Wavelets on the 2-Hyperboloid

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1 Introduction

The continuous wavelet transform is already a well established procedure for analysing data. Its main advantages over the classical Fourier transform are its local and multiresolution nature, which provide the interesting properties of a mathematical microscope. Its theory is well known in the case of the line and plane. However, the representation and analysis of signals in non-Euclidean geometry is a recurrent problem in many scientific domains. Not only certain data are constrained by nature on curved surfaces, but a lot of detectors collect information via interfaces which are geometrically complicated. Because of these demands, spherical wavelets [Antoine and Vandergheynst, 1999] were recently developed and applied, for example in Cosmology [Martinez-Gonzalez et al., 2002, Cayon et al., 2003].

Although the sphere is a manifold most desirable for applications, the mathematical analysis made so far invites us to consider other manifolds with similar geometrical properties, and first of all, other Riemmanian symmetric spaces of constant curvature. Among them, the two-sheeted hyperboloid stands as a very interesting case. For instance, in quantum mechanics such a manifold may be a particular example of a phase space [Gazeau et al., 1989]. Other examples come from physical systems constrained on a hyperbolic manifold, for instance, an open expanding model of the universe. A completely different example of application is provided by the emerging field of catadioptric image processing [Makadia and Daniilidis, 2003, Daniilidis et al.,

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2002]. In this case, a normal (flat) sensor is overlooking a curved mirror in order to obtain an omnidirectional picture of the physical scene. An efficient system is obtained using a hyperbolic mirror, since it has a single effective viewpoint. Finally, from a purely conceptual point of view, having already built the CWT for data analysis in Euclidean spaces and on the sphere, it is natural to raise the question of its existence and form on the dual manifold.

In general, for constructing a CWT on H^2, few basic requirements should be satisfied

- wavelets and signals must "live" on the hyperboloid;
- the transform must involve dilations of some kind; and
- the CWT on H^2, should reduce locally to the usual CWT on the plane.

The paper is organized as follows. In Section 2 we sketch the geometry of the two-sheeted hyperboloid H^2. In Section 3 we define affine transformations on the upper sheet H^2_+ of H^2. There are two fundamental operations : dilations and hyperbolic motions represented by the group $SO_0(1,2)$. Then, the action of the dilation on the hyperboloid is derived in Section 4. In Section 5, harmonic analysis on the hyperboloid is introduced by means of the Fourier-Helgason transform : this is a central tool for constructing and studying the wavelet transform. Section 6 really constitutes the core of this paper. First we define the CWT on H^2_+ through a hyperbolic convolution. Then we prove a hyperbolic convolution theorem which allows us to work conveniently in the Fourier-Helgason domain. Theorems 6.2.1 and 6.2.2 are our main results. We would like to state them roughly here in order to wet our readers' appetite since these results are reminiscent of their Euclidean counterparts. The first one states a generic *admissibility condition* for the existence of hyperbolic wavelets

Theorem 1.0.1 *Let ψ be a square integrable function on H^2_+ whose Fourier-Helgason coefficients satisfy :*

$$0 < \mathcal{A}_\psi(\nu) = \int_0^\infty |\widehat{\psi_a}(\nu)|^2 \, \frac{\mathrm{d}a}{a^3} < +\infty.$$

Then the hyperbolic wavelet transform is a bounded operator from $L^2(H^2_+)$ to a subset of $L^2(\mathbb{R}^+_* \times SO_0(1,2)$ that is invertible on its range.

Our second featured theorem shows that the admissibility condition simplifies to a zero-mean condition and really motivates the wavelet terminology.

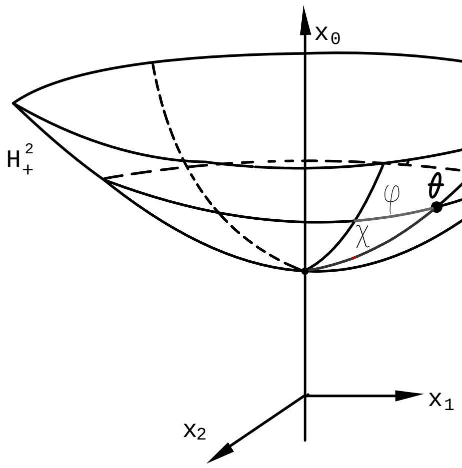

Figure 1: Geometry of the 2-hyperboloid.

Theorem 1.0.2 *A square integrable function on H^2_+ is a wavelet if its integral vanished, that is*

$$\int_{H^2_+} \mathrm{d}\mu(\chi,\varphi)\psi(\chi,\varphi) = 0.$$

Finally we conclude this paper with illustrating examples of hyperbolic wavelets and wavelet transforms and give directions for future work.

2 Geometry of the two-sheeted hyperboloid. Projective structures.

We start by recalling basic facts about the upper sheet of the two-sheeted hyperboloid of radius ρ, $H^2_{+\rho}$. Let χ, φ be a system of polar coordinates for $H^2_{+\rho}$. To each point $\theta = (\chi, \varphi)$ we shall associate the vector $x = (x_0, x_1, x_2)$ of $\mathbb{R}^3$ given by

$$\begin{aligned} x_0 &= \rho\cosh\chi, \\ x_1 &= \rho\sinh\chi\cos\varphi, \quad \rho > 0, \quad \chi \geqslant 0, \quad 0 \leq \varphi < 2\pi, \\ x_2 &= \rho\sinh\chi\sin\varphi, \end{aligned}$$

where $\chi \geqslant 0$ is the arc length from the pole to the given point on the hyperboloid, while φ is the arc length over the equator, as shown on Figure 1. The meridians ($\varphi = const$) are geodesics.

The squared metric element in hyperbolic coordinates is:

$$(\mathrm{ds})^2 = -\rho^2\bigg((\mathrm{d}\chi)^2 + \sinh^2\chi(\mathrm{d}\varphi)^2\bigg), \tag{2.1}$$

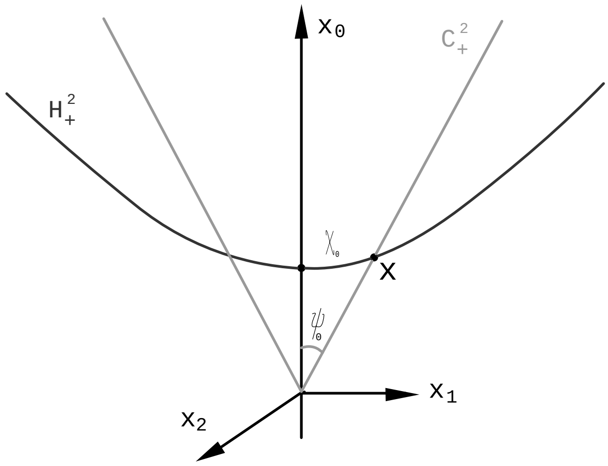

Figure 2: Geometry of the conic projection.

called *Lobachevskian metric*, whereas the measure element on the hyperboloid is

$$d\mu = \rho^2 \sinh\chi d\chi d\varphi. \tag{2.2}$$

In the sequel, we shall designate the unit hyperboloid $H^2_{+\rho=1}$ by H^2_+.

2.1 Conic projection

Various projections can be used to endow H^2_+ with a local euclidean structure. Let us consider a half null cone $C^2_+ \in \mathbb{R}^3$ of equation $(x_0)^2 - \frac{1}{\tan\psi_0}((x_1)^2 + (x_2)^2) = 0, x_0 \geqslant 0$. This cone C^2_+ has Euclidean nature (metric (2.1) vanishes). The cone surface unrolled is a circular sector. All points of H^2_+ will be mapped onto C^2_+ using a specific conic projection. The characteristic parameter of a conic projection is the constant of the cone $m = \cos\psi_0$, where ψ_0 is the Euclidean angle of inclination of the generatrix of the cone as shown on Figure 2. The relation with the hyperbolic angle χ_0 of the parallel touching the cone and ψ_0 is :

$$\cos\psi_0 = \frac{1}{\sqrt{1+\tanh^2\chi_0}} \equiv m$$

We will only consider radial conic projection and it is more convenient to use a radius r defined by the euclidean distance of the point on the cone to the x_0-axis:

$$r = f(\chi), \quad dr = f'(\chi)d\chi \quad \text{with} \quad \left.\frac{dr}{d\chi}\right|_{\chi=0} = 1. \tag{2.3}$$

Each suitable projection is determined by a specific choice of $f(\chi)$. The metric and measure can be expressed in the new coordinates by:

$$(\mathrm{d}s)^2 = -\Big((\mathrm{d}\chi)^2 + \sinh^2\chi(\mathrm{d}\varphi)^2\Big) = -\left(\left(\frac{\mathrm{d}r}{f'(\chi)}\right)^2 + \sinh^2\chi(\mathrm{d}\varphi)^2\right) \quad (2.4)$$

$$\mathrm{d}\mu = \sinh\chi \mathrm{d}\chi \mathrm{d}\varphi = \frac{\sinh\chi}{f'(\chi)}\mathrm{d}r\mathrm{d}\varphi. \quad (2.5)$$

3 Affine transformations on the 2-hyperboloid

We recall that our purpose is to build a total family of functions in $L^2(H^2_+, \mathrm{d}\mu)$ by picking wavelet or probe $\psi(\chi)$ with suitable localization properties and applying on it hyperbolic motions, belonging to the group $SO_0(1,2)$, and appropriate dilations

$$\psi(x) \to \lambda(a,x)\psi(d_{1/a}g^{-1}x) \equiv \psi_{a,g}(x), \quad g \in SO_0(1,2). \quad (3.1)$$

Dilations d_a will be studied bellow. Hyperbolic rotations and motions, $g \in SO_0(1,2)$, act on x in the following way.

