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Constrained paths in the flip-graph of regular triangulations

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Abstract

We describe particular paths in the flip-graph of regular triangulations in any dimension. It is shown that any pair of regular triangulations is connected by a path along which none of their common faces are destroyed. As a consequence, we obtain the connectivity of the flip-graph of regular triangulations that share the same vertex set.

Keywords: Regular triangulations; Flip-graph; Secondary polytope; Secondary fan

1. Introduction

Triangulations of point configurations in $\mathbb{R}^d$ are fundamental geometric objects that play a central role in computation and modeling. Quite deservedly they have thus attracted much research attention for over a century already. As their hidden faces are being illuminated, more intriguing—structural and algorithmic—questions about them arise and one realizes how fascinatingly complex these seemingly simple objects really are. Rather than studying individual triangulations, this note is about traveling between them. Among all triangulations, regular triangulations stand out as being of particular interest in applications. Bistellar flips, based on Radon's decomposition lemma, are a popular means to construct and explore the set of triangulations of a given point configuration. Gel'fand et al. [6,7] have introduced the so-called secondary polytope of a point configuration whose vertices and edges correspond precisely to the regular triangulations of a point configuration and the bistellar flips between pairs of them, respectively. Hence an edge following path on the secondary polytope visiting successive vertices (i.e. a path on the 1-skeleton of that polytope) is analogous to the path described by successive iterations of the simplex method. Such a path on the secondary polytope encounters a sequence of regular triangulations obtained by successive bistellar flips corresponding to pivot steps in linear programming.

The above 1-skeleton, also called *flip-graph of regular triangulations* of a point configuration can alternatively be defined as the sub-graph induced by all regular triangulations in the *flip-graph of all triangulations* of a point configuration. While the connectivity of the flip-graph of regular triangulations naturally follows from the mere existence of the secondary polytope, that of the flip-graph of all triangulations has only been established in two

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dimensions [9]. In dimensions five and six, point configurations with disconnected flip-graphs of all triangulations have been found [16,18]. For point sets in dimensions three and four, the connectivity of this graph is still subject to investigation, though there are some results bounding the degree of its vertices [1,16,17].

In this note we study paths in the 1-skeleton of the secondary polytope. It is shown that between any pair of regular triangulations, there is a path along which none of their common faces is destroyed. As a consequence, we obtain the connectivity of the flip-graph of regular triangulations that share the same vertex set. We thus settle an open question, namely showing that it is always possible to find a sequence of flips between two given regular triangulations sharing the same vertex set without adding nor deleting any vertex on the way, all intermediate triangulations being regular. Up to now it was only known that there is an incremental flip-algorithm exclusively visiting regular triangulations to find any given regular triangulation of a point set [4].

To make this note self-contained, we first briefly review some of the underlying mathematical notions and objects and then prove our main result. To conclude, we sketch an algorithm that flips a regular triangulation to another regular triangulation with the same vertex set. This algorithm is an alternative to the incremental construction and it has proven useful in our applications of grain flow modeling.

Comprehensive and more accessible treatments of the theory of the secondary polytope [6,7] can be found in [2,10].

2. Regular triangulations and geometric bistellar operations

In this section we give mathematical definitions for the objects and notions mentioned in the introduction. If A and B are collections of subsets of $\mathbb{R}^d$, the set $\{\mathrm{conv}(p \cup q)\colon (p, q) \in A \times B\}$ will be denoted by $A \star B$. The Euclidean scalar product of two vectors x and y of $\mathbb{R}^d$ will be denoted $x.y$. The vertex set of a polyhedron p will be called $\mathcal{V}(p)$.

