Systems Science

Vol. 26 2000 No. 4

BÁLINT KISS*,**, JEAN LÉVINE**,
PHILIPPE MULLHAUPT**

# MODELING AND MOTION PLANNING FOR A CLASS OF WEIGHT HANDLING EQUIPMENT***

A unified framework for the modeling of a class of weight handling equipment (WHE) is presented. The dynamic equations are obtained using Lagrange multipiers associated to geometric constraints between generalized coordinates. This approach provides a simple way to show differential flatness for all WHEs of the class. The flatness property can then be exploited for motion planning.

## 1. INTRODUCTION

Many different types of weight handling equipment (WHE), and in particular cranes, are used in various industries including construction and naval transport [14]. The aim of their control [1], [5]–[8], [10], [12] is to increase productivity and operational security by assisting the human crane operator.

Such devices can be decomposed into a fully actuated, articulated mechanical structure with in general one or two degrees of freedom (e.g., a crane with a rotating platform and a boom or a gantry crane with moving bridge), and a hoisting system composed of ropes, winches and pulleys.

During opertion, a duty cycle of a WHE consists in moving the load from its initial position to its desired final destination in its working space along a trajectory, avoiding obstacles and sway [9], [13]. This requires motion planning for the position of the load.

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* On postgraduate stay in France funded by the Ministry of Foreign Affairs of the French Government.

** Centre Automatique et Systemes, École Nationale Supérieure des Mines de Paris, 35, rue Saint-Honoré, F-77305 Fontainebleau, France. E-mail:{kiss,levine,mullhaupt}@cas.ensmp.fr

*** Research partially supported by the Nonlinear Control Network (NCN), funded by the European Commission's Training and Mobility of Researchers (TMR) Programme, Research Network#ERB FMRXCT-970137.

Our goal is to give a systematic way to obtain dynamic models of a class of WHEs and to show how to find trajectories corresponding to a duty cycle exploiting the flatness property [2]–[4] of the dynamic model.

To this aim, the derived model of the class of WHEs involves Lagrange multipliers associated to geometric constraints on the generalized coordinates. This contrasts with choosing a minimal number of coordinates and eliminating the constraints.

The form of the deduced model shows that each member of the class is differentially flat and the coordinates of the load constitute all or part of the components of a flat output, depending on the number of motors. Thus the solution of the motion planning problem becomes an interpolation problem using sufficiently smooth functions (e.g., polynomials).

The remaining part of the paper is organized as follows. The next section gives three examples of WHEs. Section 3 describes the general model for two and three dimensional WHEs. Flatness of the models is proven in Section 4. Then the solution of the motion planning problem is provided in Section 5. The example of the 3D US-Navy crane is studied in detail in Section 6 and some simulation results on its real reduced size model are presented in Section 7.

## 2. INTRODUCTORY EXAMPLES

We will present three examples of WHEs and describe some of their common points. This will then lead to general systematic modeling procedure to be introduced in the next section.

Figures 1 and 2 represent a 2D overhead crane, a 3D cantilever crane and a 3D US-Navy crane. The following characteristics are noteworthy:

– The load moves in a working space of either dimension $p = 2$, such as the overhead crane of Fig.1, or $p = 3$ as portrayed in Fig. 2.

– All WHEs considered comprise the following elements:

• A working load of mass $m$ whose coordinates are $x_i$, $i = 1, \ldots, p$.

• A hoisting system composed of motors winding ropes and pulleys. The motors are supposed to be torque controlled and each one delivers a torque noted $T_j$ where $j$ numbers the actuator. The different rope lengths are denoted by $L_j$.

• A fully articulated structure on which there are attached the motors winching the ropes. For the overhead crane depicted in Fig. 1 it is a rail structure without articulation and for the 3D cranes of Fig. 2 it corresponds to a pole that can rotate under motor actuation (the mechanical structure has one articulation).

• A mobile or main pulley whose coordinates are $x_{0i}$ $i = 1, \ldots, p$.

• A rail constraining the movement of the mobile pulley might (see the overhead crane and the cantilever crane in Figs. 1 and 2, respectively) or might not be present (see the US-Navy crane in Fig. 2).

