Polar cographs

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Abstract. Polar graphs are a natural extension of some classes of graphs like bipartite graphs, split graphs and complements of bipartite graphs. A graph is (s,k)-polar if there exists a partition A, B of its vertex set such that A induces a complete s-partite graph (i.e., a collection of at most s disjoint stable sets with complete links between all sets) and B a disjoint union of at most k cliques (i.e., the complement of a complete k-partite graph).
Recognizing a polar graph is known to be NP-complete. We provide a polynomial time algorithm for finding a largest polar induced subgraph in cographs (graphs without induced path on four vertices). A characterization of polar cographs in terms of forbidden subgraphs is given. We examine also the monopolar cographs which are the (s,k)-polar cographs where $\min(s,k) \leq 1$. A characterization of these graphs by forbidden subgraphs is given. Some open questions related to polarity are discussed.

Keywords polar graphs, cographs, split graphs, threshold graphs.

1 Introduction

Polar graphs are a natural extension of some classes of graphs which include bipartite graphs, split graphs (i.e., graphs whose vertex set can be partitioned into a clique and a stable set) and complements of bipartite graphs.
Following [2], a graph $G = (V, E)$ is called *polar* if its vertex set V can be partitioned into (A, B) (A or B may possibly be empty) such that A induces a complete multipartite graph (it is a join of stable sets) and B a (disjoint) union of cliques (i.e., the complement of a join of stable sets).
We shall say that G is (s,k)*-polar* if there exists a partition (A, B) where A is a join of at most s stable sets and B a union of at most k cliques. Thus polar graphs are just the (∞,∞)-polar graphs. Notice that every graph is not polar: the graphs N_1 and N_2 in Figure 1 are not polar as can be checked, but if any vertex is removed, the remaining graph is polar. Observe also that the complement $\overline{G}$ of an (s,k)-polar graph is a (k,s)-polar graph. Notice that $(1,1)$-polar graphs are just split graphs. In [2] it was shown that recognizing whether an arbitrary graph is polar is NP-complete. Some polynomial time recognition problems are also discussed in [2] for the case where the largest size of the stable sets and of the cliques in the partition (A, B) are bounded.
Besides this, [5] gives a general framework for partitioning the vertex set of

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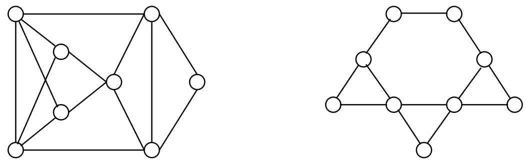


N_1 N_2

Fig. 1. Some minimal non polar graphs.

graphs with requirements on the links between the subsets of the partitions. Some of the results in [5] will be used here.

In this paper we shall concentrate mainly on cographs, i.e., graphs without induced P_4 (path on four vertices). We study polar cographs and give some polynomial time recognition algorithms as well as some characterizations by forbidden subgraphs. It is well-known [3] that for a cograph G, either G or $\overline{G}$ is disconnected. Subsequently, a tree can be constructed with cograph G as the root. Children of each vertex represent the components of either the graph at the parent vertex (in which case the parent vertex is labeled 0-vertex), or the complement of the graph at the parent vertex (in which case the parent is labeled 1-vertex). This tree is known as the cotree and can be constructed in linear time [3]. Cotree will be used in our algorithms.

We will also examine a subclass of polar graphs called *monopolar* graphs; these are the (s,k)-polar graphs where $\min(s,k) \leq 1$. In other words for such graphs, a partition (A,B) exists with at most one stable set in A or at most one clique in B. A characterization of monopolar cographs by forbidden subgraphs will be derived.

In addition some remarks on the recognition of (s,k)-polar perfect graphs will be provided together with some open questions related to other classes of perfect polar graphs in Section 4.

In what follows, we denote by P_l, C_l and K_l respectively a path, a chordless cycle and a clique on l vertices. Given two graphs G_1, G_2, $G_1 \oplus G_2$ denotes their join (with complete links) and $G_1 \cup G_2$ their disjoint union. Let x, y be two vertices, then xy and $\overline{xy}$ mean respectively that they are adjacent and non-adjacent.

