Scaling Laws for One and Two-Dimensional Random Wireless Networks in the Low Attenuation Regime

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Abstract

The capacity scaling of extended two-dimensional wireless networks is known in the *high attenuation regime*, i.e. when the power path loss exponent α is greater than 4. This has been accomplished by deriving information theoretic upper bounds for this regime that match the corresponding lower bounds. On the contrary, not much is known in the so-called *low attenuation regime* when $2 \leq \alpha \leq 4$. (For one-dimensional networks, the uncharacterized regime is $1 \leq \alpha \leq 2.5$.) The dichotomy is due to the fact that while communication is highly power-limited in the first case and power-based arguments suffice to get tight upper bounds, the study of the low attenuation regime requires a more precise analysis of the degrees of freedom involved. In this paper, we study the capacity scaling of extended wireless networks with an emphasis on the low attenuation regime and show that in the absence of small scale fading, the low attenuation regime does not behave significantly different from the high attenuation regime.

I. Introduction

In their seminal study [1] of the capacity of ad hoc wireless networks, P. Gupta and P. R. Kumar consider n nodes randomly located on the unit disk. The nodes are randomly paired into $n/2$ source destination pairs and each source wants to communicate to its destination at a common rate R_n. They show under certain assumptions on the physical layer that derive from state-of-the art communications practice, that the maximally achievable rate per source-destination pair R_n decreases like $R_n \leq \frac{K}{\sqrt{n}}$ as the system size increases. Although their original set-up considers a fixed area network, their result can be readily extended to the case where the network area scales with n. For a d-dimensional *extended network*, where the area (or the volume) occupied by the network increases linearly in n keeping the density of the users constant, their result yields

$$R_n \leq \frac{K}{n^{\frac{1}{d}}}. \tag{1}$$

It is not apriori clear whether this limitation is a consequence of the assumptions on the physical layer or is inherent to the problem. With the motivation to study wireless networks without making any particular assumption on the way communication is established, there has been some subsequent work [2], [3], [4], [5] that tackles the problem from an information-theoretic point of view. These works have lead to the following partial answers to the problem.

In wireless communications, a commonly accepted model is that the transmitted signals are attenuated by a factor g of the form

$$g(r) = \frac{e^{-\gamma r}}{r^{\alpha/2}}\, h \qquad \text{with} \qquad \gamma \geq 0, \qquad \alpha \geq d, \tag{2}$$

and further corrupted by additive noise. In (2), r is the distance between the transmitter and the receiver, α is called the power path loss exponent, γ is the absorption constant and thus the first term represents the attenuation of the signal due to electromagnetic propagation. The second term h models small scale fading, usually assumed to be independent and identically distributed across node pairs. Different authors have considered different variants of the channel model given above. The results obtained can be roughly summarized as follows: for two-dimensional networks, the scaling law in (1) has been confirmed by information theoretic arguments for absorptive media ($\gamma > 0$) or under the assumption of large power path loss ($\alpha > 4$), for both the cases where h is absent (considered in [2], [4]) or present in the model (considered in [3], [5]). For one-dimensional networks, (1) has been confirmed for $\alpha > 2.5$. In the more interesting case where there is no absorption and the attenuation is moderate, the only results are due to O. Lévêque and E. Telatar in [4], where they consider the simplified channel model without small scale fading (h is absent in the model) and show that the maximally achievable rate per source-destination pair still decreases to zero in this regime with increasing system size. However, the scaling law they obtain is significantly different from (1). More precisely, they show that if $\alpha > (d \vee 2(d-2))$, then

$$R_n \leq K \frac{n^{\frac{1}{\alpha}} \log n}{n^{\frac{1}{d}}} \tag{3}$$

where $a \vee b$ is the maximum of a and b. Observe that the upper bound in (3) decreases much slower to zero than that in (1), especially when α is small.

The restriction of the above mentioned information theoretic results to the high attenuation regime seems due to the fact that they all rely on power-based arguments. Extended wireless networks are power limited in the high attenuation regime and power-based arguments suffice to get tight upper bounds. However, the study of the low attenuation regime requires a more precise analysis of the degrees of freedom. This is the aim of the current paper; by performing a detailed mathematical analysis, we strengthen the results of [4] for one and two-dimensional networks. For one-dimensional networks, we recover the scaling law in (1) up to logarithmic terms for all $\alpha > 0$. For two-dimensional networks and again $\alpha > 0$, we get a scaling law that is only slightly larger than (1) and can be regarded as a significant improvement to the already known results for the attenuation regime of interest ($2 \leq \alpha \leq 4$).

A crucial remark is worth emphasizing once more: the results in [4] as well as the results in the present paper are obtained by considering the simplified channel model with no small scale fading. This assumption leads to an interesting subsequent mathematical analysis of the problem, involving in particular the study of determinants of Cauchy matrices. The reader should be aware that the results below do not apply to the channel model with small scale fading. While the model we are studying can be of interest in its own right, one should not discard the potential impact of small scale fading in general, since it may bring some extra degrees of freedom to the problem, that may allow to overcome the bottlenecks in wireless networks in the low attenuation regime. This remains as one of the important open problems concerning the capacity scaling of wireless networks.

II. Main Result

We consider a network of n users (or nodes) independently and uniformly distributed on a domain D whose area increases linearly with n. The nodes are randomly paired up into $n/2$ source-destination pairs without any consideration on their respective locations. Each source has the same traffic rate R_n to send to its destination node and a common average transmit power constraint P applies for all nodes. The transmitted signals are attenuated by a factor

$$g(r) = \frac{1}{r^{\alpha/2}} \tag{4}$$

over distance r.[1] Let us also introduce the following classical notation for a sequence of random variables (A_n) and numbers (b_n):

$$A_n \le b_n \quad \text{with high probability}$$

if

$$\lim_{n\to\infty} \mathbb{P}(A_n \le b_n) = 1.$$

Our main results are the following.

Theorem 2.1: Consider a one-dimensional extended random network and the channel model given in (4) with $\alpha > 0$. For any $\varepsilon > 0$, there exists $K_1 > 0$ independent of n such that the maximally achievable rate R_n per source-destination pair in the network is bounded above by

$$R_n \le K_1 \frac{(\log n)^{3+\varepsilon}}{n}, \quad \text{with high probability.}$$

Theorem 2.2: Consider a two-dimensional extended random network and the channel model given in (4) with $\alpha > 0$. For any $\varepsilon > 0$, there exists $K_2 > 0$ independent of n such that the maximally achievable rate R_n per source-destination pair in the network is bounded above by

$$R_n \le K_2 \frac{n^{\frac{1}{\alpha+8}+\varepsilon}}{\sqrt{n}}, \quad \text{with high probability.}$$

Although the results presented in this section as well as the discussions in the previous section are in terms of rate per source-destination pair, the corresponding results on sum-rate defined as nR_n, also referred to as aggregate throughput in the literature, are immediate. It is usually also not difficult to convert results on rate per source-destination pair to transport capacity or vice-versa. The transport capacity, first introduced in [1] for wireless networks, measures the total bit-meters per second a network can reliably support and is formally defined as

$$T_c(n) = \sup \sum_{\substack{i,j \\ i\neq j}} R_{ij} r_{ij}, \tag{5}$$

where the supremum is taken over all feasible $n \times n$ rate matrices $\{R_{ij}\}$. The quantity R_{ij} is the rate of transmission from node i to node j and r_{ij} is the distance between the two nodes.

The above results yield the following upper bounds on the transport capacity of one and two-dimensional random networks.

Theorem 2.3: Let $\alpha > 0$. For any $\varepsilon > 0$, the transport capacity of a one-dimensional extended random network is bounded above by

$$T_c(n) \le K_3\, n\, (\log n)^{3+\varepsilon}, \quad \text{with high probability;}$$

similarly, the transport capacity of a two-dimensional extended random network is bounded above by

$$T_c(n) \le K_4\, n n^{\frac{1}{(\alpha+8)}+\varepsilon}, \quad \text{with high probability,}$$

where $K_3 > 0$ and $K_4 > 0$ are constants independent of n.

The paper is organized as follows: in the following section, we introduce our approach by first concentrating on two-dimensional networks and then pointing out the simplifications arising for the one-dimensional case. (The approach has already been introduced in [4] for two-dimensional networks and in [6] for one-dimensional networks.) We prove Theorem 2.1 in Section IV and Theorem 2.2 in Section V. The corresponding results for the transport capacity in Theorem 2.3 are discussed in Section VI. Section VII contains our conclusions.

[1]Following the general convention, we use the power path loss exponent α to describe the attenuation in Sections I and II. We will switch to $\delta = \alpha/2$ in the following sections, which turns out to be more handy for the analysis.

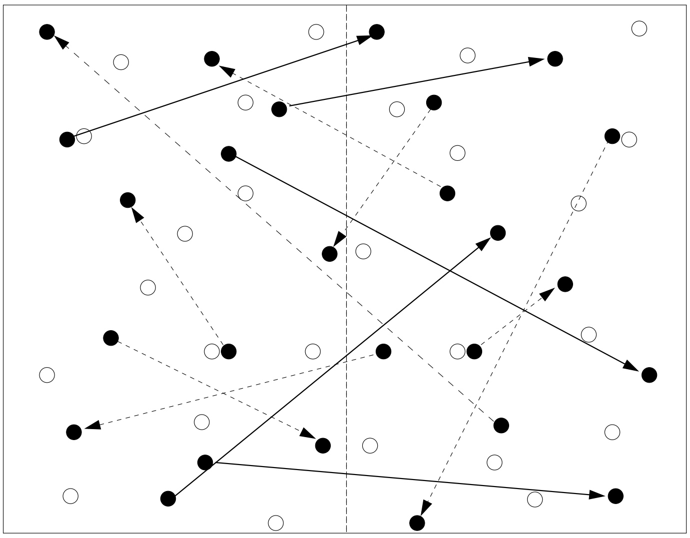

Fig. 1. The network is depicted after the introduction of “mirror” users. Note that the original nodes are depicted in black while the “mirror” users are depicted in white. Our approach concentrates on the traffic requests that pass the imaginary boundary on the y-axis from left to right, depicted in bold lines in the figure.