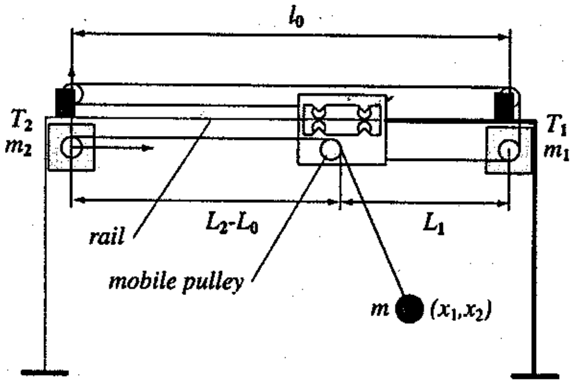


Fig. 1. Overhead crane

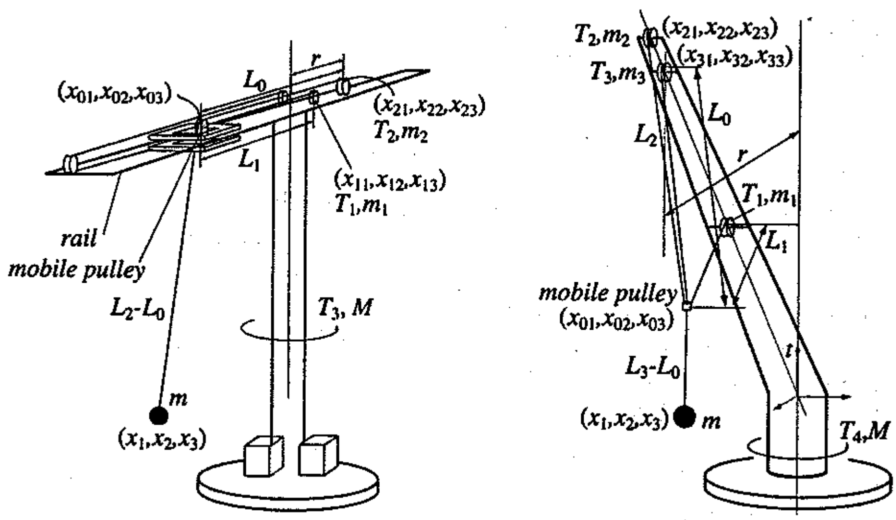


Fig. 2. 3D cranes: (i) Cantilever crane; (ii) US-Navy crane

## 3. GENERAL FORMULATION FOR 2D AND 3D WHEs

### 3.1. WHE DESCRIPTION

Let $p$ be the dimension of the working space with $p \in \{2, 3\}$.

DEFINITION 1 (WHE)

A WHE is constituted by the following elements: (i) a rigid articulated actuated mechanical structure with $d \in \{0, 1\}$ degrees of freedom, (ii) motors, (iii) ropes, (iv) pulleys, (v) a load, and enjoys the following topographic properties:

1. There is at least one motor fixed on the articulated structure. Let $s + 1$ be the number of such motors, $s \geq 0$.

2. There are as many ropes as motors.

3. One motor is linked to a pulley or to the load with a rope.

4. $s$ ropes end on a unique pulley, called the mobile pulley. If $s = 0$ there is no mobile pulley. All pulleys but the mobile pulley are fixed to the structure.

5. There is a unique rope going through the mobile pulley and ending on the load.

6. Between the load and the mobile pulley there is no other pulley.

Moreover, the following physical property is assumed. The mobile pulley moves in a manifold of dimension $n \in (p-1, p)$. This manifold is determined thanks to the constraints imposed by the ropes and by possibly restricting the mobile pulley to move along a rail. If $n = p - 1$ the manifold is transversal to the gravitational field.

Let us enumerate and order the fixed pulleys on the structure along each rope starting from the motor winding the rope to the mobile pulley or to the load. This is possible due to the previous definition. Denote by $r_i$ the number of fixed pulleys along the $i$-th rope ($i = 1, \ldots, s + 1$).

### 3.2. WHE MODELING

We present here a Lagrangian approach to the WHE modeling. Hence, we start with the choice of generalized coordinates, then express the Lagrangian and the geometric constraints. The model is given in Theorem 1 below.

Consider an inertial base frame such that its $p$-th axis is pointed in the direction opposite to $g$, the gravity acceleration. We introduce the following coordinates:

1. position of the working load: $(x_1, \ldots, x_p)$,
2. position of the mobile pulley (if it exists): $(x_{01}, \ldots, x_{0p})$,
3. position of the motors: $(x_{i1}, \ldots, x_{ip})$ for $i = 1, \ldots, s + 1$,
4. positions of the fixed pulleys: $(w_{ij1}, \ldots, w_{ijp})$ for $i = 1, \ldots, s + 1$ and $j = 1, \ldots, r_i$,
5. rope lengths: $L_i$ for $i = 1, \ldots, s + 1$,
6. rope length $L_0$ between the mobile pulley (if it exists) and the motor winching the working load.

