# REGULARITY IN METRIC SPACES

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# Abstract

Using arguments developed by De Giorgi in the 1950's, it is possible to prove the regularity of the solutions to a vast class of variational problems in the Euclidean space. The main goal of the present thesis is to extend these results to the more abstract context of metric spaces with a measure. In particular, working in the axiomatic framework of Gol'dshtein - Troyanov, we establish both the interior and the boundary regularity of quasi-minimizers of the $p$-Dirichlet energy. Our proof works for quite general domains, assuming some natural hypotheses on the (axiomatic) $D$-structure. Furthermore, we prove analogous results for extremal functions lying in the class of Sobolev functions in the sense of Hajłasz - Koskela, i.e. functions characterized by the single condition that a Poincaré inequality be satisfied.

Our strategy to prove these regularity results is first to show that, in a very general setting, the (Hölder) continuity of a function is a consequence of three specific technical hypotheses. This part of the argument is the essence of the De Giorgi method. Then, we verify that for a function $u$ which is a quasi-minimizer in an axiomatic Sobolev space or an extremal Sobolev function in the sense of Hajłasz - Koskela, these technical hypotheses are indeed satisfied and $u$ is thus (Hölder) continuous.

In addition to that, we establish the Harnack's inequality for these extremal functions, and we show that the Dirichlet semi-norm of a piecewise-extremal function is equivalent to the sum of the Dirichlet semi-norms of its components.

## Key words

Analysis on metric spaces, Sobolev spaces, Quasi-minima, Interior and boundary regularity, De Giorgi's method.

# Résumé

En utilisant des arguments développés par De Giorgi dans les années 1950, on peut démontrer la régularité des solutions de nombreux problèmes variationnels dans l'espace euclidien. Le but principal de cette thèse est d'étendre ces resultats de régularité au cadre plus abstrait des espaces métriques mesurés. En particulier, en travaillant dans le cadre axiomatique développé par Gol'dshtein et Troyanov, on démontre la régularité intérieure et la régularité au bord des fonctions qui sont quasi-minimisantes pour la $p$-energie de Dirichlet. Notre preuve est valide pour des domaines assez généraux, en supposant que quelques conditions naturelles sur la $D-$structure (axiomatique) sont satisfaites. Nous démontrons aussi des résultats analogues pour les fonctions extrémales dans la classe des fonctions de Sobolev étudiée par Hajłasz et Koskela, i.e. des fonctions qui sont caractérisées par une inégalité de Poincaré.

Pour établir ces résultats nous employons la stratégie suivante: Nous montrons d'abord que, dans un cadre très général, une fonction qui vérifie trois hypothèses techniques est (Hölder) continue. Cette partie de l'argument forme l'essence de la méthode de De Giorgi. Puis nous vérifions que pour toute fonction $u$ qui est quasi-minimisantes pour la $p$-energie de Dirichlet dans un espace de Sobolev axiomatique ou qui est une fonction de Sobolev extrémale au sens de Hajłasz et Koskela, nos trois hypothèses techniques sont en effet vérifiées. La continuité Hölderienne de $u$ en découle.

En conclusion de cette thèse, nous établissons l'inégalité de Harnack pour ces fonctions extrémales et nous prouvons que la semi-norme de Dirichlet d'une fonction extrémale par morceaux est équivalente à la somme des semi-normes de Dirichlet de ses composants.

## Mots-clés

Analyse sur les espaces métriques, Espaces de Sobolev, Fonctions quasi-minimisantes, Régularité intérieure et au bord, Méthode de De Giorgi.

# Contents

# Chapter 1

# Introduction

## 1.1 On the history of the regularity problem in the calculus of variations on $\mathbb{R}^n$

The problem of the regularity of solutions to partial differential equations with prescribed boundary values and of regular variational problems constitutes one of the most interesting chapters in analysis, which has its origins mostly starting from the year 1900, when D. Hilbert formulated his famous 23 problems in an address delivered before the International Congress of Mathematicians at Paris. The essential parts of the twentieth problem on existence of solutions and its related nineteenth problem about the regularity itself read as follows:

*19th problem:* "Are the solutions of regular problems in the calculus of variations always necessarily analytic?"

*20th problem:* "Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also, if need be, that the notion of a solution shall be suitably extended?"

By a regular variational problem Hilbert meant a problem of minimizing a variational integral of the type

$$J[u] = \int_\Omega F(x, u(x), \nabla u(x))dx$$

in a set of functions $u : \overline{\Omega} \to \mathbb{R}$ of the class $\mathcal{C}^1(\Omega)$ satisfying the Dirichlet type boundary condition

$$u(x) = \varphi(x) \quad \text{for} \quad x \in \partial\Omega\,,$$

for a prescribed continuous boundary values function $\varphi$ on $\partial\Omega$. This problem is called the Dirichlet problem for the functional $J$. Here $\Omega$ is open in $\mathbb{R}^n$ and the given function (integrand) $F(x,u,p)$ satisfies the regularity ("convexity") condition:

$$F \in \mathcal{C}^2(\overline{\Omega})\,, \quad \left(\frac{\partial^2 F}{\partial p_i \partial p_j}\right) > 0 \quad \text{for} \quad x \in \overline{\Omega},\ u \in \mathbb{R},\ p \in \mathbb{R}^n\,.$$

This problem is linked to partial differential equations by means of its Euler-Lagrange equation. Namely, if $u$ minimizes the integral $J[u]$ and if it is sufficiently smooth, then $u$ satisfies the following partial differential equation

$$\sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i \partial p_i}(x,u,p) = \frac{\partial F}{\partial u}(x,u,p),$$

or

$$\sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial p_i \partial p_j}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\left(\frac{\partial^2 F}{\partial p_i \partial u}\frac{\partial u}{\partial x_i} + \frac{\partial^2 F}{\partial p_i \partial x_i}\right) = \frac{\partial F}{\partial u}$$

(obtained from the first equation after differentiation), which, under the regularity condition mentioned above, is a quasi-linear elliptic equation of second order.

In particular, for the Dirichlet p-energy integral

$$\int_\Omega |\nabla u(x)|^p dx$$

the corresponding Euler-Lagrange equation, for $1 < p < \infty$, is

$$\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = 0.$$

In 1904 S. Bernstein proved that a $\mathcal{C}^3$ solution of a nonlinear second order elliptic equation in the plane,

$$\mathcal{F}(x,y,u,Du,D^2u) = 0$$

is analytic whenever $\mathcal{F}$ is analytic. Several years later he has obtained also the existence of solutions of analytic quasilinear second order equations in two variables. To carry through his proofs, S. Bernstein established estimates for derivatives of any eventual solution. These kind of estimates, namely estimates valid for all possible solutions of a class of problems, even if the hypotheses do not guarantee the existence of such solutions, have the name "a priori estimates".

The fundamental role of the "a priori estimates" in the existence and regularity problems for general elliptic equations was fully understood and clarified in the works of Leray and Schauder in 1934. In particular, applying these estimates it was proved by them and by other authors in the 30's that every sufficiently smooth, say $\mathcal{C}^{0,\alpha}$ (Hölder continuous), solution of the Dirichlet problem is analytic, provided that $F$ is analytic.

Another approach to the existence problem is provided by the so-called "Direct Methods in the Calculus of Variations". While this tool is very powerful and quite general (though applied primarily to the variational case), the solutions which are obtained have derivatives only in a generalized sense and satisfy the equation only in a correspondingly weak form.

Thus arose the problem of proving that such "generalized solutions" are "regular", namely possess enough smoothness so as to satisfy the differential equation in a classical sense. In this respect, Hilbert's twentieth problem of existence of classical solutions becomes precisely the problem of regularity of generalized solutions.

This *problem of regularity*, by which we now mean the problem to show that solutions, or extremals, which belong to a Sobolev space, are in fact Hölder continuous, resisted many attempts, but finally in 1957, E. De Giorgi [4] and J. Nash [22], independently of each other, provided a proof of it. Later, in 1960, J. Moser [21], by entirely different methods, gave another proof of their result. The Moser's argument was later extended by J. Serrin, N. S. Trudinger and by others. While this approach (known as Moser's iteration technique), which is based on differential equation, has proved to be very useful for investigating various problems in the Euclidian spaces, it is not readily generalized to the case when one wants to deal with regularity questions on a general metric space (see, however, [2]), since the concept of a partial derivative is (generally) meaningless on a metric space, and thus there is no differential (Euler-Lagrange) equation. However, since it is possible to define a substitute for the modulus of the usual gradient to the case of general metric spaces, the approach of De Giorgi, which is essentially a variational one, can be used. This approach was developed and generalized to certain cases of non-linear equations by O. Ladyzhenskaya, N. Ural'tseva, G. Stampacchia and by others. Later, in the 80s, M. Giaquinta [6] (see also [7]), and then, in the 90s, J. Malý, W. P. Ziemer [18] have tried to give to the method of De Giorgi a more transparent form.

The primary purpose of all the above mentioned results was to investigate the behavior of weak solutions in the interior of a domain. In 1924 N. Wiener [27] established a criterion to characterize continuity at the boundary for harmonic functions. For the more general case of elliptic equations the first steps to find a similar criterion were made by W. Littman, G. Stampacchia and H.F. Weinberger who proved that a point in the boundary of an arbitrary domain was simultaneously regular for harmonic functions and weak solutions of linear equations with bounded, measurable coefficients. In his work concerning the local behavior of solutions of quasilinear equations, J. Serrin discovered that a capacity, now known as the $p$-capacity, was the appropriate measurement for describing removable sets for weak solutions. Later, V.G. Maz'ya [19], [20] discovered a Wiener-type expression involving this capacity which provided a sufficient condition for continuity at the boundary of weak solutions of equations whose structure is similar to that of the $p$-Laplacian. Utilizing different techniques, R. Gariepy and W.P. Ziemer have shown that the Maz'ya's condition was also sufficient for boundary continuity for solutions of a large class of quasilinear equations in divergence form. After some time, Ziemer [28] generalized this result for quasiminimizers, a concept generalizing the notion of solutions of elliptic equations and variational problems.

## 1.2 New development: Analysis on metric measure spaces

The subject of analysis on metric measure spaces has become a topic of intensive study in the last decade and presents now quite a rich theory. The sources [1], [10], [14], [15], [16] are good references to the subject. A generalization of the classical theory of Sobolev spaces has been motivated by diverse applications to singular Riemannian manifolds, analysis on graphs, subelliptic equations, quasiconformal mappings on Loewner spaces etc.

Various notions of Sobolev spaces on a metric measure space have been introduced and studied in recent years. Among the most important ones are the Sobolev spaces of Hajłasz [12], the Sobolev spaces via upper gradients [16], the axiomatic Sobolev spaces of Gol'dshtein - Troyanov [10],[11] and the Sobolev spaces based on a Poincaré inequality [14]. Let us mention that the last two are more general ones including the two first.

The concept of the Sobolev space in the sense of Hajłasz consists in the following: Given a metric measure space $(X, d, \mu)$ and a locally integrable function $u : X \to \mathbb{R}$, a measurable function $g : X \to \mathbb{R}_+$ is said to be a *Hajłasz pseudo-gradient* of the function $u$, if

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$$

for almost all $x, y \in X$. The Sobolev space of Hajłasz is then the set of functions integrable on $X$ and having an integrable pseudo-gradient.

In 1998, J. Heinonen and P. Koskela [16] proposed an alternative notion of gradient on the metric measure spaces: Let $(X, d, \mu)$ be a metric measure space and $u : X \to \mathbb{R}$ be a continuous function. A Borel measurable function $g : X \to \mathbb{R}_+$ is an *upper gradient* of $u$, if for every rectifiable curve parameterized by the arc-length $\gamma : [0, l_\gamma] \to X$ we have

$$|u(\gamma(l_\gamma)) - u(\gamma(0))| \leq \int_\gamma g\, ds\,.$$

The basic idea of the axiomatic description of a Sobolev space on metric spaces is the following: "Given a metric space $X$ with a measure $\mu$, one associates (by some unspecified mean) to each function $u : X \to \mathbb{R}$ a set $D[u]$ of functions called the *pseudo-gradients* of $u$ (intuitively, a pseudo-gradient $g \in D[u]$ is a function which exerts some control on the variation of $u$). Instead of specifying how the pseudo-gradients are actually defined, one requires them to satisfy some axioms. A function $u \in L^p(X)$ belongs then to $W^{1,p}(X)$ if it admits a pseudo-gradient $g \in D[u] \cap L^p(X)$" (see [10]).

In order to briefly describe the approach to the Sobolev spaces based on a Poincaré inequality, let us quote the following phrases from [14]: "(Given a metric space $X$), it is natural to regard a pair $u, g$ that satisfies a $p$-Poincaré inequality in $X$ as a Sobolev function and its gradient. In this sense we develop the theory of Sobolev functions on metric spaces with "gradient" in $L^p$ for all $p > 0$."

The axiomatic approach of Gol'dshtein - Troyanov and the Sobolev spaces based on a Poincaré inequality will constitute the general settings of the present thesis and will be recalled in more details in Chapters 2 and 3.

## 1.3 Main results

The principal aim of the present thesis is to extend the aforementioned results on regularity in $\mathbb{R}^n$ to the context of a general metric space with a measure. Namely we would like to treat the following

**Problem:** *Prove interior and boundary regularity for certain class of variational problems on a metric space.*

The question of the interior regularity on a general metric space appeared for the first time in the paper [17] of J. Kinnunen and N. Shanmugalingam. The boundary regularity on metric spaces was treated by J. Björn in her paper [3]. In these two papers the authors applying the De Giorgi's method have studied the Hölder continuity of the quasi-minimizers of the $p$-Dirichlet integral on general metric spaces using the notion of upper gradients. Note however that this approach to Sobolev spaces is restricted to length spaces or quasi-convex metric spaces, the spaces which have sufficiently many rectifiable curves, which excludes fractals and graphs.

In this thesis we extend the regularity results to the context of the Sobolev spaces based on a Poincaré inequality and for the axiomatic Sobolev spaces.

As to the approach of Hajłasz note that, due to the "global" nature of his Sobolev space, it seems that it would not be possible to establish the regularity results we want at all.

We are now in position to state the main results contained in the present work. These are Theorems A, B, C and D below. A standing assumption of these theorems is the doubling condition for the measure $\mu$ on $X$, which means that there exists a constant $C > 0$ such that

$$\mu(2B) \le C\mu(B)$$

whenever $B$ is a ball in $X$ and $2B$ is the ball with the same center as $B$ and with radius twice that of $B$.

### 1.3.1 Regularity for Sobolev functions in the sense of Hajłasz - Koskela

**Definition** Let $X$ be a metric space with a measure $\mu$. We say that a function $u \in L^1_{loc}(X)$ is a *Sobolev function in the sense of Hajłasz - Koskela* if there exists a function $0 \le g \in L^q(X)$, $1 < q < \infty$, and two constants $\sigma \ge 1$ and $C_P > 0$ such that the following a $(1,q)$-*Poincaré inequality*

$$\left( \fint_B |u - u_B| d\mu \right) \le C_P \, r \left( \fint_{\sigma B} g^q d\mu \right)^{1/q} \tag{1.1}$$

holds on every ball $B \subset X$, where $r$ is the radius of B. Here and in what follows, we use the notation

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u d\mu.$$

We denote by $PW^{1,q}(X)$ the set of all Sobolev functions $u \in L^1_{loc}(X)$ in the sense of Hajłasz - Koskela.

The pair $(u, g)$ may satisfy some additional important properties. In particular, one says that $(u, g)$ has the *truncation property* if when we truncate the function $u$, the obtained function and the same truncation of $g$ still satisfy the $(1,q)$-Poincaré inequality. Another property is the *p-De Giorgi condition*, which is a kind of reverse Poincaré inequality (see Chapter 3 for precise definitions).

**Theorem A** *Let $u, -u \in PW^{1,q}(X)$ (i.e. there exist two functions $g^+, g^- \in L^q(X)$ such that both $(u, g^+)$ and $(-u, g^-)$ satisfy a $(1,q)$-Poincaré inequality). Assume that any pair of functions satisfying a $(1,q)$-Poincaré inequality in $X$ has the truncation property. If the pairs $(u, g^+)$ and $(-u, g^-)$ enjoy the p-De Giorgi condition, $p > q$, then the function $u$ is locally Hölder continuous.*

(see Theorem 7.1)

**Theorem B** *Let $\Omega \subset X$ be an arbitrary open set and $x_0 \in \partial\Omega$ a boundary point. Assume that the following "Wiener type" condition*

$$\liminf_{\rho \to 0} \frac{1}{|\log \rho|} \int_\rho^1 \exp\left( -C \left( \frac{\mu(B(x_0,R))}{\mu(B(x_0,R)\setminus\Omega)} \right)^{\frac{(q-1)p}{p-q}} \right) \frac{dR}{R} > 0\,,$$

*holds for some constant $C > 0$. Suppose also that the functions $u, -u \in PW^{1,q}(X)$ satisfy the p-De Giorgi condition, $p > q$, and that any pair of functions satisfying a $(1,q)$-Poincaré inequality in $X$ has the truncation property. Then, if $u$ coincides a.e. in the complement of $\Omega$ with a Hölder continuous at $x_0$ function $\vartheta$, the function $u$ itself is Hölder continuous at $x_0$.*

(see Theorem 7.2)

The examples of metric spaces that support a $(1,q)$-Poincaré inequality with the validity of the truncation property are Riemannian manifolds, topological manifolds, Carnot-Carathéodory spaces and others (see Sections 10 and 11 in [14]).

### 1.3.2 Regularity in axiomatic Sobolev spaces

**Definition** A *D-structure* on a metric measure space $(X, d, \mu)$ is an operation which associates to each function $u \in L^p_{loc}(X)$ a collection $D[u]$ of measurable functions $g : X \to \mathbb{R}_+ \cup \{\infty\}$ (called the *pseudo-gradients* of $u$), satisfying a number of axioms (see Axioms A1 - A5 in Section 2.1).

*Strong locality* and a $(1,q)$-*Poincaré inequality* are additional properties which a $D$-structure may possess. These are notions similar to the truncation property and the $(1,q)$-Poincaré inequality respectively for Sobolev functions in the sense of Hajłasz - Koskela (see Sections 2.2 and 2.3 for precise definitions).

The *Dirichlet p-energy* of a function $u$ is defined as

$$\mathcal{E}_p(u) := \inf\left\{ \int_X g^p d\mu \,\middle|\, g \in D[u] \right\}$$

and the *variational capacity* of a bounded set $F \subset X$ by

$$\mathrm{Cap}_p(F) := \inf\{\mathcal{E}_p(u) \mid u \in \mathcal{A}_p(F)\}\,,$$

where $\mathcal{A}_p(F) := \{u \mid u \geq 1 \text{ near } F,\ u \geq 0 \text{ a.e. and } \operatorname{supp}(u) \Subset X\}$ (see corresponding Definitions in Sections 2.2 and 2.4).

**Theorem C** *Let a metric measure space $X$ be equipped with a $D$-structure. If the $D$-structure is strongly local and supports a $(1,q)$-Poincaré inequality for some $q$, $q < p$, then a quasi-minimizer $u$ of the variational $p$-capacity of certain set $F \subset X$ is locally Hölder continuous on the set $X \setminus F$.*

(see Theorem 6.2)

**Theorem D** *Assume that the hypotheses of Theorem C hold. Let $x_0 \in \partial F$ a boundary point. If in addition the following condition is satisfied*

$$\liminf_{\rho \to 0} \frac{1}{|\log \rho|} \int_\rho^1 \exp\left( -C \left( \frac{R^{-q}\mu(B(x_0,R))}{\operatorname{Cap}_q\left(B(x_0, \frac{1}{2}R) \cap F, B(x_0,R)\right)} \right)^{\frac{p}{p-q}} \right) \frac{dR}{R} > 0\,,$$

*for some constant $C > 0$, then the quasi-minimizer $u$ is Hölder continuous at $x_0$.*

(see Theorem 6.5)

The strong locality is a generalization of the notion of strict locality introduced in [10]. The main examples of strictly local axiomatic Sobolev spaces which support a $(1,q)$-Poincaré inequality are weighted Sobolev spaces, Sobolev spaces on Riemannian and sub-riemannian manifolds (Carnot groups) and others (see Section 2 in [10]).

### 1.3.3 Strategy of proving the main results

One of the objects of this thesis is to show that the De Giorgi's method might be applied to a very general situation, when there is not any (analog of) Sobolev space to deal with. In particular, in the present work we try to further formalize the method reducing it to the form when, for checking the Hölder continuity (both in the interior and in the boundary points of a set) of a function $u$ on a metric measure space, it is sufficient only to verify some natural hypotheses for this function and the eventual Sobolev space of functions we are going to work with. These hypotheses (see Hypotheses H1 - H3 in Section 4.1 for the interior regularity and in addition to Hypotheses H1, H2, the hypothesis H3(b) in Chapter 5 for the boundary regularity) are expressed in terms which do not assume that $u$ belongs to a class of Sobolev functions.