We will also need the notion of *threshold graphs* which are split graphs (i.e, $(1,1)$-polar graphs), where for any two vertices v, w in the stable set S, the sets of neighbors satisfy $N(v) \supseteq N(w)$ or $N(w) \supseteq N(v)$. A graph G is a threshold graph if and only if it does not contain $2K_2, C_4$ or P_4 as induced subgraphs. Properties of threshold graphs are studied in [6].

It will be convenient to denote by $K^*(s,k)$ a (s,k)-polar graph with partition (A,B), which is the join of A and B (i.e., with complete links between A and B). This graph is called a *complete* (s,k)-polar graph.

For graph theoretical terms not defined here, the reader is referred to [1].

2 Largest polar subgraph in cographs

In this section, we describe how to find an induced polar subgraph of maximum size in a cograph using its cotree representation. Given a cograph G let us denote by $MC(G)$ a maximum clique in G, by $MS(G)$ a maximum stable set in G, by $MT(G)$ a maximum threshold graph in G, by $MUC(G)$ a maximum (sized) union of cliques in G, by $MJS(G)$ a maximum (sized) join of stable sets in G, by $MMPS(G)$ a maximum $(1,k)$-polar subgraph for some k (maximum monopolar subgraph with one stable set) in G, by $MMPC(G)$ a maximum $(s,1)$-polar subgraph for some s (maximum monopolar subgraph with one clique) in G and finally by $MP(G)$ a maximum polar subgraph in G. $n(MP(G))$ denotes the size of $MP(G)$ and the sizes of all other maximum subgraphs are denoted in a similar way. All maximum subgraphs mentioned below are represented by a pair (A,B) as described in the introduction.
In what follows, we assume that the cotree representation of the cograph is given. If x denotes a vertex in the cotree, then $c_1x, c_2x, \ldots$ are the children of x. Each vertex x has a type $t(x)$ which is 0 or 1 and x will also represent the subgraph associated with vertex x of the cotree. This should be clear from the context.
First of all, note that given a cotree, $MS(x)$ and $MC(x)$ can be found in linear time for any x in the cotree [3]. Also, it has been shown in [4] that a maximum threshold subgraph in cographs is obtained by the union of any maximum stable set and any maximum clique since no pair of maximum stable set and maximum clique is disjoint. Therefore, for any vertex x of the cotree, we have $MT(x) = MC(x) \cup MS(x)$ and $n(MT(x)) = n(MC(x)) + n(MS(x)) - 1$. In what follows, we assume for the sake of simplicity that $MC(x), MS(x)$ and $MT(x)$ are known for any vertex x of the cotree.
Note that the following lemmas are established for computing a maximum subgraph at a 0-vertex assuming that all parameters on children needed for the computation are already known. Consequently, the maximum subgraph M computed at a 0-vertex has a polar partition (A,B) such that; if M is the result of a union of maximum subgraphs M_i with polar partition (A_i,B_i) where A_i induces one stable set in child c_ix, then $A = \cup_i A_i$ (inducing one stable set) and $B = \cup_i B_i$ (inducing the union of cliques of each M_i); if M is a subgraph M_j with polar partition (A_j,B_j) realizing the maximum of some quantity, then $A = A_j$ and $B = B_j$ (knowing that a sum in a maximum operator should be interpreted as a union of the subgraphs under consideration).