The load mass is $m$ and the mobile pulley mass is $m_0$. To each motor fixed on the structure there is a corresponding equivalent mass $m_i$, $i = 1, \ldots, s + 1$. The coordinate $L_0$ is not associated to any mass. We assume that the moving part of the rigid articulated mechanical structure with at most one degree of freedom has an equivalent mass $M$ (including the mass of the motors and pulleys fixed to it) and its position is given by the coordinates of the motor winching the load, namely by $(x_{(s+1)1}, \ldots, x_{(s+1)p})$.

The reader can easily check that all fixed pulleys along each rope can be virtually eliminated by placing the corresponding motor at the position of the last fixed pulley with an equivalent mass obtained by adding to its own equivalent mass the sum of equivalent masses of all the pulleys removed. Each rope length is then reduced by the sum of the *constant* rope distances between the pulleys removed along that rope. For notational convenience, $L_i$'s stand for these new lengths. Let us make the following assumptions which are satisfied by most of the WHEs used in practice. These assumptions also allow further notational simplification which do not impart on generality.

**Assumptions**

(A1) The mobile pulley is present. Consequently, $s \geq 1$.

(A2) The angular velocities of the fixed pulleys are small enough to neglect their quadratic effects (if any) w.r.t. the motions of the mechanical structure. Note that in all of our three examples no such quadratic effect appears.

(A3) We suppose that all the motors are located on the structure along a line determined by the origin of the base frame and by the position of the motor winching the load: $x_{ji} = \alpha_j x_{(s+1)i}$ for $j = 1, \ldots, s$ and $i = 1, \ldots, p$.

(A4) If the mobile pulley moves along a rail, the rail coincides with the above line. Let us introduce a parameter $c$ such that $c = 1$ if the rail is present and $c = 0$ otherwise.

(A5) The crane has no redundant actuator or motor: $s = p - d - c$.

(A6) If $d = 1$ the origin of the base frame is on the joint axis of the articulated mechanical structure. The articulated mechanical structure consists of either a rotational joint, in which case the joint axis is collinear with $g$, or a prismatic joint, in which case the joint axis is orthogonal to $g$. This assumption eliminates the variable $x_{(s+1)p}$. (The vertical position of the motor winching the load remains constant.)

Table 1

Parameter values compatible with the assumptions

| $p$ | $d$ | $c$ | $s$ | $d+s+1$ |
|---|---|---|---|---|
| 2 | 0 | 0 | 2 | 3 |
| 2 | 0 | 1 | 1 | 2 |
| 3 | 1 | 0 | 2 | 4 |
| 3 | 1 | 1 | 1 | 3 |

The number of actuators (i.e., the actuator of the articulated structure and the motors winding ropes taken together) equals $d + s + 1$. Table 1 gives the possible values of the parameters $p$ (dimension of the working space), $d$ (number of articulations of the mechanical structure), $c$ (number of constraints for the mobile pulley), and $s$ (number of ropes attached to the mobile pulley) compatible with the assumptions.

The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\left( m\sum_{i=1}^{p} \dot{x}_i^2 + m_0 \sum_{i=1}^{p} \dot{x}_{0i}^2 + M\sum_{i=1}^{p-1} \dot{x}_{(s+1)i}^2 + m_i \sum_{i=1}^{s+1} \dot{L}_i^2 \right) - g(mx_p + m_0 x_{0p}).$$

Constraints on the rope lengths are present either due to ropes terminating at the mobile pulley

$$C_j(x_{01},\dots,x_{0p},x_{(s+1)1},\dots,x_{(s+1)p-1},L_j)=0 \quad j=1,\dots,s, \tag{1}$$

or due to the rope terminating at the working load. Two constraints are obtained for the latter rope, one for the total length between the mobile pulley and the corresponding motor fixed to the structure and one for the length between the load and the mobile pulley

$$C_{s+1}(x_{01},\dots,x_{0p},x_{(s+1)1},\dots,x_{(s+1)p-1},L_0)=0, \tag{2}$$

$$C_{s+2}(x_{01},\dots,x_{0p},x_1,\dots,x_p,L_0,L_{s+1})=0. \tag{3}$$

An additional constraint is imposed by the motion compatible with the degree of freedom of the structure if $d > 0$. In view of the above assumptions, this constraint exists only if $p = 3$ (see Table 1),

$$C_{s+3}(x_{(s+1)1},\dots,x_{(s+1)p-1})=0. \tag{4}$$

The motion of the mobile pulley along the rail (if it is present) is of the form

$$C_{s+p+k}(x_{0k},x_{0p},x_{(s+1)k})=0 \quad k=1,\dots,p-1. \tag{5}$$

Denote by $l$ the total number of constraints. If (5) is present, $l = s + 2p - 1$ and $l = s + p$ otherwise.