The obtained "machinery" (checking Hypotheses H1 - H3) is our strategy to prove Theorems A, B, C and D. Namely, to verify the results of Theorems we first show the regularity of a function $u$ in an abstract setting (see Chapter 4 for the interior regularity and Chapter 5 for the boundary regularity) assuming only our technical hypotheses. Then we prove that these hypotheses are indeed satisfied under conditions of an appropriate theorem (Theorem A, B, C or D), either in the case of Sobolev functions in the sense of Hajłasz - Koskela (see Chapter 7) or in the case of axiomatic Sobolev spaces (see Chapter 6).

Note that Hypotheses H2 and H3 are the characteristics of the Sobolev space of functions we work with, whereas Hypothesis H1 is the property of some particular functions, the functions whose regularity we establish.

### 1.3.4 Some additional results

After the main problems of the thesis we prove some complementary results which we do not use in the main part, but which are interesting by themselves. In particular, we show that extremal functions of Theorems A and C satisfy the following Harnack's inequality

$$\sup_B u \leq C \inf_B u$$

for every sufficiently small ball $B \Subset \Omega$, where $C > 0$ is a constant independent of the ball $B$ and the function $u$ (see Theorem 8.3).

Furthermore, working in the framework of axiomatic approach to the theory of Sobolev spaces on metric spaces, we prove that the Dirichlet semi-norm of a piecewise-extremal function (see Definition 9.2) is equivalent to the sum of the Dirichlet semi-norms of its components, i.e. we show that for the function $u = \sum\limits_{k=1}^{l} b_k u_k$ the following chain of inequalities

$$\sum_{k=1}^{l} |b_k| \, \mathcal{E}_p(u_k) \leq \mathcal{E}_p(u) \leq C \sum_{k=1}^{l} |b_k| \, \mathcal{E}_p(u_k)$$

holds for some constant $C > 0$ (see Theorem 9.3).

The thesis is organized as follows.

After the introduction in the first chapter, in the second chapter we give some preliminaries on axiomatic approach to the theory of Sobolev spaces on a metric space. In particular, in the first section we define the notion of a $D$-structure, then in Section 2.2 we list some of its properties and give the definition of an axiomatic Sobolev space. In the next section we investigate some locality properties of this Sobolev space. In Section 2.4 we recall the proof of the existence of a minimizer of the $p$-Dirichlet energy.

We recall the concept of the "Sobolev space" on general metric spaces based on a Poincaré inequality in Chapter 3. Note that all the definitions and results of this and the previous chapters, except Definition 2.10, Propositions 2.11 and 2.13, are taken from [10], [11] and [14] (the definitions of Chapter 3).

In the first section of Chapter 4, we formulate the above mentioned hypotheses H1 - H3 we are going to work with. After this, in Section 4.2, we show that if a pair of functions $(u, g)$, $u, g \in L^p(X)$, satisfies Hypotheses H1 and H2, then the function $u$ of the pair is locally bounded in a set $\Omega \subset X$. In the next section we prove that if, in addition, Hypothesis H3 is satisfied for the functions $u$ and $g$, then the function $u$ is locally Hölder continuous in the interior of $\Omega$.

In the third chapter, we mimic the strategy of the previous chapter to show that a function $u \in L^p(X)$ satisfying with some function $g \in L^p(X)$ Hypotheses H1, H2 and Hypothesis H3(b) stated at the beginning of the chapter and which coincides a.e. with a Hölder continuous function outside a set, is Hölder continuous at a boundary point of this set, provided certain condition for the set is satisfied in this point. This condition (5.2) applied to the concrete situations in the next chapters will give an analog of the famous Wiener criterion (the sufficiency part of it) for the continuity at a boundary point. Note that the "criterion" (6.3) we obtain in Section 6.2 is very similar to the one obtained by J. Björn in [3] for the upper-gradients approach.

Chapter 6 is devoted to the regularity, both interior and boundary, of a quasi-minimizer of the $p$-energy functional in the axiomatic setting. In Section 6.1, we prove that if the $D$-structure in the sense of Gol'dshtein - Troyanov on a metric space $X$ equipped with a Borel regular doubling measure $\mu$ is strongly local and supports a weak $(1, q)$-Poincaré inequality for some $q$, $q < p$, then a quasi-minimizer, the function minimizing on a set $\Omega \subset X$ the Dirichlet $p$-

energy up to a constant, satisfies our hypotheses H1 - H3 in the pair with its minimal pseudo-gradient and thus is Hölder continuous inside the set $\Omega$. In the second section of the chapter, we show that under the same assumptions Hypotheses H1, H2 and H3(b) are satisfied by a quasi-minimizer and its minimal pseudo-gradient in a boundary point of $\Omega$, and, therefore, with the condition (6.3) held, that the quasi-minimizer is Hölder continuous at this point.

In the next chapter, we show that on a metric measure space $X$ with a doubling measure $\mu$, a function $u$ from the class $PW^{1,q}(X)$ of the Poincaré-Sobolev functions on $X$, which has an additional property ($p$-De Giorgi condition), satisfies Hypotheses H1 - H3, provided that $PW^{1,q}(X)$ has the truncation property, and, thus, is Hölder continuous.

In Chapter 8, we show that the extremal functions in the axiomatic Sobolev spaces and in the class of Poincaré-Sobolev functions satisfy the Harnack's inequality. This result which is not used in other parts of the thesis, is interesting by itself. The Harnack's inequality for quasi-minimizers in upper-gradient's approach is proven in [17].

Finally, in the last chapter, we prove that the Dirichlet semi-norm of a piecewise-extremal function is equivalent to the sum of the Dirichlet semi-norms of its components. This problem is motivated by some considerations in the theory of homeomorphisms with bounded $p$-distortion on metric spaces (see [8], [25], [26]).

# Chapter 2

# Preliminaries on Axiomatic Sobolev Spaces

Let $(X, d)$ be a metric space equipped with a Borel regular outer measure $\mu$ such that $0 < \mu(B) < \infty$ for any ball $B = B(R) = B(z, R) = \{x \in X : d(x, z) < R\}$ in $X$ of positive radius. If $\sigma > 0$ and $B = B(z, R)$ is a ball, we let $\sigma B$ denote the ball $B(z, \sigma R)$.

In the sequel, for convenience we will suppose that the space $X$ is locally compact and separable. For $1 \leq p < \infty$, $L^p_{loc}(X) = L^p_{loc}(X, d, \mu)$ is the space of measurable functions on $X$ which are $p$-integrable on every relatively compact subset of $X$.

In this chapter we recall basic definitions and give a brief summary of the axiomatic theory of Sobolev spaces developed by V.M. Gol'dshtein and M. Troyanov in [10], which will constitute the general setup of our study in Chapters 5,6,7 and 8. We refer the reader to this paper and to the paper [11] for more details on the axiomatic theory of Sobolev spaces.

## 2.1 $D$-structure on a metric measure space

**Definition 2.1 ($D$-structure)** *A $D$-structure on $(X, d, \mu)$ is an operation which associates to each function $u \in L^p_{loc}(X)$ a collection $D[u]$ of measurable functions $g : X \to \mathbb{R}_+ \cup \{\infty\}$ (called the pseudo-gradients of $u$). The correspondence $u \to D[u]$ is supposed to satisfy the following axioms A1-A5:*

**Axiom A1 (Non triviality)** If $u : X \to \mathbb{R}$ is non-negative and $k$-Lipschitz, then the function

$$g := k\chi_{\mathrm{supp}(u)} = \begin{cases} k & \text{on} \quad \mathrm{supp}(u) \\ 0 & \text{on} \quad X \setminus \mathrm{supp}(u) \end{cases}$$

belongs to $D[u]$.

**Axiom A2 (Upper linearity)** If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$ and $g \geq |\alpha|g_1 + |\beta|g_2$ almost everywhere, then $g \in D[\alpha u_1 + \beta u_2]$.

**Axiom A3 (Strong Leibnitz rule)** Let $u \in L^p_{loc}(X)$. If $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function

$$h(x) = (|\varphi|g(x) + \mathrm{Lip}(\varphi)|u(x)|)$$

belongs to $D[\varphi u]$.

**Axiom A4 (Lattice property)** Let $u := \max\{u_1, u_2\}$ and $v := \min\{u_1, u_2\}$ where $u_1, u_2 \in L^p_{loc}(X)$. If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$, then

$$g := \max\{g_1, g_2\} \in D[u] \cap D[v]\,.$$

**Axiom A5 (Completeness)** Let $\{u_i\}$ and $\{g_i\}$ be two sequences of functions such that $g_i \in D[u_i]$ for all $i$. Assume that $u_i \to u$ in $L^p_{loc}(X)$ topology and $(g_i - g) \to 0$ in $L^p$ topology, then $g \in D[u]$.

**Remark** Originally, in [10] in the place of Axiom A3 stated here one postulates the following

**Axiom A3$^*$(Leibnitz rule)** Let $u \in L^p_{loc}(X)$. If $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function

$$h(x) = (\sup|\varphi|g(x) + \mathrm{Lip}(\varphi)|u(x)|)$$

belongs to $D[\varphi u]$ (The absolute value of $\varphi$ is replaced by $\sup|\varphi|$).

This “weaker” version of the Leibnitz rule allows the authors to include in the class of axiomatic Sobolev spaces such spaces as graphs (combinatorial Sobolev spaces) and Sobolev spaces of Hajłasz. Note, however, that these “global” spaces do not satisfy certain localization properties without which it is not clear how it would be possible to achieve the regularity results of the present thesis.

## 2.2 Some properties of $D$-structure. Axiomatic Sobolev space.

**Definition 2.2 (Poincaré inequality)** *One says that a $D$-structure on a metric measure space $X$ supports a weak $(s,q)$-Poincaré inequality, $s,q \geq 1$, if there exist two constants $\sigma \geq 1$ and $C_P > 0$ such that*

$$\left(\fint_B |u - u_B|^s d\mu\right)^{1/s} \leq C_P\, r \left(\fint_{\sigma B} g^q d\mu\right)^{1/q} \tag{2.1}$$

*for any ball $B \subset X$, any $u \in L^p_{loc}(X)$ and any $g \in D[u]$. Here $r$ is the radius of $B$. Recall that*

$$u_B = \fint_B u\, d\mu = \frac{1}{\mu(B)} \int_B u d\mu.$$

By the Hölder inequality, a weak $(s,q)$-Poincaré inequality implies weak $(s',q')$-Poincaré inequalities with the same $\sigma$ for all $s' \leq s$ and $q' \geq q$. On the other hand, by Theorem 5.1 in [14], a weak $(1,q)$-Poincaré inequality implies a weak $(s,q)$-Poincaré inequality for some $s > q$ and possibly a new $\sigma$.

We define a notion of energy and the associated Sobolev space as follows:

**Definition 2.3 (Energy and Sobolev space)** *The p-Dirichlet energy of a function $u \in L^p_{loc}(X)$ is defined to be*

$$\mathcal{E}_p(u) = \inf\left\{\int_X g^p d\mu : g \in D[u]\right\},$$

*and the p-Dirichlet space is the space $\mathcal{L}^{1,p}(X)$ of functions from $L^p_{loc}(X)$ with finite p-energy. The Sobolev space is then the space*

$$W^{1,p}(X) := \mathcal{L}^{1,p}(X) \cap L^p(X).$$

**Theorem 2.4** *$W^{1,p}(X)$ is a Banach space with norm*

$$\|u\|_{W^{1,p}(X)} = \left(\int_X |u|^p d\mu + \mathcal{E}_p(u)\right)^{1/p}.$$

**Proof** See [10].

**Proposition 2.5** *Assume that* $1 < p < \infty$*. Then for any function* $u \in \mathcal{L}^{1,p}(X)$*, there exists a unique function* $g_u \in D[u]$ *such that* $\int_X g_u^p d\mu = \mathcal{E}_p(u)$*.*

**Proof** See [10].

The function $g_u$ is called the *minimal pseudo-gradient* of $u$.

## 2.3 Locality in axiomatic Sobolev space

**Definition 2.6 (Locality)** *We say that a* $D$*-structure is* local *if, in addition to Axioms A1-A5, the following property holds:* If $u$ is constant a.e. on a relatively compact subset A$\subset X$, then $\mathcal{E}_p(u|A) = 0$, where

$$\mathcal{E}_p(u|A) := \inf \left\{ \int_A g^p d\mu \,\middle|\, g \in D[u] \right\}$$

is the local $p$-Dirichlet energy of $u$.

**Definition 2.7 (Strict locality)** *We say that a* $D$*-structure is* strictly local *if, in addition to Axioms A1-A5, we have* $(g\chi_{\{v>0\}}) \in D[v^+]$ *for any* $v \in \mathcal{L}^{1,p}(X)$ *and* $g \in D[v]$*, where* $v^+ = \max\{v, 0\}$ *and* $\chi_{\{v>0\}}$ *is the characteristic function of the set* $\{v > 0\}$*.*

**Lemma 2.8** *If the* $D$*-structure is strictly local, then it is local.*

**Proof** See [10].

**Lemma 2.9** *If the* $D$*-structure is strictly local and a pair of functions* $u, v \in L^p_{loc}(X)$ *is such that* $u = v$ *on a relatively compact set* $A \subset X$*, then*

$$\mathcal{E}_p(v|A) = \mathcal{E}_p(u|A)\,.$$

**Proof** See [10].

For the proofs of the theorems 6.2 and 6.5 we will need a still stronger notion of locality which we give in the following

**Definition 2.10 (Strong locality)** *We say that a $D$-structure is* strongly local *if, in addition to Axioms A1-A5, the following property holds:*
Let $u_1, u_2 \in \mathcal{L}^{1,p}(X)$. If $g_1 \in D[u_1], g_2 \in D[u_2]$ and

$$g(x) = \begin{cases} g_1(x) & \text{if} \quad u_1(x) < u_2(x) \\ g_2(x) & \text{if} \quad u_1(x) > u_2(x) \\ \min\{g_1(x), g_2(x)\} & \text{if} \quad u_1(x) = u_2(x)\,, \end{cases}$$

then $g \in D[\min\{u_1, u_2\}]$.

Note that if we take one of the functions $u_1$ or $u_2$ to be identically zero in the definition 2.10 of the strong locality of a $D$-structure, we obtain the strict locality of the $D$-structure.

**Proposition 2.11** *Let $u \in \mathcal{L}^{1,p}(X)$ and $A \subset X$ be a relatively compact set. If the $D$-structure on the space $X$ is strongly local, then*

$$\mathcal{E}_p(u|A) = \int_A g_u^p d\mu\ ,$$

*in particular, if $u_1, u_2 \in \mathcal{L}^{1,p}(X)$ are such that $u_1 = u_2$ a.e. on $A$, then*

$$\int_A g_{u_1}^p d\mu = \int_A g_{u_2}^p d\mu$$

**Proof** The result will easily follow if we would show that $\forall g \in D[u]$

$$g_u \leq g \quad \text{a.e. on } X.$$

Suppose that the last assertion is not true, i.e. there exist a subset $A \subset X$, $\mu(A) > 0$, and $g \in D[u]$ such that $g < g_u$ on $A$. Let $B$, $A \subset B \subset X$, be the subset of $X$ such that $g_u \leq g$ on $X \setminus B$ and $g < g_u$ on $B$. From the strong locality of the $D$-structure it will follow then that the function $h = \min\{g_u, g\}$ belongs to $D[u = \min\{u, u\}]$ and we will have

$$\int_X h^p\, d\mu = \int_B g^p\, d\mu + \int_{X\setminus B} g_u^p\, d\mu < \int_B g_u^p\, d\mu + \int_{X\setminus B} g_u^p\, d\mu = \int_X g_u^p\, d\mu\,,$$

which contradicts the minimality of $g_u$.

Q.E.D.

To prove most of the results of the thesis we will need to impose an additional condition for the measure $\mu$ on $X$.

**Definition 2.12 (Doubling measure)** *The measure $\mu$ is called* **doubling** *if there exists a constant $C_d \geq 1$ such that for all balls $B \subset X$ we have*

$$\mu(2B) \leq C_d \mu(B).$$

*Recall that $2B$ means the ball with the same center as $B$ and with radius twice that of $B$. $C_d$ is called the* **doubling constant**.

The locality of the $D$-structure together with a Poincaré inequality imply certain connectedness of the space $X$. Namely, in the sequel we will need the following

**Proposition 2.13** *If the space $X$ admits a $D$-structure which is strictly local and supports a weak $(1,q)$-Poincaré inequality for some $0 < q < p$, then for every $z \in X$ and $0 < r < R < diam(X)/3$, we have*

$$\mu\left(B(z,R) \setminus B(z,r)\right) > 0\,.$$

*In other words, the measure of sufficiently small annuli in $X$ is positive. If, in addition, the measure $\mu$ is doubling, then there exists $\gamma$, $0 < \gamma < 1$, independent of $R$ such that*

$$\frac{\mu(B(z, \frac{R}{2}))}{\mu(B(z,R))} \leq \gamma\,.$$

**Proof** With $z \in X$ fixed, for some $\delta$, $0 < \delta < \frac{R-r}{2}$, let us denote

$$S := \left\{x \in X \mid d(x,z) = \tfrac{R+r}{2}\right\}$$

the sphere of radius $\frac{R+r}{2}$ centered at $z$, and

$$S_\delta := \{y \in X \mid d(y,x) \leq \delta \text{ for some } x \in S\}$$

its $\delta$-neighborhood. Denote also

$$E = B\left(\tfrac{R+r}{2}\right) \setminus S_\delta \quad \text{and} \quad F = X \setminus (S_\delta \cup E)\,.$$

Suppose now that the set $S_\delta$ is empty. Then the function

$$u := \begin{cases} 1 & \text{on } E \\ 0 & \text{on } F \end{cases}$$

is a $k$-Lipschitz with $k = \frac{1}{2\delta}$. Hence, by Axiom A1, the function

$$g := k\chi_{\mathrm{supp}(u)} = \tfrac{1}{2\delta}\chi_E$$

belongs to $D[u]$. Therefore, $u$ has a finite $p$-energy, i.e. $u \in \mathcal{L}^{1,p}(X)$.

As the strict locality of the $D$-structure implies its locality, it will follow then that for any $\varepsilon > 0$, there exists a function $g_1 \in D[u]$ such that

$$\int_E g_1^p d\mu < \varepsilon.$$

From the strict locality itself, it will follow further that the function

$$g_2 = g_1\chi_{\{u>0\}} \in D[u^+] = D[u],$$

since $u^+ = u$. Note that

$$g_2 = \begin{cases} g_1 & \text{on} \quad E \\ 0 & \text{on} \quad F\,. \end{cases}$$

As $R < \mathrm{diam}(X)/3$, there are points in X lying in the complement of $B(R)$. Let $R_1 > R$ be large enough so that some of these points lie inside the ball $B(R_1)$.

The right-hand side of the weak $(1,q)$-Poincaré inequality applied to the functions $u$ and $g_2 \in D[u]$ on the ball $B(R_1)$ with $\sigma > 1$ can be estimated as follows

$$\begin{aligned} C_P R_1 \left( \fint_{B(\sigma R_1)} g_2^q d\mu \right)^{1/q} &\leq C_P R_1 \left( \fint_{B(\sigma R_1)} g_2^p d\mu \right)^{1/p} \\ &= C_P R_1 \left( \frac{1}{\mu(B(\sigma R_1))} \int_E g_1^p d\mu \right)^{1/p} \\ &< C_P R_1 \left( \frac{\varepsilon}{\mu\left(B(r)\right)} \right)^{1/p}, \end{aligned}$$

and, thus, can be made arbitrarily small by varying $\varepsilon$. The Poincaré inequality will imply then that the function $u$ is a.e. constant on the ball $B(R_1)$.

This contradiction shows that the set $S_\delta$ is non-empty and, hence, there exists a point $x_0 \in S_\delta$ and we have

$$\mu\left(B(z,R) \setminus B(z,r)\right) \geq \mu(B(x_0,\rho)) > 0\,,$$

for some $\rho < \left(\frac{R-r}{2} - \delta\right)$.

Suppose now that the measure $\mu$ is doubling. Taking $r = \frac{R}{2}$ and $\delta = \frac{R}{8}$ we see that there exists a point $x_0$ in $S_{\frac{R}{8}}$. As $B(z, \frac{R}{2}) \subset B(z, R) \setminus B(x_0, \frac{R}{8})$ and $B(z, R) \subset 15\, B(x_0, \frac{R}{8})$, the doubling property of $\mu$ implies

$$\frac{\mu(B(z, \frac{R}{2}))}{\mu(B(z, R))} \leq 1 - \frac{\mu(B(x_0, \frac{R}{8}))}{\mu(B(z, R))} \leq \gamma\,,$$

where $0 < \gamma < 1$.

