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Wavelets on the 2-Sphere: A Group-Theoretical Approach

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We present a purely group-theoretical derivation of the continuous wavelet transform (CWT) on the 2-sphere S^2, based on the construction of general coherent states associated to square integrable group representations. The parameter space X of our CWT is the product of $SO(3)$ for motions and $\mathbb{R}_*^+$ for dilations on S^2, which are embedded into the Lorentz group $SO_0(3,1)$ via the Iwasawa decomposition, so that $X \simeq SO_0(3,1)/N$, where $N \simeq \mathbb{C}$. We select an appropriate unitary representation of $SO_0(3,1)$ acting in the space $L^2(S^2, d\mu)$ of finite energy signals on S^2. This representation is square integrable over X; thus it yields immediately the wavelets on S^2 and the associated CWT. We find a necessary condition for the admissibility of a wavelet, in the form of a zero mean condition. Finally, the Euclidean limit of this CWT on S^2 is obtained by redoing the construction on a sphere of radius R and performing a group contraction for $R \to \infty$. Then the parameter space goes into the similitude group of $\mathbb{R}^2$ and one recovers exactly the CWT on the plane, including the usual zero mean necessary condition for admissibility.

1. INTRODUCTION

Analyzing data with the continuous wavelet transform (CWT) is by now a well-established procedure. The most common cases are data on the line (signal processing), on the plane (image analysis), or occasionally in $\mathbb{R}^3$ (e.g., in fluid dynamics)—see [5] for a survey of applications in physics. However, there are various instances where data are given on a *sphere*. Geophysical data are the prime example, but others occur in statistical problems, computer vision, or medical imaging. The problem is to adapt the method of analysis to spherical data. Of course, this is not specific to wavelet analysis, but shows up in all methods, mostly based on Fourier techniques, and it is in general a nontrivial task

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from the numerical point of view (see [19] for a list and precise references). So the question arises, how does one extend the CWT to the sphere or a manifold?

Let us first make that statement precise. In order to obtain a genuine CWT on S^2, the following three requirements should be satisfied [24]:

- the signals and the wavelets must live *on* the sphere;
- the transform must involve (local) dilations of some kind; and
- possibly the CWT on S^2 should reduce locally to the usual CWT on the (tangent) plane (Euclidean limit).

The problem has attracted a lot of interest in the last few years and many proposals have been made, but, in our opinion, none of them is fully satisfactory. In fact they all suffer from the same defect. The general belief is that, the sphere being compact, dilations cannot be defined on it. This is, we think, a false problem, yet it pervades all the attempted solutions. To quote a few:

- A number of works extend to S^2 the discrete wavelet scheme based on a multiresolution analysis, often with numerical purposes in mind, using adapted interpolation methods and spline functions [6, 16, 30] or second generation wavelets [35, 36]. The former approach usually leads to numerical difficulties around the poles.
- Others exploit the geometry of the sphere, as encoded in the system of spherical harmonics [14, 15, 28, 31], but as a result, their analyzing functions are poorly localized. In fact they do not really resemble wavelets.
- In order to avoid the dilation problem described above, one may define a WT on the tangent bundle of the sphere [7] or instead a Gabor transform on the sphere itself [38] (since no dilation is then involved). Also Calderón's reproducing formula may be extended to S^2 [32, 33], but this does not yield an explicit CWT.
- The most satisfactory approach is that of Holschneider [20], who produces a CWT on S^2 that satisfies the three criteria above, but is based on introducing an abstract parameter that plays the role of dilation and fulfills a number of *ad hoc* assumptions; in particular, the correct Euclidean limit is obtained, but essentially put by hand.

As can be seen from this brief description, none of the proposed solutions fully qualifies for a genuine CWT on S^2. On the contrary, the construction presented here fulfills all three requirements, but, in addition:

(i) the construction is *entirely* derived from group theory, following the formalism of general coherent states developed in [1];

(ii) the Euclidean limit has also a precise group-theoretical formulation, in terms of *group contraction* [21, 34]. In a sense, we are able to derive all the assumptions of [20] from the general formalism.

Actually our construction extends in a straightforward way to higher dimensional spheres S^n and, to some extent, to other manifolds, such as a two-sheeted hyperboloid. We will give here only some brief indications on these extensions; a full treatment will be presented elsewhere [4, 39]. The paper is organized as follows. In Section 2 we identify the group of affine transformations of S^2, namely the Lorentz group $SO_0(3,1)$. In Section 3, we derive the CWT on S^2 from an appropriate unitary representation of $SO_0(3,1)$, using the coherent state machinery of [1], which is briefly described in Appendix A. Finally Section 4 is devoted to the Euclidean limit, using the theory of contraction of group representations developed by Dooley [10, 11]. Appendixes B and C collect some explicit formulas on $SO_0(3,1)$ and present a brief survey of the contraction method, with application to $SO_0(3,1) \to SIM(2)$.

2. AFFINE TRANSFORMATIONS ON THE SPHERE S^2

The usual CWT on the line is derived from the natural unitary representation of the $ax+b$ group in the space of finite energy signals $L^2(\mathbb{R}, dx)$ [17, 18]. Similarly, in two dimensions, one starts from a unitary representation of the similitude group of the plane (translations, rotations, dilations) in the space $L^2(\mathbb{R}^2, d^2x)$ [3, 27]. Note that in each case, the group acts transitively on the basis manifold, $\mathbb{R}$ or $\mathbb{R}^2$. The same scheme applies to the CWT on a general manifold, subject to the transitive action of some group of transformations that contains dilations (we present in Appendix A an outline of the general case).

Let us apply this method to the sphere S^2 and consider the space of finite energy signals $\mathcal{H} = L^2(S^2, d\mu)$, where $d\mu(\omega) = \sin\theta \, d\theta \, d\varphi$ is the usual (rotation invariant) measure on S^2. The first step for constructing a CWT on S^2 is to identify the appropriate transformations. These are of two types, displacements, also called *motions*, and dilations:

(i) Motions are given by elements of the rotation group $SO(3)$, which indeed acts transitively on S^2, and $S^2 \simeq SO(3)/SO(2)$.

(ii) Dilations may be derived in two steps:

- dilations around the North Pole are obtained by considering usual dilations in the tangent plane at the North Pole and lifting them to S^2 by inverse stereographic projection from the South Pole;
- a dilation around any other point $\omega \in S^2$ is obtained by moving ω to the North Pole by a rotation $\gamma \in SO(3)$, performing a dilation D_N as before and going back by the inverse rotation:

$$D_\omega = \gamma^{-1} D_N \gamma. \tag{2.1}$$

Clearly the dilations act also transitively on S^2.

For future use, we note that the stereographic projection $\Phi\colon S^2 \to \mathbb{C}$ (here S^2 is taken as the Riemann sphere and the tangent plane at the North Pole as the complex plane $\mathbb{C}$) is a bijection given by

$$\Phi(\omega) = \zeta = 2\tan\frac{\theta}{2}\, e^{i\varphi}, \qquad \omega \equiv (\theta, \varphi); \qquad 0 \le \theta \le \pi,\ 0 \le \varphi < 2\pi.$$

In particular, let $D_a^{\mathbb{C}}\colon \zeta \mapsto a\,\zeta$ be a usual dilation in the tangent plane $\mathbb{C}$. Then it is readily lifted back to S^2, as

$$D_a^{S^2}\colon \omega \mapsto \left(\Phi^{-1} \cdot D_a^{\mathbb{C}} \cdot \Phi\right)(\omega). \tag{2.2}$$

In polar spherical coordinates $\omega = (\theta, \varphi)$, (2.2) reads

$$D_a^{S^2}(\theta, \varphi) = (\theta_a, \varphi), \qquad \text{with } \tan\frac{\theta_a}{2} = a \cdot \tan\frac{\theta}{2}.$$

The next step is to identify a group of affine transformations on S^2. First we note that motions $\gamma \in SO(3)$ and dilations by $a \in \mathbb{R}_*^+$ do not commute. Also it is impossible to build a semidirect product $SO(3) \rtimes \mathbb{R}_*^+$, since $SO(3)$ has no outer automorphisms, and therefore the only extension of $SO(3)$ by $\mathbb{R}_*^+$ is their *direct product*. In order to find a way out, we go back to the tangent plane by the stereographic projection Φ.

- The rotation group $SO(3)$ is mapped onto $SU(2)$, with homographic action on $\mathbb{C}$:

$$\zeta \mapsto \frac{a\zeta + b}{c\zeta + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2). \tag{2.3}$$

Actually the action of $SU(2)$ on $\mathbb{C}$ is not simply transitive, only that of $SU(2)/\mathbb{Z}_2 \simeq SO(3)$ is.

- In the same realization, a dilation $\zeta \mapsto a\zeta,\ a > 0$ is represented by the diagonal matrix diag $(a^{1/2}, a^{-1/2})$.
- Combining the two types of transformations, we obtain the full group $SL(2,\mathbb{C})$, and a simply transitive action on $\mathbb{C}$ by $SL(2,\mathbb{C})/\mathbb{Z}_2 \simeq SO_0(3,1)$, the Lorentz group.

We also note that $SL(2,\mathbb{C})$ is the complexification of $SU(2)$, whereas $SO_0(3,1)$ is the conformal group of the tangent plane $\mathbb{R}^2$, and that of S^2 as well [12].

These geometric considerations may be recast in a purely group-theoretical language, by considering the *Iwasawa decomposition* of the Lorentz group $SO_0(3,1)$. Indeed, like any connected semisimple Lie group, the latter admits a decomposition into three closed subgroups, namely $G = KAN$, where K is a maximal compact subgroup, A is Abelian and N nilpotent, and both are simply connected [22]. In the case of $SO_0(3,1)$, we get:

- $K \sim SO(3)$ is the maximal compact subgroup.
- $A \sim SO_0(1,1) \sim \mathbb{R}_*^+ \sim \mathbb{R}$ is the subgroup of Lorentz boosts in the z direction.
- $N \sim \mathbb{C}$ is two-dimensional and Abelian (for a general semisimple group, N is nilpotent).

In the case of $SL(2,\mathbb{C})$, $K = SU(2)$ and N corresponds to translations $\zeta \mapsto \zeta + b$ of the plane, represented by matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad b \in \mathbb{C}.$$

Thus we get, respectively [37],

$$SO_0(3,1) = SO(3) \cdot \mathbb{R}_*^+ \cdot \mathbb{C}, \qquad SL(2,\mathbb{C}) = SU(2) \cdot \mathbb{R}_*^+ \cdot \mathbb{C}. \tag{2.4}$$

Now let again $G = KAN$ be the Iwasawa decomposition of a connected semisimple Lie group G, with finite center. Let M be the centralizer of A in K; that is, $M = \{\gamma \in K : \gamma a = a\gamma,\ \forall a \in A\}$. Then $P = MAN$ is a closed subgroup of G, called the *minimal parabolic subgroup*. In the case of $SO_0(3,1)$ and $SL(2,\mathbb{C})$, we have $M = SO(2)$ or $U(1)$, respectively, corresponding to the subgroup of rotations around the x_3 axis. The subgroup P is not invariant, but it is the stability subgroup of the North Pole, and the quotient G/P, which is isomorphic to K/M, is simply

$$S^2 \simeq SO_0(3,1)/P \simeq SO(3)/SO(2) \qquad \text{and} \qquad S^2 \simeq SL(2,\mathbb{C})/P \simeq SU(2)/U(1). \tag{2.5}$$

This shows that both $SO_0(3,1)$ and $SL(2,\mathbb{C})$ act transitively on S^2. We have previously identified K with Euclidean motions on S^2 and A with dilations, which constitute our basic operations. Thus the parameter space of our theory is the homogeneous space

$$X \equiv SO_0(3,1)/N \simeq SO(3)\cdot A.$$

In order to compute explicitly the action of $SL(2,\mathbb{C})$ on S^2, we use the Iwasawa decomposition of a generic element $g = \gamma a n$, which reads

