ON THE OUTFLOW PROCESS OF N-STATIONS MERGE SYSTEMS FOR ITEMS WITH NON-VANISHING SPACIAL EXTENSIONS

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Abstract. In this study a N-station discrete material flow merge system model feeding a buffer connected to a downstream station is presented. The processing rates of the N reliable stations are exponentially distributed. The resulting arrival process into the merging system is therefore a Poisson process. We show that when the items processed have finite spacial extensions, the Poisson structure of the arrival process is lost. The resulting non-Poisson outflow process from the merging system is explicitly calculated. Numerical simulations underline the accuracy of the method.

1. Introduction

A general production system is a network of workstations, transportation elements and buffers. The material processed by the production system is routed through the stations where appropriate operations are performed. The ubiquitous presence of noise sources within the network (failures of machines random processing times etc.) clearly affects the dynamics of the materials flow and makes their understanding, control and optimization a formidable task. The issuing large amount of research literature facing these questions dealing with stochastic production networks can be classified into discrete materials and continuous materials flow production systems [1].

In discrete materials flow production systems (e.g. automobile, integrated circuits), each item receives a specific, more or less time consuming operation at stations where it is routed through. When finite buffers are considered, a discrete materials flow production system is basically a *discrete queuing network* subject to blocking which have been extensively studied in industrial engineering, operations research, telecommunications and applied probabilities [2].

In continuous materials flow production systems (e.g. chemical industry, oil refining), the stations process a fluid or produce quantities of material flowing through the network of workstations. The continuous case seems so far to be less investigated than the discrete one even though their theoretical treatment as *fluid queue models* [3, 4] is an ongoing research topic in stochastic modelling and control. Fluid queues find also pertinent applications in the modeling of the Internet [5].

Common to both, the discrete and the continuous materials flow systems, is the fact that most of the available analytical results are based on *Markov* assumptions. This is realized by supposing that the stations failure times, stations repair times or the

Key words and phrases. Stochastic manufacturing system, production-inventory process, merging, traffic.

This work is partially supported by the Fonds Nationale Suisse pour la recherche scientifique (FNRS).

processing times are mutually independent and exponentially distributed random variables. Especially for the discrete material flow case, which is the viewpoint adopted in this paper, most of the past analytical research on stochastic production systems leads to the study of Markov chain models and in the simplest cases to the study of Jackson networks with Markov routing. The availability of analytical results in such networks is based on *i*) the classical result that the output process of a $M/M/1$ queue is a Poisson process and *ii*) the robust splitting and merging properties of the Poisson process (see e.g. example (a) in [6] Chap. XI.4).
In this paper we analytically question the use of Poisson processes downstream of a merging point in discrete materials flow production systems. We indeed show that the outflow process from a multiplexing point – fed by N independent Poisson streams – forms a Poisson process only in case of (idealized) items with vanishing spacial extensions. In all other cases, the time headway between consecutive items flowing out of the merge system is not exponential and may be interpreted as the maximum of two random variables taking into account the spacial extensions of the items. Clearly, the non-markovian behaviour propagates downstream together with the flow of items and we show that the inflow to workstations below a merge system is described by batch arrivals of workload rather than by simple Poisson streams.
The paper is organized as follows. In section 2 we state the problem of interest. In section 3 we analytically derive the stationary distribution of the outflow process and show that it tends to a Poisson process as the extensions of the items shrink to that of a material point. We compare in Section 4 the analytical results with numerical simulation studies and conclude the paper with section 5.

2. Problem Formulation

Suppose items arrive from N independent Poisson streams with intensities $\lambda_1, ..., \lambda_N$ into a collecting buffer B (i.e. into a merging system). Suppose further that items leave the buffer in the "first in first out" queueing discipline from where they are transported on a conveyor system to some workstation M. (see Fig. 1).