A motion $g \in SO_0(1,2)$ can be factorized as $g = k_1 h k_2$, where $k_1, k_2 \in SO(2), h \in SO_0(1,1)$, and the respective action of k and h are the following

$$k(\varphi_0).x(\chi,\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_0 & -\sin\varphi_0 \\ 0 & \sin\varphi_0 & \cos\varphi_0 \end{pmatrix} \begin{pmatrix} \cosh\chi \\ \sinh\chi\cos\varphi \\ \sinh\chi\sin\varphi \end{pmatrix} \quad (3.2)$$

$$= x(\chi, \varphi + \varphi_0), \quad (3.3)$$

$$h(\chi_0).x(\chi,\varphi) = \begin{pmatrix} \cosh\chi_0 & \sinh\chi_0 & 0 \\ \sinh\chi_0 & \cosh\chi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\chi \\ \sinh\chi\cos\varphi \\ \sinh\chi\sin\varphi \end{pmatrix} \quad (3.4)$$

$$= x(\chi + \chi_0, \varphi) \quad (3.5)$$

On the other hand, the dilation is a homeomorphism $d_a : H^2_+ \to H^2_+$ and we require that d_a fulfills the two conditions:

(i) it monotonically dilates the azimuthal distance between two points on H^2_+:

$$\mathrm{dist(d_a(x), d_a(x'))}, \quad (3.6)$$

where $\mathrm{dist(x, x')}$ is defined by

$$\mathrm{dist(x,x')} = \cosh^{-1}(\mathrm{x \cdot x'}), \quad (3.7)$$

where the dot product is the Minkowski product in $\mathbb{R}^3$;

(ii) it is homomorphic to the group $\mathbb{R}_*^+$;

$$\mathbb{R}_*^+ \ni a \to d_a, \quad d_{ab} = d_a d_b, \quad d_{a^{-1}} = d_a^{-1}, \quad d_1 = \mathbb{I}_d.$$

The action of motion on a point $x \in H_+^2$ is trivial: it displaces (rotates) by an hyperbolic angle $\chi \in \mathbb{R}_+$ (respectively by an angle φ). It has to be noted that, as opposed to the case of the sphere, attempting to use the conformal group $SO_0(1,3)$ for describing dilation, our requirements are not satisfied. In this paper we adopt an alternative procedure that describes different maps for dilating the hyperboloid.

4 Dilation on hyperboloid

4.1 Dilation on hyperboloid through conic dilation

Considering the null cone of equation $x_0^2 - x_1^2 - x_2^2 = 0$ there exist the $SO_0(1,2)$-motions and the obvious Euclidean dilations

$$x \in C_+^2 \to ax \in C_+^2 \equiv d_a^C(x), \tag{4.1}$$

which form a multiplicative one parameter group isomorphic to $\mathbb{R}_*^+$.

Now, it is natural to use possible *conic projections* of H_+^2 onto C_+^2

$$H_+^2 \ni x \to \Phi(x) \in C_+^2 \tag{4.2}$$

in order to lift dilation (4.1) back to H_+^2.

Various Φ are possible, of course. One of them is immediate. It suffices to *flatten* the hyperboloid onto $\mathbb{R}^2 \simeq \mathbb{C}$

$$x = x(\chi, \varphi) \to \Pi_0 \Phi(x) = \sinh \chi e^{i\varphi}. \tag{4.3}$$

The invariant metric and measure on H_+^2, respectively (2.1) and (2.2), are then transformed into

$$(\mathrm{d}s)^2 \quad \to \quad \cosh^2 \chi (\mathrm{d}\chi)^2 - \sinh^2 \chi (\mathrm{d}\varphi)^2, \tag{4.4}$$

$$\mathrm{d}\mu(\chi, \varphi) \quad \to \quad \frac{\sinh \chi \cosh \chi}{\sqrt{1 + \sinh^2 \chi}} \mathrm{d}\chi \mathrm{d}\varphi. \tag{4.5}$$

In polar coordinates r, φ, the measure (4.5) reads

$$\mathrm{d}\mu(r, \varphi) = \frac{r}{\sqrt{1 + r^2}} \mathrm{d}r \mathrm{d}\varphi. \tag{4.6}$$

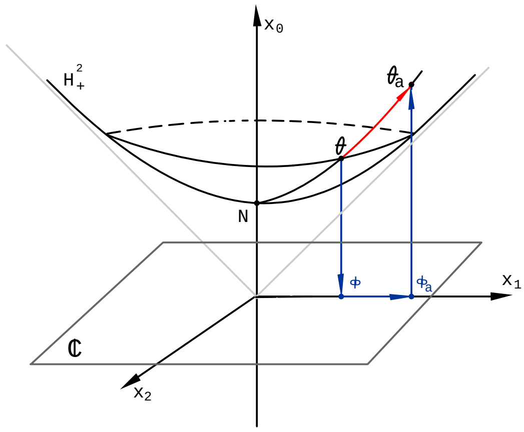

Figure 3: Action of a dilation a on the hyperboloid H^2 by "flattening".

and the action of dilation by flattening is depicted on Figure 3.

This is not what we should expect from a genuine expansion on H^2_+. Indeed, we wish to find the form of $\Pi_0\Phi$ such that, expressed in polar coordinates, the measure is

$$\mathrm{d}\mu = r\mathrm{d}r\mathrm{d}\varphi. \tag{4.7}$$

Thus dilating r will linearly dilate the measure $\mathrm{d}\mu$ as well. By expressing the measure (4.7) with the radius defined in (2.3) we obtain

$$f(\chi)f'(\chi) = \sinh\chi \quad \Longrightarrow \quad f(\chi) = 2\sinh\frac{\chi}{2}. \tag{4.8}$$

Consequently, the radius of the conic projection is $r = 2\sinh\frac{\chi}{2}$.

Thus, the conic projection $\Pi_0\Phi : H^2_+ \to C^2_+$ is a bijection given by

$$\Pi_0\Phi(x) = 2\sqrt{2}\sinh\frac{\chi}{2}e^{i\varphi},$$

with $x \equiv (\chi, \varphi), \quad \chi \in \mathbb{R}_+, \quad 0 \leq \varphi < 2\pi$. The action of $\Pi_0\Phi$ is depicted on Figure 4. Then, the lifted dilation is of the form

$$\sinh\frac{\chi_a}{2} = a\sinh\frac{\chi}{2}. \tag{4.9}$$

and the dilated point $x_a \in H^2$ is

$$x_a = (\cosh\chi_a, \sinh\chi_a\cos\varphi, \sinh\chi_a\sin\varphi). \tag{4.10}$$

The behaviour of $\mathrm{dist(x_N, x_a)}$, with x_N being the North Pole, is shown on Figure 5. We can see that this is an increasing function with respect to a.

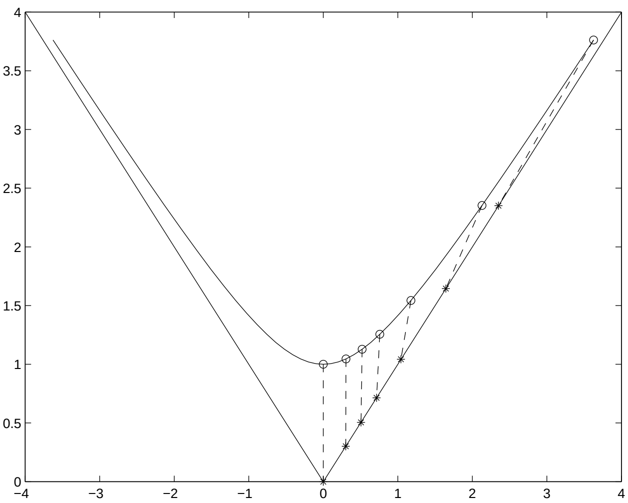

Figure 4: Projection of the hyperboloid H_+^2 onto a cone.