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \cdot \begin{pmatrix} \delta^{-1/2} & 0 \\ 0 & \delta^{1/2} \end{pmatrix} \cdot \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}, \tag{2.6}$$

where $\alpha, \beta, \zeta \in \mathbb{C}$ and $\delta \in \mathbb{R}^+_*$ and $ad - bc = 1$. Solving these equations, we obtain

$$\delta = (|a|^2 + |c|^2)^{-1}, \qquad \alpha = a\,\delta^{1/2}, \qquad \beta = -\overline{c}\,\delta^{1/2}, \qquad \zeta = a^{-1}(b + \overline{c}\,\delta). \tag{2.7}$$

Let us introduce the Euler parametrization for the elements of $SU(2)$;

$$\alpha = \cos\frac{\theta}{2}\cdot \exp -i\left(\frac{\varphi+\psi}{2}\right),$$
$$\beta = -i\sin\frac{\theta}{2}\cdot \exp i\left(\frac{\psi-\varphi}{2}\right),$$

and the Euler decomposition

$$\gamma = m(\psi)u(\theta)m(\varphi), \tag{2.8}$$

where u and m stand respectively for

$$m(\varphi) = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}, \qquad u(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{2.9}$$

For $g_0 \in SL(2,\mathbb{C})$, we get by the same decomposition

$$g_0 g = \gamma_{g_0}\, a_{g_0}\, n_{g_0}.$$

This induces a homeomorphism on the quotient. Indeed the map $\gamma \mapsto \gamma_{g_0}$ does not depend on γ, but rather on its equivalence class in $K/M \simeq S^2$. In view of the Cartan KAK decomposition [22] of $SO_0(3,1)$, it is enough to show this for elements $g_0 \equiv g_0(a) \in A$, since the action of K is trivial. Thus, for a pure dilation by a, represented by the left action $D_a\colon \gamma \mapsto a^{-1}\gamma$, we write:

$$g_0 \equiv g_0(a) = \mathrm{diag}(a^{-1/2}, a^{1/2})$$

and express $\gamma = \gamma(\psi,\theta,\varphi) \in SU(2)$ by its decomposition (2.8). By definition of $M \sim U(1)$, $g_0(a)$ commutes with $m(\varphi)$, so that

$$g_0\gamma = m(\psi)g_0 u(\theta)m(\varphi).$$

Let us compute the central product and then write the Iwasawa decomposition (2.6) of the result:

$$g_0 u(\theta) = \begin{pmatrix} a^{-1/2} & 0 \\ 0 & a^{1/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} a^{-1/2}\cos\frac{\theta}{2} & -ia^{-1/2}\sin\frac{\theta}{2} \\ -ia^{1/2}\sin\frac{\theta}{2} & a^{1/2}\cos\frac{\theta}{2} \end{pmatrix} = u(\theta_{g_0}) \begin{pmatrix} \delta_{g_0}^{-1/2} & 0 \\ 0 & \delta_{g_0}^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \zeta_{g_0} \\ 0 & 1 \end{pmatrix}.$$

Computing these factors, we find

$$\cos\theta_{g_0} = \frac{(1+a^2)\cos\theta + (1-a^2)}{(1-a^2)\cos\theta + (1+a^2)}, \tag{2.10}$$

from which it is easily seen that

$$\tan\frac{\theta_{g_0}}{2} = a \cdot \tan\frac{\theta}{2}. \tag{2.11}$$

The transformation corresponding to a pure dilation is thus exactly the usual Euclidean dilation lifted on S^2 by inverse stereographic projection. This is precisely what we obtained in (2.2) when looking for a dilating map on the sphere. Figure 1 represents the action of this transformation on a point $\omega \in S^2$. A dilation around an arbitrary point $\omega \in S^2$ is obtained, according to (2.1), by combining the dilation around the North Pole just described with an appropriate rotation. Finally, a rotation $\gamma \in SU(2)$ may be expressed by the product (2.8), acting on $\mathbb{C}$ by the homographic action (2.3), followed by an inverse stereographic projection.

3. THE CONTINUOUS WAVELET TRANSFORM ON S^2

3.1. Principal Series Representations of the Lorentz Group $SO_0(3,1)$

According to our program, as outlined in Appendix A, the next step toward constructing affine coherent states on S^2 is to find a suitable unitary irreducible representation of the Lorentz group $SO_0(3,1)$ in the Hilbert space $L^2(S^2, d\mu)$, where $d\mu(\omega)$ is the normalized $SO(3)$-invariant measure on S^2.

Let us look first at rotations only. A natural unitary representation of $SO(3)$ is the quasi-regular representation U_{qr}, defined by

$$(U_{qr}(\gamma)f)(\omega) = f(\gamma^{-1}\omega), \qquad \gamma \in SO(3),\ f \in L^2(S^2, d\mu). \tag{3.1}$$

The representation U_{qr} of $SO(3)$ is infinite dimensional, and decomposes into the direct sum of all the familiar $(2l+1)$-dimensional representations, $l = 0, 1, 2, \ldots$.

In order to incorporate dilations into this scheme, we have to lift U_{qr} from $SO(3)$ to $SO_0(3,1)$. However, the measure $d\mu$ is not dilation invariant, so that a Radon–Nikodym derivative $\lambda(g,\omega)$ must be inserted, namely

$$\lambda(g,\omega) = \frac{d\mu(g^{-1}\omega)}{d\mu(\omega)}, \qquad g \in SO_0(3,1). \tag{3.2}$$

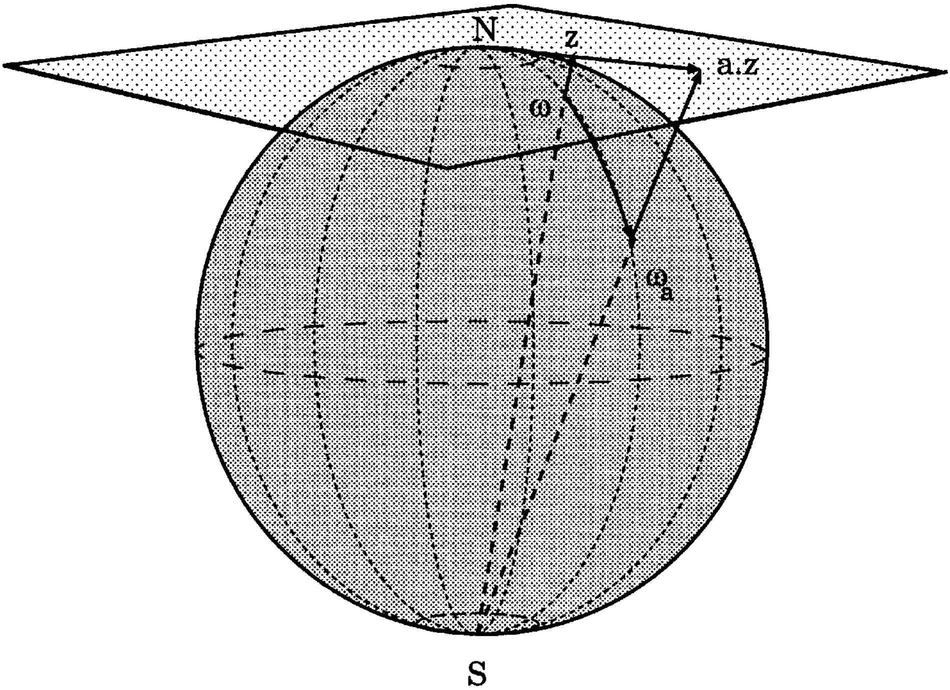

FIG. 1. The action of a pure dilation a around the North Pole of the sphere, lifted from the tangent plane by inverse stereographic projection.

The function λ is a 1-cocycle and satisfies the equation

$$\lambda(g_1 g_2, \omega) = \lambda(g_1, \omega)\lambda(g_2, {g_1}^{-1}\omega). \tag{3.3}$$

Natural candidates are the representations of continuous principal series of $SO_0(3,1)$ [22, 37], which are given by the operators

$$\left[U^s(g)f\right](\omega) = \lambda(g,\omega)^{s/2}\chi(a)f\left(g^{-1}\omega\right),$$
$$g \in SO_0(3,1),\ s \in \mathbb{C},\ f \in L^2(S^2, d\mu), \tag{3.4}$$

where $g = \gamma an$, the Iwasawa decomposition, χ is a character of A, and the multiplier $\lambda(g,\omega)$ is the Radon–Nikodym derivative given in (3.2). In particular, we will choose the subset of class I representations. The latter are induced by unitary irreducible representations of the minimal parabolic subgroup $P = MAN$, which are trivial on M (for these representations, the carrier Hilbert space contains a vector which is invariant under the action of the maximal compact subgroup $SO(3)$). In addition we take the trivial character $\chi(a) \equiv 1$.

The main properties of these principal series representations are summarized by the following theorem [37].

THEOREM 3.1. *The representation*

$$\left[U^s(g)f\right](\omega) = \lambda(g,\omega)^{s/2} f\left(g^{-1}\omega\right)$$

is a strongly continuous representation of $SO_0(3,1)$ *in* $L^2(S^2, d\mu)$. *It is cyclic when* $s \neq 0, -1, -2, \ldots$ *and it is unitary and irreducible if and only if* $\mathrm{Re}\, s = 1$.

In the following, we will always set $s = 1$ and write simply $U \equiv U^1$.

As already mentioned, we will not deal with the full Lorentz group, since we are only interested in the action of dilations and motions. We thus restrict ourselves to the corresponding homogeneous space using a suitable section $\sigma\colon\ X = KAN/N \to KAN$ in the principal fiber bundle defined by the Iwasawa decomposition. Thus we will concentrate on the reduced expression

$$\big[U(\sigma(x)) f\big](\omega) = \lambda(\sigma(x), \omega)^{1/2} f\big(\sigma(x)^{-1}\omega\big). \tag{3.5}$$

We write points of the space X as pairs $x \equiv (\gamma, a)$, with $\gamma \in SO(3)$ and $a \in A \simeq SO(1,1)$, and choose the natural (Iwasawa) section

$$\sigma_I(\gamma, a) = \gamma a. \tag{3.6}$$

We have already computed the action of dilations in (2.10) and, by $SO(3)$ invariance, it is easily seen that, with $\omega = (\theta, \varphi)$,

$$\lambda(\sigma_I(\gamma, a), \omega) \equiv \lambda(a, \theta) = \frac{4a^2}{[(a^2-1)\cos\theta + (a^2+1)]^2}. \tag{3.7}$$

In addition, from the choice of the section (3.6), we have

$$U(\sigma_I(\gamma, a)) = U(\gamma a) = U(\gamma) U(a),$$

and therefore:

PROPOSITION 3.2. *The representation* (3.5) *factorizes as*

$$\big[U(\sigma_I(\gamma, a)) f\big](\omega) = \big(U_{\mathrm{qr}}(\gamma) \circ D^a f\big)(\omega),$$

where $U_{\mathrm{qr}}(\gamma)$, $\gamma \in SO(3)$ *is the quasi-regular representation of* $SO(3)$ *in* $L^2(S^2, d\mu)$, *and* D^a, $a \in \mathbb{R}_*^+$, *is a pure dilation, that is*

$$\big(D^a f\big)(\omega) = \lambda(a, \theta)^{1/2} f(\omega_{1/a}), \qquad \textit{with } \omega_a \equiv (\theta_a, \varphi). \tag{3.8}$$

3.2. Lorentz Coherent States as Wavelets on the Sphere S^2

Following the general approach of [1], we will build in this section a system of coherent states for the Lorentz group, indexed by points of the homogeneous space $X = SO_0(3,1)/N$. Since N is not the isotropy subgroup of a particular vector in the representation Hilbert space, the resulting coherent states are not of the Gilmore–Perelomov type [29].

The CS system associated to the representation U is defined by

$$\eta_{\sigma(x)}(\omega) = \big[U(\sigma(x))\eta\big](\omega), \tag{3.9}$$

with $\eta \in L^2(S^2, d\mu)$ and $x \in X$, that is, the elements of the orbit of η under G, modulo the section σ. Then, according to the general theory [1], sketched in Appendix A, the system (3.9) is a (over)complete family in $L^2(S^2, d\mu)$ if the representation U is square integrable modulo the section σ and the subgroup N. This means that there exists a nonzero vector $\eta \in L^2(S^2, d\mu)$, called admissible, such that

$$\int_X d\nu(x) |\langle U(\sigma(x))\eta|\phi\rangle|^2 < \infty, \qquad \forall \phi \in L^2(S^2, d\mu).$$

In this expression, ν is an $SO_0(3,1)$-invariant measure on X, and the inner product in the integrand is taken in $L^2(S^2, d\mu)$. An explicit calculation (see Appendix B) yields, for the section σ_I,

$$d\nu(\gamma, a) = \frac{d\mu(\gamma)da}{a^3}, \tag{3.10}$$

where $d\mu(\gamma)$ is the invariant (Haar) measure on $SO(3)$.

