Laplacian Operator, Diffusion Flow and Active Contour on non-Euclidean Images

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Motivation

Our goal is to study image processing techniques for 360^o-degree images, which are obtained from omni-directional sensors [11, 17]. Having a curved mirror, i.e. hyperbolic, spherical or parabolic, as a lens for the corresponding catadioptric system, we obtain non-Euclidean images. One way of processing such an image is to perform a panoramic projection of this image onto a cylinder. In this way one unfolds the 360^o-image to the usual 2d image and only after this the usual image processing techniques are performed.

We propose here some basic approaches to direct processing of the 360^o-images as we take into consideration the geometry of each one. Obviously, the geometry of each of these images is a consequence of the geometry of the sensor's mirror used.

This technical report is organized as follows: In Section 1 we develop the Laplacian operator on non-Euclidean manifolds. First we start by derivation of Laplacian operator on Riemannian manifolds and than we derive it explicitly for each of the non-Euclidean manifolds of our interest, i.e. hyperboloid, sphere and paraboloid. This allows us to implement the gradient and diffusion flow on hyperbolic and spherical image. For testing this techniques, a synthetic and a real image was used in the case of hyperboloid and sphere respectively. In Section 2 we demonstrate the active contour on non-Euclidean images. First it was derived by directly minimizing the energy functional where the specific geometry of the non-Euclidean image was taken into account. Then the same was proofed through Polyakov action. We give some examples in each of the cases and so derive conclusions about the influence of the geometry for each particular case.

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1 Laplacian operator on non-Euclidean images

1.1 The Laplacian on a Riemannian manifold

By a Riemannian manifold, we roughly mean a manifold equipped with a method for measuring lengths of tangent vectors, and hence of curves. While the simplest differential operator $\frac{d}{dt}$ on the real line generalizes to the exterior derivative d on a smooth manifold, it is not possible to generalize the second derivative to manifolds without the additional structure of a Riemannian metric. Thus it can be argued that the Laplacian is the simplest, and hence the most basic, differential operator on functions on a Riemannian manifold.

1.1.1 Riemannian Metrics

Given a smooth manifold, there is no natural way to define a generalization of the Laplacian on a manifold, without as additional data a *geometry* in the form of a Riemannian metric. Certainly we should be able to measure lengths of curves on the manifold in order to do geometry. For a manifold $M^2 \subset \mathbb{R}^3$, we can measure the length of a curve $\gamma : [0, 1] \to M$ by the usual formula

$$l(\gamma) = \int_0^1 |\gamma'(t)| \mathrm{d}t. \tag{1}$$

Notice that the basic ingredients is the measure of the lenght of the tangent vector $\gamma'(t) \in \mathcal{T}_{\gamma(t)}M$. We can also use this formula for any manifold embedded in $\mathbb{R}^n$, which covers the classical cases in algebraic geometry and analysis. Such are the 2-sphere, 2-hyperboloid, 2-paraboloid etc.

Definition 1 *[14] A* Riemannian manifold (M, g) *is a smooth manifold M with a familly of smoothly varying positive definite inner products $g = g_x$ on T_xM for each $x \in M$. The family g is called a* Riemannian metric. *Two Riemannian manifolds (M, g) and (N, h) are called* isometric *if there exists a smooth diffeomorphism $f : M \to N$ such that*

$$g_x(X, Y) = h_{f(x)}(\mathrm{d}fX, \mathrm{d}fY)$$

for all $X, Y \in T_xM$, for all $x \in M$.

Given a Riemannian metric we can set the length of a curve $\gamma : [0, 1] \to M$ to be

$$l(\gamma) = \int_0^1 g_x \sqrt{(\gamma'(t), \gamma'(t))} \mathrm{d}t \tag{2}$$

On $\mathbb{R}^n$, the standard Riemannian metric is given by the standard inner product $g_x(v, w) = v \cdot w$ for all $u, w \in T_x\mathbb{R}^n$, for all $x \in \mathbb{R}^n$. Of course, we call $\mathbb{R}^n$ with this Riemannian metric *Euclidean space.* If M is a submanifold of Euclidean space, then M has a natural Riemannian metric given by $g_x(v, w) = v \cdot w$. This so called *induced metric* is the metric used in the classical theory of curves and surfaces in Euclidean three-space. By this same construction, a submanifold of a Riemannian manifold always inherits an induced metric.

To compute with Riemannian metric, we must be able to analyze it in a local coordinate chart. If $v, w \in \mathcal{T}_xM$ and $(x^1, x^2, \cdots, x^n)$ are coordinates near

x, then there exist α^i, β^i such that

$$v = \sum \alpha^i \frac{\partial}{\partial x^i}, \quad w = \sum \beta^i \frac{\partial}{\partial x^i}. \tag{3}$$

We have

$$\begin{aligned} g_x(v,w) &= g_x\Big(\sum_i \alpha^i \partial_{x^i}, \sum_j \beta^j \partial_{x^j}\Big) & (4)\\ &= \sum_{i,j} \alpha^i \beta^j g_x(\partial_{x^i}, \partial_{x^j}), & (5) \end{aligned}$$

where $\partial_{x^i} = \frac{\partial}{\partial x^i}$. Thus, g_x is determined by the symmetric, positive definite matrix $(g_{ij}(x)) = (g_x(\partial_{x^i}, \partial_{x^j}))$. Note that while the metric g is defined on all of M, the $g_{ij}(x)$ are defined only in a coordinate chart, where we can write

$$g = \sum_{i,j} g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{6}$$

If $h_{ij} = g(\partial_{y^i}, \partial_{y^j})$ is the matrix of the metric g in another coordinate system with coordinates $(y^1, y^2, \cdots y^n)$, then we have

$$g_{ij} = \sum_{k,l} \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} h_{kl}. \tag{7}$$

1.1.2 The Laplacian on functions

First, we need to define a Hilbert space of real valued functions on M, i.e $L^2(M)$, by setting $\langle f, g\rangle = \int_M f(x)g(x)$.

Next, let us define the Laplacian $\Delta : L^2(M,g) \to L^2(M,g)$. We want the Laplacian to agree with the standard Laplacian $-\Big(\frac{\partial^2}{\partial(x^1)^2}, + \cdots + \frac{\partial^2}{\partial(x^n)^2}\Big)$ on $\mathbb{R}^n$. However, this expression depends on the standard coordinates for $\mathbb{R}^n$, and we need a coordinate-free expression for our realization. This is provided by the classical equation

$$-\Big(\frac{\partial^2}{\partial(x^1)^2}, + \cdots + \frac{\partial^2}{\partial(x^n)^2}\Big) = -\mathrm{div} \circ \nabla. \tag{8}$$

In local coordinates we have

$$\nabla f = g^{ij} \partial_i f \partial_j, \tag{9}$$

where $\partial_j = \partial_{x^j} = \frac{\partial}{\partial_{x^j}}$, and g^{ij} is the inverse matrix of g_{ij}.

As for div, integration by parts applied to $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ gives

$$-\int_{\mathbb{R}^n} \partial_i X^i \cdot f = \int_{\mathbb{R}^n} \partial_i f \cdot X^i, \tag{10}$$

for functions X^i, which shows that the divergence $\partial_i X^i$ of a vector field $X = X^i \partial_i$ on $\mathbb{R}^n$ is characterized by the equation

$$\langle -\mathrm{div} X, f\rangle = \langle X, \nabla f\rangle, \tag{11}$$

where the inner products are the global inner products on functions and vector fields induced by the standard dot product.

We are interested in what does the operator div$X$ look like in local coordinates. First, we assume that a given manifold M is oriented and connected. But since we cannot integrate functions on a manifold, only n-forms transform correctly to give an integral over an n-manifold which is independent of coordinates, we are looking for an n-norm $\alpha(x)$ such that $\langle f, g\rangle = \langle f, g\rangle_M = \int_M f(x)g(x)\alpha(x)$ defines a positive definite inner product; such an α is called a *volume form.* In a positively oriented coordinate neighborhood U around $x = (x^1, \cdots x^n)$, pick a large number N of points p_j and in each tangent space $\mathcal{T}_{p_j}M$ form a box B_j with sides $(\Delta x^1)\frac{\partial}{\partial x^1}, (\Delta x^2)\frac{\partial}{\partial x^2}, \cdots, (\Delta x^n)\frac{\partial}{\partial x^n}$, for some small numbers Δx^i. Let $v_1, \cdots v_n$ be positively oriented orthonormal basis of $\mathcal{T}_{p_j}M$. Then $\partial_{x^i} = \alpha_i^k v_k$ for some matrix $\alpha_i^k = \alpha_i^k(p_j)$. Here it is used the *Einstein summation convention*, which means that an index which appears as both a superscript and subscript in an expression is summed over, e.g. $\alpha_i^k v_k = \sum_k \alpha_i^k v_k$. Thus the volume is

$$\text{vol}U = \lim_{\Delta x^i\to 0}\lim_{N\to\infty}\sum_{j=1}^{N}(\text{volume of } B_j) \tag{12}$$

$$= \lim_{\Delta x^i\to 0}\lim_{N\to\infty}\sum_{j=1}^{N}\left(\text{volume of box with } i^{th}\text{side}(\Delta x^i)\sum_{k=1}^{N}\alpha_i^k v_k\right) \tag{13}$$

$$= \lim_{\Delta x^i\to 0}\lim_{N\to\infty}\sum_{j=1}^{N}\left((\Delta x^1)(\Delta x^2)\cdots(\Delta x^n)\det(\alpha_i^k)\right) \tag{14}$$

$$= \int_U \det(\alpha_i^k)\mathrm{d}x^1\cdots\mathrm{d}x^n. \tag{15}$$

Definition 2 *[14] The* volume form *of a Riemannian metric is the top dimensional form* dvol *which in local coordinates is given by*

$$\mathrm{d}vol = \sqrt{\det g}\ \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n,$$

whenever $(\partial_{x^1}, \cdots, \partial_{x^n})$ *is a positively oriented basis of* T_xM. *The volume of* (M, g)*is set to be*

$$vol(M) = \int_M \mathrm{d}vol(x).$$

Finally, for any function $f \in \mathcal{C}^\infty(U)$ and vector field $X = X^i\partial_i \in TM$, we have

$$\begin{aligned}
\langle X, \nabla f\rangle &= \int_M \langle X, \nabla f\rangle \mathrm{d}vol & (16)\\
&= \int_U \langle X^i\partial_i, g^{kj}\partial_k f\partial_j\rangle \mathrm{d}vol & (17)\\
&= \int_U X^i(\partial_k f)g^{kj}g_{ij}\sqrt{\det g}\ \mathrm{d}x^1\cdots \mathrm{d}x^n & (18)\\
&= \int_U X^i(\partial_i f)\sqrt{\det g}\ \mathrm{d}x^1\cdots \mathrm{d}x^n & (19)\\
&= -\int_U \frac{1}{\sqrt{\det g}} f\cdot\partial_i(X^i\sqrt{\det g})\sqrt{\det g}\ \mathrm{d}x^1\cdots \mathrm{d}x^n & (20)\\
&= \langle f, -\frac{1}{\sqrt{\det g}}\ \partial_i(X^i\sqrt{\det g})\rangle. & (21)
\end{aligned}$$

From here we can see that it must be satisfied

$$\mathrm{div}X = \frac{1}{\sqrt{\det g}}\partial_i(X^i\sqrt{\det g}). \tag{22}$$

Assuming this expression is independent of choice of coordinates, we can then define the Laplacian on functions to be $\Delta = -div\circ\nabla$, a second order differential operator. In local coordinates, we get

$$\begin{aligned}
\Delta f &= -\frac{1}{\sqrt{\det g}}\partial_j(g^{ij}\sqrt{\det g}\ \partial_i f) & (23)\\
&= -g^{ij}\partial_i\partial_j f + (\text{lower order terms}). & (24)
\end{aligned}$$

We must note that this reduces to the usual expression for the Laplacian on $\mathbb{R}^n$. The last expression shows that not only is the Laplacian determined by the Riemannian metric, but the Laplacian also determines the metric. In other words, by evaluating Δ on a function which is locally $x^i x^j$, we recover g^{ij} and hence g_{ij}. We expect the spectral theory of the Laplacian to be intimately connected with the geometry of (M, g).

