructions exist, multiple levels of relays may be needed. In a multi-level AF relay network, the first level of relays amplify the received signals and forward the amplified signals to the next level of relays. The next level of relays repeats the same operation, and so on. Finally, the last level of relays forwards the signals to the destinations. Figure 2 illustrates the multi-level AF relay network.

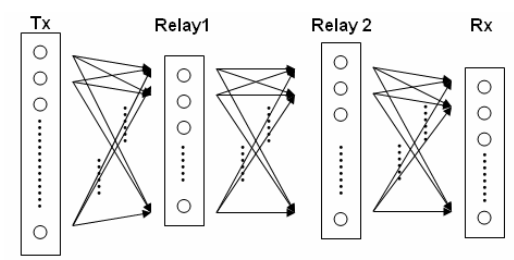

Fig. 2. 2-level AF relay network

The multi-level amplify-and-forward strategy allows saving of both computation time and energy at the relays, at the price of noise amplification at each level. This paper quantifies precisely the loss incured by this noise amplification when the number of nodes gets large, but the number of relay levels remains fixed. The capacity of the multi-level AF scheme is the expectation of the log determinant of a product of random matrices. This work needs classical tools from Random Matrix Theory throughout, as developed in [1]-[4].

The analysis of the capacity-scaling behavior of AF relay networks has also been conducted by Morgenshtern and Bolcskei in [5]-[7]. They use results in [4], but consider only one level of relays. This paper modifies Morgenshtern and Bolcskei's approach to obtain a general formula for multi-level AF relay networks. Multi-level AF networks have also been considered by Borade and Zheng in [8], but in the high SNR regime. Our contribution is a fixed SNR analysis.

II. System Model

This paper follows the same system model as in [5]-[7]. Under the simplified assumption of a flat-fading channel and perfectly synchronized transmission and reception among all terminals. Suppose there are m_T source terminals and m_R destination terminals. If the source terminals are far from the destination terminals and no direct link between them exists, there will be at least one level of relay terminals present to assist the transmission. Suppose that there are in total K levels of relays. Aversion of interference suggests a time-division transmission strategy: In every transmission cycle, there are $K+1$ time slots. At the first time slot, the source terminals transmit the signals to the first level of relays. Each level of relay terminals performs the AF operation; that is, they amplify their received signals and forward them to the next level of relays. The k^{th} level of relays transmits the messages to the $(k+1)^{th}$ level of relays at the $(k+1)^{th}$ time slot. At the $(K+1)^{th}$ time slot, the messages arrive at the destination nodes from the last level of relays.

Let l_k be the number of relays at level k for $k = 1, ..., K$ and $l_0 = m_T$, $l_{K+1} = m_R$. Let then $H_k = \{H_k^{i,j}\} \in \mathbb{C}^{l_k \times l_{k-1}}$ denote the channel gain matrix between the $(k-1)^{th}$ and k^{th} level of relays (where $k = 0$ corresponds to the source nodes and $k = K+1$ to the destination nodes). The matrices H_k are independent and their entries are i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ random variables, varying ergodically over time (fast fading assumption). Additive white Gaussian noise Z_k (with unit variance) adds at each level k. $Z = Z_{K+1}$ is thus the noise at the final destinations.

There is no cooperation nor channel-state information (CSI) at the source and relay terminals, but full cooperation and full CSI (i.e. the knowledge of the realizations of all the random matrices $H_0, \ldots, H_{K+1}$) is assumed at the destinations. The power constraint at each node in the network is inversely proportional to the number of nodes at its level.

