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“Si je devais traverser la vallée où règnent les ténèbres de la mort, je ne craindrais aucun mal, car tu es auprès de moi : ta houlette me conduit et ton bâton me protège.”

Psaume 23.4, la Bible

à Chantal, Amélie et Jonathan

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Abstract

Using an algebraic formalism based on matrices in $SL(2,\mathbb{R})$, we explicitly give the Teichmüller spaces of Riemann surfaces of signature $(0,4)$ (X pieces), $(1,2)$ ("Fish" pieces) and $(2,0)$ in trace coordinates. The approach, based upon gluing together two building blocks (Q and Y pieces), is then extended to tree-like pants decomposition for higher signatures (g,n) and limit cases such as surfaces with cusps or cone-like singularities.

Given the Teichmüller spaces, we establish a set of generators of their modular groups for signatures $(0,4)$, $(1,2)$ and $(2,0)$ in trace coordinates using transformations acting separately on the building blocks and an algorithm on dividing geodesics. The fact that these generators act particularly nice in trace coordinates gives further motivation to this choice (rather then the one of Fenchel-Nielsen coordinates).

This allows us to solve the Riemann moduli problem for X pieces, "Fish" pieces and surfaces of genus 2; i.e. to give the moduli spaces as the fundamental domains for the action of the modular groups on the Teichmüller spaces. In this context, we also give an algorithm deciding whether two Riemann surfaces of signatures $(0,4)$, $(1,2)$ or $(2,0)$ given by points in the Teichmüller space are isometric or not.

As a consequence, we show the following two results concerning simple closed geodesics:

1. On any purely hyperbolic Riemann surface (containing neither cusps nor cone-like singularities), the longest of two simple closed geodesics that intersect one another n times is of length at least l_n, a sharp constant independent of the surface. We explicitly give l_n for $n = 1, 2, 3$ and study its behaviour when n goes to infinity.
2. X pieces are spectrally rigid with respect to the length spectrum of simple closed geodesics.

Key words: Riemann surfaces, Teichmüller spaces, Fuchsian groups, modular groups, moduli spaces, simple closed geodesics, collars, length spectra.

Résumé

En utilisant un formalisme algébrique basé sur des matrices dans $SL(2,\mathbb{R})$, les espaces de Teichmüller des surfaces de Riemann de signature $(0,4)$ (pièces X), $(1,2)$ (pièces "poisson") et $(2,0)$ sont donnés explicitement dans des coordonnées de traces. L'approche, basée sur le collage de deux pièces de construction (des pièces Q et Y), est alors étendue à des décompositions en pantalons (dont les graphes sont des arbres) de surfaces de signature (g,n) ainsi qu'aux cas limites des surfaces avec "cusps" ou singularités coniques.

Donnés les espaces de Teichmüller, nous établissons un ensemble de générateurs des groupes modulaires pour les signatures $(0,4)$, $(1,2)$ et $(2,0)$ dans des coordonnées de traces en utilisant des transformations agissant séparément sur les pièces de construction et un algorithme sur les géodésiques divisentes. Le fait que ces générateurs agissent d'une manière particulièrement simple en coordonnées de traces donne une motivation supplémentaire pour ce choix (plutôt que celui des coordonnées de Fenchel-Nielsen).

Ceci nous permet de résoudre le problème des moduli de Riemann pour les pièces X, les pièces "poisson" et les surfaces de genre 2 ; c.-à-d. de donner les espaces des moduli comme domaines fondamentaux de l'action des groupes modulaires sur les espaces de Teichmüller. Dans ce contexte, nous donnons également un algorithme décidant si deux surfaces de Riemann des signatures $(0,4)$, $(1,2)$ ou $(2,0)$ données par des points dans l'espace de Teichmüller sont isométriques.

Comme conséquence, nous prouvons les deux résultats suivants concernant les géodésiques fermées simples :

1. Sur toute surface de Riemann purement hyperbolique (ne contenant ni "cusps" ni singularités coniques), la plus longue de deux géodésiques fermées simples qui s'intersectent n fois est de longueur au moins l_n, une constante optimale indépendante de la surface. Nous donnons explicitement l_n pour $n = 1, 2, 3$ et nous étudions son comportement quand n tend vers l'infini.
2. Les pièces X sont spectralement rigides par rapport au spectre des longueurs des géodésiques fermées simples.

Mots clefs : surfaces de Riemann, espaces de Teichmüller, groupes Fuchsiens, groupes modulairs, espaces des moduli, géodésiques fermées simples, colliers, spectres des longueurs.

Contents

Introduction

This dissertation deals with *Teichmüller space* and the *Riemann moduli problem* for Riemann surfaces in general and in particular for those of signature $(0, 4)$ (which we call X pieces), signature $(1, 2)$ (which we call "Fish" pieces) and signature $(2, 0)$, where signature (g, b) denotes a surface of genus g with b funnels or border geodesics.

The Teichmüller space of a Riemann surface can be defined equivalently in many different ways, for instance as the space of isotopy classes of *marked* complex structures of the Riemann surface, where two structures define the same point in Teichmüller space if there exists a holomorphic homeomorphism homotopic to the identity leading from one to the other. The Riemann moduli problem consists in describing the space of isomorphism classes of Riemann surfaces of a given signature which is the *moduli space*. If the Teichmüller space is known, the moduli space can be obtained as the quotient of the Teichmüller space by the action of the *modular group* or *extended mapping class group*, i.e. the group whose elements map a point in Teichmüller space to another if the corresponding Riemann surfaces are isometric. Note that classically, the mapping class group is defined as the group of elements that map a point in Teichmüller space to another if there is a *direct* or *orientation preserving* isometry between the corresponding Riemann surfaces. But, from a geometric point of view, it is interesting to know when two surfaces are isometric, regardless whether the isometry between them is orientation reversing or not.

We take the view in this thesis that a Riemann surface is given by a Fuchsian group, namely a discrete subgroup Γ of $SL(2, \mathbb{R})$ acting on the upper half plane $\mathbb{H}$ by Möbius transformations. By quotienting $\mathbb{H}$ by Γ we obtain the complex structure of the surface, defined up to conjugation. We choose $SL(2, \mathbb{R})$ rather than $PSL(2, \mathbb{R})$, as is usually done in the literature, because this gives easily available information on direction of geodesics (see Chapter 1). However, as we choose the traces of the generating elements to be positive, these two approaches are equivalent (see e.g. [SS92]). In this language we can define Teichmüller space as the space of endomorphisms of the Fuchsian group into $SL(2, \mathbb{R})$ up to conjugation which preserves parabolic elements and whose image is discrete. A parameterization of Teichmüller space can thus be achieved by giving the generators of the subgroup of $SL(2, \mathbb{R})$ in terms of traces.

The main results of this thesis are threefold:

1. We give explicit parameterizations in terms of traces of the fundamental groups as subgroups of $SL(2, \mathbb{R})$ and of the Teichmüller spaces as submanifolds of $\mathbb{R}^n$ given

by polynomial equations for signatures $(0,4)$, $(1,2)$ and $(2,0)$. We then extend the methods in an inductive way to surfaces of higher signatures with cusps and cone-like singularities.

2. We prove that the modular groups in trace coordinates act polynomially for X pieces and rationally for "Fish" pieces and surfaces of genus 2.