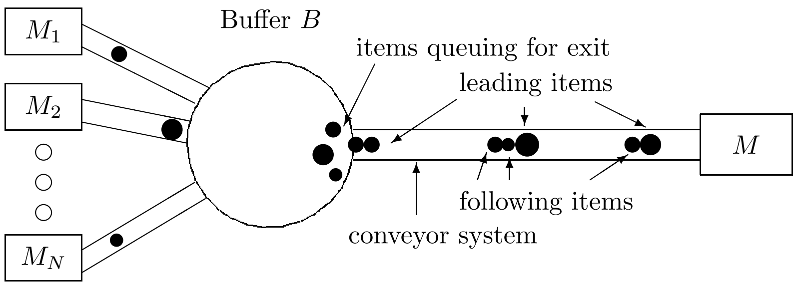

Figure 1. Merging of N Poisson streams of possibly different items into a buffer. The conveyor system transports the items from the buffer to the machine M with constant velocity v. The output process is divided in following and leading items.

As long as the buffer B is not full, the arrival process to the buffer is a Poisson process of intensity $\lambda = \sum_{i=1}^{N} \lambda_i$. The departure process however, physically realized by a conveyor system is Poisson only when the outgoing items do not interact during the outflow process (i.e. no outflow delay for items leaving the buffer due to other items queuing for exit in the buffer). In case of no interactions the distribution of the outflow process coincides with the distribution of the arrival process. When interactions occur (i.e. queuing at the buffer exit) the conveyor system produces a stream of items, separated only by a conveyor dependant minimal headway h. Streams of following items flow toward M where they are proceeded further. We now divide items flowing out of the buffer into two classes; items separated from its predecessor by the minimal headway h (following items) and items separated from its predecessor by headways exceeding h (leading items).

In this paper we address the problem of finding the distribution of following and leading items after the merging as a function of λ (i.e. the intensity of the item arrival stream into the buffer), the distribution of the possibly random finite extension (length) of the items L and the constant velocity v of the conveyor system. For ease of presentation we suppose here that the minimum headway h on the conveyor system is zero (i.e. $h = 0$). This assumption is easily removed by replacing the length L of an item by a characteristic outflow length $L + h$.

Note that it is the finite spacial extensions of the items together with the finite velocity v of the conveyor system which blocks the output process during L/v (i.e. no other item than the outflowing one can leave the buffer during L/v). This problem is clearly related to the class of counter problems of recording apparatus when the finite resolving time is taken into account. A typical example is an α-particle counter as treated in [7] and which is impeded during a certain interval of time after a particle is recorded. More pertinent however is the connection with velocity traffic models modelling the narrowing of a multi-lane road to a one-lane road as described by A.J. Koning in [8]. The paper contains, in the setting of car-traffic modelling, all the main ideas presented below.

3. Exact analysis of the output process from the buffer

Enumerate the items in the order in which they leave the buffer. This enumeration $\{n\}$ is the index set of a sequence of random variables S_n which we interpret as the minimum exit time of item n from the buffer. This minimum exit time of item n from the buffer is easily measured on the shop floor. Indeed it corresponds to the time needed of the corresponding exit event to take place. This event starts when item n reaches the conveyor system and ends at the instant item n is entirely placed on the conveyor system (see Fig. 2). Formally, $S_n = L_n/v$ with L_n being the (possibly random) characteristic length of item n and v the conveyor speed. The minimum exit times S_n are assumed to form a sequence of i.i.d. random variables with cumulative distribution function G.

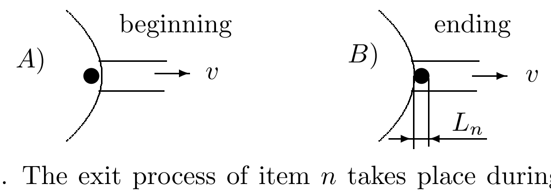

Figure 2. The exit process of item n takes place during a time interval of length L_n/v. A) Beginning of the exit event of item n from the buffer to the conveyor system with constant velocity v. B) The exit of item n from the buffer is terminated.