Q.E.D.

## 2.4 The variational capacity

Let $\Omega \subset X$ be an open set. We denote by $C_0(\Omega)$ the set of continuous functions $u : \Omega \to \mathbb{R}$ such that $\mathrm{supp}(u) \Subset \Omega$, i.e. $\mathrm{supp}(u)$ is a compact subset of $\Omega$. $\mathcal{L}_0^{1,p}(\Omega)$ is then the closure of $C_0(\Omega) \cap \mathcal{L}^{1,p}(X)$ in $\mathcal{L}^{1,p}(X)$ for the norm

$$\|u\|_{\mathcal{L}^{1,p}(\Omega,Q)} = \left(\int_Q |u|^p d\mu + \mathcal{E}_p(u)\right)^{1/p},$$

where $Q \Subset \Omega$ is a fixed relatively compact subset of positive measure.

**Definition 2.14 (Capacity)** *The variational p-capacity of a pair $F \subset \Omega \subset X$ (where $F$ is arbitrary) is defined as*

$$\mathrm{Cap}_p(F, \Omega) := \inf\{\mathcal{E}_p(u) |\, u \in \mathcal{A}_p(F, \Omega)\},$$

*where the set of admissible functions is defined by*

$$\mathcal{A}_p(F, \Omega) := \left\{u \in \mathcal{L}_0^{1,p}(\Omega) \,\middle|\, u \geq 1 \ \text{ on a neighbourhood of } F \text{ and } u \geq 0 \text{ a.e.}\right\}.$$

If $\mathcal{A}_p(F, \Omega) = \emptyset$, then we set $\mathrm{Cap}_p(F, \Omega) = \infty$. If $\Omega = X$, we simply write $\mathrm{Cap}_p(F, \Omega) = \mathrm{Cap}_p(F)$.

We now state a result about the existence and uniqueness of extremal functions for $p$-capacity. We first need two definitions:

**Definition 2.15 (a)** *A set $S \subset X$ is* p-polar *(or* p-null*) if for any pair of open relatively compact sets $\Omega_1 \subset \Omega_2 \neq X$ such that* $\operatorname{dist}(\Omega_1, X \setminus \Omega_2) > 0$, *we have* $\operatorname{Cap}_p(S \cap \Omega_1, \Omega_2) = 0$.

**(b)** *A property is said to hold* p-quasi-everywhere *if it holds everywhere except on a p-polar set.*

**Definition 2.16** *A Borel measure $\tau$ is said to be* absolutely continuous with respect to $p$-capacity *if $\tau(S) = 0$ for all $p$-polar subsets $S \subset X$*

For any Borel subset $F \subset X$ we denote by $\mathcal{M}_p(F)$ the set of all probability measures $\tau$ on $X$ which are absolutely continuous with respect to $p$-capacity and whose support is contained in $F$.

**Definition 2.17** *A subset $F$ is said to be $p$-*fat *if it is a Borel subset and $\mathcal{M}_p(F) \neq \emptyset$.*

**Theorem 2.18** *Let $F \subset X$ be a $p$-fat subset $(1 < p < \infty)$ of the space $X$, such that $\operatorname{Cap}_p(F) < \infty$. Then there exists a unique function $u^* \in \mathcal{L}_0^{1,p}(X)$ such that $u^* = 1$ $p$-quasi-everywhere on $F$ and $\mathcal{E}_p(u^*) = \operatorname{Cap}_p(F)$. Furthermore $0 \leq u^* \leq 1$ for all $x \in X$.*

The function $u^*$ is called the *capacitary function* of the condenser $F$.

The proof of this theorem could be found in [11]. However, as the minimizer given by the theorem is one of the primary objects of study in this thesis, below we give the proof of the theorem 2.18 in details:

**Proof** Let us choose a measure $\tau \in \mathcal{M}_p(F)$ and set $E := L^p(X, d\tau) \oplus L^p(X, d\mu)$. Then $E$ is a uniformly convex Banach space (i.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $x, y \in E$ with $\|x\| = \|y\| = 1$ then $\|x - y\| \geq \varepsilon$ implies $\|\frac{1}{2}(x+y)\| \leq (1-\delta)$) for the norm

$$\|(u,g)\|_E := \left( \int_X |u|^p d\tau + \int_X |g|^p d\mu \right)^{\frac{1}{p}}.$$

Let us set

$$A := \left\{ (u,g) \in E \,\middle|\, u \in T(\mathcal{A}'_p(F, X)) \text{ and } g \in D[u] \right\},$$

where $\mathcal{A}'_p(F,X)$ is the closure of $\mathcal{A}_p(F,X)$ in $\mathcal{L}_0^{1,p}(X)$ and
$T : \mathcal{L}_0^{1,p}(X) \to \mathcal{L}_0^{1,p}(X)$ is a "truncation" operator defined as follows

$$Tu(x) := \begin{cases} 0 & \text{if } u(x) < 0, \\ u(x) & \text{if } 0 \le u(x) \le 1, \\ 1 & \text{if } u(x) > 1. \end{cases}$$

Then $A$ is a convex closed subset of $E$. As $\mathrm{Cap}_p(F) < \infty$, we also have that $A \neq \emptyset$ and since in any nonempty closed convex subset $A \subset E$ of a uniformly convex Banach space $E$, there exists a unique element $x^* \in A$ with minimal norm: $\|x^*\| = \inf_{x\in A} \|x\|$, we know that there exists a unique element $(u^*, g^*) \in A$ which minimizes the norm. It is clear that $g^*$ is the minimal pseudo-gradient of $u^*$, i.e. that $\mathcal{E}_p(u^*) = \int_X |g^*|^p d\mu$.

We assert that $\mathcal{E}_p(u^*) = \mathrm{Cap}_p(F)$. Indeed, if $\mathcal{E}_p(u^*) > \mathrm{Cap}_p(F)$, then by Proposition 7.3 in [11] which states that $\mathrm{Cap}_p(F,X) := \inf\{\mathcal{E}_p(u) | \; u \in T(\mathcal{A}'_p(F,X))\}$, one could find $(u,g) \in A$ such that $\int_X |g|^p d\mu < \int_X |g^*|^p d\mu$. Since $u, u^* \in T(\mathcal{A}'_p(F,X))$, we may assume, by Proposition 7.1 in [11] that $u = u^* = 1$ $p$-quasi everywhere on $F$ and thus that $u = u^* = 1$ $\tau$-almost everywhere on $F$ because $\tau$ is absolutely continuous with respect to the $p$-capacity. Therefore,

$$\|(u,g)\|_E = \left(1 + \int_X |g|^p d\mu\right)^{\frac{1}{p}} < \left(1 + \int_X |g^*|^p d\mu\right)^{\frac{1}{p}} = \|(u^*,g^*)\|_E,$$

which contradicts the minimality of $(u^*, g^*)$.

Q.E.D.

**Definition 2.19 (Quasi-minimizer)** *A function $u \in L^p_{loc}(\Omega)$ is called a quasi-minimizer of the energy $\mathcal{E}_p$ on the set $\Omega \subset X$ if there exists a constant $K > 0$ such that for all functions $\varphi \in \mathcal{L}^{1,p}(X)$ with $supp(\varphi) \Subset \Omega$ the inequality*

$$\int_{supp(\varphi)} g_u^p d\mu \le K \int_{supp(\varphi)} g_{u+\varphi}^p d\mu$$

*holds (where, as usual, $g_{u+\varphi}$ is the minimal pseudo-gradient of $u+\varphi$). When $K = 1$, the corresponding quasi-minimizer is called the minimizer of the energy functional $\mathcal{E}_p$.*

**Proposition 2.20** *Assume that the $D$-structure on $X$ is strongly local. Then the capacitary function $u^*$ of the condenser $F$ is a minimizer of $\mathcal{E}_p$ on the set $X \setminus F$.*

**Proof** Let $\varphi \in \mathcal{L}^{1,p}(X)$ with $\operatorname{supp}(\varphi) \Subset X \setminus F$ and $v = u^* + \varphi$. Then

$$v = u^* \text{ on } X \setminus \operatorname{supp}(\varphi)\,,$$

and the strong locality implies that

$$\int_{X\setminus \operatorname{supp}(\varphi)} g_v^p d\mu = \int_{X\setminus \operatorname{supp}(\varphi)} g_{u^*}^p d\mu\,.$$

As the function $v^+ \in \mathcal{A}_p(F, X)$, by the energy minimizing property of $u^*$ we have

$$\begin{aligned}\int_{\operatorname{supp}(\varphi)} g_{u^*}^p d\mu + \int_{X\setminus \operatorname{supp}(\varphi)} g_{u^*}^p d\mu &= \int_X g_{u^*}^p d\mu \le \int_X g_{v^+}^p d\mu \\ &\le \int_X g_v^p d\mu = \int_{\operatorname{supp}(\varphi)} g_v^p d\mu + \int_{X\setminus \operatorname{supp}(\varphi)} g_v^p d\mu\,.\end{aligned}$$

Thus,

$$\int_{\operatorname{supp}(\varphi)} g_{u^*}^p d\mu \le \int_{\operatorname{supp}(\varphi)} g_{u^*+\varphi}^p d\mu$$

and $u^*$ is a minimizer.

Q.E.D.

# Chapter 3

# Sobolev functions in the sense of Hajłasz - Koskela

In this chapter we shortly recall the approach to Sobolev spaces on a metric space using Poincaré inequalities (see [14] for the definitions given below).

**Definition 3.1 (Poincaré inequality)** *Let $u \in L^1_{loc}(X)$ and $g : X \to [0,\infty]$ be Borel measurable functions. We say that the pair $(u, g)$ satisfies a $(s, q)$-Poincaré inequality in $\Omega \subset X$, $s, q \geq 1$, if there exist two constants $\sigma \geq 1$ and $C_P > 0$ such that the inequality*

$$\left( \fint_B |u - u_B|^s d\mu \right)^{1/s} \leq C_P \, r \left( \fint_{\sigma B} g^q d\mu \right)^{1/q} \tag{3.1}$$

*holds on every ball $B$ with $\sigma B \subset \Omega$, where $r$ is the radius of $B$.*

**Definition 3.2 (Sobolev functions)** *A function $u \in L^1_{loc}(X)$ for which there exists $0 \leq g \in L^q(X)$ such that the pair $(u, g)$ satisfies a $(1, q)$-Poincaré inequality in $X$, we call a Sobolev function in the sense of Hajłasz - Koskela or a Poincaré-Sobolev function. We denote by $PW^{1,q}(X)$ the set of all Poincaré-Sobolev functions.*

The Poincaré inequality (3.1) is the only relationship between the functions $u$ and $g$. Working in this setting P. Hajłasz and P. Koskela have developed in [14] quite a rich theory of these Sobolev functions on metric spaces.

In the sequel we will need also the following definitions.

Given a function $v$ and $\infty > t_2 > t_1 > 0$, we set

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}.$$

**Definition 3.3 (Truncation property)** *Let the pair $(u, g)$ satisfy a $(1, q)$-Poincaré inequality in $\Omega$. Assume that for every $b \in \mathbb{R}$, $\infty > t_2 > t_1 > 0$, and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \leq t_2\}})$, where $v = \varepsilon(u - b)$, satisfies the $(1, q)$-Poincaré inequality in $\Omega$ (with fixed constants $C_P, \sigma$). Then we say that the pair $(u, g)$ has the truncation property.*

The truncation property for Poincaré-Sobolev functions is the notion similar to the one of the strict locality in axiomatic Sobolev spaces, which also reflects some localization properties of the Sobolev space under consideration. Note that in the Euclidean space $\mathbb{R}^n$ both conditions mean that the gradient of a function, which is constant on some set, equals zero a.e. on that set.

As we will see in Section 6.1, in the case of axiomatic Sobolev spaces the quasi-minimizers of the $p$-Dirichlet energy satisfy the De Giorgi condition (Hypothesis H1). Note that this is also the case for the quasi-minimizers in the approach to Sobolev spaces on a metric space via upper gradients (see [17]). For the class of Poincaré-Sobolev functions the possible notion of energy is not consistent, in particular it is not clear how it would be possible to prove the existence of corresponding minimizers, since in this case the corresponding Sobolev space is not a Banach space (it is, in fact, only a quasi-Banach space). But the De Giorgi condition is still legitimate for the Poincaré-Sobolev functions. Thus, as it seems that there exists an intimate connection between extremal functions and the functions satisfying the De Giorgi condition, in the case of Poincaré-Sobolev functions, the functions whose regularity we are going to establish will be those who satisfy the following property:

**Definition 3.4 ($p$-De Giorgi condition)** *We say that a Poincaré-Sobolev function $u$ ($u$ satisfies a $(1, q)$-Poincaré inequality with some function $g$) satisfies the $p$-De Giorgi condition on the set $\Omega$ if for all $k \in \mathbb{R}$, $z \in X$ and $0 < \rho < R \leq diam(X)/3$, the following inequality*

$$\int_{A(k,\rho)} g^p d\mu \leq \frac{C}{(R - \rho)^p} \int_{A(k,R)} (u - k)^p d\mu, \tag{3.2}$$

*holds, provided $\mu(\Omega \setminus A(k, R)) = 0$, where $A(k, r) = B(r) \cap \{x \colon u(x) > k\}$, $p \in \mathbb{R}$, $p > q$, and $B(r)$ is the ball centered at $z$ with the radius $r$.*

# Chapter 4

# De Giorgi Argument in an Abstract Setting (interior regularity)

At the beginning of this section we want to underline that in the sequel the notation $g_{(u)}$ for a function from $L^p(X)$ means no a priori dependence of this function on the given function $u \in L^p_{loc}(X)$, whereas $g_u$ stands for the minimal pseudo-gradient of the function $u$.

Let $\Omega$ be an open subset of $X$ and $u$ be a function in $L^p(\Omega)$. In this section we prove that if the functions $u$ and $-u$ satisfy Hypotheses H1 and H2 stated below in the pairs with some functions $g_{(u)}, g_{(-u)} \in L^p(\Omega)$ respectively, and if, in addition, the pair $(u, g_{(u)})$ satisfies Hypothesis H3, then $u$ (and, of course, $-u$) is Hölder continuous inside the set $\Omega$.

Note that Hypotheses H2 and H3 are the characteristics of the Sobolev space of functions we will work with in the next chapters, whereas Hypothesis H1 is the property of some particular functions, the functions whose regularity we want to establish.

Unless otherwise stated, $C$ denotes a positive constant whose exact value is unimportant, can change even within a line and depends only on fixed parameters, such as $X$, $d$, $\mu$, $p$ and others.

## 4.1 List of hypotheses

The hypotheses for two functions $u, g_{(u)} \in L^p(\Omega)$, which we shall need are the following:

**Hypothesis H1 (De Giorgi condition)** There exist constants $C > 0$ and $k^* \in \mathbb{R}$, such that for all $k \geq k^*$, $z \in \Omega$, and $0 < \rho < R \leq \operatorname{diam}(X)/3$ so that $B(z, R) \subset \Omega$, the following Cacciopoli type inequality on the "upper-level" sets of the function $u$ holds

$$\int_{A(k,\rho)} g_{(u)}^p d\mu \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p d\mu, \tag{4.1}$$

where $A(k, r) = A_z(k, r) = \{x \in B(z, r) = B(r) \colon u(x) > k\}$ with $z \in \Omega$ being fixed.

Let $\eta$ be a $\frac{C}{(R-\rho)}$- Lipschitz (cutoff) function for some $C > 0$, such that $0 \leq \eta \leq 1$, the support of $\eta$ is contained in $B(\frac{R+\rho}{2})$ and $\eta = 1$ on $B(\rho)$.

**Hypothesis H2** There exists a constant $C > 0$ such that for functions $v = \eta\,(u-k)_+$ and $g_{(v)} = g_{(u)}\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{R-\rho}(u-k)_+$, where, as usual, $(u-k)_+ = \max\{u-k, 0\}$, and for some $t$ and $q$, $t > p > q$, we have

$$\left( \fint_{B(\frac{R+\rho}{2})} v^t d\mu \right)^{\frac{1}{t}} \leq CR \left( \fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu \right)^{\frac{1}{q}}. \tag{4.2}$$

Here $k, \rho$ and $R$ are as in Hypothesis H1.

**Hypothesis H3** There exist constants $C > 0$ and $\sigma \geq 1$, such that for all $h, k \in \mathbb{R}$, $h > k \geq k^*$ for the functions

$$w = u_k^h := \min\{u, h\} - \min\{u, k\} = \begin{cases} h-k & \text{if } \ u \geq h, \\ u-k & \text{if } \ k < u < h, \\ 0 & \text{if } \ u \leq k, \end{cases}$$

and $g_{(w)} = g_{(u)}\,\chi_{\{k<u\leq h\}}$ we have

$$\left( \int_{B(R)} w^q d\mu \right)^{\frac{1}{q}} \leq CR \left( \int_{B(\sigma R)} g_{(w)}^q d\mu \right)^{\frac{1}{q}}, \tag{4.3}$$

where $q$ is as in Hypothesis H2.

## 4.2 Boundedness

In this section we prove that a function $u \in L^p_{loc}(X)$ satisfying Hypotheses H1 and H2 with some function $g_{(u)} \in L^p(\Omega)$ is locally bounded in $\Omega$.

**Theorem 4.1** *Suppose that a pair of functions $(u, g_{(u)})$ satisfies Hypotheses* H1,H2*. If $k' \geq k^*$, then there exist constants $C > 0$ and $\theta > 1$ such that*

$$\operatorname*{ess\,sup}_{B(\frac{R}{2})} u \leq k' + C \left( ⨍_{B(R)} (u - k')_+^p d\mu \right)^{\frac{1}{p}} \left( \frac{\mu(A(k', R))}{\mu(B(\frac{R}{2}))} \right)^{\frac{\theta}{p}},$$

*for all $z \in \Omega$ and* $0 < R \leq \operatorname{diam}(X)/3$.