III. Our Approach

For simplicity, let us assume that the two-dimensional domain D is a rectangle $D = [-\sqrt{n}, \sqrt{n}] \times [0, \sqrt{n}]$. We start by dividing D into two equal parts $[-\sqrt{n}, 0] \times [0, \sqrt{n}]$ and $[0, \sqrt{n}] \times [0, \sqrt{n}]$. We first concentrate on bounding the total information flow from one half of the network to the other, or equivalently the sum of the rates of communications passing the imaginary boundary on the y-axis, say from left to right. A simple statistical argument then allows to use this result for bounding above R_n.

Since we are interested in upper bounds, we make a series of optimistic assumptions: we first introduce n additional “mirror” users that help relaying traffic, where the mirror location of (x_i, y_i) is $(-x_i, y_i)$. The mirror users are only helpers for establishing the original traffic requests and do not have their own traffic demand. Their introduction brings a useful symmetry to the problem: there are now exactly n users on each side of the domain symmetrically located with respect to the y-axis. See Figure 1. Because of the symmetry, the locations of the nodes on the right half domain are now enough to characterize the configuration of the whole network. By possibly re-numbering the nodes, we will denote by $(L_i := (x_i, y_i), i = 1, \ldots, n)$ these right-hand domain locations. Note that $(L_1, L_2, \ldots, L_n)$ form a family of two-dimensional independent random variables uniformly distributed on the set $[0, \sqrt{n}] \times [0, \sqrt{n}]$. A further optimistic assumption is that all the users on the left half domain can cooperate freely and even distribute their power resources among themselves in order to establish communication in the most efficient way with the users on the right half domain, which in turn can also cooperate at no cost. This assumption allows to bound above the total information flow from the left half of the network to the right half, by the capacity of the following point-to-point MIMO channel,

$$V_i = \sum_{j=1}^{n} G_{ij}^n U_j + Z_i, \qquad i = 1, ...n,$$

where $U_1, \ldots, U_n$ denote the signals tranmitted by the left-hand side nodes, $V_1, \ldots, V_n$ are the signals received by the right-hand side nodes and $(Z_1, \ldots, Z_n)$ is a vector of circularly symmetric complex

Gaussian random variables with unit variance. The entries of the $n \times n$ channel matrix G^n are given by

$$G_{ij}^n = \frac{1}{((x_i + x_j)^2 + (y_i - y_j)^2)^{\frac{\delta}{2}}} \tag{6}$$

where $((x_i, y_i), i = 1, \ldots, n)$ are the right-hand side node locations. Note that the left-hand side nodes are located on the symmetric locations $((-x_i, y_i), i = 1, \ldots, n)$, hence the pairwise gains in (6). Note also that $\delta = \alpha/2$. In the following sections, we will sometimes use the notation

$$G^n \{L_1, \ldots L_n\} \tag{7}$$

to refer to the $n \times n$ matrix G^n corresponding to a specific configuration $(L_1, \ldots L_n)$ or omit the argument when no confusion arises. Under the total power constraint

$$\sum_{j=1}^{n} E\left[|U_j|^2\right] \leq nP$$

the capacity of the above channel, which bounds above the maximum flow of information from one half of the network to the other, is given by

$$C_n^{2D} = \max_{P_k \geq 0 : \sum_{k=1}^n P_k \leq nP} \sum_{k=1}^{n} \log\left(1 + P_k \lambda_k^2\right)$$

where λ_k are the eigenvalues of the symmetric matrix G^n. Noticing that $P_k \leq nP$ for each k and that the λ_k are non-negative (see Appendix I), we further obtain

$$\begin{aligned} C_n^{2D} &\leq \sum_{k=1}^{n} \log\left(1 + nP\lambda_k^2\right) \leq \sum_{k=1}^{n} \log\left(1 + \sqrt{nP}\lambda_k\right)^2 \\ &= 2 \log\det\left(I + \sqrt{nP}\, G^n\right). \end{aligned} \tag{8}$$

In the one-dimensional case, we have n nodes that are uniformly and independently distributed on the line segment $[-n, n]$. Adding n "mirror" relay nodes, as in the two-dimensional case, we end up with a symmetrical configuration with respect to the origin: the n nodes on the right-hand side are uniformly and independently distributed on the line segment $[0, n]$ and those on the left are located at "mirror" positions. Since all arguments leading to the upper bound in (8) also apply in this case, the total information flow from left to right is again bounded above by

$$C_n^{1D} \leq 2 \log\det\left(I + \sqrt{nP}\, H^n\right), \tag{9}$$

however the corresponding channel matrix H^n now has the following simpler structure:

$$H_{ij}^n = \frac{1}{(x_i + x_j)^\delta} \tag{10}$$

where $x_1, \ldots, x_n$ denote the positions of the nodes on $[0, n]$.

Under the uniform traffic assumption we have made, it is trivial to show that there will be order n communication requests that need to pass the boundary from left to right. We can therefore conclude that the maximally achievable rate per communication pair in the network is bounded above by

$$R_n \leq K \frac{C_n}{n} \tag{11}$$

for a constant $K > 0$ independent of n. (We have not used a superscript for C_n, implying that using either C_n^{1D} or C_n^{2D} gives the corresponding result for one or two-dimensional networks.) In the rest of the paper, we will focus on the problem of determining the scaling of the upper bound on C_n^{1D} given by (9) and (10) and that on C_n^{2D} given by (8) and (6). Note that the structure of the one-dimensional matrix H^n in (10) is much simpler than the two-dimensional matrix G^n in (6). Indeed, for $\delta = 1$, H^n becomes a Cauchy matrix and an explicit expression for its determinant is known. A precise analysis of the determinant leads to the result in Theorem 2.1 for one-dimensional networks. This result then serves as a basic tool for tackling the two-dimensional case.

IV. One-Dimensional Networks

The outline of the proof of Theorem 2.1 goes as follows [2]: we start by considering the upper bound on C_n^{1D} given in (9). Using the following identity, valid for any $n \times n$ matrix A:

$$\det(I + A) = \sum_{J \subset \{1,\dots,n\}} \det(A(J)), \text{ where } A(J) = (a_{ij})_{i,j \in J}\,, \tag{12}$$

we obtain that the upper bound $2\,\log\det\left(I + \sqrt{nP}H^n\right)$ on C_n^{1D} can be expressed in terms of determinants of the form

$$D_\delta(\mathbf{x}_J) = \det\left(\left(\frac{1}{(x_i + x_j)^\delta}\right)_{i,j \in J}\right), \text{ where } \mathbf{x}_J = (x_i)_{i \in J}. \tag{13}$$

The first step of the proof consists in showing that for any $\delta > 0$, there exist $\tilde{K} > 0$ and $\eta > 0$ independent of m such that

$$D_\delta(\mathbf{x}_J) \le \left(\frac{m}{x_{\min}}\right)^{\tilde{K}m} D_1(\mathbf{x}_J)^\eta, \tag{14}$$

where $m = |J|$ and $x_{\min} = \min\{x_i : i = 1, \dots, n\}$. We can therefore focus on the case $\delta = 1$, for which we have the following explicit expression (see for instance [7, p. 202]):

$$D_1(\mathbf{x}_J) = \left(\prod_{\substack{i,j \in J \\ i<j}} (x_j - x_i)^2\right) \Bigg/ \left(\prod_{i,j \in J} (x_i + x_j)\right). \tag{15}$$

The second part of the proof is a detailed study of the configuration $\mathbf{x}_J$ maximizing $D_1(\mathbf{x}_J)$ and leads to the following estimate: there exists $\hat{K} > 0$ independent of m such that

$$D_1(\mathbf{x}_J) \le \frac{1}{(x_{\min})^m} \exp(-\hat{K}\, m^{3/2}). \tag{16}$$

Finally, combining estimates (14) and (16) together with formula (12) and the fact that $x_{\min}$ is typically of order 1 allows us to bound C_n^{1D} above by

$$C_n^{1D} \le K_1 \,(\log n)^{3+\varepsilon}, \quad \text{with high probability,}$$

for any $\varepsilon > 0$. Equation (11) then leads to the conclusion of Theorem 2.1.

Proof of Theorem 2.1: first step.

Lemma 4.1: Let $\delta > 0$ and $J \subset \{1, \dots, n\}$. The determinant defined in (13) satisfies the following identity:

$$D_\delta(\mathbf{x}_J) = \frac{1}{m!\,\Gamma(\delta)^m} \left(\prod_{k \in J} \int_{\mathbb{R}_+} dt_k\, t_k^{\delta-1}\right) \det\left(\left(e^{-t_i x_j}\right)_{i,j \in J}\right)^2,$$

[2] Note that this result was already proved in [6] under stronger assumptions on the attenuation factor δ.

where $m = |J|$ and Γ is the Euler Gamma function.

Notice that in order to shorten the notation, we have written

$$\left(\prod_{k\in J}\int_{\mathbb{R}_+} dt_k\, t_k^{\delta-1}\right)\,\det\left(\left(e^{-t_i\,x_j}\right)_{i,j\in J}\right)^2$$

in place of the somewhat less ambiguous expression

$$\int_{\mathbb{R}_+} dt_{k_1}\, t_{k_1}^{\delta-1}\cdots\int_{\mathbb{R}_+} dt_{k_m}\, t_{k_m}^{\delta-1}\,\det\left(\left(e^{-t_i\,x_j}\right)_{i,j\in J}\right)^2,$$

where $k_1,\ldots,k_m$ enumerate the elements of J.