Define O_n as the time instant at which item n starts to leave the buffer (output process) and suppose for convenience that $O_0 = 0$. Denote by D_n the time instant item n starts to leave the buffer if its beginning was not delayed by the exit process of item $n-1$. The crucial observation is that when the characteristic length of all items vanishes, the exit process is immediate and therefore $O_n = D_n$. Moreover in this case the output process (O_n) from the buffer and the arrival process into the buffer which by assumption is a Poisson point process with intensity λ have the same distribution (indeed, the output process equals the arrival process shifted by the deterministic time an item needs to travel through the buffer). The above observation amounts to saying that D_n is Poisson with intensity λ and we have similar as in [8] for $n \geq 1$ the relation:

$$O_n = \max\big(D_n, O_{n-1} + S_{n-1}\big), \tag{1}$$

which is to say that the nth item starts to flow out of the buffer at time D_n unless it is delayed by the exit process of item $n-1$. In the later case n starts to leave the buffer at time $O_{n-1} + S_{n-1}$. Note that under the First in First out (FIFO) buffer discipline, which is assumed throughout the paper, the ordering of O_n and D_n are the same (by definition of FIFO, the ordering given by the exit enumeration $\{n\}$ equals the ordering of the arrival process). The later ordering is clearly independent of the characteristic length and coincides therefore with the ordering given by D_n. The following auxiliary definition is now meaningful:

$$W_n = O_n - D_n \geq 0, \quad n \geq 1 \tag{2}$$

which is the time the outflow process of item n is delayed by the exit of item $n-1$. We are now ready to characterise the output process by applying Theorem 2.1. of Koning [8] which for completeness is repeted here:

Theorem. Let $(O_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that there exists a Poisson point process $(D_n)_{n\in\mathbb{N}}$ with intensity λ, and a sequence $(S_n)_{n\in\mathbb{N}}$ of independent random variables with identical distribution function G independent of $(D_n)_{n\in\mathbb{N}}$, satisfying equation (1). If $\lambda\mathbb{E}(S_n) < 1$, where $\mathbb{E}(S_n)$ is the expectation of S_n, the time Y_n between the beginnings of the outflow processes of items $n-1$ and n given by

$$Y_n = O_n - O_{n-1}, \quad n \geq 1 \tag{3}$$

has an equilibrium distribution as n tends to ∞ with cumulative distribution function

$$F(y) = \big(1 - e^{-\lambda(y-\theta)}\big)G(y) \tag{4}$$

where

$$\theta = \frac{1}{\lambda}\ln\Big(\frac{1-\lambda\mathbb{E}(S_n)}{\mathbb{E}(e^{-\lambda S_n})}\Big). \tag{5}$$

The proof is based on the observation that the sequence of delays W_n is equivalent to a queuing process induced by the sequence $S_n - (D_n - D_{n-1})$. The distribution of W_n equals therefore the waiting time distribution W_n^q of the nth customer of a $M/G/1$ queue with arrival intensity λ and service times S_n (see e.g., Feller [6] p.194, Definition 1 and Example (a)). Here "waiting time" means the time from the customer arrival to the epoch where his service *starts*. In this framework the hypotheses $\lambda\mathbb{E}(S_n) < 1$ reduces to the stability condition for the queue length not to explode. Rewritten as $\mathbb{E}(S_n) < 1/\lambda$, the stability condition tells us that the

mean minimum exit time must be smaller than the mean inter arrival times of the items into the buffer.
In view of the theorem it is now immediate to check that for items with vanishing characteristic length (i.e. $L_n \to 0$) the output process equals D_n and is in particular a Poisson point process. Indeed:

$$S_n \to 0 \Rightarrow G(y) \to 1_{\{y \geq 0\}} \Rightarrow \theta \to 0 \text{ and } F(y) = 1 - e^{-\lambda y}.$$

Hence, eq.(4) reduces to the cumulative distribution function of an exponentially distributed random variable. In the case of non vanishing characteristic length (i.e. $L_n > 0 \Leftrightarrow \theta \neq 0$) eq.(4) clearly shows that the output process is not a Poisson process.
Note that the value of θ given in (5) makes the (stationary) intensity of the outflow process O_n equal to the (stationary) intensity of the arrival process. Indeed,

$$\begin{aligned} \mathbb{E}(Y_n) &= \int \big(1 - F(y)\big)\mathrm{d}y = \int \big(1 - G(y) + G(y)e^{-\lambda(y-\theta)}\big)\mathrm{d}y \\ &= \mathbb{E}(S_n) + e^{\lambda\theta} \int G(y)e^{-\lambda y}\mathrm{d}y = \mathbb{E}(S_n) + \frac{e^{\lambda\theta}}{\lambda}\mathbb{E}(e^{-\lambda S_n}) = \frac{1}{\lambda}. \end{aligned} \tag{6}$$