The result is given by the following theorem.

THEOREM 3.3. *The representation U given in* (3.5) *is square integrable modulo the subgroup N and the section σ_I; that is, the representation space $L^2(S^2, d\mu)$ contains a nonzero vector η admissible* $\mathrm{mod}(N, \sigma_I)$, *which means that there exists a constant $c > 0$, independent of l, such that*

$$\frac{8\pi^2}{2l+1} \sum_{|m|\le l} \int_0^\infty \frac{da}{a^3} |\widehat{\eta_a}(l,m)|^2 < c, \tag{3.11}$$

where $\widehat{\eta}(l,m) = \langle Y_l^m|\eta\rangle$ stands for the Fourier coefficient of η and

$$\eta_a(\omega) = \big[U(\sigma_I(e,a))\eta\big](\omega) \equiv (D^a\eta)(\omega) = \lambda(a,\theta)^{1/2}\eta(\omega_{1/a}). \tag{3.12}$$

Proof. We have to check that, for any $\phi \in L^2(S^2, d\mu)$,

$$I \equiv \int_X d\nu(x) \big|\langle U(\sigma_I(x))\eta\big|\phi\rangle|^2 < \infty. \tag{3.13}$$

By Proposition 3.2, this simplifies into

$$\int_0^\infty \frac{da}{a^3} \int_{SO(3)} d\mu(\gamma) |\langle U_{\mathrm{qr}}(\gamma) \circ D^a\eta|\phi\rangle|^2 < \infty.$$

Introducing Wigner $\mathcal{D}$ functions by the relation

$$[\widehat{U_{\mathrm{qr}}(\gamma)\psi}](l,m) = \sum_{|n|\le l} \mathcal{D}^l_{mn}(\gamma)\widehat{\psi}(l,n),$$

and using Parseval's equation for the Fourier series on $SO(3)$, we have

$$I = \int_0^\infty \frac{da}{a^3} \int_{SO(3)} d\mu(\gamma) \sum_{l,l'} \sum_{m,m'} \sum_{n,n'} \overline{\mathcal{D}^l_{mn}(\gamma)} \cdot \mathcal{D}^{l'}_{m'n'}(\gamma)$$
$$\times \overline{\widehat{\eta_a}(l,n)} \cdot \widehat{\eta_a}(l',n') \cdot \widehat{\phi}(l,m) \cdot \overline{\widehat{\phi}(l',m')}$$

$$= \int_0^\infty \frac{da}{a^3} \Big\{ \sum_{\text{all indices}} \overline{\widehat{\eta_a}(l,n)} \cdot \widehat{\eta_a}(l',n') \cdot \widehat{\phi}(l,m) \cdot \overline{\widehat{\phi}(l',m')}$$
$$\times \int_{SO(3)} d\mu(\gamma) \overline{\mathcal{D}^l_{mn}(\gamma)} \cdot \mathcal{D}^{l'}_{m'n'}(\gamma) \Big\}.$$

Using the orthogonality relations for the Wigner functions in the last integral, we end up with

$$I = \sum_{l=0}^{\infty} \frac{8\pi^2}{2l+1} \sum_{|m|\le l} |\widehat{\phi}(l,m)|^2 \sum_{|n|\le l} \int_0^\infty \frac{da}{a^3} |\widehat{\eta_a}(l,n)|^2.$$

Now, putting

$$S_l = \sum_{|m|\le l} |\widehat{\phi}(l,m)|^2, \qquad G_l = \frac{8\pi^2}{2l+1} \sum_{|n|\le l} \int_0^\infty \frac{da}{a^3} |\widehat{\eta_a}(l,n)|^2, \tag{3.14}$$

we see that $(S_l) \in l^1(\mathbb{N})$, since $\sum_l |S_l| = \|\phi\|^2$. Then the admissibility condition is rephrased as

$$\sum_l S_l G_l < \infty, \qquad \forall S_l \in l^1(\mathbb{N}),$$

which converges absolutely if and only if $(G_l) \in l^\infty(\mathbb{N})$ [23]. Finally, η is admissible if and only if

$$\frac{8\pi^2}{2l+1} \sum_{|n|\le l} \int_0^\infty \frac{da}{a^3} |\widehat{\eta_a}(l,n)|^2 < c, \qquad \text{with } c \text{ independent of } l.$$

Clearly there are many functions $\eta \in L^2(S^2, d\mu)$ that satisfy this condition. In fact, the functions satisfying it form a dense set in $L^2(S^2, d\mu)$. ■

Notice that, once η is admissible, (3.13) may be written as

$$I = \int_X d\nu(x) |\langle U(\sigma_I(x))\eta|\phi\rangle|^2 \le c\|\phi\|^2. \tag{3.15}$$

This means that the family $\{\eta_{\sigma_I(x)},\ x \in X\}$ is a continuous family of CS, but in fact we have more:

PROPOSITION 3.4. *For any admissible vector η such that $\int_0^{2\pi} d\varphi\, \eta(\theta,\varphi) \neq 0$ (for instance, if $\eta \neq 0$ is axisymmetric), the family $\{\eta_{\sigma_I(x)},\ x \in X\}$ is a continuous frame; that is, there exist constants $A > 0$ and $B < \infty$ such that*

$$A\|\phi\|^2 \le \int_X d\nu(x) |\langle \eta_{\sigma_I(x)}|\phi\rangle|^2 \le B\|\phi\|^2, \qquad \forall \phi \in L^2(S^2, d\mu). \tag{3.16}$$

We begin by an easy lemma.

LEMMA 3.5. (1) *The correspondence $\Theta\colon L^2(S^2, d\mu) \to L^2(\mathbb{R}^2, r^{-1} dr\, d\varphi)$ defined by*

$$\Theta\colon f(\theta,\varphi) \mapsto \frac{2r}{1+r^2} f\left(\arccos \frac{1-r^2}{1+r^2}, \varphi\right) \tag{3.17}$$

is an isometry (and in fact a unitary map).

(2) *Let* $D_a^+ \colon\ f(r,\varphi) \mapsto f(a^{-1}r,\varphi)$ *denote the usual dilation on* $\mathbb{R}^2$*, and* D^a *the spherical dilation* (3.8). *Then one has the intertwining relation:* $\Theta D^a = D_a^+ \Theta$.

Both statements follow immediately from an explicit calculation. One may notice that Θ coincides with the unitary map $I_{1/2}$ defined in (4.3) below.

Proof of Proposition 3.4. It remains only to prove the lower bound. We start from the quantity G_l defined in (3.14), which is clearly nonnegative. We claim that $G_l > 0$ for all l. Assume indeed that $G_{l_0} = 0$ for some l_0. This is possible only if $\widehat{\eta_a}(l_0,n) = 0$ for all a and all $n = -l_0, \dots, l_0$. Let us rewrite this Fourier coefficient, with the help of Lemma 3.5:

$$\widehat{\eta_a}(l_0,n) = \langle Y_{l_0}^n | D^a \eta\rangle_{L^2(S^2)} = \langle \Theta Y_{l_0}^n | \Theta D^a \eta\rangle_{L^2(\mathbb{R}^2)} = \langle \Theta Y_{l_0}^n | D_a^+ \Theta\eta\rangle_{L^2(\mathbb{R}^2)}$$
$$= \int_0^{2\pi} d\varphi \int_0^\infty \frac{dr}{r} \overline{[\Theta Y_{l_0}^n](r,\varphi)} [\Theta\eta]\left(\frac{r}{a},\varphi\right).$$

Since $[\Theta Y_{l_0}^n](r,\varphi) \propto [\Theta P_{l_0}^n](r)\, e^{in\varphi}$, one gets

$$0 = \widehat{\eta_a}(l_0,n) = \int_0^\infty \frac{dr}{r}\, \overline{[\Theta P_{l_0}^n](r)}\, [\Theta\eta^{(n)}]\left(\frac{r}{a}\right),$$

where the last factor is the nth Fourier coefficient of $\Theta\eta$. Since this integral is a convolution in $L^2(\mathbb{R}_*^+, r^{-1}dr)$, it vanishes for every a only if one of the functions is identically zero, which implies $\Theta\eta^{(n)} = 0$ for all $n = -l_0, \dots, l_0$. For $n = 0$, this implies $\eta \equiv 0$, by assumption. Therefore $G_l > 0$ for every l.

In fact, G_l is not only positive, but bounded from below. To see that, we fix $L \gg 1$. Then, on one hand, we have $c_L = \min_{l \leq L} G_l > 0$. On the other hand, for $l > L$, we may write

$$\widehat{\eta_a}(l,n) = \langle Y_l^n | D^a \eta\rangle$$
$$= \langle D^{1/a} Y_l^n | \eta\rangle$$
$$= \int d\mu\, \lambda(a^{-1},\theta)^{1/2}\, Y_l^n(\theta_a,\varphi)\, \eta(\theta,\varphi).$$

Now, because of the limit

$$\lim_{l\to\infty} \sqrt{\frac{4\pi}{2l+1}}\, Y_l^m\left(\frac{\theta}{l},\varphi\right) = J_m(\varphi)\, e^{im\varphi},$$

it follows that, for $l > L \gg 1$, the only contribution to the integral over θ comes from the region $a \sim 1/l \ll 1$ (here we use the fact that, for $a \ll 1$, one has $\theta_a \sim a\theta$), so that we get a lower bound for $G_l, l > L$, by taking the integral over a from 0 to $1/L$ and replacing the spherical harmonic Y_l^n by its limiting value. As a result, the dependence on l disappears and one gets $G_l > c(\eta)$, independently of l.

Combining these two results yields the lower bound in the frame condition (3.16). ■

Thus, for most admissible vectors η, we get a continuous frame, but not necessarily a tight frame, in the terminology of [1] (see Appendix A). To get a tight frame would require an equality in (3.15), with $c = c(\eta)$ the analogue of the admissibility constant of the vector η. Equivalently, we would need $G_l = G(\eta)$, independently of l. We conjecture this is in fact *not* true; that is, the frame operator A_σ of Appendix A has spectrum in a nontrivial interval. However, if the same analysis is redone on a sphere of radius R, we expect that this interval will contract to a point in the limit $R \to \infty$. Indeed, as we will see in Section 4, this is the Euclidean limit, and in that limit, the family $\{\eta_{\sigma_I(x)},\ x \in X\}$ will converge to a tight frame. A further clue to that statement is that, as shown in the proof of Proposition 3.4, the quantity G_l indeed becomes independent of l for l large enough, and only large values of l contribute for R large enough, as shown in [20]. We may notice that in this paper Holschneider constructs tight frames, but only for very special functions η (essentially eigenvectors of the operator $A_\sigma \equiv A_\sigma^\eta$, which of course depends strongly on η).

