A conjecture about Minkowski additions of convex polytopes *

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Abstract

This is a short paper on different proofs for special cases of a conjecture about Minkowski sums of polytopes.

1 Introduction

Any face of a Minkowski sum of polytopes can be *decomposed* uniquely into a sum of faces of the summands. We will say that the decomposition is *exact* when the dimension of the sum is equal to the sum of the dimensions of the summands. When all facets have an exact decomposition, we will say the summands are *relatively in general position.*

This is our main conjecture:

Conjecture 1 *Let P_1 and P_2 be d-dimensional polytopes relatively in general position, and $P = P_1 + P_2$ their Minkwoski sum. Then*

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}k(f_k(P) - f_k(P_1) - f_k(P_2)) = 0$$

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Note at the form is rather similar to Euler's Equation:

$$\sum_{k=0}^{d-1}(-1)^{d-1-k} f_k(P) = 1 - (-1)^d$$

By using Euler's Equation, we can write the conjecture slightly differently:

Corollary 1 *Let P_1 and P_2 be d-dimensional polytopes relatively in general position, and $P = P_1 + P_2$ their Minkwoski sum. Then*

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}(k+a)(f_k(P) - f_k(P_1) - f_k(P_2)) = -a + a(-1)^d$$

The conjecture has an interesting application when used in conjunction with this theorem:

Theorem 1 *Let P be a perfectly centered polytope. A subset H of $P + P^*$ is a nontrivial face of $P + P^*$ if and only if $H = G + F^D$ for some ordered nontrivial faces $G \subseteq F$ of P.*

A polytope and its dual satisfy the general position condition posed by the main conjecture, which makes it a general statement about lattices of polytopes, or at least lattices of polytopes which can be made perfectly centered. We have as yet no idea whether some polytopes can't be made perfectly centered or all of them can.

2 Proof for particular cases

Here is the proof a few few special cases.

2.1 Proof for zonotopes sums

Zonotopes f-vectors are completely known. Since the sum of two zonotopes is again a zonotope, it is possible to prove the following:

Theorem 2 *Let $Z_d^{m_1}$ and $Z_d^{m_2}$ be two d-dimensional zonotope in generated by m_1 respectively m_2 segments in general position, then the main conjecture is true for their Minkowski sum.*

Proof. As stated in [3] the f-vector of Z_d^m is given by:

$$f_k(Z_d^m) = 2\binom{m}{k}\sum_{h=0}^{d-k-1}\binom{m-k-1}{h}$$

Using the identity $k \begin{pmatrix} m \\ k \end{pmatrix} = m \begin{pmatrix} m-1 \\ k-1 \end{pmatrix}$, and defining $d' = d-1$, $k' = k-1$ and $m' = m-1$, we can write:

$$\sum_{k=0}^{d-1} (-1)^{d-1-k} k f_k(Z_d^m) =$$

$$\sum_{k=1}^{d-1} (-1)^{d-1-k} k 2 \begin{pmatrix} m \\ k \end{pmatrix} \sum_{h=0}^{d-k-1} \begin{pmatrix} m-k-1 \\ h \end{pmatrix} =$$

$$\sum_{k'=0}^{d'-1} (-1)^{d'-1-k'} m 2 \begin{pmatrix} m' \\ k' \end{pmatrix} \sum_{h=0}^{d'-k'-1} \begin{pmatrix} m'-k'-1 \\ h \end{pmatrix} =$$

$$m \sum_{k'=0}^{d'-1} (-1)^{d'-1-k'} f_{k'}(Z_{d'}^{m'})$$

Since $Z_{d'}^{m'}$ is a polytope, Euler's formula tells us that the alternating sum of its f-vector is equal to $1-(-1)^{d'}$, and so

$$\sum_{k=0}^{d-1} (-1)^{d-1-k} k f_k(Z_d^m) = m(1+(-1)^d)$$

Obviously, $(m_1+m_2)(1+(-1)^d) - m_1(1+(-1)^d) - m_2(1+(-1)^d) = 0$ ∎

2.2 Proof for nesterov roundings of perfectly centered polytopes

We have managed to have proof for the particular case of perfectly centered polytopes summed with their own dual. We will use for this the Dehn-Sommerville relations for extended f-vectors:

Lemma 1 *Let P be an Eulerian poset of rank d, $S \subset \{0,\ldots,d-1\}$, $\{i,k\} \subseteq S \cup \{-1,d\}$, $i < k-1$, and S contains no j so that $i < j < k$. Then*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup j}(P) = f_S(P)(1-(-1)^{k-i-1}).$$

These relations have been proved in [1]. It has also been proved in [4] that convex polytopes lattices are Eulerian.