**Proof** Suppose that the functions $u, g_{(u)} \in L^p(\Omega)$ satisfy the conditions of the theorem, $k \in \mathbb{R}$, $\rho, R \in \mathbb{R}$ are such that $0 < \rho < R \leq \operatorname{diam}(X)/3$ and $B(R) \subset \Omega$. Replacing $\rho$ by $\frac{R+\rho}{2}$ and $C$ by $C/2^p$ we may rewrite the inequality (4.1) in the form

$$\int_{A(k, \frac{R+\rho}{2})} g_{(u)}^p d\mu \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p d\mu,$$

which is equivalent to

$$\int_{B(\frac{R+\rho}{2})} g_{(u)}^p \, \chi_{A(k, \frac{R+\rho}{2})} d\mu \leq \frac{C}{(R-\rho)^p} \int_{B(R)} (u-k)_+^p d\mu. \tag{4.4}$$

Let $\eta, v, g_{(v)}$ be as in Hypothesis H2, i.e. $\eta$ is Lipschitz, $v = \eta\,(u-k)_+$ and $g_{(v)} = g_{(u)}\, \chi_{A(k, \frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u-k)_+$. The Minkowski inequality and the inequality (4.4) imply that

$$\begin{aligned}
&\left( \int_{B(\frac{R+\rho}{2})} g_{(v)}^p d\mu \right)^{\frac{1}{p}} \\
&\quad \leq \left( \int_{B(\frac{R+\rho}{2})} g_{(u)}^p \, \chi_{A(k, \frac{R+\rho}{2})} d\mu \right)^{\frac{1}{p}} + \frac{C}{(R-\rho)} \left( \int_{B(\frac{R+\rho}{2})} (u-k)_+^p d\mu \right)^{\frac{1}{p}} \\
&\quad \leq \left( \frac{C}{(R-\rho)^p} \int_{B(R)} (u-k)_+^p d\mu \right)^{\frac{1}{p}} + \frac{C}{(R-\rho)} \left( \int_{B(\frac{R+\rho}{2})} (u-k)_+^p d\mu \right)^{\frac{1}{p}} \\
&\quad \leq \frac{C}{(R-\rho)} \left( \int_{B(R)} (u-k)_+^p d\mu \right)^{\frac{1}{p}}.
\end{aligned}$$

From this last inequality, the inequality (4.2) and the Hölder inequality we obtain (recall that $q < p < t$)

$$\begin{aligned}
\left(\int_{B(\rho)} (u-k)_+^t d\mu\right)^{\frac{1}{t}} &\leq \left(\int_{B(\frac{R+\rho}{2})} v^t d\mu\right)^{\frac{1}{t}} \\
&\leq C\, R\, \frac{\mu(B(\frac{R+\rho}{2}))^{\frac{1}{t}}}{\mu(B(\frac{R+\rho}{2}))^{\frac{1}{q}}} \left(\int_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}} \\
&\leq C\, R\, \left(\mu(B(\tfrac{R+\rho}{2}))\right)^{\frac{1}{t}-\frac{1}{p}} \left(\int_{B(\frac{R+\rho}{2})} g_{(v)}^p d\mu\right)^{\frac{1}{p}} \\
&\leq C\, R\, \left(\mu(B(\tfrac{R+\rho}{2}))\right)^{\frac{1}{t}-\frac{1}{p}} \frac{C}{(R-\rho)} \left(\int_{B(R)} (u-k)_+^p d\mu\right)^{\frac{1}{p}} \\
&\leq C\, \frac{R}{(R-\rho)}\, \left(\mu(B(\tfrac{R+\rho}{2}))\right)^{\frac{1}{t}-\frac{1}{p}} \left(\int_{B(R)} (u-k)_+^p d\mu\right)^{\frac{1}{p}}. \qquad (4.5)
\end{aligned}$$

The Hölder inequality implies that

$$\left(\int_{B(\rho)} (u-k)_+^p d\mu\right)^{\frac{1}{p}} \leq \mu(A(k,\rho))^{\frac{1}{p}-\frac{1}{t}} \left(\int_{B(\rho)} (u-k)_+^t d\mu\right)^{\frac{1}{t}}.$$

Therefore, the inequality (4.5) gives us

$$\begin{aligned}
&\left(\int_{B(\rho)} (u-k)_+^p d\mu\right)^{\frac{1}{p}} \\
&\qquad \leq C\, \frac{R}{(R-\rho)} \left(\frac{\mu(A(k,\rho))}{\mu(B(\frac{R+\rho}{2}))}\right)^{\frac{1}{p}-\frac{1}{t}} \left(\int_{B(R)} (u-k)_+^p d\mu\right)^{\frac{1}{p}}. \qquad (4.6)
\end{aligned}$$

If $h > k \geq k^*$, then $h-k \leq u-k$ on $A(h,\rho)$. Therefore, as $A(h,\rho) \subset A(k,\rho)$, we conclude that

$$\begin{aligned}
(h-k)^p \mu(A(h,\rho)) &= \int_{A(h,\rho)} (h-k)^p d\mu \\
&\leq \int_{A(h,\rho)} (u-k)^p d\mu \leq \int_{A(k,\rho)} (u-k)^p d\mu. \qquad (4.7)
\end{aligned}$$

Let

$$a(h,\rho) = \mu(A(h,\rho)) \qquad \text{and} \qquad u(h,\rho) = \int_{A(h,\rho)} (u-h)^p d\mu.$$

Note that if $h \leq k$ and $\rho \leq r$, then $a(k,\rho) \leq a(h,r)$ and $u(k,\rho) \leq u(h,r)$.

Let $h > k \geq k^*$ and $R > \rho > 0$. Then, by inequality (4.7) we have

$$a(h,\rho) \leq \frac{1}{(h-k)^p} u(k,\rho) \leq \frac{1}{(h-k)^p} u(k,R),$$

and by inequality (4.6) we obtain

$$u(h,\rho) \leq u(k,\rho) \leq C \left(\frac{R}{R-\rho}\right)^p \left(\frac{\mu(A(k,\rho))}{\mu(B(\frac{R}{2}))}\right)^{1-\frac{p}{t}} u(k,R)\,.$$

Let $\alpha$ be the positive solution of the equation $(t-p)\alpha^2 - t(\alpha+1) = 0$ . From the last two inequalities we have

$$u(h,\rho)^\alpha a(h,\rho) \leq C \left(\frac{R}{R-\rho}\right)^{p\alpha} \left(\frac{\mu(A(k,\rho))}{\mu(B(\frac{R}{2}))}\right)^{\alpha(1-\frac{p}{t})} \frac{1}{(h-k)^p}\, u(k,R)^{\alpha+1}\,.$$

Let

$$\phi(h,\rho) := u(h,\rho)^\alpha\, a(h,\rho).$$

Then, by the above, we conclude that

$$\phi(h,\rho) \leq C \left(\frac{R}{R-\rho}\right)^{p\alpha} \mu(B(\tfrac{R}{2}))^\theta \frac{1}{(h-k)^p}\, \phi(k,R)^\theta\,,$$

with $\theta = \alpha\left(1 - \frac{p}{t}\right) = 1 + \frac{1}{\alpha} > 1$.

Now, for some $k' \geq k^*$ and $j \in \mathbb{N}$, set

$$\rho_j := \frac{R}{2}\left(1 + \frac{1}{2^j}\right)\,,$$

$$k_j := k' + d - \frac{d}{2^j} \geq k',$$

where

$$d^p = C2^{p(1+\alpha)^2+p\alpha}\mu(B(\tfrac{R}{2}))^\theta\phi(k',R)^{\theta-1}.$$

Since $p,\alpha>0$,

$$\begin{aligned}\phi(k_j,\rho_j) &\le C\left(\frac{1+\frac{1}{2^{j-1}}}{\frac{1}{2^j}}\right)^{p\alpha}\mu(B(\tfrac{R}{2}))^\theta\frac{1}{\left(\frac{d}{2^j}\right)^p}\,\phi(k_{j-1},\rho_{j-1})^\theta\\ &\le \frac{C\,2^{p(j\alpha+\alpha+j)}}{d^p}\mu(B(\tfrac{R}{2}))^\theta\phi(k_{j-1},\rho_{j-1})^\theta\\ &= 2^{\frac{\beta}{\alpha}(j-1-\alpha)}\,\phi(k',R)^{1-\theta}\,\phi(k_{j-1},\rho_{j-1})^\theta,\end{aligned}$$

with $\beta=p\alpha(\alpha+1)$.

By induction, conclude that

$$\phi(k_j,\rho_j)\le\frac{\phi(k',R)}{2^{\beta j}}.$$

Letting $j\to\infty$ we obtain

$$a(k'+d,R/2)\,u((k'+d,R/2)^\alpha=\phi(k'+d,R/2)=0.$$

It follows that either $u(k'+d,R/2)=0$ or $a(k'+d,R/2)=0$. Thus,

$$\operatorname*{ess\,sup}_{B(\frac{R}{2})}u\le k'+d=k'+\,C\left(\int_{B(R)}(u-k')_+^p d\mu\right)^{\frac{1}{p}}\frac{\mu(A(k',R))^{\frac{\theta-1}{p}}}{\mu(B(\frac{R}{2}))^{\frac{\theta}{p}}}.$$

Q.E.D.

**Corollary 4.2** *Suppose that the measure $\mu$ is doubling and the functions $u$ and $-u$ satisfy Hypotheses* H1 *and* H2 *with some functions $g_{(u)}$ and $g_{(-u)}$ respectively, $u,g_{(u)},g_{(-u)}\in L^p(\Omega)$. Then there exist some constants $C>0$ and $k^*\in\mathbb{R}$ such that*

$$\operatorname*{ess\,sup}_{B(\frac{R}{2})}|u|\le k+C\left(\fint_{B(R)}|u|^p d\mu\right)^{\frac{1}{p}},\tag{4.8}$$

*for all $z\in\Omega$, $k\ge k^*$ and $0<R\le diam(X)/3$.*

## 4.3 Hölder continuity

The goal of this section is to prove the Hölder continuity of a function satisfying all of Hypotheses H1 - H3. We have the following

**Theorem 4.3** *Suppose that the measure $\mu$ on $X$ is doubling. If in addition to Hypotheses* H1 *and* H2*, Hypothesis* H3 *is satisfied for the functions $u$ and $g_{(u)}$ or for the functions $-u$ and $g_{(-u)}$, then the function $u \in L^p(\Omega)$ is locally Hölder continuous.*

**Proof** Using the inequality (4.3) for our auxiliary functions $w$ and $g_{(w)}$ with some $h$ and $k$, $h > k \geq k^*$, we obtain

$$\begin{aligned}
(h-k)\mu(A(h,R)) &= \int_{A(h,R)} w\,d\mu \ \leq \int_{B(R)} w\,d\mu \\
&\leq \left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&\leq C\,R \left(\int_{B(\sigma R)} {g_{(w)}}^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&= C\,R \left(\int_{B(\sigma R)} {g_{(u)}}^q\, \chi_{\{k<u\leq h\}} d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&= C\,R \left(\int_{A(k,\sigma R)\setminus A(h,\sigma R)} {g_{(u)}}^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned}$$

Hence, by Hölder inequality we have

$$\begin{aligned}
(h-k)\mu(A(h,R)) \leq C\,R &\left(\int_{A(k,\sigma R)} {g_{(u)}}^p d\mu\right)^{\frac{1}{p}} \\
&\times \left(\mu(A(k,\sigma R)) - \mu(A(h,\sigma R))\right)^{\frac{1}{q}-\frac{1}{p}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned}$$

As the functions $u$ and $g_{(u)}$ satisfy the inequality (4.1), we conclude that

$$\begin{aligned}
(h-k)\mu(A(h,R)) \leq C &\left(\int_{A(k,2\sigma R)} (u-k)^p d\mu\right)^{\frac{1}{p}} \\
&\times \left(\mu(A(k,\sigma R)) - \mu(A(h,\sigma R))\right)^{\frac{1}{q}-\frac{1}{p}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned} \tag{4.9}$$

Let

$$m(R) = \operatorname*{ess\,inf}_{B(R)} u \qquad \text{and} \qquad M(R) = \operatorname*{ess\,sup}_{B(R)} u.$$

Denote

$$M = M(2\sigma R), \;\; m = m(2\sigma R) \;\text{ and }\; k_0 = \frac{(M+m)}{2}.$$

By Corollary 4.2, $m$ and $M$ are finite for small enough $R$. Using Theorem 4.1 with $k'$ replaced by $k_\nu = M - 2^{-\nu-1}(M-m)$, $\nu = 0, 1, 2, \ldots$, we get

$$M(R/2) \leq k_\nu + C(M - k_\nu) \left( \frac{\mu(A(k_\nu, R))}{\mu(B(\frac{R}{2}))} \right)^{\frac{\theta}{p}} .$$

By Proposition 4.4 stated after the proof, it is possible to choose an integer $\nu$, independent of $z$, $R$ and $u$, large enough so that

$$C \left( \frac{\mu(A(k_\nu, R))}{\mu(B(\frac{R}{2}))} \right)^{\frac{\theta}{p}} < \frac{1}{2} .$$

Hence

$$M(R/2) < k_\nu + \frac{1}{2}(M - k_\nu) = M - \frac{M-m}{2^{\nu+2}} ,$$

and therefore

$$M(R/2) - m(R/2) \leq M(R/2) - m < (M-m)\left(1 - 2^{-(\nu+2)}\right) .$$

Let

$$\operatorname{osc}(r) = M(r) - m(r)$$

denote the oscillation of $u$ on $B(z, r)$. Then by the above inequality

$$\operatorname{osc}(R/2) < \lambda \operatorname{osc}(2\sigma R), \tag{4.10}$$

where $\lambda = 1 - 2^{-(\nu+2)} < 1$.

To complete the proof we iterate the inequality (4.10). Choose an integer $j \geq 1$ so that $(4\sigma)^{j-1} \leq \frac{R}{r} < (4\sigma)^j$. Inequality (4.10) implies that

$$\operatorname{osc}(r) \leq \lambda^{j-1} \operatorname{osc}((4\sigma)^{j-1} r) \leq \lambda^{j-1} \operatorname{osc}(R).$$

By the choice of $j$ we conclude that

$$\lambda^{j-1} = (4\sigma)^{(j-1)(\log\lambda)/\log(4\sigma)} \leq (4\sigma)^{\alpha}\left(\frac{R}{r}\right)^{-\alpha},$$

where $\alpha = -(\log\lambda)/\log(4\sigma)$. Note that $0 < \alpha \leq 1$.

Finally, we have

$$\operatorname{osc}(r) \leq (4\sigma)^{\alpha}\left(\frac{r}{R}\right)^{\alpha}\operatorname{osc}(R) \leq H\, r^{\alpha},$$

with $H = (4\sigma)^{\alpha}\sup\limits_{\varrho\in(\frac{R}{4\sigma},R)}\dfrac{\operatorname{osc}(\varrho)}{\varrho^{\alpha}}$.

Therefore, after a redefinition on a set of measure zero, $u$ is locally Hölder continuous on $\Omega$.

Q.E.D.

**Proposition 4.4** *Under the conditions of Theorem* 4.3 *there exists a sequence* $\{\alpha_\nu\}\subset\mathbb{R}$, *such that* $\alpha_\nu\to 0$ *when* $\nu\to\infty$, *and*

$$\frac{\mu(A(k_\nu,R))}{\mu(B(\frac{R}{2}))} \leq \alpha_\nu\,.$$

**Proof** Let

$$k_i = M - 2^{-(i+1)}(M-m),\ i = 0,1,2,\ldots$$

Then $k_i \nearrow M$ as $i\to\infty$ and $k_0 = \frac{(M+m)}{2}$. Note that

$$M - k_{i-1} = 2^{-i}(M-m) \quad\text{and}\quad k_i - k_{i-1} = 2^{-(i-1)}(M-m)\,.$$

By the inequality (4.9) we have

$$(k_i-k_{i-1})\mu(A(k_i,R)) \leq C\left(\int_{A(k_{i-1},2\sigma R)}(u-k_{i-1})^p d\mu\right)^{\frac{1}{p}}$$
$$\times\left(\mu(A(k_{i-1},\sigma R)) - \mu(A(k_i,\sigma R))\right)^{\frac{1}{q}-\frac{1}{p}}\mu(B(R))^{1-\frac{1}{q}}.$$

Therefore, as $u - k_{i-1} \leq M - k_{i-1}$ on $A(k_{i-1},2\sigma R)$, we conclude that

$$2^{-(i+1)}\,(M-m)\,\mu(A(k_i,R))$$
$$\leq C\,\mu(B(2\sigma R))^{1-\frac{1}{q}+\frac{1}{p}}\,2^{-i}(M-m)(\mu(A(k_{i-1},\sigma R)) - \mu(A(k_i,\sigma R)))^{\frac{1}{q}-\frac{1}{p}}.$$

Note that if $\nu \geq i$, then $\mu(A(k_\nu, R)) \leq \mu(A(k_i, R))$. Hence

$$\mu(A(k_\nu, R)) \leq 2\, C\, \mu(B(2\sigma R))^{1-\frac{1}{q}+\frac{1}{p}} \left(\mu(A(k_{i-1}, \sigma R)) - \mu(A(k_i, \sigma R))\right)^{\frac{1}{q}-\frac{1}{p}}.$$

Now raising the last inequality to the power $\frac{pq}{p-q}$ and then summing the result over $i = 1, 2, ..., \nu$, we get

$$\begin{aligned} \nu \mu(A(k_\nu, R))^{\frac{pq}{p-q}} &\leq C\, \mu(B(2\sigma R))^{\frac{pq}{p-q}-1} \left(\mu(A(k_0, \sigma R)) - \mu(A(k_\nu, \sigma R))\right) \\ &\leq C\, \mu(B(2\sigma R))^{\frac{pq}{p-q}}. \end{aligned}$$

Dividing both parts of the last inequality by $\mu(B(\frac{R}{2}))^{\frac{pq}{p-q}}$ and using the doubling property of $\mu$, we obtain the result.

Q.E.D.

# Chapter 5

# De Giorgi Argument in an Abstract Setting (regularity at the boundary)

Let $x_0 \in \partial\Omega$, a boundary point of the set $\Omega$, be fixed. Let also $u$ be a function in $L^p_{loc}(X)$ and $\vartheta \in L^p(X)$ be such that $u = \vartheta$ a.e. on $X \setminus \Omega$. Suppose that the function $\vartheta$ is Hölder continuous at the point $x_0 \in \partial\Omega$. In this section we will prove that if the functions $u$ and $-u$ satisfy at the point $x_0$ Hypotheses H1 and H2 of Chapter 4 and Hypothesis H3(b) stated below in the pairs with some functions $g_{(u)}, g_{(-u)} \in L^p(X)$ respectively then the function $u$ is Hölder continuous at the boundary point $x_0$.

The hypothesis H3(b) for the functions $u$ and $g_{(u)}$, which we mean is the following:

**Hypothesis H3(b)** There exists a function $\Phi : 2^X \times X \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for fixed $\Omega \subset X$ and $x_0 \in X$, $\frac{\Phi(\Omega,x_0,R)}{R}$ is bounded for all $R \in \mathbb{R}_+$, $\Phi$ is not constant for all of its three arguments, and for all $h, k \in \mathbb{R}$, $h > k \geq k^*$, for the functions $w = u_k^h$ and $g_{(w)} = g_{(u)}\,\chi_{\{k<u\leq h\}}$ we have

$$\left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \leq \Phi(\Omega, x_0, R) \left(\int_{B(\sigma R)} g^q_{(w)} d\mu\right)^{\frac{1}{q}}, \tag{5.1}$$

where $\sigma > 1$ is a constant and $q$ is as in Hypothesis H2. Note that the balls of Hypothesis H3(b) as well as the balls of Hypotheses H1 and H2 considered in this chapter are centered at the point $x_0$, a boundary point of $\Omega$.

A typical example of the function $\Phi$ is given (up to a constant) by the following quantity

$$\Phi(\Omega, x_0, R) = \left( \frac{\mu(B(x_0, R))}{\mathrm{C}_q\left(B(x_0, \frac{1}{2}R) \setminus \Omega\right)} \right)^{\frac{1}{q}},$$

where $\mathrm{C}_q$ is the Sobolev $q$-capacity of the axiomatic setting (see Proposition 6.6 and Remark following the proof it). Another example of $\Phi$ is (up to a constant) the function

$$\Phi(\Omega, x_0, R) = R \left( \frac{\mu(B(x_0, R))}{\mu(B(x_0, R) \setminus \Omega)} \right)^{1-\frac{1}{q}},$$

(see Proposition 7.3).

**Theorem 5.1** *Suppose that a pair of functions $(u, g_{(u)})$ satisfies Hypotheses* H1 *and* H2 *at the boundary point $x_0$. Then for all $k \geq k^*$ there exists a constant $C > 0$ such that*

$$\operatorname*{ess\,sup}_{B(x_0, \frac{R}{2})} u \leq k + C \left( ⨍_{B(x_0,R)} (u-k)_+^p d\mu \right)^{\frac{1}{p}}.$$

**Proof** repeats literally for our case the proof of Theorem 4.1.

**Theorem 5.2** *If the pairs $(u, g_{(u)})$ and $(-u, g_{(-u)})$ satisfy Hypotheses* H1, H2 *and* H3$(b)$ *(with some $g_{(-u)} \in L^p(X)$), the function $\vartheta \in L^p(X)$ is Hölder continuous at $x_0 \in \partial\Omega$ and the following condition is satisfied*

$$\liminf_{\rho \to 0} \frac{1}{|\log \rho|} \int_\rho^1 \exp\left( -C \left( \frac{\Phi(R)}{R} \right)^{\frac{pq}{p-q}} \right) \frac{dR}{R} > 0\,, \tag{5.2}$$

*for some constant $C > 0$, then the function $u$ is Hölder continuous at $x_0$.*

With $\Omega \subset X$ and $x_0 \in \partial\Omega$ being fixed, here and in the sequel, for simplicity, we indicate the dependance of the function $\Phi : (\Omega, x_0, R) \mapsto \Phi(\Omega, x_0, R)$ only on its third argument, i.e. instead of $\Phi(\Omega, x_0, R)$ we write $\Phi(R)$.