Theorem 3.3 yields the basic ingredient for writing the CWT on S^2. Given an admissible wavelet $\psi \in L^2(S^2, d\mu)$, our wavelets on the sphere are the functions $\psi_{\gamma,a} = U(\sigma_I(\gamma, a))\psi$, and the CWT reads, with $U_S(\gamma, a) \equiv U(\sigma_I(\gamma, a))$,

$$\begin{aligned} S(\gamma, a) &= \langle U(\sigma_I(\gamma, a))\psi | s\rangle \\ &= \int_{S^2} d\mu(\omega) \overline{[U_S(\gamma, a)\psi](\omega)} s(\omega) \\ &= \int_{S^2} d\mu(\omega) \overline{\psi_a(\gamma^{-1}\omega)} s(\omega). \end{aligned} \tag{3.18}$$

A natural question is that of the covariance of this spherical CWT under motions on S^2 and dilations. In the flat case, the usual 2-D CWT is fully covariant with respect to translations, rotations, and dilations, and this property is essential for applications, in particular the covariance under translations (often called improperly “shift invariance”). In fact, covariance is a general feature of all CS systems directly derived from a square integrable representation [1]. The present case is slightly more complicated, because the representation of $SO_0(3,1)$ is only square integrable on the quotient $X = SO_0(3,1)/N$, and then no general theorem is available. Thus we resort to a direct calculation, with the following result:

- The spherical CWT (3.18) is covariant under motions on S^2: for any $\gamma_0 \in SO(3)$, the transform of the rotated signal $s(\gamma_0^{-1}\omega)$ is the function $S(\gamma_0^{-1}\gamma, a)$.
- But it is *not* covariant under dilations. Indeed the wavelet transform of the dilated signal $\lambda(a_0, \omega)^{1/2} s(a_0^{-1}\omega)$ is $\langle U(g)\psi|s\rangle$, with $g = a_0^{-1}\gamma a$, and the latter, while a well-defined element of $SO_0(3,1)$, is *not* of the form $\sigma_I(\gamma', a')$. In fact, one can compute the Iwasawa decomposition of g and get $g = \gamma' a' n'$, where, in the Euler decomposition (2.8), $\gamma' = \gamma(\psi, \theta', \varphi)$, with $\tan\theta'/2 = a_0 \tan\theta/2$, and a' and n' are complicated functions of (γ, a). In particular $n' = 0$ iff γ is a rotation around the x_3 axis.

For applications, of course, it is the covariance under motions that is essential, since it reduces to translation covariance in the Euclidean limit, as we shall see in Section 4. As for dilations, the negative result reflects the fact that the parameter space X of the spherical CWT is not a group. Again we meet here a general feature of coherent state systems based on homogeneous spaces.

Condition (3.11), which was derived in [20] in a different way, is necessary and sufficient for the admissibility of η, but it is somewhat complicated to use in practice, since it requires the evaluation of nontrivial Fourier coefficients. Instead, there a simpler, although only necessary, condition.

PROPOSITION 3.6. *A function $\eta \in L^2(S^2, d\mu)$ is admissible only if it satisfies the condition*

$$\int_{S^2} d\mu(\theta,\varphi) \frac{\eta(\theta,\varphi)}{1+\cos\theta} = 0. \tag{3.19}$$

Proof. We have to compute

$$\int_0^\infty \frac{da}{a^3} \left| \langle Y_l^m | D^a \eta \rangle \right|^2. \tag{3.20}$$

Let us assume first that the support of η is bounded away from the South Pole; that is

$$\eta(\theta,\varphi) = 0 \qquad \text{if } \theta > \tilde{\theta}, \text{ for some } \tilde{\theta} < \pi.$$

Then we have

$$\begin{aligned} \langle Y_l^m | D^a \eta \rangle &= \int_{S^2} d\mu(\theta,\varphi)\, \overline{Y_l^m(\theta,\varphi)} \lambda(a,\theta)^{1/2} \eta(\theta_{1/a},\varphi) \\ &= \int_0^{2\pi} d\varphi \int_0^{\tilde{\theta}_a} d\theta \sin\theta \overline{Y_l^m(\theta,\varphi)} \lambda(a,\theta)^{1/2} \eta(\theta_{1/a},\varphi). \end{aligned}$$

Let us now split (3.20) into three parts:

$$\int_0^\infty \frac{da}{a^3} = \underbrace{\int_0^\epsilon \frac{da}{a^3}}_{\text{I}} + \underbrace{\int_\epsilon^{1/\epsilon} \frac{da}{a^3}}_{\text{II}} + \underbrace{\int_{1/\epsilon}^\infty \frac{da}{a^3}}_{\text{III}}. \tag{3.21}$$

Let us start with the first term. Making the change of variables $\theta' = \theta_{1/a}$, the Fourier coefficients become

$$\int_0^{2\pi} d\varphi \int_0^{\tilde{\theta}} d\theta' \sin\theta' \overline{Y_l^m(\theta'_a,\varphi)} \lambda(a,\theta'_a)^{1/2} \lambda(a^{-1},\theta') \eta(\theta',\varphi). \tag{3.22}$$

Using the cocycle property (3.3), we have

$$\lambda(a^{-1},\theta')^{1/2} \lambda(a,\theta'_a)^{1/2} = 1. \tag{3.23}$$

Inserting this into (3.22), we end up with

$$\langle Y_l^m | D^a \eta \rangle = \int_0^{2\pi} d\varphi \int_0^{\tilde{\theta}} d\theta' \sin\theta' \overline{Y_l^m(\theta'_a,\varphi)} \lambda(a^{-1},\theta')^{1/2} \eta(\theta',\varphi). \tag{3.24}$$

Coming back to (I) above with ϵ small enough, so that a is small, since $0 \le a \le \epsilon$, we have

$$\theta'_a \simeq 0 \qquad \text{and} \qquad \lambda(a^{-1},\theta')^{1/2} \simeq \frac{2a}{1+\cos\theta'}.$$

Then, because of the behavior of spherical harmonics at $\theta = 0$, $Y_l^m(0,\varphi) = \delta_{m,0}\sqrt{(2l+1)/(4\pi)}$, the integral over small scales converges only if we impose the condition

$$\int_{S^2} d\mu(\theta,\varphi) \frac{\eta(\theta,\varphi)}{1+\cos\theta} = 0. \tag{3.25}$$

(Of course, the result must be independent of the support of the particular function η chosen; thus we have to integrate over the full sphere S^2.)

The second integral (II) is handled easily because D^a is a strongly continuous operator and thus, by continuity of the scalar product, the integrand is a bounded continuous function on $[\epsilon, 1/\epsilon]$. For the last part (III), we first rewrite the Fourier coefficients for large scales

$$\int_0^{2\pi} d\varphi \int_0^{\tilde{\theta}_{1/a}} d\theta \sin\theta \overline{Y_l^m(\theta,\varphi)} \frac{2a}{1+\cos\theta} \eta(\theta_a, \varphi). \tag{3.26}$$

The only large scale divergence in (III) will never be reached because of the support property of η and this finally ensures convergence of the last term as well.

Now, if we drop the restriction on the support of η, condition (3.19) is *a fortiori* necessary, which proves the statement. ■

This necessary condition is the exact equivalent of the usual necessary condition for wavelets in the plane, $\int d^2x\, \psi(x) = 0$. And, as we shall see in Section 4, it reduces to the latter in the Euclidean limit. The interesting point is that (3.19) is a zero mean condition, as in the flat case. As such it will play the same role, namely it ensures that the CWT on S^2 given in (3.18) acts as a local filter. This is crucial for applications and it is one of the main reasons of the efficiency of the CWT. Thus our spherical CWT will have a comparable behavior. One should notice that the poles do not play any particular role in this CWT, since the sphere S^2 is a homogeneous space under $SO(3)$: all points of S^2 are really equivalent, despite the appearance to the contrary given by (3.19). One may also wonder what should be added to condition (3.19) to make it also sufficient. By analogy with the limiting flat case, we expect that a slightly faster vanishing at the South Pole will do, but this remains to be proven.

A further advantage of the simplified admissibility condition (3.19) is that it allows in a straightforward way the requirement of vanishing moments. It suffices to formulate the condition in the tangent plane, namely

$$\int d^2x\, x_1^{\alpha}\, x_2^{\beta}\, \psi(x) = 0, \qquad 0 \le \alpha + \beta \le N, \tag{3.27}$$

and to lift it to the sphere by inverse stereographic projection. To that effect, one introduces polar coordinates (r, φ) in the plane and uses the correspondence (which is a unitary map between the respective L^2 spaces, see (4.4) below)

$$(I_1^{-1} f)(\theta,\varphi) = \frac{1}{1+\cos\theta}\, f\left(2\tan\frac{\theta}{2}, \varphi\right), \qquad f \in L^2(\mathbb{R}^2, d^2x). \tag{3.28}$$

Thus, after some algebra, we get $N+1$ conditions for the vanishing of all moments of order up to N,

$$\int_{S^2} d\mu(\theta,\varphi) \frac{(1-\cos\theta)^N}{(1+\cos\theta)^{N+1}} e^{i2\nu\varphi}\, \eta(\theta,\varphi) = 0, \tag{3.29}$$

where $\nu = \pm 2k, \pm(2k-2), \ldots, 0$, if $N = 2k$, and $\nu = \pm(2k+1), \pm(2k-1), \ldots, 1$, if $N = 2k+1$.

3.3. An Example of Spherical Wavelets

We conclude this section by presenting an explicit class of admissible vectors, that is, spherical wavelets. First of all, we need the following result.

PROPOSITION 3.7. *Let $\phi \in L^2(S^2, d\mu)$. Then*

$$\int_{S^2} d\mu(\theta,\varphi) \frac{D^a \phi(\theta,\varphi)}{1+\cos\theta} = a \int_{S^2} d\mu(\theta,\varphi) \frac{\phi(\theta,\varphi)}{1+\cos\theta}.$$

Proof. By a simple computation, followed by the change of variables $\theta' = \theta_{1/a}$, we get

$$I \equiv \int_{S^2} d\mu(\theta,\varphi) \frac{D^a \phi(\theta,\varphi)}{1+\cos\theta} = \int_{S^2} d\mu(\theta,\varphi) \frac{\lambda(a,\theta)^{1/2}\phi(\theta_{1/a},\varphi)}{1+\cos\theta}$$
$$= \int_{S^2} d\mu(\theta',\varphi)\lambda(a,\theta'_a)^{1/2}\lambda(a^{-1},\theta') \frac{\phi(\theta',\varphi)}{1+\cos\theta'_a}.$$

Using again (3.23), we end up with

$$I = \int_{S^2} d\mu(\theta',\varphi)\lambda(a^{-1},\theta')^{1/2} \frac{\phi(\theta',\varphi)}{1+\cos\theta'_a}. \tag{3.30}$$

Then since

$$1+\cos\theta'_a = \frac{2\,(1+\cos\theta')}{(1-a^2)\cos\theta' + (1+a^2)},$$

Eq. (3.30) becomes

$$I = a \int_{S^2} d\mu(\theta',\varphi) \frac{\phi(\theta',\varphi)}{1+\cos\theta'},$$

and this proves the statement. ■

With the result just proved, it is easy to build a "difference wavelet," similar to those commonly used in vision (the "difference-of-Gaussians" or DOG wavelet, for instance) [2, 13]. Given a square integrable function ϕ , we define

$$\eta_\phi^{(\alpha)}(\theta,\varphi) = \phi(\theta,\varphi) - \frac{1}{\alpha} D^\alpha \phi(\theta,\varphi) \qquad (\alpha > 1).$$

Then it is easily checked that $\eta_\phi^{(\alpha)}$ satisfies the admissibility condition (3.19); that is, it is a spherical wavelet. Figure 2 shows a typical difference wavelet for the choice

$$\phi(\theta,\varphi) = \exp\left(-\tan^2\frac{\theta}{2}\right),$$

which is the inverse stereographic projection of a Gaussian in the tangent plane. The resulting spherical wavelet is shown for different values of the scale a and the position (θ,φ) on the sphere. Note that here "η at scale a" means that the function being plotted is $D^a\eta$; i.e., one must always use the covariant dilation operator D^a.