Here, the functions $C_1$, …, $C_l$ are quadratic functions of all their arguments. Moreover, $C_1$, …, $C_{s+2}$ contain no product involving $L_j$, for $j = 0, \dots, s + 1$. Their exact form is not needed in the sequel (see Remark 2 below).

In place of obtaining an explicit differential model, we prefer an implicit formulation with additional variables, known as *Lagrange multipliers*.

THEOREM 1

*Assume that the constraints are independent in an open subset of the generalized coordinate space. The dynamical model associated to a WHE corresponding to Definition* 1 *reads:*

$$m\ddot{x}_i=\lambda_{s+2}\frac{\partial C_{s+2}}{\partial x_i}-\delta_{ip}mg \quad i=1,\dots,p, \tag{6}$$

$$m_0\ddot{x}_{0i}=\sum_{j=1}^{l}\lambda_j\frac{\partial C_j}{\partial x_{0i}}-\delta_{ip}m_0g \quad i=1,\dots,p, \tag{7}$$

$$0=\sum_{j=1}^{l}\lambda_j\frac{\partial C_j}{\partial L_0}, \tag{8}$$

$$m_i\ddot{L}_i=\sum_{j=1}^{l}\lambda_j\frac{\partial C_j}{\partial L_i}+T_i \quad i=1,\dots,s+1, \tag{9}$$

$$M\ddot{x}_{(s+1)i} = \sum_{j=1}^{l} \lambda_j \frac{\partial C_j}{\partial x_{(s+1)i}} + F_i(T_{s+2}) \quad i = 1, ..., p-1, \tag{10}$$

*subject to Constraints* (1)–(5), *where* $\delta_{ip} = 1$ *if* $i = p$ *and* $\delta_{ip} = 0$ *otherwise.* $T_1, \ldots, T_{s+1}$ *are the torques produced by the motors on the structure and* $T_{s+2}$ *the one produced by the structure actuator.* $F_1, \ldots, F_{p-1}$ *are the generalized external forces depending on the torque delivered by the structure actuator.*

*Proof*: We compute $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = F_q + \tau_q$ where $q = (x_1, \ldots, x_p, x_{01}, \ldots, x_{0p}, L_0, L_1, \ldots, L_{s+1}, x_{(s+1)1}, \ldots, x_{(s+1)(p-1)})^T$, $\tau_q$ are the constraint forces. We have

$$F_q = (\underbrace{0, ..., 0}_{2p+1}, T_1, ..., T_{s+1}, F_1(T_{s+2}), ..., F_{(p-1)}(T_{s+2}))^T .$$

Taking total differential of the constraints leads to $\Sigma_{j=1}^{\dim q} \frac{\partial C_i}{\partial q_j} dq_j = 0,\ i = 1, ..., l,$ expressing that virtual displacements are in ker $dC$, where $dC$ is the matrix whose entries are $\frac{\partial C_i}{\partial q_j}$. Since the constraint forces compatible with the virtual displacements are workless we have $\Sigma_{i=1}^{\dim q} \tau_i dq_i = 0$. Therefore $\tau = (\tau_1, \ldots, \tau_{\dim q})$ is a linear combination of the lines of $dC$

$$\tau_i = \sum_{j=1}^{l} \lambda_j \frac{\partial C_j}{\partial q_i} \quad i = 1, ..., \dim q \tag{11}$$

and the theorem is proved. □

Remark 1: Note that the left hand side $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$ of the model (6)–(10) is independent of the specific topography of the WHE, whereas the right hand side consists of the exterior forces $F_q$ plus gravity terms $\frac{\partial \mathcal{L}}{\partial q}$ and the terms given by (11) which sum up the topographic specificity.

Remark 2: The exact form of the constraints $C_j, j = 1, ..., l$ are

$$C_j = \frac{1}{2}\sum_{i=1}^{p} (x_{0i} - \alpha_j x_{(s+1)i})^2 - \frac{L_j^2}{2} = 0 \quad j = 1, ..., s, \tag{12}$$

$$C_{s+1} = \frac{1}{2}\sum_{i=1}^{p-1} (x_{0i} - x_{(s+1)i})^2 - \frac{L_0^2}{2} = 0, \tag{13}$$

$$C_{s+2} = \frac{1}{2}\sum_{i=1}^{p}(x_i - x_{0i})^2 - \frac{(L_{s+1} - L_0)^2}{2} = 0, \tag{14}$$

