Supersymmetry in random two velocity processes

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Abstract

We discuss a random two-velocity process on the line with space dependent exogenous drift. For this process, the probability density and the associated "probability current" are shown to be in a supersymmetric relation.

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Introduction

Consider the inhomogeneous diffusion of particles on $\mathbb{R}$ described by the stochastic differential equation (i.e. Langevin equation) in the form:

$$dX_t = g(X_t)dB_t \qquad (S), \tag{1}$$

where dB_t stands for the standard Brownian motion, $g(x) > 0$ controls the diffusion process and where S means that the underlying stochastic integral is interpreted in its Stratonovitch form. The probability density $u(x,t)$ associated with the stochastic process Eq.(1) obeys to the Fokker-Planck equation [1]:

$$\partial_t u(x,t) = \frac{1}{2}\partial_x\Big[g(x)\partial_x[g(x)u(x,t)]\Big]. \tag{2}$$

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The *parabolic nature* of Eq.(2) implies that probability propagates with an infinite velocity. This feature may lead to difficulties for the physical interpretation see for instance [2,3].

To remove this structural difficulty, several alternatives to Eq.(1) have been proposed. Among the simplest possibilities, K.P. Hadeler [2] emphasizes that a tractable alternative which takes into account inertia and correlations are random velocity (RV) jump processes. Basically, these models lead to motion with finite propagation speeds and approaches Brownian motion in a diffusive limit [4]. The simplest model belonging to the RV-class is based on a two-velocity process on $\mathbb{R}$ given by the Langevin-type equation:

$$\dot{X}_t = g(X_t) I_t, \tag{3}$$

where I_t is now a dichotomous noise taking values in the set of velocities $\{-v, v\}$ and having exponentially distributed holding times with parameters $\lambda > 0$ (from v to $-v$) and $\mu > 0$ (from $-v$ to v). Langevin equations with this type of telegraphic noise are thoroughly studied in [5]. It is established that the probability densities $u^+ = u^+(x,t)$ (resp. $u^- = u^-(x,t)$) of a particle going to the right (resp. left) and subject to the dynamics Eq.(3) obey to the hyperbolic system of partial differential equations:

$$\partial_t u^+ + v\partial_x[g(x)u^+] = -\lambda u^+ + \mu u^-, \tag{4}$$
$$\partial_t u^- - v\partial_x[g(x)u^-] = +\lambda u^+ - \mu u^-. \tag{5}$$

From Eqs. (4) and (5), we can draw the following elementary remarks:

- The system given by Eqs.(4) and (5) exhibits a *hyperbolic structure*. Accordingly, the propagation of probability occurs at finite speed in contrast with the parabolic structure of Eq.(2).
- In both Eqs.(1) and (3), we can, supposing the integrability of $x \mapsto g^{-1}(x)$, introduce the new variable $Y = \int^X g^{-1}(x)\, dx$ and then consider, in terms of the variable Y, the associated (homogeneous) transport problem having a constant diffusion coefficient. Accordingly, without explicit mention, we shall always consider the case where $g(x) \equiv 1$.

An alternative interpretation of Eqs.(4) and (5) is given by S. Goldstein [6] who investigates particles performing independently persistent random walks on a lattice. A suitable continuum limit of this transport process also yields Eqs.(4) and (5) for the particles densities going to the right respectively to the left. The first direct treatment of the telegraph equation in continuous time as given in eq.(3) goes back to M. Kac [7] (see, [8] for a recent comprehensive overview on the history of random evolutions). As emphasised in [2,3], the persistent random walk provides a better description of spatial spread in population dynamics than Brownian motion. Defining the total density $P(x,t)$ and the current $Q(x,t)$ as:

$$P(x,t) = u^+(x,t) + u^-(x,t), \tag{6}$$
$$Q(x,t) = u^+(x,t) - u^-(x,t), \tag{7}$$

one finds from Eqs.(4) and (5) the one-dimensional Cattaneo-like system [3]:

$$\partial_t P(x,t) + v\partial_x Q(x,t) = 0, \tag{8}$$
$$\partial_t Q(x,t) + v\partial_x P(x,t) = [\mu - \lambda]\, P(x,t) - [\lambda + \mu]\, Q(x,t), \tag{9}$$

which describes a macroscopic spatial spread of particles on $\mathbb{R}$. Separating the fields $P(x,t)$ and $Q(x,t)$ by differentiating Eqs.(8) and (9) with respect to t and x, it is immediate to obtain that $P(x,t)$ and $Q(x,t)$ both satisfy the same dissipative wave equation:

$$\partial_t^2\phi(x,t) + (\lambda+\mu)\partial_t\phi(x,t) = v^2\partial_x^2\phi(x,t) + (\lambda-\mu)v\frac{\partial}{\partial x}\phi(x,t). \tag{10}$$

When $\lambda = \mu$, Eq.(10) reduces to the standard telegraphist equation [6,7]. Since these pioneering works, numerous alternative derivations of Eq.(10) have been performed [9,10]. A recent and comprehensive review devoted to this topic is delivered by G.H. Weiss [11] who emphasizes several relevant physical aspects of such transport processes.

While the space inhomogeneity in the Langevin Eqs.(1) and (3) is introduced via the $g(x)$ term as a noise *amplitude modulation* (see e.g., [12]), it is important to emphasize that Eq.(3) does also offer the possibility to consider inhomogeneity due to noise *spectral modulations* via spatial dependence of the terms $\lambda(x) > 0$ and $\mu(x) > 0$. Relatively little attention has so far been devoted to these spectral modulation cases. Noticeable exceptions being *i*) first passage time problems considered in [13] where inhomogeneities of the spectral type occur naturally and *ii*) non-Markovian dichotomous processes considered in [14] where the non-Markovian character of the holding times is translated into the dependence of the switching rates λ and μ on x. In addition the relevance of spectral modulation for flagellated bacteria such as *E. coli* or more generally for *chemotaxis* in living systems has also been recently pointed out in [15].

The aim of the present paper is to show that for a special class of noise spectral inhomogeneities (i.e. when $\lambda(x) + \mu(x) = \text{const.}$) the resulting density field $P(x,t)$ and its associated current field $Q(x,t)$ are connected via a supersymmetric relation similar to the one arising in quantum mechanics. This exceptional structure offers the possibility to apply powerful algebraic tools to discuss the relations between the transient behaviors of the fields $P(x,t)$ and $Q(x,t)$ for these inhomogeneous transport problems.

The presentation is organized as follows: In section 1 we introduce the general inhomogeneous random velocity process and derive formally the associated stationary probability measures. In section 2, the supersymmetric structure connecting the dynamics of $P(x,t)$ and $Q(x,t)$ is explicitly unveiled and a simple illustration is given.

1 Velocity process with inhomogeneous dichotomous noise

We consider as in [5] a stochastic process $\{X_t\}_{t\in\mathbb{R}^+}$ defined on a standard probability space $(\Omega,\mathcal{F},\mathbb{P})$ with state space $(\mathbb{R},\mathcal{B})$ and whose dynamical evolution is given by the piecewise deterministic evolution:

$$\dot{X}_t = h(X_t) + g(X_t)I_t(X_t) \tag{11}$$

where h and g are given functions of class $\mathcal{C}^1(\mathbb{R})$ with $g>0$ such that $x\mapsto 1/g(x)$ is integrable and where $I_t(X_t)$ is a state dependant dichotomous noise with values in $\{-1,+1\}$ (for simplicity velocity v is now set to 1). The random holding time of $I_t(x)$ in state 1 (resp. -1) is governed by a *space inhomogeneous* probability density of exponential type with parameters $\lambda(x)\in\mathcal{C}^1(\mathbb{R})$ (resp. $\mu(x)\in\mathcal{C}^1(\mathbb{R})$). For all $x\in\mathbb{R}$, $\lambda(x)$ (resp. $\mu(x)$) is strictly positive and gives the average frequency of switching from $\{1\}$ to $\{-1\}$ (resp. from $\{-1\}$ to $\{1\}$).