Remark. Exactly as in the flat case, it is often useful to replace the so-called L^2 normalization used so far, which guarantees the unitarity of the representation (3.5),

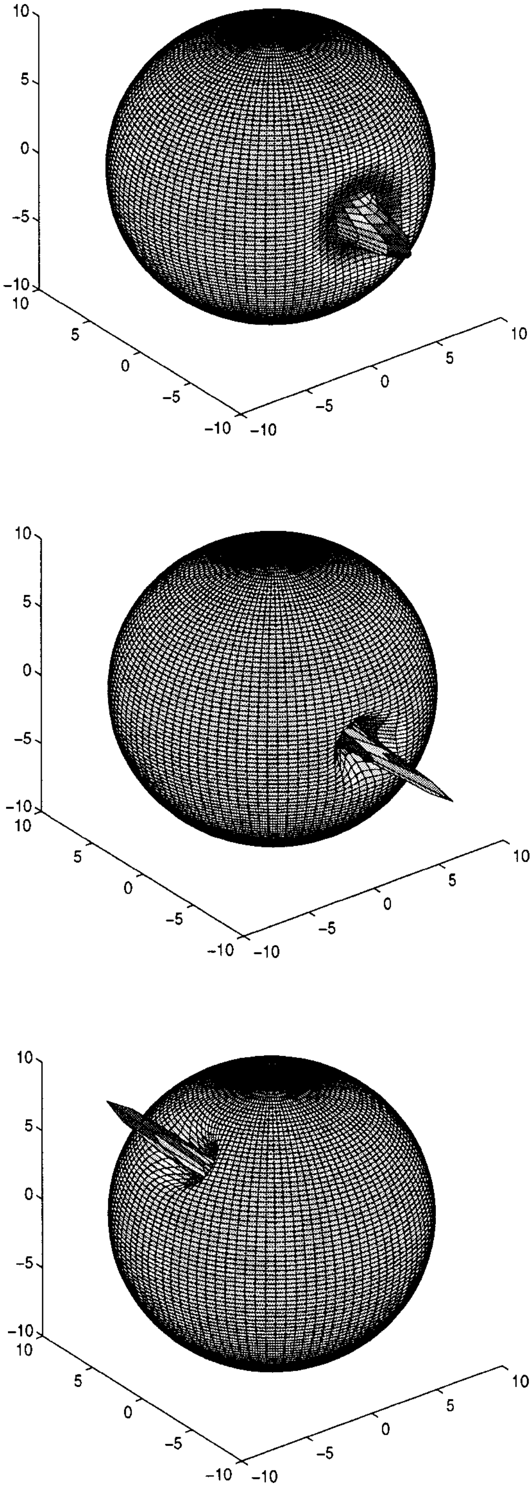

FIG. 2. The difference wavelet built from the "spherical Gaussian" $\phi(\theta, \varphi) = \exp(-\tan^2(\theta/2))$, for $\alpha = 2$. (Top) At scale $a = 0.125$ and position $\theta = 90°$, $\varphi = 0°$; (middle) at scale $a = 0.0625$ and position $\theta = 90°$, $\varphi = 0°$ (bottom), at scale 0.0625 and position $\theta = 135°$, $\varphi = 90°$. As mentioned in the text, "at scale a" means the function $D^a \eta_\phi^{(\alpha)}$.

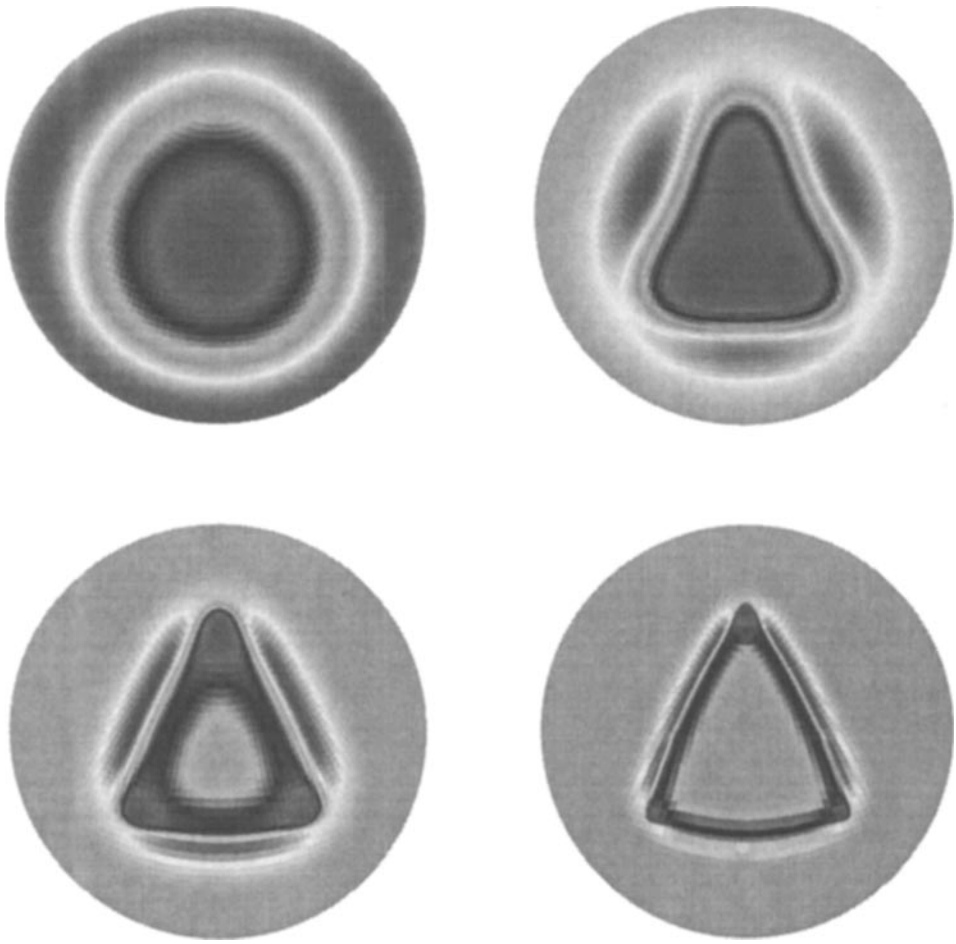

FIG. 3. The wavelet transform of the characteristic function of a spherical triangle with apex at the North Pole, $0 \leq \theta \leq \pi/3$, $\pi/3 \leq \varphi \leq 2\pi/3$, obtained with the modified Gaussian difference wavelet $\widetilde{\eta}_{\phi}^{(\alpha)}$. The transform is shown at four successive scales, $a = 0.5, 0.2$ (top row) and $a = 0.1, 0.05$ (lower row). For a small enough, the transform vanishes inside the triangle, as expected, and presents a "wall" along the contour, with a negative value just outside, a sharp positive maximum just inside, and positive peaks at each corner.

by the L^1 normalization. Although it hides the group-theoretical origin of the spherical CWT, this alternative choice has the effect of enhancing further the small scales, thus the singularities in the signal. The only change required is to replace the dilation operator D^a defined in (3.8) by the modified operator

$$\left(\widetilde{D}^a f\right)(\omega) = \lambda(a, \theta) f(\omega_{1/a}). \tag{3.31}$$

Thus the modified WT reads

$$\widetilde{S}(\gamma, a) = \int_{S^2} d\mu(\omega) \overline{\widetilde{\psi}_a(\gamma^{-1}\omega)} s(\omega), \qquad \widetilde{\psi}_a \equiv \widetilde{D}^a \psi. \tag{3.32}$$

With this definition one recovers some of the results obtained in [14] by a direct calculation. Further details may be found in [39]. Then the function η_a appearing in the admissibility condition (3.11) is replaced by the corresponding function $\widetilde{\eta}_a$, and as a consequence, the necessary condition (3.19) becomes simply

$$\int_{S^2} d\mu(\theta,\varphi)\,\eta(\theta,\varphi) = 0, \tag{3.33}$$

that is, a zero mean condition exactly as in the flat case. Accordingly, the difference wavelet $\eta_\phi^{(\alpha)}$ becomes

$$\widetilde{\eta}_\phi^{(\alpha)}(\theta,\varphi) = \phi(\theta,\varphi) - \widetilde{D}^\alpha \phi(\theta,\varphi) \qquad (\alpha > 1).$$

It remains to compute explicit wavelet transforms. We give in Fig. 3 the transform of the characteristic function of a spherical triangle on S^2, with one of the corners sitting at the North Pole. The wavelet used is the modified spherical Gaussian $\widetilde{\eta}_\phi^{(\alpha)}$, and the transform is shown at four successive scales, from $a = 0.5$ to $a = 0.05$. The spherical WT behaves here exactly as, in the flat case, the WT of the characteristic function of a square, as shown in [3]. For large a, the WT sees only the object as a whole, thus allowing one to determine its position on the sphere. When a decreases, increasingly finer details appear; in this simple case, only the contour remains, and it is perfectly seen at $a = 0.05$. The transform vanishes in the interior of the triangle, as it should; only the "walls" remain, with a negative value just outside, a zero-crossing right on the boundary, and a sharp positive maximum just inside. In addition, each corner gives a neat peak, which is positive, since the corner is convex [3]. Notice that the three corners are alike, so that indeed the poles play no special role (except that the numerical integration with *Mathematica* automatically provides more mesh points around the poles).

4. THE EUCLIDEAN LIMIT

According to Holschneider [20], a good wavelet transform on the sphere should satisfy a geometrical constraint expressing its asymptotic Euclidean behavior. Since the sphere is locally flat, the associated wavelet transform should match the usual 2-D CWT in the plane at small scales or, what amounts to the same thing, for large values of the radius of curvature. In this section, we will give a precise mathematical meaning to these notions using the technique of group contractions. The main definitions concerning contraction of Lie algebras and Lie groups may be found in Appendix C.

4.1. Contracting the Lorentz Group and Its Homogeneous Spaces

The method proceeds in three steps. In the first stage, we reformulate the theory described so far on a sphere of radius R and let $R \to \infty$. In this limit, the Lorentz group $SO_0(3,1)$ is contracted along its minimal parabolic subgroup $P = SO(2) \cdot \mathbb{R}_*^+ \cdot N$ into a semidirect product,

$$G_1 = SO_0(3,1) \overset{R\to\infty}{\longrightarrow} G_2 = \mathbb{R}^2 \rtimes SIM(2),$$

where $SIM(2) = \mathbb{R}^2 \rtimes (\mathbb{R}_*^+ \times SO(2))$ is the similitude group of $\mathbb{R}^2$, that is, precisely the invariance group of the 2-D CWT. The detailed calculation is presented in Appendix C.

Next we have to quotient out the nilpotent subgroup N, which is preserved during the contraction. Indeed, the parameter space of the spherical CWT is $X = G_1/N \simeq SO(3) \cdot A$, which is not a group (and this forced us to use the general formalism of [1]). After contraction, we get $G_2/N \simeq SIM(2)$, which is the parameter space of the 2-D CWT (notice that we use here the isomorphism between the two forms of the contracted group G_2; see Section C.2). Thus the missing group structure is restored by the contraction!

We have now to formulate the contraction directly on the two parameter spaces; that is, we must restrict the contraction map (C.7) to the respective homogeneous spaces $SIM(2) = G_2/N$ and $X = SO_0(3,1)/N$. To that purpose, we introduce a section $\tilde{\sigma}\colon SIM(2) \to N \rtimes SIM(2)$ by the relation

$$\begin{aligned} \tilde{\sigma}\colon \big(\boldsymbol{b}, (a, \psi)\big) &\mapsto \big(\boldsymbol{n}(\boldsymbol{b}), (\boldsymbol{b}, (a, \psi))\big), \\ \boldsymbol{b} \in \mathbb{R}^2,\ a \in A,\ \psi &\in [0, 2\pi),\ \boldsymbol{n}(\boldsymbol{b}) \in N \sim \mathbb{R}^2. \end{aligned} \tag{4.1}$$

Combining this with the canonical projection of the Iwasawa bundle,

$$\Gamma\colon KAN \to X \simeq KA,$$

we may define the restricted contraction maps $\tilde{\Pi}_R\colon SIM(2) \to X$ by

$$SIM(2) \ni g \mapsto \tilde{\Pi}_R(g) = \Gamma\big(\Pi_R(\tilde{\sigma}(g))\big),$$

where $\Pi_R\colon G_2 \to G_1$ is the contraction map (C.7). Altogether we have the following commutative diagram:

$$\begin{array}{ccc} G_2 = \mathbb{R}^2 \rtimes SIM(2) & \xrightarrow{\Pi_R} & G_1 = SO_0(3,1) \\ \uparrow \tilde{\sigma} & & \downarrow \Gamma \\ G_2/N \simeq SIM(2) & \xrightarrow{\tilde{\Pi}_R} & SO_0(3,1)/N = X \end{array} \tag{4.2}$$

Finally we notice that the homogeneous spaces $S^2 = G_1/MAN$ and $\mathbb{R}^2 = G_2/MAN$, which carry the respective CWT, are also related through contraction. Thus the geometrical picture is fully coherent.

4.2. The Euclidean Limit of the Spherical CWT

We are now prepared for the third step, namely the Euclidean limit itself, which will be formulated as a contraction at the level of group representations.

Whereas contractions of Lie algebras and Lie groups are relatively ancient and well known [21, 34], the extension of the procedure to group representations is rather recent [25]. A rigorous version has been given by Dooley [10, 11], which we follow. The additional difficulty here is that the representation space itself varies during the procedure.