A *triangulation* of a point configuration $\mathcal{A}$ in $\mathbb{R}^d$ is a polyhedral subdivision T of $\mathcal{A}$ whose faces are simplices. For a face s of T, we call *link* of s in T the set $\{p \in T\colon \mathrm{conv}(s \cup p) \in T, s \cap p = \emptyset\}$. Let $\mathcal{A}$ be a point configuration of $\mathbb{R}^d$. A *height function* on $\mathcal{A}$ is any function $w : \mathcal{A} \to \mathbb{R}$. Height functions induce particular polyhedral subdivisions of their underlying point configurations according to the following proposition:

Proposition 1. *Let $\mathcal{A}$ be a point configuration of $\mathbb{R}^d$ and w a height function on $\mathcal{A}$. There is a unique polyhedral subdivision $T(\mathcal{A}, w)$ of $\mathcal{A}$ so that for all $p \in T(\mathcal{A}, w)$, there exists $y \in \mathbb{R}^d$ satisfying the following two statements*:

(i) *For all $a \in \mathcal{V}(p)$, $a.y = w(a)$,*
(ii) *For all $a \in \mathcal{A} \setminus \mathcal{V}(p)$, $a.y < w(a)$.*

For all $a \in \mathcal{A}$, call a^w the point $(a, w(a)) \in \mathbb{R}^{d+1}$. The faces of $T(\mathcal{A}, w)$ can actually be obtained by projecting the lower hull of the polytope $\mathrm{conv}(\{a^w\colon a \in \mathcal{A}\})$ back on $\mathbb{R}^d$. Polyhedral subdivisions that can be constructed in this way exhibit particular regularity properties [2], which earns them to be denominated accordingly:

Definition 2. A polyhedral subdivision T of a point configuration $\mathcal{A}$ is called regular if there exists a height function w such that $T = T(\mathcal{A}, w)$. In this case we say that w realizes T.

A flip is a local transformation of a triangulation, some instances of which are shown in Fig. 1. The two-dimensional example of Fig. 1a) consists in exchanging the diagonals of a convex quadrilateral. Observe that the flip shown in Fig. 1b) exhibits a different structure, as it makes a vertex appear or disappear. Degenerate cases may occur as well, like the flip of Fig. 1c) involving five vertices, three of them being aligned, and one of the latter appearing or disappearing depending on the triangulation in which the flip is performed. The simplest three-dimensional flip, shown in Fig. 1d) consists in exchanging two tetrahedra and a triangle for three tetrahedra, three triangles and an edge. Of course, flips analogous to those of Figs. 1b) and 1c) also exist in three dimensions. In this section, we give a definition of those geometric bistellar operations valid in any dimension, thus gathering the flips of Fig. 1 into a unique description.

A *circuit* is any minimal affinely dependent subset Z of $\mathbb{R}^d$. The set $\{a, b, c, d\}$ is a circuit in Figs. 1a) and 1b) and $\{b, d, e\}$ is one in Fig. 1c). A circuit admits exactly two triangulations. This comes from the existence of the so-called *Radon partition* of circuits, according to the following theorem:

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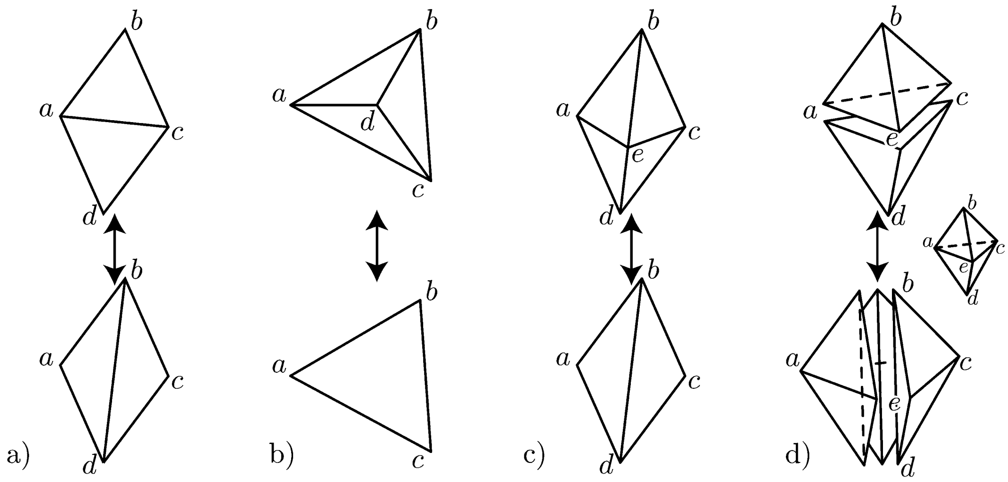

Fig. 1. Some flips.