Therefore stationary first order performance measures (i.e. performance measures involving only first order moments) such as the mean throughput of the downstream machine M, will not be affected by the non-markovian behaviour of the flows due to the spacial extensions of the products. This is not so for performance measures involving higher order moments. We elucidate this point by applying standard results available for the $M/G/1$ queue (with arrival rate λ and service S_n) to the merging process in the buffer. For example, exploiting the Pollaczek-Kinchine formula (see e.g., [9] pp.259) we may establish under equilibrium conditions:
(a) *the waiting time as a function of items sizes.* The simple relation between the mean delays for exponential exits S_n and general exit distributions S_n (denoted resp. by $\mathbb{E}(W_n)[M/M/1]$ and $\mathbb{E}(W_n)[M/G/1]$):

$$\mathbb{E}(W_n)[M/G/1] = \mathbb{E}(W_n)[M/M/1]\frac{1 + c_v^2}{2}. \tag{7}$$

where $c_v = \sqrt{\mathrm{Var}(S_n)}/\mathbb{E}(S_n)$ is the coefficient of variation of S_n. Hence delays will increase with increasing variability of the extensions of the circulating items.
(b) *the population in the merge system as a function of items sizes.* The probability generating function $V(s)$ of the number of items in the buffer at the departure times $O_n + S_n$ of items:

$$V(s) = \frac{(1 - \lambda\mathbb{E}(S_n))(1 - s)\mathbb{E}(e^{(\lambda - \lambda s)S_n})}{\mathbb{E}(e^{(\lambda - \lambda s)S_n}) - s}. \tag{8}$$

As an illustration suppose for instance that the minimal exit time distribution S_n is erlangian with parameter $k \in \mathbb{N}$ (including the markovian case $k = 1$ and the deterministic case $k \to \infty$) we have:

$$V(s) = \frac{(1 - \lambda\mathbb{E}(S_n))(1 - s)}{1 - s\big(1 + (1 - s)\frac{\lambda\mathbb{E}(S_n)}{k}\big)^k} \tag{9}$$

from where it directly follows that the mean number of items in the buffer at the departure instants is:

$$M_k := \lim_{s\to 1} \frac{\mathrm{d}}{\mathrm{d}s} V(s) = \frac{\lambda \mathbb{E}(S_n)}{1-\lambda\mathbb{E}(S_n)}\Big(1 - \frac{\lambda\mathbb{E}(S_n)}{2}(1-\frac{1}{k})\Big). \tag{10}$$

M_k is visibly decreasing in k and indicates how a shrinking randomness in the exit process (variance of S_n decreases to zero as $k \to \infty$) can allow for smaller buffer design.

(c) *the outflow delays from the merge system as a function of items sizes.* The Laplace-Stielties Transform (LST) $W_q^*(s)$ of the outflow delays is:

$$W_q^*(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{d}W_n(t) = \frac{s(1-\lambda\mathbb{E}(S_n)}{s-\lambda\big(1-\mathbb{E}(e^{-sS_n})\big)} \tag{11}$$

which permits in particular to unveil the probabilistic information contained in θ. Indeed, for $s = \lambda$ we have:

$$W_q^*(\lambda) = \mathbb{E}(e^{-\lambda W_n}) = \frac{1-\lambda\mathbb{E}(S_n)}{\mathbb{E}\big(e^{-\lambda S_n}\big)} = e^{\lambda\theta}; \tag{12}$$

rewriting eq.(12) in the form:

$$\mathbb{E}(e^{-\lambda(W_n+\theta)}) = 1 \tag{13}$$

shows that θ is a measure for the mean sample path delay which is, thanks to the exponential form, very sensitive to large values of W_n.