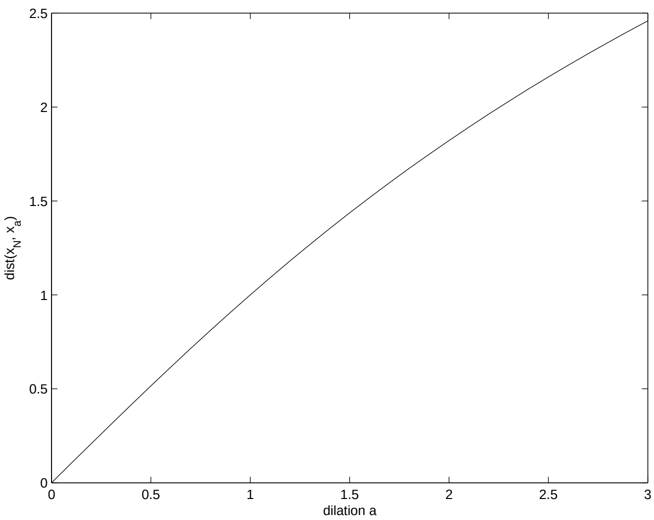

Figure 5: Analysis of the distance (3.6) as a function of dilation a, with x_N being the North Pole.

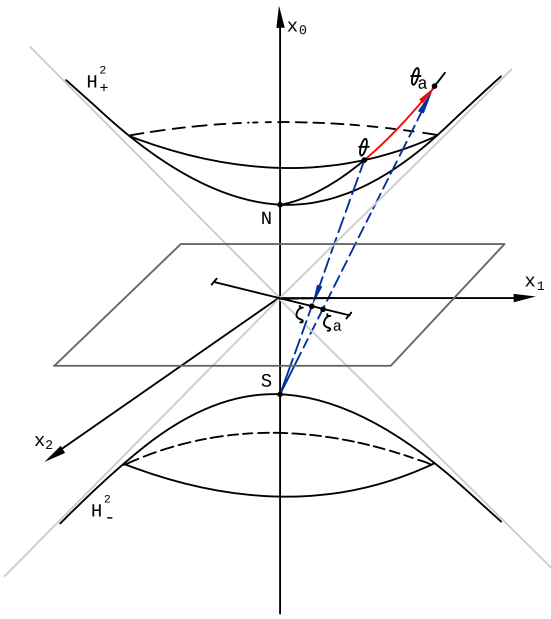

Figure 6: Action of a dilation a on the hyperboloid H_+^2 through a stereographic projection.

It is also interesting to compute the action of dilations in the bounded model of H_+^2. The latter is obtained by applying the stereographic projection from the South Pole of H^2 and it maps the upper sheet H_+^2 inside the unit disc in the equatorial plane:

$$x = x(\chi, \varphi) \to \Phi(x) = \tanh \frac{\chi}{2} e^{i\varphi}. \tag{4.11}$$

Using (4.9) and basic trigonometric relations we obtain

$$\tanh \frac{\chi_a}{2} = \sqrt{\frac{a^2 \tanh^2 \frac{\chi}{2}}{1 + (a^2 - 1) \tanh^2 \frac{\chi}{2}}} \equiv \zeta. \tag{4.12}$$

In this case, the dilation leaves invariant both $\zeta = 0$ and $\zeta = 1$. Figure 6 depicts the action of this transformation on a point $x \in H_+^2$. A dilation around the North Pole (D_N) is considered as a dilation in the unit disc in equatorial plane and lifted back to H^2 by inverse stereographic projection from the South Pole. A dilation around any other point $x \in H_+^2$ is obtained by moving x to the North Pole by a rotation $g \in SO_0(1, 2)$, performing dilation D_N and going back by inverse rotation:

$$D_x = g^{-1} D_N g.$$

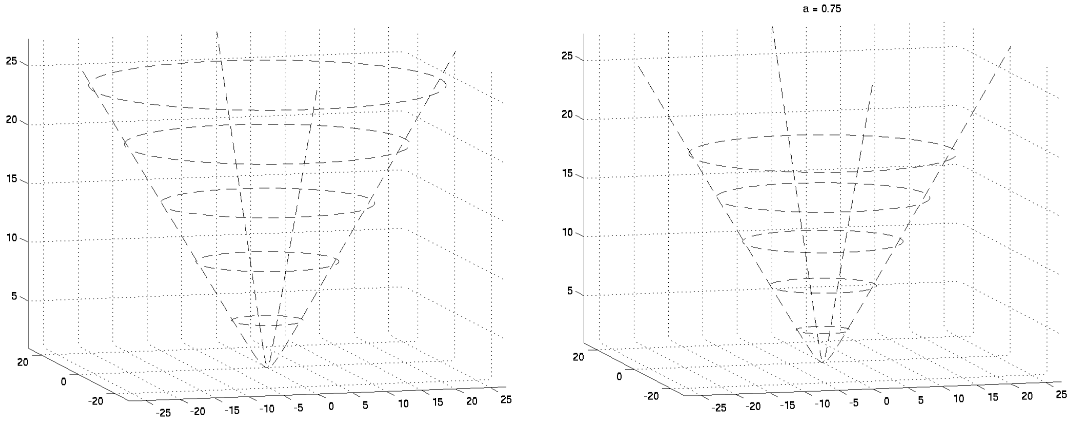

Figure 7: Visualization of the dilation on a hyperboloid H^2_+.

The visualization of the dilation on the hyperboloid H^2_+ is provided on Figure 7. Here, each circle represents points on the hyperboloid at constant χ and is dilated at scale $a = 0.75$.

5 Harmonic analysis on the 2-hyperboloid

5.1 Fourier-Helgason Transform

This integral transform is the precise analog of the Fourier-Plancherel transform on $\mathbb{R}^n$. It consists of an isometry between two Hilbert spaces

$$\mathcal{FH} : L^2(H^2_+, \mathrm{d}\mu) \longrightarrow L^2(\mathcal{L}, \mathrm{d}\eta), \tag{5.1}$$

where the measure $\mathrm{d}\mu$ is the $SO_0(1,2)$-invariant measure on H^2_+ and $L^2(\mathcal{L}, \mathrm{d}\eta)$ denotes the Hilbert space of sections of a line-bundle $\mathcal{L}$ over another suitably defined manifold, so-called Helgason-dual of H^2_+ and denoted by Ξ. We note here that the Helgason-dual of $\mathbb{R}^n$ is just its own dual.

Let us see what is the concrete realization of the dual space Ξ. Most of the following discussion can be found in [Ali and Bertola, 2002], and we summerize it here for convenience. In fact Ξ can be realized as the *projective half null-cone* asymptotic to $H^2_{+\rho}$ times the positive real line

$$\begin{aligned} C^2_+ &= \left\{ \xi \in \mathbb{R}^3 : \xi \cdot \xi = \xi_0^2 - \xi_1^2 - \xi_2^2 = 0, \quad \xi_0 > 0 \right\} && (5.2)\\ \Xi &= \mathbb{R}_+ \times \mathbb{P}C_+ && (5.3)\\ r &= (\nu, \vec{\xi}) \in \Xi, && (5.4) \end{aligned}$$

where $\mathbb{P}C_+$ denotes the projectivized forward cone $\{\xi \in C_+^2 \mid \lambda\xi \equiv \xi,\ \lambda > 0,\ \xi_0 > 0\}$ (the set of "rays" on the cone). A convenient realization of $\mathbb{P}C_+$ makes it diffeomorphic to the 1-sphere S^1 as follows

$$\begin{aligned} \mathbb{P}C_+ &\simeq \{\vec{\xi} \in \mathbb{R}^2 : \|\vec{\xi}\| = 1\} \sim S^1 && (5.5)\\ \xi \equiv (\xi_0, \xi_1, \xi_2) &\mapsto \frac{1}{\xi_0}\vec{\xi}. && (5.6)\end{aligned}$$

The Fourier - Helgason transform, is defined in an way similar to ordinary Fourier transform by using the eigenfunctions of the invariant differential operator of second order, i.e. the Laplacian on H_+^2. In our case, the functions of the (unique) invariant differential operator (the Laplacian) are named *hyperbolic plane waves* [Bros et al., 1994]

$$\begin{aligned} \mathcal{E}_{\nu,\xi}(x) &= (\xi \cdot x)^{-\frac{1}{2}-i\nu}, && (5.7)\\ \xi &\in \Xi^C, \quad C_+^2 = \left\{\xi \equiv (\xi_0, \vec{\xi}) \in \mathbb{R}^3, \quad \xi \cdot \xi = 0, \quad \xi_0 > 0\right\}. && (5.8)\end{aligned}$$

These waves are not defined on $\mathbb{R}_+ \times \mathbb{P}C_+^2$ but rather on $\mathbb{R}_+ \times C_+^2$; however the action of $\mathbb{R}_+$ on C_+^2 just rescales them by a factor which is constant in $x \in H_+^2$. In other words, they are sections of an apropriate line bundles over Ξ which we denote by $\mathcal{L}$ and C_+^2 is thought of as total space of $\mathbb{R}_+$ over $\mathbb{P}C_+$. As well, we note that the inner product $\xi \cdot x$ is positive on the product space $C_+^2 \times H_+^2$, so that the complex exponential is uniquely defined.