3. We give an explicit description of the moduli spaces for signatures $(0,4)$, $(1,2)$ and $(2,0)$.

Our approach, which is based upon an algebraic formalism studied for genus 2 in [Sem88], is quite similar to the one taken in [Kee77] (based upon previous work by Keen and the original ideas of Fricke and Klein; cf. [Kee65, Kee66, Kee71, Kee73, FK12]) where Keen gives rough fundamental domains for the action of modular groups on the Teichmüller spaces for various signatures. However, our manner to show that there are no two points in our fundamental domain that correspond to isometric surfaces (based upon shortest dividing geodesics; see Chapters 4 and 5) is radically different from the techniques developed by Keen.

Note that the polynomial relation governing X pieces has already been studied in [BG99]. In that article, the authors consider the character variety of free groups on 3 generators, based upon [Mag80], where Magnus treats rings of Fricke characters of representations of free groups on n free generators into $SL(2,\mathbb{C})$ and their automorphisms. However, this approach has two disadvantages: firstly, the modular group (group of automorphisms) of the ring of Fricke characters is *not* the modular group acting on the Teichmüller space of surfaces of signature $(0,n+1)$ because n free generators do not necessarily generate such a surface. Indeed, the fundamental group of surfaces of signature $(0,n+1)$ as well as the one of surfaces of signature $(1,n-1)$ is a free group on n generators, yet these surfaces are not homeomorphic. Secondly, his approach does not allow to give the subgroup of $SL(2,\mathbb{R})$ isomorphic to the fundamental group of the surface nor its action on the upper half plane explicitly.

Nevertheless, some of the results regarding the parameterization of the Teichmüller spaces are already given in [Luo98] (previous related results can be found also in [SS89, SS88, SS86]) and thus not new. Indeed, Luo gives a set of conditions for a function f over the isotopy classes of essential unoriented simple closed curves on an orientable surface to be the geodesic length function of a hyperbolic metric on the surface. As these conditions are polynomial equations in $\cosh(f/2)$, the parameterization of the Teichmüller Space is the same as the one we obtain (up to a factor of 2 because we choose half-traces), but uses heavy machinery such as the Maskit Combination Theorem (see [Mas65]) and does not give the explicit parameterizations of the fundamental groups which are essential in order to obtain the modular group and solve the Riemann moduli problem.

This dissertation is structured as follows:

In the Preliminaries, we first expose an algebraic formalism using quaternions treating geometric objects as well as isometries of the upper half plane; then we present Poincaré's

Polyhedron Theorem. Both of these tools will be used intensively throughout the thesis.

Chapter 2 deals with surfaces having purely loxodromic two generator fundamental groups (Q and Y pieces) and will be the building blocks for more complicated surfaces.

We study surfaces that arise when we join two building blocks along a boundary geodesic and construct their Teichmüller spaces in Chapter 3. Given the Teichmüller spaces of surfaces of signatures $(0,4)$, $(1,2)$ and $(2,0)$ explicitly in terms of traces, we show how to obtain explicitly the Teichmüller spaces of surfaces of signatures (g,n) using tree-like pants decompositions and give the Teichmüller spaces of limit cases with cusps or cone-like singularities.

In Chapter 4, we show that the modular groups in trace coordinates are polynomial for signature $(0,4)$ and rational for signatures $(1,2)$ and $(2,0)$, giving explicit generators and using an algorithm on dividing geodesics.

The Riemann moduli problem for these signatures is then solved in Chapter 5 giving the moduli space, seen as a fundamental domain for the modular group acting on its corresponding Teichmüller space.

Finally, in the last chapter, we use the Teichmüller and moduli spaces to show the following two results concerning simple closed geodesics:

1. On any purely hyperbolic Riemann surface (no elliptic or parabolic elements in its fundamental group.), the longest of two simple closed geodesics that intersect one another n times is of length at least l_n, a sharp constant independent of the surface. We explicitly give l_n for $n = 1, 2, 3$ and study its behaviour when n goes to infinity.

2. X pieces are spectrally rigid with respect to the length spectrum of simple closed geodesics.

Note that many of the results concerning the X piece have already been published in [GS05]. The results concerning the lengths of intersecting simple closed geodesics are the subject of a forthcoming article [GP06] by Parlier and the author of this thesis.

Chapter 1

Preliminaries

This chapter deals mainly with the upper half plane $\mathbb{H} := \{z = x+iy \in \mathbb{C} \mid y > 0\}$ endowed with the standard metric of curvature -1 (i.e. the hyperbolic metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$). We first recall an algebraic formalism studied in [Sem88] to treat geometric objects as well as isometries of $\mathbb{H}$ and that we will use throughout the rest of the thesis. We then also recall Poincaré's Polyhedron Theorem as it is formulated in [Mas88, p.73-75], which will be used in order to know if the quotient ${}^{\mathbb{H}}\!/_{\Gamma}$ is a Riemann surface for some given group of isometries Γ.

1.1 $SL(2, \mathbb{R})$ and the Upper Half Plane

We consider Möbius transformations leaving the upper half plane invariant, and parameterize them by elements of $SL(2, \mathbb{R}) \subset M(2, \mathbb{R})$, a vector space over $\mathbb{R}$ which we give the basis of quaternions $\{\mathbf{1}, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Notation. We call H_0 the vector subspace generated by $\mathbf{I}, \mathbf{J}$ and $\mathbf{K}$. We call *trace* of an element α of $SL(2, \mathbb{R})$ its $\mathbf{1}$-component and note it $tr(\alpha)$ (it is actually half of the standard trace of the matrix α). If an element of $SL(2, \mathbb{R})$ is written as $(a+A)$ we mean the element $a\mathbf{1} + A$, with $a \in \mathbb{R}$ and $A \in H_0$.

1.1.1 The Products

Definition 1.1 *The* "scalar" product *over H_0 is the symmetric bilinear form*

$$(.\,,.) : H_0 \times H_0 \to \mathbb{R}$$

defined by

$$(A, B) = a_1 b_1 + a_2 b_2 - a_3 b_3$$

where $A = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$ and $B = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$.

Definition 1.2 *The* $\wedge$-product *is the unique antisymmetric bilinear form*

$$\wedge : H_0 \times H_0 \to H_0$$

satisfying

$$\mathbf{I} \wedge \mathbf{J} = \mathbf{K}, \ \mathbf{I} \wedge \mathbf{K} = \mathbf{J} \text{ and } \mathbf{J} \wedge \mathbf{K} = -\mathbf{I},$$

i.e $A \wedge B = (a_3 b_2 - a_2 b_3)\mathbf{I} + (a_1 b_3 - a_3 b_1)\mathbf{J} + (a_1 b_2 - a_2 b_1)\mathbf{K}$.

Remark 1.3 The usual matrix product of two elements $(a + A)$ and $(b + B)$ of $SL(2, \mathbb{R})$ can be written as

$$(a + A) * (b + B) = ab + (A, B) + aB + bA + A \wedge B.$$

1.1.2 The Geometric Elements of $SL(2, \mathbb{R})$

At first sight, the products in 1.1.1 operate on matrices only and have no geometric meaning. Using Möbius transformations we can give them a geometric sense:

Definition 1.4 *Let* $\alpha = (a + A) \in SL(2, \mathbb{R})$, $A = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$ *and* m_α *the Möbius transformation* $z \mapsto \frac{(a+a_1)z+(a_2+a_3)}{(a_2-a_3)z+(a-a_1)}$. *We define the geometric object corresponding to* $(a+A)$*:*

- *if* $(A, A) > 0$*, the geodesic passing through the fixed points of* m_α *(in this case, we say that* α *is* hyperbolic *and call the geodesic its* axis*);*
- *if* $(A, A) = 0$*, the (infinite) fixed point of* m_α *(*α *is* parabolic*);*
- *if* $(A, A) < 0$*, the fixed point of* m_α *that has a positive imaginary part (*α *is* elliptic*).*

Remark 1.5 An Euclidean half-circle centered on the real axis is indeed a geodesic of the upper half plane endowed with the distance d, defined by

$$\cosh(d(\alpha, \beta)) = \frac{|(A, B)|}{\sqrt{(A, A)(B, B)}}$$

for $\alpha = (a + A)$ and $\beta = (b + B) \in SL(2, \mathbb{R})$ such that $(A, A) < 0$ and $(B, B) < 0$ (cf. [Sem88]). This distance yields the standard metric of the upper half plane.