For convenience, we will sometime use the following notations:

$$f_{a,b,c} := f_{\{a,b,c\}} \quad f_{a,a,b} = f_{a,b}$$

(We can define $f_{a,a,b}$ as the number of chains of three faces F_1, F_2 and F_3 of respective dimension a, a and b, so that $F_1 \subseteq F_2 \subseteq F_3$. This is obviously equal to $f_{a,b}$.)

If we examine the special case of D-S relations where $S = \{i,k\} \subseteq \{-1,\ldots,d\}$, we can write the following equation:

Lemma 2

$$\sum_{j=i}^{k} (-1)^j f_{i,j,k}(P) = 0$$

Using Thm 1, since G and F^D are subsets of orthogonal spaces, we can write a formula for the f-vector of $P+P^*$ using the extended f-vector of P.

Theorem 3 *Let P be a perfectly centered polytope, then the f-vector of $P+P^*$ can be written as:*

$$f_k(P+P^*) = \sum_{i=0}^{k} f_{i,i+d-1-k}, \quad \forall k = 0,\ldots,d-1$$

Proof. Let P be a perfectly centered polytope. For every k, the sumber of k-faces of $P+P^*$ is equal to the number of pairs of faces (F,G) of P, $F \subseteq G$ so that $dim(F) + dim(G^D) = k$, which means $dim(F) + d - 1 - k = dim(G)$. Which is the number of chains of two non-trivial faces of dimensions i and $i+d-1-k$. ∎

Theorem 4 *Let P be a perfectly centered polytope. Then the conjecture is true for the Minkowski sum $P+P^*$.*

Proof. Let P be a perfectly centered polytope. We have that

$$\sum_{k=0}^{d-1} (-1)^{d-1-k} k f_k(P) = -\sum_{i=0}^{d-1} (-1)^{d+i} i f_{i,d}(P)$$

By using $k' = d-1-k$:

$$\sum_{k=0}^{d-1} (-1)^{d-1-k} k f_k(P^*) = -\sum_{k'=0}^{d-1} (-1)^{k'} (d-1-k') f_{-1,k'}(P)$$

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}kf_k(P+P^*) = \sum_{k=0}^{d-1}\sum_{i=0}^{k}(-1)^{d-1-k}kf_{i,i+d-1-k}(P)$$

We will change the sum to replace k by $k' = i + d - 1 - k$:

$$= \sum_{i=0}^{d-1}\sum_{k'=i}^{d-1}(-1)^{k'-i}(i+d-1-k')f_{i,k'}(P)$$

$$= \sum_{i=0}^{d-1}\sum_{k'=i}^{d-1}(-1)^{k'+i}if_{i,k}(P) + (-1)^{k'+i}(d-1-k')f_{i,k}(P)$$

$$= \sum_{i=0}^{d-1}\sum_{k'=i}^{d-1}(-1)^{k'+i}if_{i,k}(P) + \sum_{i=0}^{k'}\sum_{k'=0}^{d-1}(-1)^{k'+i}(d-1-k')f_{i,k}(P)$$

Composing the three, we get:

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}kf_k(P+P^*) - f_k(P) - f_k(P^*) =$$

$$\sum_{i=0}^{d-1}\sum_{k=i}^{d}(-1)^{k+i}if_{i,k}(P) + \sum_{i=-1}^{k}(-1)^{k'+i}\sum_{k'=0}^{d-1}(d-1-k')f_{i,k}(P)$$

$$= \sum_{i=0}^{d-1} i \underbrace{\sum_{k=i}^{d}(-1)^{k+i}f_{i,k,d}(P)}_{0} + \sum_{k=0}^{d-1}(d-1-k)\underbrace{\sum_{i=-1}^{k}(-1)^{k+i}f_{-1,i,k}(P)}_{0} = 0$$

By Dehn-Sommerville relations (2). ∎

2.3 Old proofs

I include here old proofs which have been made useless by 4

We will use for some proofs the Binomial Theorem:

Theorem 5

$$\sum_{k=0}^{d} a^k b^{(d-k)} \binom{d}{k} = (a+b)^d$$

and a variant of the latter :