$$C_{s+3} = \begin{cases} \frac{1}{2}\sum_{i=1}^{p-1} x_{(s+1)i}^2 - r^2 = 0 & \text{for rotational joint}, \\ t_1 x_{(s+1)2} - x_{(s+1)1} t_2 = 0 & \text{for prismatic joint}, \end{cases} \tag{15}$$

$$C_{s+p+k} = x_{0k} x_{(s+1)p} - x_{(s+1)k} x_{0p} = 0 \quad k = 1, \dots, p-1, \tag{16}$$

where $t = (t_1, \dots, t_p)^T$ is the vector of joint axis of the articulated structure and $r$ is the constant distance between the joint axis and the motor winching the load in the case of rotational joint. Note that these formulas are not needed to state and prove our main results.

## 4. FLATNESS

DEFINITION 2 (flatness)
The system

$$f(\dot{x}, x, u) = 0 \tag{17}$$

with $x \in \mathrm{R}^n$ and $u \in \mathrm{R}^m$ is differentially flat if one can find a set of variables, called flat output,

$$y = h(x, u, \dot{u}, \ddot{u}, \dots, u^{(r)}), \quad y \in \mathrm{R}^m. \tag{18}$$

with $r$ finite integer, such that

$$\begin{aligned} x &= \alpha(y, \dot{y}, \ddot{y}, \dots, y^{(q)}) \\ u &= \beta(y, \dot{y}, \ddot{y}, \dots, y^{(q+1)}) \end{aligned} \tag{19}$$

with $q$ a finite integer, and such that the system equations

$$0 = f\left(\frac{d\alpha}{dt}(y, \dot{y}, \ddot{y}, \dots, y^{(q+1)}), \alpha(y, \dot{y}, \ddot{y}, \dots, y^{(q)}), \beta(y, \dot{y}, \ddot{y}, \dots, y^{(q+1)})\right)$$

are identically satisfied.

Assume that we exclude free fall trajectories of the load, namely such that $\ddot{x}_p = -g$, an such that $\frac{\partial C_{s+2}}{\partial x_p} \neq 0$.

THEOREM 2

*WHEs defined by Definition* 1 *and satisfying* (A1)–(A6) *are differentially flat. The flat output, denoted by* **x** *in the sequel, can be chosen as* $(x_1, \dots, x_p)$*, the coordinates of the load, and* $s + d + 1 - p$ *coordinates of the mobile pulley.*

*Proof*: In view of the assumptions we need to distinguish the four cases of Table 1. We provide the proof for $p = 3$, the simplest cases with $p = 2$ are left to the reader. (Recall that $p = 2$ implies $d = 0$).

Assume first that $s = 2 = p - 1$ and consider $\mathbf{x} = (x_1, \ldots, x_p, x_{0p})$ as a candidate flat output. Combining the $p$-th equation of (6) and (3) and the fact that the $C_i$'s contain no cross-terms involving $L_0$, $L_{s+2}$ by assumption, one obtains $\lambda_{s+2}$ as a function of $x_p$, $\ddot{x}_p$ and $x_{0p}$ since $\frac{\partial C_{s+2}}{\partial x_p} \neq 0$. Next, as long as $\lambda_{s+2} \neq 0$ which is guaranteed by the assumption that $\ddot{x}_p = -g$, the $p - 1$ first equations of (6) express the remaining coordinates $x_{01}, \ldots, x_{0(p-1)}$ as functions of $x_j$, $\ddot{x}_j$, $j = 1, \ldots, p$, and $x_{0p}$. Next, we use the $2p + 1$ equations (2)–(4), (7) and (8) to express the $2p + 1$ variables $L_0$, $L_{s+1}$, $x_{(s+1)1}, \ldots, x_{(s+1)p-1}$, $\lambda_1, \ldots, \lambda_p$ as functions of $x_{01}, \ldots, x_{0p}$, $x_1, \ldots, x_p$, $\lambda_{s+2}$ and derivatives up to order 2, which in turn can be expressed as functions of $\mathbf{x}$ and derivatives up to order 4. Now, by (1), one can express $L_1, \ldots, L_s$ as functions of the previous ones. By (9), $T_1, \ldots, T_p$ are also obtained as functions of the previous ones and derivatives up to order 6, and finally, $T_{s+2}$ and $\lambda_{s+3}$ are obtained in a similar way by (10), which proves that $\mathbf{x} = (x_1, \ldots, x_p, x_{0p})$ is a flat output.