Let G_2 be a contraction of G_1, defined by the contraction map $\Pi_R\colon G_2 \to G_1$ (see Appendix C) and let U be a representation of G_2 in a Hilbert space $\mathcal{H}$. Suppose that, for each $R \in [1, \infty)$, we have a representation $\{\mathcal{H}_R, U_R\}$ of G_1, a dense subspace $\mathcal{D}_R$ of $\mathcal{H}$, and a linear injective map $I_R\colon \mathcal{H}_R \to \mathcal{D}_R$. Then one says that the representation U of G_2 is a contraction of the family of representations $\{U_R\}$ of G_1 if there exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ such that, for all $\phi \in \mathcal{D}$ and $g \in G_2$, one has:

- for every R large enough, $\phi \in \mathcal{D}_R$ and $U_R(\Pi_R(g))I_R^{-1}\phi \in I_R^{-1}(\mathcal{D}_R)$,
- $\lim_{R\to\infty} \| I_R U_R(\Pi_R g) I_R^{-1}\phi - U(g)\phi\|_{\mathcal{H}} = 0$.

Using this definition, we will show that the CWT on the sphere S^2 converges to the usual 2-D CWT on $\mathbb{R}^2$ in the geometrical limit of large radius. The point will be established by proving that the associated series of square integrable representations of $SO_0(3,1)$ contract to the usual wavelet representation of $SIM(2)$, defined in [3, 27].

Let $\mathcal{H}_R = L^2(S_R^2, d\mu_R)$ be the Hilbert space of square integrable functions on a sphere of radius R,

$$\int_{S_R^2} |f(\theta,\varphi)|^2 R^2 \sin\theta \, d\theta \, d\varphi < \infty,$$

and $\mathcal{H} = L^2(\mathbb{R}^2, d^2\boldsymbol{x})$. The choice of the map I_R is forced by geometry. Since we are trying to approximate functions in the plane, we will map a function $\phi \in \mathcal{H}_R$ to a function in $\mathcal{H}$ by stereographic projection. With a suitable convergence factor, we obtain an isometry $I_R\colon \mathcal{H}_R \to \mathcal{H}$,

$$(I_R f)(r,\varphi) = \frac{4R^2}{4R^2 + r^2} f\left(2\arctan\frac{r}{2R}, \varphi\right), \tag{4.3}$$

where we have used polar coordinates (r,φ) in the plane. Then one checks that $\|I_R f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}_R}$. Injectivity of I_R is thus ensured. The inverse map reads

$$(I_R^{-1} f)(\theta,\varphi) = \frac{2}{1+\cos\theta} f\left(2R\tan\frac{\theta}{2}, \varphi\right), \tag{4.4}$$

and a closer inspection shows that it is also an isometry. Thus I_R is unitary.

For each R, we choose $\mathcal{D}_R = \mathcal{D} = \mathcal{C}_0(\mathbb{R}^2)$, the space of continuous functions of compact support, which is dense in $\mathcal{H}$. Now let U be the usual wavelet representation of $SIM(2)$ in $\mathcal{H}$ and U_R the representation (3.5) of $SO_0(3,1)$ realized in $\mathcal{H}_R$.

THEOREM 4.1 (Euclidean Limit). *The representation U of $SIM(2)$ is a contraction of the family of representations U_R of $SO_0(3,1)$ as $R \to \infty$.*

Proof. For every $g \in SIM(2)$, we have to prove the strong limit

$$\lim_{R\to\infty} \left\| I_R U_R\left(\tilde{\Pi}_R(g)\right) I_R^{-1}\phi - U(g)\phi \right\|_{\mathcal{H}} = 0. \tag{4.5}$$

We first look at pointwise convergence. The subgroups A and $M = SO(2)$ are preserved by contraction, and it is easily seen that the associated operators commute with I_R. Thus it is enough to consider elements $g \in SIM(2)$ of the form $g = (\boldsymbol{v}, (1,0))$, with $\boldsymbol{v} = (0, v_y)$, that is, $t = 0$, $\psi = 0$, $\varphi = \pi/2$. Then, according to (C.8), one has

$$\tilde{\Pi}_R(g) = \hat{u}\left(\frac{v_y}{R}\right),$$

from which one readily checks that, pointwise,

$$\lim_{R\to\infty} \left| I_R U_R\left(\tilde{\Pi}_R(g)\right) I_R^{-1}\phi - U(g)\phi \right| = 0.$$

Next, since I_R^{-1} is a unitary operator, we can compute the strong limit in $\mathcal{H}_R$. Define, for $\phi \in \mathcal{D}$, $C = \max_{\boldsymbol{x}\in\mathbb{R}^2} \phi(\boldsymbol{x})$. Then, using the relation (4.4), we obtain the following bound, uniformly for all $R \geq 1$,

$$\begin{aligned}\left|U_R\left(\tilde{\Pi}_R(g)\right) I_R^{-1}\phi - I_R^{-1}U(g)\phi\right| &< \left|U_R\left(\tilde{\Pi}_R(g)\right) I_R^{-1}\phi\right| + \left|I_R^{-1}U(g)\phi\right| \\ &< 2C \cdot \max\left(1 + \tan^2\frac{\theta}{2}\right),\end{aligned}$$

where the maximum is taken over all θ such that $(2R\tan\theta/2, \varphi) \in \operatorname{supp}\phi$. Finally, the l.h.s. of this inequality is uniformly bounded and tends pointwise to zero as $R \to \infty$; thus Lebesgue's dominated convergence theorem gives the result. ■

This theorem yields the expected result that local wavelet analysis on the sphere, as defined here, is equivalent, in the limit of large radius, to local wavelet analysis in the plane. Indeed the whole structure on the sphere S_R^2 goes into the corresponding one in $\mathbb{R}^2$ as $R \to \infty$:

- the Hilbert spaces: $L^2(S_R^2, d\mu_R) \to L^2(\mathbb{R}^2, d^2\boldsymbol{x})$
- the group structures: $SO_0(3,1)/N \simeq SO(3) \cdot \mathbb{R}_*^+ \to SIM(2)$, together with their respective action
- the group representations: $U_R \to U$.

Thus the matrix elements of the corresponding representations also converge to one another, and therefore the square integrability condition (3.13) converges into the corresponding one for the CWT in the plane,

$$\int_{\mathbb{R}^2} d^2k \frac{|\hat{\psi}(k)|^2}{|k|^2} < \infty.$$

Admissible wavelets on S^2 converge to admissible wavelets on $\mathbb{R}^2$. For instance, the spherical DOG wavelet described in Section 3.3 converges to the usual DOG wavelet. Also, because the renormalizing factor in (4.4) is exactly the one that links the two invariant measures under stereographic projection, it follows that the necessary condition (3.19) also goes into the corresponding necessary condition for wavelets in the plane, $\int d^2x\, \psi(x) = 0$. The striking fact is that this Euclidean limit is entirely built in the group-theoretical structure of the theory.

5. CONCLUSION

The construction presented here fulfills all the requirements stated in the Introduction for a continuous wavelet transform on the sphere. It is entirely derived from group theory, following the formalism of general coherent states developed in [1]. In addition, the Euclidean limit is valid, with a precise group-theoretical formulation. Thus the formulas (3.18) yield a genuine CWT on the sphere, which has none of the defects of the other versions mentioned in the Introduction. Preliminary tests, with the spherical DOG wavelet, show that it has the expected capability of detecting discontinuities, whether or not they lie at one of the poles of the sphere. The only remaining problem is of a computational nature. Indeed Eq. (3.18) requires a pointwise convolution on the sphere, which is very time-consuming. However, this is not specific to wavelet analysis, it simply reflects the lack of an efficient convolution algorithm on the sphere, and in particular the difficulty of finding an appropriate discretization of the latter. Several methods have been proposed in the literature [14, 19, 26], but none of them is fully satisfactory. However, it seems reasonable to hope that faster algorithms will be available soon [39].

APPENDIX A: THE CWT ON A MANIFOLD

In this appendix, we briefly sketch the method of construction of coherent states (CS) associated to a group representation. Further details may be found, for instance, in the review paper [1].

Let Y be a manifold. For instance, Y could be space $\mathbb{R}^n$, the 2-sphere S^2, space–time $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2 \times \mathbb{R}$, etc. In order to construct coherent states on Y, one typically needs two ingredients:

- the class of finite energy signals living on Y, i.e., the space $L^2(Y, d\mu) \equiv \mathcal{H}$; and
- a (locally compact) group G of transformations acting (transitively) on Y, i.e., $y \mapsto g[y]$, with $g[g'[y]] = gg'[y]$, $e[y] = y$, and for any pair $y, y' \in Y$, there is at least one $g \in G$ such that $g[y] = y'$.

From this one obtains a natural unitary representation of G in the space $L^2(Y, d\mu)$ (we assume that μ is G-invariant, but this can be relaxed):

$$[U(g)f](y) = f(g^{-1}y). \tag{A.1}$$

Then a system of CS on Y associated to G may be defined if U is a *square integrable* representation of G; that is, U is irreducible (cyclic would suffice) and there exists a nonzero vector $\eta \in L^2(Y, d\mu)$, called *admissible*, such that the matrix element $\langle U(g)\eta|\eta\rangle$ is square integrable as a function on G, with respect to the (left or right) invariant Haar measure on G. When this is the case, the corresponding CS, indexed by G, are obtained as the vectors in the orbit of the admissible vector η under U:

$$\eta_g = U(g)\eta, \qquad g \in G. \tag{A.2}$$

Quite often, however, the representation U is not square integrable in the strict sense just described (it would be a discrete series representation, and many groups have no discrete series—a case in point is the Lorentz group $SO_0(3,1)$). However, it may become square integrable when restricted to a homogeneous space $X = G/H$, for some closed subgroup H. By this we mean the following. Let $\sigma\colon X \to G$ be a Borel section. Then the nonzero vector $\eta \in L^2(Y, d\mu)$ is said to be *admissible* mod(H, σ), and the representation U square integrable mod(H, σ), if the following condition holds:

$$\int_X |\langle U(\sigma(x))\eta|\phi\rangle|^2 \, d\nu(x) < \infty, \qquad \forall \phi \in \mathcal{H} \tag{A.3}$$

(we assume that ν is a G-invariant measure on X, but again this is not really a restriction). Then CS indexed by X may be defined as

$$\eta_{\sigma(x)} = U(\sigma(x))\eta, \qquad x \in X, \tag{A.4}$$

and they form a total (or overcomplete) set $\mathcal{S}_\sigma$ in $\mathcal{H}$, with essentially the same properties as in the restricted case described before.

The condition (A.3) may also be rewritten as

$$0 < \int_X |\langle \eta_{\sigma(x)}|\phi\rangle|^2 \, d\nu(x) = \langle \phi | A_\sigma \phi \rangle < \infty, \qquad \forall \phi \in \mathcal{H}, \tag{A.5}$$

where A_σ is a positive, bounded, invertible operator [1]. If the operator A_σ^{-1} is also bounded, the family $\mathcal{S}_\sigma = \{\eta_{\sigma(x)},\, x \in X\}$ is called a *frame*, and a *tight frame* if $A_\sigma = \lambda I$, for some $\lambda > 0$. This terminology is familiar in the discrete case, for instance, in wavelet or Gabor analysis [8, 9].

Here are some familiar examples of this construction:

(1) The $ax + b$ group acting on $\mathbb{R}$ yields the usual 1-D continuous wavelets.

(2) The Weyl–Heisenberg group, also acting on $\mathbb{R}$, gives the windowed Fourier transform, or Gabor transform. Here all vectors are admissible.

(3) The similitude group of $\mathbb{R}^n$, consisting of translations $b \in \mathbb{R}^n$, rotations $R \in SO(n)$, and dilations $a > 0$, yields the n-dimensional wavelets. For an axisymmetric wavelet η, the isotropy group H is $SO(n-1)$ and so $X = \mathbb{R}^n \cdot \mathbb{R}_*^+ \cdot S^{n-1} \sim \mathbb{R}^{2n}$.

(4) Coherent states on the Galilei group or the Poincaré group, both inaccessible to the standard Gilmore–Perelomov method [1].