**Proof** Using the inequality (5.1) for our auxiliary functions $w$ and $g_{(w)}$ with some $h$ and $k$, $h > k \geq k^*$, we obtain

$$\begin{aligned}
(h-k)\mu(A(h,R)) &= \int_{A(h,R)} w d\mu \ \leq \int_{B(R)} w d\mu \\
&\leq \left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&\leq \Phi(R) \left(\int_{B(\sigma R)} {g_{(w)}}^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&= \Phi(R) \left(\int_{B(\sigma R)} {g_{(u)}}^q \, \chi_{\{k<u\leq h\}} d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&= \Phi(R) \left(\int_{A(k,\sigma R)\setminus A(h,\sigma R)} {g_{(u)}}^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned}$$

Hence, by Hölder inequality we have

$$\begin{aligned}
(h-k)\mu(A(h,R)) \leq \Phi(R) &\left(\int_{A(k,\sigma R)} {g_{(u)}}^p d\mu\right)^{\frac{1}{p}} \\
&\times \left(\mu(A(k,\sigma R)) - \mu(A(h,\sigma R))\right)^{\frac{1}{q}-\frac{1}{p}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned}$$

Since the functions $u$ and $g_{(u)}$ satisfy the inequality (4.1), we conclude that

$$\begin{aligned}
(h-k)\mu(A(h,R)) \leq C\frac{\Phi(R)}{R} &\left(\int_{A(k,2\sigma R)} (u-k)^p d\mu\right)^{\frac{1}{p}} \\
&\times \left(\mu(A(k,\sigma R)) - \mu(A(h,\sigma R))\right)^{\frac{1}{q}-\frac{1}{p}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned} \tag{5.3}$$

With $x_0 \in \partial\Omega$ fixed, let us denote for $R_0$, $0 < 2\sigma R \leq R_0 \leq \operatorname{diam}(X)/3$,

$$M(R, R_0) = (\operatorname*{ess\,sup}_{B(x_0,R)} u - \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta)_+.$$

Let us also define $M = M(2\sigma R, R_0)$ and

$$k_j = \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta + M(1-2^{-j}), \quad j \in \mathbb{N}.$$

Replacing now $h$ by $k_{j+1}$ and $k$ by $k_j$ in the inequality (5.3) we obtain

$$(k_{j+1}-k_j)\mu(A(k_{j+1},R)) \le C\frac{\Phi(R)}{R}\left(\int_{A(k_j,2\sigma R)}(u-k_j)^p d\mu\right)^{\frac{1}{p}} \times (\mu(A(k_j,\sigma R))-\mu(A(k_{j+1},\sigma R)))^{\frac{1}{q}-\frac{1}{p}}\,\mu(B(R))^{1-\frac{1}{q}},$$

Noting that $k_{j+1}-k_j=\frac{M}{2^{j+1}}$ and denoting

$$T_j(\sigma R)=(\mu(A(k_j,\sigma R))-\mu(A(k_{j+1},\sigma R)))^{\frac{1}{q}-\frac{1}{p}}$$

we further have

$$\begin{aligned}
&\frac{M}{2^{j+1}}\mu(A(k_{j+1},R)) \le \\
&\quad\le C\frac{\Phi(R)}{R}\left(\int_{A(k_j,2\sigma R)}(u-k_j)^p d\mu\right)^{\frac{1}{p}} T_j(\sigma R)\;\mu(B(R))^{1-\frac{1}{q}} \\
&\quad\le C\frac{\Phi(R)}{R}\;\mu(B(2\sigma R))^{\frac{1}{p}}\,(\operatorname*{ess\,sup}_{B(2\sigma R)} u-k_j)_+\;T_j(\sigma R)\;\mu(B(R))^{1-\frac{1}{q}} \\
&\quad\le C\frac{\Phi(R)}{R}\;\mu(B(R))^{1-\frac{1}{q}+\frac{1}{p}}\;\frac{M}{2^j}\;T_j(\sigma R).
\end{aligned}$$

In the last inequality we have used the doubling condition of the measure $\mu$. Dividing both parts of the last inequality by $\frac{M}{2^{j+1}}$ and recalling the expression of $T_j(\sigma R)$ we obtain

$$\mu(A(k_{j+1},R)) \le C\frac{\Phi(R)}{R}\;\mu(B(R))^{1-\frac{1}{q}+\frac{1}{p}}\;(\mu(A(k_j,\sigma R))-\mu(A(k_{j+1},\sigma R)))^{\frac{1}{q}-\frac{1}{p}}\,.$$

If $n\ge j+1$, then the set $A(k_{j+1},R)$ on the left-hand side can be replaced by $A(k_n,R)$ and the inequality remains true. We have

$$\mu(A(k_n,R))^{\frac{pq}{p-q}} \le C\left(\frac{\Phi(R)}{R}\right)^{\frac{pq}{p-q}}\mu(B(R))^{\frac{pq}{p-q}-1}\,(\mu(A(k_j,\sigma R))-\mu(A(k_{j+1},\sigma R)))\,.$$

Now summing up over $j=0,1,\dots,n-1$ and using the doubling property of $\mu$, we obtain

$$\mu(A(k_n,R))^{\frac{pq}{p-q}} \le \frac{C}{n}\left(\frac{\Phi(R)}{R}\right)^{\frac{pq}{p-q}}\;\mu(B(R))^{\frac{pq}{p-q}},$$

or

$$\left(\frac{\mu(A(k_n,R))}{\mu(B(R))}\right)^{\frac{pq}{p-q}} \leq \frac{C}{n}\left(\frac{\Phi(R)}{R}\right)^{\frac{pq}{p-q}} . \tag{5.4}$$

Theorem 5.1 with $k$ replaced by $k_n$ and the fact that $u - k_n \leq 2^{-n}M$ on $B(R)$ give

$$\begin{aligned}
\operatorname*{ess\,sup}_{B(x_0,\frac{R}{2})} u &\leq k_n + C\left(\fint_{B(R)} (u-k_n)_+^p \, d\mu\right)^{\frac{1}{p}} \\
&= \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta + M(1-2^{-n}) + C\left(\frac{1}{\mu(B(R))}\int_{A(k_n,R)} (u-k_n)^p \, d\mu\right)^{\frac{1}{p}} \\
&\leq \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta + M(1-2^{-n}) + \frac{CM}{2^n}\left(\frac{\mu(A(k_n,R))}{\mu(B(R))}\right)^{\frac{1}{p}} . \qquad (5.5)
\end{aligned}$$

As the function $\frac{\Phi(R)}{R}$ is bounded for all $R \in \mathbb{R}_+$, using the estimate (5.4) we see that the last term on the right-hand side in (5.5) is at most $2^{-n-1}M$, whenever

$$n \geq n(R) = C\left(\frac{\Phi(R)}{R}\right)^{\frac{pq}{p-q}} . \tag{5.6}$$

Inserting the smallest integer $n \geq n(R)$ into (5.5) gives the following inequality

$$\operatorname*{ess\,sup}_{B(x_0,\frac{R}{2})} u \leq \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta + M\left(1 - \frac{1}{2^{n+1}}\right) .$$

Noting that if

$$\operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta \geq \operatorname*{ess\,sup}_{B(x_0,\frac{R}{2})} u \ ,$$

then

$$M(\tfrac{1}{2}R, R_0) = (\operatorname*{ess\,sup}_{B(x_0,\frac{R}{2})} u - \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta)_+ = 0 \ ,$$

we see that it follows from the last inequality that

$$M(\tfrac{1}{2}R, R_0) \leq (1 - 2^{-n(R)-2})\, M(2\sigma R, R_0) \ . \tag{5.7}$$

Let $C > 0$ and $n(R)$ be as in (5.6), and

$$\omega(R) = 2^{-n(2R)} = \exp\left(-C_0\left(\frac{\Phi(2R)}{2R}\right)^{\frac{pq}{p-q}}\right) ,$$

with $C_0 = C\log 2$.

For $m = 1, 2$, we divide the interval $(0, R_0)$ into two disjoint subsets as follows

$$I_m = \bigcup_{j=1}^{\infty} \left[(4\sigma)^{m-2j-1}R_0, (4\sigma)^{m-2j}R_0\right).$$

Then $I_1 \cup I_2 = (0, R_0)$, and hence for some $m$,

$$\int_\rho^{R_0} \omega(R)\frac{dR}{R} \le 2\int_{(\rho,R_0)\cap I_m} \omega(R)\frac{dR}{R} \tag{5.8}$$

For $j = 1, 2, \dots$, choose $R_j \in [(4\sigma)^{m-2j-1}R_0, (4\sigma)^{m-2j}R_0)$ so that

$$\begin{aligned}\int_{(4\sigma)^{m-2j-1}R_0}^{(4\sigma)^{m-2j}R_0} \omega(R)\frac{dR}{R} &\le \frac{\omega(R_j)}{R_j}\int_{(4\sigma)^{m-2j-1}R_0}^{(4\sigma)^{m-2j}R_0} dR \\ &\le \frac{\omega(R_j)}{R_j}(4\sigma-1)(4\sigma)^{m-2j-1}R_0 \le (4\sigma-1)\omega(R_j)\,.\end{aligned} \tag{5.9}$$

We take some $\rho$, $0 < \rho < R_0$. Then there exists a number $N \in \mathbb{N}$ such that $\rho \in \left[(4\sigma)^{m-2N-1}R_0, (4\sigma)^{m-2N}R_0\right)$. Summing up the inequality (5.9) for $j = 1, 2, \dots, N$, and using the fact that $\omega(R) < 1$ for all $R > 0$, we get

$$\begin{aligned}\int_{(\rho,R_0)\cap I_m} \omega(R)\frac{dR}{R} &\le \int_{((4\sigma)^{m-2j-1}R_0,R_0)\cap I_m} \omega(R)\frac{dR}{R} \\ &\le \int_{((4\sigma)^{m-2j}R_0,R_0)\cap I_m} \omega(R)\frac{dR}{R} + \int_{((4\sigma)^{m-2j-1}R_0,(4\sigma)^{m-2j}R_0)\cap I_m} \omega(R)\frac{dR}{R} \\ &\le (4\sigma-1)\sum_{\rho\le R_j\le R_0} \omega(R_j) + \ln(4\sigma).\end{aligned} \tag{5.10}$$

Now we apply the inequality (5.7) for $R_j$, $j = 1, 2, \dots$, to obtain

$$\begin{aligned}M\left((4\sigma)^{m-2j-1}R_0, R_0\right) &\le M(R_j, R_0) \le \left(1 - 2^{-n(2R_j)-2}\right)M(4\sigma R_j, R_0) \\ &\le \left(1 - \frac{\omega(R_j)}{4}\right)M(4\sigma R_j, R_0) \le \left(1 - \frac{\omega(R_j)}{4}\right)M((4\sigma)^{m-2j+1}R_0, R_0).\end{aligned}$$

Iterating this estimate, we get for $0 < \rho < R_0$,

$$M(\rho, R_0) \leq \prod_{\rho \leq R_j \leq R_0} \left(1 - \frac{\omega(R_j)}{4}\right) M(R_0, R_0). \tag{5.11}$$

Using the fact that $\log(1-t) \leq -t$ for $t < 1$ and noting that $\omega(R) < 1$, we have

$$\begin{aligned}
\prod_{\rho \leq R_j \leq R_0} \left(1 - \frac{\omega(R_j)}{4}\right) &= \exp \log \prod_{\rho \leq R_j \leq R_0} \left(1 - \frac{\omega(R_j)}{4}\right) \\
&= \exp \sum_{\rho \leq R_j \leq R_0} \log \left(1 - \frac{\omega(R_j)}{4}\right) \\
&\leq \exp \left( - \sum_{\rho \leq R_j \leq R_0} \frac{\omega(R_j)}{4} \right).
\end{aligned}$$

Which gives us together with the inequality (5.11) the following estimate

$$M(\rho, R_0) \leq \exp \left( -\frac{1}{4} \sum_{\rho \leq R_j \leq R_0} \omega(R_j) \right) M(R_0, R_0).$$

This inequality and the inequalities (5.10) and (5.8) imply that

$$M(\rho, R_0) \leq C M(R_0, R_0) \exp \left( -\frac{1}{8(4\sigma - 1)} \int_\rho^{R_0} \omega(R) \frac{dR}{R} \right),$$

or, recalling the expression of $\omega(R)$, that

$$\begin{aligned}
&M(\rho, R_0) \leq \\
&\leq C M(R_0, R_0) \exp \left( -\frac{1}{8(4\sigma - 1)} \int_\rho^{R_0} \exp \left( -C_0 \left( \frac{\Phi(2R)}{2R} \right)^{\frac{pq}{p-q}} \right) \frac{dR}{R} \right),
\end{aligned} \tag{5.12}$$

where $C = (4\sigma)^{\frac{1}{4(4\sigma-1)}}$.

Because the function $\vartheta$ is continuous at $x_0$, without loss of generality, we can assume that $\vartheta(x_0) = 0$.

As the function $-u$ satisfies in the pair with the function $g_{(-u)}$ Hypotheses H1, H2 and H3(b), in the rest of the proof it suffices to estimate $(u(x)-\vartheta(x_0))_+ = u_+(x)$ in the ball $B(x_0,\rho)$. The same estimate will hold for the function $u_-$.

For $R_0 > 0$, we have

$$M(R_0,R_0) = (\operatorname*{ess\,sup}_{B(x_0,R_0)} u - \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta)_+ \le (\operatorname*{ess\,sup}_{B(x_0,R_0)} u - \vartheta(x_0))_+ = \operatorname*{ess\,sup}_{B(x_0,R_0)} u_+\,,$$

and from Theorem 5.1 it follows that

$$M := \operatorname*{ess\,sup}_{B(x_0,R_0)} u_+ < \infty\,.$$

For $0 < \rho < R_0$ the inequality (5.12) gives

$$\begin{aligned} \operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ &\le \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta_+ + M(\rho,R_0) \\ &\le \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta_+ + \qquad\qquad\qquad\qquad (5.13) \\ &+ CM \exp\left(-\frac{1}{8(4\sigma-1)}\int_\rho^{R_0} \exp\left(-C_0\left(\frac{\Phi(2R)}{2R}\right)^{\frac{pq}{p-q}}\right)\frac{dR}{R}\right). \end{aligned}$$

As the function $\vartheta$ is Hölder continuous at $x_0$ and $\vartheta(x_0)=0$, there exist some $C', \beta > 0$ such that for all sufficiently small $\rho$ and $R_0$ we have

$$\operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta_+ \le C' R_0^\beta\,.$$

By the assumption (5.2) of the theorem, there exists (sufficiently small) $\alpha > 0$ such that

$$\int_\rho^1 \exp\left(-C_0\left(\frac{\Phi(2R)}{2R}\right)^{\frac{pq}{p-q}}\right)\frac{dR}{R} \ge \alpha|\log\rho|\,.$$

Note also that for all $0 < R_0 < 1$,

$$\int_{R_0}^1 \exp\left(-C_0\left(\frac{\Phi(2R)}{2R}\right)^{\frac{pq}{p-q}}\right)\frac{dR}{R} \le \int_{R_0}^1 \frac{dR}{R} = |\log R_0|\,.$$

From the inequality (5.13) and the last three inequalities, for sufficiently small $\rho$ and $R_0$, we have

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ \le C' R_0^{\beta} + CM\rho^{\frac{\alpha}{8(4\sigma-1)}} R_0^{-\frac{1}{8(4\sigma-1)}} .$$

Choosing now $R_0 = \rho^{\alpha'}$ with $\alpha' = \frac{\alpha}{8(4\sigma-1)\beta+1}$, we obtain

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ \le H\rho^{\gamma} ,$$

where $H = C' + CM$ and $\gamma = \alpha'\beta$. Note that with an appropriate choice of $\alpha$, $0 < \gamma \le 1$.

As the same estimate holds for the function $u_-$, we have

$$\operatorname*{osc}_{B(x_0,\rho)} u = \operatorname*{ess\,sup}_{B(x_0,\rho)} u - \operatorname*{ess\,inf}_{B(x_0,\rho)} u \le 2H\rho^{\gamma} ,$$

and thus, after a redefinition on a set of measure zero, the function $u$ is Hölder continuous at $x_0$.

Q.E.D.

# Chapter 6

# Regularity of Quasi-minimizers in Axiomatic Sobolev Spaces

In this chapter we assume that the metric measure space $(X, d, \mu)$ is equipped with a $D$-structure.

For the proofs of Propositions 6.3 and 6.6 of this chapter we will need the following

**Lemma 6.1** *Let $f(r)$ be a nonnegative function defined on the interval $[R_1, R_2]$, where $R_1 \geq 1$. Suppose that for all $R_1 \leq r_1 < r_2 \leq R_2$,*

$$f(r_1) \leq \theta f(r_2) + \frac{A}{(r_2 - r_1)^\alpha} + B\,,$$

*where $A, B \geq 0$, $\alpha > 0$ and $0 \leq \theta < 1$. Then there exists $C > 0$ depending only on $\alpha$ and $\theta$ such that for all $R_1 \leq r_1 < r_2 \leq R_2$,*

$$f(r_1) \leq C\left(\frac{A}{(r_2 - r_1)^\alpha} + B\right)\,.$$

**Proof** See, e.g., Lemma 5.1 in [7].

## 6.1 Interior regularity

In this section we derive the Hölder continuity of a quasi-minimizer of the $p$-Dirichlet energy of the axiomatic setting of Chapter 1 in an interior point of the domain $\Omega \subset X$. The existence of quasi-minimizers on the set $\Omega$ is proven in Theorem 2.18. We have the following

**Theorem 6.2** *Assume that the $D$-structure on $X$ is strongly local. If it also supports a weak $(1, q)$-Poincaré inequality for some $q$, $q < p$, then a quasi-minimizer $u^*$ of the energy functional $\mathcal{E}_p$ on the set $\Omega$ is locally Hölder continuous inside the set $\Omega \subset X$.*

The result of this theorem follows from Theorem 4.3 and the following

**Proposition 6.3** *Under the conditions of Theorem* 6.2*, either the quasi-minimizer $u^*$ and its minimal pseudo-gradient $g_{u^*}$ satisfy Hypotheses* H1 - H3*, or these three hypotheses are satisfied by the pair $(-u^*, g_{u^*})$.*

**Proof** Hypothesis H1: Let $B(z, R) \subset \Omega$, $0 < \rho < R$ and $\eta$ be a $\frac{1}{(R-\rho)}-$Lipschitz cutoff function so that $0 \leq \eta \leq 1$, $\eta = 1$ on $B(z, \rho)$ and the support of $\eta$ is contained in $B(z, R)$. Set

$$v = u^* - \eta \max\{u^* - k, 0\},$$

where $k \geq k^*$, $k^*$ will be chosen in the proof of Hypothesis H3. Observe that

$$v = \begin{cases} (1-\eta)(u^* - k) + k & \text{on} \quad A(k, R) \\ \qquad u^* & \text{on} \quad \Omega \setminus A(k, R)\,, \end{cases}$$

where $A(k, R) = \{x \in B(R) : \ u^*(x) > k\}$.

Note that obviously $v = u^* + (v - u^*)$ and that $(v - u^*) = 0$ on $\Omega \setminus A(k, R)$. As $u^*$ is a quasi-minimizer we have

$$\int_{A(k,R)} g_{u^*}^p d\mu \leq K \int_{A(k,R)} g_v^p d\mu\,,$$

where $K$ is the constant in the definition the quasi-minimizer $u^*$.

From the strong locality of the $D$-structure it follows (see Proposition 2.11) that

$$\int_{A(k,R)} g_v^p d\mu = \int_{A(k,R)} g_{(1-\eta)(u^*-k)+k}^p d\mu\,,$$

Note that $\frac{1}{(R-\rho)} \in D[1 - \eta]$. Axioms A1, A2 and A3 imply

$$(u^* - k)\frac{1}{(R-\rho)} + (1 - \eta)g_{u^*} \in D[(1 - \eta)(u^* - k) + k].$$

From this and the last two inequalities we obtain

$$\begin{aligned}\int_{A(k,\rho)} g_{u^*}^p d\mu &\le K \int_{A(k,R)} \left( (u^* - k)\frac{1}{(R-\rho)} + (1-\eta) g_{u^*} \right)^p d\mu \\ &\le \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u^* - k)^p d\mu + C \int_{A(k,R)\setminus A(k,\rho)} g_{u^*}^p d\mu,\end{aligned}$$

where $C = K\, 2^{p-1}$. Here we used the fact that $1 - \eta = 0$ on $A(k, \rho)$. Adding the term $C \int_{A(k,\rho)} g_{u^*}^p d\mu$ to the left and right hand sides of the inequality above, we see that

$$(1 + C) \int_{A(k,\rho)} g_{u^*}^p d\mu \le C \int_{A(k,R)} g_{u^*}^p d\mu + \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u^* - k)^p d\mu,$$

or

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \frac{C}{1+C} \int_{A(k,R)} g_{u^*}^p d\mu + \frac{C}{1+C} \frac{1}{(R-\rho)^p} \int_{A(k,R)} (u^* - k)^p d\mu.$$

Hence, if $\rho < r \le R$, then

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \frac{C}{1+C} \int_{A(k,r)} g_{u^*}^p d\mu + \frac{C}{1+C} \frac{1}{(r-\rho)^p} \int_{A(k,R)} (u^* - k)^p d\mu.$$

From the last inequality and Lemma 6.1 we conclude that there is a constant $C$ depending on $p$ and $K$ only so that

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u^* - k)^p d\mu$$

and hence the pair $u^*$ and $g_{u^*}$ satisfies Hypothesis H1.