In conclusion, on a smooth manifold, the differential $\mathrm{d}f$ encodes all the first derivative information of a function f in a coordinated-free manner; equivalently, on a Riemannian manifold, the gradient ∇f encodes this information. To keep track of second derivative information, $\mathrm{d}^2 f$ certainly won't do, and Δf is a complicated combination of second derivative information(and lower order terms).

In the following, we will consider three types of images: hyperbolic, spherical and parabolic. Each kind is obtained from a cata-dioptric systems with the corresponding mirror. Each of them is obtained from a system where the mirror is respectively hyperbolic, spherical and parabolic [8, 1, 6].

1.2 Laplacian on a hyperbolic image

Consider the 2-hyperboloid in $\mathbb{R}^3$, with coordinates (x_0, x_1, x_3), in terms of which the Lobachevskian metric has the form

$$\mathrm{d}l^2 = \mathrm{d}x_0^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2. \tag{25}$$

By 2-hyperboloid of radius R, is meant the set of points satisfying the equation

$$x_0^2 - x_1^2 - x_2^2 = R^2. \tag{26}$$

In spherical coordinates (ρ, χ, φ), the metric (25) takes the form

$$\mathrm{d}l^2 = -\rho^2(\mathrm{d}\chi^2 + \sinh^2\chi \mathrm{d}\varphi^2) + \mathrm{d}\rho^2. \tag{27}$$

For $\rho = R$ we distinguish the upper sheet of the hyperboloid, and for $\rho = -R$, the lower one. Since ρ is constant on the hyperboloid, the metric can be written as

$$-\mathrm{d}l^2 = R^2(\mathrm{d}\chi^2 + \sinh^2\chi \mathrm{d}\varphi^2). \tag{28}$$

We can define a stereographic projection of the hyperboloid onto the plane. It maps the upper sheet of the hyperboloid onto the open disc $x_1^2 + x_2^2 < R^2$. If a point $P \in H_+^2$ has coordinates (x_0, x_1, x_2), and its projection on the plane has coordinates (u, v), then

$$\frac{x_1}{u} = \frac{x_0 + R}{R}, \quad \frac{x_2}{v} = \frac{x_0 + R}{R}, \tag{29}$$

whence

$$x_1 = u\Big(1 + \frac{x_0}{R}\Big), \quad x_2 = v\Big(1 + \frac{x_0}{R}\Big). \tag{30}$$

Substituting these in (26) and solving the resulting equation for $x_0 > 0$, we get

$$x_0 = -R\Big(1 + \frac{2R^2}{u^2 + v^2 - R^2}\Big), \tag{31}$$

from where we obtain

$$x_1 = \frac{2R^2 u}{R^2 - u^2 - v^2}, \quad x_2 = \frac{2R^2 v}{R^2 - u^2 - v^2}. \tag{32}$$

Thus,we can express the induced metric in terms of the coordinates (u, v). But first, let us recall

$$x_0 = R\cosh\chi \tag{33}$$

$$x_1 = R\sinh\chi\cos\varphi \tag{34}$$

$$x_2 = R\sinh\chi\sin\varphi \tag{35}$$

From the equivalence of (33) and (31), and putting $u^2 + v^2 = r^2$, we get

$$\cosh\chi = -\Big(1 + \frac{2R^2}{(r^2 - R^2)}\Big), \tag{36}$$

which, after differentiation, leads to

$$\sinh\chi \mathrm{d}\chi = \frac{4R^2 r}{(r^2 - R^2)^2}\mathrm{d}r. \tag{37}$$

On the other hand, from (32), (34) and (35) we get

$$\sinh^2\chi = \frac{x_1^2 + x_2^2}{R^2} = \frac{4R^2 r^2}{(R^2 - r^2)^2}. \tag{38}$$

From (28) and using (37) and (38) we get to

$$-\mathrm{d}l^2 = \frac{4R^4}{(R^2 - r^2)^2}(\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2), \tag{39}$$

that is

$$-\mathrm{d}l^2 = \frac{4R^4}{(R^2 - u^2 - v^2)^2}(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{40}$$

We can see that the metric on the hyperboloid is obtained from the metric on the Euclidean plane by multiplying the latter of a function, i.e. these two metric are "proportional".

If we take out the minus sign in (28) we obtain

$$\mathrm{d}l^2 = R^2(\mathrm{d}\chi^2 + \sinh^2\chi\mathrm{d}\varphi^2), \tag{41}$$

which is the metric on the upper sheet of the hyperboloid.

In terms of the coordinates in the disk x, y, the metric on the upper sheet of the unit hyperboloid , i.e $R = 1$, takes the form

$$\mathrm{d}l^2 = \frac{4}{(1 - x^2 - y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2), \tag{42}$$

where $x^2 + y^2 < 1$. The open disc with the metric (42) is called the Poincaré model of Lobachevsky's geometry. Taking into account that $\mathrm{d}l^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ with $x^1 = x$ and $x^2 = y$, we directly obtain the metric tensor :

$$(g)_{ij} = \begin{pmatrix} \frac{4}{(1-x^2-y^2)^2} & 0 \\ 0 & \frac{4}{(1-x^2-y^2)^2} \end{pmatrix} = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix}, \tag{43}$$

and accordingly its contra-variant (inverse) metric:

$$(g)^{ij} = \begin{pmatrix} \frac{(1-x^2-y^2)^2}{4} & 0 \\ 0 & \frac{(1-x^2-y^2)^2}{4} \end{pmatrix}. \tag{44}$$

Let us develop the Laplacian operator (23) for this particular case. We obtain

$$\Delta_{D_+} f = \frac{1}{\sqrt{detg}}\Big(\frac{\partial}{\partial_x}\sqrt{\det g}\; g^{xx}\frac{\partial f}{\partial x} + \frac{\partial}{\partial_y}\sqrt{\det g}\; g^{yy}\frac{\partial f}{\partial y}\Big) \tag{45}$$

$$= \frac{(1 - x^2 - y^2)^2}{4}\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big) \tag{46}$$

$$= \frac{(1 - x^2 - y^2)^2}{4}\Delta_{\mathbb{R}^2} f. \tag{47}$$

We try this technique on a synthetic hyperbolic image, which is shown on Figure 1. This is an Escher tiling of the hyperbolic plane. The equivalence between the hyperboloid and the disk is by the stereographic projection through the South Pole.

Accordingly, for the gradient of a hyperbolic image we have:

$$(\nabla_{D_+} f)_x = g^{xx}\frac{\partial f}{\partial x} + g^{xy}\frac{\partial f}{\partial y} = g^{xx}\frac{\partial f}{\partial x} \tag{48}$$

$$(\nabla_{D_+} f)_y = g^{yx}\frac{\partial f}{\partial x} + g^{yy}\frac{\partial f}{\partial y} = g^{yy}\frac{\partial f}{\partial y} \tag{49}$$

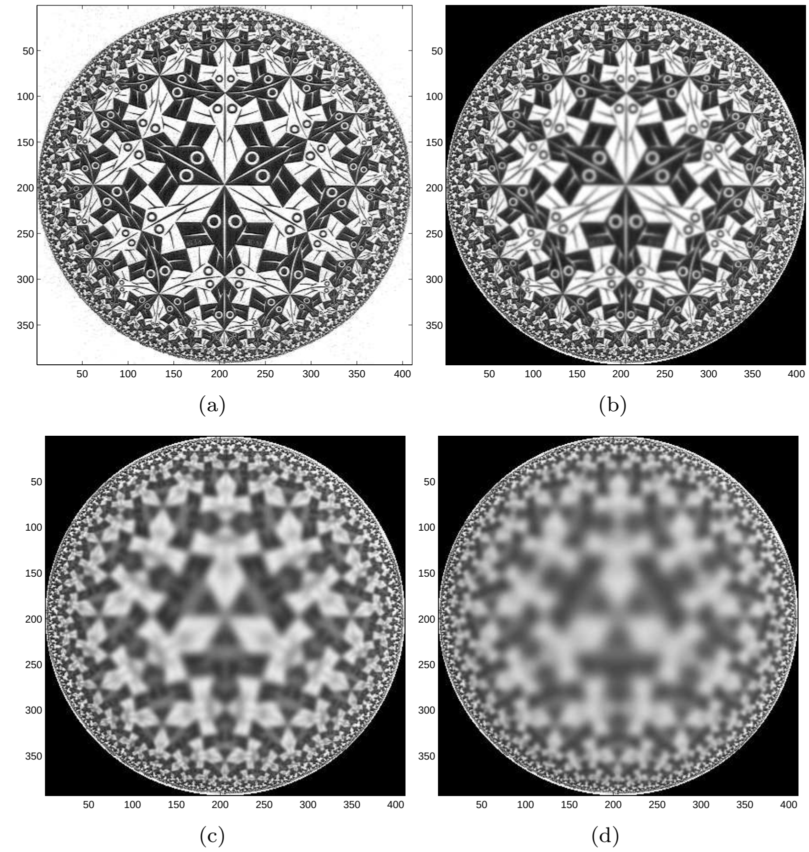
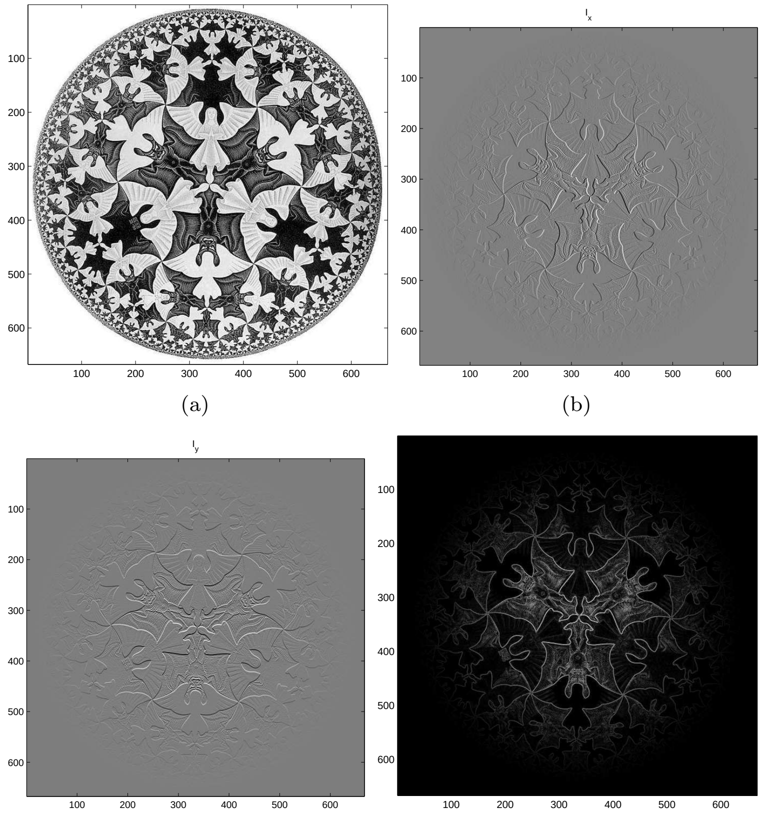