Lemma 1. *Given a cotree, $MUC(x)$ and $MJS(x)$ can be computed for any 0-vertex x in time linear in the number of children of x.*

Proof. Clearly, we have $MUC(x) = \cup_i MUC(c_ix)$ since cliques of different children are not linked at all. On the other hand, $MJS(x)$ is the set realizing the maximum of $\left[\max_i n(MJS(c_ix)); \sum_i n(MS(c_ix))\right]$; in fact if at least two children contribute then no more than one stable set can be taken from each child since the children of x are not linked at all. □

Lemma 2. *Given a cotree, $MMPS(x)$ and $MMPC(x)$ can be computed for any 0-vertex x in time $O(k^2)$ where k is the number of children of x.*

Proof. Obviously, we have $MMPS(x) = \cup_i MMPS(c_i x)$ since the union of one stable set from each child yields one stable set and cliques of different children remain disjoint. For $MMPC(x)$, it is the subgraph realizing the maximum of $\left[n(MT(x)); \max_i n(MMPC(c_i x)); \max_{i,j}\big(n(MJS(c_i x)) + n(MC(c_j x))\big)\right]$. In fact, a maximum $(s,1)$-polar subgraph at a 0-vertex is either a threshold graph having the clique part in one component and the stable set part being the union of maximum stable sets in each child, or the largest maximum $(s,1)$-polar subgraph among the children, or the largest union among the children, of a maximum join of stable sets in one child and a maximum clique in another child (if both are coming from the same child then it amounts to be a maximum $(s,1)$-polar subgraph of the child under consideration).
The time complexity is due to the third term where the maximum is taken over any possible pair of children. □

Lemma 3. *Given a cotree, $MP(x)$ can be computed for any 0-vertex x in time $O(k^2)$ where k is the number of children of x.*

Proof. A maximum polar subgraph is obtained by either taking the union of a maximum polar graph in one child and maximum union of cliques in other children or taking the union of the maximum of a threshold graph, a maximum union of cliques and a maximum $(1,k)$-polar subgraph from each child. It follows that $MP(x)$ is the subgraph realizing the maximum of $\left[\max_i(n(MP(c_i x)) + \sum_{j\neq i} n(MUC(c_j x)); \sum_i \max\big(n(MT(c_i x)); n(MUC(c_i x)); n(MMPS(c_i x))\big)\right]$.
Note that the time complexity is decided by the first term where the maximum is taken on all possible pairs of children. □

Theorem 1. *For any cograph G given by its cotree, $MP(G)$ can be computed in time $O(n^2)$.*

Proof. One may think of an algorithm searching the cotree from the leaves to the root and computing for each vertex of the cotree a maximum polar subgraph; the one computed at the root provides $MP(G)$. By Lemma 3, one can compute a maximum polar subgraph at a 0-vertex. On the other hand, at a 1-vertex x, we know that the complement of the subgraph remaining under this vertex is a disconnected graph which can be represented by a cotree with a root of type 0. Then applying Lemma 3 and taking the complement of the resulting subgraph (thus stable sets and cliques are interchanging roles) gives a maximum polar subgraph at x.
The initialization of this algorithm is done by the following assignments: for a vertex x which is a leaf representing the vertex v, $MUC(x)$ and $MMPC(x)$ are in the form (A,B) where $A = \emptyset, B = \{v\}$, $MJS(x)$ and $MMPS(x)$ are in the form (A,B) where $A = \{v\}, B = \emptyset$ and $MP(x)$ is in one of these forms.
The complexity is $O(n^2)$ since Lemma 2 and Lemma 3 are applied for all vertices of the cotree. □

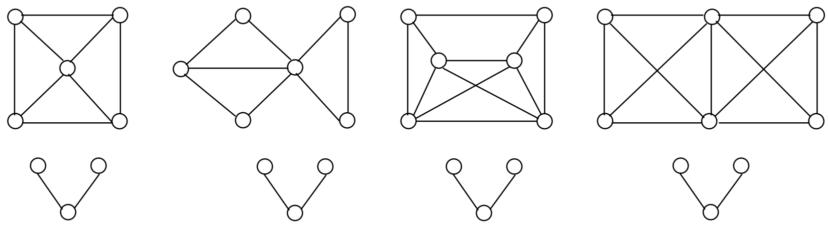
We remark that in a cograph G with weighted vertices, a maximum weighted polar subgraph can be found in exactly the same way as previously; it suffices to replace in all lemmas the size of a subgraph by its weight which is the sum of the weights of the vertices in the subgraph.