(d) *the busy period of the merge system as a function of items sizes.* By Takàcs integral equation we know also the LST of the busy period T of the merge system (loosely speaking this is the period of time a connected outflow stream of items from the buffer is observed on the conveyor system; it will be defined more accurately below). In particular for the first two moments:

$$\mathbb{E}(T) = \frac{\mathbb{E}(S_n)}{1-\lambda\mathbb{E}(S_n)}, \tag{14}$$

$$\mathbb{E}(T^2) = \frac{\mathbb{E}(S_n^2)}{(1-\lambda\mathbb{E}(S_n))^3} = \mathbb{E}(T)^2\frac{c_v^2+1}{1-\lambda\mathbb{E}(S_n)}. \tag{15}$$

Therefore the busy period of the merge system increases with increasing variability of the extensions of the circulating items.

(e) *the number of items in a busy period as a function of items sizes.* The probability generating function $P(s)$ of the number of items which flow out during a busy period N satisfies the following functional equation:

$$P(s) = s\int_0^\infty \mathrm{e}^{-(\lambda-\lambda P(s))t}\mathrm{d}G(t) \tag{16}$$

In particular for the first two moments:

$$\mathbb{E}(N) = \frac{1}{1-\lambda\mathbb{E}(S_n)}, \tag{17}$$

$$\mathbb{E}(N^2) = \frac{1-\lambda\mathbb{E}(S_n)^2+\lambda^2\mathbb{E}(S_n^2)}{(1-\lambda\mathbb{E}(S_n))^3} \tag{18}$$

and the number of outflowing items during a busy period of the merge system is seen to be increasing with increasing variability of the extensions of the circulating items.

4. Beyond the merge

Let us now study more closely the distribution on the conveyor system below the merge. Recall that an item n finishes its outflow process at $A_n + S_n$; then it is proceeded on the conveyor system with constant speed v towards M. This process is supposed to be deterministic and independent of the downstream stage M. In particular, we suppose that M (or the buffer system of M) is capable to absorb the flow of products without inducing jamming on the conveyor system. Under this assumptions M will receive batches of workloads as illustrated in figure 1 rather than simple poisson flows of products. To make this statement more precise we emphasise the following corollary which is a direct consequence of the above theorem:

Corollary. The time headway between successive items flowing out of a buffer which itself is filled by a Poisson process of intensity λ is *not* in general *a Poisson process* but may be interpreted as the maximum of a shifted exponential random variable with parameter λ and a minimal time headway S_n. More precisely: there exists a sequence of mutually independent and exponentially distributed random variables $(T_n)_{n\in\mathbb{N}}$ with parameter λ, independent of $(S_n)_{n\in\mathbb{N}}$ such that for all $n \geq 1$, $Y_n = \max\big(T_n + \theta, S_n\big)$ almost surely and where θ is given by eq.(5).

The corollary allows the following simple characterization of leading and following items:

1) Item n is a leading item iff $T_n + \theta > S_n$. In that case the leading (time) headway is $T_n + \theta$.
2) Item n is a following item iff $T_n + \theta \leq S_n$. In that case the following (time) headway is S_n.

Using this characterization eq.(4) can be rewritten as:

$$
\begin{aligned}
F(y) &= \mathbb{P}(\max\big(T_n + \theta, S_n\big) \leq y) \\
&= \mathbb{P}(T_n + \theta \leq y \mid T_n + \theta > S_n)\, \mathbb{P}(T_n + \theta > S_n) \\
&\quad + \mathbb{P}(S_n \leq y \mid T_n + \theta \leq S_n)\, \mathbb{P}(T_n + \theta \leq S_n) \\
&= F_L(y)(1-\rho) + F_F(y)\rho
\end{aligned}
\tag{19}
$$

with the obvious definitions:

$$
\begin{aligned}
F_L(y) &:= \mathbb{P}(T_n + \theta \leq y \mid T_n + \theta > S_n) \\
F_F(y) &:= \mathbb{P}(S_n \leq y \mid T_n + \theta \leq S_n) \\
\rho &:= \mathbb{P}(T_n + \theta \leq S_n).
\end{aligned}
$$