Let $Y_k = [y_{k1}, y_{k2}, \ldots, y_{kl_k}]$ and $X_k = [x_{k1}, x_{k2}, \ldots, x_{kl_k}]$ be the signals received and transmitted by the k^{th} level of relay, respectively. Then the K-level AF relay channel can be modeled as:

$$\begin{aligned} Y_1 &= H_1 X + Z_1 \\ Y_k &= H_k X_{k-1} + Z_k, \quad k = 2, \ldots, K \\ Y &= H_{K+1} X_K + Z \end{aligned}$$

For each level of relays, the received signals are scaled according to the power constraint at this level. Let the scaling factor of the k^{th} level be α_k. Therefore, the scaling is

$$X_k = \alpha_k Y_k, \quad k = 1, \ldots, K$$

The total power constraint at the k^{th} level is P_k. For the first level,

$$P_1 = \mathbb{E}[X_1^* X_1] = \alpha_1^2 \mathbb{E}[Y_1^* Y_1] = \alpha_1^2 \mathbb{E}[(H_1 X + Z_1)^*(H_1 X + Z_1)]$$

Since X, H_1 and Z_1 are independent of each other,

$$\begin{aligned} P_1 &= \alpha_1^2 (\mathbb{E}[X^* \mathbb{E}[H_1^* H_1] X] + \mathbb{E}[Z_1^* Z_1]) \\ &= \alpha_1^2 (\mathbb{E}[X^* (l_1 I) X] + l_1) = \alpha_1^2 l_1 (P+1) \end{aligned}$$

Similarly, for other levels:

$$P_k = \mathbb{E}[X_k^* X_k] = \alpha_k^2 l_k (P_{k-1} + 1), \qquad k = 2, \ldots, K$$

Define $\beta_1 = \frac{P+1}{P_1}$; $\beta_k = \frac{P_{k-1}+1}{P_k}$, $k = 2, 3, \ldots, K$. The scaling factors can then be written as $\alpha_k = \frac{1}{\sqrt{\beta_k l_k}}$, $k = 1, \ldots, K$. For completeness, $\beta_0 = \frac{1}{P}$ and $\alpha_0 = \frac{1}{\sqrt{\beta_0 l_0}} = \sqrt{\frac{P}{m_T}}$.

A recursive definition of a new series of matrices is $\{G_k\}_{k=0}^{K+1}$, where $G_{K+1} = I$ and

$$G_k = \alpha_k G_{k+1} H_{k+1}, \quad k = K, \ldots, 0 \tag{1}$$

Therefore, the overall channel can be expressed via G_k as:

$$\begin{aligned} Y &= G_K Y_K + Z = G_{K-1} Y_{K-1} + G_K Z_K + Z \\ &= \ldots = G_1 H_1 X + \textstyle\sum_{k=1}^K G_k Z_k + Z \end{aligned} \tag{2}$$

$G_1 H_1 X$ is *the signal part* and $\sum_{k=1}^K G_k Z_k + Z$ is *the noise part*. Perfect receiver channel-state information is assumed, so the destination terminals know all the H_k's. For ease of notation, a new series of matrices $\{\Sigma_k\}_{k=1}^{K+1}$ is introduced, where $\Sigma_{K+1} = I$ and

$$\Sigma_k = \mathbb{E}((Z + \sum_{i=k}^K G_i Z_i)\,(Z + \sum_{i=k}^K G_i Z_i)^* | H_1, \ldots, H_{K+1})$$

for $k = 1, \ldots, K$. The covariance matrix of *the noise part* is then Σ_1. Also, since $Z_1, Z_2, ...Z_K$ and Z have i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ components, the matrices $\{\Sigma_k\}_{k=1}^K$ obey the following recursive relationship

$$\Sigma_k = \Sigma_{k+1} + G_k G_k^*, \quad k = K, \ldots, 1 \tag{3}$$

Capacity computation requires knowledge of both the *noise* and the *signal* covariance matrices. Based on the Gaussian channel assumption, the capacity is achieved when the entries of X are jointly Gaussian. Suppose the covariance matrix of X is Q. Since by assumption, there is no CSI at the sources and the entries of H_1 are i.i.d. Gaussian and independent of G_1, [9] relates that the optimal X is distributed according to $\mathcal{N}_{\mathbb{C}}(0, \frac{P}{m_T} I)$. Thus, the covariance matrix of the *signal part* is $\frac{P}{m_T} G_1 H_1 H_1^* G_1^* = G_0 G_0^*$.