3 Characterization of polar cographs by forbidden subgraphs

The following theorem provides a forbidden subgraph characterization of polar cographs.

Theorem 2. *For a cograph G, the following statements are equivalent:*

a) G is polar.
b) Neither G nor $\overline{G}$ contains any one of the graphs $H_1, \ldots, H_4$ of Figure 2 as induced subgraphs.



H_1 H_2 H_3 H_4

Fig. 2. Forbidden subgraphs for polar cographs.

Proof. *a)* $\Rightarrow$ *b)* Since every induced subgraph of a polar graph is polar and the complement of a polar graph is polar, it is enough to show that $H_1, \ldots, H_4$ are non-polar. Suppose $H_1, \ldots, H_4$ are polar. Since a complete join of stable sets is a connected subgraph, it follows that in each of the four graphs, one of the components must induce a disjoint union of cliques. Clearly, it is not the case for any of the graphs. Hence they are non-polar.
b) $\Rightarrow$ *a)* Suppose G is non-polar. Assume without loss of generality that G is minimal non-polar. Assume also without loss of generality that the cograph G is disconnected (otherwise take its complement). Let (A, B) be a partition of its vertex set into non-empty sets without edges between A and B. By the minimality of G, both $G[A]$ and $G[B]$ are polar. If $G[A]$ contains no induced P_3, then it is a disjoint union of cliques and hence G is polar. So we may assume that both A and B contain three vertices inducing a P_3. If both $G[A]$ and $G[B]$ have polar partitions with single stable sets S_A and S_B respectively, then G has a polar partition with single stable set $S_A \cup S_B$. So assume that $G[A]$ has at least two stable sets in every polar partition. Let $A' \subset A$ induce the connected component containing the join of stable sets. $A \setminus A'$ induces a disjoint union of cliques. Since G is a cograph, A' is partitioned into (C, D) with complete join between C and D. We consider two cases.

<u>*Case 1 :*</u> An induced P_3 in A is completely contained in D.
If C contains a non-edge, then the non-edge along with the P_3 in D and the P_3 in B induce an H_1 in G, a contradiction. So C must induce a clique. If C contains an edge, then D is $2K_2$-free for otherwise G contains an H_4. D is also C_4-free (else G contains H_1) and P_4-free. Thus D induces a threshold graph. It follows that $G[A]$ has polar partition with at most one stable set and many cliques, a contradiction. It follows that C must consist of a single vertex.
Let S be a maximal stable set in D containing both the ends of P_3 in D. Let c be the center of the P_3.

Claim 1. *For any $a \in D \setminus S$, ac, i.e., $D \setminus S \subseteq N(c)$.*

Proof. S being a maximal stable set, a has a neighbor in S. If $a \notin N(c)$, then D must contain a P_4, C_4 or $P_3 \cup K_2$, a contradiction since G is P_4-free, H_1-free and H_2-free. □

Claim 2. *For any $a \in D \setminus S$, ax, for some $x \in N(c) \cap S$.*

Proof. Similar to the proof of Claim 1. □

Claim 3. *$N(c) \cap S$ is linearly ordered by domination in $N(c) \setminus S$, i.e., there are no two vertices $x, y \in N(c) \cap S$ such that for some $a, b \in N(c) \setminus S$, $xa, \overline{xb}, \overline{ya}$ and yb.*

Proof. Since $a, b \in N(c) \setminus S$, ac and bc. If $\overline{ab}$ then a, b, c, x, y and the vertex in C along with P_3 in B induce H_4. If ab, then G contains a P_4. □

Claim 4. *There exists $d \in N(c) \cap S$ such that da for all $a \in N(c) \setminus S$.*

Proof. Follows from Claim 2 and 3. □

Claim 5. *For $x \in S \setminus \{d\}$ and for $a, b \in N(c) \setminus S$, if $\overline{ab}$ then $\overline{xa}$ and $\overline{xb}$.*

Proof. Since da, db it follows that if xa and/or xb, then D induces a C_4 or P_4, a contradiction. □