Note also that for $\alpha = (a + A)$ and $\beta = (b + B) \in SL(2, \mathbb{R})$ with $(A, A) > 0$ and $(B, B) > 0$ such that their corresponding geodesics intersect, we get the formula

$$\cos(\sphericalangle(\alpha, \beta)) = \frac{|(A, B)|}{\sqrt{(A, A)(B, B)}}$$

for the angle between the two geodesics (cf. [Sem88]).

Definition 1.6 *If $(a + A) \in SL(2,\mathbb{R})$ is of positive trace and $(A, A) > 0$, we give its corresponding geodesic the following direction:*

- *if $a_2 - a_3 > 0$, the geodesic is directed towards the right,*
- *if $a_2 - a_3 < 0$, towards the left,*
- *if $a_2 - a_3 = 0$ and $|a + a_1| > 1$, upwards,*
- *if $a_2 - a_3 = 0$ and $|a + a_1| < 1$, downwards.*

If $(a+A) \in SL(2,\mathbb{R})$ is of negative trace and $(A, A) > 0$, we give its corresponding geodesic the direction of the geodesic $(-a - A)$ defined as before.

Remark 1.7 Note that the orientation of a geodesic corresponding to $(a+A) \in SL(2,\mathbb{R})$, $a < 0$ is well defined because $(a + A)$ and $(-a - A)$ correspond to the same Möbius transformation.

The definition gives a geodesic that is directed from the repulsive towards the attractive fixed point of m_α if $tr(\alpha) > 0$.

Note also that $(a + A)$ and $(a + A)^{-1} = (a - A)$ correspond to the same geodesic but have opposite orientation.

Lemma 1.8 *Let $\alpha = (a + A)$ and $\beta = (b + B) \in SL(2,\mathbb{R})$ such that $(A, B) = 0$.*

1. *If α and β correspond to two geodesics, then they intersect perpendicularly.*
2. *If α corresponds to a geodesic and β to a point, then the point is part of the geodesic (it can be one of the fixed points of m_α).*
3. *If α and β correspond to two points, then they correspond to the same infinite point (i.e. the fixed point of $m_\alpha = m_\beta$, where α and β are parabolic).*

Proof. We distinguish the three cases of the lemma:

1. We have to prove that the two geodesics intersect and that this intersection is perpendicular. It is thus enough to show that square of the Euclidian distance between the centers of the two half-circles is equal to the sum of the squares of their radii, i.e.

$$\left(\frac{a_1}{a_2 - a_3} - \frac{b_1}{b_2 - b_3}\right)^2 = \frac{(A, A)}{(a_2 - a_3)^2} + \frac{(B, B)}{(b_2 - b_3)^2}.$$

But $$\frac{(A, A)}{(a_2 - a_3)^2} + \frac{(B, B)}{(b_2 - b_3)^2} - \left(\frac{a_1}{a_2 - a_3} - \frac{b_1}{b_2 - b_3}\right)^2 = \frac{2(A, B)}{(a_2 - a_3)(b_2 - b_3)} = 0.$$

2. The same calculation proves that the square of the Euclidian distance between the point corresponding to β and the center of the circle corresponding to α is equal to the square of its radius:

$$\frac{-(B,B)}{(b_2-b_3)^2}+\left(\frac{a_1}{a_2-a_3}-\frac{b_1}{b_2-b_3}\right)^2=\frac{(A,A)}{(a_2-a_3)^2}.$$

3. Without loss of generality, we can assume that $(a_2-a_3)(b_2-b_3)\geq 0$ because $\alpha=(a+A)$ and $\alpha^{-1}=(a-A)$ correspond to the same point. The square of the Euclidian distance between the points corresponding to α and β is

$$\left(\frac{a_1}{a_2-a_3}-\frac{b_1}{b_2-b_3}\right)^2+\left(\frac{\sqrt{-(A,A)}}{a_2-a_3}-\frac{\sqrt{-(B,B)}}{b_2-b_3}\right)^2=-\frac{2\sqrt{(A,A)(B,B)}}{(a_2-a_3)(b_2-b_3)}\leq 0.$$

Thus they correspond to the same point and $(A,A)=(B,B)=0$.

□

Proposition 1.9 *Let $\alpha=(a+A)$ and $\beta=(b+B)$ and $\gamma=(c+C)$ be elements of $SL(2,\mathbb{R})$ such that $(c+C)=(\sqrt{1+(A\wedge B,A\wedge B)}+A\wedge B)$. Then we have the following results:*

1. *If α and β correspond to two geodesics that do not intersect, then γ corresponds to the geodesic perpendicular to both α and β.*

2. *If α and β correspond to two geodesics that do intersect, then γ corresponds to the point of intersection.*

3. *If α (resp. β) corresponds to a geodesics and β (resp. α) to a point that is not a fixed point of m_α (resp. m_β), then γ corresponds to the geodesic perpendicular to α (resp. β) containing β (resp. α).*

4. *If α (resp. β) corresponds to a geodesics and β (resp. α) to a fixed point of m_α (resp. m_β), then γ corresponds to the same fixed point.*

5. *If α and β correspond to two distinct points, then γ corresponds to the geodesic containing the two points.*

Proof. As $(A,A\wedge B)=(B,A\wedge B)=0$ we can use Lemma 1.8 and conclude. □

1.1.3 Isometries of $\mathbb{H}$

Up to now we treated the correspondence between matrices (in $SL(2,\mathbb{R})$) and geometric objects of the upper half plane $\mathbb{H}$. But we know that there is a homomorphism of groups from $SL(2,\mathbb{R})$ to Möbius transformations which leads to the following definition:

Definition 1.10 *For each matrix $\alpha = (a + A) \in SL(2,\mathbb{R})$ we define the endomorphism*

$$h_\alpha : SL(2,\mathbb{R}) \to SL(2,\mathbb{R}) : \beta \mapsto \alpha * \beta * \alpha^{-1}.$$

Proposition 1.11 *The restriction of h_α to $\{(b + B) \in SL(2,\mathbb{R}) | 0 < b < 1\}$ is also a homomorphism $(0 < tr(h_\alpha(\beta)) < 1)$ and even an isometry of $\mathbb{H}$.*

Proof. We have $tr(h_\alpha(b + B)) = b$ by direct calculation using $(a + A)^{-1} = (a - A)$. Thus $h_\alpha(b + B) = b + \alpha * B * \alpha^{-1}$. As $(B, C) = tr(B * C)$ we have also

$$\begin{aligned} (B, C) &= tr(B * C) &&= tr(h_\alpha(B * C)) \\ &= tr(\alpha * B * C * \alpha^{-1}) &&= tr(\alpha * B * \alpha^{-1} * \alpha * C * \alpha^{-1}) \\ &= (\alpha * B * \alpha^{-1}, \alpha * C * \alpha^{-1}) &&= (h_\alpha(B), h_\alpha(C)). \end{aligned}$$

Therefore h_α leaves the "scalar" product invariant and thus is an isometry of $\mathbb{H}$ endowed with the distance d given in Remark 1.5, that yields the standard metric of the upper half plane. □

Remark 1.12 This isometry h_α acting on matrices is actually the same transformation as the Möbius transformation m_α acting on the points of the upper half plane; in the sense that if $\gamma = (c + C)$ corresponds to the point z in $\mathbb{H}$, then the matrix $h_\alpha(\gamma)$ corresponds to the point $m_\alpha(z)$ and $h_\alpha \circ h_\beta(\gamma) = h_{\alpha*\beta}(\gamma)$ corresponds to $m_\alpha(m_\beta(x))$.

