Effects of Motion and Configural Complexity on Color Transparency Perception

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Abstract

We tested whether motion and configural complexity affect perceived transparency. A series of five coherent chromatic transformations in color space was applied across a figure: translation, convergence, shear, divergence and rotation. The stimuli consisted of a bipartite or a checkerboard configuration (10x10 deg), with a central static or moving overlay (5x5 deg). Three different luminance conditions (the plane of chromatic transformation oriented toward higher, lower or equal luminances) were also tested for each of three modulation depths. For each stimulus, the observer judged whether the overlay appeared transparent or not. The main results indicated an interaction between the type of chromatic transformation and stimulus motion and complexity. For example, convergences are judged to appear transparent significantly more often when motion is added for bipartite configurations, or when they are generated in a checkerboard configuration. Surprisingly, shears which have been reported to appear opaque, are more frequently reported to appear transparent with short vector lengths and when combined with motion. Other transformations are also affected by motion, although the **effectiveness** of figural complexity on transparency seems to depend on both the type of color shifts and the presence of motion. The results indicate that adding motion and stimulus complexity are not necessarily neutral with respect to the chromatic shifts evoking transparency. Thus, studies that have used motion to enhance transparency may yield different results about the color shifts supporting transparency perception from those that did not. The same might be supposed for stimulus complexity under some conditions.

Keywords: Color vision, transparency perception, motion, configural complexity, psychophysics

1   Introduction

Color transparency provides an intriguing example of a segmentation phenomenon in perception.

A surface presented in direct view and at the same time behind a transparent color filter is

identified as one and the same surface, in spite of the chromatic changes across the retinal image.

The interpretation of surface cues leading to transparency perception requires a global integration

of local color differences in the image (Dojat et al., in this issue; Knoblauch **&** Dojat, 2003). The

relevance of this phenomenon to color constancy has widely been recognized, as well (D'Zmura

et al., 2000; Khang & Zaidi, 2002; Koffka, 1935; Westland & Ripamonti, 2000). Several studies

have suggested that ensembles of color changes across a region that describe a translation and/or

a convergence in a linear color space lead to the perception of transparency, but other

transformations, such as shear and rotation do not (Chen & D'Zmura, 1998; D'Zmura et al, 1997;

D'Zmura et al., 2000; Hagedorn & D'Zmura, 2000). Since color changes that describe a

translation in color space can be considered to converge to a point at infinity, one speaks generally

of the Convergence Model of transparency. Alternative proposals are that transparency perception

arises when cone ratios are preserved across a region (Westland & Ripamonti, 2000; Ripamonti &

Westland, 2003) or when the color changes can be described in terms of subtractive color

mixtures (Faul & Ekroll, 2002). Both of these models, however, can be viewed as special cases of

the Convergence Model (Faul & Ekroll, 2002). For example, as noted above, the Convergence

Model is derived from the combination of convergent and translational changes across a set of

color boundaries, with convergent change represented as

$$\mathbf{a_P} = \alpha \mathbf{a_A} + (1 - \alpha)\mathbf{g} \tag{1}$$

where $\mathbf{a_P}$ and $\mathbf{a_A}$ are the tristimulus vectors of adjacent regions (see, for example, the surfaces

labelled A and P in Figure 1a and the vector diagram in Figure 2b), $\mathbf{g}$ the tristimulus values of a

point of convergence and α a real scalar in the interval [0, 1], and translational change as

3

$$\mathbf{a_P} = \mathbf{a_A} + \mathbf{t}, \tag{2}$$

where $\mathbf{t}$ is the vector of tristimulus values of the translation (see the vector diagram of Figure 2a).