Let us express the plane waves in polar coordinates for a point $x \equiv (x_0, \vec{x}) \in H_+^2$

$$\begin{aligned} \mathcal{E}_{\nu,\xi}(x) &= (\xi \cdot x)^{-\frac{1}{2}-i\nu} && (5.9)\\ &= \left(\cosh\chi - \frac{\vec{\xi}\cdot\vec{x}}{\xi_0}\right)^{-\frac{1}{2}-i\nu} && (5.10)\\ &= (\cosh\chi - (\hat{n}\cdot\hat{x})\sinh\chi)^{-\frac{1}{2}-i\nu}, && (5.11)\end{aligned}$$

where $\hat{n} \in S^1$ is a unit vector in the direction of $\vec{\xi}$ and $\hat{x} \in S^1$ is the unit vector in the direction of $\vec{x}$. Applying any rotation $\varrho \in SO(2) \in SO_0(1,2)$ on this wave, it immediately follows

$$R(\varrho) : \mathcal{E}_{\nu,\xi}(x) \to \mathcal{E}_{\nu,\xi}(\varrho^{-1}\cdot x) = \mathcal{E}_{\nu,\varrho\cdot\xi}(x). \quad (5.12)$$

Finally, the Fourier - Helgason transform $\mathcal{FH}$ and its inverse $\mathcal{FH}^{-1}$ are defined as

$$\begin{aligned} \hat{f}(\nu,\xi) \equiv \mathcal{FH}[f](\nu,\xi) &= \int_{H^2} f(x)(x\cdot\xi)^{-\frac{1}{2}+i\nu}\mathrm{d}\mu(x), \quad \forall f \in \mathcal{C}_0^\infty(H^2), && (5.13)\\ \mathcal{FH}^{-1}[g](x) &= \int_{i\Xi} g(\nu,\xi)(x\cdot\xi)^{-\frac{1}{2}-i\nu}\mathrm{d}\eta(\nu,\xi), \quad \forall g \in \mathcal{C}_0^\infty(\mathcal{L}), && (5.14)\end{aligned}$$

where $\mathcal{C}_0^\infty(\mathcal{L})$ denotes the space of compactly supported smooth sections of the line-bundle $\mathcal{L}$. The integration in (5.14) is performed along any smooth embedding $\mathbf{j}\Xi$ into the total space of the line-bundle $\mathcal{L}$ and the measure $\mathrm{d}\eta$ is given by

$$\mathrm{d}\eta(\nu, \xi) = \frac{\mathrm{d}\nu}{|\mathbf{c}(\nu)|^2}\mathrm{d}\sigma_0, \tag{5.15}$$

with $\mathbf{c}(\nu)$ being the Harish-Chandra $\mathbf{c}$-function [Helgason, 1994]

$$\mathbf{c}(\nu) = \frac{2^{i\nu}\Gamma(1)\Gamma(i\nu)}{\sqrt{\pi}\Gamma(\frac{1}{2} + i\nu)}. \tag{5.16}$$

The factor $|\mathbf{c}(\nu)|^2$ can be simplified to

$$|\mathbf{c}(\nu)|^{-2} = \frac{\nu \sinh(\pi\nu)|\Gamma(\frac{1}{2} + i\nu)|^2}{\Gamma^2(1)}. \tag{5.17}$$

The 1-form $\mathrm{d}\sigma_0$ in the measure (5.15) is defined on the null cone C_+^2, it is closed on it and hence the integration is independent of the particular embedding of Ξ. Thus, such an embedding can be the following

$$\begin{aligned} \mathbf{j} : \Xi \quad &\longrightarrow \quad \mathbb{R}_+ \times C_+^2 && (5.18) \\ (\nu, \xi) \quad &\mapsto \quad (\nu, (1, \frac{\xi_1}{\xi_0}, \frac{\xi_2}{\xi_0})) \mapsto (\nu\hat{\xi}) && (5.19) \end{aligned}$$

Note that the transform $\mathcal{FH}$ maps functions on H_+^2 to sections of $\mathcal{L}$ and the inverse transform maps sections to functions. Thus, we have

Proposition 5.1.1 *[Helgason, 1994] The Fourier - Helgason transform defined in equations (5.13, 5.14) extends to an isometry of $L^2(H^2, \mathrm{d}\mu)$ onto $L^2(\mathcal{L}, \mathrm{d}\eta)$ so that we have*

$$\int_{H^2} |f(x)|^2 \mathrm{d}\mu(x) = \int_{\mathbf{j}\Xi} |\hat{f}(\xi, \nu)|^2 \mathrm{d}\eta(\xi, \nu). \tag{5.20}$$

6 Continuous Wavelet Transform on the Hyperboloid

One way of constructing the CWT on the hyperboloid H_+^2 would be to find a suitable group containing both $SO_0(1, 2)$ and the group of dilations, and then find its square-integrable representations in the Hilbert space $\psi \in L^2(H_+^2, \mathrm{d}\mu)$, where $\mathrm{d}\mu$ is the normalized $SO_0(1, 2)$-invariant measure on H_+^2.

We will take another approach, by directly studying the following wavelet transform

$$\int f(x)\psi_{a,g}(x)\mathrm{d}\mu(x) = \langle f, \psi_{a,h}\rangle.$$

Looking at pseudo-rotations (motions) only, we have

$$[\mathcal{U}_g\psi](x) = f(g^{-1}x), \quad g \in SO_0(1,2), \quad \psi \in L^2(H_+^2, \mathrm{d}\mu), \tag{6.1}$$

where $\mathcal{U}_g$ is a quasi-regular representation of $SO_0(1,2)$ on $L^2(H_+^2)$.

We now have to incorporate the dilation. However, the measure $d\mu$ is not dilation invariant, so that a Radon-Nikodym derivative $\lambda(g,x)$ must be inserted, namely:

$$\lambda(g,x) = \frac{\mathrm{d}\mu(g^{-1}x)}{\mathrm{d}\mu(x)}, \quad g \in SO_0(1,2). \tag{6.2}$$

The function λ is a 1-cocycle and satisfies the equation

$$\lambda(g_1g_2, x) = \lambda(g_1, x)\lambda(g_2, g_1^{-1}x). \tag{6.3}$$

In the case of dilating the hyperboloid through conic dilation, we simply have

$$\lambda(a,\chi) = \frac{\mathrm{d}\cosh\chi_a}{\mathrm{d}\cosh\chi} = a^{-2}. \tag{6.4}$$

Thus, the action of the dilation operator on the function is

$$D_a\psi(x) \equiv \psi_a(x) = \lambda^{\frac{1}{2}}(a,\chi)\psi(d_a^{-1}x) = \lambda^{\frac{1}{2}}(a,\chi)\psi(x_{\frac{1}{a}}) \tag{6.5}$$

with $x_a \equiv (\chi_a, \varphi) \in H_+^2$ and it reads

$$\psi_a(x) = \frac{1}{a}\psi(x_{\frac{1}{a}}).$$

Finally, the *hyperbolic wavelet function* can be written as

$$\psi_{a,g}(x) = \mathcal{U}_g D_a\psi(x) = \mathcal{U}_g\psi_a(x).$$

Accordingly, the *hyperbolic continuous wavelet transform* of a signal (function) $f \in L^2(H_+^2)$ is defined as:

$$\begin{aligned} W_f(a,g) &= \langle\psi_{a,g}|f\rangle & (6.6)\\ &= \int_{H_+^2} \overline{[\mathcal{U}_g D_a\psi](x)} f(x)\mathrm{d}\mu(x) & (6.7)\\ &= \int_{H_+^2} \overline{\psi_a(g^{-1}x)} f(x)\mathrm{d}\mu(x) & (6.8) \end{aligned}$$

where $x \equiv (\chi, \varphi) \in H_+^2$ and $g \in SO_0(1,2)$.