Figure 1: Laplace operator on a hyperbolic image (a) original (b) N=10, (c) N=100 , (d) N=300.

We can easily see that

$$\nabla_{D_+} f = \frac{(1 - x^2 - y^2)^2}{4} \nabla_{\mathbb{R}^2} f, \tag{50}$$

i.e. the gradient of the disk is a scaled gradient of the plane.



(a) (b) (c) (d)

Figure 2: Gradient of a hyperbolic image: (a) original image (b) gradient in direction of x, (c) gradient in direction of y , (d) magnitude of the gradient.

1.3 The Laplacian on a spherical image

The equation of the sphere $S^2 \subset \mathbb{R}^3$ of radius R is

$$x_0^2 + x_1^2 + x_2^2 = R^2. \tag{51}$$

In spherical coordinates r, θ, φ we have

$$\begin{aligned} x_0 &= r \cos\theta && (52) \\ x_1 &= r \sin\theta \sin\varphi, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi && (53) \\ x_2 &= r \sin\theta \cos\varphi, && (54) \\ . & && (55) \end{aligned}$$

Thus a point X on the sphere is the vector(x_0, x_1, x_2) as shown on Figure 3(a) In terms of spacial coordinates, the Euclidean metric takes the form

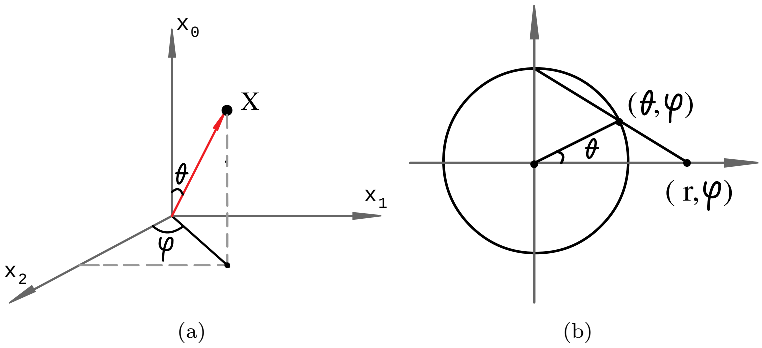


Figure 3: Geometry of the 2-sphere: (a) spherical polar coordinates (b) stereographic projection.

$$\mathrm{d}l^2 = \mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2). \tag{56}$$

On the surface $r = R$, the differential $\mathrm{d}r = 0$, so the metric induced on the sphere is given by

$$\mathrm{d}l^2 = R^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \tag{57}$$

We consider the stereographic projection of the sphere onto the plane shown on Figure 3(b). It sends a point (θ, φ) on the sphere to the point with polar coordinates (r, φ) in the plane, for which is fulfilled $\varphi = \varphi, r = R\cot\theta/2$. In terms of these new coordinates the metric becomes

$$\mathrm{d}l^2 = \frac{4R^4}{(R^2 + r^2)^2}(\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2). \tag{58}$$

But in terms of the usual Euclidean coordinates $(x_1, x_2) \equiv (x, y) \in \mathbb{R}^2$ in the plane, where $r^2 = x^2 + y^2$, we obtain

$$\mathrm{d}l^2 = \frac{4R^4}{(R^2 + x^2 + y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{59}$$

We can see that in this case as well, the metric on the sphere is obtained from the Euclidean metric on the plane by multiplying the latter by the function $\frac{4R^2}{(R^2+x^2+y^2)^2}$:

$$\mathrm{d}l_{S^2}^2 = \frac{4R^4}{(R^2 + x^2 + y^2)^2}\mathrm{d}l_{\mathbb{R}^2}^2. \tag{60}$$

Accordingly, the metric on the unit sphere, i.e. $R = 1$, is derived as

$$(g)_{ij} = \begin{pmatrix} \frac{4}{(1+x^2+y^2)^2} & 0 \\ 0 & \frac{4}{(1+x^2+y^2)^2} \end{pmatrix}, \tag{61}$$

and respectively the inverse metric is

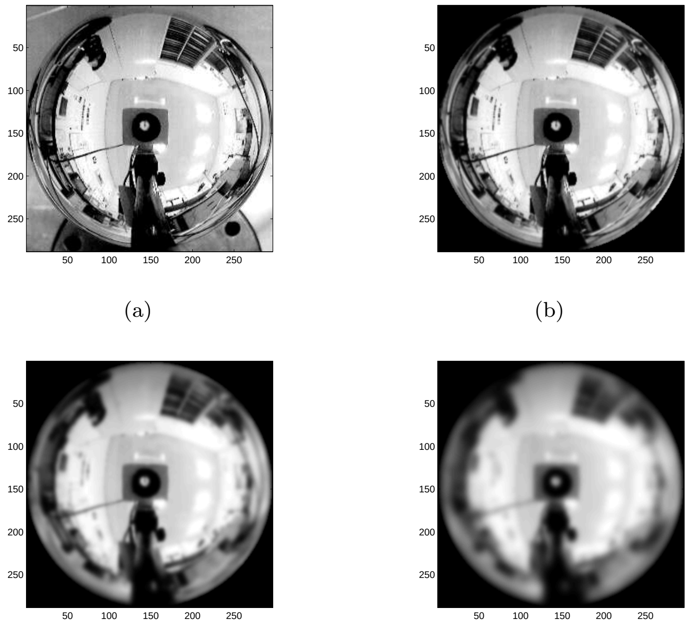
$$(g)^{ij} = \begin{pmatrix} \frac{(1+x^2+y^2)^2}{4} & 0 \\ 0 & \frac{(1+x^2+y^2)^2}{4} \end{pmatrix}. \tag{62}$$

We develop the Laplacian operator (23) on the sphere and obtain

$$\Delta_{S^2} f = \frac{1}{\sqrt{detg}}\Big(\frac{\partial}{\partial_x}\sqrt{\det g}\; g^{xx}\frac{\partial f}{\partial x} + \frac{\partial}{\partial_y}\sqrt{\det g}\; g^{yy}\frac{\partial f}{\partial y}\Big) \tag{63}$$

$$= \frac{(1+x^2+y^2)^2}{4}\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big) \tag{64}$$

$$= \frac{(1+x^2+y^2)^2}{4}\Delta_{\mathbb{R}^2} f. \tag{65}$$



(a) (b) (c) (d)

Figure 4: Laplacian Operator on a spherical image: (a) original image (b) $N = 10$, (c) $N = 100$, (d) $N = 300$.

The gradient of a spherical image is

$$\nabla_{S^2} f = \frac{(1+x^2+y^2)^2}{4}\nabla_{\mathbb{R}^2} f, \tag{66}$$

1.4 Laplacian operator on parabolic images

The paraboloid is a quadratic surface which can be expressed by the Cartesian equation

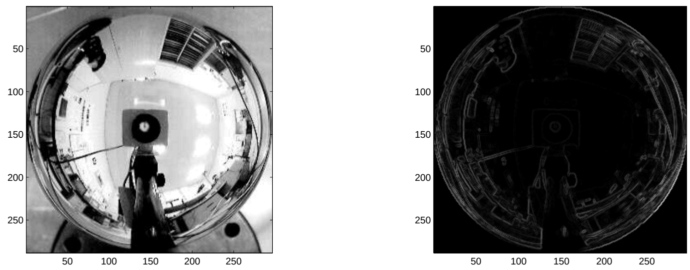
$$z = b(x^2+y^2). \tag{67}$$

In polar coordinates (r, φ) we express it as

$$x_0 = r^2, \tag{68}$$

$$x_1 = r\cos\varphi, \quad 0 \le \varphi < 2\pi, \quad r \ge 0 \tag{69}$$

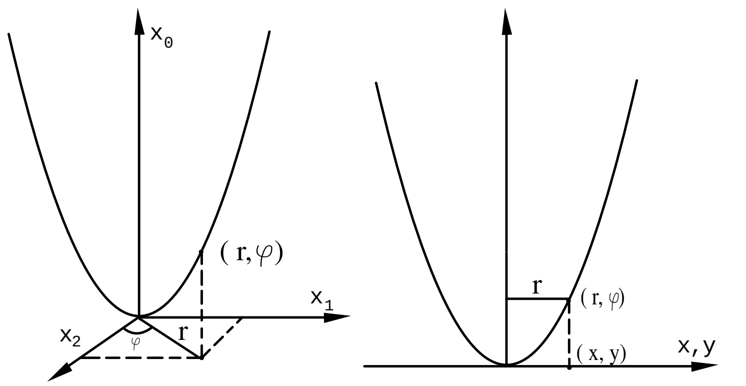
$$x_2 = r\sin\varphi. \tag{70}$$



(a) (b)

Figure 5: Gradient of a real spherical image: (a) original image (b) spherical gradient.