The composition of these two chromatic changes yields either,

$$\mathbf{a_P} = \alpha\mathbf{a_A} + (1 - \alpha)\mathbf{g} + \mathbf{t} = \alpha\mathbf{a_A} + \mathbf{b}, \tag{3}$$

or

$$\mathbf{a_P} = \alpha(\mathbf{a_A} + \mathbf{t}) + (1 - \alpha)\mathbf{g} = \alpha\mathbf{a_A} + \mathbf{b}', \tag{4}$$

where $b = \mathbf{t} + (1 - \alpha)\mathbf{g}$ and $b' = \alpha\mathbf{t} + (1 - \alpha)\mathbf{g}$, depending on whether convergence precedes

translation or vice versa. In the former case, if $(1 - \alpha)\mathbf{g} = -t$, then $b = 0$; in the latter, if

$(1 - \alpha)\mathbf{g} = -\alpha t$, then $b' = 0$. Both of these special cases yield

$$\mathbf{a_P} = \alpha\mathbf{a_A}, \tag{5}$$

which preserves invariant cone ratios. A more general case, $\mathbf{a_P} = \Lambda\mathbf{a_A}$, where Λ is a diagonal

matrix with different diagonal elements, also preserves cone ratios if $\mathbf{a_P}$ and $\mathbf{a_A}$ are coordinates

in a cone excitation space. When the more general case is combined with a translation, it is

referred to as the Generalized Convergence Model (GCM, see, e.g., D'Zmura, Rinner &

Gegenfurtner, **2000** and Ripamonti, Westland & Da Pos, 2004). While the above derivation

permits the contributions of convergent and translatory changes to be considered separately, it can

be bypassed by treating translation as convergence to a point at infinity. Thus,

$$\lim_{\substack{\beta \to 0 \\ \mathbf{g} \to \infty}} (1 - \beta)\mathbf{a_A} + \beta\mathbf{g} = \mathbf{a_A} + \mathbf{t}, \tag{6}$$

4

with $\beta = 1 - \alpha$ and $\mathbf{t} = \beta\mathbf{g}$, i.e., the limit is approached along a hyperbolic trajectory for each component of $\mathbf{t}$.

Faul and Ekroll (2002) contrasted additive and subtractive models of transparency. They recognized both types of models as special cases of the GCM and proposed that while the GCM may include a class of models that describe several types of perceptual transparency, the fact that it includes many cases that do not appear transparent, limits its use as a general model of the phenomenon.

All of the studies discussed above have concentrated on conditions within the scope of the GCM. Incidental observations suggest that chromatic changes outside of this model, such as shearing transformations, do not appear transparent (D'Zmura et al., 1997) but systematic observations of such stimuli were not reported. In fact, physical transparency can occur in situations that would be described as neither additive nor subtractive, for example, when either the filter or the underlying surfaces are fluorescent. Recently, Khang and Zaidi (2004) showed that segmentation based on motion can override conflicting luminance and color cues. The authors used backgrounds simulating a wide variety of spectral reflectances, spectrally reflective filters and equal energy light. Background materials were simulated as overlaying a circular region and moving along a circle. They pointed out that a moving filter has the advantage of covering a larger sample background of material than a static filter of the same size and noted as well that the movement of filters greatly enhances the perception of a transparent layer. We studied the effect of motion and configural complexity on the appearance of transparency, with a variety of systematic chromatic changes, some of which included chromatic changes inconsistent with the GCM.

2 Methods

2.1 Stimulus

All experiments were conducted with a BARCO PCD-321 monitor connected to a DELL PC (Precision 330). The monitor had a resolution of 1280 x 1024 pixels and and a refresh rate of 75 Hz. Calibrations were performed with a Minolta CS1000 spectroradiometer, and luminance was linearized using lookup tables. The experimental stimuli were created with Matlab v.6.0.088 R12 (Mathworks, MA., U.S.A.), and displayed with OpenGL. The stimuli consisted of a bipartite or a 6x6 checkerboard configuration (10x10 deg) (Figure 1), with a central static or moving overlay (5x5 deg). Each check was 1.67 deg on a side. The overlay performed the same circular movement (2 deg radius) for all motion trials, with a speed of 120 deg/s. The rotation was clockwise in all cases. The luminances of the surround fields were assigned randomly to one of four levels (40, 45, 50 or 55 cd/m^2).