In the next section, we show that this expression can be conveniently interpreted and studied as an hyperbolic convolution.

6.1 Convolutions on H^2

Since H^2_+ is a homogeneous space of $SO_0(1,2)$, one can easily define a convolution. Indeed, let $f \in L^2(H^2_+)$ and $s \in L^1(H^2_+)$, their *hyperbolic convolution* is the function of $g \in SO_0(1,2)$ defined as

$$(f * s)(g) = \int_{H^2_+} f(g^{-1}x)s(x)\mathrm{d}\mu(x). \tag{6.9}$$

Then $f * s \in L^2(SO_0(1,2), \mathrm{d}g)$, where $\mathrm{d}g$ stands for the left Haar measure on the group and

$$\|f * s\|_2 \leq \|f\|_2\|s\|_1, \tag{6.10}$$

by the Young convolution inequality.

In this paper however, we will deal with a simpler definition where the convolution is a function defined on H^2_+. Let $[\cdot] : H^2_+ \longrightarrow SO_0(1,2)$ be a section in the fiber bundle defined by the group and its homogeneous space. In the following we will make use of the *Euler section*, whose construction we now highlight. Recall from Section 3 that any $g \in SO_0(1,2)$ can be uniquely decomposed as a product of three elements $g = k(\varphi)h(\chi)k(\psi)$. Using this parametrization, we thus define :

$$\begin{array}{rcl}[\cdot] & : & H^2_+ \longrightarrow SO_0(1,2) \\ [\cdot] & : & x(\chi,\varphi) \mapsto g = k(\varphi)h(\chi)\end{array}$$

The hyperbolic convolution, restricted to H^2_+, thus takes the following form:

$$(f * s)(y) = \int_{H^2_+} f([y]^{-1}x)s(x)\mathrm{d}\mu(x),\ y \in H^2_+$$

We will mostly deal with convolution kernels that are axisymmetric (or rotation invariant) functions on H^2_+ (i.e. functions of the variable χ alone). The Fourier-Helgason transform of such an element has a simpler form as shown by the following proposition.

Proposition 6.1.1 *If f is a rotation invariant function, i.e. $f(\varrho^{-1}x) = f(x)$, $\forall \rho \in SO(2)$, its Fourier-Helgason transform $\hat{f}(\xi,\nu)$ is a function of ν alone, i.e. $\hat{f}(\nu)$.*

Proof : Applying the Fourier-Helgason transform on a rotation-invariant function we write:

$$
\begin{aligned}
\hat{f}(\xi,\nu) &= \int_{H_+^2} f(x)\,\mathcal{E}_{\xi,\nu}(x)\mathrm{d}\mu(x) && (6.11)\\
&= \int_{H_+^2} f(\varrho^{-1}x)(\xi\cdot\nu)^{-\frac{1}{2}-i\nu}\mathrm{d}\mu(x), \quad \xi\in\mathbb{P}C_+,\ \rho\in SO(2) && (6.12)\\
&= \int_{H_+^2} f(x')(\xi\cdot\varrho x')^{-\frac{1}{2}-i\nu}\mathrm{d}\mu(x') && (6.13)\\
&= \hat{f}(\varrho^{-1}\xi,\nu), && (6.14)
\end{aligned}
$$

and so $\hat{f}(\xi,\nu)$ does not depend on ξ. ■

We now have all the basic ingredients for formulating a useful convolution theorem in the Fourier-Helgason domain. As we will now see the FH transform of a convolution takes a simple form, provided one of the kernels is rotation invariant.

Theorem 6.1.2 (Convolution) *Let f and s be two measurable functions; $f,s\in L^2(H_+^2)$ and s be rotation invariant. The convolution $(s*f)(y)$ is in $L^1(H_+^2)$ and its Fourier-Helgason transform satisfies*

$$
\widehat{(s*f)}(\nu,\xi) = \hat{f}(\nu,\xi)\,\hat{s}(\nu). \tag{6.15}
$$

Proof : The convolution of s and f is given by:

$$
(s*f)(y) = \int_{H_+^2} s([y]^{-1}x)f(x)\mathrm{d}\mu(x).
$$

Since s is $SO(2)$-invariant, we write its argument in this equation in the following way :

$$
\begin{pmatrix} \cosh\chi & \sinh\chi & 0\\ \sinh\chi & \cosh\chi & 0\\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} x_0\\ x_1\\ 0 \end{pmatrix}
=
\begin{pmatrix} \cosh\chi x_0 + \sinh\chi x_1\\ \sinh\chi x_1 + \cosh\chi x_1\\ 0 \end{pmatrix}, \tag{6.16}
$$

where $x=(x_0,x_1,x_2)$ and we used polar coordinates for $y=y(\chi,\varphi)$. On the other hand we can also write this equation in a symmetric form :

$$
\begin{pmatrix} \cosh\chi x_0 + \sinh\chi x_1\\ \sinh\chi x_1 + \cosh\chi x_1\\ 0 \end{pmatrix}
=
\begin{pmatrix} x_0 & x_1 & 0\\ x_1 & x_0 & 0\\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \cosh\chi\\ \sinh\chi\\ 0 \end{pmatrix}. \tag{6.17}
$$

Thus we have

$$s([y]^{-1}x) = s([x]^{-1}y). \tag{6.18}$$

Therefore, the convolution with a rotation invariant function is given by

$$\begin{aligned}(s * f)(y) &= \int_{H^2_+} f(x)s([y]^{-1}x)\,\mathrm{d}\mu(x) &(6.19)\\ &= \int_{H^2_+} f(x)s(x\cdot y)\,\mathrm{d}\mu(x). &(6.20)\end{aligned}$$

On the other hand, applying the Fourier-Helgason transform on $s * f$ we get

$$\begin{aligned}\widehat{(s * f)}(\nu,\xi) &= \int_{H^2_+} (s*f)(y)(y\cdot\xi)^{-\frac{1}{2}+i\nu}\mathrm{d}\mu(y)\\ &= \int_{H^2_+}\mathrm{d}\mu(y)\int_{H^2_+}\mathrm{d}\mu(x)s([y]^{-1}x)f(x)(y\cdot\xi)^{-\frac{1}{2}+i\nu}\\ &= \int_{H^2_+}\mathrm{d}\mu(x)f(x)\int_{H^2_+}\mathrm{d}\mu(y)s([y]^{-1}x)(y\cdot\xi)^{-\frac{1}{2}+i\nu}\\ &= \int_{H^2_+}\mathrm{d}\mu(x)f(x)\int_{H^2_+}\mathrm{d}\mu(y)s([x]^{-1}y)(y\cdot\xi)^{-\frac{1}{2}+i\nu}\\ &= \int_{H^2_+}\mathrm{d}\mu(x)f(x)\int_{H^2_+}\mathrm{d}\mu(y)s(y)([x]y\cdot\xi)^{-\frac{1}{2}+i\nu}\\ &= \int_{H^2_+}\mathrm{d}\mu(x)f(x)\int_{H^2_+}\mathrm{d}\mu(y)s(y)(y\cdot[x]^{-1}\xi)^{-\frac{1}{2}+i\nu}.\end{aligned}$$

Since ξ belong to the projective null cone, we can write

$$(y\cdot[x]^{-1}\xi) = ([x]^{-1}\xi)_0\left(y\cdot\frac{[x]^{-1}\xi}{([x]^{-1}\xi)_0}\right), \tag{6.21}$$

and using $([x]^{-1}\xi)_0 = (x\cdot\xi)$, we finally obtain

$$\begin{aligned}\widehat{(s * f)}(\nu,\xi) &= \int_{H^2_+}\mathrm{d}\mu(x)f(x)(x\cdot\xi)^{-\frac{1}{2}+i\nu}\int_{H^2_+}\mathrm{d}\mu(y)s(y)\left(y\cdot\frac{[x]^{-1}\xi}{([x]^{-1}\xi)_0}\right)^{-\frac{1}{2}+i\nu}\\ &= \hat{f}(\nu,\xi)\hat{s}(\nu)\end{aligned}$$

where we used the rotation invariance of s. ■

Based on Theorem 6.1.2, we can write the hyperbolic continuous wavelet transform of a function f with respect to an axisymmetric wavelet ψ as

$$W_f(a,g) \equiv W_f(a,[x]) = \big(\bar{\psi}_a * f\big)(x). \tag{6.22}$$

6.2 Wavelets on the hyperboloid

We now come to the heart of this paper : we prove that the hyperbolic wavelet transform is a well-defined invertible map, provided the wavelet satisfy an admissibility condition.