Claim 6. *$N(c) \setminus S$ is $2K_2$-free.*

Proof. Any $2K_2$ in $N(c) \setminus S$, along with cd, the vertex in C and a P_3 from B would induce H_4 in G, a contradiction. □

Claim 7. *$G[A]$ has a polar partition with a single stable set.*

Proof. By Claim 6, $(N(c) \setminus S)$ is $2K_2$-free. Also D is C_4-free. Hence $(N(c) \setminus S)$ induces a threshold graph. Let (S', K) be a polar partition of $(N(c) \setminus S) \cup C$ with S' the single stable set and K the single clique. Then $(S' \cup S \setminus \{d\}, K \cup \{d\})$ is a polar partition of $G[A]$ with a single stable set, by Claims 4 and 5. □

It follows that Case 1 is impossible.
Case 2 : Every P_3 of A intersects both C and D.
Since both C and D are P_3-free, each one induces a disjoint union of cliques. We can assume without loss of generality that C consists of either a single clique or a single stable set, for otherwise, i.e., if both C and D are neither a single clique nor a single stable set, G contains H_3. If C consists of a single stable set, then $G[A]$ has a polar partition with one stable set. If C consists of a clique of size at least 2, then D has at most one clique of size at most 2 (else G has H_4). It follows that the rest of D forms a single stable set and $G[A]$ has a polar partition with a single stable set and many cliques. Thus this case is also impossible.
It follows that G must be polar. □

4 Recognition of polar cographs

Indeed, Theorem 1 of Section 2 implies a polynomial time recognition algorithm for polar cographs; given a cograph $G = (V, E)$ where $|V| = n$, G is polar if and only if $n(MP(G)) = n$. Here we give a simpler algorithm with a better time complexity deciding whether a given cograph is polar or not and building a polar partition if there is one. The main idea of the algorithm is that at a 0-vertex of the cotree, the underlying graph is polar if and only if there is a polar partition of each connected component C_i with at most 1 stable set. First, let us establish the following lemma:

Lemma 4. *A connected $(1,\infty)$-polar cograph which is not a clique or a threshold graph, is a complete $(1,\infty)$-polar graph $K^*(1,\infty)$. This can be recognized in linear time.*

Proof. Consider a $(1,\infty)$-polar cograph G which is not a clique. If G admits a polar partition with only one clique then it is a threshold graph which can be recognized in linear time; a consequence of a result of [4] is that a cograph is a threshold graph if and only if removing any maximum stable set leaves a clique. Now, assume that every polar partition of G has more than one clique. Then the stable set is linked to all cliques since G is connected. Moreover, one can verify that the links are complete otherwise there are P_4's. To recognize $K^*(1,\infty)$, we will repetitively eliminate one of the *real twins*, i.e., adjacent vertices having the same neighborhood, and label the remaining one with u. Note that real twins have to be in a same clique in all polar partitions and they can be found in linear time on the cotree. Consequently, G is a $K^*(1,\infty)$ if and only if at the end of the process of twin elimination, the remaining graph is a complete bipartite graph where the vertices labeled with u form one stable set. This later condition is necessary; observe that applying the twin elimination process to the graph of Figure 3 a) yields the complete bipartite graph of Figure 3 b) but all vertices with label u are not in a same stable set, hence the original graph is not a $K^*(1,\infty)$. Moreover, the polar partition (A, B) is such that A induces the stable set with no labeled vertices and the cliques induced by B are obtained by keeping track of the twins. Note that this can be done in linear time in the number of vertices. □

a) b)

Fig. 3. Labeling real twins.