Figure 1 about here

In recent studies on transparency perception, different color spaces have been exploited: CIE XYZ (D'Zmura et al., 1997), DKL (Chen and D'Zmura, 1998; D'Zmura et al., 2000; Hagedorn and D'Zmura, 2000) and Munsell and cone excitation (Westland and Ripamonti, 2000; Ripamonti and Westland, 2003; Faul and Ekroll, 2002; Khang and Zaidi, 2002, 2004). In this experiment, CIE LUV (a space in which discrimination steps have been made more uniform with respect to the CIE XYZ spaces) was chosen to control the length, ΔE^*_{uv}, of each vector. Five chromatic transformations were considered: translation, convergence, shear, divergence and rotation in CIE XYZ space. These were generated as follows. Consider a field A whose extension under a transparent region is a field P (Figure1a). Let the tristimulus coordinates of each of these fields be denoted by $\mathbf{a}_A$ and $\mathbf{a}_P$, respectively. Their chromatic difference is a vector $\mathbf{a}_A - \mathbf{a}_P$. Suppose another pair of surfaces, B and Q, in a similar relation with vector of chromatic change $\mathbf{a}_B - \mathbf{a}_Q$.

Figures 2a-e show with respect to the double bipartite field of Figure 1a how the tristimulus

coordinates of the fields were related to generate each coherent chromatic change evaluated.

These rules were extended in a natural fashion to the checkerboard stimulus (Figure 1b) to obtain

the same type of chromatic changes across the test overlay region.

Figure 2 about here

In addition to the 5 chromatic variations described above and the 2 levels of configural

complexity (bipartite or checkerboard), several other factors were varied, as well, including

presence or absence of motion (2 levels), 3 levels of chromatic modulation (or vector length,

corresponding to $\Delta E^*_{uv} = 8, 20, 32$) and 3 levels of luminance elevation of the plane in which

chromatic variations were constrained (equiluminant, luminance increment or illuminant-like and

luminance decrement or filter-like changes). Crossing these 5 factors yields 180 stimulus

conditions. Each was repeated 4 times in a session and each session repeated 4 times generating

2880 stimuli for each of 3 observers or 8640 total.

2.2 Subjects and Procedures

Three color normal, naïve observers were tested (2 females and 1 male). The color discrimination

of each observer was verified with the FM-100 Hue test. The psychophysical experiments were

performed in a dark room. Subjects sat in front of the monitor, 50 cm from the screen. The

observer classified the central patch of each stimulus as either transparent or opaque. The

observer received the instructions: "You will be presented with a pattern of fields that contains an

interior square region that appears to be either transparent or opaque. Press the left key if this

region appears to be transparent and the right if it appears to be opaque." The set of all patches

was presented in a randomized sequence. Stimulus duration was 2 sec. A training set was

presented before each session, the results of which were discarded. No feedback was provided.

2.3   Analysis

The raw data consisted of the counts of the number of transparent and not-transparent reponses to each stimulus condition. Count data are typically analyzed with log-linear models (Bishop et al., 1975; Knoke and Burke, 1980) which treat the data as arising from a multinomial distribution. The sufficient statistics, however, are identical under the assumption that the data are distributed according to a Poisson distribution (Bishop et al., 1975). The analysis is simplified by fitting the data using a Generalized Linear Model with a Poisson family and a logarithmic link function. Diagnostic plots based on the model fits provided no evidence of overdispersion or systematic patterns in the residuals which would invalidate this approach.