Eq.(19) interprets the distribution of outflowing items as a convex combination of the outflow of following and leading items. The somewhat lengthy calculations of Appendix A show that the ratio of following items ρ (i.e. the probability that n is a following item) is simply:

$$\rho = \lambda \mathbb{E}(S_n) \tag{20}$$

and that the distributions of leading and following items are given by:

$$F_L(y) = \frac{\int_0^y \lambda e^{-\lambda t} G(t)\mathrm{d}t}{\int_0^\infty e^{-\lambda t}\mathrm{d}G(t)}, \quad y \geq 0 \tag{21}$$

$$F_F(y) = \frac{G(y) - \int_0^y e^{-\lambda(t-\theta)}\mathrm{d}G(t)}{\lambda \int_0^\infty t\mathrm{d}G(t)}, \quad y \geq 0. \tag{22}$$

We define now in analogy to the $M/G/1$ queue the *busy period* and the *free period* of the merge system. The busy period commences at an instant a *leading* item, say item n, starts the outflow process and terminates at $A_{n+k} + S_{n+k}$ where $k \in \mathbb{N}$ is the smallest integer such that n_{n+k+1} is a leading item (i.e. it terminates at the instant the last *following* item connected to the stream of items induced by n is placed on the conveyor system).
The free period is complementary to the busy period and formally given, using the same notations as above, by $A_{n+k+1} - A_{n+1}$.
From queueing theory we know that the epochs of arrivals of (lucky) customers finding the server unoccupied constitute a renewal process and that the free periods form a sequence of iid random variables (see e.g., Feller [6] p.197). Hence, the leading items initiate a busy period of the merge system and the arrival times of leading items form a renewal process. Its equilibrium distribution T can be found via Takàcs' integral equation (see e.g., Medhi [9] p.278) and the first two moments are given in eqs.(14,15). Moreover, the number of items flowing out during a busy period follows the distribution given in eq.(16) and depends visibly on the distribution G of the spacial extensions of the items. In conclusion, the downstream station M receives batches of workload which forms a renewal process (called hereafter "batch arrival process"). The arrival of the first item in the batch at M is a renewal point of the batch arrival process. The outflow process (leading *and* following items) however is not a renewal process in general as eq.(4) indicates (see also the remark in [8] eq.(2.13)). The batch sizes depend on the spacial extensions of the items with the mean and the variance explicitly given in eqs.(17,18). In case of vanishing spacial extensions, every item is a leading item, all batches have size one and the outflow process (which is now a renewal process) coincides in distribution with the batch arrival process.

5. Numerical simulation

To appreciate the accuracy of the approach we compared the analytical distributions F, F_L and F_F with the results of a discrete event simulation software (here, "Taylor ED"). We simulated the outflow events of about 60000 items from a merge buffer in the case where S_n is i) constant and ii) exponentially distributed. The empirical distributions presented in the figures below are in excellent agreement with the theoretical distributions.

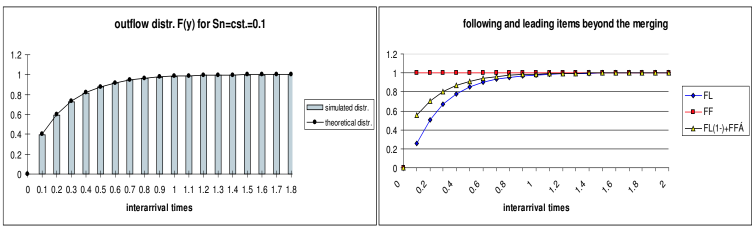

Figure 3. Left: Empirical and theoretical cumulated distributions for a constant minimum exit time ($S_n = 0.1$). Right: Theoretical cumulated distributions of the outflow times of leading and following items ($S_n = 0.1$).

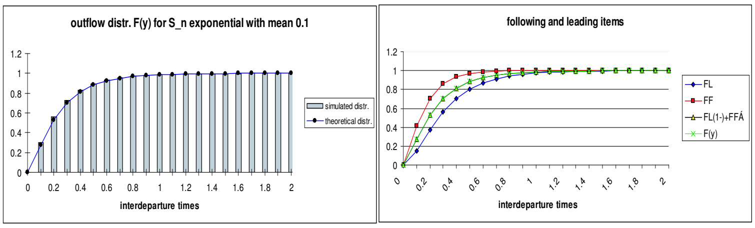

FIGURE 4. Left: Empirical and theoretical cumulated distributions for a exponentially distributed minimum exit time ($\mathbb{E}(S_n) = 0.1$). Right: Theoretical cumulated distributions of the outflow times of leading and following items ($\mathbb{E}(S_n) = 0.1$).