Hypothesis H2 Let $\eta$ be the Lipschitz function from Hypothesis H2 and $v = \eta\,(u^* - k)_+$. Axioms A1, A2, A3 and the strict locality of $D$-structure imply that function $\eta\, g_{u^*}\, \chi_{\{u^*>k\}} + \frac{C}{(R-\rho)}(u^* - k)_+ \in D[v]$. Obviously, $v = v\chi_{\{v>0\}}$. Hence, it follows from the strict locality that

$$\left(\eta\, g_{u^*}\, \chi_{\{u^*>k\}} + \tfrac{C}{(R-\rho)}(u^* - k)_+\right) \chi_{\{v>0\}} \in D[v]\,.$$

Therefore, since the $D$-structure of the space $X$ supports a weak $(1, q)$-Poincaré inequality and thus a weak $(t, q)$-Poincaré inequality for some $t$,

$t > p > q$, and $\tau > 1$ (see Section 2.2), we have by the Minkowski inequality and the facts that $g_{u^*} \geq 0$, $0 \leq \eta \leq 1$,

$$\begin{aligned}
\left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} &\\
\leq \left(\fint_{B(R+\rho)} |v - v_{B(R+\rho)}|^t d\mu\right)^{\frac{1}{t}} &+ |v_{B(R+\rho)}| \\
\leq C(R+\rho)\left(\fint_{B(\tau(R+\rho))}\left(\eta\, g_{u^*}\,\chi_{\{u^*>k\}} + \frac{C}{(R-\rho)}(u^*-k)_+\right)^q \chi_{\{v>0\}}\, d\mu\right)^{\frac{1}{q}} &\\
+ |v_{B(R+\rho)}| &\\
\leq C(R+\rho)\left(\fint_{B(\tau(R+\rho))}\left(g_{u^*}\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u^*-k)_+\right)^q \chi_{\{v>0\}}\, d\mu\right)^{\frac{1}{q}} &\\
+ |v_{B(R+\rho)}| &
\end{aligned}$$

$$\leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}} + |v_{B(R+\rho)}|. \tag{6.1}$$

In the last inequality we denoted $g_{(v)} = g_{u^*}\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u^*-k)_+$ and used the doubling property of $\mu$ and the fact that $\{v > 0\} \subset B(\frac{R+\rho}{2})$.

Since $v = \eta\,(u-k)_+$ is non-negative, by the Hölder inequality we obtain

$$\begin{aligned}
|v_{B(R+\rho)}| &= \frac{1}{\mu(B(R+\rho))}\int_{B(R+\rho)} v d\mu = \frac{1}{\mu(B(R+\rho)}\int_{B(R+\rho)} v\,\chi_{\{v>0\}} d\mu \\
&\leq \left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}}\left(\frac{\mu\left(\{x \in B\,(R+\rho) : v(x) > 0\}\right)}{\mu\left(B\,(R+\rho)\right)}\right)^{1-\frac{1}{t}}.
\end{aligned}$$

Since $\operatorname{supp} v \subset B\left(\frac{R+\rho}{2}\right)$, the property of the measure $\mu$ proved under assumptions of the theorem in Proposition 2.13 implies that

$$\frac{\mu\left(\{v>0\}\right)}{\mu\left(B\,(R+\rho)\right)} \leq \frac{\mu\left(B\left(\frac{R+\rho}{2}\right)\right)}{\mu\left(B\,(R+\rho)\right)} \leq \gamma$$

for some $\gamma$, $0 < \gamma < 1$.

Hence from the previous inequality and the inequality (6.1) we obtain

$$(1-\gamma^{1-\frac{1}{t}})\left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}}.$$

From the doubling property of $\mu$ finally we will have

$$\left(\fint_{B(\frac{R+\rho}{2})} v^t d\mu\right)^{\frac{1}{t}} \leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}},$$

for some constant $C > 0$.

Note that since by Axiom A2, $D[-u] = D[u]$ for any function $u \in L^p_{loc}(X)$, the function $-u^*$ is also a quasi-minimizer of the energy functional $\mathcal{E}_p$. Thus Hypotheses H1 and H2 are also true for the function $-u^*$ and the pseudo-gradient $g_{u^*}$.

Hypothesis H3: By Corollary 4.2 the function $u^*$ is now locally bounded. Let

$$k^* := \frac{\operatorname{ess\,sup}_{B(\sigma R)} u^* + \operatorname{ess\,inf}_{B(\sigma R)} u^*}{2}.$$

For $h > k \geq k^*$, the function $g_{(w)} := g_{u^*}\,\chi_{\{k<u^*\leq h\}}$ belongs to the $D[w]$ for $w = u_k^h = \min\{u^*, h\} - \min\{u^*, k\} = h - k - (h - k - (u^* - k)_+)_+$. Indeed, from the strict locality of $D$-structure, we have $g_{u^*}\,\chi_{\{u^*>k\}} \in D[(u^* - k)_+]$, and thus $g_{u^*}\,\chi_{\{u^*>k\}}\,\chi_{\{h-k-(u^*-k)_+>0\}} \in D[(h-k-(u^*-k)_+)_+]$. Axioms A1 and A2 imply finally that $g_{u^*}\,\chi_{\{k<u^*\leq h\}} \in D[w]$, as $\chi_{\{u^*>k\}}\,\chi_{\{h-k-(u^*-k)_+>0\}} = \chi_{\{k<u^*<h\}}$ and $g_{u^*}\,\chi_{\{k<u^*\leq h\}} \geq g_{u^*}\,\chi_{\{k<u^*<h\}}$.

If $\mu(\{x \in B(R) : u^*(x) > k^*\}) > \frac{1}{2}\mu(B(R))$, then

$$\mu(\{x \in B(R) : -u^*(x) \leq -k^*\}) > \tfrac{1}{2}\mu(B(R)).$$

Consequently we have

$$\mu(\{x \in B(R) : -u^*(x) > -k^*\}) \leq \tfrac{1}{2}\mu(B(R)),$$

and hence we could consider $-u^*$ rather then $u^*$ in our discussion. Then if we prove that $-u^*$ is Hölder continuous, obviously the function $u^*$ will be

Hölder continuous too. Therefore, without loss of generality, we may assume that $\mu(\{x \in B(R) : u^*(x) > k^*\}) \leq \frac{1}{2}\mu(B(R))$, and thus, for $k \geq k^*$, that

$$\begin{aligned}\mu(\{x \in B(R) : w > 0\}) &\leq \mu(\{x \in B(R) : u^*(x) > k\}) \\ &\leq \mu(\{x \in B(R) : u^*(x) > k^*\}) \leq \tfrac{1}{2}\mu(B(R)). \qquad (6.2)\end{aligned}$$

Since $g_{(w)} = g_{u^*}\,\chi_{\{k<u^*\leq h\}} \in D[w]$ and because a weak $(t,q)$-Poincaré inequality, $t > q$, implies a weak $(q,q)$-Poincaré inequality, we have

$$\begin{aligned}\left(\fint_{B(R)} w^q d\mu\right)^{\frac{1}{q}} &\leq \left(\fint_{B(R)} |w - w_{B(R)}|^q d\mu\right)^{\frac{1}{q}} + |w_{B(R)}| \\ &\leq C_P R \left(\fint_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}} + |w_{B(R)}|.\end{aligned}$$

By the Hölder inequality we obtain

$$\begin{aligned}|w_{B(R)}| &= \frac{1}{\mu(B(R))}\int_{B(R)} w d\mu = \frac{1}{\mu(B(R))}\int_{B(R)} w\,\chi_{\{w>0\}} d\mu \\ &\leq \left(\fint_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \left(\frac{\mu\left(\{x \in B(R) : w(x) > 0\}\right)}{\mu(B(R))}\right)^{1-\frac{1}{q}}.\end{aligned}$$

The inequality (6.2) will then imply

$$\left(\fint_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \leq C_w R \left(\fint_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}},$$

where the constant $C_w = C_P/(1 - \left(\frac{1}{2}\right)^{1-\frac{1}{q}})$. Thus, finally we obtain

$$\left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \leq C_w R \left(\int_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}},$$

and hence Hypothesis H3 is satisfied.

Q.E.D.

## 6.2 Boundary regularity

Theorem 2.18 gives the existence of the capacitary function of the $p$-Dirichlet energy in the axiomatic setting. For a $p$-fat subset $F$ of $X$ (see Definition 2.17 of a $p$-fat set in Section 2.4) this function equals $p$-quasi-everywhere to 1 on $F$ and minimizes the energy in the complement of $F$.

**Definition 6.4 (Quasi-minimizer with boundary data)** *Let $\vartheta \in L^p_{loc}(X)$. We say that a function $u \in L^p_{loc}(X)$ is a quasi-minimizer of the $p$-energy integral $\mathcal{E}_p$ on a set $\Omega \subset X$ with boundary data $\vartheta$ if $\mu(\Omega \setminus supp(u-\vartheta)) = 0$ and there exists a constant $K > 0$ such that for all functions $\varphi \in \mathcal{L}^{1,p}(X)$ with $\mu(\Omega \setminus supp(\varphi)) = 0$ the inequality*

$$\int_{\varphi \neq 0} g_u^p d\mu \leq K \int_{\varphi \neq 0} g_{u+\varphi}^p d\mu$$

*holds. Here, as usual, $g_{u+\varphi}$ is the minimal pseudo-gradient of $u+\varphi$. When $K=1$, the corresponding quasi-minimizer is called the minimizer with boundary data $\vartheta$ of the energy functional $\mathcal{E}_p$.*

When the $D$-structure is strongly local the capacitary function of the condenser $F$ is a minimizer of $\mathcal{E}_p$ on the set $X \setminus F$ with boundary data 1 (cf. Proposition 2.20)

In this section we show that under certain conditions a quasi-minimizer $u^*$ of the energy functional $\mathcal{E}_p$ on the set $X \setminus F$ with boundary data $\vartheta$ is Hölder continuous at a boundary point of the set $F$.

For this we adapt the notations of Chapter 5 on the regularity at the boundary in an abstract setting and show that the quasi-minimizer $u^*$ satisfies Hypotheses H1, H2 of Chapter 4 and Hypothesis H3(b) of Chapter 5.

Namely, we set $\Omega = X \setminus F$ and we suppose the quasi-minimizer $u^*$ coincides a.e. with the function $\vartheta$ on the set $X \setminus \Omega = F$ and that $\vartheta$ is Hölder continuous at a boundary point $x_0$ of $\Omega$.

We have the following

**Theorem 6.5** *Assume that the $D$-structure on $X$ is strongly local and supports for some $q$, $q < p$, a weak $(1,q)$-Poincaré inequality. If, in addition,* *the following condition is satisfied*

$$\liminf_{\rho\to 0} \frac{1}{|\log \rho|} \int_\rho^1 \exp\left(-C\left(\frac{R^{-q}\mu(B(x_0,R))}{Cap_q\left(B(x_0,\frac{1}{2}R)\setminus\Omega, B(x_0,R)\right)}\right)^{\frac{p}{p-q}}\right)\frac{dR}{R} > 0\,, \tag{6.3}$$

*for some constant $C > 0$, then the quasi-minimizer $u^*$ is Hölder continuous at $x_0$.*

The result of this theorem follows from Theorem 5.2 and the following

**Proposition 6.6** *If the D-structure on $X$ is strongly local and supports a weak $(1,q)$-Poincaré inequality, then the quasi-minimizer $u^*$ and its minimal pseudo-gradient $g_{u^*}$, as well as $-u^*$ and $g_{u^*}$, satisfy in a neighborhood of $x_0$ Hypotheses* H1, H2 *and* H3$(b)$. *In this case the function $\Phi$ of Hypothesis* H3$(b)$ *can be chosen to be*

$$\Phi(\Omega, x_0, R) = C\frac{\mu(B(x_0,R))^{\frac{1}{q}}}{\mathrm{Cap}_q\left(B(x_0,\frac{1}{2}R)\setminus\Omega, B(x_0,R)\right)^{\frac{1}{q}}}\,,$$

*with some $C > 0$.*

**Proof** Hypothesis H1: For the point $x_0 \in \partial\Omega$ and $0 < \rho < R \le \mathrm{diam}(X)/3$, let $\eta$ be a $\frac{1}{(R-\rho)}$−Lipschitz cutoff function so that $0 \le \eta \le 1$, $\eta = 1$ on $B(x_0,\rho)$ and the support of $\eta$ is contained in $B(x_0,R)$.
Set

$$v = -\eta\, \max\{u^* - k, 0\},$$

where $k \ge k^* := \operatorname{ess\,sup}_{B(x_0,R)} \vartheta$. Then

$$u^* + v = (1-\eta)\,(u^* - k) + k$$

on $A(k,R) = \{x \in B(x_0,R):\ u^*(x) > k\}$.

Note that as $u^* = \vartheta := 1$ a.e. on $B(x_0,R)\setminus\Omega$, $v = 0$ a.e. on $B(x_0,R)\setminus\Omega$ and $u^* = \vartheta \le k$ a.e. on the set $X\setminus\Omega$. Moreover, $v = 0$ outside $B(x_0,R)$ by the definition of $\eta$. Therefore, $\mu(\Omega\setminus \mathrm{supp}(v)) = 0$, and by the energy quasi-minimizing property of $u^*$, we have

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \int_{v\neq 0} g_{u^*}^p d\mu \le K\int_{v\neq 0} g_{u^*+v}^p d\mu \le K\int_{A(k,R)} g_{u^*+v}^p d\mu\,,$$

where $K$ is the constant in the definition the quasi-minimizer $u^*$.

From the strong locality of the $D$-structure it follows (see Proposition 2.11) that

$$\int_{A(k,R)} g^p_{u^*+v} d\mu = \int_{A(k,R)} g^p_{(1-\eta)(u^*-k)+k} d\mu \,,$$

Note that $\frac{1}{(R-\rho)} \in D[1-\eta]$. Axioms A1, A2 and A3 imply

$$(u^* - k)\tfrac{1}{(R-\rho)} + (1-\eta)g_{u^*} \in D[(1-\eta)(u^*-k)+k].$$

From this and the last two inequalities we obtain

$$\begin{aligned}\int_{A(k,\rho)} g^p_{u^*} d\mu &\leq K \int_{A(k,R)} ((u^*-k)^p \frac{1}{(R-\rho)^p} + (1-\eta)^p g^p_{u^*}) d\mu \\ &\leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u^*-k)^p d\mu + C \int_{A(k,R)\setminus A(k,\rho)} g^p_{u^*} d\mu,\end{aligned}$$

where $C = K\,2^{p-1}$. Here we used the fact that $1-\eta = 0$ on $A(k,\rho)$. Adding the term $C\int_{A(k,\rho)} g^p_{u^*} d\mu$ to the left and right hand sides of the inequality above, we see that

$$(1+C)\int_{A(k,\rho)} g^p_{u^*} d\mu \leq C \int_{A(k,R)} g^p_{u^*} d\mu + \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u^*-k)^p d\mu,$$

or

$$\int_{A(k,\rho)} g^p_{u^*} d\mu \leq \frac{C}{1+C} \int_{A(k,R)} g^p_{u^*} d\mu + \frac{C}{1+C}\frac{1}{(R-\rho)^p} \int_{A(k,R)} (u^*-k)^p d\mu.$$

Hence, if $\rho < r \leq R$, then

$$\int_{A(k,\rho)} g^p_{u^*} d\mu \leq \frac{C}{1+C} \int_{A(k,r)} g^p_{u^*} d\mu + \frac{C}{1+C}\frac{1}{(r-\rho)^p} \int_{A(k,R)} (u^*-k)^p d\mu.$$

From the last inequality and Lemma 6.1 we conclude that there is a constant $C > 0$ depending on $p$ and $K$ only so that

$$\int_{A(k,\rho)} g^p_{u^*} d\mu \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u^*-k)^p d\mu$$

and hence the pair $u^*$ and $g_{u^*}$ satisfies Hypothesis H1.

Hypothesis H2: The proof of this hypothesis at the point $x_0 \in \partial\Omega$ for the pair $(u^*, g_{u^*})$ repeats without any changes the proof of Hypothesis H2 at an interior point of $\Omega$ for the functions $u^*$ and $g_{u^*}$. Here we do not use the fact that $u^*$ is a quasi-minimizer of the Dirichlet energy and, thus, the proof does not depend on the region where $u^*$ is minimal. As it was noted in Section 4.1, Hypothesis H2 is a characteristic of the whole Sobolev space of functions $W^{1,p}(X)$ and does not depend on the behavior of its particular members.

Hypothesis H3(b): Because the function $\Phi$ of Hypothesis H3(b) does depend on the domain of minimization $\Omega$, the previous remark on the proof of Hypothesis H2 for the boundary case is not suitable for the proof of Hypothesis H3(b).

As $h$ and $k$ in the definition of the function $w$ are such that $h > k > k^* = \operatorname{ess\,sup}_{B(x_0,R)} \vartheta$, we have that $w = 0$ a.e. on $B(x_0,R) \setminus \Omega$. In fact, it is not difficult to check that $w = u_k^h = (u-k)_+ - (u-h)_+$ and as $u = \vartheta$ a.e. on the complement of $\Omega$, in particular on $B(x_0,R) \setminus \Omega$, we have $(u-k)_+ = (u-h)_+ = 0$ a.e. on this set.

Let

$$\bar{w} = \left( \fint_{B(x_0,R)} w^q \, d\mu \right)^{\frac{1}{q}}$$

and $\eta$ be a $\frac{2}{R}$-Lipschitz function vanishing outside $B(x_0,R)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on $B(x_0, \frac{1}{2}R)$. Then the function $f = \eta(1 - \frac{w}{\bar{w}})_+$ is admissible for the capacity $\operatorname{Cap}_q\big(B(x_0, \frac{1}{2}R) \setminus \Omega, B(x_0,R)\big)$. From Axioms A1,A2 and A3 and from the strict locality it follows that $\frac{1}{\bar{w}} g_w \chi_{\{\frac{w}{\bar{w}}<1\}} + \frac{2}{R}(1 - \frac{w}{\bar{w}})_+ \in D[w]$. Hence

$$\begin{aligned}
\operatorname{Cap}_q\big(B(\tfrac{1}{2}R) \setminus \Omega, B(R)\big) &\le \int_{B(R)} g_f^q \, d\mu \\
&\le \int_{B(R)} \left( \tfrac{1}{\bar{w}} g_w \chi_{\{\frac{w}{\bar{w}}<1\}} + \tfrac{2}{R}(1 - \tfrac{w}{\bar{w}})_+ \right)^q d\mu \\
&\le \frac{2^{q-1}}{\bar{w}^q} \int_{B(R)} g_w^q \, d\mu + \frac{2^{q-1} 2^q}{R^q \bar{w}^q} \int_{B(R)} |w - \bar{w}|^q \, d\mu \,.
\end{aligned}$$

Denoting the ball $B(R)$ by $B$, by the Minkowski inequality we have

$$\left(\int_B |w-\bar{w}|^q d\mu\right)^{\frac{1}{q}} \le \left(\int_B |w-w_B|^q d\mu\right)^{\frac{1}{q}} + |\bar{w}-w_B|\mu(B)^{\frac{1}{q}}\,.$$

Using a weak $(q,q)$-Poincaré inequality and the doubling condition of $\mu$, we estimate the second term of the last inequality as follows

$$\begin{aligned}|\bar{w}-w_B|\mu(B)^{\frac{1}{q}} &= |\|w_B\|_{L^q(B)} - \|w\|_{L^q(B)}| \\ &\le \|w-w_B\|_{L^q(B)} \le CR\left(\int_{\sigma B} g_w^q d\mu\right)^{\frac{1}{q}},\end{aligned}$$

where $C>0$. The last three inequalities, a weak $(q,q)$-Poincaré inequality and the doubling condition now give

$$\operatorname{Cap}_q\left(\tfrac{1}{2}B\setminus\Omega, B\right) \le \frac{C}{\bar{w}^q}\int_{\sigma B} g_w^q d\mu\,,$$

or

$$\bar{w}^q = \fint_B w^q d\mu \le \frac{C}{\operatorname{Cap}_q\left(\frac{1}{2}B\setminus\Omega, B\right)}\int_{\sigma B} g_w^q d\mu\,.$$

In the proof of Hypothesis H3 for the functions $u^*$ and $g_{u^*}$ it was shown that the function $g_{(w)} := g_{u^*}\,\chi_{\{k<u^*\le h\}}$ belongs to the set $D[w]$ of the pseudo-gradients of $w$. Therefore, from the last inequality we have

$$\int_B w^q d\mu \le \frac{C\mu(B)}{\operatorname{Cap}_q\left(\frac{1}{2}B\setminus\Omega, B\right)}\int_{\sigma B} g_{(w)}^q d\mu\,,$$

and, thus, as a function $\Phi$ of Hypothesis H3(b) we can take the function

$$\Phi(\Omega, x_0, R) = C\frac{\mu(B(x_0,R))^{\frac{1}{q}}}{\operatorname{Cap}_q\left(B(x_0,\frac{1}{2}R)\setminus\Omega, B(x_0,R)\right)^{\frac{1}{q}}}\,,$$

with some $C>0$.