Theorem 3. *Let $Z \subset \mathbb{R}^d$ be a circuit. There exists a unique partition (Z_-, Z_+) of Z so that* $\text{conv}(Z_-) \cap \text{conv}(Z_+) \neq \emptyset$.

Let (Z_-, Z_+) be the Radon partition of a circuit Z and consider the two subsets T_- and T_+ of $\mathcal{P}_Z = \{\text{conv}(s)\colon s \subset Z\}$ defined by:

$$T_- = \{p \in \mathcal{P}_Z\colon Z_- \not\subset \mathcal{V}(p)\} \quad \text{and} \quad T_+ = \{p \in \mathcal{P}_Z\colon Z_+ \not\subset \mathcal{V}(p)\}$$

Every simplex in $\mathcal{P}_Z$ belongs either to T_- or to T_+. Moreover, one can check that both T_- and T_+ are triangulations by using the unicity of (Z_-, Z_+) as a partition of Z so that $\text{conv}(Z_-) \cap \text{conv}(Z_+) \neq \emptyset$. This proves that Z admits T_- and T_+ as its only two triangulations. While knowing that a circuit admits exactly two triangulations is not strictly required to proceed with the definition of flips, it helps to understand the structure of circuits, which are the minimal point configurations admitting more than one triangulation.

Definition 4. Let T be a triangulation of a point configuration $\mathcal{A}$. Suppose the two following statements hold for some circuit $Z \subset \mathcal{A}$:

(i) Some triangulation T_- of Z is a subcomplex of T,
(ii) All cells of T_- have the same link L in T.

Then, we say that Z is a *flippable* circuit in T. Moreover, a triangulation T' of $\mathcal{A}$ can be obtained replacing $T_- \star L$ by $T_+ \star L$ in T. This operation is called a *geometric bistellar flip* and we say that T and T' are *geometric bistellar neighbors*.

Observe in Figs. 1a) and 1b) that $\{a, b, c, d\}$ are flippable circuits, the link L stated in (ii) being empty. In Fig. 1c), however, $\{b, d, e\}$ is a flippable circuit with $L = \{a, c\}$. Actually, statement (ii) will only be useful when the circuit to be flipped is not full-dimensional, the flip itself being degenerate as that of Fig. 1c).

3. Constrained connectivity of the graph of regular triangulations

Let $\mathcal{A}$ be a configuration of n points in $\mathbb{R}^d$. Since the space of height functions on $\mathcal{A}$ is a vector space of dimension n, we will identify it with $\mathbb{R}^n$ from now on. For a given regular polyhedral subdivision T of $\mathcal{A}$, we denote by $\mathcal{C}_\mathcal{A}(T)$ the set $\{w \in \mathbb{R}^n\colon T = T(\mathcal{A}, w)\}$ of all height functions realizing T. Finally $\mathcal{C}_\mathcal{A}$ denotes the collections of sets $\mathcal{C}_\mathcal{A}(T)$ corresponding to all regular polyhedral subdivisions T of $\mathcal{A}$. The following proposition is proven in [6]:

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Proposition 5. *The set $\mathcal{C}_{\mathcal{A}}$ is a complete polyhedral fan (partition of $\mathbb{R}^n$ into a finite collection of convex cones). Moreover, for a regular polyhedral subdivision T of $\mathcal{A}$, the polyhedral cone $\mathcal{C}_{\mathcal{A}}(T)$ is full-dimensional if and only if T is a triangulation.*

