EDGE ELEMENTS AND COCHAIN-BASED FIELD FUNCTION APPROXIMATION

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Abstract

Edge Elements (EE) have received in recent times much attention from the Finite Element community. The present contribution analyzes the role played by EE in the preservation of the fundamental properties of physical equations submitted to discretization. A short review of these properties is presented, where it is emphasized the presence in the equations, of intrinsically discrete terms that can be represented in a most natural way using the concept of cochain. It is then shown how EE are instrumental in the bridging of the gap represented by terms which cannot be exactly discretized. It is maintained that the role of EE lies in their ability to provide a simple machinery to build a continuous representation (ideally, as a differential form) for a field starting from its discrete representation in terms of a cochain, an operation for which the name of cochain-based field function approximation is suggested. The interpolation practices of Finite Elements and Finite Volumes are considered under this light to clear away some confusion and show the way to further generalizations.

INTRODUCTION

A fundamental distinction can be drawn in the realm of electromagnetic field equations between *Maxwell equations* and *constitutive equations*. This distinction is based on the fact that Maxwell equations possess an intrinsic discrete nature, whereas constitutive equations do not. Note in this respect that most of the physical principles expressed by Maxwell equations, were originally intended as relations between global electromagnetic quantities. Indeed, in their most mature form, Maxwell equations can be derived from two conservation principles: the conservation of electric charge and the conservation of magnetic flux [1, 2], both global quantities. On the other hand, constitutive equations must be written in terms of local field quantities, since only in very particular cases (uniform fields, homogeneous materials...) a constitutive equation allows an exact rendering in terms of global quantities.

This distinction reflects on the discretization strategies which can be adopted in numerical electromagnetics. Due to their intrinsic discreteness, Maxwell equations can be enforced exactly in methods such as finite volumes and finites elements. A good reason for doing so is the possibility to preserve in the process the fundamental algebraic properties of these equations. The enforcement of constitutive equations is a much tougher problem. In general, once Maxwell equations are exactly enforced, it can be regarded as a problem of approximation aiming at the optimization of the discretization error with respect to the computational effort [3, 4].

The present contribution develops further this line of thought, maintaining that to fully exploit the benefits that can derive to a numerical method from the intrinsic discrete nature of Maxwell equations, discrete description tools for both domains and fields should be used as basic building blocks in the numerical method. It is then argued that where non-discrete descriptions are needed - i.e. in the enforcement of constitutive equations - an effort must be made to link these descriptions to the discrete framework in a way as smooth as possible. A subset of the large variety of edge elements that can be found in the literature is suggested as the proper tool to perform this transition form discrete to non-discrete representations.

MAXWELL EQUATIONS AS CONSERVATION LAWS

Conservation of Electric Charge and definition of Charge-Current Potential

The first conservation law of electromagnetics is the conservation of electric charge

$$\int_{t_1}^{t_2}\iint_{\partial V} \mathbf{J}\cdot \mathrm{d}\mathbf{s}\ \mathrm{d}t + \iiint_{V|t_2} \rho\ \mathrm{d}v - \iiint_{V|t_1} \rho\ \mathrm{d}v = 0 \qquad (1)$$

which can be written as

$$\oiiint_{V_c} J = 0 \qquad (2)$$

where J is a twisted differential 3-form, the *charge-current* form, and V_c is an arbitrary closed volume in space-time. Equation (2) is a global statement about electric charge, which holds for volumes of arbitrary extension. From (2) follows the existence of a twisted differential 2-form G, the *charge-current potential* form, such that

$$\iint_{\partial V} G = \iiint_{V} J \qquad (3)$$

where V is an arbitrary volume. Equation (3) corresponds to the following pair of Maxwell equations

$$\oiint_{\partial V} \mathbf{D}\cdot \mathrm{d}\mathbf{s} = \iiint_{V} \rho\ \mathrm{d}\,v \qquad (4)$$

$$\int_{t_1}^{t_2}\int_{\partial S} \mathbf{H}\cdot \mathrm{d}\mathbf{l}\ \mathrm{d}t - \iint_{S|t_2} \mathbf{D}\cdot \mathrm{d}\mathbf{s} + \iint_{S|t_1} \mathbf{D}\cdot \mathrm{d}\mathbf{s} = \int_{t_1}^{t_2}\iint_{S} \mathbf{J}\cdot \mathrm{d}\mathbf{s}\ \mathrm{d}t \qquad (5)$$

where V is an arbitrary volume and $[t_1,t_2]$ an arbitrary time interval. Being (3) another global statement, we conclude that this pair of Maxwell equations has an intrinsic discrete nature.