6. Conclusion

In the setting of discrete materials flow production systems we considered the merging of N Poisson streams of items with finite spacial extensions into a single stream. The outflow process from the merging system is a Poisson point process only in the limit case of vanishing extensions of the items lengths. When the extensions of the processed items are finite and independently drawn from a random variable L, the resulting outflow process may be interpreted as the maximum of a shifted exponential random variable and a minimal time headway $S = L/v$ where v is the conveyor speed governing the outflow process.
A distinction between items with minimal time headway (following items) and items exceeding the minimal headway (leading items) is drawn and their distributions are explicitly given. Leading items form a renewal process and the downstream stage receives at the renewal events (and under some hypotheses concerning the downstream stage) batches of workload with a size distribution depending on the extensions of the items. The size distribution is equivalent to the (well known) distribution of the number of served clients during a busy period of a simple $M/G/1$ queueing system.
The results constructively show the limited use of the standard Markov chain analysis at merging points of production systems when the circulating items are supposed to have finite spacial extensions. We analyse the non-markovian outflow process (A_n) by comparing it with a fictive poissonian outflow process (D_n). The method is different from the more common perturbation analysis of Markov processes and might find further impacts on non-markovian production flow modelling.

7. Appendix

Given the independent random variables $S, T : \Omega \to \mathbb{R}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that T is exponentially distributed with parameter λ and such that the positive random variable S has the given cumulative distribution G we show that:

$$\rho \quad := \quad \mathbb{P}(T+\theta \le S) = \lambda\mathbb{E}(S) \tag{23}$$

$$F_L(y) \quad := \quad \mathbb{P}(T+\theta \le y \mid T+\theta > S) = \frac{\int_0^y \lambda e^{-\lambda t} G(t)\mathrm{d}t}{\int_0^\infty e^{-\lambda t}\mathrm{d}G(t)}, \quad y \ge 0 \tag{24}$$

$$F_F(y) \quad := \quad \mathbb{P}(S \le y \mid T+\theta \le S) = \frac{G(y) - \int_0^y e^{-\lambda(t-\theta)}\mathrm{d}G(t)}{\lambda\int_0^\infty t\mathrm{d}G(t)}, \quad y \ge 0 \tag{25}$$

where θ is given by eq.(5). Note that θ is negative. Indeed, based on the well known inequality $e^{-x} \ge 1-x$ it is immediate to see that $1 > \frac{1-\lambda\mathbb{E}(S)}{\mathbb{E}(e^{-\lambda S})}$ and hence that $\theta < 0$. This result is used twice in the following calculations where we replace the lower integration boundary $\max(\theta,0) \overset{\text{not.}}{=} \theta \vee 0$ by 0.
We have:

$$\begin{aligned} \mathbb{P}(T+\theta \le S) &= \int_0^\infty \mathbb{P}(S \ge x+\theta)\lambda e^{-\lambda x}\mathrm{d}x \\ &= \int_0^\infty (1-G(x+\theta))\lambda e^{-\lambda x}\mathrm{d}x \\ &= 1 - \int_0^\infty \lambda e^{-\lambda x}\big(\int_0^{x+\theta}\mathrm{d}G(t)\big)\mathrm{d}x \\ &= 1 - \int_0^\infty \big(\int_0^\infty \lambda e^{-\lambda x} 1_{x \ge t-\theta}\mathrm{d}x\big)\mathrm{d}G(t) \\ &= 1 - \int_0^\infty \lambda\big(\int_{t-\theta}^\infty e^{-\lambda x}\mathrm{d}x\big)\mathrm{d}G(t) \\ &= 1 - \int_0^\infty e^{-\lambda(t-\theta)}\mathrm{d}G(t) \end{aligned} \tag{26}$$