It is shown on Figure 6(a). The metric in this coordinates is:



(a) (b)

Figure 6: Gradient of a real spherical image: (a) original image (b) spherical gradient.

$$\mathrm{d}l^2 = \mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 = (1 + 4r^2)\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2. \tag{71}$$

After projecting it on the plane $(x, y) \in \mathbb{R}^2$, as depicted on Figure 6(b), we have $r^2 = x^2 + y^2$ and $\varphi = \arctan \frac{y}{x}$. Performing this change of variable, we get

$$\mathrm{d}l^2 = (1 + 4x^2)\mathrm{d}x^2 + 8xy\mathrm{d}x\mathrm{d}y + (1 + 4y^2)\mathrm{d}y^2, \tag{72}$$

which corresponds to the following metric:

$$(g)_{ij} = \begin{pmatrix} 1 + 4x^2 & 4xy \\ 4xy & 1 + 4y^2 \end{pmatrix}. \tag{73}$$

The inverse metric is:

$$(g)^{ij} = \frac{1}{1 + 4x^2 + 4y^2} \begin{pmatrix} 1 + 4y^2 & -4xy \\ -4xy & 1 + 4x^2 \end{pmatrix}. \tag{74}$$

Let us explicetly write the calculation for Laplacian on the 2-paraboloid. We start we Eq.(23), which reads:

$$\Delta f_{P^2} = -\frac{1}{\sqrt{g}}\Big(\partial_1(g^{11}\sqrt{g}\partial_1 f + g^{21}\sqrt{g}\partial_2 f) + \partial_2(g^{12}\sqrt{g}\partial_1 f + g^{22}\sqrt{g}\partial_2 f)\Big). \quad (75)$$

Here we distinguish $g = \det g$, and we put the indexes $1 \equiv x, 2 \equiv y$ and then obtain:

$$\begin{aligned}\Delta f_{P^2} = -\frac{1}{\sqrt{g}} \quad & \Big[\quad \frac{\partial}{\partial x}\Big(\frac{1+4y^2}{\sqrt{1+4x^2+4y^2}}\frac{\partial f}{\partial x} - \frac{4xy}{\sqrt{1+4x^2+4y^2}}\frac{\partial f}{\partial y}\Big) + \\ & + \quad \frac{\partial}{\partial y}\Big(\frac{1+4x^2}{\sqrt{1+4x^2+4y^2}}\frac{\partial f}{\partial y} - \frac{4xy}{\sqrt{1+4x^2+4y^2}}\frac{\partial f}{\partial x}\Big)\Big].\end{aligned}$$

Extracting the derivatives, we finally obtain:

$$\begin{aligned}\Delta f_{P^2} = -\frac{1}{1+4x^2+4y^2} \quad & \Big[\quad (1+4y^2)\frac{\partial^2 f}{\partial x^2} - 4xy\frac{\partial^2 f}{\partial x \partial y} + (1+4x^2)\frac{\partial^2 f}{\partial y^2} - \\ & - \quad \frac{8x(1+2x^2+2y^2)}{(1+4x^2+4y^2)}\frac{\partial f}{\partial x} - \frac{8y(1+2x^2+2y^2)}{(1+4x^2+4y^2)}\frac{\partial f}{\partial y}\Big].\end{aligned}$$

2 Active contours on non-Euclidean images

2.1 Geodesic Active Contour

The geodesic/geometric active contour (GAC) model is a variational model which is widely used in image processing applications to extract objects of interest by deforming a contour curve toward the edges of these objects.

To develop the formalism of active contours in non-Euclidean spaces we use its counterpart in Euclidean space which was first worked out in [4]. It is clear that what distinguishes both cases is their geometry.

The evolution equation of the parametric disk-curve $\mathcal{C}(h) = (x(h), y(h)) \in \Omega, h \in [0,1]$ is given by minimizing an energy functional.

$$\begin{aligned}\mathcal{E}(\mathcal{C}) &= \alpha \int_0^1 \Big|\frac{\partial \mathcal{C}(h)}{\partial h}\Big|^2 \mathrm{d}h + \lambda \int_0^1 f(|\nabla I(\mathcal{C}(h))|)^2 \mathrm{d}h \quad (76) \\ &= \int_0^1 (E_{int}(\mathcal{C}(h)) + E_{ext}(\mathcal{C}(h)))\mathrm{d}h. \quad (77)\end{aligned}$$

The goal is to minimize E for $\mathcal{C}$ in a certain allowed space of curves. This functional is not intrinsic since it depends on the parametrization h, which for the moment is arbitrary.

The classical theory explains when the solution to the energy problem is given by a curve of minimal "weighted distance" between given points. Distance is measured in the given Riemannian space with the first fundamental form g_{ij}. This form defines the metric or distance measurement in the space. Thus minimizing $E(\mathcal{C})$ as in (77) is equivalent to minimizing

$$\int_0^1 \sqrt{g_{ij}\mathcal{C}'_i\mathcal{C}'_j}\mathrm{d}h, \quad (i,j = 1,2), \quad (78)$$

or

$$\int_0^1 \sqrt{g_{11}\mathcal{C}_1'^2 + 2g_{12}\mathcal{C}_1'\mathcal{C}_2' + g_{22}\mathcal{C}_2'^2}\mathrm{d}h, \tag{79}$$

with $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$.

It is shown in [4] that this minimization is equivalent minimizing the intrinsic problem

$$\int_0^1 f(|\nabla I(\mathcal{C}(h))|)|\mathcal{C}'(h)|\mathrm{d}h, \tag{80}$$

where f is the edge-detecting function. The geodesic is computed by the calculus of variations.But first, taking into account that we are looking for a minimization in a specific geometry, the non-Euclidean one, we cannot just simplify the second term in this integral. In our case the functional to be minimized is

$$\begin{aligned}
\mathcal{E}(\mathcal{C}) &= \int_0^1 f(|\nabla I(C(h))|)\sqrt{(g)_{ij}\mathcal{C}_i'\mathcal{C}_j'}\mathrm{d}h \\
&= \int_0^1 f(|\nabla I(C(h))|)\sqrt{g^{1/2}\delta_{ij}\mathcal{C}_i'\mathcal{C}_j'}\mathrm{d}h \\
&= \int_0^1 g^{1/4} f(|\nabla I(C(h))|)\Big|\frac{\partial \mathcal{C}}{\partial h}\Big|,
\end{aligned}$$

where g denotes the derivative of $(g)_{ij}$.

Let us introduce an artificial time t and consider the family of curves $\mathcal{C}(t)$ in the disk D. The first variation of the energy $\mathcal{E}(\mathcal{C})$ is then $\frac{\mathrm{d}\mathcal{E}(\mathcal{C})}{\mathrm{d}t}$, and it reads :

$$\frac{\mathrm{d}\mathcal{E}(\mathcal{C})}{\mathrm{d}t} = \int_0^1 \frac{\mathrm{d}f_D(|\nabla I(C)|)}{\mathrm{d}t}\Big|\frac{\partial \mathcal{C}}{\partial h}\Big|\mathrm{d}h + \int_0^1 f_D(|\nabla I(C)|)\frac{\mathrm{d}}{\mathrm{d}t}\Big|\frac{\partial \mathcal{C}}{\partial h}\Big|\mathrm{d}h, \tag{81}$$

with $f_D = g^{1/4} f(|\nabla I(C)|)$.

Let us first develop the derivatives separately. We start by the term

$$\frac{\mathrm{d}f_D(|\nabla I(\mathcal{C})|)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} f_D(\mathcal{C}_1, \mathcal{C}_2) = \frac{\partial f_D}{\partial u_1}\frac{\partial \mathcal{C}_1}{\partial t} + \frac{\partial f_D}{\partial u_2}\frac{\partial \mathcal{C}_2}{\partial t} = \langle \nabla f_D, \frac{\partial \mathcal{C}}{\partial t}\rangle, \tag{82}$$

where the contour was considered as a vector $\mathcal{C} \equiv (\mathcal{C}_1, \mathcal{C}_2)$ and for obtaining its derivative the chain rule was applied.

Next, we look at the term:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|\frac{\partial \mathcal{C}}{\partial h}\Big| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\Big(\frac{\partial \mathcal{C}_1}{\partial h}\Big)^2 + \Big(\frac{\partial \mathcal{C}_2}{\partial h}\Big)^2} \tag{83}$$

$$= \frac{1}{2\sqrt{\Big(\frac{\partial \mathcal{C}_1}{\partial h}\Big)^2 + \Big(\frac{\partial \mathcal{C}_2}{\partial h}\Big)^2}} 2\Big(\frac{\partial \mathcal{C}_1}{\partial h}\frac{\partial^2 \mathcal{C}_1}{\partial t \partial h} + \frac{\partial \mathcal{C}_2}{\partial h}\frac{\partial^2 \mathcal{C}_2}{\partial t \partial h}\Big) \tag{84}$$

$$= \langle \frac{\frac{\partial \mathcal{C}}{\partial h}}{\Big|\frac{\partial \mathcal{C}}{\partial h}\Big|}, \frac{\partial^2 \mathcal{C}}{\partial t \partial h}\rangle \tag{85}$$

Then the energy variation in (81) becomes

$$\frac{\mathrm{d}\mathcal{E}(\mathcal{C})}{\mathrm{d}t} = \int_0^1 \langle \nabla f_D, \frac{\partial \mathcal{C}}{\partial t}\rangle\Big|\frac{\partial \mathcal{C}}{\partial h}\Big|\mathrm{d}h + \int_0^1 f_D\langle \frac{\frac{\partial \mathcal{C}}{\partial h}}{\Big|\frac{\partial \mathcal{C}}{\partial h}\Big|}, \frac{\partial^2 \mathcal{C}}{\partial t \partial h}\rangle\mathrm{d}h \tag{86}$$

We integrate by parts with respect to h the second integral and thus obtain

$$\frac{\mathrm{d}\mathcal{E}(\mathcal{C})}{\mathrm{d}t} = \int_0^1 \langle \nabla f_D, \frac{\partial \mathcal{C}}{\partial t} \rangle \Big|\frac{\partial \mathcal{C}}{\partial h}\Big| \mathrm{d}h - \int_0^1 \langle \frac{\partial \mathcal{C}}{\partial t}, f_D \frac{\partial}{\partial h}\Big(\frac{\frac{\partial \mathcal{C}}{\partial h}}{\big|\frac{\partial \mathcal{C}}{\partial h}\big|}\Big) + \frac{\frac{\partial \mathcal{C}}{\partial h}}{\big|\frac{\partial \mathcal{C}}{\partial h}\big|} \langle \nabla f_D, \frac{\partial \mathcal{C}}{\partial h} \rangle \rangle \mathrm{d}h.$$

This equation can be rewritten as

$$\frac{\mathrm{d}\mathcal{E}(\mathcal{C})}{\mathrm{d}t} = \int_0^1 \Big|\frac{\partial \mathcal{C}}{\partial h}\Big| \langle \frac{\partial \mathcal{C}}{\partial t}, \nabla f_D - \frac{f_D}{\big|\frac{\partial \mathcal{C}}{\partial h}\big|} \frac{\partial}{\partial h}\Big(\frac{\frac{\partial \mathcal{C}}{\partial h}}{\big|\frac{\partial \mathcal{C}}{\partial h}\big|}\Big) - \frac{\frac{\partial \mathcal{C}}{\partial h}}{\big|\frac{\partial \mathcal{C}}{\partial h}\big|} \langle \nabla f_D, \frac{\frac{\partial \mathcal{C}}{\partial h}}{\big|\frac{\partial \mathcal{C}}{\partial h}\big|} \rangle \rangle \mathrm{d}h \quad (87)$$

By definition, for the tangent vector we have:

$$\mathcal{T} = \frac{\frac{\partial \mathcal{C}}{\partial h}}{\big|\frac{\partial \mathcal{C}}{\partial h}\big|}. \quad (88)$$

The normal vector $\mathcal{N}$ is perpendicular to the tangent and by its definition it reads

$$\frac{1}{\big|\frac{\partial \mathcal{C}}{\partial h}\big|} \frac{\partial}{\partial h} \mathcal{T} = \kappa \mathcal{N}, \quad (89)$$

with κ being the curvature.