Conservation of Magnetic Flux

The other pair of Maxwell equations, written in integral form, can be considered as the space-like and time-like part, respectively, of a more general space-time conservation law for magnetic flux.

$$\oiint_{S_c} \mathbf{B}\cdot \mathrm{d}\mathbf{s} = 0 \qquad (6)$$

$$\int_{t_1}^{t_2}\int_{\partial S} \mathbf{E}\cdot \mathrm{d}\mathbf{l}\ \mathrm{d}t + \iint_{S|t_2} \mathbf{B}\cdot \mathrm{d}\mathbf{s} - \iint_{S|t_1} \mathbf{B}\cdot \mathrm{d}\mathbf{s} = 0 \qquad (7)$$

Here S_c is an arbitrary closed surface in space, S a surface with boundary ∂S, and $[t_1,t_2]$ an arbitrary time interval. Note that it is usually the similarity of equations (7) and (5) which is stressed. We suggest instead to consider equation (7) as the magnetic counterpart of the conservation law (1). To appreciate this similarity, note that equations (6) and (7) can both be rendered as the following counterpart of equation (2):

$$\oiint_{S_c} F = 0 \qquad (8)$$

Here F is a differential 2-form defined in space-time, the *electromagnetic field* form, and S_c is an arbitrary closed surface in space-time. From this observation follows that this pair of Maxwell equations too has an intrinsic discrete nature since (8) is a *global* statement about magnetic flux which holds for surfaces of arbitrary extension.
Summing up, Maxwell equations can be *exactly* written in discrete form in terms of the *global electromagnetic quantities* Φ (magnetic flux) and Q (electric charge). The additional global quantities V (electric tension), I (electric current), Ψ (electric flux), and H (magnetic tension), along with V (electric tension impulse) and H (magnetic tension impulse), can be defined to ease the separate treatment of the space-like and time-like parts of the equations.

DISCRETE REPRESENTATION OF FIELDS

Global Quantities and Cochains.

The main fact about Maxwell equations written in their primitive, integral form, is that they link *global quantities*. The main fact about global quantities is that they are naturally associated with *oriented chrono-geometric objects* (points, lines, surfaces, volumes, hypervolumes, time instants and time intervals, endowed with internal or external orientation) [5]. In order to enforce exactly Maxwell equations in the discrete framework of numerical methods, it is therefore advisable to employ a representation in terms of global quantities. Moreover this representation should respect the association with chrono-geometric objects. A representation which fulfills these requests exists in algebraic topology and is called a *cochain*. Given an oriented cell complex in the domain of definition of a given field, a cochain is the way in which the field manifests itself on the discretized domain. Strictly speaking a cochain is the law which associates global quantities with oriented cells. However, from the more practical perspective of numerical methods*, a cochain can be thought of simply as a collection of global quantities associated with all the cells of the grid having a given dimension and orientation* (e.g. all the externally oriented two-dimensional cells in the case of the electric flux cochain; internally oriented one-dimensional cells for the electric tension cochain, and so on). In other words, a cochain is a collection of global quantities *plus* some additional property such as linearity with respect to cell union and sign change for inversion of cell orientation, but since these additional properties derive naturally from the physics of global quantities there is no need to worry about them here. We stress this point and chose to italicize the sentence above, to prevent readers from feeling uneasy about the simple concept represented by cochains, in consequence of their unfamiliar name.
With the concept of cochain at hand we can rephrase the conclusion of the last section, adapting it to the case of numerical methods: Maxwell equations are best (and, of course, still *exactly*) written in discrete form in terms of the electromagnetic cochains i.e. in terms of relations between global electromagnetic quantities associated with oriented chrono-geometric objects.

CONSTITUTIVE EQUATIONS AND LOCAL REPRESENTATIONS

Constitutive equations are the mathematical representation of the behavior of the materials (vacuum included) which fill the domain of the problem. It can be shown, following their derivation from phenomenological observations, that constitutive equations do not admit, in general, an exact discrete representation. In other words they cannot be written as relations between global quantities but by accepting some approximation error. An easy way to ascertain this is by considering that, given that a numerical solution is usually an approximate one and that Maxwell equation can be discretized without error, some approximation must be introduced by the only other term present in discretization: the constitutive equations.

$$\begin{aligned} \mathbf{D} &= \varepsilon(\mathbf{E}) \\ \mathbf{B} &= \mu(\mathbf{H}) \qquad (9) \\ \mathbf{J} &= \sigma(\mathbf{E}) \end{aligned}$$

We know on the other hand that constitutive equations written in *local* form - i.e. as functional relations between differential forms (usually rendered as scalar or vector fields, as in (9)) - do not suffer from this limitation. To discretize the constitutive equations consistently with the stance adopted in the previous section, it is therefore tempting to build a local representation for the fields starting from their discrete representation, applying then to this local representation the local constitutive equations. This is indeed how finite element and finite volume methods usually proceed [6]. From our former discussion concerning the correct discrete representation for fields we can conclude that this reconstruction of a local representation should start from cochains, i.e. from global quantities associated with chrono-geometric objects. We can therefore call this operation *cochain-based field function approximation* or *cochain approximation* in short.