1.2 Poincaré's Polyhedron Theorem

If we have a group of isometries Γ, we can (sometimes) build the quotient surface $\mathbb{H}/\Gamma$. When Γ is appropriate, it is preferable to dispose of a fundamental domain D in which we can represent points, geodesics, etc. of the quotient surface. The following theorem tells us if a domain D is a fundamental domain for Γ and gives us a representation of Γ in terms of generators and relations as it is presented in [Mas88, p.73-75] but adapted to the less general case of an m-gon in $\mathbb{H}$:

Theorem 1.13 (Poincaré's Polyhedron Theorem) *Let $D \subset \mathbb{H}$ be an m-gon such that each edge s is associated to an isometry g_s called* side pairing transformation *verifying the following conditions:*

(a) *For each side s of D, there is a side s' such that $g_s(s) = s'$.*

(b) *$g_{s'} = g_s^{-1}$, if s and s' are such that $g_s(s) = s'$.*

Observe that if there is a side s such that $s' = s$, the two first conditions imply the relation $g_s^2 = id_{\mathbb{H}}$ called reflection relation*.*

(c) *For each edge s we have $g_s(D) \cap D = \emptyset$.*

(d) *Let D^* be the space of equivalence classes, with the usual topology, so that the projection $p : \bar{D} \to D^*$ is continuous and open. Then, $\forall\, z \in D^*$, $p^{-1}(z)$ is a finite set.*

(e) *The vertices of D (intersections of two edges) are cyclically identified by the side pairing transformations. If $\{e_1, e_2, \ldots, e_k\}$ are the vertices of such a cycle, s_i an edge containing e_i, g_i the isometry associated to s_i and $h = (g_k \circ \cdots \circ g_2 \circ g_1)$ the* cycle transformation*, then for all finite vertices there is $t \in \mathbb{N}$ so that $h^t = id_{\mathbb{H}}$.*
This equality will be called cycle relation*.*

(f) *At each vertex, two edges meet at a well defined angle measured from inside D. The angles of a cycle form an angle of $\frac{2\pi}{t}$:*

$$\sum_{i=1}^{k} \alpha(e_i) = \frac{2\pi}{t}.$$

(g) *D^* is complete.*

Then the group Γ generated by the side pairing transformations is discrete, D is a fundamental domain for Γ and the reflection relations and cycle relations form a complete set of relations for the group.

Remark 1.14 In [Mas88, p.73-75], the theorem is presented in the more general case of a polyhedron D in $\mathbb{H}^n$, $\mathbb{E}^n$ or $\mathbb{S}^n$. In our case it is possible to simplify some of the conditions (cf. [Mas88, p.78-80]): Supposing the three first conditions to be satisfied, Maskit shows that in dimension 2 (our case):

- the condition (d) is automatically verified if each cycle of vertices is finite, in particular if D only has a finite number of vertices (as in our case of an m-gone);
- the condition (e) is a consequence of condition (f).

Maskit also proves the following:

- If $D \subset \mathbb{H}^n$ is a finite sided polyhedron satisfying the conditions (a)-(d), then the condition (g) is also satisfied if and only if every infinite cycle transformation h at every infinite edge[1] is parabolic.

[1]In dimension 2, we usually speak of an ideal vertex: it's the "intersection" of two infinite edges at "infinity".

Remark 1.15 It is important to give the side pairing transformations explicitly as shows the following example:

Let $D = \{a{+}ib \in \mathbb{H} \mid 1 < a < 2\}$. The domain D has the sides $s_1 = \{a{+}ib \in \mathbb{H} \mid a = 1\}$ and $s_2 = \{a{+}ib \in \mathbb{H} \mid a = 2\}$ that meet in the infinite vertex $e = \infty = s_1 \cap s_2$ at a zero angle. We consider now the side pairing transformations g_{s_1} and g_{s_2}, where $g_{s_1}(s_1) = g_{s_2}^{-1}(s_1) = s_2$ and distinguish the following two cases:

1. $g_{s_1} = \omega_1 : z \mapsto z + 1$.

   This Möbius transformation is parabolic and the cycle transformation h associated to the vertex e is the transformation $h = \omega_1$ of infinite order. It is easy to see that the conditions (a)-(d) of Poincaré's Polyhedron Theorem are satisfied (the conditions (e) and (f) concern only finite vertices). The last point of the previous remark shows that D^* is complete. Thus $\langle\omega_1\rangle$ is discrete, D is a fundamental domain for $\langle\omega_1\rangle$ and the quotient $\mathbb{H}/_{\langle\omega_1\rangle}$ is therefore a surface topologically equivalent to a cylinder (geometrically, it is a cusp).

2. $g_{s_1} = \omega_2 : z \mapsto 2z$.

   This Möbius transformation is elliptic and the cycle transformation h associated to the vertex e is the transformation $h = \omega_2$ of infinite order. Again, it is easy to see that the conditions (a)-(d) of Poincaré's Polyhedron Theorem are satisfied. But, by the last point of the previous remark, D^* is not complete and we cannot apply Poincaré's Polyhedron Theorem to D.

   Nevertheless, $\langle\omega_2\rangle$ is discrete and admits the fundamental domain $\tilde{D} = \{a + ib \in \mathbb{H} \mid 1 < a^2{+}b^2 < 4\}$ as shows Poincaré's Polyhedron Theorem applied to $\tilde{D}$. The quotient surface $\mathbb{H}/_{\langle\omega_2\rangle}$ is again topologically equivalent to a cylinder but geometrically it is a (double-)funnel and not a cusp.

Chapter 2

The Building Blocks

A Riemann surface S of finite signature (g, b) (where g is the genus and b the number of funnels or border geodesics of the surface[1]) is a geometric object that may be quite complicated. However, any given Riemann surface can be decomposed into several simpler surfaces (*Y pieces* and *Q pieces*). The geometry of S can then be derived from the geometry of these building blocks together with the instructions of how to constitute the whole surface from them.

In our approach of Riemann surfaces, they arrive as the quotients of the upper half plane by discrete groups of isometries. These groups can be classified by the number of its generators and we can thus look at the family of two generator groups. A lot of research has been done on these two generator groups (see e.g. [Bea83, Bin00, Bus92, Gil95, GM91, Mas88, SS92]). The general result is, that depending on the two generators, the quotient surfaces are either of signature $(0, 3)$ (called pairs of pants, three-holed spheres or Y pieces) or of signature $(1, 1)$ (called one-holed tori or Q pieces).

Note that the Q and Y pieces defined as these quotients are non compact Riemann surfaces with funnels. If we cut off these funnels along geodesics that lie in the boundary of the Nielsen kernel of the surface (along border or boundary geodesics) we get a compact surface with boundary that contains all closed geodesics of the original surface and that we will sometimes also call a Q or a Y piece.