Using these two definitions, we write for the energy variation:

$$\frac{\mathrm{d}\mathcal{E}(\mathcal{C})}{\mathrm{d}t} = \int_0^1 \Big|\frac{\partial \mathcal{C}}{\partial h}\Big| \langle \frac{\partial \mathcal{C}}{\partial t}, \nabla f_D - \kappa f_D \mathcal{N} - \langle \mathcal{T}, \nabla f_D \rangle \mathcal{T} \rangle \mathrm{d}h. \quad (90)$$

Decomposing the vector $\nabla f_D = \langle \nabla f_D, \mathcal{N} \rangle \mathcal{N} + \langle \nabla f_D, \mathcal{T} \rangle \mathcal{T}$, we obtain:

$$\frac{\mathrm{d}\mathcal{E}(\mathcal{C})}{\mathrm{d}t} = \int_0^1 \Big|\frac{\partial \mathcal{C}}{\partial h}\Big| \langle \frac{\partial \mathcal{C}}{\partial t}, \langle \nabla f_D, \mathcal{N} \rangle \mathcal{N} - \kappa f_D \mathcal{N} \rangle \mathrm{d}h. \quad (91)$$

From here it is obvious that the direction of the strongest energy variations correspond to

$$\frac{\partial \mathcal{C}}{\partial t} = (\kappa f_D - \langle \nabla f_D, \mathcal{N} \rangle) \mathcal{N}, \quad (92)$$

which is the equation of the active geodesic contour, representing an evolving curve in the direction of its normal vector and under the action of a force $\mathcal{F}$:

$$\frac{\partial \mathcal{C}}{\partial t} = \mathcal{F} \mathcal{N}. \quad (93)$$

In the specific case $f_D = 1$, we get an evolving curve which minimizes the mean curvature:

$$\frac{\partial \mathcal{C}}{\partial t} = \kappa \mathcal{N}. \quad (94)$$

The f_D, which lives on the disk image and it depends on the its metric. But since the geometry of the disk image depends on the mirror used, then according the metric of the mirror manifold we will update this function. Thus, we distinguish the following stoping functions:

$$f_{H_+^2} = \frac{2}{1 - x^2 - y^2} f_{\mathbb{R}^2}, \quad (95)$$

$$f_{S^2} = \frac{2}{1 + x^2 + y^2} f_{\mathbb{R}^2}, \quad (96)$$

where x, y are the coordinates of the disk-image on the Euclidean plane. Once again, the stoping function on the hyperbolic and spherical images is "proportional" to the one on the Euclidean (flat) images. But this is not the case of the parabolic image. And this is obvious from its metric tensor-it is symmetric but not diagonal.

2.1.1 GAC on the 2-hyperboloid

The specific evalution of the active contour on the hyperboloid is shown on Figure 7. This evolution is analogical to the Euclidean plane: the given initial contour shrinks to the center of the hyperbolic image.

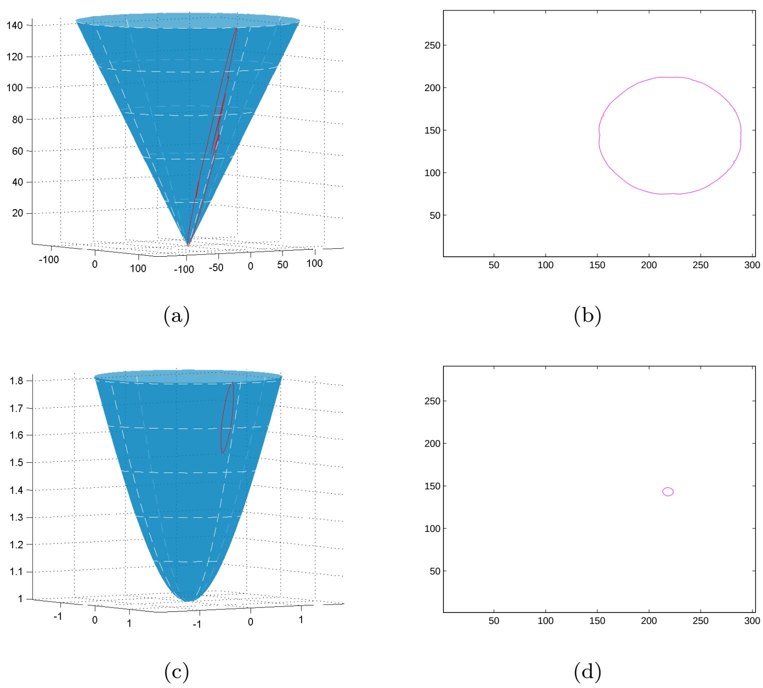
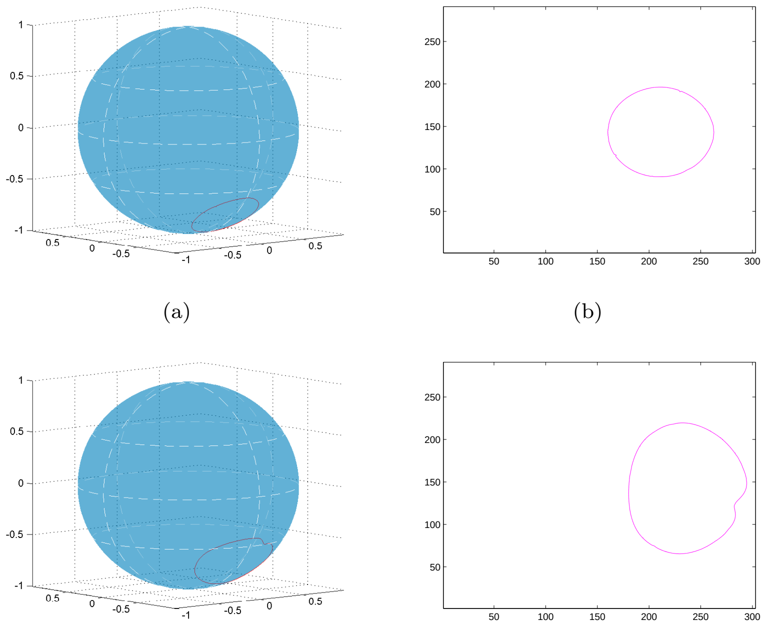


Figure 7: evolution of GAC on the 2-hyperboloid: (a) , (b) , (c) , (d) .

2.1.2 GAC on the 2-sphere

On Figure 8 is shown the evolution of an contour on the sphere. It is interesting to notice that opposite to the hyperbolic case, here the contour evolves towards the border of the disk. Such an evolution is natural and it comes directly from the action of the spherical geometry on the contour.



(a) (b)

(c) (d)

Figure 8: Evolution of GAC on the 2-sphere: (a)AC in the initial state, (b) , (c) evolution of the , (d).

2.1.3 GAC on the paraboloid

2.2 GAC on non-Euclidean manifolds through Polyakov action

Following the first model of active contours developed by Kass *et al.* in [7], Caselles *et al.* in [4] and Kichenassamy *et al.* in [9] proposed the following minimization problem invariant w.r.t. the curve parametrization. The intrinsic energy functional is as follows:

$$F_{GAC}(C) = \int_0^1 f(|\nabla I_0(C(p))|)|C_p|dp = \int_0^{L(C)} f(|\nabla I_0(C(s))|)ds, \qquad (97)$$

where GAC stands for Geodesic/Geometric Active Contour, p is the curve parametrization parameter, ds is the Euclidean element of length and $L(C)$ is the Euclidean length of the curve C defined by $L(C) = \int_0^1 |C_p|dp = \int_0^{L(C)} ds$. Hence, the functional (97) is actually a new length obtained by weighting the Euclidean element of length ds by the function f which contains information concerning the boundaries of objects [4]. The function f is an edge detecting function defined e.g. by

$$f(I_0) = \frac{1}{1 + |\nabla I_0|^2}, \qquad (98)$$

where I_0 is the original image.

Caselles *et al.* proved in [4] that the curve minimizing F_{GAC} is actually a geodesic in a Riemannian space which metric tensor is:

$$a_{ij} = f^2(|\nabla I_0|)\delta_{ij} \qquad (99)$$

where f is the edge detecting function. This geodesic is computed by the calculus of variations. Let us introduce an artificial time t and let us consider a family of curves $C(t)$ such that:

$$F_{GAC}(t) = \int_0^1 f(|\nabla I_0(C(p,t))|)|C_p(p,t)|dp. \qquad (100)$$

The first variation of the energy F_{GAC} is then:

$$\frac{dF_{GAC}}{dt} = \int_0^1 \left\langle \frac{\partial C}{\partial t}, \langle \nabla f, \mathcal{N} \rangle \mathcal{N} - \kappa f \mathcal{N} \right\rangle |C_p|dp, \qquad (101)$$

see [4] for details. Hence, the direction for which F_{GAC} decreases most rapidly provides us the following minimization flow:

$$\frac{\partial C}{\partial t} = \left(\kappa f - \langle \nabla f, \mathcal{N} \rangle\right) \mathcal{N}, \qquad (102)$$

where $\mathcal{N}$ is the unit normal to the curve C and κ is its curvature. The right hand side of the equation (102) corresponds to the Euler-Lagrange of Energy (97). The first term is the mean curvature motion, also called curve shortening flow, weighted by the edge detecting function f. It smoothes the curve shape by decreasing its total length as fast as possible. The second term of (102) attracts the curve toward the boundaries of objects by creating an attraction valley centered on the edges. Hence, the function f does not need to be equal to zero to stop the evolution of the snake on the contours of objects.

Osher and Sethian have introduced in [12] the implicit and intrinsic level set representation of contours to deal with topological changes and to efficiently solve the contour propagation from a numerical point of view. Equation (102) can be written in the level set form as follows:

$$\frac{\partial \phi}{\partial t} = \left(\kappa f + \langle \nabla f, \frac{\nabla \phi}{|\nabla \phi|} \rangle \right) |\nabla \phi|, \tag{103}$$

where ϕ is the level set function embedding the active contour C.