In what follows $P(G)$ denotes a polar partition (A, B) of G. Now if $P(G) = (A, B)$ then we define $\overline{P(G)} = (\overline{B}, \overline{A})$ meaning that stable sets (resp. cliques) of A (resp. B) become cliques (resp. stable sets). Clearly $\overline{P(G)} = P(\overline{G})$ and it is also a polar partition. b is the number of connected components having at least 2 stable sets in every polar partition. Also, updating (A, B) means that according to the polar partition of the connected component under consideration, we add its stable set(s) in A and its clique(s) in B.

```
Polar cograph recognition
input :  a cograph G and its cotree
output :  a polar partition P(G) = (A, B) of G
          or a negative answer ``G is not polar''
Begin
A := ∅;  B := ∅;
If G is disconnected with components CC_1, ..., CC_p then
   b := 0;
   For i = 1 to p Do
        If CC_i is a stable set or a clique or a threshold graph
           or a K*(1, k) for some k then update (A, B)
           Else b := b + 1;
        Fi
   Od
   If b = 0 then return P(G) = (A, B)
      Else If b = 1 and ∀i ∈ [0, ... p], i ≠ j CC_i is a clique then
              P(G) := (A, B) ∪ P(CC_j)‾
              Else return ``G is not a polar graph ''
          Fi
   Fi
   Else P(G) := P(G‾)‾
Fi
End.
```

Theorem 3. *For any cograph G, the algorithm* `Polar cograph recognition` *can recognize whether G is a polar graph in time $O(n \log n)$.*

Proof. The algorithm gives a negative answer when there is at least one connected component with exactly one stable set in every polar partition and another connected component which is neither a stable set, nor a clique, nor a threshold graph nor a $K^*(1,k)$. We know by Lemma 4 that these are all possible cases for a $(1,k)$-polar cograph therefore there is no possible polar partition for such a graph. On the other hand, if G is a polar cograph then `Polar cograph recognition` provides a polar partition.
The complexity is provided by Lemma 4 and the fact that these linear operations are repeated at most h times where h is the height of the cotree, i.e., $O(\log n)$. □

Now, let us mention a more general remark on the recognition of polar graphs. Although it is NP-hard to recognize polar graphs in general, it becomes polynomially solvable under some circumstances.
In [5], $\mathcal{S}$ and $\mathcal{D}$ are defined as two classes of graphs, called *sparse* and *dense* respectively, satisfying the following conditions: both $\mathcal{S}$ and $\mathcal{D}$ are hereditary classes and there exists a constant c such that the intersection $S \cap D$ has at most c vertices for any $S \in \mathcal{S}$ and $D \in \mathcal{D}$. A *sparse-dense partition* of a graph G with respect to the classes $\mathcal{S}$ and $\mathcal{D}$, is a partition of the vertex set of G into two parts where one induces a sparse graph and the other one induces a dense graph.

Theorem 4. *[5] All sparse-dense partitions of a graph can be found in time $O(n^{2c+2}T(n))$ where $T(n)$ is the time for recognizing sparse and dense graphs.*

Corollary 1. *For any perfect graph G and for fixed s, k, it can be recognized in polynomial time whether G admits a (s,k)-polar partition.*

Proof. First, note that a join of s stable sets is a sparse graph and that a union of k cliques is a dense graph. Then, one can observe that for fixed s, k, there can be at most $c = \min(s,k)$ vertices in the intersection of a $(s,0)$-polar graph and a $(0,k)$-polar graph. Furthermore, if G is perfect then $(s,0)$-polar and $(0,k)$-polar graphs can be recognized in polynomial time; G is $(s,0)$-polar if and only if $\chi(G) = s$ and G does not contain an edge and an isolated vertex as induced subgraph, G is $(0,k)$-polar if and only if $\overline{G}$ is $(k,0)$-polar. Note that the complexity is no more polynomial if s and k are not fixed. □

5 Characterization of monopolar cographs by forbidden subgraphs

Theorem 5. *For a cograph G, the following are equivalent.*

a) G is monopolar.
b) Neither G nor $\overline{G}$ contains any one of the graphs $G_1, \ldots, G_9$ of Figure 4 as an induced subgraph.
c) G or $\overline{G}$ is a disjoint union of threshold graphs and complete $(1,\infty)$-polar graphs.