Proof: Let us first recall that

$$\frac{1}{(x_i+x_j)^\delta} = \frac{1}{\Gamma(\delta)}\int_{\mathbb{R}_+} dt\, t^{\delta-1}\, e^{-t\,(x_i+x_j)}.$$

For a matrix $A(J) = (a_{ij})_{i,j\in J}$, the following formula holds:

$$\begin{aligned}\det(A(J)) &= \sum_{\sigma\in\mathcal{S}(J)}\varepsilon(\sigma)\prod_{i\in J}a_{i,\sigma(i)}\\ &= \frac{1}{m!}\sum_{\sigma,\tau\in\mathcal{S}(J)}\varepsilon(\sigma)\,\varepsilon(\tau)\prod_{i\in J}a_{\sigma(i),\tau(i)},\end{aligned}$$

where $\mathcal{S}(J)$ is the set of permutations of J and $\varepsilon(\sigma)$ is the signature of the permutation σ. Therefore,

$$\begin{aligned}&D_\delta(\mathbf{x}_J)\\ &= \frac{1}{m!\,\Gamma(\delta)^m}\sum_{\sigma,\tau\in\mathcal{S}(J)}\varepsilon(\sigma)\,\varepsilon(\tau)\prod_{i\in J}\left(\int_{\mathbb{R}_+} dt_i\, t_i^{\delta-1}\, e^{-t_i\,(x_{\sigma(i)}+x_{\tau(i)})}\right)\\ &= \frac{1}{m!\,\Gamma(\delta)^m}\left(\prod_{k\in J}\int_{\mathbb{R}_+} dt_k\, t_k^{\delta-1}\right)\left(\sum_{\sigma\in\mathcal{S}(J)}\varepsilon(\sigma)\prod_{i\in J}e^{-t_i\,x_{\sigma(i)}}\right)^2\\ &= \frac{1}{m!\,\Gamma(\delta)^m}\left(\prod_{k\in J}\int_{\mathbb{R}_+} dt_k\, t_k^{\delta-1}\right)\det\left(\left(e^{-t_i\,x_j}\right)_{i,j\in J}\right)^2,\end{aligned}$$

■

Lemma 4.2: Let $\gamma,\delta>0$ be such that either $\gamma<\delta<1$ or $1<\delta<\gamma$. Let also $f:\mathbb{R}_+\to\mathbb{R}$ be a measurable function such that both $t^{\gamma-1}\,f(t)^2$ and $f(t)^2$ are integrable on $\mathbb{R}_+$. Then

$$\int_{\mathbb{R}_+} dt\, t^{\delta-1}\, f(t)^2 \le \left(\int_{\mathbb{R}_+} dt\, t^{\gamma-1}\, f(t)^2\right)^{\frac{\delta-1}{\gamma-1}}\left(\int_{\mathbb{R}_+} dt\, f(t)^2\right)^{\frac{\gamma-\delta}{\gamma-1}}$$

Proof: Let us write $t^{\delta-1}\,f(t)^2 = u(t)\,v(t)$, where

$$u(t) = t^{\delta-1}\,f(t)^{\frac{2(\delta-1)}{\gamma-1}}\quad\text{and}\quad v(t) = f(t)^{\frac{2(\gamma-\delta)}{\gamma-1}}.$$

Using Hölder's inequality with $p=\frac{\gamma-1}{\delta-1}\in\,]1,\infty[$ and $q=\frac{\gamma-1}{\gamma-\delta}\in\,]1,\infty[$ (so that $\frac{1}{p}+\frac{1}{q}=1$), we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+} dt\, u(t)\, v(t)\\ &\le \left(\int_{\mathbb{R}_+} dt\, u(t)^{\frac{\gamma-1}{\delta-1}}\right)^{\frac{\delta-1}{\gamma-1}}\left(\int_{\mathbb{R}_+} dt\, v(t)^{\frac{\gamma-1}{\gamma-\delta}}\right)^{\frac{\gamma-\delta}{\gamma-1}}\\ &= \left(\int_{\mathbb{R}_+} dt\, t^{\gamma-1}\, f(t)^2\right)^{\frac{\delta-1}{\gamma-1}}\left(\int_{\mathbb{R}_+} dt\, f(t)^2\right)^{\frac{\gamma-\delta}{\gamma-1}}.\end{aligned}$$

■

Lemma 4.3: Let $\gamma, \delta > 0$ be such that either $\gamma < \delta < 1$ or $1 < \delta < \gamma$. Then for all $J \subset \{1, \ldots, n\}$, we have

$$D_\delta(\mathbf{x}_J) \le \left(\frac{\Gamma(\gamma)^{\frac{\delta-1}{\gamma-1}}}{\Gamma(\delta)} \right)^m D_\gamma(\mathbf{x}_J)^{\frac{\delta-1}{\gamma-1}} \, D_1(\mathbf{x}_J)^{\frac{\gamma-\delta}{\gamma-1}},$$

where $m = |J|$.

Proof: Let $\mathbf{t}_J = (t_i)_{i \in J}$. A multidimensional version of Lemma 4.2 shows that for a measurable function $f : \mathbb{R}_+^m \to \mathbb{R}$,

$$\begin{aligned} &\left(\left(\prod_{k \in J} \int_{\mathbb{R}_+} dt_k t_k^{\delta-1} \right) f(\mathbf{t}_J)^2 \right) \\ &\le \left(\left(\prod_{k \in J} \int_{\mathbb{R}_+} dt_k t_k^{\gamma-1} \right) f(\mathbf{t}_J)^2 \right)^{\frac{\delta-1}{\gamma-1}} \left(\left(\prod_{k \in J} \int_{\mathbb{R}_+} dt_k \right) f(\mathbf{t}_J)^2 \right)^{\frac{\gamma-\delta}{\gamma-1}} \end{aligned}$$

provided that both integrals on the right-hand side are finite. From this and Lemma 4.1, we deduce that

$$\begin{aligned} & D_\delta(\mathbf{x}_J) \\ &= \frac{1}{m! \, \Gamma(\delta)^m} \left(\prod_{k \in J} \int_{\mathbb{R}} dt_k \, t_k^{\delta-1} \right) \det \left(\left(e^{-t_i x_j} \right)_{i,j \in J} \right)^2 \\ &\le \frac{1}{m! \, \Gamma(\delta)^m} \left(\left(\prod_{k \in J} \int_{\mathbb{R}} dt_k \, t_k^{\gamma-1} \right) \det \left(\left(e^{-t_i x_j} \right)_{i,j \in J} \right)^2 \right)^{\frac{\delta-1}{\gamma-1}} \\ &\qquad \cdot \left(\left(\prod_{k \in J} \int_{\mathbb{R}} dt_k \right) \det \left(\left(e^{-t_i x_j} \right)_{i,j \in J} \right)^2 \right)^{\frac{\gamma-\delta}{\gamma-1}} \\ &= \frac{1}{m! \, \Gamma(\delta)^m} \left(m! \, \Gamma(\gamma)^m \, D_\gamma(\mathbf{x}_J) \right)^{\frac{\delta-1}{\gamma-1}} \left(m! \, \Gamma(1)^m \, D_1(\mathbf{x}_J) \right)^{\frac{\gamma-\delta}{\gamma-1}} \\ &= \left(\frac{\Gamma(\gamma)^{\frac{\delta-1}{\gamma-1}}}{\Gamma(\delta)} \right)^m D_\gamma(\mathbf{x}_J)^{\frac{\delta-1}{\gamma-1}} \, D_1(\mathbf{x}_J)^{\frac{\gamma-\delta}{\gamma-1}}, \end{aligned}$$

since $\Gamma(1) = 1$. ■

Lemma 4.4: For any $\delta > 0$, there exist $\tilde{K} > 0$ and $\eta > 0$ independent of m such that

$$D_\delta(\mathbf{x}_J) \le \left(\frac{m}{x_{\min}} \right)^{\tilde{K} m} D_1(\mathbf{x}_J)^\eta,$$

where $x_{\min} = \min\{x_i : i = 1, \ldots, n\}$.

Proof: Let us recall the definition of the *permanent* of an $m \times m$ matrix $A(J) = (a_{ij})_{i,j \in J}$:

$$\operatorname{perm}(A(J)) = \sum_{\sigma \in \mathcal{S}(J)} \prod_{i \in J} a_{i,\sigma(i)}.$$

Let us also define for $\gamma > 0$ and $J \subset \{1, \ldots, n\}$:

$$P_\gamma(\mathbf{x}_J) = \operatorname{perm} \left(\left(\frac{1}{(x_i + x_j)^\gamma} \right)_{i,j \in J} \right).$$

Noticing that $D_\gamma(\mathbf{x}_J) \le P_\gamma(\mathbf{x}_J)$, we obtain using Lemma 4.3 that for any $\delta > 0$, there exists $\gamma > 0$ such that either $\gamma < \delta < 1$ or $1 < \delta < \gamma$, and

$$D_\delta(\mathbf{x}_J) \le \left(\frac{\Gamma(\gamma)^{\frac{\gamma-1}{\delta-1}}}{\Gamma(\delta)} \right)^m P_\gamma(\mathbf{x}_J)^{\frac{\delta-1}{\gamma-1}} D_1(\mathbf{x}_J)^{\frac{\gamma-\delta}{\gamma-1}}. \tag{17}$$

By definition of the permanent, we have the trivial bound

$$P_\gamma(\mathbf{x}_J) = \sum_{\sigma \in \mathcal{S}(J)} \prod_{i \in J} \frac{1}{(x_i + x_{\sigma(i)})^\gamma} \le \sum_{\sigma \in \mathcal{S}(J)} \prod_{i \in J} \frac{1}{x_i^\gamma} \le \frac{m!}{(x_{\min})^{\gamma m}}.$$

So (17) finally implies that

$$D_\delta(\mathbf{x}_J) \le \left(\frac{\Gamma(\gamma)^{\frac{\gamma-1}{\delta-1}}}{\Gamma(\delta)} \right)^m \left(\frac{m!}{(x_{\min})^{\gamma m}} \right)^{\frac{\delta-1}{\gamma-1}} D_1(\mathbf{x}_J)^{\frac{\gamma-\delta}{\gamma-1}}.$$

Setting $\tilde{K} = (2 \vee \gamma)\, \frac{\delta-1}{\gamma-1}$ and $\eta = \frac{\gamma-\delta}{\gamma-1}$ completes the proof of the lemma.