It is important to notice that the geodesic/geometric active contour model defined in Equations (97) and (102) is designed to extract objects on flat images. Indeed, this image segmentation method can not be directly used to extract significant objects on non-Euclidean images such as omni-directional images acquired with hyperbolic, spherical or parabolic mirrors. The desired segmentation model should consider the curved geometry of hyperbolic, spherical and parabolic manifolds.

Recently, Bresson-Vandergheynst-Thiran consider in [2, 3] the general case of *an active hypersurface evolving on any Riemannian manifold.* We propose to apply this formalism to define the evolution equation of active contours in hyperbolic, spherical and parabolic manifolds.

2.3 Active Contours on Hyperbolic, Spherical and Parabolic Manifolds

Based on the work of Sochen-Kimmel-Malladi ([15, 16]), it was proposed in [2] and [3] the following functional:

$$\left\{ \begin{array}{ccl} P_f(X, \Sigma, M) & = & \int d^{n_\Sigma}\varsigma \; f(X, g_{\mu\nu}, h_{ij}) \; g^{1/2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j h_{ij}, \\ X & : & (\Sigma, g_{\mu\nu}) \rightarrow (M, h_{ij}), \end{array} \right. \tag{104}$$

which corresponds to the Polyakov action [13] weighted by the function f. The Polyakov action is basically a functional that measures the weight of a mapping X between an embedded manifold Σ and the embedding manifold M (see Figure 9).

More precisely, $g_{\mu\nu}$ is the metric tensor/first fundamental form [10] of the manifold Σ, $g^{\mu\nu}$ is the inverse metric of $g_{\mu\nu}$, g is the determinant of $g_{\mu\nu}$, n_Σ is the dimension of Σ, $\mu, \nu = 1, ..., n_\Sigma$, $d^{n_\Sigma}\varsigma \; g^{1/2}$ is the volume element with respect to (w.r.t.) the local coordinates on Σ, h_{ij} is the metric tensor of the embedding space M, n_M the dimension of M, $i, j = 1, ..., n_M$, $\partial_\mu X^i = \partial X^i / \partial \varsigma^\mu$

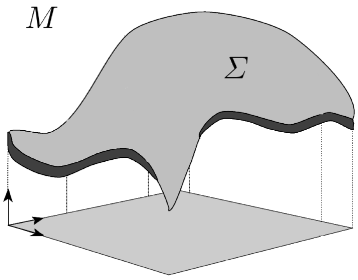


Figure 9: The manifold Σ embedded in M, reproduced from [16].

and $g^{\mu\nu}\partial_\mu X^i \partial_\nu X^j h_{ij}$ is the generalization of the magnitude of the gradient to maps between Riemannian manifolds. We observe that the volume element as well as the rest of the expression is re-parametrization invariant. In other words, they are invariant under a smooth transformation. Thus, this action depends on the geometrical objects and not on the way we describe them via our parametrization of the coordinates. Finally, when identical indices appear one up and one down in Equation (104), they are summed over according to the Einstein summation convention.

The calculus of variations gives us the Euler-Lagrange equation of the functional (104) w.r.t. the l-th embedding coordinate X^l, $g_{\mu\nu}$ and h_{ij} being fixed:

$$\begin{aligned} 0 &= f \cdot \left(g^{-1/2}\partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu X^l) + \Gamma^l_{jk} \partial_\mu X^j \partial_\nu X^k g^{\mu\nu} \right) \\ &\quad + \partial_k f g^{\mu\nu} \partial_\mu X^k \partial_\nu X^l - \frac{n_M}{2} h^{lk} \partial_k f g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j h_{ij}, \end{aligned} \tag{105}$$

And the flow minimizing P_f w.r.t. X^l is thus as follows:

$$\begin{aligned} \frac{\partial X^l}{\partial t} &= f \cdot \left(g^{-1/2}\partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu X^l) + \Gamma^l_{jk} \partial_\mu X^j \partial_\nu X^k g^{\mu\nu} \right) \\ &\quad + \partial_k f g^{\mu\nu} \partial_\mu X^k \partial_\nu X^l - \frac{n_M}{2} h^{lk} \partial_k f g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j h_{ij}, \end{aligned} \tag{106}$$

for $1 \leq l \leq n_M$, $g^{-1/2}\partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu X^l)$ is the Beltrami operator which generalizes the Laplace operator to non-Euclidean manifolds and $\Gamma^l_{jk} = \frac{1}{2} h^{li} (\partial_j h_{ik} + \partial_k h_{ji} - \partial_i h_{jk})$ is the Levi-Civita connection coefficients [10].

If the metric tensor $g_{\mu\nu}$ of the embedded manifold Σ is chosen to be the induced metric tensor: $g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j h_{ij}$, then the map X are harmonic maps such as geodesics and minimal surfaces and the weighted Polyakov action is reduced to the weighted Euler functional/Nambu action that describes the (hyper-)area of a (hyper-)surface Σ:

$$S_f = \int d^{n_\Sigma} \varsigma \; f \; g^{1/2}. \tag{107}$$

The induced metric tensor is also introduced in the flow (106), which yields to:

$$\begin{cases} \frac{\partial X^l}{\partial t} & = & f \cdot \mathcal{H}^l + \partial_k f g^{\mu\nu} \partial_\mu X^k \partial_\nu X^l - \frac{n_M . n_\Sigma}{2} \partial_k f h^{kl}, \\ \mathcal{H}^l & = & \left(g^{-1/2} \partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu X^l) + \Gamma^l_{jk} \partial_\mu X^j \partial_\nu X^k g^{\mu\nu} \right)_{g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j h_{ij}} \end{cases} \quad (108)$$

for $1 \leq l \leq n_M$ and $\mathcal{H}$ is the mean curvature vector generalized to any embedding manifold M.

The functional (107) and the minimization flow (108) define the energy and the evolution equation for the active contour model defined in a general Riemannian manifold such as hyperbolic, spherical or parabolic manifolds.

In a first approach, we propose to recover the classical geodesic/geometric active contour model, i.e. when the embedding manifold is the Euclidean space.

Geodesic/Geometric Active Contours: The geodesic/geometric active contour model evolving in a 2-D Euclidean space proposed in [4, 9] can be recovered/revisited if we choose $X := C : p \to (C^1(p), C^2(p))$ and $h_{ij} = \delta_{ij}$ (Euclidean metric tensor), which means that the metric tensor of Σ is as follows:

$$g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j h_{ij} = \partial_\mu C^i \partial_\nu C^j \delta_{ij} \quad (109)$$

$$= (\partial_p C^1)^2 + (\partial_p C^2)^2 = |C_p|^2 = g_{pp} \quad (110)$$

with $\mu = \nu = p$ and $C_p := \frac{dC}{dp}$. Thus, the determinant of $g_{\mu\nu}$ is $g = |C_p|^2$ and the energy functional (107) is equal to

$$S_f = \int d^{n_\Sigma} \varsigma \ f \ g^{1/2} = \int dp \ f \ |C_p| = \int f ds = F_{GAC}(C), \quad (111)$$

which corresponds to the energy of the geodesic/geometric active contour model defined in Equation (97). Moreover, the minimization flow (102) can also be recovered. The mean curvature vector is equal to

$$\mathcal{H}^l = \left(g^{-1/2} \partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu X^l) + \Gamma^l_{jk} \partial_\mu X^j \partial_\nu X^k g^{\mu\nu} \right)_{g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j h_{ij}} \quad (112)$$

$$= \frac{1}{|C_p|} \partial_p \left(|C_p| \frac{1}{|C_p|^2} \partial_p C^l \right) + 0 \quad (113)$$

$$= \frac{1}{|C_p|} \partial_p \left(\frac{\partial_p C^l}{|C_p|} \right), \quad (114)$$

We remember that the unit tangeant vector to the curve parametrized by p is $\mathcal{T} = C_p / |C_p|$ and the derivative of $\mathcal{T}$ is equal to $\partial_p \mathcal{T} = |C_p| \partial_s \mathcal{T} = |C_p| \kappa \mathcal{N}$ where s, κ and $\mathcal{N}$ are respectively the arc-length, the curvature and the normal to the curve. Thus,

$$\mathcal{H} = \frac{1}{|C_p|} \partial_p \left(\frac{\partial_p C}{|C_p|} \right) = \frac{1}{|C_p|} \partial_p \mathcal{T} = \kappa \mathcal{N}. \quad (115)$$

The second part of the flow is equal to:

$$\partial_k f g^{\mu\nu} \partial_\mu X^k \partial_\nu X^l - \frac{n_M . n_\Sigma}{2} \partial_k f h^{kl} = \frac{1}{|C_p|^2} \partial_k f \cdot \partial_p C^k \partial_p C^l - \partial_k f \delta^{kl} \quad (116)$$

$$= \langle \nabla f, \frac{C_p}{|C_p|} \rangle \frac{C_p^l}{|C_p|} - \partial_l f \quad (117)$$

$$= \langle \nabla f, \mathcal{T} \rangle \mathcal{T}^l - \partial_l f, \quad (118)$$

under the vectorial form, we have:

$$\partial_k f g^{\mu\nu} \partial_\mu X^k \partial_\nu X^l - \frac{n_M . n_\Sigma}{2} \partial_k f h^{kl} = \langle \nabla f, \mathcal{T} \rangle \mathcal{T} - \nabla f, \quad (119)$$

$$= -\langle \nabla f, \mathcal{N} \rangle \mathcal{N}. \quad (120)$$

Finally, the flow (108) is as follows:

$$\partial_t C = f \kappa \mathcal{N} - \langle \nabla f, \mathcal{N} \rangle \mathcal{N}, \quad (121)$$

which is the well-known flow of the geodesic/geometric active contour model defined in Equation (102).