All calculations were performed within the R statistical environment (R Development Core Team, 2004). Model selection was performed with an automated procedure that stepped systematically through the terms of the model and retained or eliminated them based on Akaike's An Information Criterion (AIC), defined as minus twice the log likelihood plus twice the number of parameters (Akaike, 1973, Venables and Ripley, 2002). More parameters will always increase the likelihood. The AIC penalizes the likelihood by the number of parameters so in some sense represents a balance between optimizing the likelihood and the parsimony of the model. Lower AIC values correspond to a better model. In the simplest model, the response variable will be independent of the explanatory variables. The inclusion of interaction terms with the response variable implies that the explanatory variable differentially affects the perception of transparency. The modelling process is termed hierarchical because the exclusion of a factor or an interaction term results in the exclusion of all higher order interactions in which it participates (Bishop et al., 1975).

3   Results

The final model required no terms including interactions between the responses and the observers, indicating that observers performed similarly. Therefore, observer responses were combined in the results described below.

Figures 3–7 regroup by type of chromatic variation across the overlay the graphs showing the number of transparent responses for each stimulus condition. As each condition was repeated 4 times in each of 4 sessions for each of 3 observers, the maximum number of responses is 48. Under the Poisson hypothesis, the standard deviations are equal to the square root of the observed counts. Each individual graph shows the number of responses for the three types of luminance elevation. Each column indicates the results for a modulation depth (or vector length, ΔE^*_{uv}). Each row of graphs indicates one of the four combinations of motion and configuration complexity tested ((Static [Bip.]), (Motion [Bip.]), (Static [Check.]) and (Motion [Check.])).

Figures 3–7 about here

For static-bipartite overlays, convergent changes are less often classified as transparent as the modulation depth of the convergence increases. This trend appears for static-bipartite overlays for the other four chromatic variations, as well, in Figures 4–7. In the case of convergence, the overlay tends toward opacity as the modulation depth increases. The reduction in transparency of equiluminant translations for longer chromatic modulations was described by Chen and D'Zmura (1998). Motion of the bipartite overlay or increased configural complexity each increase the number of transparent responses for convergence. The combination of motion with increased configural complexity results in nearly 100% transparent classification. A similar trend is seen for translations, although, interestingly, not for low modulation depth equiluminant translations (Figure 4).

A significant interaction between type of chromatic variation, presence of motion and complexity with response in the best-fitting model ($\chi^2 = 11.9$, $p = 0.018$) suggests that the

pattern of interaction of motion and complexity depends on the type of chromatic variation. Indeed, for shearing transformations, chromatic variation across the overlay resulted in a different global pattern of responses (Figure 5). For most conditions, shearing transformations were infrequently classified as transparent. Only for short modulation depths did the number of transparent responses increase. The counts, however, hovered near 50% suggesting an ambiguity in the stimulus. Only for equiluminant shifts did motion lead to systematic increases while greater stimulus complexity decreased the number of transparent responses.

With divergence, the overlay in both static conditions is rarely classified as transparent for larger chromatic modulations and more frequently for smaller ones (Figure 6). One might expect that the overlay itself would not appear transparent for divergent chromatic variations, but instead it would appear as an aperture in a transparent surround. Nevertheless, the addition of motion, increases the number of transparent responses. Perceptually, these conditions were reported to appear like the transparency that arises from convergent chromatic variations, that is, changes in contrast in the overlay region.

Rotations are more often classified as transparent for smaller chromatic modulations, but motion and especially combined with configural complexity increases the number of transparent responses to nearly maximal (Figure 7). Interestingly, increasing chromatic modulation disadvantages transparent responses most at equiluminance.

4 Discussion

To summarize, neither bipartite nor checkerboard static overlays produced by convergence or translation were always classified as transparent. These results replicate observations of Chen and D'Zmura (1998) and Faul and Ekroll (2002) that demonstrated limitations in the convergence model in describing color transparency. Nevertheless, motion was able to override color cues, generating nearly unanimous classification. In particular, equiluminant overlays with large

chromatic modulations, the conditions that Chen and D'Zmura identified as violating the convergence model, were more likely to be classified as transparent with these additional cues. Complexity only increased transparency responses when paired with motion or an appropriate chromatic change, as indicated by the fact that its only significant interaction occurred in terms with these two variables.