■

Proof of Theorem 2.1: second step.

For notational simplicity, we rewrite the vector $\mathbf{x}_J$ as $\mathbf{x}_J = x_{\min}\, \mathbf{z}$, where $\mathbf{z} = (z_1, \ldots, z_m)$ and $1 \le z_1 \le \ldots \le z_m$. We therefore have

$$D_1(\mathbf{x}_J) = \frac{1}{(x_{\min})^m} D_1(\mathbf{z}). \tag{18}$$

We are looking for a uniform upper bound on $D_1(\mathbf{z})$ over all vectors $\mathbf{z} \in \mathcal{Z} = \{\mathbf{z} \in [1, \infty[^m \colon z_1 \le \ldots \le z_m\}$, so we may as well assume that $z_1 = 1$; indeed, if $z_1 > 1$, then $\mathbf{z}' = \frac{1}{z_1}\mathbf{z}$ is such that $z_1' = 1$ and

$$D_1(\mathbf{z}) = \left(\frac{1}{z_1} \right)^m D_1(\mathbf{z}') < D_1(\mathbf{z}').$$

Notice that

$$D_1(\mathbf{z}) = \left(\prod_{1 \le i < j \le m} \frac{z_j - z_i}{z_i + z_j} \right)^2 \prod_{1 \le i \le m} \frac{1}{2 z_i} \le \prod_{1 \le i \le m} \frac{1}{2 z_i},$$

so D_1 reaches its supremum on the set $\mathcal{Z}$. We define next the function V_1 by

$$\begin{aligned} V_1(\mathbf{z}) &= -\log D_1(\mathbf{z}) \\ &= 2 \sum_{1 \le i < j \le m} \log\left(\frac{z_i + z_j}{z_j - z_i} \right) + \sum_{1 \le i \le m} \log(z_i) + m \log(2). \end{aligned}$$

Lemma 4.5: Let $\mathbf{z} \in \mathcal{Z}$ be such that $V_1(\mathbf{z})$ reaches its infimum at $\mathbf{z}$. If $1 \le p \le k < l \le m$, then

$$(k - p + 1)\,(l - k)\, \frac{z_p z_l}{z_l^2 - z_p^2} \le \frac{m-k}{4}. \tag{19}$$

Proof: If $g(t) = V_1(z_1, \ldots, z_k, t z_{k+1}, \ldots, t z_m)$, then

$$\begin{aligned} g'(t) &= 2 \sum_{\substack{1 \le i \le k \\ k < j \le m}} \left[\log\left(\frac{t z_j + z_i}{t z_j - z_i} \right) \right]' + \frac{m-k}{t} \\ &= 2 \sum_{\substack{1 \le i \le k \\ k < j \le n}} \left(\frac{z_j}{t z_j + z_i} - \frac{z_j}{t z_j - z_i} \right) + \frac{m-k}{t}, \end{aligned}$$

so

$$g'(1) = -4 \sum_{\substack{1\le i\le k\\ k<j\le m}} \frac{z_i z_j}{z_j^2 - z_i^2} + m - k.$$

Since V_1 is minimum in $\mathbf{z}$, we have $g'(1) = 0$, i.e.

$$\sum_{\substack{1\le i\le k\\ k<j\le m}} \frac{z_i z_j}{z_j^2 - z_i^2} = \frac{m-k}{4}. \tag{20}$$

But $\frac{z_j z_i}{z_j^2 - z_i^2} = \frac{1}{\frac{z_j}{z_i} - \frac{z_i}{z_j}}$ increases if i increases or j decreases, so

$$\frac{z_p z_l}{z_l^2 - z_p^2} \le \frac{z_i z_j}{z_j^2 - z_i^2}, \quad \text{whenever} \quad p \le i \le k \quad \text{and} \quad k < j \le l.$$

Therefore,

$$\frac{m-k}{4} \ge \sum_{\substack{p\le i\le k\\ k<j\le l}} \frac{z_i z_j}{z_j^2 - z_i^2} \ge (k-p+1)\,(l-k)\,\frac{z_p z_l}{z_l^2 - z_p^2}.$$

■

Corollary 4.6: Let $\mathbf{z} \in \mathcal{Z}$ be such that $V_1(\mathbf{z})$ reaches its infimum at $\mathbf{z}$. If $1 \le p \le k < l \le m$, then

$$\frac{z_l}{z_p} \ge \frac{4(k-p+1)(l-k)}{m-k}. \tag{21}$$

Proof: Let $x > 0$ and $\beta > 0$. If $x - \frac{1}{x} \ge 2\beta$, then $x \ge \beta + \sqrt{1+\beta^2} \ge 2\beta$. The corollary follows by combining this fact with inequality (19) (taking $x = \frac{z_l}{z_p}$ and $\beta = \frac{2(k-p+1)(l-k)}{m-k}$). ■

The proof of Theorem 2.1 relies on the following key estimate.

Lemma 4.7: There exists $\hat{K} > 0$ independent of m such that

$$V_1(\mathbf{z}) \ge \hat{K}\, m^{3/2}, \quad \text{or equivalently,} \quad D_1(\mathbf{z}) \le \exp(-\hat{K}\, m^{3/2}), \tag{22}$$

for all $\mathbf{z} \in \mathcal{Z}$.

Remark 4.8: One can actually show that (22) is not only an asymptotic upper bound, but is also tight. More precisely, there exists $\bar{K} > 0$ independent of m such that

$$V_1(\mathbf{z}) \le \bar{K}\, m^{3/2}, \quad \text{if } z_j = \exp\left(\frac{j-1}{\sqrt{m}}\right),\ j = 1, \dots, m.$$

Proof: (Lemma 4.7)

Let $\mathbf{z} \in \mathcal{Z}$ be the point where V_1 reaches its infimum. Let $r = 1 + \lfloor\sqrt{m-1}\rfloor$, where $\lfloor\sqrt{m-1}\rfloor$ is the integer part of $\sqrt{m-1}$. By (21), we have

$$\frac{z_{k+r}}{z_{k-r+1}} \ge \frac{4r^2}{m-k} \ge \frac{4r^2}{m-1} \ge 4, \quad \text{if} \quad r \le k \le m-r,$$

i.e.

$$\frac{z_{s+2r-1}}{z_s} \ge 4, \quad \text{if} \quad 1 \le s \le m+1-2r, \quad \text{so} \quad z_{t(2r-1)+1} \ge 4^t.$$

Let us write $m = (2r-1)t + q$ (q and t integers, $0 \le q < 2r-1$). We obtain that

$$\begin{aligned}\sum_{1\le i\le m} \log z_i &\ge (2r-1)\log 1 + (2r-1)\log 4 + \dots \\ &\quad + (2r-1)\log(4^{t-1}) + q\log(4^t) \\ &\ge (2r-1)\,\frac{t(t-1)}{2}\,\log 4.\end{aligned}$$

So for all $\varepsilon > 0$, there exists $m_0 \ge 1$ such that

$$\sum_{1\le i\le m} \log z_i \ge m^{3/2}\,(1-\epsilon)\,\frac{\log 4}{4}, \quad \text{whenever} \quad m \ge m_0.$$

This concludes the proof, since $V_1(\mathbf{z}) \ge \sum_{1\le i\le m} \log z_i + m\,\log(2)$.

■

Proof of Theorem 2.1: conclusion.

Let us now gather together all estimates. By (9) and (12), we have

$$\begin{aligned}&\exp(C_n/2) \\ &\le \det\left(I + \sqrt{nP}\,H\right) = \sum_{J\subset\{1,\dots,n\}} \det\left(\sqrt{nP}\,H(J)\right) \\ &= \sum_{J\subset\{1,\dots,n\}} (nP)^{|J|/2}\, D_\delta(\mathbf{x}_J) \\ &= \sum_{m=0}^{n} (nP)^{m/2} \sum_{\substack{J\subset\{1,\dots,n\}\\ |J|=m}} D_\delta(\mathbf{x}_J).\end{aligned}$$

Using successively (14) and (16), we obtain

$$\begin{aligned}&\exp(C_n/2) \\ &\le \sum_{m=0}^{n} (nP)^{m/2} \left(\frac{m}{x_{\min}}\right)^{\tilde{K}m} \sum_{\substack{J\subset\{1,\dots,n\}\\ |J|=m}} D_1(\mathbf{x}_J)^\eta \\ &\le \sum_{m=0}^{n} (nP)^{m/2} \left(\frac{m}{x_{\min}}\right)^{(\tilde{K}+1)m} \sum_{\substack{J\subset\{1,\dots,n\}\\ |J|=m}} \exp(-\hat{K}\eta\, m^{3/2}).\end{aligned}$$