Active Contours on Hyperbolic and Spherical manifolds:
We now develop the model for active contours evolving on non-Euclidean manifolds such as the hyperbolic and spherical manifolds. The metric tensors for both manifolds were defined previously in Equations (43) and (61):

$$(h_{ij})_{H_+^2} = \begin{pmatrix} \frac{4}{(1-x^2-y^2)^2} & 0 \\ 0 & \frac{4}{(1-x^2-y^2)^2} \end{pmatrix}, \quad (122)$$

$$(h_{ij})_{S^2} = \begin{pmatrix} \frac{4}{(1+x^2+y^2)^2} & 0 \\ 0 & \frac{4}{(1+x^2+y^2)^2} \end{pmatrix}, \quad (123)$$

Both tensors are diagonal with the same components, which allows us to write them in this compact expression:

$$(h_{ij})_\xi = h_\xi^{1/2} \, \delta_{ij} \text{ with } \xi = H_+^2 \text{ or } S^2, \quad (124)$$

where h_ξ is the determinant of $(h_{ij})_\xi$. Thus, the metric tensors of the active contour, called Σ in the Polyakov framework, embedded on a hyperbolic or a spherical manifold, called M, are as follows:

$$g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j h_{ij} = \partial_p C^i \partial_p C^j h_\xi^{1/2} \delta_{ij} \quad (125)$$

$$= h_\xi^{1/2} |C_p|^2 = g_{pp} = g \quad (126)$$

Thus, the energy functional (107) of the active contours embedded on a hyperbolic or a spherical manifold is equal to

$$S_f = \int d^{n_\Sigma}\varsigma \; f \; g^{1/2} = \int f \; h_\xi^{1/4}|C_p|dp = \int f h_\xi^{1/4} ds = F_{ACHSM}(C), \quad (127)$$

where $ACHSM$ stands for Active Contours on Hyperbolic and Spherical Manifolds. Let us now compute the evolution equation for the active contours on a hyperbolic and a spherical manifold. The Beltrami part is equal to:

$$\begin{aligned} g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu X^l) &= h_\xi^{-1/4}\frac{1}{|C_p|}\partial_p\left(h_\xi^{1/4}|C_p|g_\xi^{-1/2}\frac{1}{|C_p|^2}\partial_p C^l\right) && (128)\\ &= h_\xi^{-1/4}\frac{1}{|C_p|}\partial_p\left(h_\xi^{-1/4}\frac{\partial_p C^l}{|C_p|}\right) && (129)\\ &= h_\xi^{-1/2}\kappa\mathcal{N}^l + h_\xi^{-1/4}\frac{1}{|C_p|}\partial_p\left(h_\xi^{-1/4}\right)\frac{\partial_p C^l}{|C_p|} && (130)\\ &= h_\xi^{-1/2}\kappa\mathcal{N}^l - \frac{1}{4}h_\xi^{-3/2}\langle\nabla h_\xi, \mathcal{T}\rangle\mathcal{T}^l. && (131)\end{aligned}$$

The Levi-Civita connection coefficients for the hyperbolic and spherical manifolds are equal to:

$$\begin{aligned} \Gamma^l_{jk} &= \frac{1}{2}h^{li}(\partial_j h_{ik} + \partial_k h_{ji} - \partial_i h_{jk}), && (132)\\ (\Gamma^1_{jk})_\xi &= \frac{1}{4}h_\xi^{-1}\begin{pmatrix} \partial_x h_\xi & \partial_y h_\xi \\ \partial_y h_\xi & -\partial_x h_\xi \end{pmatrix} && (133)\\ (\Gamma^2_{jk})_\xi &= \frac{1}{4}h_\xi^{-1}\begin{pmatrix} -\partial_y h_\xi & \partial_x h_\xi \\ \partial_x h_\xi & \partial_y h_\xi \end{pmatrix}, && (134)\end{aligned}$$

which gives us the second term of the mean curvature vector:

$$\begin{aligned} \Gamma^l_{jk}\partial_\mu X^j\partial_\nu X^k g^{\mu\nu} &= \Gamma^l_{jk}\partial_p C^j\partial_p C^k g^{pp}, && (135)\\ &= h_\xi^{-1/2}\Gamma^l_{11}\frac{\partial_p C^1}{|C_p|}\frac{\partial_p C^1}{|C_p|} + h_\xi^{-1/2}\Gamma^l_{12}\frac{\partial_p C^1}{|C_p|}\frac{\partial_p C^2}{|C_p|}\\ &\quad + h_\xi^{-1/2}\Gamma^l_{21}\frac{\partial_p C^2}{|C_p|}\frac{\partial_p C^1}{|C_p|} + h_\xi^{-1/2}\Gamma^l_{22}\frac{\partial_p C^2}{|C_p|}\frac{\partial_p C^2}{|C_p|} && (136)\end{aligned}$$

$$\Gamma^1_{jk}\partial_p X^j\partial_p X^k g^{\mu\nu} = \frac{1}{4}h_\xi^{-3/2}\left(\partial_x h_\xi\mathcal{T}^1\mathcal{T}^1 + 2\partial_y h_\xi\mathcal{T}^1\mathcal{T}^2 - \partial_x h_\xi\mathcal{T}^2\mathcal{T}^2\right) \quad (137)$$

$$\Gamma^2_{jk}\partial_p X^j\partial_p X^k g^{\mu\nu} = \frac{1}{4}h_\xi^{-3/2}\left(-\partial_y h_\xi\mathcal{T}^1\mathcal{T}^1 + 2\partial_x h_\xi\mathcal{T}^1\mathcal{T}^2 + \partial_y h_\xi\mathcal{T}^2\mathcal{T}^2\right), \quad (138)$$

which is as follows under the vectorial form:

$$\Gamma_{jk}\partial_\mu X^j\partial_\nu X^k g^{\mu\nu} = \frac{1}{4}h_\xi^{-3/2}(\langle\nabla h_\xi, \mathcal{T}\rangle\mathcal{T} - \langle\nabla h_\xi, \mathcal{N}\rangle\mathcal{N}), \quad (139)$$

if we consider $\mathcal{T} = (\mathcal{T}^1, \mathcal{T}^2)$ and $\mathcal{N} = (\mathcal{T}^2, -\mathcal{T}^1)$. Thus, the mean curvature vector is equal to:

$$
\begin{aligned}
\mathcal{H} &= \left(g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu X) + \Gamma_{jk}\partial_\mu X^j\partial_\nu X^k g^{\mu\nu}\right)_{g_{\mu\nu}=\partial_\mu X^i\partial_\nu X^j h_{ij}} && (140)\\
&= h_\xi^{-1/2}\kappa\mathcal{N} - \frac{1}{4}h_\xi^{-3/2}\langle\nabla h_\xi, \mathcal{T}\rangle\mathcal{T} \\
&\quad + \frac{1}{4}h_\xi^{-3/2}(\langle\nabla h_\xi, \mathcal{T}\rangle\mathcal{T} - \langle\nabla h_\xi, \mathcal{N}\rangle\mathcal{N}) && (141)\\
&= h_\xi^{-1/2}\kappa\mathcal{N} - \frac{1}{4}h_\xi^{-3/2}\langle\nabla h_\xi, \mathcal{N}\rangle\mathcal{N} && (142)
\end{aligned}
$$

Then, the second part of the flow (108) is equal to:

$$
\begin{aligned}
\partial_k f g^{\mu\nu}\partial_\mu X^k\partial_\nu X^l &- \frac{n_M.n_\Sigma}{2}\partial_k f h^{kl} && (143)\\
&= h_\xi^{-1/2}\frac{1}{|C_p|^2}\partial_k f\cdot\partial_p C^k\partial_p C^l - \partial_k f h_\xi^{-1/2}\delta^{kl} && (144)\\
&= h_\xi^{-1/2}\left(\langle\nabla f, \mathcal{T}\rangle\mathcal{T}^l - \partial_l f\right), && (145)\\
&= -h_\xi^{-1/2}\langle\nabla f, \mathcal{N}\rangle\mathcal{N}^l, && (146)
\end{aligned}
$$

Finally, the flow (108) of the active contours embedded on a hyperbolic or spherical manifold is as follows:

$$
\partial_t C = f h_\xi^{-1/2}\kappa\mathcal{N} - \frac{1}{4}f h_\xi^{-3/2}\langle\nabla h_\xi, \mathcal{N}\rangle\mathcal{N} - h_\xi^{-1/2}\langle\nabla f, \mathcal{N}\rangle\mathcal{N}. \quad (147)
$$

It is interesting to notice that this result coud also be obtained with the classical geodesic/geometric active contour model. As we said previously (Section **??**), Caselles *et al.* [**?**] defined the active contour problem as the determination of a geodesic, i.e. a curve of minimal weighted distance/length, between two points in a non-Euclidean space. The minimal length between two points in a space defined by a metric tensor a_{ij} is given by the following formula:

$$
\int_0^1 \sqrt{a_{ij}C_i'C_j'}dp = \int_0^1 \sqrt{a_{11}C_1'^2 + 2a_{12}C_1'C_2' + a_{22}C_2'^2}dp, \quad (148)
$$

with $C = (C_1, C_2)$. Caselles *et al.* considered the following metric tensor for the embedded space (Equation (99)):

$$
a_{ij} = f^2\delta_{ij}, \quad (149)
$$

where f is the edge detecting function, to recover the active contour energy:

$$
\int_0^1 \sqrt{f^2\delta_{ij}C_i'C_j'}dp = \int_0^1 f\sqrt{\delta_{ij}C_i'C_j'}dp = \int_0^1 f|C_p|dp = \int f ds = F_{GAC}(C). \quad (150)
$$

Then, they determined the minimization flow (102) with the calculus of variations and the Euler-Lagrange equation technique.

The functional $\int \sqrt{a_{ij}C_i'C_j'}dp$ gives us the opportunuity to change the metric tensor of the embedding space. Thus, it is possible to consider the metric tensors of the hyperbolic and spherical manifolds, $(h_{ij})_{H_+^2}$ and $(h_{ij})_{S^2}$, defined in Equations (122) and (123), if we consider the following metric tensor:

$$a_{ij} = f^2(h_{ij})_\xi, \quad \xi := H_+^2 \text{ or } S^2, \tag{151}$$

then the new weighted distance is equal to:

$$\int \sqrt{a_{ij}C_i'C_j'}dp = \int fh_\xi^{1/4}|C_p|dp = \int fh_\xi^{1/4}ds = F_{ACHSM}(C), \tag{152}$$

which is exactly the energy of the active contours on hyperbolic and spherical manifolds as defined in Equation (127). Caselles *et al.* proves in [?] that a functional with the form $\int fh_\xi^{1/4}ds = \int f_\xi' ds$ has the following minimization flow (see Appendix **??**):

$$\begin{aligned} \partial_t C &= f_\xi' \kappa \mathcal{N} - \langle \nabla f_\xi', \mathcal{N}\rangle \mathcal{N} & (153)\\ &= (fh_\xi^{1/4})\kappa\mathcal{N} - \langle \nabla (fh_\xi^{1/4}), \mathcal{N}\rangle\mathcal{N}, & (154)\end{aligned}$$

Finally, it is easy to show that the Euler-Lagrange equation of the previous flow (153) is equivalent to the one defined in Equation (147) because we do not change the minimization solution by multiplying the Euler-Lagrange equation by a strictly positive function. Thus, let us multiply the Euler-Lagrange equation of Equation (153) by the strictly positive function $h_\xi^{-3/4}$:

$$\begin{aligned} 0 &= \left(f_\xi'\kappa\mathcal{N} - \langle\nabla f_\xi', \mathcal{N}\rangle\mathcal{N}\right) \times h_\xi^{-3/4} & (155)\\ &= \left(fh_\xi^{1/4}\kappa\mathcal{N} - \langle\nabla(fh_\xi^{1/4}),\mathcal{N}\rangle\mathcal{N}\right) \times h_\xi^{-3/4} & (156)\\ &= \left(fh_\xi^{1/4}\kappa\mathcal{N} - f\langle\nabla h_\xi^{1/4},\mathcal{N}\rangle\mathcal{N} - h_\xi^{1/4}\langle\nabla f,\mathcal{N}\rangle\mathcal{N}\right) \times h_\xi^{-3/4} & (157)\\ &= \left(fh_\xi^{1/4}\kappa\mathcal{N} - \frac{1}{4}fh_\xi^{-3/4}\langle\nabla h_\xi,\mathcal{N}\rangle\mathcal{N} - h_\xi^{1/4}\langle\nabla f,\mathcal{N}\rangle\mathcal{N}\right) \times h_\xi^{-3/4} & (158)\\ &= fh_\xi^{-1/2}\kappa\mathcal{N} - \frac{1}{4}fh_\xi^{-3/2}\langle\nabla h_\xi,\mathcal{N}\rangle\mathcal{N} - h_\xi^{-1/2}\langle\nabla f,\mathcal{N}\rangle\mathcal{N}, & (159)\end{aligned}$$

which is exactly the Euler-Lagrange equation of Equation (147). Thus, the two minimization solutions by flows (147) and (153) are strictly equal, which means that we can choose one of the two flows indistinctly.