Observe that the pair process (X_t,I_t) is a Markov process. Its transition probability density is denoted by $\mathbb{P}(x,i,t|y,j,s)$, $x,y\in\mathbb{R}$, $i,j\in\{-1,1\}$, $0<s<t$. Fix a starting point $X_0=x_0$ and an initial velocity $i_0\in\{-1,1\}$ and set:

$$\begin{aligned} u^+(x,t|0) &:= \mathbb{P}(X_t=x, I_t=+1, t|X_0=x_0, I_0=i_0, 0),\\ u^-(x,t|0) &:= \mathbb{P}(X_t=x, I_t=-1, t|X_0=x_0, I_0=i_0, 0). \end{aligned}$$

It is easy to establish that, exactly as in the case of constant switching rates (see e.g., [5] p.260), the time evolution of $u^+(x,t|0)$ and $u^-(x,t|0)$ reads:

$$\begin{aligned} \partial_t u^+(x,t|0) = {} & -\partial_x\Big[\Big(h(x)+g(x)\Big)u^+(x,t|0)\Big]\\ & -\lambda(x)u^+(x,t|0)+\mu(x)u^-(x,t|0), \end{aligned} \tag{12}$$
$$\begin{aligned} \partial_t u^-(x,t|0) = {} & -\partial_x\Big[\Big(h(x)-g(x)\Big)u^-(x,t|0)\Big]\\ & +\lambda(x)u^+(x,t|0)-\mu(x)u^-(x,t|0). \end{aligned} \tag{13}$$

Integrating out the initial conditions, one directly sees that these equations still hold for the unconditioned joint probabilities respectively denoted by $u^+(x,t)$ and $u^-(x,t)$. We further define a probability density $P(x,t)$ and the associated probability flow $Q(x,t)$ in the form:

$$P = P(x,t) := u^{+}(x,t) + u^{-}(x,t), \tag{14}$$
$$Q = Q(x,t) := u^{+}(x,t) - u^{-}(x,t). \tag{15}$$

Using Eqs.(12) and (13), the resulting Cattaneo-like system for these new fields P and Q reads:

$$\partial_t P + \partial_x \Big[h(x)P + g(x)Q \Big] = 0, \tag{16}$$
$$\partial_t Q + \partial_x \Big[g(x)P + h(x)Q \Big] = -(\lambda(x) - \mu(x))P - (\lambda(x) + \mu(x))Q. \tag{17}$$

For inhomogeneous rates $\lambda(x)$ and $\mu(x)$, the elimination from Eqs.(16) and (17) of one of the fields P or Q to obtain a simple hyperbolic system of the telegraphist type is not possible in general. However, supposing that $g(x)^2 > h(x)^2$ for all $x \in \mathbb{R}$, the stationary solutions $P_s(x)$ and $Q_s(x)$ can be formally obtained in the closed form:

$$Q_s(x) = -\frac{h(x)}{g(x)} P_s(x) + \frac{C}{g(x)}, \tag{18}$$
$$\begin{aligned} P_s(x) = & \frac{g(x)}{g(x)^2 - h(x)^2} \exp\Big(\int_0^x \mathrm{d}y \frac{(\mu(y)+\lambda(y))h(y) + (\mu(y)-\lambda(y))g(y)}{g(y)^2 - h(y)^2} \Big) \\ \times & \Big[N - C \int_0^x \big[\mathrm{d}y \frac{\mu(y) + \lambda(y) + h'(y) - h(y)g'(y)/g(y)}{g(y)} \\ & \quad \times \exp\big(- \int_0^y \mathrm{d}z \frac{(\mu(z)+\lambda(z))h(z) + (\mu(z)-\lambda(z))g(z)}{g(z)^2 - h(z)^2} \big) \big] \Big] \end{aligned} \tag{19}$$

with N and C being two constants. When interpreting $P_s(x)$ as a probability measure (and not as a particle density function), N stands for a normalization constant which exists whenever we can find a positive K such that:

$$(\mu(x) + \lambda(x))h(x) < (\lambda(x) - \mu(x))g(x), \quad \forall x > K, \tag{20}$$
$$(\mu(x) + \lambda(x))h(x) > (\lambda(x) - \mu(x))g(x), \quad \forall x < -K. \tag{21}$$