Since

$$\sum_{\substack{J\subset\{1,\dots,n\}\\ |J|=m}} 1 = \begin{pmatrix} n \\ m \end{pmatrix} \le n^m,$$

we obtain that for some $\tilde{K}_1$, $\hat{K}_1 > 0$:

$$\exp(C_n/2) \le \sum_{m=0}^{n} \exp(\tilde{K}_1 m \log(n/x_{\min}) - \hat{K}_1 m^{3/2}).$$

Choosing $m_0 = \left(\frac{\tilde{K}_1}{\hat{K}_1}\log(n/x_{\min})\right)^2$, we moreover have

$$\begin{aligned}
\exp(C_n/2) &\leq \sum_{m=0}^{m_0-1}\exp(\tilde{K}_1 m\log(n) - \hat{K}_1 m^{3/2}) \\
&\quad + \sum_{m=m_0}^{n}\exp(\tilde{K}_1 m\log n - \hat{K}_1 m^{3/2}) \\
&\leq \sum_{m=0}^{m_0-1}\exp(\tilde{K}_1 m_0\log n) + \sum_{m=m_0}^{n} 1 \\
&= m_0\,\exp(\tilde{K}_1 m_0\log n) + n.
\end{aligned}$$

Notice that for any $\zeta > 0$, we have

$$P\left(x_{\min} \leq \frac{1}{n^\zeta}\right) \leq n\,P\left(x_1 \leq \frac{1}{n^\zeta}\right) = n\,\frac{1}{n^{1+\zeta}} \underset{n\to\infty}{\rightarrow} 0,$$

which means that

$$x_{\min} \geq \frac{1}{n^\zeta}, \quad \text{with high probability.}$$

Although any $\zeta > 0$ is sufficient to prove Theorem 2.1, let us choose $\zeta = 3$ to establish a stronger result that will be needed in Section VI. With $\zeta = 3$ we have,

$$P\left(x_{\min} \leq \frac{1}{n^3}\right) \leq \frac{1}{n^3}. \tag{23}$$

Since $m_0 = \left(\frac{\tilde{K}_1}{\hat{K}_1}(\log(n) + \log(1/x_{\min}))\right)^2$, we conclude that for all $\varepsilon > 0$, there exists $K_1 > 0$ such that for sufficiently large n,

$$\exp(C_n/2) \leq \exp(K_1\,(\log n)^{3+\varepsilon}), \quad \text{with high probability,}$$

which completes the proof of Theorem 2.1. ■

V. Two-Dimensional Networks

Our approach for two-dimensional networks is to divide the planar network into horizontal strips and make use of the result obtained for one-dimensional networks. The motivation behind is that the capacity of a strip should not be asymptotically different from a linear network since the degrees of freedom provided in one of the dimensions is strictly limited for the strip and will be exhausted with increasing number of users. Although we are unable to show that a strip and a one-dimensional network behave similarly asymptotically, the approach is still useful and gives us improved upper bounds for planar networks[3].

Proof of Theorem 2.2: Let us start by dividing the domain D into $N = \frac{\sqrt{n}}{\epsilon}$ equal strips, namely

$$S_i = [-\sqrt{n}, \sqrt{n}] \times [(i-1)\epsilon, i\epsilon] \qquad \text{for} \qquad i = 1, 2 \ldots N. \tag{24}$$

Let us denote the total number of users in the horizontal strip S_i by the random variable $2n_i$. Recall that due to the introduction of "mirror" users, each strip has symmetric left and right-hand side configuration. See Figure 2. We recall the generalized Hadamard's Inequality (or Fischer's Inequality). See [9, Thm

[3]Part of this work was presented in [8].

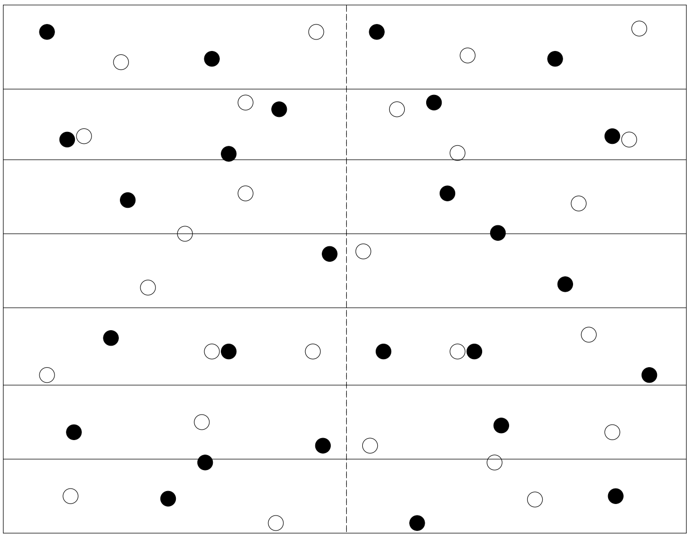

Fig. 2. The division of the two-dimensional network into horizontal strips.

9.C.1]. If $A^{(n)}$ is $n \times n$ Hermitian non-negative definite matrix and $(A^{m_i}, i = 1, \ldots, p)$ are the diagonal blocks of A of given sizes $\{m_i\}$ (such that $n = \sum_{i=1}^{p} m_i$) then

$$\det(A) \leq \prod_{i=1}^{p} \det\left(A^{m_i}\right).$$

By possibly re-numbering the nodes, we can apply this inequality to the non-negative definite matrix $(I + \sqrt{nP} G^n)$ with the diagonal blocks referring to the N strips we have introduced. This yields

$$C_n^{2D} \leq 2 \log\det(I + \sqrt{nP} G^n) \leq \sum_{i=1}^{N} 2 \log\det(I + \sqrt{nP} G^{n_i}\left\{\mathcal{S}_i\right\}). \tag{25}$$

where $\mathcal{S}_i$ refers to the configuration of the strip S_i. Let us now consider the expected value of this upper bound over random node locations, thus

$$\begin{aligned} E_{(x_1,y_1),\ldots,(x_n,y_n)} & \left[\log\det(I + \sqrt{nP} G^n)\right] \\ & \leq E_{n_1,\ldots,n_N}\left[E_{X(\mathcal{S}_1),\ldots,X(\mathcal{S}_n);Y(\mathcal{S}_1),\ldots,Y(\mathcal{S}_n)}\left[\sum_{i=1}^{N} \log\det(I + \sqrt{nP} G^{n_i}\left\{\mathcal{S}_i\right\})\right]\right] \\ & = \sum_{i=1}^{N} E_{n_i}\left[E_{X(\mathcal{S}_i)}\left[E_{Y(\mathcal{S}_i)}\left[\log\det(I + \sqrt{nP} G^{n_i}\left\{\mathcal{S}_i\right\})\right]\right]\right] \end{aligned} \tag{26}$$

where the subscripts denote the variables with respect to which the expectation is performed. $X(\mathcal{S}_i)$ and $Y(\mathcal{S}_i)$ refers to the collection $(x_1, \ldots, x_{n_i})$ and $(y_1, \ldots, y_{n_i})$ denoting the x and y-coordinates of the nodes in S_i respectively. It is easy to see that the terms in (26) governing different strips S_i are equal. Without loss of generality, we concentrate on the strip S_1 with number of users n_1 and configuration $\mathcal{S}_1$. For notational convenience, we denote the matrix $G^{n_1}\left\{\mathcal{S}_1\right\}$ by G^{n_1}, $X(\mathcal{S}_1)$ by X and $Y(\mathcal{S}_1)$ by Y, however we keep in mind that the node locations (x_i, y_i), $1 \leq i \leq n_1$ are now uniformly and independently distributed on the set $S_1 = [0, \sqrt{n}] \times [0, \epsilon]$. Considering the inner most expectation for given n_1 and a set of X and recalling that $\log\det(\cdot)$ is a concave function on the set of positive definite matrices, we apply Jensen's Inequality to obtain

$$E_Y\left[\log\det(I+\sqrt{nP}G^{n_1})\right] \leq \log\det\left(I+\sqrt{nP}E_Y\left[G^{n_1}\right]\right). \tag{27}$$

Let us recall the expression for the entries of the matrix G^{n_1} given by (6), that is

$$G_{ij}^{n_1} = \frac{1}{((x_i+x_j)^2+(y_i-y_j)^2)^{\frac{\delta}{2}}}. \tag{28}$$

Note that given $(x_i, i=1,\ldots,n_1)$, each y_i, $1\leq i\leq n_1$ is uniformly and independently distributed in the interval $[0,\epsilon]$ and the random variable $y=(y_i-y_j)^2$ has a distribution $p_y(y)$ supported on the interval $[0,\epsilon^2]$ when $i\neq j$. Thus the entries of the matrix $E_Y\left[G^{n_1}\right]$ are given by

$$\begin{aligned} E_Y\left[G_{\substack{ij\\ i\neq j}}^{n_1}\right] &= \int_0^{\epsilon^2} p_y(y)\frac{1}{((x_i+x_j)^2+y)^{\frac{\delta}{2}}}dy \\ E_Y\left[G_{ii}^{n_1}\right] &= \frac{1}{(2x_i)^{\delta}}. \end{aligned} \tag{29}$$

The matrix $E_Y\left[G^{n_1}\right]$ can be written as a sum of two matrices

$$E_Y\left[G^{n_1}\right] = D'^{n_1} + G'^{n_1}$$

where G'^{n_1} is the matrix whose entries are given by

$$G'^{n_1}_{ij} = \int_0^{\epsilon^2} p_y(y)\frac{1}{((x_i+x_j)^2+y)^{\frac{\delta}{2}}}dy$$

and D'^{n_1} is the diagonal matrix that compensates the difference between the diagonal entries of G'^{n_1} and $E_Y\left[G^{n_1}\right]$. Thus,

$$D'^{n_1}_{ii} = \frac{1}{(2x_i)^{\delta}}\int_0^{\epsilon^2} p_y(y)\left(1-\left(1+\frac{y}{(2x_i)^2}\right)^{-\frac{\delta}{2}}\right)dy.$$