Theorem 6.2.1 (Admissibility condition) *Let $\psi \in L^1(H^2_+)$ be an axisymmetric function and let m, M be two constants such that*

$$0 < m \leq \mathcal{A}_\psi(\nu) = \int_0^\infty |\hat{\psi}_a(\nu)|^2 \, \alpha(a)\mathrm{d}a \leq M < +\infty. \tag{6.23}$$

Then the linear operator A_ψ defined by :

$$A_\psi f(x') = \int_{\mathbb{R}^*_+} \int_{H^2_+} W_f(a,x)\psi_{a,x}(x')\mathrm{d}x \, \alpha(a)\mathrm{d}a, \tag{6.24}$$

is bounded and with bounded inverse. More precisely A_ψ is univocally characterized by the following Fourier-Helgason multiplier :

$$\widehat{A_\psi}\hat{f}(\nu,\varphi) \equiv \widehat{A_\psi f}(\nu,\varphi) = \hat{f}(\nu,\varphi) \int_0^\infty |\hat{\psi}_a(\nu)|^2 \, \alpha(a)\mathrm{d}a = \mathcal{A}_\psi(\nu)\hat{f}(\nu,\varphi).$$

Proof : Let the wavelet transform W_f be defined as in equation (6.8) and consider the following quantity :

$$\Delta_a(x') = \int_{H^2_+} W_f(a,x)\psi_{a,x}(x')\mathrm{d}x. \tag{6.25}$$

A close inspection reveals that $\Delta_a(x')$ is itself a convolution. Taking the Fourier - Helgason transform on both sides of (6.25) and applying Theorem 6.1.2 twice, we thus obtain:

$$\widehat{\Delta_a}(\nu,\varphi) = |\hat{\psi}_a(\nu)|^2 \hat{f}(\nu,\varphi)\,.$$

Finally, integrating over all scales we obtain :

$$\int_{\mathbb{R}^*_+} \alpha(a)\mathrm{d}a \, \widehat{\Delta_a}(\nu,\varphi) = \hat{f}(\nu,\varphi) \int_{\mathbb{R}^*_+} \alpha(a)\mathrm{d}a \, |\hat{\psi}_a(\nu)|^2 \tag{6.26}$$

which is the expected result. ■

There are three important remarks concerning this result. First, Theorem 6.2.1 shows that the wavelet familly $\{\psi_{a,x}, a \in \mathbb{R}^+_*, \, x \in H^2_+\}$ forms a continuous frame [Ali et al., 2000] provided the admissibility condition (6.23) is satisfied. In this case, the wavelet transform W_f of any f can be inverted in the following way. Let $\widetilde{\psi_{a,x}}$ be a *reconstruction wavelet* defined by :

$$\widehat{\widetilde{\psi_{a,x}}}(\nu) = \mathcal{A}_\psi^{-1}(\nu)\widehat{\psi_{a,x}}(\nu).$$

As a direct consequence of Theorem 6.2.1, the inversion formula, to be understood in the strong sense in $L^2(H_+^2)$, reads :

$$f(x') = \int_{\mathbb{R}_+^*} \int_{H_+^2} W_f(a,x)\widetilde{\psi_{a,x}}(x')\mathrm{d}x\, \alpha(a)\mathrm{d}a\,. \tag{6.27}$$

As a second remark, the reader can check that Theorem 6.2.1 does not depend on choice of dilation! This is not exactly true, actually. The *architecture* of the proof does not depend on the explicit form of the dilation operator, but the admissibility condition explicitly depends on it. As we shall see later, it will be of crucial importance when trying to construct admissible wavelets. Finally the third remark concerns the somewhat arbitrary choice of measure $\alpha(a)$ in the formulas. The reader may easily check that the usual 1-D wavelet theory can be formulated along the same lines, keeping an arbitrary scale measure. In that case though, the choice $\alpha = a^{-2}$ leads to a tight continuous frame, i.e. the frame operator A_ψ is a constant. The situation here is more complicated in the sense that no choice of measure would yield to a tight frame, a particularity shared by the continuous wavelet transform on the sphere [Antoine and Vandergheynst, 1999]. Some choices of measure though lead to simplified admissibility conditions as we will now discuss.

Theorem 6.2.2 *Let $\alpha(a)\mathrm{d}a$ be a homogeneous measure of the form $a^{-\beta}\mathrm{d}a$, $\beta > 0$. If D_a is the conic dilation defined by equations* (4.9), (6.4) *and* (6.5), *then an axisymmetric function $\psi \in L^2(H_+^2, \mathrm{d}\mu(\chi,\varphi))$ is admissible if one of the two following conditions is satisifed :*

- $1 < \beta \leq 2$ *and ψ is integrable, or*
- $\beta > 2$ *and ψ satisfies the zero-mean condition*

$$\int_{H_+^2} \psi(\chi,\varphi)\mathrm{d}\mu(\chi,\varphi) = 0. \tag{6.28}$$

Proof : Let us assume $\psi(x)$ belongs to $\mathcal{C}_0(H_+^2)$, i.e. it is compactly supported

$$\psi(x) = 0 \quad \text{if} \quad \chi > \tilde{\chi}, \quad \tilde{\chi} < const,$$

continuous and decays when $\chi \to +\infty$. We wish to prove that

$$\int_0^\infty |\langle \mathcal{E}_{\xi,\nu}|D_a\psi\rangle|^2\, \alpha(a)\mathrm{d}a < \infty. \tag{6.29}$$

First, we compute the Fourier-Helgason coefficients of the dilated function ψ:

$$\begin{aligned}
\langle \mathcal{E}_{\xi,\nu}(\chi,\varphi)|D_a\psi(\chi,\varphi)\rangle &= \int_{H_+^2} D_a\psi(\chi,\varphi)\,\overline{\mathcal{E}_{\xi,\nu}(\chi,\varphi)}\,\mathrm{d}\mu(\chi,\varphi) \\
&= \int_0^{2\pi}\!\!\int_0^{\tilde{\chi_a}} \lambda^{\frac{1}{2}}(a,\chi)\psi(\chi_{\frac{1}{a}},\varphi)\overline{\mathcal{E}_{\xi,\nu}(\chi,\varphi)}\sinh\chi\mathrm{d}\chi\mathrm{d}\varphi.
\end{aligned}$$

By performing the change of variable $\chi' = \chi_{\frac{1}{a}}$, we get $\chi = \chi'_a$ and $\mathrm{d}\cosh\chi = \mathrm{d}\cosh\chi'_a = \lambda(a^{-1},\chi')\mathrm{d}\cosh\chi'$. The Fourier-Helgason coefficients become

$$\langle \mathcal{E}_{\xi,\nu}|D_a\psi\rangle = \int_0^{2\pi}\!\!\int_0^{\tilde{\chi}} \lambda^{\frac{1}{2}}(a,\chi'_a)\psi(\chi',\varphi)\overline{\mathcal{E}_{\xi,\nu}(\chi'_a,\varphi)}\lambda(a^{-1},\chi')\sinh\chi'\mathrm{d}\chi'\mathrm{d}\varphi. \tag{6.30}$$

Using the cocycle property

$$\lambda^{\frac{1}{2}}(a^{-1},\chi')\ \lambda^{\frac{1}{2}}(a,\chi'_a) = 1,$$

we express

$$\lambda^{\frac{1}{2}}(a,\chi'_a) = \frac{1}{\lambda^{\frac{1}{2}}(a^{-1},\chi')} = a. \tag{6.31}$$

Substituting this in (6.30) we get

$$\begin{aligned}
\langle \mathcal{E}_{\xi,\nu}|D_a\psi\rangle &= \int_0^{2\pi}\!\!\int_0^{\tilde{\chi}} \lambda^{\frac{1}{2}}(a^{-1},\chi')\psi(\chi',\varphi)\overline{\mathcal{E}_{\xi,\nu}(\chi'_a,\varphi)}\sinh\chi'\mathrm{d}\chi'\mathrm{d}\varphi \qquad (6.32)\\
&= a\int_0^{2\pi}\!\!\int_0^{\tilde{\chi}} \psi(\chi',\varphi)\overline{\mathcal{E}_{\xi,\nu}(\chi'_a,\varphi)}\sinh\chi'\mathrm{d}\chi'\mathrm{d}\varphi. \qquad (6.33)
\end{aligned}$$