The Eqs.(20) and (21) express, for $h \neq 0$, the deterministic stability condition introduced in [5]. For $h = 0$, which is the case we will focus on, the following observations can be given:

A finite stationary solution $P_s(x)$ exists only when the switching rates satisfy $\mu(x) < \lambda(x)$ for all sufficiently large x and $\mu(x) > \lambda(x)$ for all sufficiently large $|x|$, $x < 0$. Therefore the noise spectral modulation can generate noise induced spatial structures. Moreover, when the conditions Eqs.(20) and (21) are satisfied, the integration constant C vanishes (see [5] p. 266) and hence $Q_s(x) \equiv 0$.

2 Transient behavior and Supersymmetry

The importance of studying transient behaviour of probability densities associated with stochastic differential equations is largely commented in far from equilibrium processes (see for instance the recent works devoted to Brownian ratchets and stochastic resonance [16]). In particular, diffusion processes are abundantly described and explicit transient solutions of Fokker-Planck equations have been derived using, among other approaches, the connection with supersymmetric quantum mechanics [17]. Explicit transient solutions of the Chapmann-Kolmogorov equation associated to stochastic differential equations of the type given in Eq.(11) are so far much less discussed. Exceptions worthwhile mentioning are the cases i) $h(x) = 0$ with homogenous rates λ and μ leading to the telegraphist eq.(10) and ii) $h(x) = -\gamma x$ with γ a constant and with homogenous rates λ and μ discussed in [18] and more recently in [14] and [19]. As pointed out in [18], the basic difficulty when $h \neq 0$ is due to the lack of self-adjointness in the system of equations (12) and (13). Here we consider the case $h(x) = 0$ together with space inhomogeneous $\lambda(x)$ and $\mu(x)$ with the restriction $\lambda(x) + \mu(x) = \beta =$ constant. The resulting class of Langevin-type equations (indexed by $\beta \in \mathbb{R}^+$) enjoys the following remarkable properties:

i) The probability densities $P(x,t)$ and the associated probability flows $Q(x,t)$ obey second order PDE's with a spatial part similar to Fokker-Planck equations corresponding to diffusive processes with a drift term.
ii) The probability density $P(x,t)$ and the associated probability flow $Q(x,t)$ are in a supersymmetric relation.

To exhibit these properties, we introduce the potential $V(x) = \int_{-\infty}^{x}(\lambda(y) - \mu(y))\mathrm{d}y$, and write the Cattaneo-like system Eqs.(16) and (17) as:

$$\partial_t P + \partial_x Q = 0, \tag{22}$$

$$Q(x,t) + \frac{1}{\beta}\partial_t Q = -\frac{1}{\beta}\Big(\partial_x P + P\partial_x V\Big). \tag{23}$$

Observe that the above system is identical to the modified Smoluchowski diffusion equation discussed in [20] which approximately describes the motion of a particle moving in a potential $V(x)$ subject to Brownian movement at a constant temperature. This observation leads us to introduce:

a) a drift force $W(x) := \mu(x) - \lambda(x) = -\partial_x V$ and
b) a constant parameter $\beta := \mu(x) + \lambda(x)$ which plays the role of an effective temperature [20].

With these definitions and when separating the fields P and Q by differentiating Eqs.(22) and (23) with respect to t and x, one ends with the following damped wave equations:

$$\partial_t^2 P + \beta\partial_t P = \partial_x^2 P - \partial_x[W(x)P], \tag{24}$$
$$\partial_t^2 Q + \beta\partial_t Q = \partial_x^2 Q - W(x)\partial_x Q. \tag{25}$$

Note that Eqs.(24) and (25) are identical when $W(x)$ = constant. Moreover due to the underlying probabilistic interpretation, the conservation of the positivity of the solutions of Eqs.(24) is guaranteed even for inhomogeneous $\lambda(x)$ and $\mu(x)$. A purely analytical approach to establish the positivity has been discussed in [21] and is the basis for the associated H-theorem exposed in [22].