2.1 The Y Piece

Definition 2.1 *We say that the matrices $\alpha = (a + A)$ and $\beta = (b + B) \in SL(2, \mathbb{R})$ satisfy the condition H_Y if they have traces greater than 1 $(a, b > 1)$ and if they are such that there is $\gamma = (c + C) \in SL(2, \mathbb{R})$ with $c > 1$ and $\alpha * \beta * \gamma = -1$.*

Lemma 2.2 *Let $\alpha = (a + A)$ and $\beta = (b + B) \in SL(2, \mathbb{R})$ satisfy H_Y. Then the geodesics corresponding to α, β and $\gamma = -(\alpha * \beta)^{-1}$ form a domain whose borders are three geodesics*

[1] If S has also s cusps and r cone-like singularities of orders $m_1, \ldots, m_r$, we will use the notation of [Sin72] for the signature, i.e. $(g; m_1, \ldots, m_r; s; b)$.

that do not intersect, i.e they are disjoint and no one of them separates $\mathbb{H}$ into two regions that contain each one of the others.

Proof. See Appendix 2.A, where we give the same elementary proof as in [Gau01]. □

Proposition 2.3 *If α and $\beta \in SL(2,\mathbb{R})$ satisfy H_Y, then there exist only two situations for the hyperbolic triangle formed by α, β and $\gamma = -(\alpha * \beta)^{-1}$ in the unit disc model*[2] *of the upper half plane $\mathbb{H}$:*

A) The triangle is positively oriented (counter-clockwise) and the axes are as in Figure 2.1:

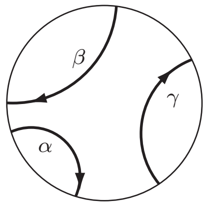

Figure 2.1

B) The triangle is negatively oriented (clockwise) and the axes are as in Figure 2.2:

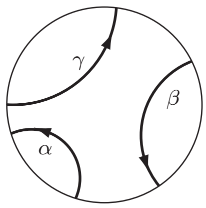

Figure 2.2

Proof. By the previous lemma we know that the axes form a triangle.

Let two of the "edges" (axes) be positively oriented. We can suppose that their names are α and β by cyclic permutation of the names. We can also suppose that (in the upper half plane model) they are both directed towards the right and that $(a + A)$ is on the left of $(b + B)$ $(\frac{a_1}{a_2-a_3} < \frac{b_1}{b_2-b_3})$ by conjugation. But in this case we have

$$\frac{c_2-c_3}{(a_2-a_3)(b_2-b_3)} = \frac{a}{a_2-a_3} - \frac{a_1}{a_2-a_3} + \frac{b}{b_2-b_3} + \frac{b_1}{b_2-b_3} > \frac{a}{a_2-a_3} + \frac{b}{b_2-b_3} > 0$$

and $(c + C)$ is also directed towards the right. This means that we are in situation A) because in all other cases we can construct (by cyclic permutation of the names and conjugation) a case where $(c + C)$ is directed towards the left.

Analogously, we are in situation B) if two of the "edges" are negatively oriented. □

[2]If we choose an arbitrary point p of the unit circle, then there exists a natural isometry between $\mathbb{D}^2$ and $\mathbb{H}$ such that $\mathbb{S}\backslash\{p\}$ is mapped to $\mathbb{R}$.

Proposition 2.4 *The domain whose boundaries $s_1, \ldots, s_4$ are each perpendicular to two of α, β, $\gamma = -(\alpha * \beta)^{-1}$ and $h_{\beta^{-1}}(\alpha) = \beta^{-1} * \alpha * \beta$ (see Figure 2.3) is a fundamental domain for a Y piece.*

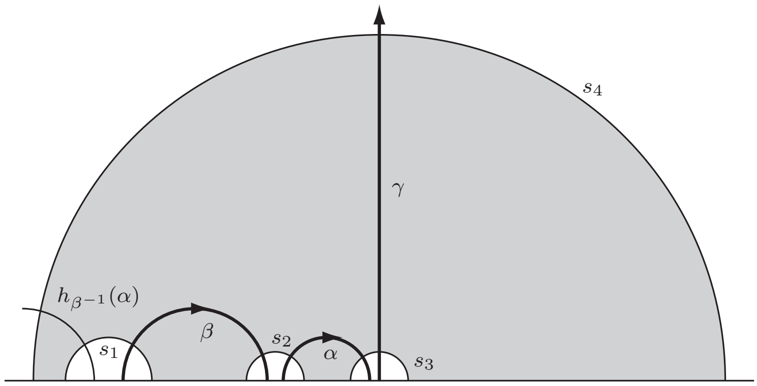

Figure 2.3

Proof. Using the side pairing transformations $g_{s_1} = h_\beta$, $g_{s_2} = h_\beta^{-1}$, $g_{s_3} = h_\gamma$ and $g_{s_4} = h_\gamma^{-1}$ we verify the conditions of Poincaré's Polyhedron Theorem. Therefore, the quotient $\mathbb{H}/\langle h_\beta, h_\gamma \rangle = \mathbb{H}/\langle h_\alpha, h_\beta \rangle$ is a Riemann surface and its signature is $(0, 3)$ as can easily be seen from the fundamental domain and the following figure (funnels are dropped). $\square$

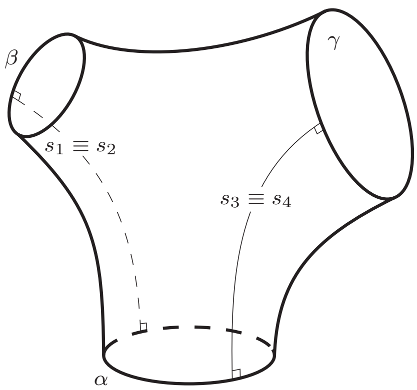

Figure 2.4

2.2 The Q Piece

Definition 2.5 *We say that the matrices $\varrho = (r + R)$ and $\sigma = (s + S) \in SL(2, \mathbb{R})$ satisfy the condition H_Q if they have traces greater than 1 ($r, s > 1$) and if they are such that $(R, S)^2 - (R, R)(S, S) < -1$.*

Lemma 2.6 *Let $\varrho = (r+R)$ and $\sigma = (s+S) \in SL(2,\mathbb{R})$ satisfy H_Q. Then, $\tau = (t+T) = (\varrho * \sigma)^{-1} \in SL(2,\mathbb{R})$ is such that $t > 1$ and the geodesics corresponding to ϱ, σ and τ intersect two by two. The commutator $\gamma = (c + C) = -[\varrho, \sigma]$ is such that $c > 1$ and corresponds to a geodesic that does not intersect the geodesics corresponding to ϱ, σ and τ.*

Proof. See Appendix 2.A, where we give the same elementary proof as in [Gau01]. □

Proposition 2.7 *If ϱ and σ satisfy H_Q, then there exist only two situations for the figure formed by ϱ, σ and $\gamma = -[\varrho, \sigma]$ in the unit disc model of $\mathbb{H}$:*

A) The geodesic corresponding to ϱ intersects the one corresponding to σ "positively" [3] *and the geodesic corresponding to γ is situated between the attracting fixed points and oriented clockwise as in in Figure 2.5:*

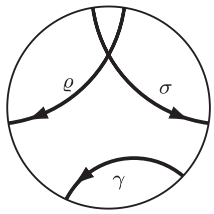

Figure 2.5

B) The geodesic corresponding to ϱ intersects the one corresponding to σ "negatively" and the geodesic corresponding to γ is situated between the attracting fixed points and oriented counter-clockwise as in Figure 2.6:

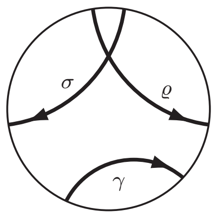

Figure 2.6

Proof. By the previous lemma we know that the axes of γ and ϱ as well as γ and σ are disjoint.