Potentially more problematic is that divergent and rotational chromatic shifts across the overlay could also produce systematic transparency classification. These responses were more frequent at lower chromatic modulations and were also enhanced by motion and, in some instances, when motion was paired with configural cues. Interestingly, motion and configural complexity had little effect on the appearance of divergent and rotational overlays at equiluminance. Finally, overlays produced by shear were classified as transparent for about half of the stimuli at the lowest chromatic modulation level tested.

Transparency is a multi-determined phenomenon that can be evoked also by X-junctions between surfaces (Anderson, 1997) or motion (Adelson and Movson, 1982), alone. Increasing configural complexity increases the number of X-junctions between the overlay and the background, and motion generates motion of X-junctions. These cues would be expected, in general, to enhance transparency and probably explain the response increases observed for convergences and translations. Such an enhancement has been previously noted in the case of motion (D'Zmura et al., 2000; Khang and Zaidi, 2004) and configural complexity (Ripamonti et al., 2004).

Few published studies, however, have examined the effects on appearance when these cues are in conflict. Robilotto and Zaidi (2004), for example, showed that replacing moving X-junctions by T-junctions rendered an overlay opaque. Some of the patterns of response observed in this study might be explained by such conflicts. For example, the lack of transparent responses for large equiluminant shears could indicate that X-junctions are not sufficient to overcome inappropriate color transitions for transparency. Motion only slightly increased the

number of transparent responses in this case. Only when the color cues are weaker, at low chromatic modulations, could X-junctions influence perception. A similar explanation would apply in the case of rotations.

Adding motion and stimulus complexity are not neutral with respect to the chromatic shifts evoking transparency. Thus, studies that have used motion to enhance transparency may yield different results about the color shifts supporting transparency perception from those that did not. The same might be supposed for stimulus complexity when combined with motion and/or the appropriate color shifts. Our results highlight the need for quantitative investigations of the nature of cue interactions in transparency perception. Transparency perception can be an important component of surface perception and object segmentation. Understanding how cues combine will aid in specifying the role of transparency in these phenomena and the nature of the mechanisms involved.

Acknowledgment

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Figure Legends

Figure 1.   Stimulus configuration used to test transparency perception. (a) Bipartite configuration. The outer bipartite field is 10x10 deg and the inner one is 5x5 deg. The letters symbolize different fields of the image and are used in the text to define the different types of chromatic changes used. A and B indicate opaque underlying surfaces. P and Q indicate surfaces as seen through a transparent overlay. (b) Checkerboard configuration.

Figure 2.   Examples of each of the 5 chromatic variations used to define the color change in the test overlay. The tails of each vector, a_A and a_B, correspond to tristimulus coordinates outside the overlay, and the heads, a_P and a_Q, within. The equations define the rules relating interior to exterior tristimulus coordinates. In the above examples, t is a constant vector.

Figure 3.   Grouped results for convergent chromatic variations across the test overlay. The bars indicate the total number of transparent responses. Each graph shows the counts for the three elevations with respect to the equiluminance plane. Each column indicates a different vector length in Luv space of the chromatic modulations. Each row indicates one of four combinations of motion and configuration complexity, from top to bottom: i) static overlay, bipartite field, ii) moving overlay, bipartite field, iii) static overlay, checkerboard field, iv) moving overlay, checkerboard field. Under the hypothesis that the data are distributed according to a Poisson distribution, the standard deviations are the square root of the counts.

Figure 4.   Grouped results for translatory chromatic variations across the test overlay. Other details are as in Figure 3.

Figure 5.   Grouped results for shearing chromatic variations across the test overlay. Other details are as in Figure 3.

Figure 6.   Grouped results for divergent chromatic variations across the test overlay. Other details are as in Figure 3.