As mentioned in point i) above, the RHS of Eq.(24) (resp. Eq. (25)) formally coincide with the Fokker Planck forward equation (resp. backward equation) associated with the diffusion process:

$$dX_t = \frac{1}{\beta}W(X_t)\mathrm{d}t + \frac{1}{\sqrt{\beta}}dB_t, \tag{26}$$

resp.

$$dX_t = -\frac{1}{\beta}W(X_t)\mathrm{d}t + \frac{1}{\sqrt{\beta}}dB_t \tag{27}$$

with $\mathrm{d}B_t$ being standard Brownian motion and $\pm W(x)/\beta$ drift terms.

Let us now unveil the supersymmetric relation between P and Q. We write:

$$\mathbf{\Psi} = \begin{pmatrix} \psi^-(x,t) \\ \psi^+(x,t) \end{pmatrix} := \exp\left[\frac{1}{2}\Big(V(x) - V(x_0)\Big)\right] \begin{pmatrix} Q(x,t) \\ P(x,t) \end{pmatrix}, \tag{28}$$

where $x_0 \in \mathbb{R}$ is a fixed starting point of the particle. With this notations, Eqs.(24) and (25) can be written as:

$$\begin{pmatrix} \partial_{t,t}^2 + \beta\partial_t & 0 \\ 0 & \partial_{t,t}^2 + \beta\partial_t \end{pmatrix} \mathbf{\Psi} = -\begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \mathbf{\Psi}, \tag{29}$$

with

$$H_\pm := -\partial_{x,x}^2 + \frac{1}{4}W(x)^2 \pm \frac{1}{2}W'(x) \tag{30}$$

and where $W'(x)$ stands for the derivation with respect to x. This is precisely the formalism of supersymmentry (SUSY) applied in quantum mechanics (QM) (see [17,23] for recent reviews). In particular $H_\pm$ are the SUSY partner Hamiltonians acting on $L^2(\mathbb{R})$ and the drift terms $\pm W(x)$ are the so-called SUSY partner potentials. Hence the operators H_+ resp. H_- appear to be partner Hamiltonians similar to those in Wittens model of SUSY QM

[24] and Eq.(29) establishes the mentioned supersymmetric relation between P and Q.

Recall that the SUSY partner Hamiltonians $H_\pm$ with spectrum $\mathrm{Spec}(H_\pm)$ are related by means of the differential operator A and its adjoint $A^\dagger$ by:

$$H_+ = AA^\dagger, \quad H_- = A^\dagger A, \tag{31}$$

where:

$$A := \partial_x + \frac{1}{2}W(x), \quad A^\dagger = -\partial_x + \frac{1}{2}W(x). \tag{32}$$

It follows that the partner Hamiltonians are positive and essentially isospectral (i.e. the strictly positive eigenvalues of H_- and H_+ coincide). Hence, the transient behaviour of the probability density P and the flow Q are identical and the relaxation to the equilibrium is governed by the value of β and the smallest non zero eigenvalue of the Hamiltonians. Indeed, for a large class of time independent potentials W the spectrum of $H_\pm$ is of the form (see e.g., [5] Chapt. 6.7):

$$\mathrm{Spec}\Big(H_\pm\Big) \setminus \{0\} = \{\nu_1, ..., \nu_n, ...\} \cup [a, \infty[\tag{33}$$

where $[a, \infty[$, $a \in \mathbb{R} \cup \{\infty\}$ is the continuous (possibly empty) range of eigenvalues of $H_\pm$ and where $\{\nu_1, ..., \nu_n, ...\}$ is the countable (possibly empty or finite) set of eigenvalues of $H_\pm$ satisfying $0 < \nu_1 \leq ...\nu_n \leq ... \leq a$. For given initial conditions $\psi^\pm(x, 0)$ and $\psi_t^\pm(x, 0)$ the solution to Eqs.(29) can formally be expanded in a series of eigenfunctions as