The overall capacity is then

$$\begin{aligned} C &= \frac{1}{K+1} \mathbb{E} \log\det\left(I + \frac{P}{m_T} \Sigma_1^{-\frac{1}{2}} G_1 H_1 H_1^* G_1^* \Sigma_1^{-\frac{1}{2}}\right) \\ &= \frac{1}{K+1} \mathbb{E} \log\det\left(I + \frac{P}{m_T} H_1^* G_1^* \Sigma_1^{-1} G_1 H_1\right) \end{aligned} \tag{4}$$

(the $\frac{1}{K+1}$ term comes from the use of the time-division scheme). We analyze the above capacity when m_T, m_R, and all l_k's tend to infinity, and they also tend to some given ratios while going to infinity, say, $\frac{l_1}{m_T} \to c_1$; $\frac{l_k}{l_{k-1}} \to c_k, k = 2, \ldots, K$; $\frac{m_R}{l_K} \to c_{K+1}$.

III. Capacity Analysis

A powerful tool for analyzing the limiting eigenvalue distribution (LED) of large dimensional random matrices is the Stieltjes transform. A thorough discussion of its applications can be found in [2]. Let F be a distribution on $\mathbb{R}$ (here and in the rest of the paper, one identifies a distribution on $\mathbb{R}$ with its cumulative distribution function). Then its Stieltjes transform is defined as

$$g(z) \equiv \int_{-\infty}^{\infty} \frac{1}{x-z}\, dF(x),\;\; z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\} \tag{5}$$

with the inversion formula

$$\lim_{\epsilon \downarrow 0} \int_{x_1}^{x_2} \frac{1}{\pi} \,\mathrm{Im}(g(x+i\epsilon))dx = F(x_2) - F(x_1) \tag{6}$$

and the useful fact that a sequence of distributions converges to a limit if and only if the corresponding sequence of Stieltjes transforms converges. The following result is a straightforward consequence of a result by Silverstein [4]. Recall that the empirical eigenvalue distribution of an $n \times n$ Hermitian matrix A_n with real eigenvalues $\lambda_1, \ldots, \lambda_n$ is given by $F_{A_n}(x) = \frac{1}{n}\sum_{k=1}^{n} 1_{\{x \geq \lambda_k\}}$.

Theorem 1: Let $n, N \geq 1$ and let us assume that:
(a) $X_n = (X_{ij}^n)$ is an $n \times N$ random matrix with i.i.d. entries such that $\mathbb{E}\,|X_{11}^n - \mathbb{E}X_{11}^n|^2 = 1$;
(b) $N = N(n)$ with $n/N \to c > 0$ as $n \to \infty$;
(c) T_n is an $n \times n$ random Hermitian non-negative definite matrix such that its empirical eigenvalue distribution F_{T_n} converges almost surely, as $n \to \infty$, to a (deterministic) distribution F_T, with corresponding Stieltjes transform g_T;
(d) X_n and T_n are independent.
Let $A_n = \frac{1}{N} X_n^* T_n X_n$. Then its empirical eigenvalue distribution F_{A_n} converges almost surely, as $n \to \infty$, to a (deterministic) distribution F_A, whose Stieltjes transform g_A satisfies

$$z g_A(z) + 1 = c\left(\frac{-1}{g_A(z)}\, g_T\left(\frac{-1}{g_A(z)}\right) + 1\right) \tag{7}$$

in the sense that, for each $z \in \mathbb{C}^+$, $g = g_A(z)$ is the unique solution to (7) such that $g \in \mathbb{C}^+$.

In the particular case where $T_n = I$, $F_T(x) = I_{\{x \geq 1\}}$ and $g_T(z) = 1/(1-z)$, so the above equation reads

$$z g_A(z) + 1 = \frac{c g_A(Z)}{1 + g_A(z)}, \tag{8}$$

and its solution is the Stieltjes transform of the well-known Marčenko-Pastur distribution [1].

In order to illustrate why Silverstein's theorem can be used to compute the capacity expression (4), the single-level AF relay network will be examined first. The single-level case provides general guidance for computing the general multi-level problem. We then derive a set of recursive equations that can be used to compute the Stieltjes transform of the LED of the multi-level AF relay network.