Figure 7.   Grouped results for rotatory chromatic variations across the test overlay. Other details are as in Figure 3.
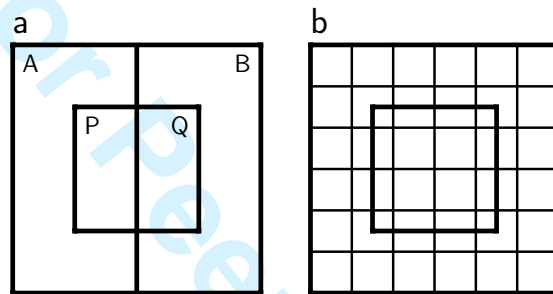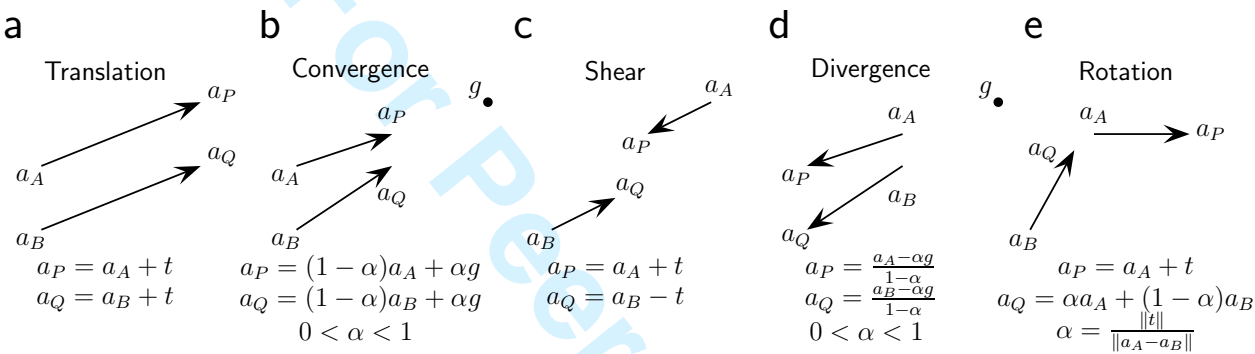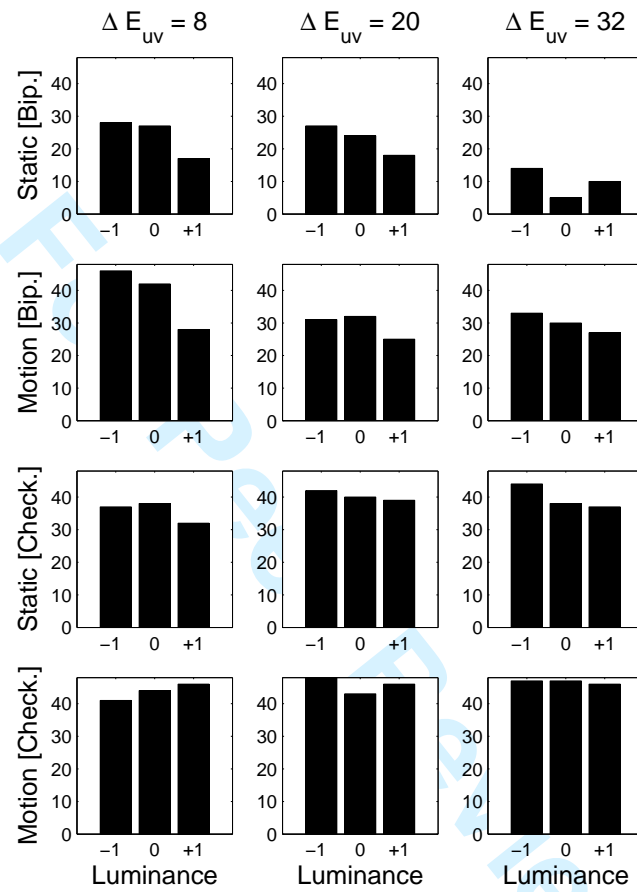
Figure 1