$$\psi^\pm(x, t) = \sum c_\nu(t)\phi_\nu^\pm(x) + \int_a^\infty c_\nu(t)\phi_\nu^\pm(x)\mathrm{d}\nu \tag{34}$$

where the summation is taken over all discrete eigenvalues, the integral is taken over the continuous range of eigenvalues and where the square integrable eigenfunctions $\phi_\nu^\pm$ are supposed to be normalized. Depending on the sign of $\Delta_\nu := \beta^2 - 4\nu$ the $c_\nu(t)$, solving the characteristic equation $c''(t) + \beta c'(t) + \nu c(t) = 0$, are given by:

i) $c_\nu(t) = \exp(-\frac{\beta}{2}t)\Big(A_\nu \cosh(t\sqrt{\Delta_\nu}/2) + B_\nu \sinh(t\sqrt{\Delta_\nu}/2)\Big)$ if $\Delta_\nu > 0$

ii) $c_\nu(t) = \exp(-\frac{\beta}{2}t)\Big(A_\nu \cos(t\sqrt{-\Delta_\nu}/2) + B_\nu \sin(t\sqrt{-\Delta_\nu}/2)\Big)$ if $\Delta_\nu < 0$

iii) $c_\nu(t) = \exp(-\frac{\beta}{2}t)\Big(A_\nu + B_\nu t\Big)$ if $\Delta_\nu = 0$

and where the A_ν's and B_ν's are determined by the initial conditions. If the smallest non-zero eigenvalue $\nu_1 \in \mathrm{Spec}\Big(H_\pm\Big) \setminus \{0\}$ is less than $\beta^2/4$ (hence c_{ν_1} is not oscillating and of the form given in i)) the relaxation to stationary state is governed by $\Big(-\beta + \sqrt{\beta^2 - 4\nu_1}\Big)/2$. In the contrary ($\nu_1 \geq \beta^2/4$) the approach to equilibrium is governed by $\beta/2$ and oscillates (when the inequality is strict) with a frequency $\sqrt{\nu_1 - \beta^2/4}$.

In the language of SUSY QM, the case where zero is an eigenvalue (i.e., $0 \in \mathrm{Spec}(H_-) \cup \mathrm{Spec}(H_+)$) is coined "good SUSY" and when all eigenvalues are strictly positive SUSY is "broken". Recall that for a non vanishing asymptotic behaviour of the drift force $W(x)$, the dichotomy between "good" and "broken" SUSY can be discussed using:

$$\text{broken SUSY} \Leftrightarrow \mathrm{sign}\Big(\lim_{x\to\infty} W(x)\Big) = \mathrm{sign}\Big(\lim_{x\to-\infty} W(x)\Big).$$

and

$$\text{good SUSY} \Leftrightarrow \mathrm{sign}\Big(\lim_{x\to\infty} W(x)\Big) \neq \mathrm{sign}\Big(\lim_{x\to-\infty} W(x)\Big).$$

The two previous alternatives allow to draw the following conclusions concerning the existence of stationary solutions:

1a) In the case of good SUSY, with

$$\mathrm{sign}\Big(\lim_{x\to\infty} W(x)\Big) = -1 \quad \text{and} \quad \mathrm{sign}\Big(\lim_{x\to-\infty} W(x)\Big) = 1$$

we have:

$$0 = \inf\{\nu \mid \nu \in \mathrm{Spec}(H_+)\} < \inf\{\nu \mid \nu \in \mathrm{Spec}(H_-)\},$$

and therefore a non-trivial stationary distribution for P (solving $A^\dagger P = 0$) exists but no (non-trivial) stationary flow Q (i.e. $C = 0$) does exist. The solution P equals P_s given by Eq.(19) with N being the normalisation constant.