Then, we split (6.29) in three parts:

$$\int_0^\infty \alpha(a)\mathrm{d}a = \underbrace{\int_0^\sigma \alpha(a)\mathrm{d}a}_{I_1} + \underbrace{\int_\sigma^{\frac{1}{\sigma}} \alpha(a)\mathrm{d}a}_{I_2} + \underbrace{\int_{\frac{1}{\sigma}}^\infty \alpha(a)\mathrm{d}a}_{I_3}. \tag{6.34}$$

Let us focus on the first integral. Developing the Fourier-Helgason kernel $\mathcal{E}_{\xi,\nu}$ in (6.33), we obtain :

$$I_1 = \int_0^\sigma \alpha(a)a^2\mathrm{d}a\Big|\int_0^{\tilde{\chi}}\int_0^{2\pi} \mathrm{d}\mu(\chi,\varphi)\psi(\chi')(\cosh\chi'_a - \sinh\chi'_a\cos\varphi)^{-\frac{1}{2}+i\nu}\Big|^2.$$

Using the explicit form of χ'_a, we have :

$$I_1 = \int_0^\sigma \alpha(a) a^2 \mathrm{d}a \Big| \int_0^{\tilde{\chi}} \int_0^{2\pi} \mathrm{d}\mu(\chi,\varphi)\psi(\chi')$$
$$\big(1 + 2a^2 \sinh^2 \frac{\chi'}{2} - 2a\sqrt{1 + a^2 \sinh^2 \frac{\chi'}{2}} \sinh \frac{\chi'}{2} \cos\varphi\big)^{-\frac{1}{2}+i\nu}\Big|^2. \quad (6.35)$$

Since we are interested in the small scale behaviour of this quantity, we can focus on the leading term in the expansion as powers of a, which yields :

$$\begin{aligned} I_1 &\sim \int_0^\sigma \alpha(a) a^2 \mathrm{d}a \Big| \int_0^{\tilde{\chi}} \int_0^{2\pi} \mathrm{d}\mu(\chi,\varphi)\psi(\chi')\big(1 - 2a \sinh \frac{\chi'}{2} \cos\varphi\big)^{-\frac{1}{2}+i\nu}\Big|^2 \\ &\sim \int_0^\sigma \alpha(a) a^2 \mathrm{d}a \Big| \int_0^{\tilde{\chi}} \int_0^{2\pi} \mathrm{d}\mu(\chi,\varphi)\psi(\chi')\big(1 - (-1+2i\nu) a \sinh \frac{\chi'}{2} \cos\varphi\big)\Big|^2. \end{aligned}$$

Finally, integrating over φ and using the rotation invariance of ψ, we obtain :

$$I_1 \sim \int_0^\sigma \alpha(a) a^2 \mathrm{d}a \Big| \int_0^{\tilde{\chi}} \sinh\chi' \mathrm{d}\chi' \, \psi(\chi')\Big|^2. \quad (6.36)$$

The second subintegral (I_2) is straightforward, since the operator D_a is strongly continuous and thus the integrand is bounded on $[\sigma, \frac{1}{\sigma}]$. As for the last term, I_3, we use a similar strategy. First write it as :

$$I_3 = \int_{\frac{1}{\sigma}}^{+\infty} \alpha(a) a^2 \mathrm{d}a \Big| \int_0^{\tilde{\chi}} \int_0^{2\pi} \mathrm{d}\mu(\chi,\varphi)\psi(\chi') a^{-1+2i\nu}$$
$$\big(\frac{1}{a^2} + 2\sinh^2 \frac{\chi'}{2} - 2\sqrt{\frac{1}{a^2} + \sinh^2 \frac{\chi'}{2}} \sinh \frac{\chi'}{2} \cos\varphi\big)^{-\frac{1}{2}+i\nu}\Big|^2. \quad (6.37)$$

Since we are interested in the large scale behaviour this time, we keep the leading term in the expansion as powers of $1/a$ and obtain :

$$I_3 \sim \int_{\frac{1}{\sigma}}^{+\infty} \alpha(a) \mathrm{d}a \Big| \int_0^{\tilde{\chi}} \sinh\chi' \mathrm{d}\chi' \, \big(2\sinh^2 \frac{\chi'}{2}\big)^{-\frac{1}{2}+i\nu} \psi(\chi')\Big|^2. \quad (6.38)$$

The convergence of I_1 and I_3 clearly depends on the choice of measure in the integral over scales. Restricting ourselves to homogeneous measures $\alpha(a) = a^{-\beta}$, we can distinguish the following cases :

- $\beta \leq 1$: In this case I_3 does not converge and there are no admissible wavelets.
- $1 < \beta \leq 2$: In this case both I_1 and I_3 converge with only mild assumptions on ψ, namely that it is integrable.
- $\beta > 2$: In this case I_1 diverges except when $\int_{H_+^2} \psi = 0$.

■

6.3 An example of Hyperbolic Wavelet

For concluding this section, we need to find a class of admissible vectors, which satisfy the admissibility condition.

Proposition 6.3.1 *Let* $\psi \in L^2(H^2_+, \mathrm{d}\mu)$. *Then*

$$\int_{H^2_+} D_a\psi(\chi,\varphi)\mathrm{d}\mu(\chi,\varphi) = a\int_{H^2_+} \psi(\chi,\varphi)\mathrm{d}\mu(\chi,\varphi). \tag{6.39}$$

Proof: We have to compute the following integral

$$I = \int_{H^2_+} D_a\psi(\chi,\varphi)\mathrm{d}\mu(\chi,\varphi) = \int_{H^2_+} \lambda^{\frac{1}{2}}(a,\chi)\psi(\chi_{\frac{1}{a}},\varphi)\mathrm{d}\mu(\chi,\varphi).$$

By change of variable $\chi_{\frac{1}{a}} = \chi'$, we get

$$\begin{aligned} I &= \int_{H^2_+} \lambda^{\frac{1}{2}}(a,\chi'_a)\,\psi(\chi',\varphi)\,\lambda(a^{-1},\chi')\mathrm{d}\mu(\chi',\varphi) \\ &= \int_{H^2_+} \lambda^{\frac{1}{2}}(a^{-1},\chi')\psi(\chi',\varphi)\mathrm{d}\mu(\chi',\varphi), \end{aligned}$$

and having $\lambda^{\frac{1}{2}}(a^{-1},\chi') = a$, which follows directly from (6.4), we get

$$I = a\int_{H^2_+} \psi(\chi',\varphi)\mathrm{d}\mu(\chi',\varphi),$$

which proves the proposition. ■

Using this result, we can build the hyperbolic "difference" wavelet (difference-of-Gaussian, or DOG wavelet). Thus, given a square-integrable function ψ, we define

$$f^\beta_\psi(\chi,\varphi) = \psi(\chi,\varphi) - \frac{1}{\beta}D_\beta\psi(\chi,\varphi), \quad \beta > 1.$$

More precisely, using the hyperbolic function $\psi = e^{-\sinh^2\frac{\chi}{2}}$, we dilate it through earlier specified conic projection and obtain

$$D_\beta\psi = \frac{1}{\beta}e^{-\frac{1}{\beta^2}\sinh^2\frac{\chi}{2}}, \tag{6.40}$$

we get:

$$f^\beta_\psi(\chi,\varphi) = e^{-\sinh^2\frac{\chi}{2}} - \frac{1}{\beta^2}e^{-\frac{1}{\beta^2}\sinh^2\frac{\chi}{2}}. \tag{6.41}$$

Now, applying a dilation operator on (6.41) we get

$$D_a f^\beta = \frac{1}{a} e^{-\frac{1}{a^2}\sinh^2\frac{\chi}{2}} - \frac{1}{a\beta^2} e^{-\frac{1}{a^2\beta^2}\sinh^2\frac{\chi}{2}}. \tag{6.42}$$

One particular example of *hyperbolic DOG wavelet* at $\beta = 2$ is:

$$f_\psi^2(\chi, \varphi) = \frac{1}{a} e^{-\frac{1}{a^2}\sinh^2\frac{\chi}{2}} - \frac{1}{4a} e^{-\frac{1}{4a^2}\sinh^2\frac{\chi}{2}}.$$

The resulting hyperbolic DOG wavelet is shown for different values of the scale a and the position (χ, φ) on the hyperboloid.