In examples (1), (2), and (3), one has $A_\sigma = 1$, but in case (4), A_σ is in general a nontrivial operator.

Let us normalize the admissible vector η by $c(\eta) = \langle \eta | A_\sigma \eta \rangle = 1$, and assume that it generates a frame; that is, A_σ^{-1} is bounded (otherwise domain problems arise). Define the linear map $W_\eta \colon \mathcal{H} \to L^2(X, d\nu)$ by

$$(W_\eta \phi)(x) = \langle \eta_{\sigma(x)} | \phi \rangle, \qquad \phi \in \mathcal{H}. \tag{A.6}$$

The map W_η is called the *CS map* or the *wavelet transform* associated to η. Its range, $\mathcal{H}_\eta$, is complete with respect to the scalar product $\langle \Phi | \Psi \rangle_\eta \equiv \langle \Phi | W_\eta \, A_\sigma^{-1} \, W_\eta^{-1} \Psi \rangle$ and W_η is unitary from $\mathcal{H}$ onto $\mathcal{H}_\eta$. As a consequence, the map W_η may be inverted on its range by the adjoint operator, which yields the reconstruction formula

$$\phi = W_\eta^{-1} \Phi = \int_X d\nu(x) \Phi(x) A_\sigma^{-1} \, \eta_{\sigma(x)}, \qquad \Phi \in \mathcal{H}_\eta. \tag{A.7}$$

In other words, the signal ϕ is expanded in terms of CS $A_\sigma^{-1} \eta_{\sigma(x)}$, the (wavelet) coefficients being $\Phi(x) = (W_\eta \phi)(x)$.

If we particularize these statements (with $A_\sigma = 1$) to examples (1), (2), and (3) above, we recognize the familiar formulas of wavelet or Gabor analysis.

APPENDIX B: SOME EXPLICIT FORMULAS FOR $SL(2, \mathbb{C})$ AND $SO_0(3, 1)$

In this appendix, we collect some explicit formulas for the Lorentz group $SO_0(3, 1)$ and its double (in fact, universal) covering $SL(2, \mathbb{C})$. Both are semisimple Lie groups, thus unimodular (the left and right Haar measures coincide). Therefore, as explained in

Section 2, they have an Iwasawa decomposition into three closed subgroups, namely [37]

$$SO_0(3,1) = SO(3)\cdot\mathbb{R}^+_*\cdot\mathbb{C}, \qquad SL(2,\mathbb{C}) = SU(2)\cdot\mathbb{R}^+_*\cdot\mathbb{C}.$$

The explicit decomposition of a generic element of $SL(2,\mathbb{C})$ was given in (2.6). Moreover, the matrices $m(\varphi)$ and $u(\theta)$ appearing in the Euler decomposition (2.8) of a element of $SU(2)$ are given in (2.9). As for the elements of $A \sim \mathbb{R}^+_*$, they will be written with $\delta = e^{-t}$:

$$d(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}. \tag{B.1}$$

Next, let us recall the standard homomorphism between the two groups $SL(2,\mathbb{C})$ and $SO_0(3,1)$. For $(x_0,x_1,x_2,x_3)\in\mathbb{R}^4$, consider the following Hermitian matrix:

$$X = \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix}.$$

Any element $g\in SL(2,\mathbb{C})$ specifies a unique linear transformation $X\mapsto X' = gX\overline{g}^t$, which in turn induces a Lorentz transformation in $\mathbb{R}^4$. The explicit correspondence reads:

$$\begin{aligned}
m(\varphi) \mapsto \widehat{m}(\varphi) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
u(\theta) \mapsto \hat{u}(\theta) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \\
d(t) \mapsto \hat{d}(t) &= \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}.
\end{aligned} \tag{B.2}$$

The conformal group of the sphere S^2 can thus be identified with $SO_0(3,1)$. In particular, the last relation shows that dilations in $SL(2,\mathbb{C})$ correspond to pure Lorentz boosts.

It remains to compute explicitly an $SL(2,\mathbb{C})$- or $SO_0(3,1)$-invariant measure on X. First, since both $SO_0(3,1)$ and N are unimodular, their quotient X necessarily possesses a unique invariant measure ν, which may be derived from the relation $dg = d\nu(x)\,dn$, where dg and dn are the invariant measures on $SO_0(3,1)$ and N, respectively [22]. Next, $d\nu(x)$ may be computed explicitly from the Iwasawa decomposition. Indeed, for $G = KAN$, write the generic element of A as $\exp tQ$, $t\in\mathbb{R}$, with Q the infinitesimal generator of the dilation subgroup A. Then one has

$$dg = e^{2\rho_A(tQ)} dk\,dt\,dn,$$

where $2\rho_A$ is the sum of the positive roots of the Lie algebra and dk, dt, and dn the Haar measures on the three (unimodular) components. In our case, $dk = d\mu(\gamma)$, the invariant measure on $SO(3)$, dt is the Lebesgue measure on $\mathbb{R}$, and $2\rho_A(tQ) = 2t$, so that we get

$dg = e^{2t}\, d\mu(\gamma)\, dt\, dn$ [37]. Writing $a = e^{-t} \in \mathbb{R}_*^+$, we obtain finally

$$d\nu(\gamma, a) = \frac{d\mu(\gamma)\, da}{a^3}.$$

APPENDIX C: CONTRACTIONS OF LIE ALGEBRAS AND LIE GROUPS

APPLICATION TO THE LORENTZ GROUP

C.1: Contraction of Lie Algebras and Lie Groups

We begin by recalling some basic facts concerning the process of contraction, for both Lie algebras and Lie groups. Let $\mathfrak{g}_1 = (V, [\cdot,\cdot]_1)$ and $\mathfrak{g}_2 = (V, [\cdot,\cdot]_2)$ be two Lie algebras on the same vector space V. We say that $\mathfrak{g}_2$ is a *contraction* of $\mathfrak{g}_1$ if there exists a one-parameter family of invertible linear mappings ϕ_R, $R \in [1, \infty)$, from V to V such that

$$\lim_{R\to\infty} \phi_R^{-1}[\phi_R X, \phi_R Y]_1 = [X, Y]_2, \qquad \forall\, X, Y \in V. \tag{C.1}$$

The limit (C.1) defines a new Lie algebra structure on V, which is not isomorphic to the original one. A special case is the Inönü–Wigner contraction [21], in which a particular subalgebra of $\mathfrak{g}_1$ is conserved throughout the process. More precisely, suppose that there exists a subalgebra $\mathfrak{s}$ in $\mathfrak{g}_1$ and a vector subspace $\mathfrak{v}_c$, complement of $\mathfrak{s}$ in $\mathfrak{g}_1$, that is

$$\mathfrak{g}_1 = \mathfrak{s} + \mathfrak{v}_c\ , \tag{C.2}$$

such that

$$[\mathfrak{s}, \mathfrak{s}]_2 \subset \mathfrak{s}, \qquad [\mathfrak{v}_c, \mathfrak{v}_c]_2 = 0, \qquad [\mathfrak{s}, \mathfrak{v}_c] \subset \mathfrak{v}_c. \tag{C.3}$$

Using (C.2) we can decompose any $X \in V$ as

$$X = X_{\mathfrak{s}} + X_c, \qquad X_{\mathfrak{s}} \in \mathfrak{s},\ X_c \in \mathfrak{v}_c,$$

and define the contraction mappings

$$\phi_R(X) = X_{\mathfrak{s}} + \frac{1}{R} X_c.$$

Then applying (C.1) does not affect the subalgebra $\mathfrak{s}$. We say in this case that we have a contraction of $\mathfrak{g}_1$ *along* $\mathfrak{s}$.

The contraction process may be lifted to the corresponding Lie groups [10, 11]. Let again $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras such that $\mathfrak{g}_2$ is a contraction of $\mathfrak{g}_1$. Let G_1 be the simply connected Lie group with Lie algebra $\mathfrak{g}_1$. Let S be the subgroup of G_1 whose Lie algebra is $\mathfrak{s}$ in the decomposition (C.2). Defining the semidirect product

$$G_2 \equiv V_c \rtimes S, \qquad V_c = \exp \mathfrak{v}_c \simeq \mathfrak{v}_c,$$

one easily checks that $\mathfrak{g}_2 = (V, [\cdot,\cdot]_2)$ is the Lie algebra of G_2. Consider now the family of maps $\Pi_R\colon\ G_2 \to G_1$ given by

$$\Pi_R\colon\ (v, s) \mapsto \left(\exp_{G_1} R^{-1} v\right) \cdot s. \tag{C.4}$$

They play essentially the same role as the maps ϕ_R of (C.1) at the level of the corresponding groups, namely

$$\lim_{R\to\infty} \Pi_R^{-1}\big(\Pi_R(g) \overset{1}{\circ} \Pi_R(g')\big) = g \overset{2}{\circ} g', \qquad \forall\, g, g' \in G_2, \tag{C.5}$$

where $\overset{1}{\circ}$ and $\overset{2}{\circ}$ denote the product in G_1 and G_2, respectively. Indeed one easily checks that $T_e\Pi_R = \phi_R$, where T_e is the derivative of Π_R evaluated at the neutral element of G_1. It is easily seen on (C.4) that the subgroup S is preserved during the contraction.

C.2: Contraction of the Lorentz Lie Algebra

Let us now focus on the Lie algebra $\mathfrak{so}(3,1)$. Its Iwasawa decomposition gives

$$\mathfrak{g}_1 \equiv \mathfrak{so}(3,1) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

where $\mathfrak{k}$ is the maximal compact subalgebra $\mathfrak{so}(3)$ with generators $\{X_1, X_2, X_3\}$ and

$$[X_i, X_j]_1 = \epsilon_{ijk} X_k.$$

We denote by Q and N_1, N_2 the generators of $\mathfrak{a}$ and $\mathfrak{n}$, respectively, and give the complete set of commutation relations:

$$\begin{array}{lll}
[X_1, X_2]_1 = X_3, & [X_1, X_3]_1 = -X_2, & [X_2, X_3]_1 = X_1 \\
[X_1, Q]_1 = N_1 - X_1, & [X_2, Q]_1 = N_2 - X_2, & [X_3, Q]_1 = 0 \\
[X_1, N_1]_1 = -Q, & [X_2, N_1]_1 = -X_3, & [X_3, N_1]_1 = -N_2 \\
[X_1, N_2]_1 = X_3, & [X_2, N_2]_1 = -Q, & [X_3, N_2]_1 = -N_1 \\
[N_1, Q]_1 = N_1, & [N_2, Q]_1 = N_2, & [N_1, N_2]_1 = 0.
\end{array}$$

We then introduce the contraction mappings $\phi_R\colon \mathbb{R}^6 \to \mathbb{R}^6$, as follows,

$$\begin{array}{ll}
\phi_R(X_1) = R^{-1}X_1, & \phi_R(Q) = Q \\
\phi_R(X_2) = R^{-1}X_2, & \phi_R(N_i) = N_i \\
\phi_R(X_3) = X_3, &
\end{array}$$

and study the singular limit (C.1). This defines a new Lie algebra structure that we can compute explicitly:

$$\begin{array}{lll}
[X_1, X_2]_2 = 0, & [X_1, X_3]_2 = -X_2, & [X_2, X_3]_2 = X_1 \\
[X_1, Q]_2 = -X_1, & [X_2, Q]_2 = -X_2, & [X_3, Q]_2 = 0 \\
[X_1, N_1]_2 = 0, & [X_2, N_1]_2 = 0, & [X_3, N_1]_2 = N_2 \\
[X_1, N_2]_2 = 0, & [X_2, N_2]_2 = 0, & [X_3, N_2]_2 = -N_1 \\
[N_1, Q]_2 = N_1, & [N_2, Q]_2 = N_2, & [N_1, N_2]_2 = 0.
\end{array}$$

On these relations, we notice that $\{X_1, X_2, X_3, Q\}$ is a subalgebra isomorphic to $\mathfrak{sim}(2)$ and $\{X_3, Q, N_1, N_2\}$ is another one, inherited from the minimal parabolic subalgebra of

$\mathfrak{so}(3,1)$, which is invariant by contraction. Let us make more precise the structure of $\mathfrak{g}_2$. In accordance with (C.2), we put

$$\mathfrak{s} = \text{span}\{X_3, Q, N_1, N_2\},$$

which is preserved, and

$$\mathfrak{v}_c = \text{span}\{X_1, X_2\}.$$

We see that $\mathfrak{s}$ and $\mathfrak{v}_c$ satisfy the commutation relations (C.3). Therefore, the group G_2 associated with $\mathfrak{g}_2$ is the semidirect product

$$G_2 \equiv V_c \rtimes S, \qquad V_c \equiv \mathbb{R}^2, S \equiv SIM(2).$$

It is crucial to remark that the quotient G_2/V_c is isomorphic to the similitude group of the plane $SIM(2)$. One should also notice that the decomposition (C.3) is not unique. One could have chosen instead

$$\mathfrak{s}' = \text{span}\{X_1, X_2, X_3, Q\}, \qquad \mathfrak{v}'_c = \text{span}\{N_1, N_2\}, \tag{C.6}$$

and obtained an isomorphic structure for $\mathfrak{g}_2$, except that $\mathfrak{s}'$ is no more preserved by contraction.