The entries of the diagonal matrix $D'^{n_1}_{ii}$ can be bounded above by making use of the relation

$$1-(1+x)^{-\gamma} = \int_0^x \gamma(1+z)^{-\gamma-1}dz \leq \gamma x \sup_{z\in[0,x]}(1+z)^{-\gamma-1} \leq \gamma x$$

which yields

$$D'^{n_1}_{ii} \leq \frac{\delta/2}{(2x_i)^{\delta+2}}\int_0^{\epsilon^2} y p_y(y)dy \leq \frac{\delta\epsilon^2}{2(2x_i)^{\delta+2}} = D_{ii}^{n_1} \tag{30}$$

where D^{n_1} is defined as the upper bounding diagonal matrix. In Appendix II, we prove that the difference matrix $H^{n_1}-G'^{n_1}$ whose entries are given by

$$H_{ij}^{n_1} - G'^{n_1}_{ij} = \frac{1}{(x_i+x_j)^{\delta}} - \int_0^{\epsilon^2} p_y(y)\frac{1}{((x_i+x_j)^2+y)^{\frac{\delta}{2}}}dy.$$

is non-negative definite. This fact together with (30) implies that $D^{n_1} + H^{n_1} - D'^{n_1} - G'^{n_1}$ is a non-negative definite matrix. Recalling that the $\log\det(\cdot)$ is not only concave, but also increasing on the set of non-negative definite matrices (see [9], 16.E) gives

$$\begin{aligned}\log\det\left(I + \sqrt{nP}E_Y\left[G^{n_1}\right]\right) &= \log\det\left(I + \sqrt{nP}D'^{n_1} + \sqrt{nP}G'^{n_1}\right)\\ &\le \log\det\left(I + \sqrt{nP}D^{n_1} + \sqrt{nP}H^{n_1}\right)\\ &\le \log\det\left(I + \sqrt{nP}D^{n_1}\right) + \log\det\left(I + \sqrt{nP}H^{n_1}\right)\end{aligned} \tag{31}$$

The last inequality in (31) is a property of non-negative definite matrices and can be obtained by specializing the following entropy relation to independent Gaussian vectors.

Lemma 5.1: If X, Y and Z are independent random variables (or vectors) then,

$$h(Y + X + Z) + h(X) \le h(Y + X) + h(X + Z).$$

Proof: First note that the random variables Y, $Y + X$ and $Y + X + Z$ form a Markov chain and the data processing theorem gives us

$$I(Y; Y + X + Z) \le I(Y; Y + X).$$

Expanding this inequality gives the desired result:

$$\begin{aligned}I(Y; Y + X + Z) &\le I(Y; Y + X)\\ h(Y + X + Z) - h(Y + X + Z \mid Y) &\le h(Y + X) - h(Y + X \mid Y)\\ h(Y + X + Z) - h(X + Z) &\le h(Y + X) - h(X)\\ h(Y + X + Z) + h(X) &\le h(Y + X) + h(X + Z)\end{aligned}$$

■

Looking closer at the two terms in (31), we notice that the second term resembles the upper bound (9) governing one-dimensional networks except that H^{n_1} is now $n_1 \times n_1$ matrix with $n_1 \le n$. However, by the interlacing property of symmetric matrices (see [10, Thm 4.3.8]) the n_1 largest eigenvalues of the matrix H^n that has H^{n_1} as an upper left submatrix dominate the eigenvalues of H^{n_1}. Using also the fact that $I + \sqrt{nP}H^n$ has all its eigenvalues larger than 1 since H^n is non-negative definite (see Appendix I), we have

$$\begin{aligned}\log\det\left(I + \sqrt{nP}H^{n_1}\right) &\le \log\det\left(I + \sqrt{nP}H^{n}\right)\\ &\le K_1(\log n)^{3+\varepsilon}, \quad \text{with high probability,}\end{aligned} \tag{32}$$

for any $\varepsilon > 0$.

For the first term in (31), let us consider the expectation over $(x_i, i = 1, \dots, n_1)$ denoted by X in the following equation, thus

$$\begin{aligned}E_X\left[\log\det\left(I + \sqrt{nP}D^{n_1}\right)\right] &= E_X\left[\sum_{i=1}^{n_1}\log\left(1 + \sqrt{nP}\frac{\delta\epsilon^2}{2(2x_i)^{\delta+2}}\right)\right]\\ &= n_1\int_0^{\sqrt{n}}\frac{1}{\sqrt{n}}\log\left(1 + \frac{\delta\sqrt{P}}{2}\frac{n^{\frac{1}{2}-2\eta}}{(2x)^{\delta+2}}\right)dx\end{aligned} \tag{33}$$

where (33) is obtained by choosing $\epsilon = n^{-\eta}$ with $\eta > 0$. We have the following lemma from [4, Lemma 2.2] which states that for any $C, p > 0$ and $\gamma > 1$, there exists a constant $K' > 0$ such that for all sufficiently large n,

$$\int_0^n dx \log\left(1 + \frac{Cn^p}{x^\gamma}\right) \leq K' n^{\frac{p}{\gamma}\wedge 1} \log n$$

where $a \wedge b$ is the minimum of a and b. Applying this lemma to (33) and performing the last expectation in (26) with respect to n_1 yields

$$E_{n_1}\left[E_X\left[\log\det\left(I + \sqrt{nP}D^{n_1}\right)\right]\right] \leq \epsilon K' n^{\frac{1-4\eta}{2(\delta+2)}} \log n$$

since the expected number of nodes in S_1 is $\frac{n}{N} = \epsilon\sqrt{n}$.

Combining all the results we have obtained until now yields the following expectation result:

$$\begin{aligned} E_{(x_1,y_1),\ldots,(x_n,y_n)}\left[\log\det(I + \sqrt{nP}G^n)\right] &\leq \sum_{i=1}^{N}\left(\epsilon K' n^{\frac{1-4\eta}{2(\delta+2)}} \log n + K''(\log n)^{3+\varepsilon}\right) \\ &= K'\sqrt{n} n^{\frac{1-4\eta}{2(\delta+2)}} \log n + K''\sqrt{n} n^{\eta} (\log n)^{3+\varepsilon} \\ &\leq K\sqrt{n} n^{\frac{1}{2(\delta+4)}+\varepsilon} \end{aligned} \tag{34}$$

for any $\varepsilon > 0$, by choosing $\eta = \frac{1}{2(\delta+4)}$.

There remains to prove that there is concentration around the expectation and that the sublinear behavior of this upper bound on C_n^{2D} takes place almost surely. For notational convenience, let us define Φ^n to be the following real-valued function of node locations $(L_1, L_2, \ldots, L_n)$,

$$\Phi^n(L_1, L_2, \ldots, L_n) := \log\det(I + \sqrt{nP}G^n\{L_1, L_2, \ldots, L_n\}).$$

We set out to prove the following proposition.

Proposition 5.2: For any $\varepsilon > 0$ and $\lambda > 0$, there exists a constant K independent of n such that

$$\mathbb{P}\left(|\Phi^n(L_1, L_2, \ldots, L_n) - E\left[\Phi^n(L_1, L_2 \ldots, L_n)\right]| \geq \lambda n^{\frac{1}{2}+\varepsilon}\right) \leq \frac{K}{n^3}.$$

Before the proof of Proposition 5.2, we introduce a concentration inequality due to McDiarmid [11] that will be used in the proof of the proposition.

Theorem 5.3: Let $(L_1, L_2, \ldots, L_n)$ be a family of independent random variables with L_k taking values in a set A_k for each k. Suppose that the real-valued function f defined on ΠA_k satisfies

$$\sup_{l_1\in A_1\ldots l_n\in A_n,\, l'_k\in A_k} |f(l_1, \ldots, l_k, \ldots, l_n) - f(l_1, \ldots, l_{k-1}, l'_k, l_{k+1}, \ldots, l_n)| \leq c_k.$$

Then, for any $t \geq 0$

$$P\left(|f(L_1, L_2, \ldots, L_n) - E[f(L_1, L_2, \ldots, L_n)]| \geq t\right) \leq 2e^{-2t^2/\sum c_k^2}.$$

The proof of Proposition 5.2 is based on applying Theorem 5.3 to the function Φ^n. The crucial step here is to properly bound the amount of change in the value of the function Φ^n due to a change in one of its parameters. Notice however that Φ^n is unbounded since the x_i's (the random variables denoting the horizontal positions of the right-hand side nodes) can be arbitrarily close to zero, which implies that the corresponding diagonal elements of G^n go to infinity. This rather technical problem can be overcome by showing that the x_i's are all bounded away from zero with high probability as n goes to infinity and that under the condition that they are all bounded away from zero, the amount Φ^n can be affected from a change in one of the node positions is bounded by $\log n$. The concentration inequality in Theorem 5.3 can then be used to show that Φ^n concentrates around its mean with deviation of order less than $n^{\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$.

Proof of Proposition 5.2: Let us fix $\mu > 0$. As in the one-dimensional case, the probability that any of the x_i's is smaller than $n^{-(\frac{1}{2}+\mu)}$ is bounded above by

$$P\left(x_{min} \leq n^{-(\frac{1}{2}+\mu)}\right) \leq nP\left(x_1 \leq n^{-(\frac{1}{2}+\mu)}\right) = n^{-\mu},$$

since the x_i's are uniformly and independently distributed on $[0, \sqrt{n}]$. On the other hand, under the condition that $x_{min} \geq n^{-(\frac{1}{2}+\mu)}$, $(L_1, L_2, \ldots, L_n)$ is still a family of independent random variables where each L_i is now uniformly distributed on the set $\left[n^{-(\frac{1}{2}+\mu)}, \sqrt{n}\right] \times [0, \sqrt{n}]$.