Theorem 6

$$\sum_{k=0}^{d} k a^k b^{d-k} \binom{d}{k} = ad(a+b)^{d-1}$$

Proof.

$$\sum_{k=0}^{d} k a^k b^{d-k} \binom{d}{k} =$$

$$\sum_{k=1}^{d} a^k b^{d-k} k \frac{d!}{k!d-k!} =$$

$$\sum_{k=1}^{d} a^{k-1} b^{d-k} ad \frac{d-1!}{k-1!d-k!} =$$

$$ad \sum_{k=0}^{d-1} a^k b^{d-1-k} \binom{d-1}{k} = ad(a+b)^{d-1}$$

∎

Here is the proof for the Nesterov rounding of simplices:

Theorem 7 *Let Δ_d be the d-dimensional simplex, then the main conjecture is true for the Minkowski Sum of Δ_d and its dual Δ_d^*.*

Proof.

As stated in [2] the f-vector of the Nesterov Rounding of a simplex is given by:

$$f_k(\Delta_d + \Delta_d^*) = \left(\begin{array}{c} d+1 \\ k+2 \end{array} \right) \left(2^{k+2} - 2\right), \quad \text{for } 0 \leq k \leq d-1.$$

So we can write:

$$\sum_{k=0}^{d-1} (-1)^{d-1-k} (k+2) f_k(\Delta_d + \Delta_d^*) =$$

$$\sum_{k=0}^{d-1} (-1)^{d-1-k} (k+2) \left(2^{k+2} - 2\right) \left(\begin{array}{c} d+1 \\ k+2 \end{array} \right).$$

Using the identity $(k+2) \left(\begin{array}{c} d+1 \\ k+2 \end{array} \right) = (d-k) \left(\begin{array}{c} d+1 \\ k+1 \end{array} \right)$ and by using $d' = d+1$ and $k' = k+1$, we can write

$$= \sum_{k'=1}^{d'-1} (-1)^{d'-1-k'} (d'-k') \left(2^{k'+1} - 2\right) \left(\begin{array}{c} d' \\ k' \end{array} \right)$$

$$= \sum_{k'=0}^{d'} (-1)^{d'-1-k'} (d'-k') \left(2^{k'+1}-2\right) \binom{d'}{k'}.$$

Using $k'' = d' - k'$, we get

$$= 2 \sum_{k''=0}^{d'} (-1)^{k''-1} k'' \left(2^{d'-k''}-1\right) \binom{d'}{k''}$$

Using the Binomial Theorem, we get

$$= -2(-1)d' = 2(d+1).$$

We have that $f_k(\Delta_d) = f_k(\Delta_d^*) = \binom{d+1}{k+1}$.

$$\sum_{k=0}^{d-1} (-1)^{d-1-k} (k+2) \binom{d+1}{k+1} =$$

$$\sum_{k'=1}^{d'-1} (-1)^{d'-1-k'} (k'+1) \binom{d'}{k'} =$$

$$(d'+1) - (-1)^{d'-1} + \underbrace{\sum_{k'=0}^{d'} (-1)^{d'-1-k'} (k'+1) \binom{d'}{k'}}_{0} = (d+2) - (-1)^d$$

And so,

$$\sum_{k=0}^{d-1} (-1)^{d-1-k} (k+2) (f_k(\Delta_d + \Delta_d^*) - f_k(\Delta_d) - f_k(\Delta_d^*) =$$

$$2(d+1) - 2\left((d+2) - (-1)^d\right) = -2 + 2(-1)^d$$

∎

Here is the proof for the Nesterov rounding of cubes:

Theorem 8 *Let $\Box_d$ be the d-dimensional cube, then the main conjecture is true for the Minkowski Sum of $\Box_d$ and its dual $\Box_d^*$.*

Proof. As stated in [2] the f-vector of the Nesterov Rounding of a simplex is given by:

$$f_k(\Box_d + \Box_d^*) = \binom{d}{k+1} 2^{d-k-1} \left(3^{k+1}-1\right), \quad \text{for } 0 \leq k \leq d-1.$$

So we can write:

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}(k+1)f_k(\Box_d+\Box_d^*) = \sum_{k=0}^{d-1}(-2)^{d-1-k}(k+1)(3^{k+1}-1)\left(\begin{array}{c} d \\ k+1 \end{array}\right)$$