1b) In the case of good SUSY, with

$$\mathrm{sign}\Big(\lim_{x\to\infty} W(x)\Big) = 1 \quad \text{and} \quad \mathrm{sign}\Big(\lim_{x\to-\infty} W(x)\Big) = -1$$

we have:

$$0 = \inf\{\nu \mid \nu \in \mathrm{Spec}(H_-)\} < \inf\{\nu \mid \nu \in \mathrm{Spec}(H_+)\},$$

and therefore a non-trivial stationary distribution for Q (solving $AQ = 0$) exists ($C \neq 0$) but no (non-trivial) stationary distribution P does exist. The solution Q equals Q_s given by Eq.(18).

2) In the case of broken SUSY, there is no (non-trivial) stationary distribution neither for P nor for Q.

Remark. It is worthwhile noting that in the SUSY formalism the "deterministic stability condition", expressed in Eqs.(20) and (21), correspond exactly to the case 1a) above and expresses the fact that only H_+ possesses a normalizable zero energy ground state eigenfunction.

Example. As an illustration of case 1a, consider the transition rates:

$$\lambda(x) = \frac{1}{2} + \frac{1}{2}\tanh(x), \quad \mu(x) = \frac{1}{2} - \frac{1}{2}\tanh(x). \tag{39}$$

The physical relevance of this example for inhomogeneous transmission lines is given in [25]. Note that the interchange of the transition rates yields an example for case 1b. We have $\beta = 1$ and the resulting exogenous drift function $W(x) = -\tanh(x)$ is a special case of the shape invariant, good SUSY, Rosen-Morse II potential (see e.g., table 4.1 in [23]). Apart from the ground state eigenvalue $0 \in \mathrm{Spec}(H_+)$ the spectrum is purely continuous and is given by $\nu = \frac{1}{4} + \tilde{\nu}^2, \quad \tilde{\nu} \geq 0$. We specify the initial conditions by setting $P(x,0) = Q(x,0) = \delta_0(x)$ (corresponding to $x_0 = 0$ and $i_0 = +1$ in Eqs.(12,13)) where $\delta_0(x)$ is the ordinary delta-function and by setting $P_t(x,0) = Q_t(x,0) = 0$. The solution for P, calculated in [26] (see also [27] model B) reads:

$$P(x,t) = \frac{1}{\cosh(x)}\Big(\frac{1}{\pi} + \frac{\mathrm{e}^{-t/2}}{2\pi}\int_0^\infty \Big[\cos(\tilde{\nu}t) + \frac{1}{2\tilde{\nu}}\sin(\tilde{\nu}t)\Big]\Big(\phi^+_{-\tilde{\nu}}(x)\phi^+_{\tilde{\nu}}(0) + \phi^+_{\tilde{\nu}}(x)\phi^+_{-\tilde{\nu}}(0)\Big)\,\mathrm{d}\tilde{\nu}\Big) \tag{40}$$

where the $\phi^+_{\tilde{\nu}}$ are given in terms of the hypergeometric functions:

$$\phi^+_{\tilde{\nu}}(x) = \exp(i\tilde{\nu}x)\cosh(x)\,{}_2F_1\Big(-1/2, 3/2; 1 + i\tilde{\nu}; \frac{1+\tanh(x)}{2}\Big). \tag{41}$$

The SUSY-structure implies that $Q = P_s\overline{Q}$ with $\overline{Q}$ solving Eq.(24) wherein W is replaced by $-W$. Hence, for Q we obtain (see [27] model C):

$$Q(x,t) = \frac{\mathrm{e}^{-t/2}}{2\pi}\int_0^\infty \Big[\cos(\tilde{\nu}t) + \frac{1}{2\tilde{\nu}}\sin(\tilde{\nu}t)\Big]\Big(\phi^-_{-\tilde{\nu}}(x)\phi^-_{\tilde{\nu}}(0) + \phi^-_{\tilde{\nu}}(x)\phi^-_{-\tilde{\nu}}(0)\Big)\,\mathrm{d}\tilde{\nu} \tag{42}$$

where the $\phi^-_{\tilde{\nu}}$ are given in terms of the hypergeometric functions:

$$\phi^-_{\tilde{\nu}}(x) = \exp(i\tilde{\nu}x)\cosh(x)\,{}_2F_1\Big(1/2, 1/2; 1 + i\tilde{\nu}; \frac{1+\tanh(x)}{2}\Big). \tag{43}$$

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References

[1] C.W. Gardiner. *Handbook of Stochastic Methodes.* Springer Verlag, 1983.