We continue the rest of our analysis conditioned on $x_{min} \geq n^{-(\frac{1}{2}+\mu)}$. Let $(L_1, \ldots, L_{n-1}, L_n)$ and $(L_1, \ldots, L_{n-1}, L'_n)$ be two configurations that differ only in the last coordinate, that is, the position of the nth node. Let Ψ^{n-1} be defined as the following function of $n-1$ node locations:

$$\Psi^{n-1}(L_1, \ldots, L_{n-1}) := \log\det(I + \sqrt{nP}G^{n-1}\{L_1, \ldots, L_{n-1}\}).$$

Next, we bound the variation of the function Φ^n between the two configurations:

$$\begin{aligned} &|\Phi^n(L_1, \ldots, L_{n-1}, L_n) - \Phi^n(L_1, \ldots, L_{n-1}, L'_n)| \\ &\quad \leq |\Phi^n(L_1, \ldots, L_{n-1}, L_n) - \Psi^{n-1}(L_1, \ldots, L_{n-1})| \\ &\qquad + |\Psi^{n-1}(L_1, \ldots, L_{n-1}) - \Phi^n(L_1, \ldots, L_{n-1}, L'_n)|. \end{aligned} \tag{35}$$

Concentrating on the first term in (35), let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of the $n \times n$ symmetric matrix $I + \sqrt{nP}G^n\{L_1, \ldots, L_{n-1}, L_n\}$ and $\hat{\lambda}_1 \leq \cdots \leq \hat{\lambda}_{n-1}$ be the eigenvalues of the $n-1 \times n-1$ symmetric matrix $I + \sqrt{nP}G^{n-1}\{L_1, \ldots, L_{n-1}\}$. Note that by the interlacing property of symmetric matrices, we have

$$1 \leq \lambda_1 \leq \hat{\lambda}_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \hat{\lambda}_{n-1} \leq \lambda_n.$$

Moreover, all the eigenvalues are greater than 1 since G is a non-negative definite matrix. Expressing the functions Φ^n and Ψ^{n-1} in terms of these eigenvalues and recalling that the logarithm function is monotonically increasing yields

$$\begin{aligned} |\Phi^n(L_1, \ldots, L_n) - \Psi^{n-1}(L_1, \ldots, L_{n-1})| &= \Big|\sum_{i=1}^{n}\log\lambda_i - \sum_{i=1}^{n-1}\log\hat{\lambda}_i\Big| \\ &= \sum_{i=1}^{n}\log\lambda_i - \sum_{i=1}^{n-1}\log\hat{\lambda}_i \\ &\leq \log\lambda_n. \end{aligned}$$

The largest eigenvalue λ_n of $I + \sqrt{nP}G^{\delta,n}$ can be bounded by the trace of the matrix and the condition $x_{min} \geq n^{-(\frac{1}{2}+\mu)}$ implies that

$$\log\lambda_n \leq \log\sum_{i=1}^{n}\left(1 + \frac{\sqrt{nP}}{(2x_i)^\delta}\right) \leq c_1(\mu) + c_2(\mu)\log n$$

where $c_1(\mu), c_2(\mu) > 0$ are constants independent of n. The argument for the first term in (35) holds similarly for the second term. Furthermore, since the numbering of the nodes is arbitrary, the same bound applies whenever the two configurations differ in a single node location, this single node being any of the n nodes. We can therefore apply Theorem 5.3 and obtain

$$P\left(|\Phi^n - E[\Phi^n]| \geq \lambda n^{\frac{1}{2}+\varepsilon}\right) \leq P\left(x_{min} \geq n^{-(\frac{1}{2}+\mu)}\right) + P\left(|\Phi^n - E[\Phi^n]| \geq \lambda n^{\frac{1}{2}+\varepsilon} \;\middle|\; x_{min} \geq n^{-(\frac{1}{2}+\mu)}\right)$$
$$\leq n^{-\mu} + 2\ \exp\left(-\frac{2\lambda^2 n^{2\varepsilon}}{4\left(c_1(\mu) + c_2(\mu)\log n\right)^2}\right)$$

for all $\lambda > 0$. Choosing $\mu = 3$ completes the proof of the proposition. ■

Combining the expectation result in (34) with Proposition 5.2 yields

$$C_n^{2D} \leq K\sqrt{n} n^{\frac{1}{2(\delta+4)}+\varepsilon}, \quad \text{with high probability},$$

which completes the proof of Theorem 2.2. ■

VI. Transport Capacity of Random Networks

Let us first consider one-dimensional networks, where the nodes are arbitrarily located on a line and numbered in increasing order along the line. The sum in (5) can be split into two parts, one rendering the information transfer from left to right and the other from right to left

$$\sum_{\substack{i,j\\ i\neq j}} R_{ij} r_{ij} = \sum_{\substack{i,j\\ i<j}} R_{ij} r_{ij} + \sum_{\substack{i,j\\ i>j}} R_{ij} r_{ij}. \tag{36}$$

Considering only the information transfer from left to right and noticing that the same arguments apply for the second term in (36), we have

$$\sum_{\substack{i,j\\ i<j}} R_{ij} r_{ij} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} R_{ij} r_{ij} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} R_{ij} \sum_{k=i}^{j-1} r_{k,k+1} = \sum_{k=1}^{n-1} r_{k,k+1}\left(\sum_{i=1}^{k}\sum_{j=k+1}^{n} R_{ij}\right). \tag{37}$$

Notice that the expression in brackets in the last equality in (37) refers to the total information flow from left to right across a cut passing between the nodes *k* and *k+1*. Considering both of the terms in (36) yields,

$$\sum_{\substack{i,j\\ i\neq j}} R_{ij} r_{ij} = \sum_{k=1}^{n-1} r_{k,k+1}\left(\sum_{\substack{i,j\\ i\leq k<j}} R_{ij} + \sum_{\substack{i,j\\ j\leq k<i}} R_{ij}\right). \tag{38}$$

Equation (38) allows to get a simple upper bound on the transport capacity of arbitrary networks, that is

$$T_c(n) = \sup \sum_{\substack{i,j\\ i\neq j}} R_{ij} r_{ij} \leq 2L \max_{1\leq k\leq n-1} C_n(\mathcal{N}_k) \tag{39}$$

where L is the length of the one-dimensional network, $\mathcal{N}_k$ is a cut between nodes k and $k+1$ and $C_n(\mathcal{N}_k)$ is the total information flow through this cut in either direction, whichever is maximal. Equation (39) implies that if one can uniformly bound above the total information flow through any cut of an arbitrary network, then the transport capacity is simply bounded above by twice this bound times the total length of the network. The idea can be extended to random networks as follows: it is a well known fact that in a linear extended network, the minimum distance between any two nodes in the network is larger than $\frac{1}{n^{1+\gamma}}$ with high probability for any $\gamma > 0$. Hence, if we consider a family of $n^{2+\gamma}$ cuts of the line segment $[-n, n]$ such that successive cuts are separated by a distance $\frac{1}{n^{1+\gamma}}$, with high probability this family contains cuts

$\mathcal{N}_k, k = 1, \ldots, n-1$. Using (23) and the union bound, for any $\varepsilon > 0$ the information flow through all these $n^{2+\gamma}$ cuts is bounded above by

$$C_n^{1D} \le K_1 (\log n)^{3+\varepsilon}$$

with probability $1 - \frac{n^{2+\gamma}}{n^3} = 1 - \frac{1}{n^{1-\gamma}}$. Choosing $0 < \gamma < 1$, we have

$$T_c(n) \le K_3 n (\log n)^{3+\varepsilon}$$

with high probability for a one-dimensional random network.

It is not difficult to apply the same argument to planar networks. Starting with the sum in (5),

$$\sum_{\substack{i,j \\ i \neq j}} R_{ij} r_{ij} \le \sum_{\substack{i,j \\ i \neq j}} R_{ij}(x_{ij} + y_{ij}) = \sum_{\substack{i,j \\ i \neq j}} R_{ij} x_{ij} + \sum_{\substack{i,j \\ i \neq j}} R_{ij} y_{ij} \tag{40}$$

where x_{ij} and y_{ij} refer to the positive horizontal and vertical distances between nodes *i* and *j* respectively and inequality follows from triangle inequality. Note that the resulting terms in (40) have an interesting interpretation: the supremum of each term over all feasible rate matrices will yield the directional transport capacity of the network in the given direction x or y. With the bound in (40), we are back to the one-dimensional case. Substituting the expression in (38) for each of the terms in (40) yields

$$\sum_{\substack{i,j \\ i \neq j}} R_{ij} r_{ij} \le \sum_{k=1}^{n-1} x_{k,k+1} \left(\sum_{\substack{i,j \\ i \le k < j}} R_{ij} + \sum_{\substack{i,j \\ j \le k < i}} R_{ij} \right) + \sum_{k=1}^{n-1} y_{k,k+1} \left(\sum_{\substack{i,j \\ i \le k < j}} R_{ij} + \sum_{\substack{i,j \\ j \le k < i}} R_{ij} \right). \tag{41}$$