Active Contours on a Parabolic manifold:

Let us now develop the model for active contours evolving on a parabolic manifold. The metric tensor for a parabolic manifold was defined in Equation (73):

$$(h_{ij})_{par} = \begin{pmatrix} 1+4x^2 & 4xy \\ 4xy & 1+4y^2 \end{pmatrix}, \tag{160}$$

and the inverse tensor, given in Equation (74), is as follows:

$$(h^{ij})_{par} = h^{-1}\left(\begin{array}{cc} 1+4y^2 & -4xy \\ -4xy & 1+4x^2 \end{array}\right), \tag{161}$$

where $h = 1 + 4x^2 + 4y^2$ is the determinant of h_{ij}.

Thus, the metric tensor of the active contour, called Σ in the Polyakov framework, embedded on a parabolic manifold, called M, is as follows:

$$\begin{aligned}
g_{\mu\nu} &= \partial_\mu X^i \partial_\nu X^j h_{ij} && (162)\\
&= \partial_p C^i \partial_p C^j h_{ij} && (163)\\
&= C_p^1 C_p^1 h_{xx} + 2C_p^1 C_p^2 h_{xy} + C_p^2 C_p^2 h_{yy} && (164)\\
&= |C_p|^2\left(\frac{C_p^1}{|C_p|}\frac{C_p^1}{|C_p|}h_{xx} + 2\frac{C_p^1}{|C_p|}\frac{C_p^2}{|C_p|}h_{xy} + \frac{C_p^2}{|C_p|}\frac{C_p^2}{|C_p|}h_{yy}\right) && (165)\\
&= |C_p|^2\left(\mathcal{T}^1\mathcal{T}^1 h_{xx} + 2\mathcal{T}^1\mathcal{T}^2 h_{xy} + \mathcal{T}^2\mathcal{T}^2 h_{yy}\right) && (166)\\
&= |C_p|^2 \mathcal{T}^T (h_{ij})_{par}\mathcal{T} = |C_p|^2 |\mathcal{T}|_{(h_{ij})_{par}} && (167)\\
&= g_{pp} = g && (168)
\end{aligned}$$

where $\mathcal{T}^T$ means the transpose of $\mathcal{T}$ and $|\mathcal{T}|_{(h_{ij})_{par}}$ is the norm of the tangent vector on the parabolic manifold. Thus, the energy functional (107) of the active contours embedded on a parabolic manifold is equal to

$$S_f = \int d^{n_\Sigma}\varsigma\ f\ g^{1/2} = \int f\ |\mathcal{T}|_{(h_{ij})_{par}}|C_p|dp = \int f|\mathcal{T}|_{(h_{ij})_{par}} ds = F_{ACPM}(C) \tag{169}$$

where ACP stands for Active Contours on Parabolic Manifold. Let us now compute the evolution equation for the active contours on a parabolic manifold. The Beltrami part is equal to:

$$\begin{aligned}
g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu X^l) &= g^{-1/2}\partial_p\left(g^{-1/2}|C_p|\frac{\partial_p C^l}{|C_p|}\right) && (170)\\
&= g^{-1}|C_p|\partial_p(\mathcal{T}^l) + g^{-1/2}\partial_p(g^{-1/2})|C_p|\mathcal{T}^l && (171)\\
&= g^{-1}|C_p|^2\kappa\mathcal{N}^l - \frac{1}{2}g^{-2}|C_p|^2\langle\nabla g, \mathcal{T}\rangle\mathcal{T}^l. && (172)
\end{aligned}$$

The Levi-Civita connection coefficients are equal to:

$$\begin{aligned}
\Gamma^l_{jk} &= \frac{1}{2}h^{li}(\partial_j h_{ik} + \partial_k h_{ji} - \partial_i h_{jk}), && (173)\\
\Gamma^1_{jk} &= 4h^{-1}x\ \delta_{jk} && (174)\\
\Gamma^2_{jk} &= 4h^{-1}y\ \delta_{jk}, && (175)
\end{aligned}$$

which gives us the second term of the mean curvature vector:

$$\Gamma^l_{jk}\partial_\mu X^j \partial_\nu X^k g^{\mu\nu} = \Gamma^l_{jk}\partial_p C^j \partial_p C^k g^{pp}, \quad (176)$$

$$\Gamma^1_{jk}\partial_p C^j \partial_p C^k g^{pp} = 4h^{-1}x\ \delta_{jk}\partial_p C^j \partial_p C^k g^{-1}, \quad (177)$$
$$= 4h^{-1}g^{-1}|C_p|^2 x, \quad (178)$$

$$\Gamma^2_{jk}\partial_p C^j \partial_p C^k g^{pp} = 4h^{-1}y\ \delta_{jk}\partial_p C^j \partial_p C^k g^{-1}, \quad (179)$$
$$= 4h^{-1}g^{-1}|C_p|^2 y, \quad (180)$$

which is as follows under the vectorial form:

$$\Gamma_{jk}\partial_\mu X^j \partial_\nu X^k g^{\mu\nu} = 4h^{-1}g^{-1}|C_p|^2 \mathbf{x}, \quad (181)$$

where $\mathbf{x} = (x, y)$. Thus, the mean curvature vector is equal to:

$$\mathcal{H} = \Big(g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu X) + \Gamma_{jk}\partial_\mu X^j \partial_\nu X^k g^{\mu\nu}\Big)_{g_{\mu\nu}=\partial_\mu X^i \partial_\nu X^j h_{ij}} \quad (182)$$
$$= g^{-1}|C_p|^2\kappa\mathcal{N} - \frac{1}{2}g^{-2}|C_p|^2\langle\nabla g, \mathcal{T}\rangle\mathcal{T} + 4h^{-1}g^{-1}|C_p|^2\mathbf{x} \quad (183)$$

Then, the second part of the flow (108) is equal to:

$$\partial_k f g^{\mu\nu}\partial_\mu X^k \partial_\nu X^l = \partial_k f g^{pp}\partial_p C^k \partial_p C^l \quad (184)$$
$$= g^{-1}|C_p|^2 \partial_k f \frac{\partial_p C^k}{|C_p|}\frac{\partial_p C^l}{|C_p|} \quad (185)$$
$$= g^{-1}|C_p|^2\langle\nabla f, \mathcal{T}\rangle\mathcal{T}^l, \quad (186)$$

and the term $-\frac{n_M.n_\Sigma}{2}\partial_k f h^{kl}$ is equal for $l = 1, 2$ to:

$$-\begin{pmatrix} \partial_x f h^{xx} + \partial_y f h^{xy} \\ \partial_x f h^{xy} + \partial_y f h^{yy} \end{pmatrix} = -\begin{pmatrix} h^{xx} & h^{xy} \\ h^{xy} & h^{yy} \end{pmatrix}\begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = -(h^{ij})\nabla f. \quad (187)$$

Finally, the flow (108) for the active contours embedded on a parabolic manifold is a follows:

$$\partial_t C = f\left[g^{-1}|C_p|^2\kappa\mathcal{N} - \frac{1}{2}g^{-2}|C_p|^2\langle\nabla g, \mathcal{T}\rangle\mathcal{T} + 4h^{-1}g^{-1}|C_p|^2\mathbf{x}\right]$$
$$+g^{-1}|C_p|^2\langle\nabla f, \mathcal{T}\rangle\mathcal{T} - (h^{ij})\nabla f. \quad (188)$$

Epstein-Gage showed in [5] that the geometry of the curve deformation is not affected by the tangential velocity $\mathcal{T}$. This result is due to the fact that the tangential velocity does not change the geometry of the curve but its parametrization. Hence, Equation (191) can be replaced by

$$\partial_t C = f\left[g^{-1}|C_p|^2\kappa\mathcal{N} + 4h^{-1}g^{-1}|C_p|^2\langle \mathbf{x},\mathcal{N}\rangle\mathcal{N}\right] - \langle (h^{ij})\nabla f,\mathcal{N}\rangle\mathcal{N} \quad (189)$$

$$= g^{-1}|C_p|^2(f\kappa\mathcal{N} + 4h^{-1}\langle \mathbf{x},\mathcal{N}\rangle\mathcal{N}) - \langle (h^{ij})\nabla f,\mathcal{N}\rangle\mathcal{N}, \quad (190)$$

As we said previously, we do not change the minimization solution if we multiply the Euler-Lagrange equation by a strictly positive function. Thus, we finally obtain:

$$\partial_t C = f\kappa\mathcal{N} - |\mathcal{T}|_{(h_{ij})_{par}}\langle (h^{ij})\nabla f,\mathcal{N}\rangle\mathcal{N} + 4h^{-1}\langle \mathbf{x},\mathcal{N}\rangle\mathcal{N}, \quad (191)$$

because $g|C_p|^{-2} = |\mathcal{T}|_{(h_{ij})_{par}}$.

Summary:

It is possible to write a general formula for all active contour models developed in this section. Indeed, the active contour flow evolving on the Euclidean, hyperbolic (Equation (122)), spherical (Equation (123)) and parabolic (Equation (160)) manifolds defined by a metric tensor h_{ij} has the following general expression:

$$\partial_t C = f\kappa\mathcal{N} - |\mathcal{T}|_{h_{ij}}\langle (h^{ij})\nabla f,\mathcal{N}\rangle\mathcal{N} - |\mathcal{T}|_{h_{ij}}\langle \Gamma_{jk}C_p^j C_p^k g^{pp},\mathcal{N}\rangle\mathcal{N}, \quad (192)$$

where $|\mathcal{T}|_{h_{ij}}$ is the norm of the tangent vector $\mathcal{T}$ defined on the manifold h_{ij}. Thus, Equation (192) is a general expression for Equations (121), (147) and (191).

3 Conclusions and future work

In this technical report we have presented some basic image processing techniques for non-Euclidean images (360^0-images). These are Laplacian operator, gradient and active contour. It is interesting to develop the multiscale active contour for such images but this is a subject of future work.

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