7 Euclidean limit

Since the hyperboloid is locally flat, the associated wavelet transform should match the usual 2-D CWT in the plane at small scales, i. e, for large scales of radius of curvature. In this section we give a precise mathematical meaning to those notions.

Let $\mathcal{H}_\rho \equiv L^2(H_\rho^2, \mathrm{d}\mu_\rho)$ be the Hilbert space of square integrable functions on an hyperboloid of radius ρ,

$$\int_{H_\rho^2} |f(\chi, \varphi)|^2 \rho^2 \sinh\chi \mathrm{d}\chi \mathrm{d}\varphi < \infty, \tag{7.1}$$

and $\mathcal{H} = L^2(\mathbb{R}^2, \mathrm{d}^2\vec{x})$ be the square integrable functions on the plane.

We write the function $\mathcal{E}_{\nu,\xi}(x)$ for any ρ:

$$\mathcal{E}_{\nu,\xi}^\rho(x) = \left(\frac{x_0 - \hat{n}\vec{x}}{\rho}\right)^{-\frac{1}{2} - i\nu\rho}, \tag{7.2}$$

for $x \in H_+^2$, $(x^2 = \rho^2)$. The Inönü-Wigner contraction limit of the Lorentz to the Euclidean group $SO(2,1)_+ \to ISO(2)_+$ is the limit at $\rho \to \infty$ for (7.2) with $x_0 \approx \rho$, $\vec{x}^2 \ll \rho^2$, i.e

$$\begin{aligned} \lim_{\rho\to\infty} \mathcal{E}_{\nu,\xi}^\rho(x) &= \lim_{\rho\to\infty} \left(\frac{x_0 - \hat{n}\vec{x}}{\rho}\right)^{-\frac{1}{2} - i\nu\rho} && (7.3) \\ &\approx \lim_{\rho\to\infty} \left(1 - \frac{\hat{n}\vec{x}}{\rho}\right)^{-i\nu\rho} = \exp(i\nu\hat{n}\vec{x}). && (7.4) \end{aligned}$$

Having the Fourier-Helgason transform on the hyperboloid at radius ρ

$$\hat{\psi}^\rho(\nu, \xi) = \frac{\rho}{2\pi} \int_{\vec{x}} \psi(\vec{x}) \mathcal{E}_{\nu,\xi}(\vec{x}) \frac{\mathrm{d}^2\vec{x}}{x_0} \tag{7.5}$$

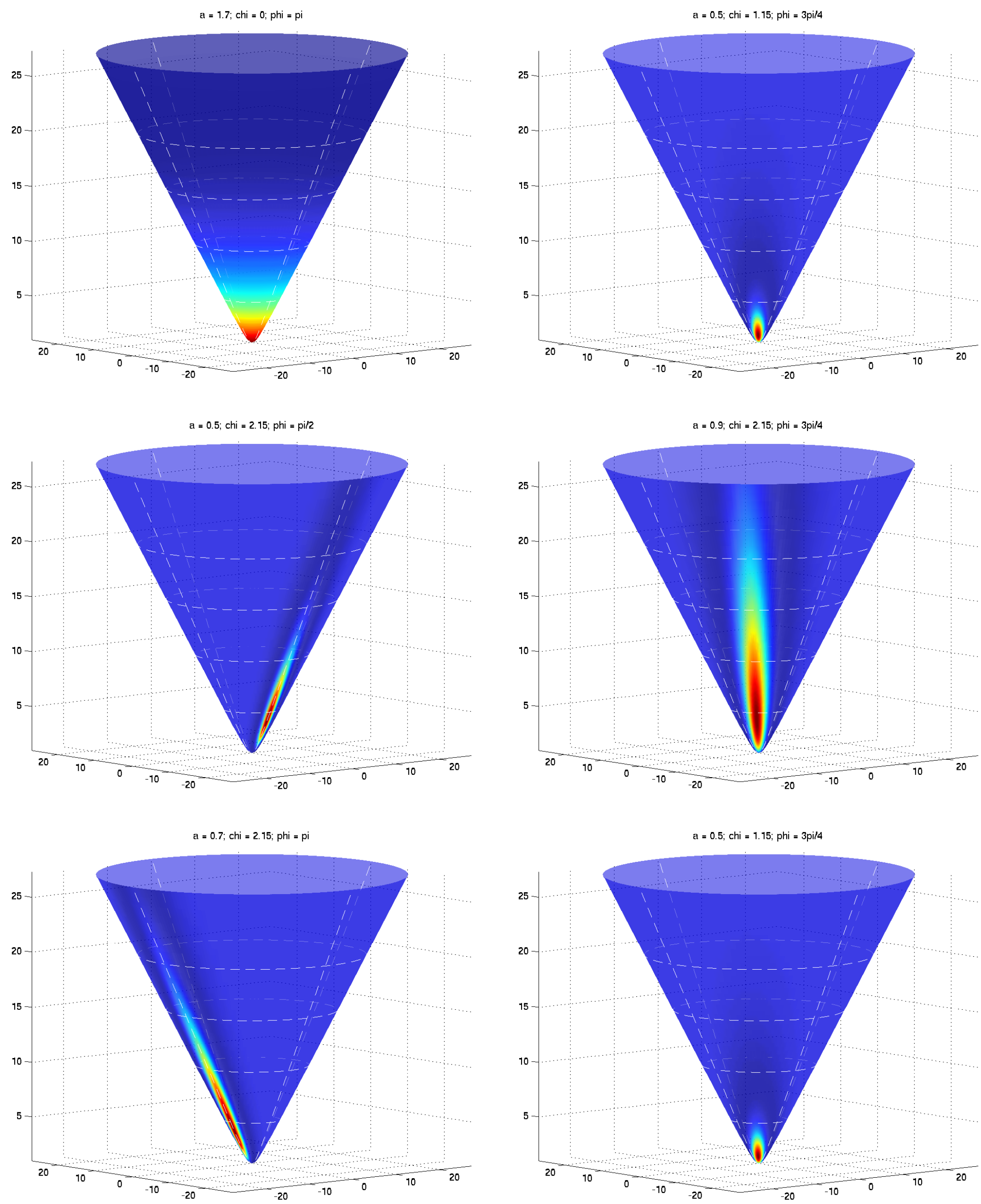

Figure 8: The hyperbolic DOG wavelet f_ψ^β, for $\beta = 2$ at different scales a and positions (χ, φ).

and since $x_0 \approx \rho$ for $\rho \to \infty$, we obtain

$$\lim_{\rho\to\infty} \hat{\psi}^{\rho}(\nu, \xi) = \frac{1}{2\pi}\int_{\vec{x}} \psi(\vec{x}) \exp\left(i\nu\hat{n}\vec{x}\right)\mathrm{d}^2\vec{x} \tag{7.6}$$

$$= \hat{\psi}(\vec{k}), \tag{7.7}$$

which is the Fourier transform in the plane.

Thus, the admissibility condition of the hyperbolic wavelet for radius $\rho \to \infty$ is

$$\lim_{\rho\to\infty} \int_{\mathbb{R}_+^*} \frac{|\hat{\psi}_a^{\rho}(\nu)|^2}{a^2}\mathrm{d}a \to \int_{\mathbb{R}^2} \frac{|\hat{\psi}(\vec{k})|^2}{|\vec{k}|^2}\mathrm{d}\vec{k}, \tag{7.8}$$

and it contracts to the admissibility condition of 2-D CWT.

Consequently, the necessary condition of the hyperbolic wavelet contracts to the one in the plane:

$$\lim_{\rho\to\infty} \int_{H^2} \psi^{\rho}(\chi, \varphi)\mathrm{d}\mu(\chi, \varphi) \to \int_{\mathbb{R}^2} \psi(\vec{x})\mathrm{d}^2\vec{x}. \tag{7.9}$$

8 Conclusions

In this paper we have presented a constructive theory of continuous wavelet transform on the hyperboloid $H_+^2 \in \mathbb{R}_+^3$. First we start by deffining the affine transformations on the hyperboloid and proposing different scheams for dilating a point from which we choose the dilation through conic projection. Then we introduce the notion of convolution on this manifold. Having defined dilations and motions together with the hyperbolic convolution we construct the continuous wavelet transform and the corresponding admissibility condition is derived. An example of hyperbolic DOG wavelet is given. Finally, we use the Inönü-Wigner contraction limit of the Lorentz to the Euclidean group $SO_0(2,1)_+ \to ISO(2)_+$ to consider the contraction of the CWT on the hyperboloid to the one on the plane.

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