a
Translation

a_P

a_A a_Q

a_B

$$a_P = a_A + t$$
$$a_Q = a_B + t$$

b
Convergence

g

a_P

a_A a_Q

a_B

$$a_P = (1-\alpha)a_A + \alpha g$$
$$a_Q = (1-\alpha)a_B + \alpha g$$
$$0 < \alpha < 1$$

c
Shear

a_A

a_P

a_Q

a_B

$$a_P = a_A + t$$
$$a_Q = a_B - t$$

d
Divergence

a_A

a_P a_B

a_Q

$$a_P = \frac{a_A - \alpha g}{1-\alpha}$$
$$a_Q = \frac{a_B - \alpha g}{1-\alpha}$$
$$0 < \alpha < 1$$

e
Rotation

g

a_A

a_Q a_P

a_B

$$a_P = a_A + t$$
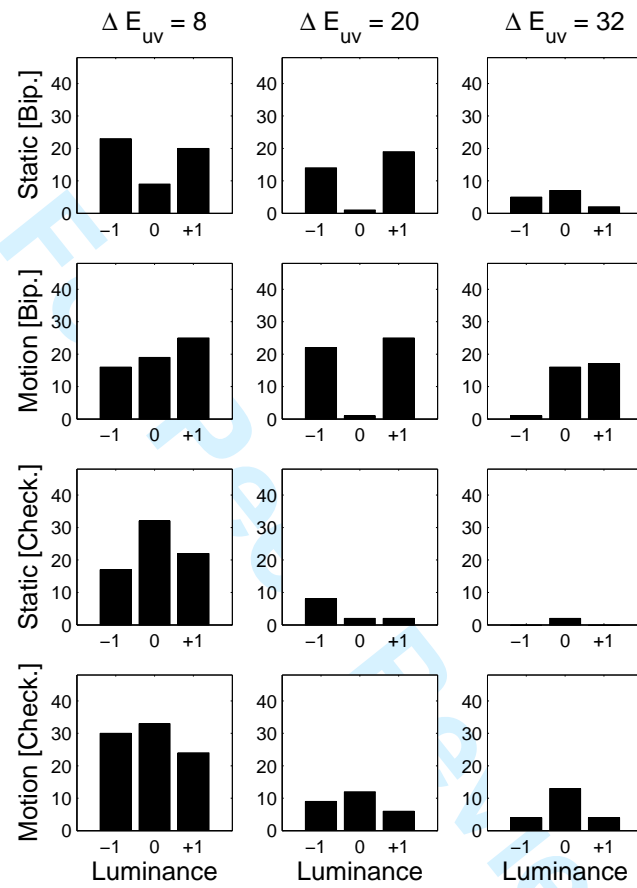$$a_Q = \alpha a_A + (1-\alpha)a_B$$
$$\alpha = \frac{\|t\|}{\|a_A - a_B\|}$$

Figure 2

Figure 3

Figure 4

19

Figure 5

$\Delta E_{uv} = 8$ $\Delta E_{uv} = 20$ $\Delta E_{uv} = 32$

Figure 6

Figure 7

a

A		B
	P	Q

b

a

Translation

a_P

a_A

a_Q

a_B

$a_P = a_A + l$
$a_Q = a_B + l$

b

Convergence

$g_\bullet$

a_P

a_A

a_Q

a_B

$a_P = (1-\alpha)a_A + \alpha g$
$a_Q = (1-\alpha)a_B + \alpha g$
$0 < \alpha < 1$

c

Shear

a_A

a_P

a_Q

a_B

$a_P = a_A + l$
$a_Q = a_B - l$

d

Divergence

$g_\bullet$

a_A

a_P

a_B

a_Q

$a_P = \frac{a_A - \alpha g}{1-\alpha}$
$a_Q = \frac{a_B - \alpha g}{1-\alpha}$
$0 < \alpha < 1$

e

Rotation

a_A

a_Q

a_P

a_B

$a_P = a_A + l$
$a_Q = \alpha a_A + (1-\alpha)a_B$
$\alpha = \frac{\|l\|}{\|a_A - a_B\|}$