1) Single-Level AF Relay Network: The single level AF relay network has also been analyzed in [5]-[7]. This paper suggests another perspective on the formulation of the problem. The advantage is extension to the multi-level case.

When $K = 1$, $G_1 = \alpha_1 H_2$ and $\Sigma_1 = I + G_1 G_1^*$. Let $T_1 = G_1^* \Sigma_1^{-1} G_1$, $M_1 = \sqrt{\frac{1}{m_T}} T_1^{\frac{1}{2}} H_1$ and $M_2 = \sqrt{\frac{1}{l_1}} H_2$. Then, from (4),

$$C_{\text{1-level AF}} = \frac{1}{2}\mathbb{E} \log\det(I + P M_1^* M_1)$$

where

$$M_1^* M_1 = \frac{1}{m_T} H_1^* T_1 H_1 \tag{9}$$

$$T_1 = M_2^* (\beta_1 + M_2 M_2^*)^{-1} M_2 \tag{10}$$

$$M_2^* M_2 = \frac{1}{l_1} H_2^* H_2 \tag{11}$$

Capacity computation first finds the LED of $M_1^* M_1$. An equation for the Stieltjes transform of the LED of $M_1^* M_1$ is given in the following theorem. From now on, notation $g_{M_k^* M_k}$ abbreviates g_k.

Theorem 2: For matrices M_1, M_2 and T_1 satisfying (9), (10) and (11) with $\frac{l_1}{m_T} \to c_1$ and $\frac{m_R}{l_1} \to c_2$, the Stieltjes transform g_1 of the LED of $M_1^* M_1$ satisfies the following equations:

$$z g_1(z) + 1 = \frac{c_1 g_1(z)}{1 + g_1(z)} \left(\frac{-\beta_1}{1 + g_1(z)}\, g_2\left(\frac{-\beta_1}{1 + g_1(z)}\right) + 1\right) \tag{12}$$

$$z g_2(z) + 1 = \frac{c_2 g_2(z)}{1 + g_2(z)} \tag{13}$$

where $g_1 : \mathbb{C}^+ \to \mathbb{C}^+$ and $g_2 : \mathbb{C}^+ \to \mathbb{C}^+$.

Proof: The basic proof idea goes as follows. From (9), if T_1 is random Hermitian nonnegative definite and independent of H_1, with its eigenvalue distribution converging almost surely as $m_T \to \infty$, then Theorem 1 applies to compute the LED of $M_1^* M_1$. On the other hand, (11) implies that the eigenvalue distribution of $M_2^* M_2$ converges a.s., with its Stieltjes transform g_2 satisfying (13) with $g_2 : \mathbb{C}^+ \to \mathbb{C}^+$. Therefore, the missing link is the relationship between the eigenvalues of T_1 and $M_2^* M_2$. This relationship is given in the following lemma, whose proof is straightforward.

Lemma 1: Consider two matrices $T_1 \in \mathbb{C}^{l_1 \times l_1}$ and $M_2 \in \mathbb{C}^{l_1 \times m_R}$ satisfying (10), and denote by t_i and m_i the eigenvalues of T_1 and $M_2^* M_2$ respectively, where $t_1 \leq t_2 \leq \ldots \leq t_{l_1}$ and $m_1 \leq m_2 \leq \ldots \leq m_{l_1}$. Then

$$t_k = \frac{m_k}{\beta_1 + m_k} \quad k = 1, \ldots, l_1 \tag{14}$$

Therefore, their eigenvalue distributions satisfy

$$F_{M_2^* M_2}(x) = F_{T_1}\left(\frac{x}{\beta_1 + x}\right) \tag{15}$$

and their corresponding Stieltjes transforms satisfy

$$z g_{T_1}(z) + 1 = \frac{1}{1-z}\left(\frac{\beta_1 z}{1-z} g_2\left(\frac{\beta_1 z}{1-z}\right) + 1\right) \tag{16}$$