The fan $\mathcal{C}_{\mathcal{A}}$ is called *secondary fan* of $\mathcal{A}$, and for a regular polyhedral subdivision T of $\mathcal{A}$, the cone $\mathcal{C}_{\mathcal{A}}(T)$ is referred to as *secondary cone* of T. The following theorem states a crucial property of the secondary fan. Its proof can be found in [2]:

Theorem 6. *Two regular triangulations T and T' of a point configuration $\mathcal{A}$ are geometric bistellar neighbors if and only if their secondary cones share a common facet.*

To these results, we now add another one which will be used in the proof of Theorem 8 to make sure that the sequence of triangulations we search for respects the face constraints. For any polytope p we call $\mathcal{K}_{\mathcal{A}}(p)$ the set $\{w \in \mathbb{R}^n\colon\ p \in T(\mathcal{A}, w)\}$ of all height functions whose induced polyhedral subdivisions of $\mathcal{A}$ admit p as a face.

Lemma 7. *Let $\mathcal{A} \subset \mathbb{R}^d$ be a point configuration and p a polytope. The set $\mathcal{K}_{\mathcal{A}}(p)$ is convex.*

Proof. Let w and w' be two elements of $\mathcal{K}_{\mathcal{A}}(p)$ and λ an element of $[0, 1]$. We will show that $w'' = \lambda w + (1 - \lambda)w'$ still belongs to $\mathcal{K}_{\mathcal{A}}(p)$. According to Proposition 1, there exist two vectors y and y' in $\mathbb{R}^d$ so that for all $a \in \mathcal{V}(p)$, $y.a = w(a)$ and $y'.a = w'(a)$ while for all $a \in \mathcal{A} \setminus \mathcal{V}(p)$, $y.a < w(a)$ and $y'.a < w'(a)$. Call y'' the vector $\lambda y + (1 - \lambda)y' \in \mathbb{R}^d$. By linearity of the scalar product, one finds that for all $a \in \mathcal{V}(p)$, $y''.a = w''(a)$ while for all $a \in \mathcal{A} \setminus \mathcal{V}(p)$, $y''.a < w''(a)$. It follows from Proposition 1 that p is a face of $T(\mathcal{A}, w'')$, which proves that $\mathcal{K}_{\mathcal{A}}(p)$ is convex. □

Actually, for a polytope p the set $\mathcal{K}_{\mathcal{A}}(p)$ is a polyhedral cone. However, we only need its convexity here which explains the way Lemma 7 has been stated. We are now ready to prove our main result:

Theorem 8. *Let $\mathcal{A} \subset \mathbb{R}^d$ be a point configuration. Let T and T' be regular triangulations of $\mathcal{A}$. Then there exists a finite sequence $T_0, \ldots, T_n$ of regular triangulations of $\mathcal{A}$ so that $T = T_0$ and $T' = T_n$, and:*

(i) *For all $i \in \{0, \ldots, n-1\}$, T_i and T_{i+1} are geometric bistellar neighbors,*
(ii) *For all $i \in \{0, \ldots, n\}$, $T \cap T' \subset T_i$.*

Proof. According to Proposition 5, the cones $\mathcal{C}_{\mathcal{A}}(T)$ and $\mathcal{C}_{\mathcal{A}}(T')$ are full-dimensional and as such, their interiors are non-empty. Observe that it is then possible to choose two height functions w and w' in the interiors of $\mathcal{C}_{\mathcal{A}}(T)$ and $\mathcal{C}_{\mathcal{A}}(T')$ respectively so that all faces of $\mathcal{C}_{\mathcal{A}}$ intersected by segment $\mathrm{conv}(w, w')$ are either facets or cells. Let (w, w') be such a pair of height functions.