Consider now the case with $s = c = 1$ (i.e., the rail constrains (5) are present) and let $\mathbf{x} = (x_1, \ldots, x_p)$ be the candidate flat output. First, we use the $2p$ equations (4)–(5) and (6) to express $2p$ variables $x_{01}, \ldots, x_{0p}$, $\lambda_{s+2}$, $x_{(s+1)1}, \ldots, x_{(s+1)p-1}$ in function of $x_j$, $\ddot{x}_j$, $j = 1, \ldots, p$. We proceed using equations (2), (1), (3) and (8) to express the rope lengths $L_0$, $L_1$, $L_2$ and $\lambda_{s+1}$ in function of $\mathbf{x}$, $\ddot{\mathbf{x}}$. Next, we use equation (7) to obtain $\lambda_s$, $\lambda_{s+p+1}$, $\lambda_{s+p+2}$ as functions of $\mathbf{x} = (x_1, \ldots, x_p)$ and their derivatives up to order 4. Finally, we use equations (9) and (10) to express $T_1, \ldots, T_{s+2}$ and $\lambda_{s+3}$ in function of $\mathbf{x}$ and their derivatives up to order 6 which proves that $\mathbf{x} = (x_1, \ldots, x_p)$ is a flat output. □

## 5. MOTION PLANNING

Assume that the position, velocity, acceleration, jerk and all derivatives up to 6-th order of the flat output (including the position of the load) at the start time $t_I$ are given by $(\mathbf{x}_I, \dot{\mathbf{x}}_I, \ddot{\mathbf{x}}_I, \ldots, \mathbf{x}_I^{(5)}, \mathbf{x}_I^{(6)})$ and the desired final configuration of the flat output at time $t_F$ is $(\mathbf{x}_F, \dot{\mathbf{x}}_F, \ddot{\mathbf{x}}_F, \ldots, \mathbf{x}_F^{(5)}, \mathbf{x}_F^{(6)})$.

The flatness property implies that for any trajectory connecting the initial and final points, the motor torques can be calculated without integration of the model equations. It is enough to follow the steps of the proof of Theorem 2. Such trajectory can be obtained for example using polynomial interpolation.

To see this, suppose that the motion planning has to be done between two different equilibria of the load such that $\mathbf{x}_I = \dot{\mathbf{x}}_I$, $\mathbf{x}_I = \ddot{\mathbf{x}}_I = \ldots = \mathbf{x}_I^{(5)} = \mathbf{x}_I^{(6)} = 0$ and $\mathbf{x}_F = \dot{\mathbf{x}}_F$, $\mathbf{x}_F = \ddot{\mathbf{x}}_F = \ldots = \mathbf{x}_F^{(5)} = \mathbf{x}_F^{(6)} = 0$. Then we can construct a 13-th degree polynomial,

$$x_{ic}(t) = \bar{x}_{Ii} + (\bar{x}_{Fi} - x_{Ii}) \sum_{j=1}^{13} a_{ji} \left( \frac{t - t_I}{t_F - t_I} \right)^j \tag{20}$$

where $x_{ic}$ are the reference trajectories of the variables of the flat output $\mathbf{x}$. This polynomial satisfies all initial conditions and the coefficients $a_{ji}$ are computed by solving linear equations, whose entries are combinations of the final conditions.

## 6. EXAMPLE

Let us illustrate our approach by giving the resulting equations for the US-Navy crane. The constraints can be easily obtained using equations (12)–(16) and the notations of Fig. 2.

EXAMPLE: 3D US-Navy crane

The parameters are: $p = 3$, $d = 1$, $c = 0$, $s = 2$ and the vector of generalized coordinates is $q = \{x_1, x_2, x_3, x_{31}, x_{32}, x_{01}, x_{02}, x_{03}, L_0, L_1, L_2, L_3\}$. Note that by Assumption A3, $x_{ji} = \alpha_j x_{3i}$ for $j = 1, \ldots, 2$ and $i = 1, \ldots, 3$.

The kinetic energy reads

$$W_k = \frac{1}{2} \sum_{i=1}^{3} (m\dot{x}_i^2 + m_0 \dot{x}_{0i}^2) + \frac{1}{2} \sum_{i=1}^{2} M \dot{x}_{3i}^2 + \frac{1}{2} \sum_{i=1}^{3} m_i \dot{L}_i^2 \tag{21}$$

and the potential energy is given by

$$W_p = mgx_3 + m_0 g x_{03} \,. \tag{22}$$

Define the Lagrangian by $L = W_k - W_p$. The constraints read

$$\frac{1}{2} \left( \sum_{i=1}^{3} (x_i - x_{0i})^2 - (L_3 - L_0)^2 \right) = 0,$$

$$\frac{1}{2} \left( \sum_{i=1}^{3} (x_{0i} - \alpha_1 x_{3i})^2 - L_1^2 \right) = 0,$$