Since $M_2^* M_2$ is Hermitian nonnegative definite, T_1 is also Hermitian nonnegative definite, and its eigenvalue distribution converges a.s. Therefore, Theorem 1 applies to find the Stieltjes transform g_1: combining (7) and (16) yields the conclusion that g_1 satisfies (12) with $g_1 : \mathbb{C}^+ \to \mathbb{C}^+$. ■

We can find g_1 by solving the overall 4^{th}-order equation combining (12) and (13). Only one of the roots of the 4^{th} order equation satisfies both $g_1 : \mathbb{C}^+ \to \mathbb{C}^+$ and $g_2 : \mathbb{C}^+ \to \mathbb{C}^+$. From g_1, the inverse Stieltjes transform (6) provides the corresponding LED F_1. The single-level AF channel capacity is then given by

$$C_{\text{1-level AF}} \sim \frac{m_T}{2} \int \log\left(1 + Px\right)\, dF_1(x) \tag{17}$$

as $m_T \to \infty$. Therefore, as already noticed in [7], the capacity increases linearly with the number of nodes at each level.

2) Multi-Level AF Relay Network: Generalization of the previous result to the multi-level case requires a recursive transformation of the matrices for iterative application of Lemma 1 and Theorem 1 to compute the Stieltjes transform of the LED of the matrix in (4).

In addition to the two series of matrices $\{G_k\}$ and $\{\Sigma_k\}$ defined in (1) and (3), two new series of matrices are:

$$\begin{aligned} T_k &= G_k^* \Sigma_k^{-1} G_k \\ M_k &= \sqrt{\beta_{k-1}} \Sigma_k^{-\frac{1}{2}} G_{k-1}, \quad k = 1, \ldots, K+1 \end{aligned}$$

Derivation of a recursive relationship between $\{T_k\}$ and $\{M_k\}$ requires the following matrix-inversion lemma, whose proof is straightforward.

Lemma 2: If Σ is positive definite, then

$$(\Sigma + GG^*)^{-1} = \Sigma^{-\frac{1}{2}} (I + \Sigma^{-\frac{1}{2}} GG^* \Sigma^{-\frac{1}{2}})^{-1} \Sigma^{-\frac{1}{2}} \tag{18}$$

Use the above lemma and the recursive formulas (1),(3) for G_k, Σ_k, leads to the following lemma.

Lemma 3: Let $T_{K+1} = I$. The recursive relationship between T_k and M_k can be written as

$$M_k^* M_k = \frac{1}{l_{k-1}} H_k^* T_k H_k, \quad k = 1, \ldots, K+1 \tag{19}$$

$$T_k = M_{k+1}^* \left(\beta_k I + M_{k+1} M_{k+1}^*\right)^{-1} M_{k+1}, \; k = 1, \ldots, K \tag{20}$$

Proof: (19) comes directly from the definition of T_k and M_k and the fact that $G_{k-1} = \alpha_{k-1} G_k H_k$:

$$\begin{aligned} M_k^* M_k &= \beta_{k-1} G_{k-1}^* \Sigma_k^{-1} G_{k-1} \\ &= \beta_{k-1} \alpha_{k-1}^2 H_k^* G_k^* \Sigma_k^{-1} G_k H_k = \frac{1}{l_{k-1}} H_k^* T_k H_k \end{aligned}$$

The derivation of (20) involves the recursion $\Sigma_k = \Sigma_{k+1} + G_k G_k^*$ and Lemma 2:

$$\begin{aligned} T_k &= G_k^* \Sigma_k^{-1} G_k = G_k^* (\Sigma_{k+1} + G_k G_k^*)^{-1} G_k \\ &= G_k^* \Sigma_{k+1}^{-\frac{1}{2}} (I + \Sigma_{k+1}^{-\frac{1}{2}} G_k G_k^* \Sigma_{k+1}^{-\frac{1}{2}})^{-1} \Sigma_{k+1}^{-\frac{1}{2}} G_k \\ &= M_{k+1}^* \left(\beta_k I + M_{k+1} M_{k+1}^*\right)^{-1} M_{k+1} \end{aligned}$$

■

Again, capacity computation is through the LED of $M_1^* M_1 = \frac{1}{l_0} H_1^* T_1 H_1 = \frac{1}{m_T} H_1^* G_1^* \Sigma_1^{-1} G_1 H_1$. The similarity between (9), (10) and (19), (20) suggests a recursive way to compute the Stieltjes transform of the LED of $M_1^* M_1$ for the multi-level case.