Using the identity $(k+1)\left(\begin{array}{c} d \\ k+1 \end{array}\right) = d\left(\begin{array}{c} d-1 \\ k \end{array}\right)$ and by using $d' = d-1$, we can write

$$= d\sum_{k=0}^{d'}(-2)^{d'-k}(k+1)(3^{k+1}-1)\left(\begin{array}{c} d' \\ k \end{array}\right).$$

And using 5 and 6, we get:

$$= d\left(3\cdot 1-(-1)^{d'}\right) = 3d+(-1)^d d$$

We have that

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}(k+1)f_k(\Box_d) =$$

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}(k+1)2^{d-k}\left(\begin{array}{c} d \\ k \end{array}\right) =$$

$$-\sum_{k=0}^{d-1}(-2)^{d-k}(k+1)\left(\begin{array}{c} d \\ k \end{array}\right) =$$

$$-\sum_{k=0}^{d}(-2)^{d-k}(k+1)\left(\begin{array}{c} d \\ k \end{array}\right) + (d+1)$$

Using 5 and 6, we get:

$$= -(d(-1)^{d-1}+(-1)^d)+(d+1) = (d-1)(-1)^d+(d+1)$$

We also have that

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}(k+1)f_k(\Box_d^*) = \sum_{k=0}^{d-1}(-1)^{d-1-k}(k+1)2^{k+1}\left(\begin{array}{c} d \\ k+1 \end{array}\right)$$

Using the identity $(k+1)\left(\begin{array}{c} d \\ k+1 \end{array}\right) = d\left(\begin{array}{c} d-1 \\ k \end{array}\right)$ and by using $d' = d-1$, we can write

$$= 2d\underbrace{\sum_{k=0}^{d'}(-1)^{d'-k}2^k\left(\begin{array}{c} d' \\ k \end{array}\right)}_{1} = 2d$$

And so,

$$\sum_{k=0}^{d-1}(-1)^{d-1-k}(k+1)(f_k(\Box_d+\Box_d^*)-f_k(\Box_d)-f_k(\Box_d^*) =$$

$$3d+(-1)^d d-2d-((d-1)(-1)^d+(d+1)) = -1+(-1)^d$$

■

Here is the proof for the Nesterov rounding of 4-dimensional perfectly centered polytopes:

Theorem 9 *Let P be a 4-dimensional perfectly centered polytope. Then the conjecture is true for the Minkowski sum $P+P^*$.*

Proof. As stated in [2], the f-vector of the faces of the Nesterov Rounding of a perfectly centered polytope P can be determined by the face lattice of P:

> **Theorem** Let P be a perfectly centered polytope. A subset H of $P+P^*$ is a nontrivial face of $P+P^*$ if and only if $H=G+F^D$ for some ordered nontrivial faces $G \subseteq F$ of P.

So in 4 dimensions, we will have

$$\mathcal{F}_3(P+P^*) = \{F+F^D \ : \ F \text{ non-trivial face of } P\}$$

$$\mathcal{F}_2(P+P^*) = \{F+G^D \ : \ F \subset G,\ dim(F)+1 = dim(G)\}$$

$$\mathcal{F}_1(P+P^*) = \{F+G^D \ : \ F \subset G,\ dim(F)+2 = dim(G)\}$$

And so we have the following:

$$f_3(P+P^*)-f_3(P)-f_3(P^*) =$$

$$(f_0(P)+f_1(P)+f_2(P)+f_3(P))-f_3(P)-f_0(P) = f_1(P)+f_2(P)$$

$$f_2(P+P^*)-f_2(P)-f_2(P^*) =$$

$$(f_{0,1}(P)+f_{1,2}(P)+f_{2,3}(P))-f_2(P)-f_1(P) = f_1(P)+f_2(P)+f_{1,2}(P)$$

$$f_1(P+P^*)-f_1(P)-f_1(P^*) =$$

$$(f_{0,2}(P)+f_{1,3}(P))-f_1(P)-f_2(P) = 2f_{1,2}(P)-f_1(P)-f_2(P)$$

Again, we have that

$$3(f_3(P+P^*)-f_3(P)-f_3(P^*))$$

$$-2(f_2(P+P^*)-f_2(P)-f_2(P^*))$$

$$+(f_1(P+P^*)-f_1(P)-f_1(P^*)) = 0.$$

■

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