We denote by $T_0, \ldots, T_n$ the sequence of regular triangulations of $\mathcal{A}$ so that $\mathcal{C}_{\mathcal{A}}(T_0), \ldots, \mathcal{C}_{\mathcal{A}}(T_n)$ are those cells of $\mathcal{C}_{\mathcal{A}}$ successively met when $\mathrm{conv}(w, w')$ is traversed from w to w'. Observe that $T_0 = T$ and $T_n = T'$. From to the way w and w' were chosen, it follows that for any $i \in \{0, \ldots, n-1\}$ the secondary cones $\mathcal{C}_{\mathcal{A}}(T_i)$ and $\mathcal{C}_{\mathcal{A}}(T_{i+1})$ share a common facet. Theorem 6 then guarantees that triangulations $T_0, \ldots, T_n$ satisfy statement (i).

Let p be an element of $T \cap T'$. According to Lemma 7, $\{w \in \mathbb{R}^n\colon\ p \in T(\mathcal{A}, w)\}$ is convex. This implies that for all $i \in \{0, \ldots, n\}$, p is a face of T_i. Triangulations $T_0, \ldots, T_n$ then satisfy statement (ii) and the theorem is proven. □

According to Theorem 8, it is possible to flip a regular triangulation into another without destroying any of their common faces. Equivalently, it is possible to constrain the paths in the flip-graph so that they preserve a given subset of the faces of the triangulations they meet. This result has obvious applications in the context of constrained triangulations. Moreover, as stated in Corollary 9 below, it also settles the regular case of the following open problem [11] formulated in the early nineties by several authors [3,8]: *is it possible to change a triangulation into another by performing flips that do not add, nor remove vertices?*

Corollary 9. *Let $\mathcal{A}$ be a point configuration in $\mathbb{R}^d$. The flip-graph of those regular triangulations of $\mathcal{A}$ that share $\mathcal{A}$ as a common vertex set is connected.*

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4. Algorithmic implications

The above result not only settles a particular case of an open question but also turns out to be useful in the practical problem of regularizing triangulations. The usual way to build a regular triangulation is incremental. Its vertices are added one by one and each time a vertex is added (using a flip that adds a vertex for example) the triangulation is regularized using flips around this vertex [4,8]. This way, one can build any regular triangulation of a given point configuration $\mathcal{A}$ from scratch.

In many situations however, a triangulation T of $\mathcal{A}$ which may be regular for some height function is given. One is interested in finding a regular triangulation T' of $\mathcal{A}$ for a another height function w' for which T is not necessarily regular. We call transforming T into T' *regularizing* (rather than *reregularizing* which would be more to the point but also uglier). An example where one repeatedly has to (re)regularize given triangulations comes up in methods using *dynamic regular triangulations* to detect contacts in a set of moving bodies [5,12,14].

Another example is the simulation of polycrystal growth, where dynamic regular triangulations appear as duals of power diagrams (the polycrystals) that have to be updated frequently in the course of the simulation process, see e.g. [19]. This application heavily depends on the fact that there are times at which individual cells disappear, making room for others that are growing. Thus, in this application flips removing vertices are crucial. However, such deletions obey physical reasons rather than algorithmic needs. Now, returning to the application of moving bodies contact detection, here the vertices are interpreted as the centers of a family of spheres enclosing the bodies whose pairwise contacts have to be detected. Two such spheres can only be in contact if their centers are neighbors in T'. Deleting them while changing triangulations would therefore be undesirable. The appropriate height function values for this application are equal to the squared euclidean norms of the center coordinate vectors of the spheres from which the respective squared radii are subtracted. Without going into further details, the process can be subsumed as follows: during the motion of the body system, the spheres change positions, and as a rule the new triangulation of their centers will no longer be regular with respect to the new height function. This is when regularization takes place.

For the general regularization process assume that the vertex sets of T and T' are identical. The algorithm then tries to transform T into T' with a sequence of flips that do not add or remove vertices. While there is no proof that such a sequence of flips exists in the general case, the above corollary tells us that there is one as soon as T is regular.