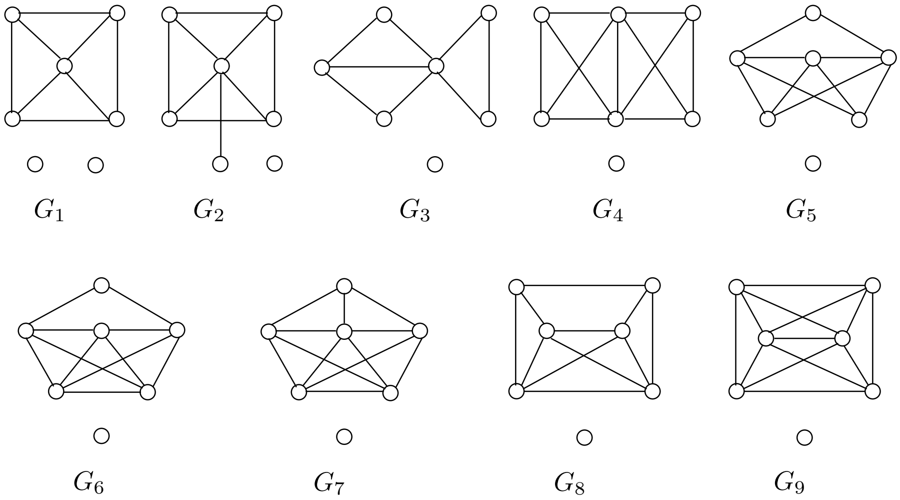


Fig. 4. Forbidden subgraphs for monopolar cographs.

Proof. *a*) $\Rightarrow$ *b*) Since the complement of a monopolar graph and every induced subgraph of a monopolar graph are also monopolar, it is enough to show that $G_1, \ldots, G_9$ are not monopolar. Since the non-trivial component in each of these graphs is not a disjoint union of cliques, it must contain the join of stable sets in any polar partition. It is routine to verify that any polar partition of these graphs must be the join of at least 2 stable sets and the union of at least 2 cliques. Hence they are not monopolar.
c) $\Rightarrow$ *a*) Since a threshold graph has a polar partition into a single stable set and a single clique, and since disjoint union of stable sets is a single stable set, it follows that if G is a disjoint union of threshold graphs and complete $(1, \infty)$-polar graphs, then G is monopolar with a single stable set and a disjoint union of cliques in a polar partition.
b) $\Rightarrow$ *c*) Since G is a cograph, assume without loss of generality that G is disconnected. It is enough to show that each non-trivial component of G is either a threshold graph or a complete $(1, \infty)$-polar graph. Let G' be any non-trivial component of G. Further assume that G' is a join of A, B (i.e., $G' = A \oplus B$). The non-empty graphs A, B exist since G' is a connected cograph with at least 2 vertices. We consider several cases.
<u>*Case 1 :*</u> A contains an induced C_4 with vertices a, b, c, d and edges ab, bc, cd and ad.
B must be a stable set, for otherwise G contains $G_9 = (C_4 \oplus K_2) \cup K_1$. Let x be any other vertex of A. Then

i) x must be joined to at least one vertex of the C_4, for otherwise G contains $G_2 = ((C_4 \cup K_1) \oplus K_1) \cup K_1$,

ii) x may not be joined to exactly 3 vertices of the C_4, for otherwise G contains G_7,
iii) x may not be joined to all 4 vertices of the C_4, for otherwise G contains $G_9 = ((C_4 \oplus K_1) \oplus K_1) \cup K_1$, and
iv) x may not miss 2 adjacent vertices of the C_4 for otherwise G' contains P_4.