Note that since by Axiom A2, $D[-u]=D[u]$ for any function $u\in L^p_{loc}(X)$, the function $-u^*$ is also a quasi-minimizer of the energy functional $\mathcal{E}_p$.Moreover the function $-u^*$ coincides with the function $-\vartheta=-1$ q.e. outside the set

$\Omega = X \setminus F$. Therefore, the reasoning similar to the one for the pair $(u^*, g_{u^*})$ shows that Hypotheses H1, H2 and H3(b) are also true for the function $-u^*$ and the pseudo-gradient $g_{u^*}$.

Q.E.D.

**Remark** The condition (6.3) in Theorem 6.5 is an analog, for our case, of the classical Wiener criterion for continuity at a boundary point of a domain in $\mathbb{R}^n$. Note, however, that the capacity which appears in many Wiener criteria for solutions of various classes of elliptic equations is, in fact, the Sobolev capacity, not the variational capacity which we have in the condition (6.3). But, due to Lemma 6.8 below, the variational capacity $\mathrm{Cap}_q$ of the criterion (6.3) could be replaced by the Sobolev capacity $\mathrm{C}_q$. In this case, the Sobolev capacity $\mathrm{C}_q$ is described by the following

**Definition 6.7 (Sobolev capacity)** *The Sobolev $q$-capacity of a set $F \subset X$ is defined by*

$$C_q(F) := \inf \left\{ \|u\|^q_{W^{1,q}(X)} \;\middle|\; u \in W^{1,q}(X), u \geq 1 \text{ near } F \text{ and } u \geq 0 \text{ a.e.} \right\} .$$

**Lemma 6.8** *Under the hypotheses of Theorem 6.5, for a set $E \subset \frac{1}{2}B = B(x_0, \frac{1}{2}R)$, we have*

$$\mathrm{Cap}_q(E, B) \geq \frac{C_q(E)}{C(1 + R^q)}$$

*and*

$$\mathrm{Cap}_q(E, B) \geq \frac{\mu(E)}{CR^q} ,$$

*for some constant $C > 0$.*

**Proof** Let $v$ be a function admissible for the variational capacity $\mathrm{Cap}_q(E, B)$. Then

$$\left( \fint_{2B} v^q d\mu \right)^{\frac{1}{q}} \leq \left( \fint_{2B} |v - v_{2B}|^q d\mu \right)^{\frac{1}{q}} + |v_{2B}| .$$

By the Hölder inequality and the fact that $v \in \mathcal{L}^{1,q}_0(B)$, we have

$$|v_{2B}| \leq \fint_{2B} v \chi_B d\mu \leq \left( \fint_{2B} v^q d\mu \right)^{\frac{1}{q}} \left( \frac{\mu(B)}{\mu(2B)} \right)^{1-\frac{1}{q}} \leq \left( \fint_{2B} v^q d\mu \right)^{\frac{1}{q}} \gamma_0 ,$$

for some $\gamma_0$, $0 < \gamma_0 < 1$ (see Proposition 2.13).

The last two inequalities, a weak $(q, q)$-Poincaré inequality, the fact that $v \in \mathcal{L}_0^{1,q}(B)$ and the locality of the $D$-structure give

$$\int_B v^q d\mu \le CR^q \int_B g_v^q d\mu \,,$$

for some constant $C > 0$.

As $v$ is admissible for the capacity $\mathrm{Cap}_q(E, B)$, we further have

$$\mu(E) \le \int_B v^q d\mu \le CR^q \int_B g_v^q d\mu$$

and

$$\mathrm{C}_q(E) \le \int_X v^q d\mu + \int_X g_v^q d\mu \le C(1 + R^q) \int_B g_v^q d\mu \,.$$

Taking the infimum over all admissible $v$ completes the proof.

Q.E.D.

**Corollary 6.9** *The condition (6.3) is also satisfied if the complement of $\Omega$ has a corkscrew at $x_0$, i.e. if the set $B(x_0, R) \setminus \Omega$ contains a ball with the radius $CR$, for some $C > 0$, or, more generally, if*

$$\mu(B(x_0, R) \setminus \Omega) \ge C\mu(B(x_0, R)) \,.$$

# Chapter 7

# Regularity in the Class of Poincaré-Sobolev Functions

## 7.1 Interior regularity

In this section we impose the following condition on the measure $\mu$. For every $z \in X$ and $0 < R \leq \operatorname{diam}(X)/3$ we assume that there exists $\gamma$, $0 < \gamma < 1$, such that

$$\frac{\mu(B(z, \frac{R}{2}))}{B(z, R)} \leq \gamma \, .$$

Note that in the axiomatic setting this condition is proved in Proposition 2.13.

We will also assume that any pair $(u, g)$, $u \in L^1_{loc}(X)$, $g \in L^q(X)$, satisfying a $(1, q)$-Poincaré inequality in $X$ has the truncation property.

We have the following

**Theorem 7.1** *Let $u$ be a Poincaré-Sobolev function (there exists $g \in L^q(X)$ such that the pair $(u, g)$ satisfies a $(1, q)$-Poincaré inequality). Suppose that the function $-u$ is also a Poincaré-Sobolev function (enjoying a $(1, q)$-Poincaré inequality with some function $g^- \in L^q(X)$ ). If the pairs $(u, g)$ and $(-u, g^-)$ satisfy the $p$-De Giorgi condition on the set $\Omega$, then one of the pairs $(u, g)$ or $(-u, g^-)$ satisfies Hypotheses* H1 - H3 *and, thus, the function $u$ is locally Hölder continuous in $\Omega$.*

**Proof** Hypothesis H1 is the definition of the $p$-De Giorgi condition.

Hypothesis H2: Let $\eta$ be the Lipschitz function as in Hypothesis H2 and $v = \eta(u-k)_+$ ($k, \rho$ and $R$ are fixed). Since the pair $(u, g)$ satisfies a $(1, q)$-Poincaré inequality, by the truncation property, for every $h \in \mathbb{R}, k < h < \infty$, the functions

$$u_k^h = \min\{\max\{0, u-k\}, h-k\} = \begin{cases} h-k & \text{if} \quad u \geq h, \\ u-k & \text{if} \quad k < u < h, \\ 0 & \text{if} \quad u \leq k, \end{cases}$$

and $g\,\chi_{\{k<v\leq h\}}$ satisfy this $(1, q)$-Poincaré inequality as well. Hence they satisfy a $(t, q)$-Poincaré inequality for some $t$, $t > q$, and thus a $(q, q)$-Poincaré inequality (see [14] and Section 2.2). Let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers such that $h_i > k, i \in \mathbb{N}$, and $h_i \to \infty$ as $i \to \infty$. Denote $u_i := u_k^{h_i}$. Then the sequence of functions $\{u_i\}_{i\in\mathbb{N}}$ converges in $L^q_{loc}$ topology to the function $(u-k)_+$. Indeed, for any $i \in \mathbb{N}$,

$$0 \leq u_i \leq (u-k)_+$$

and the fact follows from the dominated convergence theorem. Similarly, the functions $g_i := g\,\chi_{\{k<v\leq h_i\}}$ converge in $L^q_{loc}$ topology to the function $g\,\chi_{\{u>k\}}$. As for every $i \in \mathbb{N}$ the pair $(u_i, g_i)$ satisfies a $(q, q)$-Poincaré inequality, it follows that the pair $((u-k)_+, g\,\chi_{\{u>k\}})$ also satisfies it.
Denote $\varphi := (u-k)_+$. For all $x, y \in \Omega$ and some ball $B \subset \Omega$ we have

$$\begin{aligned} |\eta(x)\varphi(x) - (\eta\varphi)_B| &\leq |\eta(x)\varphi(x) - \eta(x)\varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| \\ &\leq \sup|\eta||\varphi(x) - \varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| \\ &\leq |\varphi(x) - \varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B| =: \Psi(x). \end{aligned}$$

Integrating the last expression $\Psi(x)$ to the power $q$ and using classical inequalities and the definition of Lipschitz functions we get

$$\begin{aligned} &⨍_B \Psi(x)^q d\mu(x) \\ &\quad = ⨍_B \{|\varphi(x) - \varphi_B| + |\eta(x)\varphi_B - (\eta\varphi)_B|\}^q d\mu(x) \\ &\quad = ⨍_B \left\{|\varphi(x) - \varphi_B| + \left|\eta(x)⨍_B \varphi(y)d\mu(y) - ⨍_B \eta(y)\varphi(y)d\mu(y)\right|\right\}^q d\mu(x) \\ &\quad = ⨍_B \left\{|\varphi(x) - \varphi_B| + \left|⨍_B (\eta(x)\varphi(y)) - \eta(y)\varphi(y))d\mu(y)\right|\right\}^q d\mu(x) \end{aligned}$$

$$
\begin{aligned}
&\leq \fint_B \left\{ |\varphi(x) - \varphi_B| + \fint_B |\varphi(y)||\eta(x) - \eta(y)| d\mu(y) \right\}^q d\mu(x) \\
&\leq \fint_B \left\{ |\varphi(x) - \varphi_B| + \mathrm{Lip}(\eta)\mathrm{diam}(B) \fint_B |\varphi(y)| d\mu(y) \right\}^q d\mu(x) \\
&\leq 2^{q-1} \fint_B \left\{ |\varphi(x) - \varphi_B|^q + (\mathrm{Lip}(\eta)\mathrm{diam}(B))^q \left( \fint_B |\varphi(y)| d\mu(y) \right)^q \right\} d\mu(x) \\
&= 2^{q-1} \fint_B |\varphi(x) - \varphi_B|^q d\mu(x) + 2^{q-1} (\mathrm{Lip}(\eta)\mathrm{diam}(B))^q \left( \fint_B |\varphi(y)| d\mu(y) \right)^q .
\end{aligned}
$$

Now, the pair $(\varphi = (u-k)_+, g\,\chi_{\{u>k\}})$ satisfies the following $(q,q)$-Poincaré inequality

$$
\fint_B |\varphi - \varphi_B|^q d\mu \leq \left( C_P \frac{\mathrm{diam}(B)}{2} \right)^q \fint_{\tau B} \left( g\,\chi_{\{u>k\}} \right)^q d\mu,
$$

where $\tau \geq 1$.

Hence

$$
\begin{aligned}
&\fint_B \Psi(x)^q d\mu(x) \\
&\qquad \leq C(\mathrm{diam}(B))^q \fint_{\tau B} \left\{ \left( g\,\chi_{\{u>k\}} \right)^q + (\mathrm{Lip}(\eta)|\varphi|)^q \right\} d\mu,
\end{aligned}
$$

where the constant $C$ depends only on $\tau, q$, $C_P$ and on the doubling constant $C_d$. We thus have proved that

$$
\begin{aligned}
\left( \fint_B |\eta(x)(u(x)-k)_+ - (\eta(u-k)_+)_B|^q d\mu(x) \right)^{1/q} &\leq \left( \fint_B \Psi(x)^q d\mu(x) \right)^{1/q} \\
\leq C\mathrm{diam}(B) \left( \fint_{\tau B} \left( g\,\chi_{\{u>k\}} + \mathrm{Lip}(\eta)(u-k)_+ \right)^q d\mu \right)^{1/q} &.
\end{aligned}
$$

In particular, recalling that $v = \eta(u-k)_+$, for the ball $B\,(R+\rho)$ we will have

$$
\begin{aligned}
&\left( \fint_{B(R+\rho)} |v - v_{B(R+\rho)}|^q d\mu \right)^{1/q} \\
&\leq C(R+\rho) \left( \fint_{B(\tau(R+\rho))} (g\,\chi_{\{u>k\}} + \frac{C}{R-\rho}(u-k)_+)^q d\mu \right)^{1/q},
\end{aligned}
$$

for some $C > 0$.

Obviously, $v = v\chi_{\{v>0\}}$. Repeating the argument in the very beginning of the proof of Hypothesis H2, it is easy to show that the truncation property implies that a $(q, q)$-Poincaré inequality holds for the pair of $v$ and $(g\,\chi_{\{u>k\}} + \frac{C}{R-\rho}(u-k)_+)\chi_{\{v>0\}}$.

A $(t, q)$-Poincaré inequality also holds for the functions $v$ and $(g\,\chi_{\{u>k\}} + \frac{C}{R-\rho}(u-k)_+)\chi_{\{v>0\}}$ and thus we have

$$\begin{aligned}
&\left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \\
&\quad \leq \left(\fint_{B(R+\rho)} |v - v_{B(R+\rho)}|^t d\mu\right)^{\frac{1}{t}} + |v_{B(R+\rho)}| \\
&\quad \leq C(R+\rho)\left(\fint_{B(\lambda(R+\rho))} \left(g\,\chi_{\{u>k\}} + \frac{C}{(R-\rho)}(u-k)_+\right)^q \chi_{\{v>0\}}\, d\mu\right)^{\frac{1}{q}} \\
&\qquad\qquad + |v_{B(R+\rho)}| \\
&\quad \leq C(R+\rho)\left(\fint_{B(\lambda(R+\rho))} \left(g\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u-k)_+\right)^q \chi_{\{v>0\}}\, d\mu\right)^{\frac{1}{q}} \\
&\qquad\qquad + |v_{B(R+\rho)}| \\
&\qquad \leq CR\left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu\right)^{\frac{1}{q}} + |v_{B(R+\rho)}|, \qquad (7.1)
\end{aligned}$$

where $\lambda > 0$. In the last inequality we denoted $g_{(v)} = g\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{(R-\rho)}(u-k)_+$ and used the doubling property of $\mu$ and the fact that $\{v > 0\} \subset B(\frac{R+\rho}{2})$.

By the Hölder inequality we obtain

$$\begin{aligned}
|v_{B(R+\rho)}| &= \frac{1}{\mu(B(R+\rho))}\int_{B(R+\rho)} v d\mu = \frac{1}{\mu(B(R+\rho)}\int_{B(R+\rho)} v\,\chi_{\{v>0\}} d\mu \\
&\leq \left(\fint_{B(R+\rho)} v^t d\mu\right)^{\frac{1}{t}} \left(\frac{\mu\left(\{x \in B\left(R+\rho\right) : v(x) > 0\}\right)}{\mu\left(B\left(R+\rho\right)\right)}\right)^{1-\frac{1}{t}}.
\end{aligned}$$

Then, the condition for the measure $\mu$ stated at the beginning of this section implies that

$$\frac{\mu\left(\{v>0\}\right)}{\mu\left(B\left(R+\rho\right)\right)}\le\frac{\mu\left(B\left(\frac{R+\rho}{2}\right)\right)}{\mu\left(B\left(R+\rho\right)\right)}\le\gamma\,,$$

for some $\gamma$, $0<\gamma<1$.

Hence from the previous inequality and the inequality (7.1) we obtain

$$(1-\gamma^{1-\frac{1}{t}})\left(\fint_{B(R+\rho)}v^t d\mu\right)^{\frac{1}{t}}\le CR\left(\fint_{B(\frac{R+\rho}{2})}g_{(v)}^q d\mu\right)^{\frac{1}{q}}.$$

From the doubling property of $\mu$ finally we will have

$$\left(\fint_{B(\frac{R+\rho}{2})}v^t d\mu\right)^{\frac{1}{t}}\le CR\left(\fint_{B(\frac{R+\rho}{2})}g_{(v)}^q d\mu\right)^{\frac{1}{q}},$$

for some $C>0$.

And thus Hypothesis H2 is verified.

Hypothesis H3 follows from the truncation property, the doubling condition and the fact that a $(1,q)$-Poincaré inequality on the doubling metric measure space implies a $(t,q)$-Poincaré inequality with some $t>q$ and, thus, a $(q,q)$-Poincaré inequality. Indeed, take in the definition 3.3 of the truncation property $\varepsilon=1$, $b=0$, $t_1=k$, $t_2=h$ and note that, in this case,

$$v_{t_1}^{t_2}=u_k^h=\min\{\max\{0,u-k\},h-k\}=\min\{u,h\}-\min\{u,k\}=w.$$

Then we repeat the proof of Hypothesis H3 in Proposition 6.3 with the functions $u^*$ replaced by $u$ and $g_{u^*}$ replaced by $g$.

Q.E.D.

## 7.2 Boundary regularity

The condition (6.3) for the Hölder continuity at a boundary point $x_0$ of Theorem 6.5 is expressed in terms of the capacities of certain sets related to this point. As it was already underlined in Section 3 the notion of a capacity is meaningless in the class of the Poincaré-Sobolev functions. Nevertheless, it is still possible to consider the problem of regularity at a boundary point of a set in these spaces. In this case, in the criterion of regularity some quantities other than capacities will figure (cf. Lemma 6.8 and Corollary 6.9).

Let $\Omega$ be a subset of $X$, $x_0 \in \partial\Omega$, a boundary point of $\Omega$, be fixed and a function $\vartheta \in L^p(X)$ be given. Suppose also that the function $\vartheta$ is Hölder continuous at a point $x_0 \in \partial\Omega$.

The main goal of this section is to prove the following

**Theorem 7.2** *Let $u, -u \in PW^{1,q}(X)$ satisfy the p-De Giorgi condition in $\Omega$ $(p > q)$. If $u = \vartheta$ a.e. on $X \setminus \Omega$ and the following condition*

$$\liminf_{\rho\to 0} \frac{1}{|\log\rho|} \int_\rho^1 \exp\left(-C\left(\frac{\mu(B(x_0,R))}{\mu(B(x_0,R)\setminus\Omega)}\right)^{\frac{(q-1)p}{p-q}}\right)\frac{dR}{R} > 0\,,$$

*holds for some constant $C > 0$, then the function $u$ is Hölder continuous at $x_0$.*

The result of this theorem follows from Theorem 5.2 and the following

**Proposition 7.3** *If the pairs $(u, g)$ and $(-u, g^-)$ satisfy the requirements of Theorem* 7.2*, then the pairs $(u, g)$ and $(-u, g^-)$ satisfy Hypotheses* H1*,* H2 *and* H3$(b)$*. In this case the function $\Phi$ of Hypothesis* H3$(b)$ *can be chosen to be*

$$\Phi(\Omega, x_0, R) = CR\left(\frac{\mu(B(x_0,R))}{\mu(B(x_0,R)\setminus\Omega)}\right)^{1-\frac{1}{q}}\,,$$

*with some $C > 0$.*

**Proof** Hypothesis H1: For the point $x_0 \in \partial\Omega$ and $0 < \rho < R \le \mathrm{diam(X)}/3$, choose $k \ge k^* := \operatorname{ess\,sup}_{B(x_0,R)} \vartheta$. Then for all $k \ge k^*$, we have $\mu(\Omega \setminus A(k, R)) = 0$, where $A(k, R) = B(x_0, R) \cap \{u > k\}$, since $u = \vartheta$ a.e. on $X \setminus \Omega$. Thus, we can apply the $p$-De Giorgi condition (3.2) for the functions $u$ and $g$, which gives Hypothesis H1 for the pair $(u, g)$.

Hypothesis H2: As it is remarked in the proof of Hypothesis H2 at the boundary point $x_0$ for the quasi-minimizer $u^*$ in Proposition 6.6, this hypothesis does not depend on the set $\Omega$ and, thus, the proof of it repeats without any changes the proof of Hypothesis H2 at an interior point of $\Omega$ for the functions $u$ and $g$ in Theorem 7.1.

Hypothesis H3(b): As $u = \vartheta$ a.e. on the complement of $\Omega$, for $h > k > k^* = \operatorname{ess\,sup}_{B(x_0,R)} \vartheta$, we have that $\; w = 0 \;$ a.e. on the set $B(x_0, R) \setminus \Omega$.