From Cochains to Differential Forms through Edge Elements

Edge elements are primarily known by the finite element community as a mean to build a field function with the right kind of interelement continuity for the physics of the problem at hand, allowing for example the discontinuity of some field component across adjacent elements. But if we consider some simple edge elements, for example Whitney elements [4], we realize that they are indeed the tool which performs the function advocated in the last section, namely the building of a differential form starting from global quantities or, more precisely, from a cochain. In other words, edge elements appear as the ideal tool for the implementation of cochain-based field function approximation. They are in this sense the counterpart of the familiar nodal interpolation functions. Just like these provide a set of functions each of which has value one on a particular node of the grid and zero on all other nodes, Whitney elements provide functions which associate with a given line (or surface, or volume) of the cell-complex a global quantity with value one and associate zero with all other similar objects of the complex. Note that the resulting field function is better considered a differential form than a scalar or vector function, since the concept of differential form preserves some remnant of the original association of global quantities with oriented chrono-geometric objects.

Edge Elements for Internally and Externally Oriented Geometric Objects

Chrono-geometric objects come in many dimensions (points lines, surfaces, ...), so we can expect to have many kinds of edge elements, each suiting cochain approximation on a particular object. Moreover, chrono-geometric objects occur endowed with internal or external orientation [5]. We must therefore have edge elements suited to both kinds of oriented objects (Figures 1 and 2 illustrate this point for the case of lines on a two-dimensional domain). On the other hand, a global quantity never occurs associated with an object which has simultaneously both kinds of orientation. Consider in this respect the edge element sketched in Figure 3, which results from the quest for edge elements providing higher order approximations [7]. It is apparent that it involves a mix of both kinds of orientation and as such it is not suited to our definition of cochain approximation. Therefore, of the many kinds of edge elements that were introduced in the literature, not all of them can be considered instrumental to the exact enforcement of Maxwell equations in terms of global quantities, followed by cochain approximation.

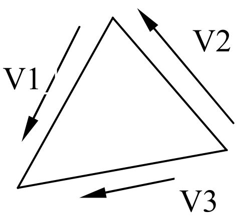

Figure 1: An edge element for global quantities associated with internally oriented lines.

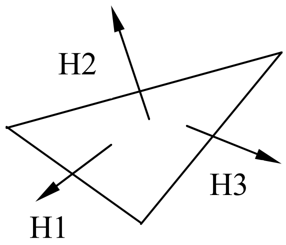

Figure 2: An edge element for global quantities associated with externally oriented lines.

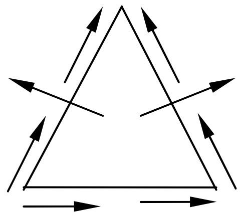

Figure 3: An edge element which does not suit the philosophy of cochain approximation

The Way to Higher Order Approximations

The elements shown in Figures 1 and 2 are consistent with cochain approximation but allow the assembly of field functions of low order only. On the other hand the element of Figure 3, which has many more degrees of freedom and gives rise to higher order functions, does not suit cochain approximation. Let us show how higher order cochain approximations can be achieved. The key point is to provide an increased number of chrono-geometric objects of the right kind as starting point for the approximation. Figure 4 shows how the same number of degrees of freedom of Figure 3 can be provided for cochain approximation. Note that we are simply adapting to cochain approximation the concept of *element*, upon which the local approximation takes place and which contains more than one cell of the cell complex used as discretization grid. The main difference with respect to the element of Figure 3 is the presence of additional geometric objects to house the additional global quantities. Note that the enforcement of Maxwell equations on these new variables is straightforward.

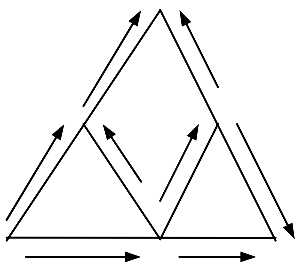

Figure 4: An edge element for higher order cochain approximation. Note that here, contrary to figure 3, all physical quantities are associated with geometric objects having the same orientation.

A Glimpse of other Physical Theories

In electromagnetics the global quantities are scalars (values of charge and magnetic flux). Therefore the cochain approximation is always based on scalar quantities, no matter on what kind of geometric object it is performed. For this reason the nodal interpolation of vector quantities does not find place in the philosophy of cochain-based numerical electromagnetics. There is however no taboo in cochain approximation against nodal interpolation of vector or more complex quantities. In theories were the global values are vectors (e.g. fluid-dynamics) the cochain approximation will be based on vectors, be they associated with points, lines or with any other chrono-geometric object. The corresponding edge elements will give rise to vector-valued differential forms. They will therefore constitute a new class of edge elements that could be tentatively called vector-valued edge elements.

CONCLUSIONS

It is often said that the advantage of edge elements in the numerical treatment of field problems lies in their providing the right kind of interelement continuity for the physics of the problem at hand. The cochain-based perspective presented in this paper shows that there is more in edge elements than interelement continuity. The possibility to write conservation equations in their proper discrete form and still easily enforce constitutive equations seems to be an even stronger point in favor of edge elements. The fact that this approach requires a tighter definition of edge element appears a minor constraint of a philosophy of discretization which opens the door to a more accurate numerical rendering of physical laws.

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