It follows that each vertex of A other than a, b, c, d is joined either to a and c or else is joined to b and d. Let N_a be the set of all neighbors of a in A and N_b be the set of all neighbors of b in A. Clearly N_a and N_b form stable sets by i)-iv) and are also completely joined, to avoid induced P_4. Thus G' is a join of 3 stable sets. Now G may contain at most one other component which must be a clique, for otherwise G contains G_1. It follows that the complement $\overline{G}$ is a complete $(1, 3)$-polar graph in this case, as required.
Case 2 : A contains an induced $2K_2$.
B must form a stable set, for otherwise G contains $G_4 = (2K_2 \oplus K_2) \cup K_1$. If A contains an induced P_3, then to avoid P_4, it must contain P or Q of Figure 5 below as an induced subgraph. If A contains P, then G contains $G_4 = (P \oplus K_1) \cup K_1$

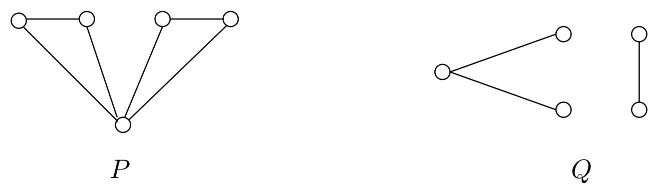


Fig. 5. Case 2 of Theorem 5.

and if A contains Q, then G contains $G_3 = (Q \oplus K_1) \cup K_1$. It follows that A is P_3-free and hence induces a disjoint union of cliques. Hence $G' = A \oplus B$ is a complete $(1, \infty)$-polar graph as required.
We may now assume by symmetry, that both A and B do not contain induced $C_4, 2K_2$ and P_4 and hence form threshold graphs.
Case 3 : A is a threshold graph containing a triangle.

i) If A contains a $K_4 \setminus e$, then B must be a clique or else G contains $G_9 = ((K_4 \setminus e) \oplus 2K_2) \cup K_1$. Since a threshold graph joined to a clique is a threshold graph, G' is a threshold graph in this case, as required.
ii) If A contains a vertex joined to exactly one vertex of the triangle, then too B must be a clique or else G contains G_7. Hence G' is a threshold graph as required.
   It follows that A induces a clique and isolated vertices.
iii) If A forms a clique and at least one isolated vertex, then B contains no non-edge, or else G contains G_6. Thus B is a clique and hence G' is a threshold graph as required.

iv) If A forms a single clique, then B being a threshold graph, G' too is a threshold graph as required.

Case 4 : Both A and B induce threshold graphs with no triangles.

i) If A contains an induced $P_3 \cup K_1$, then B must be a clique, for otherwise G contains $G_5 = ((P_3 \cup K_1) \oplus 2K_1) \cup K_1$. Hence G' is a threshold graph, as required. So we may assume that A is either $K_{1,n}$ for some $n > 1$, or P_3-free.
ii) If A is $K_{1,n}$ with $n > 1$, then B may not contain an induced P_3 (to avoid $G_9 = (P_3 \oplus P_3) \cup K_1$) and may not contain an induced $K_2 \cup K_1$ (to avoid $G_7 = (P_3 \oplus (K_2 \cup K_1)) \cup K_1$). Thus B is a single clique or a single stable set. If B is a clique, then G' is a threshold graph as required and if B is a stable set with at least 2 vertices, then G may contain only one other component which is a clique, or else G contains G_1. Hence $\overline{G}$ is a complete $(1, 3)$-polar graph as required.
iii) Hence, by symmetry, we may assume that both A and B may not contain $K_2 \cup K_1$ for otherwise G contains $G_8 = ((K_2 \cup K_1) \oplus (K_2 \cup K_1)) \cup K_1$. Thus, one of A and B, say B is a clique or a stable set. If B is a clique then since A may not contain $2K_2$ (otherwise G contains $G_4 = (2K_2 \oplus K_2) \cup K_1$), G' is a threshold graph as required. If B is a stable set, then G' is a complete $(1, \infty)$-polar graph since A is a disjoint union of cliques.

Thus in all cases, either the complement $\overline{G}$ is a complete $(1, 3)$-polar graph or G is a disjoint union of threshold graphs and complete $(1, \infty)$-polar graphs.

6 Final remarks

We have provided algorithms and characterizations related to polar cographs. There are many questions that still remain to be answered. Among those a characterization of $(2, 2)$-polar cographs by forbidden subgraphs would be a natural continuation. Also one should explore more general subclasses of perfect graphs to characterize their polarity. Further research could focus on permutation graphs or line graphs of bipartite graphs.

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