Redundancy in Non-Orthogonal Transforms

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Abstract — **Compression efficiency is mainly driven by redundancy of the overcomplete set of functions chosen for non-orthogonal signal decompositions. Redundancy is an important criteria in the design of dictionaries, whose size only provides a first indication without however taking into account the distribution of the atoms. This paper provides a new formulation for the structural redundancy of an overcomplete set of functions. The structural redundancy factor directly drives the energy compaction properties of non-orthogonal transforms in frame expansion [1] or Matching Pursuit [2].**

I. Structural Redundancy

Signal transforms are generally based on inner products to compute the contribution of each basis function or atoms into the signal reconstruction. Hence, the structural redundancy β can be interpreted as the cosine of the maximum possible angle between any direction in $\mathcal{H}$ and the closest direction of any atom of the dictionary [2]. It characterizes the redundancy of the dictionary and tends to one when the size S of the overcomplete dictionary increases.

For each dictionary vector $g_{\gamma_i}, i \in [1..S]$, one can define its projection neighborhood as the subspace of $\mathcal{H}$ whose any direction has g_{γ_i} as closest direction among the dictionary vectors.

Definition 1 *The projection neighborhood $\mathcal{V}_{\gamma_i}$ of the vector g_{γ_i} is the subspace of $\mathcal{H}$ defined by*

$$\mathcal{V}_{\gamma_i} = \{x \in \mathcal{H} \mid |\langle x|g_{\gamma_i}\rangle| \geq |\langle x|g_{\gamma_j}\rangle|, \forall j \neq i\} . \quad (1)$$

The projection neighborhood $\mathcal{V}_{\gamma_i}$, as represented in Fig. 1, corresponds to the intersection of couples of infinite convex polyhedral cones situated symmetrically with their apexes at the origin.

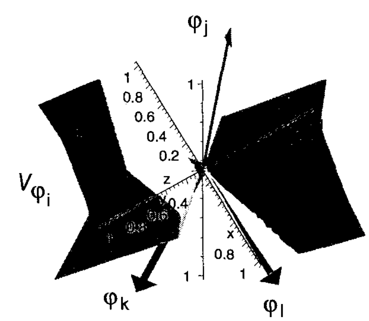

Figure 1: Representation of the projection neighborhood $\mathcal{V}_{\gamma_i}$ in $\mathbb{R}^3$.

The structural redundancy β thus corresponds to the cosine of the maximum possible angle, over all dictionary vectors, between g_{γ_i} and any direction in its projection neighborhood $\mathcal{V}_{\gamma_i}$. It can be written as

$$\beta = \min_i \inf_{x \in \mathcal{V}_{\gamma_i}} \langle x \mid g_{\gamma_i} \rangle . \quad (2)$$

II. Application in Matching Pursuit

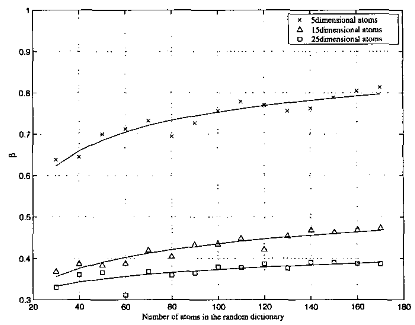

Figure 2: Structural redundancy factor β versus S.

Fig. 2 represents the evolution of the redundancy factor β with the size of random atom dictionaries. We can see that the structural redundancy factor obviously increases with the size of the dictionary. We can conjecture that the evolution of β is exponential with the number of vectors N_v. In other words, $\beta = 1 - A\ N_v^B$.

The approximation error decay rate in Matching Pursuit has been shown to be bounded by an exponential [2]. In other words, the norm of the residue converges exponentially to zero when the iteration number N tends to infinity. Fig. 3 shows that the residual energy is clearly upper-bounded by the exponential curve computed from the structural redundancy factor β.

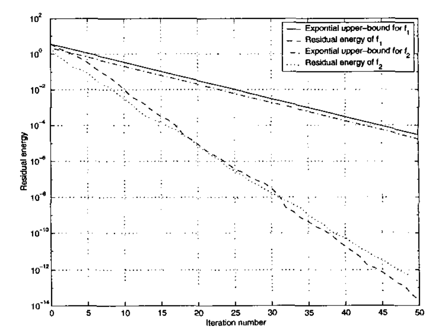

Figure 3: Matching Pursuit on random signals (f_1, f_2) of length 10 ($S = 50$).

References

[1] Goyal V.K., Vetterli M. and Thao N.T., "Quantized Overcomplete Expansions in R^N: Analysis, Synthesis and Algorithms," *IEEE Transactions on Information Theory*, vol. 44, no. 1, pp. 16–31, January 1998.

[2] Mallat S.G. and Zhang Z., "Matching Pursuits With Time-Frequency Dictionaries," *IEEE Transactions on Signal Processing*, vol. 41, no. 12, pp. 3397–3415, December 1993.

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