where on the first line we used the independence of S and T and on the forth resp. fifth line Fubinis' theorem resp. a usual integration by parts formula.
Hence replacing θ into the above formula we find:

$$\begin{aligned} \rho &= 1 - \exp\big(\ln\frac{1-\lambda\mathbb{E}(S)}{\mathbb{E}(e^{-\lambda S})}\big)\int_0^\infty e^{-\lambda t}\mathrm{d}G(t) \\ &= 1 - \frac{1-\lambda\mathbb{E}(S)}{\mathbb{E}(e^{-\lambda S})}\mathbb{E}e^{-\lambda S} = \lambda\mathbb{E}(S) \end{aligned}$$

thereby establishing the first formula eq.(23). Next we have:

$$F_L(y) \quad := \quad \mathbb{P}(T+\theta \le y \mid T+\theta > S) = \frac{\mathbb{P}(S < T+\theta \le y)}{\mathbb{P}(T+\theta > S)}.$$

Using eq.(26) we see that the above denominator is $e^{\lambda\theta}\int_0^\infty e^{-\lambda t}\mathrm{d}G(t)$. For the numerator we have:

$$\begin{aligned} \mathbb{P}(S < T+\theta \le y) &= \int_0^\infty \mathbb{P}(S < x+\theta \le y)1_{\{x \le y-\theta\}}\lambda e^{-\lambda x}\mathrm{d}x \\ &= \int_0^{y-\theta} G(x+\theta)\lambda e^{-\lambda x}\mathrm{d}x \\ &= e^{\lambda\theta}\int_{\theta\vee 0}^y G(t)\lambda e^{-\lambda t}\mathrm{d}t \end{aligned}$$

which establishes the second formula eq.(24). Finally,

$$F_F(y) \quad := \quad \mathbb{P}(S \leq y \mid T + \theta \leq S) = \frac{\mathbb{P}(T + \theta \leq S \leq y)}{\mathbb{P}(T + \theta \geq S)} \tag{27}$$

Using eq.(26) we see that the above denominator (which in fact is ρ) can be written as $\int_0^\infty t \mathrm{d}G(t)$. For the numerator we have:

$$\begin{aligned}
\mathbb{P}(T + \theta \leq S \leq y) &= \int_0^\infty \mathbb{P}(x + \theta \leq S \leq y) 1_{\{x \leq y - \theta\}} \lambda e^{-\lambda x} \mathrm{d}x \\
&= \int_0^{y-\theta} (G(y) - G(x + \theta)) \lambda e^{-\lambda x} \mathrm{d}x \\
&= G(y)(1 - e^{-\lambda(y-\theta)}) - \lambda \int_0^{y-\theta} G(x + \theta) e^{-\lambda x} \mathrm{d}x \\
&= G(y)(1 - e^{-\lambda(y-\theta)}) - \lambda \int_{\theta \vee 0}^{y} G(u) e^{-\lambda(u-\theta)} \mathrm{d}u \\
&= G(y)(1 - e^{-\lambda(y-\theta)}) - \lambda e^{\lambda\theta} \int_0^\infty 1_{u \leq y} \big(\int_0^\infty 1_{t < u} e^{-\lambda u} \mathrm{d}G(t) \big) \mathrm{d}u \\
&= G(y)(1 - e^{-\lambda(y-\theta)}) - \lambda e^{\lambda\theta} \int_0^\infty \big(\int_0^\infty 1_{t < u \leq y} e^{-\lambda u} \mathrm{d}u \big) \mathrm{d}G(t) \\
&= G(y)(1 - e^{-\lambda(y-\theta)}) + e^{\lambda\theta} \int_0^y \big(e^{-\lambda y} - e^{-\lambda t} \big) \mathrm{d}G(t) \\
&= G(y)(1 - e^{-\lambda(y-\theta)}) + e^{-\lambda(y-\theta)} G(y) - \int_0^y e^{-\lambda(t-\theta)} \mathrm{d}G(t) \\
&= G(y) - \int_0^y e^{-\lambda(t-\theta)} \mathrm{d}G(t)
\end{aligned}$$

establishing the third formula eq.(25).

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