$$\frac{1}{2} \left( \sum_{i=1}^{3} (x_{0i} - \alpha_2 x_{3i})^2 - L_2^2 \right) = 0,$$

$$\frac{1}{2}\left(\sum_{i=1}^{3}(x_{0i}-x_{3i})^2-L_0^2\right)=0,$$

$$\frac{1}{2}(x_{31}^2+x_{32}^2-r^2)=0,$$

which are indeed all quadratic in the generalized coordinates. The model is given by Theorem 1,

$$m\ddot{x}_1=\lambda_1(x_1-x_{01}),$$

$$m\ddot{x}_2=\lambda_1(x_2-x_{02}),$$

$$m\ddot{x}_3=\lambda_1(x_3-x_{03})-mg,$$

$$m_0\ddot{x}_{01}=-\lambda_1(x_1-x_{01})+\lambda_2(x_{01}-\alpha_1x_{31})+\lambda_3(x_{01}-\alpha_2x_{31})+\lambda_4(x_{01}-x_{31}),$$

$$m_0\ddot{x}_{02}=-\lambda_1(x_2-x_{02})+\lambda_2(x_{02}-\alpha_1x_{32})+\lambda_3(x_{02}-\alpha_2x_{32})+\lambda_4(x_{02}-x_{32}),$$

$$m_0\ddot{x}_{03}=-\lambda_1(x_3-x_{03})+\lambda_2(x_{03}-\alpha_1x_{33})+\lambda_3(x_{03}-\alpha_2x_{33})+\lambda_4(x_{03}-x_{33})-m_0g,$$

$$0=\lambda_1(L_3-L_0)-\lambda_4L_0,$$

$$m_1\ddot{L}_1=-\lambda_2L_1+T_1,$$

$$m_2\ddot{L}_2=-\lambda_3L_2+T_2,$$

$$m_3\ddot{L}_3=-\lambda_1(L_3-L_0)+T_3,$$

$$M\ddot{x}_{31}=-\lambda_2\alpha_1(x_{01}-\alpha_1x_{31})-\lambda_3\alpha_2(x_{01}-\alpha_2x_{31})-\lambda_4(x_{01}-x_{31})+\lambda_5x_{31}-T_4x_{32},$$

$$M\ddot{x}_{32}=-\lambda_2\alpha_1(x_{02}-\alpha_1x_{32})-\lambda_3\alpha_2(x_{02}-\alpha_2x_{32})-\lambda_4(x_{02}-x_{32})+\lambda_5x_{32}-T_4x_{31}.$$

One can prove, using Theorem 2 that the coordinates of the load and the height of the mobile pulley, $\mathbf{x}=(x_1, x_2, x_3, x_{03})$, form a flat output (see also [10], [11] for the special case with massless mobile pulley).

## 7. SIMULATION RESULTS

We illustrate the solution of the motion planning problem for the US-Navy crane modelled in the previous section. The parameters used are those of a reduced size model (1:80) realized in the authors' laboratory, depicted in Fig. 3. The mass of the load is 250 g.

Suppose that we wish to find an idle to idle trajectory for the load implying that the reference trajectory will have no sway. This makes the implementation of a closed-loop control law easy (not presented here), aiming to attenuate and damp the unmodeled perturbations [9].