Notice that (41) inherently carries the assumptions made for linear networks. In the first term, the nodes are assumed to be numbered in increasing x-coordinate; while in the second term, the nodes are numbered according to their y-coordinate. Equation (41) can also be viewed as considering $n-1$ vertical and $n-1$ horizontal cuts respectively, each cut passing between two successive nodes. Similarly to the one-dimensional case, the equation allows us to establish the following simple upper bound on the transport capacity of two-dimensional networks,

$$T_c(n) \le 2L_x \max_{1 \le k \le n-1} C_n(\mathcal{N}_k^x) + 2L_y \max_{1 \le k \le n-1} C_n(\mathcal{N}_k^y) \tag{42}$$

where x and y refer to two arbitrary orthogonal directions in the plane, L_x and L_y are the total lengths of the network in these directions, $\mathcal{N}_k^x$ is a cut perpendicular to the direction x and $C_n(\mathcal{N}_k^x)$ is the total flow across this cut. $\mathcal{N}_k^y$ and $C_n(\mathcal{N}_k^y)$ are defined in a similar way. The argument is extended similarly to two-dimensional random networks: the minimum horizontal (or vertical) distance between any two nodes in the network is greater than $\frac{1}{n^{3/2+\gamma}}$ with high probability for any $\gamma > 0$. A family of $n^{2+\gamma}$ vertical and $n^{2+\gamma}$ horizontal cuts, such that successive cuts are separated by $\frac{1}{n^{3/2+\gamma}}$ vertical or horizontal distance respectively, contains cuts $\mathcal{N}_k^x, k = 1, \ldots, n-1$ and $\mathcal{N}_k^y, k = 1, \ldots, n-1$ with high probability. Below, we show that for any $\varepsilon > 0$, the expected flow through any cut of the planar network is bounded above by $K\sqrt{n} n^{\frac{1}{2(\delta+4)}+\varepsilon}$ for some constant $K > 0$ independent of n. Proposition 5.2 and the union bound allows to establish concentration simultaneously for all the $2n^{2+\gamma}$ cuts considered. We conclude that for any $\varepsilon > 0$, the transport capacity of a uniformly distributed two-dimensional random network is bounded above by

$$T_c(n) \le K_4 n n^{\frac{1}{2(\delta+4)}+\varepsilon}, \quad \text{with high probability,}$$

for a constant K_4 independent of n.

The expected flow through a vertical or horizontal cut that divide the domain into two asymmetric parts can be bounded above by making slight modifications in the analysis in Section V. The idea is as follows: consider a vertical cut that divides the domain D into two asymmetric parts: $[-(2-a)\sqrt{n}, 0] \times [0, \sqrt{n}]$ and $[0, a\sqrt{n}] \times [0, \sqrt{n}]$, where $0 \le a \le 2$ and we have shifted our coordinate system to align with the cut considered. Similarly to Section V, let us concentrate on the first strip which has n_1 nodes before the introduction of the mirror users, and let us denote the number of nodes in the right-hand side of the cut by n_1', which implies that there are $n_1 - n_1'$ nodes located on the left-hand side. After the introduction of the "mirror" users ("mirror" with respect to the current cut considered), we still have two half strips with symmetrical configuration as before, but the n_1 right hand-side nodes will not be uniformly distributed over the right-half strip in this case. Instead, n_1' of the n_1 right-hand side nodes are uniformly and independently distributed on $[0, a\sqrt{n}] \times [0, \epsilon]$, and the rest $n_1 - n_1'$ are uniformly and independently distributed on $[0, (2-a)\sqrt{n}] \times [0, \epsilon]$. The analysis in Section V follows exactly until (33) where now we have to perform the expectation over random x-locations in two steps, that is

$$\begin{aligned}
&E_X\left[\log\det\left(I + \sqrt{nP}D^\delta\right)\right] \\
&= E_{n_1'}\left[E_X\left[\log\det\left(I + \sqrt{nP}D^\delta\right) \,\Big|\, n_1'\right]\right] \\
&= E_{n_1'}\left[E_X\left[\sum_{i=1}^{n_1}\log\left(1 + \sqrt{nP}\frac{\delta\epsilon^2}{2(2x_i)^{\delta+2}}\right) \,\Bigg|\, n_1'\right]\ \right] \\
&= E_{n_1'}\left[n_1'\int_0^{a\sqrt{n}} \frac{1}{a\sqrt{n}}\log\left(1 + \sqrt{nP}\frac{\delta\epsilon^2}{2(2x)^{\delta+2}}\right)dx\right. \\
&\left. + (n_1 - n_1')\int_0^{(2-a)\sqrt{n}} \frac{1}{(2-a)\sqrt{n}}\log\left(1 + \sqrt{nP}\frac{\delta\epsilon^2}{2(2x)^{\delta+2}}\right)dx\right] \\
&\le \frac{an_1}{2}\frac{1}{a\sqrt{n}}\tilde{K}n^{\frac{1-4\eta}{2(\delta+2)}}\log n + \frac{(2-a)n_1}{2}\frac{1}{(2-a)\sqrt{n}}\hat{K}n^{\frac{1-4\eta}{2(\delta+2)}}\log n
\end{aligned} \tag{43}$$

when $\epsilon = n^{-\eta}$ with $\eta > 0$. Note that the upper bound in (43) is independent of the cut considered. The rest of the analysis follows in complete analogy with Section V.

VII. Conclusion

We have established information theoretic upper bounds on the maximally achievable rate per communication pair and the transport capacity in one and two-dimensional random wireless networks when the medium is not absorptive and the attenuation is moderate. Our results show that in the absence of small scale fading, the low attenuation regime does not behave significantly different from the high attenuation regime.

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Appendix I
Non-negativity of the matrices H and G

Let us first consider the one-dimensional case. The entries of the matrix H^n are given by

$$H^n_{jk} = \frac{1}{(x_j + x_k)^{\alpha/2}} = \int_0^\infty dt\, \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)}\, e^{-(x_j+x_k)t},$$

where Γ is the Euler Gamma function. This implies that H^n is non-negative definite, since

$$\sum_{j,k=1}^{n} H^n_{jk}\, c_j\, c_k = \int_0^\infty dt\, \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)} \left(\sum_{j=1}^{n} e^{-x_j t}\, c_j \right)^2 \geq 0.$$

Let us now consider the two-dimensional case. We have the following expression for the entries of the matrix G^n:

$$G^n_{jk} = \frac{1}{\left((x_j + x_k)^2 + (y_j - y_k)^2\right)^{\alpha/4}},$$

so using the fact that the Fourier transform of

$$f_a(y) = \frac{1}{(a^2 + y^2)^{\alpha/4}}$$

is given by (see [12, formulas I.2.7 and I.18.29])

$$\hat{f}_a(\xi) \;=\; C_\alpha \left(\frac{\xi^2}{a}\right)^{\frac{\alpha-2}{4}} K_{\frac{\alpha-2}{4}}(a\,|\xi|),$$

where K_ν is the modified Bessel function of second kind and of order ν, we obtain that

$$G^n_{jk} = \frac{C_\alpha}{(2\pi)} \int_{\mathbb{R}} d\xi \left(\frac{|\xi|}{x_j + x_k}\right)^{\frac{\alpha-2}{4}} K_{\frac{\alpha-2}{4}}\left((x_j + x_k)\,|\xi|\right)\, e^{i\xi(y_j - y_k)}.$$

Since by formula 9.6.23 in [13], we have

$$\frac{1}{r^\nu}\, K_\nu(r) = \frac{\sqrt{\pi}}{2^\nu\, \Gamma(\nu + \frac{1}{2})} \int_0^\infty dt\, e^{-r\cosh(t)}\, \sinh^{2\nu}(t),$$

for $\nu > -\frac{1}{2}$, we obtain that the matrix M whose entries are given by

$$M_{jk} = \left(\frac{|\xi|}{x_j + x_k}\right)^{\frac{\alpha-2}{4}} K_{\frac{\alpha-2}{4}}\left((x_j + x_k)\,|\xi|\right)$$

is non-negative definite. So G^n is a convex combination of products of symmetric non-negative definite matrices, it is therefore itself non-negative definite.

APPENDIX II
NON-NEGATIVITY OF THE MATRIX $H - G'$

In this appendix, we prove that the matrix whose entries are given by

$$H_{jk} - G'_{jk} = \frac{1}{(x_j + x_k)^\delta} - \int_0^{\epsilon^2} p_y(y) \frac{1}{((x_j + x_k)^2 + y)^{\frac{\delta}{2}}} dy$$

is non-negative definite. Obviously, it is sufficient to prove that

$$H_{jk} - G''_{jk} = \frac{1}{(x_j + x_k)^\delta} - \frac{1}{((x_j + x_k)^2 + y)^{\frac{\delta}{2}}}$$

is a non-negative definite matrix for each y, since $H - G'$ is a convex combination of matrices of this type. The proof is actually straightforward when the following equivalent expression for the entries of G'' is considered

$$\begin{aligned} G''_{jk} &= \frac{1}{\pi\Gamma(\delta)} \int_0^\infty dt \int_R d\xi (|\xi| \sinh t)^{\delta-1} e^{-|\xi|(\cosh t)(x_j+x_k) - i\xi y} \\ &= \frac{2}{\pi\Gamma(\delta)} \int_0^\infty dt \int_0^\infty d\xi (\xi \sinh t)^{\delta-1} e^{-\xi(\cosh t)(x_j+x_k)} \cos(\xi y) \end{aligned}$$

where Γ is the Euler Gamma function. The expression is valid for $\delta > 0$ and can be obtained by considering [12, formulas I.2.7 and I.18.29] and [13, formula 9.6.23]. Noticing that

$$H_{jk} = \frac{2}{\pi\Gamma(\delta)} \int_0^\infty dt \int_0^\infty d\xi (\xi \sinh t)^{\delta-1} e^{-\xi(\cosh t)(x_j+x_k)}$$

yields

$$\sum_{j,k=1}^n \left(H_{jk} - G''_{jk}\right) c_j c_k = \frac{2}{\pi\Gamma(\delta)} \int_0^\infty dt \int_0^\infty d\xi (\xi \sinh t)^{\delta-1} \left(\sum_{j=1}^n c_j e^{-\xi x_j \cosh t} \right)^2 (1 - \cos \xi y) \geq 0$$

which proves the desired result.

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