Theorem 3: For matrix series $\{M_k\}_{k=1}^{K+1}$ and $\{T_k\}_{k=1}^{K+1}$ satisfying (19) and (20), the Stieltjes transforms g_k of the LED of $M_k^* M_k$ satisfy the following equations for $k = 1, \ldots, K$:

$$\begin{aligned} &z g_k(z) + 1 \\ &\quad = \frac{c_k g_k(z)}{1 + g_k(z)} \left(\frac{-\beta_k}{1 + g_k(z)}\, g_{k+1}\left(\frac{-\beta_k}{1 + g_k(z)} \right) + 1 \right) \end{aligned} \tag{21}$$

$$z g_{K+1}(z) + 1 = \frac{c_{K+1} g_{K+1}(z)}{1 + g_{K+1}(z)} \tag{22}$$

where $g_k : \mathbb{C}^+ \to \mathbb{C}^+$, $k = 1, \ldots, K+1$.

Proof: (20) implies that T_k is Hermitian nonnegative definite. Lemma 1 provides the relationship between the eigenvalues of $M_{k+1}^* M_{k+1}$ and T_k. As long as the eigenvalues distribution of $M_{k+1}^* M_{k+1}$ converges a.s., then the eigenvalue distribution of T_k converges a.s.
From Theorem 1, the eigenvalue distribution of $M_k^* M_k$ converges a.s. if the eigenvalue distribution of T_k converges a.s. Since $T_{K+1} = I$, the convergence of the eigenvalue distribution of $M_{K+1}^* M_{K+1}$ is ensured, and thus, the eigenvalue distribution of T_K also converges almost surely. This again ensures the convergence of the eigenvalue distribution of $M_K^* M_K$ and T_{K-1}, and so on. By induction, we conclude that the eigenvalue distribution of $M_k^* M_k$, $k = 1, ..., K+1$ converges almost surely.
Finally, combining (7), (8) and (16), we obtain the recursive equations (21) and (22) for the Stieltjes transform g_k of the LED of $M_k^* M_k$, $k = 1, ...K+1$. ■

Computing the capacity (4) requires the knowledge of the LED of $M_1^* M_1$. In order to be explicit, let us define a new sequence $\{z_k\}_{k=1}^{K+1}$, where $z_1 = z$, $z_{k+1} = \frac{-\beta_k}{1 + g_k(z_k)}$, $k = 1, \cdots, K$. We can then rewrite (21) and obtain for $k = 1, \ldots, K$:

$$\begin{aligned} g_{k+1}(z_{k+1}) &= \frac{-1}{z_{k+1}} \left(1 - \frac{1 + g_k(z_k)}{c_k g_k(z_k)} (z_k g_k(z_k) + 1) \right) \\ &= \frac{1 + g_k(z_k)}{\beta_k} \left(1 - \frac{1 + g_k(z_k)}{c_k g_k(z_k)} (z_k g_k(z_k) + 1) \right) \end{aligned}$$

Above equation suggests expression of $g_k(z_k)$, $k = 2, ..., K+1$, in terms of z and $g_1(z)$. Therefore, substitution of the expression for $g_{K+1}(z_{K+1})$ in terms of z and $g_1(z)$ into (22) yields an equation for $g_1(z)$. The overall equation for g_1 has order 2^{K+1}, and can be solved numerically. Only one of the roots satisfies $g_k(z_k) \in \mathbb{C}^+$, $\forall k = 1, \ldots, K+1$. The LED F_1 of $M_1^* M_1$ is computed by the inverse Stieltjes transform (6), and the capacity is given by the formula

$$C_{\text{K-level AF}} \sim \frac{m_T}{K+1} \int \log\left(1 + Px\right)\, dF_1(x)$$

as $m_T \to \infty$. The capacity increases linearly with the number of nodes at each level. Finally, it is possible to show that when $c_1 \to \infty$ and $\prod_{k=1}^{K+1} c_k = c$ remains fixed, F_1 converges to a classical Marčenko-Pastur distribution, whose Stieltjes transform g_1 is solution of an equation of the type (8).