Let f be an interior facet of T and b and c the vertices of T so that $\mathrm{conv}(f \cup \{b\})$ and $\mathrm{conv}(f \cup \{c\})$ are full-dimensional faces of T. Let w' be a height function that realizes T'. There exists $y \in \mathbb{R}^d$ so that for all $a \in \mathcal{V}(f) \cup \{b\}$, $a.y = w'(a)$. We say that f is *w'-illegal* if $c.y > w'(c)$. From the definition of a circuit, there exists a unique circuit $Z(f) \subset \mathcal{V}(f) \cup \{b, c\}$. We say that f is *flippable* in T if $Z(f)$ is a flippable circuit in T.

Based on those definitions, the following algorithm attempts to regularize T into T':

Regularization algorithm

```
while T contains a w'-illegal facet do
    if T admits a facet f that is simultaneously w'-illegal and flippable then
        flip f in T
    else {no facet of T is simultaneously w'-illegal and flippable}
        return failure statement
    end if
end while
```

The theoretical convergence of this algorithm does not depend on the order in which the flips are performed. As a referee pointed out, a natural order could be that in which the line segment introduced in the proof of Theorem 8 intersects the successive facets of the secondary fan. Such an order has also been proposed in [14], but has not been implemented so far.

Observe that the flips this algorithm performs will never remove a vertex. Indeed, if a vertex v is removed at any point, this is done by flipping a w'-illegal facet f, implying that v does not lie in the lower hull of $\mathrm{conv}(Z(f)^{w'})$ where $s^{w'}$ denotes the set $\{a^{w'} \in \mathbb{R}^{d+1}\colon\ a \in s\}$ for a subset s of $\mathcal{A}$. As a consequence, v will not lie in the lower hull of $\mathrm{conv}(\mathcal{A}^{w'})$ and will not be a face of T' which brings a contradiction. Since the vertex sets of T and T' are identical, neither will a vertex be added to T during this regularization.

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Provided that at each step an interior facet of T that is simultaneously illegal and flippable is found, the above algorithm will converge to T'. In order to show this, consider the polyhedral surface of $\mathbb{R}^{d+1}$ defined by $T^{w'} = \{\mathrm{conv}(s^{w'})\colon \mathrm{conv}(s) \in T\}$. Every time a w'-illegal facet of T is flipped, the volume below $T^{w'}$ strictly decreases. Since our regularization algorithm only flips illegal facets, it will not cycle. Still, it could fail if at step 5 a triangulation is reached that admits no simultaneously w'-illegal and flippable facet. In the general case, since it is not even known whether a sequence of flips exists between two triangulations with identical vertex sets, there is no proof that our regularization algorithm converges. However, if T is regular, Theorem 8 states that there is indeed a sequence of flips between T and T' that do not add, nor remove vertices. Moreover, one can deduce from the proof that all facets destroyed by those flips will be w'-illegal. Since any flip destroys at least one facet, we have the following result:

Corollary 10. *Let $\mathcal{A}$ be a point configuration, T and T' regular triangulations of $\mathcal{A}$ and w' a height function realizing T'. If the vertex sets of T and T' are identical then T admits a facet that is both w'-illegal and flippable.*

Following this, our regularization algorithm will converge if after every flip performed, triangulation T is still regular. A search for a flip that leaves T regular can then be added to this algorithm in order to guarantee convergence.

For our practical tridimensional contact detection purposes, the regularization algorithm flips illegal facets without testing the regularity of the intermediate triangulations. Neither does it test the regularity of the initial triangulation T. Dynamic triangulations were used for contact detection in the framework of granular media simulation [5,12,13,15]. For those applications, literally billions of regularizations have been performed. Further we have tested the regularization algorithm using specially constructed non-regular initial triangulations. The regularization algorithm converged in every tested case. The final triangulations returned by the algorithm were all explicitely checked and found to be the expected regular ones.

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