C.3: Contraction of the Lorentz Group

Now we turn to the contraction $R \to \infty$ at the group level. The subgroup that is preserved is the minimal parabolic subgroup $P = MAN$, $M = SO(2)$. We have $\mathfrak{v}_c = \text{span}\{X_1, X_2\}$ and so $\mathfrak{v}_c \simeq \mathbb{R}^2$. We write $\boldsymbol{v} = v_1 X_1 + v_2 X_2$ for elements of $\mathfrak{v}_c$ and $s = (\psi, (t, \vec{\mathbf{n}}(\vec{\xi})))$ for elements of P, with $\vec{\mathbf{n}}(\vec{\xi}) \in N$, $\xi \in \mathbb{R}^2$. Now we can compute explicitly the contraction mapping (C.4)

$$\Pi_R(\boldsymbol{v}, s) = \exp_{SO_0(3,1)}\left[R^{-1}(v_1 X_1 + v_2 X_2)\right] \cdot \hat{s}, \tag{C.7}$$

where $\hat{s} = \widehat{m}(\psi) \cdot \hat{d}(t) \cdot \hat{n}(\vec{\boldsymbol{\xi}})$ denotes the 4×4 matrix representation (B.2) of $s \in P = MAN$. Using this parametrization and expanding the matrix elements in powers of R^{-1}, one gets for the exponential factor

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{v_2}{R} \\ 0 & 0 & 1 & -\frac{v_1}{R} \\ 0 & -\frac{v_2}{R} & \frac{v_1}{R} & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{R^3}\right).$$

Then, computing (C.7) and dropping the higher order terms, we obtain, for $r/R \ll 1$, with $r = (v_1^2 + v_2^2)^{1/2}$,

$$
g = \Pi_R(\boldsymbol{v}, s) = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ \frac{v_2}{R}\sinh t & \cos\psi & -\sin\psi & \frac{v_2}{R}\cosh t \\ -\frac{v_1}{R}\sinh t & \sin\psi & \cos\psi & -\frac{v_1}{R}\cosh t \\ \sinh t & \frac{v_1}{R}\sin\psi - \frac{v_2}{R}\cos\psi & \frac{v_1}{R}\cos\psi + \frac{v_2}{R}\sin\psi & \cosh t \end{pmatrix} \cdot \hat{n}(\boldsymbol{\xi}).
$$

Writing the Iwasawa decomposition of the element $g = \gamma a n$ gives

$$
\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & \frac{v_2}{R} \\ 0 & \sin\psi & \cos\psi & -\frac{v_1}{R} \\ 0 & \frac{v_1}{R}\sin\psi - \frac{v_2}{R}\cos\psi & \frac{v_1}{R}\cos\psi + \frac{v_2}{R}\sin\psi & 1 \end{pmatrix}.
$$

Using polar coordinates for $\boldsymbol{v}$, $v_1 = r\cos\varphi$, $v_2 = r\sin\varphi$, the Euler decomposition of γ becomes

$$
\gamma = \gamma\left(\psi, \varphi, \frac{r}{R}\right) = \widehat{m}\left(\varphi - \frac{\pi}{2}\right) \cdot \hat{u}\left(\frac{r}{R}\right) \cdot \widehat{m}\left(\psi - \varphi + \frac{\pi}{2}\right), \qquad \frac{r}{R} \ll 1.
$$

Thus, finally,

$$
\Pi_R\big(\boldsymbol{v}, \big(\psi, (t, n(\boldsymbol{\xi}))\big)\big) = \gamma\left(\psi, \varphi, \frac{r}{R}\right) \cdot \hat{d}(t) \cdot \hat{n}(\boldsymbol{\xi}\,). \tag{C.8}
$$

This shows that, in geometrical terms, the contraction amounts to let the radius of the sphere go to infinity. In conclusion, the Lorentz group is contracted along its minimal parabolic subgroup to the semidirect product $G_2 = \mathbb{R}^2 \rtimes SIM(2)$. In addition, the Abelian subgroup N is also preserved during contraction.

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REFERENCES

1. S. T. Ali, J-P. Antoine, J-P. Gazeau, and U. A. Mueller, Coherent states and their generalizations: A mathematical overview, *Rev. Math. Phys.* **7** (1995), 1013–1104.
2. J-P. Antoine, M. Duval-Destin, R. Murenzi, and B. Piette, Image analysis with 2D wavelet transform: Detection of position, orientation and visual contrast of simple objects, *in* “Wavelets and Their Applications (Proc. Marseille 1989)” (Y. Meyer, Ed.), pp. 144–159, Masson, Paris, and Springer-Verlag, Berlin, 1991.
3. J-P. Antoine, P. Carrette, R. Murenzi, and B. Piette, Image analysis with two-dimensional continuous wavelet transform, *Signal Process.* **31** (1993), 241–272.

4. J-P. Antoine and P. Vandergheynst, Wavelets on the n-sphere and other manifolds, *J. Math. Phys.* **39** (1998), 3987–4008.
5. J. C. van den Berg (Ed.), "Wavelets in Physics," Cambridge Univ. Press, Cambridge, 1999.
6. S. Dahlke, W. Dahmen, E. Schmitt, and I. Weinreich, Multiresolution analysis and wavelets on S^2 and S^3, *Numer. Funct. Anal. Optim.* **16** (1995), 19–41.
7. S. Dahlke and P. Maass, Continuous wavelet transforms with applications to analyzing functions on spheres, *J. Fourier Anal. Appl.* **2** (1996), 379–396.
8. I. Daubechies, A. Grossmann, and Y. Meyer, Painless nonorthogonal expansions, *J. Math. Phys.* **27** (1986), 1271–1283.
9. I. Daubechies, "Ten Lectures on Wavelets," SIAM, Philadelphia, PA, 1992.
10. A. H. Dooley, Contractions of Lie groups and applications to analysis, *in* "Topics in Modern Harmonic Analysis," Vol. I, pp. 483–515, Istituto Nazionale di Alta Matematica Francesco Severi, Roma, 1983.
11. A. H. Dooley and J. W. Rice, Contractions of rotation groups and their representations, *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 509–517; On contractions of semisimple Lie groups, *Trans. Amer. Math. Soc.* **289** (1985), 185–202.
12. B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, "Modern Geometry—Methods and Applications," Vol. I, Springer-Verlag, New York/Berlin, 1984.
13. M. Duval-Destin, "Analyse Spatiale et Spatio-Temporelle de la Stimulation Visuelle à l'aide de la Transformée en Ondelettes," Thèse de Doctorat, Université d'Aix-Marseille II, 1991.
14. W. Freeden and U. Windheuser, Combined spherical harmonic and wavelet expansion—A future concept in Earth's gravitational determination, *Appl. Comput. Harmon. Anal.* **4** (1997), 1–37, and papers quoted therein.
15. W. Freeden, M. Schreiner, and T. Gervens, "Constructive Approximation on the Sphere, with Applications to Geomathematics," Clarendon Press, Oxford, 1997.
16. J. Göttelmann, Locally supported wavelets on manifolds, with applications to the 2D sphere, *Appl. Comput. Harmon. Anal.* **7** (1999), 1–33.
17. A. Grossmann, J. Morlet, and T. Paul, Integral transforms associated to square integrable representations. I. General results, *J. Math. Phys.* **26** (1985), 2473–2479.
18. A. Grossmann, J. Morlet, and T. Paul, Integral transforms associated to square integrable representations. II. Examples, *Ann. Inst. H. Poincaré* **45** (1986), 293–309.
19. D. M. Healy Jr., D. Rockmore, and S. S. B. Moore, "FFTs for the 2-Sphere—Improvements and Variations," Technical Report PCS-TR96-292, Dartmouth College, Hanover, NH, 1996.
20. M. Holschneider, Continuous wavelet transforms on the sphere, *J. Math. Phys.* **37** (1996), 4156–4165.
21. E. Inönü and E. P. Wigner, On the contraction of groups and their representations, *Proc. Nat. Acad. Sci. USA* **39** (1953), 510–524.
22. A. W. Knapp, "Lie Groups Beyond an Introduction," Birkhäuser-Verlag, Basel, 1996.
23. G. Köthe, "Topological Vector Spaces," Vol. I, Springer-Verlag, Berlin, 1969.
24. P. Maass, Remarks at the Colloquium "Wavelets on the Sphere," Potsdam, February 1996.
25. J. Mickelsson and J. Niederle, Contraction of representations of de Sitter groups, *Commun. Math. Phys.* **27** (1972), 167–180.
26. M. R. Mohlenkamp, "A Fast Transform for Spherical Harmonics," Ph.D. Thesis, Yale University, New Haven, CT, 1997; A fast transform for spherical harmonics, *J. Fourier Anal. Appl.* **5** (1999), 159–184.
27. R. Murenzi, "Ondelettes Multidimensionnelles et Applications à l'Analyse d'Images," Thèse de Doctorat, Univ. Cath. Louvain, Louvain-la-Neuve, 1990.
28. F. J. Narcowich and J. D. Ward, Nonstationary wavelets on the m-sphere for scattered data, *Appl. Comput. Harmon. Anal.* **3** (1996), 1324–1336.
29. A. M. Perelomov, "Generalized Coherent States and Their Applications," Springer-Verlag, Berlin, 1986.
30. D. Potts and M. Tasche, Interpolatory wavelets on the sphere, *in* "Approximation Theory," Vol. VIII (C. K. Chui and L. L. Schumaker, Eds.), pp. 335–342, World Scientific, Singapore, 1995.

31. D. Potts, G. Steidl, and M. Tasche, Kernels of spherical harmonics and spherical frames, *in* "Advanced Topics in Multivariate Approximation" (F. Fontanella, K. Jetter, and P-J. Laurent, Eds.), pp. 1–154, World Scientific, Singapore, 1996.

32. B. Rubin, Continuous wavelet transforms on a sphere, *in* "Signal and Image Representation in Combined Spaces" (J. Zeevi and R. Coifman, Eds.), pp. 457–476, Academic Press, San Diego, 1998.

33. B. Rubin, Spherical Radon transform and related wavelet transforms, *Appl. Comput. Harmon. Anal.* **5** (1998), 202–215.

34. J. Saletan, Contraction of Lie groups, *J. Math. Phys.* **2** (1961), 1–21.

35. P. Schröder and W. Sweldens, Spherical wavelets: Efficiently representing functions on the sphere, *in* "Computer Graphics Proc. (SIGGRAPH95)," pp. 161–175, ACM Siggraph, Los Angeles, 1995.

36. W. Sweldens, The lifting scheme: A custom-design construction of biorthogonal wavelets, *Appl. Comput. Harmon. Anal.* **3** (1996), 1186–1200.

37. R. Takahashi, Sur les représentations unitaires des groupes de Lorentz généralisés, *Bull. Soc. Math. France* **91** (1963), 289–433.

38. B. Torrésani, Phase space decompositions: Local Fourier analysis on spheres, preprint CPT-93/P.2878, Marseille, 1993; Position-frequency analysis for signals defined on spheres, *Signal Proc.* **43** (1995), 341–346.

39. P. Vandergheynst, "Ondelettes Directionnelles et Ondelettes sur la Sphère," Thèse de Doctorat, Univ. Cath. Louvain, Louvain-la-Neuve, 1998.