Suppose now that ϱ intersects σ "positively". It is impossible that γ is situated between the attracting fixed point of ϱ and the repulsive fixed point of σ because in that case $\gamma^{-1} = -[\varrho, \sigma]^{-1} = -[\sigma, \varrho]$ should be situated between the attracting fixed point of σ and the repulsive fixed point of ϱ. But this is contradictory because $\gamma^{-1} = (c+C)^{-1} = (c-C)$

[3] If we turn the arrow ϱ counter-clockwise onto the arrow σ, the turning angle is less than π.

corresponds to the same geodesic with opposite direction. By the same argument we prove that γ cannot be situated between the attracting fixed point of σ and the repulsive fixed point of ϱ.

By conjugation we can suppose that $\varrho = (r + R) = (r + \sqrt{r^2-1}\ \mathbf{I})$ and calculate the sign of $\frac{c_1}{c_2-c_3} = \frac{r\sqrt{r^2-1}}{r\sqrt{r^2-1}-(r^2-1)} \frac{s_2-s_3}{s-s_1}$. We know that $s_2 - s_3 < 0$ (direction of σ) and $\frac{r\sqrt{r^2-1}}{r\sqrt{r^2-1}-(r^2-1)} > 0$ (because $r\sqrt{r^2-1} > r^2-1$). If $s_1 \leq 0$ then $s - s_1 > 0$ and $\frac{c_1}{c_2-c_3} < 0$. In any case $s_1 - \sqrt{s^2-1} < 0$ (because $\frac{s_1+\sqrt{s^2-1}}{s_2-s_3} < 0 < \frac{s_1-\sqrt{s^2-1}}{s_2-s_3}$ as $s_2 - s_3 < 0$). Thus, if $s_1 > 0$, then $s_1^2 < s^2 - 1$ implies $(s - s_1)(s + s_1) > 0$ and therefore $s - s_1 > 0$.

It remains to prove that γ is directed towards the left:

$$c_2 - c_3 = 2(r\sqrt{r^2-1} - (r^2-1))(s - s1)(s2 - s3) < 0$$

and we are really in situation A).

Analogously, we are in situation B) if ϱ intersects σ "negatively". □

Notation. If $\lambda \in SL(2,\mathbb{R})$ corresponds to a geodesic and η is a geodesic arc (or a geodesic ray), we note $\eta \subset \lambda$ if η is part of the geodesic corresponding to λ.

Proposition 2.8 *Let $\varrho = (r+R)$ and $\sigma = (s+S) \in SL(2,\mathbb{R})$ satisfy H_Q and $\gamma = -[\varrho,\sigma]$. Let $s_1, s_2, s_3, s_{4a}, s_{4b}, s_5, s_6, s_7$ be the geodesic arcs (rays) such that γ is perpendicular to s_1, $s_2 \subset (\gamma^{-1} * \varrho)$, $s_3 \subset \sigma$, $s_{4a} \subset \varrho$, $s_{4b} \subset \varrho$, $s_5 \subset (\gamma * \sigma)$, $s_6 \subset \varrho * \gamma^{-1}$ and γ perpendicular to s_7 such that the geodesic arcs s_{4a} and s_{4b} with the common vertex w have the same length, the intersection of s_1 and s_2 is $h_\sigma(w)$ and the intersection of s_6 and s_7 is $h_{\varrho*\sigma*\varrho^{-1}}(w)$ as in Figure 2.7. Then, the domain whose boundaries are $s_1, s_2, s_3, s_{4a}, s_{4b}, s_5, s_6, s_7$ is a fundamental domain for a Q piece.*

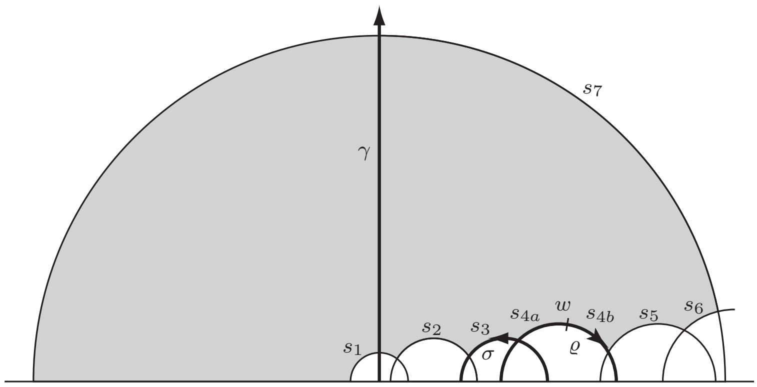

Figure 2.7

Proof. Using the side pairing transformations $g_{s_1} = h_\gamma$, $g_{s_2} = h_{\sigma^{-1}}$, $g_{s_3} = h_\varrho$, $g_{s_{4a}} = h_\sigma$, $g_{s_{4b}} = h_{\varrho*\sigma*\varrho^{-1}}$, $g_{s_5} = h_{\varrho^{-1}}$, $g_{s_6} = h_{\varrho*\sigma^{-1}*\varrho^{-1}}$ and $g_{s_7} = h_{\gamma^{-1}}$ we verify the conditions of Poincaré's Polyhedron Theorem. Thus $\mathbb{H}/\Gamma$ with $\Gamma = \langle h_\gamma, h_\varrho, h_\sigma | \gamma = -[\varrho,\sigma] \rangle = \langle h_\varrho, h_\sigma \rangle$

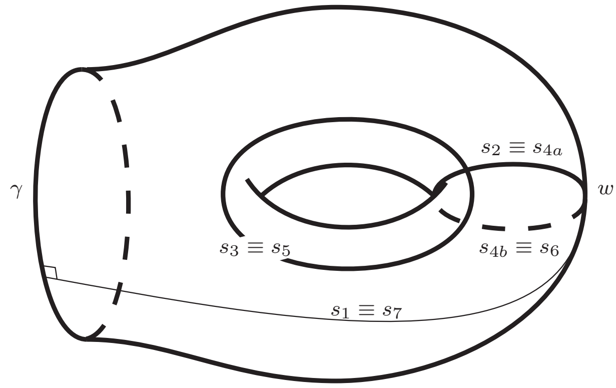

Figure 2.8

is a surface and its signature is $(1,1)$ as can be seen from the fundamental domain and Figure 2.8 (funnel dropped). □

Remark 2.9 Note that the angle condition for Poincaré's Polyhedron Theorem follows from the side pairings and the fact that γ is perpendicular to s_1 and s_7.

2.A Appendix

2.A.1 Proof of Lemma 2.2

We first show that α, β and γ are disjoint. By Proposition 1.9, we know that this is equivalent to $(A \wedge B, A \wedge B)$, $(B \wedge C, B \wedge C)$ and $(C \wedge A, C \wedge A)$ all strictly positive:

We know that $(c + C) = \big((-ab - (A,B)) + (aB + bA + A \wedge B)\big)$ and thus

$$\begin{aligned} c^2 - 1 \;=\; (C,C) &= a^2(B,B) + b^2(A,A) + 2ab(A,B) + (A \wedge B, A \wedge B) \;>\; 0 \\ \Longrightarrow\; (A \wedge B, A \wedge B) &> 2ab(-ab - (A,B)) + 2a^2b^2 - a^2(B,B) - b^2(A,A) \\ &= 2abc + a^2 + b^2 \;>\; 0. \end{aligned}$$

By cyclical permutation, α, β and γ are thus disjoint.