[2] K.P. Hadeler. *Reaction telegraph equations and random walk systems*. Springer Verlag, 1993.

[3] K.P. Hadeler. *Reaction Transport Systems in Biological Modelling*. Lecture Notes in Mathematics 1714: Springer Verlag, 1997.

[4] T. Hillen and H.G. Othmer. The diffusion limit of Transport Equations derived from Velocity-Jump Processes. *SIAM J. Appl. Math.*, 61 (3):751–775, 2000.

[5] W. Horsthemke and R. Lefever. *Noise-Induced Transitions. p.*264. Springer-Verlag, 1984.

[6] S. Goldstein. On diffusions by discontinuous movements and on the telegraph equation. *Quarterly Journal of Mechanics and Applied Mathematics*, 4 (2):129–156, 1951.

[7] M. Kac. A Stochastic Model Related to the Telegrapher's Equation. *Rocky Mount. J. Math.*, 4 :497-510, 1974.

[8] R. Hersh. The birth of random evolutions. *Mathematical Intelligencer*, 25 (1) :53-60, 2003.

[9] E. Orsingher. Hyperbolic-equations arising in random models. *Stochastic Processes and their Applications*, 21 (1):93–106, 1985.

[10] M.A Pinsky. *Lectures on Random Evolutions*. World Scientific Publishing, 1991.

[11] G.H. Weiss. Some applications of persistent random walks and the telegrapher's equation. *Physica A*, 311:381–410, 2002.

[12] N.E. Ratanov. Telegraph Evolutions in Inhomogeneous Media. *Markov Processes and Related Fields*, 5:53–68, 1999.

[13] J. Masoliver and G.H. Weiss. First passage times for a generalized telegrapher's equation. *Physica A*, 183:537–548, 1992.

[14] M. Annunziato. Non-Gaussian equilibrium distribution arising from the Langevin equation. *Physical Review E*, 65:021113, 2002.

[15] H.G. Othmer and T. Hillen. The diffusion limit of Transport Equations II: Chemotaxis equations. *SIAM J. Appl. Math.*, 62 (4):1222–1250, 2002.

[16] P. Reimann and P. Hänggi. Introduction to the physics of Brownian motors. *Appl. Phys. A: Material Science and Processing*, 75:169–178, 2002.

[17] G. Junker. *SUSY Methods in Quantum and Statistical Physics*. Springer-Verlag, 1996.

[18] P. Sibani and N.G. van Kampen. An exactly soluble Relaxation Problem. *Physica A*, 122:397–412, 1983.

[19] M.O. Cáceres. Computing a non-Maxwellian velocity distribution from first principles. *Physical Review E*, 67:016102, 2003.

[20] R.A. Sack. A modification of Smoluchowski's diffusion equation. *Physica*, XXII:917–918, 1956.

[21] M.-O. Hongler and L. Streit. Generalized Master Equations and the Telegrapher's Equation. *Physica A*, 165:196–206, 1990.

[22] J. Camacho and D. Jou. H theorem for telegrapher type kinetic equations. *Physics Letters A*, 171:26–30, 1992.

[23] F. Cooper, A. Khare, and U. Sukhatme. *SUSY in Quantum Mechanics*. World Scientific, 2001.

[24] E. Witten. Dynamical breaking of supersymmetry. *Nuc. Phys. B*, 188:513–554, 1981.

[25] M.-O. Hongler. Supersymmetry and Signal Propagation in Inhomogenous Transmission Lines. *Physica A*, 137:407–416, 1986.

[26] E. Wong. The construction of a class of stationary Markov processes. *Symp. Appl. Math.*, 16:264-276, 1964.

[27] M.O Hongler and R.C. Desai. Decay of unstable states in presence of fluctuations. *Helv.Phys. Acta.*, 59:367-389, 1986.