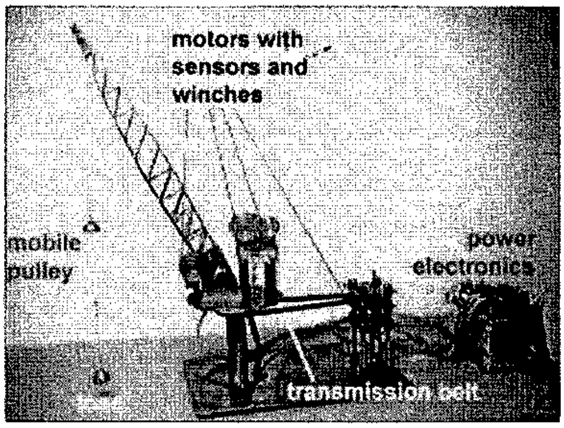


Fig. 3. Reduced size model of the US-Navy crane

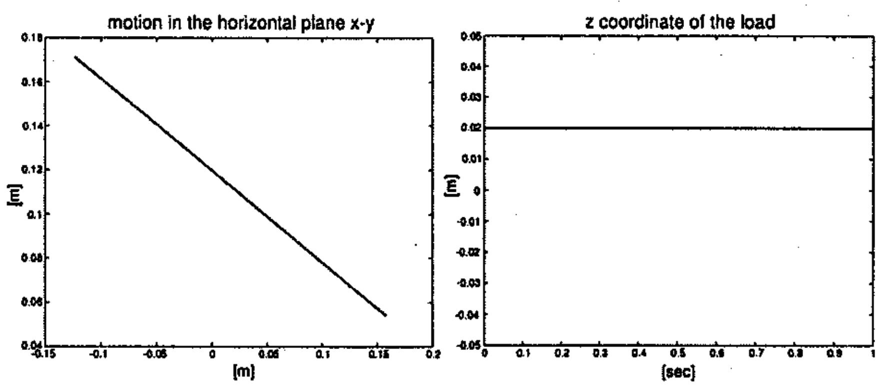


Fig. 4. Horizontal displacement of the load

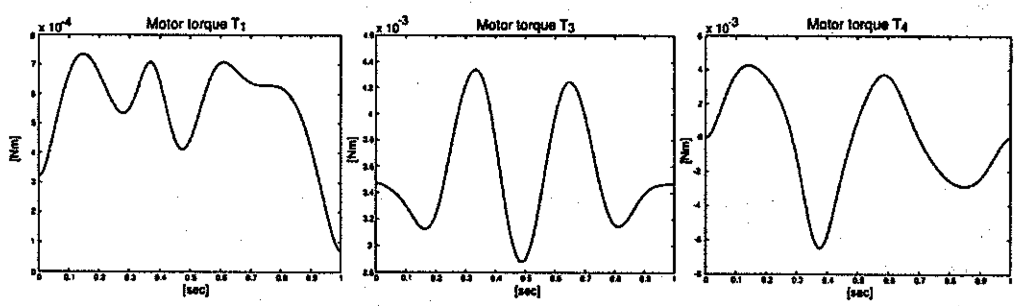


Fig. 5. Motor torques generating the horizontal displacement

The trajectory depicted in Fig. 4 is a horizontal idle to idle displacement of the load obtained using polynomial interpolation as in Section 5. Some of the corresponding motor torques are given in Fig. 5.

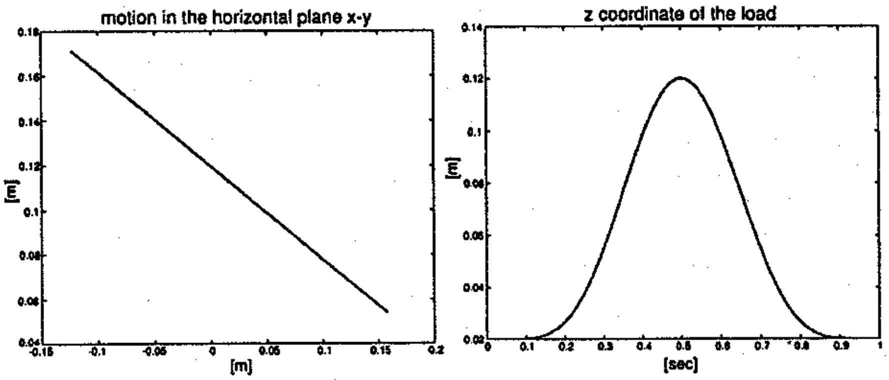


Fig. 6. Parabolic displacement of the load

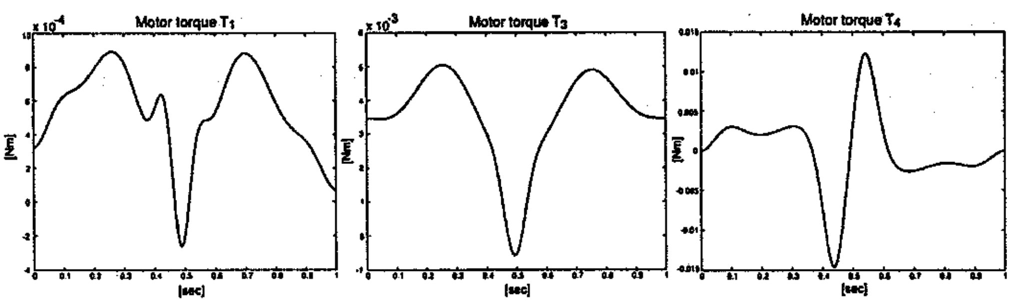


Fig. 7. Motor torques generating the parabolic trajectory

The second trajectory is again an idle to idle displacement between the same points but along a parabolic trajectory avoiding an obstacle placed between the initial and final load positions. The trajectory and the generating motor torques are depicted in Figs. 6 and 7.

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*Received January 2001*