It remains to prove that no one of α, β and γ separates the other two. By Proposition 1.9 and cyclical permutation, this is equivalent to $((A \wedge B) \wedge C, (A \wedge B) \wedge C)$ being strictly positive:

Using the properties of the "scalar" and the $\wedge$-products and the fact that $\alpha * \beta * \gamma = -1$, we get

$$\begin{aligned} ((A \wedge B) \wedge C, (A \wedge B) \wedge C) &= (A \wedge B, C)^2 - (A \wedge B, A \wedge B)(C,C) \\ &= (A \wedge B, aB + bA + A \wedge B)^2 - (A \wedge B, A \wedge B)(C,C) \\ &= (A \wedge B, A \wedge B)\,((A \wedge B, A \wedge B) - (C,C)) \\ &= (a^2 + b^2 + c^2 + 2abc - 1)(a^2 + b^2 + 2abc) > 0. \end{aligned}$$

□

2.A.2 Proof of Lemma 2.6

We first prove that $t > 1$ using the condition H_Q and the fact that $\tau = (\varrho * \sigma)^{-1}$:

In this case we have $(R,S)^2 - (R,R)(S,S) = r^2 + s^2 + t^2 - 2rst - 1 < -1$, which implies that t lies in the interval $]rs - \sqrt{r^2s^2 - r^2 - s^2}, rs + \sqrt{r^2s^2 - r^2 - s^2}[$ and thus $t > rs - \sqrt{r^2s^2 - r^2 - s^2}$. But $rs - \sqrt{r^2s^2 - r^2 - s^2} > 1$ because $(rs-1)^2 > r^2s^2 - r^2 - s^2$ which is equivalent to $(r-s)^2 > -1$.

Using Proposition 1.9, it is easy to see that the geodesics corresponding to ϱ, σ and τ intersect two by two because $(R \wedge S, R \wedge S) = (S \wedge T, S \wedge T) = (T \wedge R, T \wedge R) = h < -1$.

It remains to prove that the commutator $\gamma = (c + C) = -[\varrho, \sigma]$ does not intersect anyone of ϱ, σ and τ, using again Proposition 1.9 and the formula for $\tau = (t + T)$ and the commutator:

$$\begin{aligned} (t + T) &= rs + (R,S) - rS - sR - R \wedge S \\ (c + C) &= 1 + 2(R \wedge S, R \wedge S) \\ &\qquad + 2(rs + (R,S))\, R \wedge S + 2r\, S \wedge (R \wedge S) + 2s\, R \wedge (R \wedge S) \end{aligned}$$

$$\Longrightarrow \begin{cases} (R\wedge C, R\wedge C) &=& 4\,(r^2-(R,R))\,((R,S)^2-(R,R)(S,S))\,((R,S)^2-s^2(R,R)) \\ &=& 4(r^2+s^2+t^2-2rst-1)(s^2+t^2-2rst) \;>\; 0 \\ \\ (S\wedge C, S\wedge C) &=& 4\,(s^2-(S,S))\,((R,S)^2-(R,R)(S,S))\,((R,S)^2-r^2(S,S)) \\ &=& 4(r^2+s^2+t^2-2rst-1)(r^2+t^2-2rst) \;>\; 0 \\ \\ (T\wedge C, T\wedge C) &=& -4\,(r^2-(R,R))\,(s^2-(S,S)) \\ && ((R,S)^2-(R,R)(S,S))\,(s^2(R,R)+2rs(R,S)+r^2(S,S)) \\ &=& 4(r^2+s^2+t^2-2rst-1)(r^2+s^2-2rst) \;>\; 0 \end{cases}$$

□

Chapter 3

Pairs of Building Blocks and Teichmüller Spaces

In the previous chapter, we have investigated Q and Y pieces. In this chapter, we study surfaces and especially their Teichmüller spaces that arise when we join two such building blocks along a boundary geodesic (dropping two half-cylinders).

As we have already mentioned in the introduction, we take the point of view that a Riemann surface is determined by the group of isometries by which we quotient the upper half plane up to simultaneous conjugation. Thus, parameterizing the (marked) generators of the group is equivalent to parameterizing the Teichmüller space. The major aim of this chapter is to give this explicit parameterization for pairs of building blocks, i.e. for surfaces of signatures $(0, 4)$, $(1, 2)$ and $(2, 0)$.

Given the Teichmüller spaces of surfaces of signatures $(0, 4)$, $(1, 2)$ and $(2, 0)$, we study the Teichmüller spaces of surfaces of signatures $(0, n)$, investigate how the genus of a surface is reflected in the Teichmüller space and give the Teichmüller spaces of limit cases such as tori with two cusps or spheres with four cone-like singularities.

3.1 The X Piece

An X piece is a Riemann surface of signature (0,4) obtained by joining two Y pieces generated by $h_\alpha,\ h_\beta$ and $h_\delta,\ h_\varepsilon$ along $\gamma = -(\alpha*\beta)^{-1} = -\delta*\varepsilon$. Using Poincaré's Polyhedron Theorem, it is easy to prove that the fundamental domain of the X piece can be obtained by joining the domains of the two Y pieces along γ as in Figure 3.1 (dropping what would become two funnels or half-cylinders in the two quotient surfaces).

3.1.1 Coordinates in the Quaternion Basis

Obviously, if we take the hyperbolic elements $\alpha, \beta, \delta, \varepsilon$ such that $\alpha*\beta*\delta*\varepsilon = 1$, we cannot be sure that the axes of α and ε are situated on two different sides of γ, i.e we cannot be sure that γ is a dividing geodesic. In order to ensure this, we conjugate all the elements

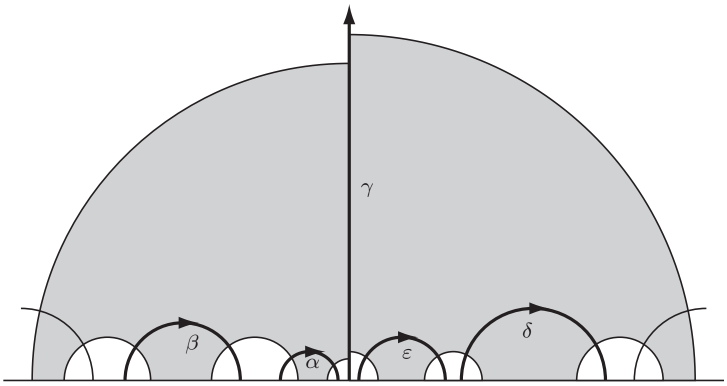

Figure 3.1

such that $\gamma = (c + \sqrt{c^2-1}\ \mathbf{I})$, and the condition for γ being a dividing geodesic is now equivalent to the condition that the attracting fixed points of α and ε are of different signs (they are real numbers!); this leads us to Proposition 3.1.

Proposition 3.1 *Let α, β, γ, δ and ε be hyperbolic elements such that h_α, h_β and h_δ, h_ε generate Y pieces and $\gamma = -(\alpha * \beta)^{-1} = -\delta * \varepsilon = (c + \sqrt{c^2-1}\ \mathbf{I})$. Suppose that the attracting fixed point of α and the repulsive fixed point of ε are $-L \in \mathbb{R}$ and $M \in \mathbb{R}$, then*