By the truncation property the pair $(w, g_{(w)})$ satisfies a $(1, q)$-inequality and, thus, a $(q, q)$-Poincaré inequality. Therefore, we have

$$\begin{aligned}\left(⨍_{B(R)} w^q d\mu\right)^{\frac{1}{q}} &\leq \left(⨍_{B(R)} |w - w_{B(R)}|^q d\mu\right)^{\frac{1}{q}} + |w_{B(R)}| \\ &\leq C_P R \left(⨍_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}} + |w_{B(R)}|.\end{aligned}$$

By the Hölder inequality we obtain

$$\begin{aligned}|w_{B(R)}| &= \frac{1}{\mu(B(R))} \int_{B(R)} w d\mu = \frac{1}{\mu(B(R))} \int_{B(R)} w\, \chi_{\{w>0\}} d\mu \\ &\leq \left(⨍_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \left(\frac{\mu(B \cap \Omega)}{\mu(B)}\right)^{1-\frac{1}{q}}.\end{aligned}$$

These two inequalities and the doubling property of the measure $\mu$ now give

$$\left(1 - \left(\frac{\mu(B \cap \Omega)}{\mu(B)}\right)^{1-\frac{1}{q}}\right) \left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \leq CR \left(\int_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}},$$

or

$$\left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \leq CR \left(\frac{\mu(B)}{\mu(B \setminus \Omega)}\right)^{1-\frac{1}{q}} \left(\int_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}},$$

for some $C > 0$. Hence, Hypothesis H3(b) is satisfied for the pair $(u, g)$ with the function

$$\Phi(\Omega, x_0, R) = CR \left(\frac{\mu(B(x_0, R))}{\mu(B(x_0, R) \setminus \Omega)}\right)^{1-\frac{1}{q}}.$$

Q.E.D.

# Chapter 8

# Harnack's Inequality in Sobolev Spaces on Metric Spaces

Let $\Omega$ be an open subset of $X$.

The goal of this section is to show that extremal functions in axiomatic Sobolev spaces and in the class of Poincaré - Sobolev functions satisfy the Harnack's inequality inside $\Omega$. To prove the main result of the section we first need the following two lemmas.

**Lemma 8.1** *Suppose that $u \geq 0$ and that the function $-u$ satisfies Hypotheses* H1 *and* H2 *of Chapter 4 in the pair with some function $g_{(-u)}$. Then there exists a constant $\gamma_0$, $0 < \gamma_0 < 1$, such that if $\mu(\{x \in B(z,R) : u(x) < \tau\}) \leq \gamma_0 \mu(B(z,R))$, $\tau \in \mathbb{R}$, then*

$$\operatorname*{ess\,inf}_{B(\frac{R}{2})} u \geq \frac{1}{2}\tau.$$

**Proof** Theorem 4.1 applied to $-u$, with $k' = -\tau$, $\tau > 0$, gives the following

$$\operatorname*{ess\,sup}_{B(\frac{R}{2})} (-u) \leq$$

$$-\tau + C\left(\int_{B(z,R)\cap\{-u>-\tau\}} (-u+\tau)^p d\mu\right)^{\frac{1}{p}} \frac{\mu(B(z,R)\cap\{-u>-\tau\})^{\frac{\theta-1}{p}}}{\mu(B(\frac{R}{2}))^{\frac{\theta}{p}}},$$

with $\theta > 1$ and $C$ some constants.

Since $\tau - u \le \tau$ on the set $\{u < \tau\}$ we have then that

$$\operatorname*{ess\,inf}_{B(\frac{R}{2})} u \ge \tau - C\left(\int_{B(z,R)\cap\{u<\tau\}} (\tau-u)^p d\mu\right)^{\frac{1}{p}} \frac{\mu(B(z,R)\cap\{u<\tau\})^{\frac{\theta-1}{p}}}{\mu(B(\frac{R}{2}))^{\frac{\theta}{p}}}$$
$$\ge \tau - C\tau\left(\frac{\mu(B(z,R)\cap\{u<\tau\})}{\mu(B(\frac{R}{2}))}\right)^{\frac{\theta}{p}} \ge \tau\left(1 - C(\gamma_0\, C_d^2)^{\frac{\theta}{p}}\right),$$

where $C_d$ is the doubling constant. Taking $\gamma_0 = (2C)^{-\frac{p}{\theta}} C_d^{-2}$ will imply the claim.

Q.E.D.

**Lemma 8.2** *Let $u$ be as in Lemma 8.1. If for some $\gamma$, $0 < \gamma < 1$,*

$$\mu(\{x \in B(z,R) :\ u(x) < \tau\}) \le \gamma\mu(B(z,R))\,,$$

*and if*

**A)** *(in the case of the axiomatic Sobolev space) the $D$-structure on the space $X$ is strongly local,*

*or*

**B)** *(in the case of the Poincaré-Sobolev functions) the pair $(-u, g_{(-u)})$ has the truncation property,*

*then there exists a constant $\lambda > 0$, independent of the ball $B(z,R)$, such that*

$$\operatorname*{ess\,inf}_{B(\frac{R}{2})} u \ge \lambda\tau.$$

**Proof** Following the arguments of Proposition 6.3 and Theorem 7.1 (the part named "Hypothesis H3") it is not difficult to see that under the conditions of the lemma, the pair $(-u, g_{(-u)})$ satisfies Hypothesis H3. Therefore, for some $k, h > 0$, such that $-k > -h$, we can apply the inequality (4.9) with $u$ replaced by $-u$, $k$ by $-k$ and $h$ by $-h$ respectively to obtain

$$
\begin{aligned}
&(k-h)\mu(\{x\in B(z,R): -u(x) > -h\}) \\
&\qquad \le C\mu(B(R))^{1-\frac{1}{q}}\left(\int_{B(z,2\sigma R)\cap\{-u>-k\}}(k-u)^p d\mu\right)^{\frac{1}{p}} \\
&\qquad\quad \times\left(\mu(B(z,\sigma R)\cap\{-u>-k\}) - \mu(B(z,\sigma R)\cap\{-u>-h\})\right)^{\frac{1}{q}-\frac{1}{p}},
\end{aligned}
$$

where $C>0$ is some constant.

Then we can repeat the proof of Proposition 4.4 for the function $-u$ with $M=0$ and $m=-\tau$, and conclude that

$$\nu\,\mu(B(z,R)\cap\{-u > -2^{-(\nu+1)}\tau\})^{\frac{pq}{p-q}} \le C\mu(B(z,2\sigma R))^{\frac{pq}{p-q}},$$

for $\nu = 1,2,\ldots$, where $C>0$ is some constant. If we now fix $\nu$ in such a way that $\left(\frac{C}{\nu}\right)^{\frac{p-q}{pq}} \le \gamma_0$, we deduce from Lemma 8.1, replacing $\tau$ by $2^{-(\nu+1)}\tau$, that

$$\operatorname*{ess\,inf}_{B(\frac{R}{2})} u \ge 2^{-(\nu+2)}\tau.$$

Q.E.D.

Now we can use the Krylov-Safonov covering theorem on a doubling metric space (see [17]) to conclude in exactly the same way as in [17] that for some $\delta$, $0<\delta<\frac{1}{C_d}<1$, and for every $t$, $0<t<\operatorname{ess\,sup}_{B(z,R)} u$, it is possible to choose $j\in\mathbb{N}$ so that

$$(C_d\delta)^j\mu(B(z,R)) \le \mu(A_{t,0}) \le (C_d\delta)^{j-1}\mu(B(z,R)),$$

where $A_{t,i} = \{x\in B(z,R): u(x)\ge t\lambda^i\}$, $i=0,1,2,\ldots$, and $\lambda$, $0<\lambda<1$, is a constant, and that in this case

$$\operatorname*{ess\,inf}_{B(z,3R)} u \ge Ct\lambda^{j-1},$$

for some constant $C$. Then for $t$, $0<t<\operatorname*{ess\,sup}_{B(z,R)} u$, we will have

$$\frac{\mu(A_{t,0})}{\mu(B(z,R))} \le Ct^{-1/\gamma}\left(\operatorname*{ess\,inf}_{B(z,3R)} u\right)^{1/\gamma},$$

where $\gamma = \log\lambda/\log(C_d\delta)$. It is easy to see that the last estimate holds true also for $t \geq \operatorname*{ess\,sup}_{B(z,R)} u$. Thus we could proceed further as in [17] to conclude that

$$\operatorname*{ess\,inf}_{B(z,R)} u \geq \operatorname*{ess\,inf}_{B(z,3R)} u \geq C \left( \fint_{B(z,R)} u^\sigma \, d\mu \right)^{\frac{1}{\sigma}}, \quad \text{where } \sigma < \frac{1}{\gamma}, \tag{8.1}$$

for every sufficiently small ball $B(z,R) \Subset \Omega$. This estimate is known as a weak Harnack's inequality for the function $u$.

Recall that Corollary 4.2 says that for some constant $C > 0$,

$$\operatorname*{ess\,sup}_{B(R/2)} |u| \leq C \left( \fint_{B(R)} |u|^p \, d\mu \right)^{\frac{1}{p}}$$

(without loss of generality, we can take the constant $k = 0$ in the inequality (4.8)).

Repeating literally the arguments of Remarks 4.4 (1), (2) from [17], it is easy to see that there is nothing particular in the factor $\frac{1}{2}$ in the last inequality and that it holds for every exponent $q > 0$. In particular, we have

$$\operatorname*{ess\,sup}_{B(\rho)} |u| \leq \frac{C}{(1-\rho/R)^Q} \left( \fint_{B(R)} |u|^q \, d\mu \right)^{\frac{1}{q}},$$

where $0 < \rho < R \leq \operatorname{diam}(X)/3$, $C > 0$ and $Q > 0$ is some constant depending on $q$ and the doubling constant only.

From the last inequality and the inequality (8.1) we obtain the following

**Theorem 8.3 (Harnack's inequality)** *Suppose that $u > 0$, the functions $u$ and $-u$ satisfy in the pairs with some functions $g_{(u)}$ and $g_{(-u)} \in L^p(X)$ Hypotheses* H1 *and* H2 *and that*

**A)** *(in the case of the axiomatic Sobolev space) the $D$-structure on the space $X$ is strongly local,*

*or*

**B)** *(in the case of the Poincaré-Sobolev functions) the pair $(-u, g_{(-u)})$ has the truncation property,* *then there exists a constant* $C \geq 1$ *such that*

$$\operatorname*{ess\,sup}_{B(R)} u \leq C \operatorname*{ess\,inf}_{B(R)} u, \tag{8.2}$$

*for every sufficiently small ball* $B(R) \Subset \Omega$. *The constant* $C$ *is independent of the ball* $B(R)$ *and the function* $u$.

**Corollary 8.4** *Under the conditions* $(A)$ *and* $(B)$ *of the theorem, the quasi-minimizers of the p-Dirichlet energy and the functions enjoying the p-De Giorgi condition satisfy the Harnack's inequality* (8.2).

# Chapter 9

# The Norm of a Piecewise-extremal Function

Working in the framework of axiomatic approach to the theory of Sobolev spaces on metric spaces, we prove that the Dirichlet semi-norm of a piecewise-extremal function is equivalent to the sum of the Dirichlet semi-norms of its components. This result might be useful for some applications in the theory of homeomorphisms with bounded $p$-distortion.

The *p-Dirichlet semi-norm* of a function $u \in \mathcal{L}^{1,p}(X)$ is defined to be

$$\|u\|_{\mathcal{L}^{1,p}(X)} = (\mathcal{E}_p(u))^{\frac{1}{p}} = \inf\{\|g\|_{L^p(X)} : g \in D[u]\}\,.$$

A *condenser* in $X$ is a pair of disjoint relatively compact non-empty sets $F_1, F_2 \subset X$. The variational $p$-capacity of such a condenser is defined by

$$\mathrm{Cap}\,(F_1, F_2, X) := \inf\left\{\mathcal{E}_p(u) \middle|\, u \in \mathcal{A}_p(F_1, F_2, X)\right\}\,,$$

where $\mathcal{A}_p\,(F_1, F_2, X)$ is the set of all functions $u \in \mathcal{L}^{1,p}(X)$ such that $u \geq 1$ on a neighborhood of $F_1$ and $u \leq 0$ on a neighborhood of $F_2$.

The following theorem can be found in [11].

**Theorem 9.1** *Let $F_1, F_2 \subset X$ be any condenser in the space $X$ such that either $F_1$ or $F_2$ is p-fat. Then there exists a unique function $u^* \in \mathcal{L}_0^{1,p}(X)$ such that $u^* = 1$ p-quasi-everywhere on $F_1$, $u^* = 0$ p-quasi-everywhere on $F_2$ and $\mathcal{E}_p(u^*) = \mathrm{Cap}_p(F_1, F_2, X)$. Furthermore $0 \leq u^* \leq 1$ for all $x \in X$.*

Consider a monotone sequence $V_0 \subset V_1 \subset \cdots \subset V_l$ of compact sets in $X$. Suppose that $\partial V_i \cap \partial V_j = \emptyset$ for all $i \neq j$ and that for every condenser $(V_{k-1}, X \setminus V_k)$ either the set $V_{k-1}$ or $X \setminus V_k$ is $p$-fat.

**Definition 9.2 (Piecewise-extremal function)** *A function $v$ is called a piecewise-extremal function associated with the sets $V_0, V_1, \ldots, V_l$ and with the real numbers $a_0, a_1, \ldots, a_l$, if*

$$v = a_0 + \sum_{k=1}^{l} (a_k - a_{k-1}) v_k,$$

*where $v_k$ is the extremal function of the pair $(V_{k-1}, X \setminus V_k)$. The functions $v_k$, $k = 1, \ldots, l$, we will call the components of the piecewise-extremal function $v$.*

In order to define a piecewise-extremal function, it suffices to know a collection of sets $\{V_k\}$ and a collection of numbers $\{a_k\}$.

The piecewise-extremal function $v$ belongs to the space $W^{1,p}(X)$, since all functions $v_k$ belong to the space and since $W^{1,p}(X)$ is a Banach space.

In the classical case of the Euclidean space, it follows immediately that the Dirichlet semi-norm of the piecewise-extremal function $v$ equals to the sum of the Dirichlet semi-norms of its components, i.e.

$$\|v\|^p_{\mathcal{L}^{1,p}(\Omega)} = \sum_{k=1}^{l} |a_k - a_{k-1}|^p \, \|v_k\|^p_{\mathcal{L}^{1,p}(\Omega)},$$

where

$$\|\psi\|_{\mathcal{L}^{1,p}(\Omega)} = \left( \int_\Omega |\nabla \psi|^p \, dx \right)^{\frac{1}{p}}$$

for $\psi \in W^{1,p}(\Omega)$, where $\Omega$ is a domain of $\mathbb{R}^n$ and $\nabla \psi = \left( \frac{\partial \psi}{\partial x_1}, \ldots, \frac{\partial \psi}{\partial x_n} \right)$.

Note, however, that in our case of the axiomatic approach this simple result is not longer true due to the fact that the definition of pseudo-gradients is not based on a linear operation, i.e. since the set $D$ of the pseudo-gradients of the functions from $L^p_{loc}(X)$ is not a linear (vector) space but only a convex set. Nevertheless, it is still possible to prove that the two quantities are equivalent. Namely, we have the following

**Theorem 9.3** *There exists a constant $C > 0$ such that*

$$\sum_{k=1}^{l} |a_k - a_{k-1}|^p \; \|v_k\|^p_{\mathcal{L}^{1,p}(X)} \leq \|v\|^p_{\mathcal{L}^{1,p}(X)} \leq C \sum_{k=1}^{l} |a_k - a_{k-1}|^p \; \|v_k\|^p_{\mathcal{L}^{1,p}(X)}. \tag{9.1}$$

**Proof** From Axioms A1 and A2 it follows that

$$\|v\|^p_{\mathcal{L}^{1,p}(X)} \leq C \sum_{k=1}^{l} |a_k - a_{k-1}|^p \; \|v_k\|^p_{\mathcal{L}^{1,p}(X)}, \tag{9.2}$$

for certain constant $C > 0$ ($C = 2^{(l-1)(p-1)}$).

From the other hand, as the function $v_k$ is extremal for the pair $(V_{k-1}, X \setminus V_k)$ we have

$$\int_X g^p_{v_k} \, d\mu \leq \int_X g^p_{v_k^*} \, d\mu \; , \tag{9.3}$$

where the function $v_k^*$ is defined as follows

$$v_k^* = \begin{cases} 1 & \text{on} \quad V_{k-1}, \\ \frac{1}{|a_k - a_{k-1}|} \, v & \text{on} \quad V_k \setminus V_{k-1}, \\ 0 & \text{on} \quad X \setminus V_k \; . \end{cases}$$

Recall that for a function $u \in \mathcal{L}^{1,p}(X)$, by $g_u$ we denote its minimal pseudo-gradient.

It is easy to see that for any function $u : X \to \mathbb{R}$ we have

$$u = \max\{u, 1\} + \min\{u, 1\} - 1$$

and thus, adding to the right hand side of this expression $\pm \min\{u, 0\}$, we have

$$u = \min\{u, 0\} + \max\{u, 1\} + \min\{u, 1\} - \min\{u, 0\} - 1.$$

In particular, for the function $v_k^*$ we have

$$v_k^* = \min\{v_k^*, 0\} + \max\{v_k^*, 1\} + \min\{v_k^*, 1\} - \min\{v_k^*, 0\} - 1.$$

Let us denote by $g^{V_k \setminus V_{k-1}}_{v_k^*}$ a pseudo-gradient of $v_k^*$ such that

$$\int_{V_k \setminus V_{k-1}} \left( g^{V_k \setminus V_{k-1}}_{v_k^*} \right)^p d\mu = \inf \left\{ \int_{V_k \setminus V_{k-1}} g^p \, d\mu : g \in D[v_k^*] \right\}.$$

As
$$\max\{v_k^*-1,0\}=\max\{v_k^*,1\}-1\,,$$
it follows from Axioms A1 and A2 and from the strict locality of the $D$-structure that
$$g_{v_k^*}^{V_k\setminus V_{k-1}}\chi_{\{v_k^*>1\}}\in D[\max\{v_k^*,1\}]$$
and also
$$g_{v_k^*}^{V_k\setminus V_{k-1}}\chi_{\{v_k^*<0\}}\in D[\min\{v_k^*,0\}]$$
and
$$0\in D[1].$$
Moreover, $g_{v_k^*}^{V_k\setminus V_{k-1}}\chi_{\{0<v_k^*<1\}}\in D[\min\{v_k^*,1\}-\min\{v_k^*,0\}]$ (see the proof of Hypothesis H3 in Proposition 6.3). Therefore, from Axiom A2 we obtain that
$$g_{v_k^*}^{V_k\setminus V_{k-1}}\chi_{\{v_k^*\neq 0,1\}}=g_{v_k^*}^{V_k\setminus V_{k-1}}\chi_{\{v_k^*<0\}}+g_{v_k^*}^{V_k\setminus V_{k-1}}\chi_{\{0<v_k^*<1\}}+g_{v_k^*}^{V_k\setminus V_{k-1}}\chi_{\{v_k^*>1\}}+0\in D[v_k^*].$$
Hence,
$$\int_X g_{v_k^*}^p\,d\mu\le\int_X\left(g_{v_k^*}^{V_k\setminus V_{k-1}}\right)^p\chi_{\{v_k^*\neq 0,1\}}\,d\mu\le\int_{V_k\setminus V_{k-1}}\left(g_{v_k^*}^{V_k\setminus V_{k-1}}\right)^p\,d\mu\;, \tag{9.4}$$
since $g_{v_k^*}$ is the minimal pseudo-gradient of the function $v_k^*$ and the set $\{v_k^*\neq 0,1\}\subset V_k\setminus V_{k-1}$.

As
$$v_k^*=\frac{1}{|a_k-a_{k-1}|}v\quad\text{on}\quad V_k\setminus V_{k-1}$$
it follows from Lemma 2.9 that
$$\int_{V_k\setminus V_{k-1}}\left(g_{v_k^*}^{V_k\setminus V_{k-1}}\right)^p\,d\mu=\inf\left\{\int_{V_k\setminus V_{k-1}}g^p\,d\mu\;:\;g\in D[\tfrac{1}{|a_k-a_{k-1}|}v]\right\}$$
$$\le\int_{V_k\setminus V_{k-1}}g^p_{\frac{1}{|a_k-a_{k-1}|}v}\,d\mu\le\frac{1}{|a_k-a_{k-1}|^p}\int_{V_k\setminus V_{k-1}}g_v^p\,d\mu \tag{9.5}$$
by Axiom A2.

From the inequalities (9.3), (9.4), (9.5) we have finally
$$\int_X g_{v_k}^p\,d\mu\le\frac{1}{|a_k-a_{k-1}|^p}\int_{V_k\setminus V_{k-1}}g_v^p\,d\mu,$$

or

$$|a_k - a_{k-1}|^p \int_X g_{v_k}^p \, d\mu \le \int_{V_k \setminus V_{k-1}} g_v^p \, d\mu.$$

Summing up the last inequalities for $k = 1, \dots, l$ we obtain

$$\sum_{k=1}^{l} |a_k - a_{k-1}|^p \|v_k\|^p_{\mathcal{L}^{1,p}(X)} \le \|v\|^p_{\mathcal{L}^{1,p}(X)}. \tag{9.6}$$

The inequalities (9.2) and (9.6) give the desired statement (9.1).

Q.E.D.

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# Curriculum Vitae

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