IV. Numerical Simulations

First, this section compares the theoretical LED that we obtain in Section III-.2 with the empirical eigenvalue distribution of a finite-dimensional matrix. Figure 3 shows that there is indeed a fairly good agreement between the two, even for a small number of nodes in the network.

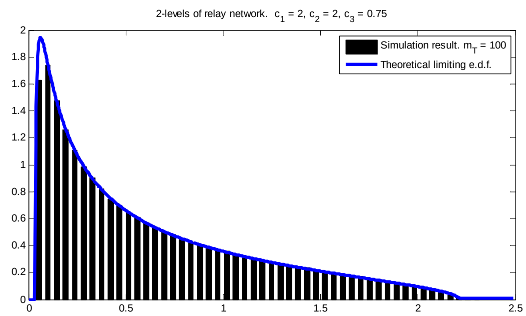

Fig. 3. Limiting versus empirical eigenvalue distribution

Next of interest is the effect of the number of relays upon the LED and the capacity. The ratio between the number of source and destination terminals is fixed, while the number of relays in between varies. When there is only one level of relays, the LED and the capacity are shown in figure 4 and 5 respectively.

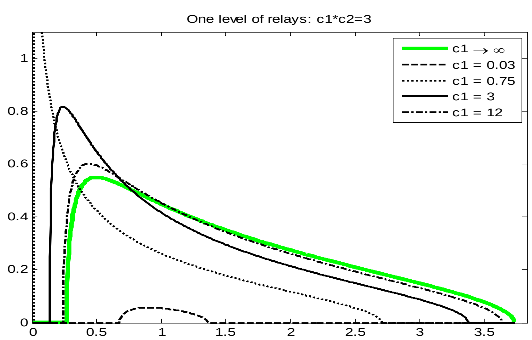

Fig. 4. Varying the number of relays in a 1-level AF relay network: LED

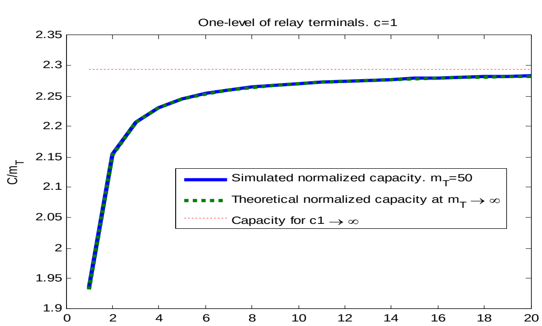

Fig. 5. Varying the number of relays in a 1-level AF relay network: capacity

Figure 6 plots the limiting eigenvalue distribution of a two-level AF relay channel. We vary the ratio between the number of relays at the first level and the number of source terminals, while the ratio between the numbers of relays in the first and second levels is fixed.

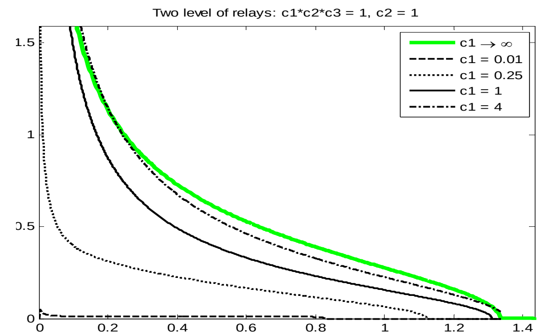

Fig. 6. Varying the number of first-level relays in a 2-level AF relay network

V. Conclusion

In this paper, the capacity of a large scale multi-level AF relay network has been analyzed. The capacity formula is expressed by means of the limiting eigenvalue distribution of a random matrix, whose Stieltjes transform is shown to satisfy a set of recursive equations. A general procedure for solving explicitly these equations has been proposed, from which one can deduce both the limiting eigenvalue distribution and the capacity.

VI. Acknowledgement

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