$$\begin{array}{rcl}(a+A) &=& a - \frac{ac+b}{\sqrt{c^2-1}}\mathbf{I} - \frac{\tilde{L}}{2(c^2-1)}(\mathbf{J}+\mathbf{K}) + \frac{h(a,b,-c)}{2\tilde{L}}(\mathbf{J}-\mathbf{K}),\\ (b+B) &=& b - \frac{bc+a}{\sqrt{c^2-1}}\mathbf{I} - \frac{\tilde{L}(c+\sqrt{c^2-1})}{2(c^2-1)}(\mathbf{J}+\mathbf{K}) + \frac{h(a,b,-c)(c-\sqrt{c^2-1})}{2\tilde{L}}(\mathbf{J}-\mathbf{K}),\\ (d+D) &=& d + \frac{cd+e}{\sqrt{c^2-1}}\mathbf{I} - \frac{\tilde{M}(c+\sqrt{c^2-1})}{2(c^2-1)}(\mathbf{J}+\mathbf{K}) + \frac{h(c,d,-e)(c-\sqrt{c^2-1})}{2\tilde{M}}(\mathbf{J}-\mathbf{K}),\\ (e+E) &=& e + \frac{ce+d}{\sqrt{c^2-1}}\mathbf{I} - \frac{\tilde{M}}{2(c^2-1)}(\mathbf{J}+\mathbf{K}) + \frac{h(c,d,-e)}{2\tilde{M}}(\mathbf{J}-\mathbf{K}),\end{array}$$

where

$$\begin{array}{rcl}\tilde{L} &=& L\sqrt{c^2-1}(b+ac+\sqrt{a^2-1}\sqrt{c^2-1}),\\ \tilde{M} &=& M\sqrt{c^2-1}(d+ce+\sqrt{c^2-1}\sqrt{e^2-1}),\\ h(u,v,w) &=& u^2+v^2+w^2-2uvw-1.\end{array}$$

Proof. Using the special form of $\gamma = (c + C) = (c + \sqrt{c^2-1}\,\mathbf{I})$ and the fact that (A, C), (B, C), (D, C) and (E, C) are known in terms of traces, we can calculate the components a_1, b_1, d_1 and e_1 (recall that the matrix without trace A can be written as $A = a_1\mathbf{I}+a_2\mathbf{J}+a_3\mathbf{K}$).

We have the attracting fixed point of α ($-L = \frac{a_1+\sqrt{a^2-1}}{a_2-a_3}$) and the repulsive fixed point of ε ($M = \frac{e_1-\sqrt{e^2-1}}{e_2-e_3}$). Thus we get $(a_2 - a_3)$ and $(e_2 - e_3)$. We know also that α and ε are normalized, i.e. $a^2 - 1 = a_1^2 + (a_2 - a_3)(a_2 + a_3)$ and $e^2 - 1 = e_1^2 + (e_2 - e_3)(e_2 + e_3)$. Therefore $(a_2 + a_3)$ and $(e_2 + e_3)$ can be expressed in terms of traces. The solution of this set of linear equations is (after simplification) a_2, a_3, e_2 and e_3.

As $(A, B) = -ab-c = a_1b_1+a_2b_2-a_3b_3$ and $(D, E) = -c-de = d_1e_1+d_2e_2-d_3e_3$, we can express b_2 and d_2 as functions of b_3 and d_3. But β and δ are normalized ($b^2-1 = b_1^2+b_2^2-b_3^2$ and $d^2 - 1 = d_1^2 + d_2^2 - d_3^2$). Thus we can replace b_2 and d_2 by their expressions in b_3 and

d_3 in the normalizing condition. We get a quadratic equation for each of b_3 and d_3 and we can exclude one of the solutions for each of b_3 and d_3 because $b_2 - b_3 > 0$ as well as $d_2 - d_3 > 0$.

After simplification, we get the result. □

Remark 3.2 This proposition, together with Proposition 2.3 proves also that every Y piece can be constructed as a quotient $\mathbb{H}/\langle h_\alpha, h_\beta \rangle$, where α and β satisfy H_Y because we can choose $\tilde{L} = 1$ by a simultaneous conjugation of α, β and γ. The Teichmüller space of Y pieces can therefore be given as

$$\mathcal{T}_{(0,3)} := \left\{ (a, b, c) \in \mathbb{R}^3 \;\middle|\; a, b, c > 1 \right\}.$$

3.1.2 Teichmüller Space

We have now expressed $\alpha, \beta, \gamma, \delta$ and ε in terms of the traces a, b, c, d and e as well as the real numbers $\tilde{M}$ and $\tilde{L}$. Clearly, a, b, c, d and e are invariant under conjugation, but $\tilde{M}$ and $\tilde{L}$ are not. We therefore introduce the elements $\xi = (x + X) = -(\alpha * \delta)^{-1}$ and $\zeta = (z + Z) = -(\beta * \delta)^{-1}$ that give us a certain measure of $\tilde{M}$ and $\tilde{L}$, and lead to the parameter space of the X piece in terms of traces only:

$$\mathcal{P}_{(0,4)} = \left\{ (a, b, c, d, e, x, z) \in \mathbb{R}^7 \;\middle|\; a, b, c, d, e, x, z > 1 \right\}$$

In fact, we shall express $\tilde{M}$ and $\tilde{L}$ in terms of these traces if we really want $\mathcal{P}_{(0,4)}$ to be the parameter space. But before that, let us consider the following lemma as well as its corollary:

Lemma 3.3 (Helling) *Let $A, B, C, D \in H_0$ and H be the matrix of "scalar" products as follows:*

$$H = \begin{pmatrix} (A,A) & (A,B) & (A,C) & (A,D) \\ (B,A) & (B,B) & (B,C) & (B,D) \\ (C,A) & (C,B) & (C,C) & (C,D) \\ (D,A) & (D,B) & (D,C) & (D,D) \end{pmatrix}$$

Then $\det(H) = 0$.

Proof. The dimension of 2 by 2 matrices without trace is 3 (they are generated by **I**, **J** and **K**). Therefore A, B, C and D are linearly dependent. Thus, the rows of H are linearly dependent and $\det(H) = 0$. □

Remark 3.4 H. Helling gives this lemma in [Hel74] in the wider context of eight elements $\alpha_1, \ldots, \alpha_4$ and $\beta_1, \ldots, \beta_4$ of an abstract group Γ and a trace function s:

$$H_s(\alpha_1, \ldots, \alpha_4; \beta_1, \ldots, \beta_4) = (h_{ij})_{1 \le i,j \le 4}; \quad h_{ij} = s(\alpha_i * \beta_j) - s(\alpha_i^{-1} * \beta_j).$$

For $s = 2\,tr$, it is easy to see that $H = \frac{1}{4} H_s(\alpha, \beta, \gamma, \delta; \alpha, \beta, \gamma, \delta)$.

Corollary 3.5 *We have*

$$\begin{aligned} Q_{(0,4)} &:= a^2+b^2+c^2+d^2+e^2+x^2+z^2+4abde-1 \\ &\quad +2c(ab+de)+2x(ad+be)+2z(ae+bd)-2cxz \;=\; 0. \end{aligned}$$

Proof. Using the fact that α, δ, ξ and β, δ, ζ constitute Y pieces, we can express the internal products in terms of the traces a, b, c, d, e, x and z. Thus

$$H = \begin{pmatrix} (A,A) & (A,B) & (A,C) & (A,D) \\ (B,A) & (B,B) & (B,C) & (B,D) \\ (C,A) & (C,B) & (C,C) & (C,D) \\ (D,A) & (D,B) & (D,C) & (D,D) \end{pmatrix} = \begin{pmatrix} a^2-1 & -ab-c & -ac-b & -ad-x \\ -ab-c & b^2-1 & -bc-a & -bd-z \\ -ac-b & -bc-a & c^2-1 & cd+e \\ -ad-x & -bd-z & cd+e & d^2-1 \end{pmatrix}$$

and therefore $\det(H) = (a^2+b^2+c^2+2abc-1)Q_{(0,4)}$.

But we know that $a^2+b^2+c^2+2abc-1>0$, which yields the result. $\square$

Proposition 3.6 *The hyperbolic elements* $(a+A)$, $(b+B)$, $(d+D)$ *and* $(e+E)$ *with the coordinates of Proposition 3.1 constitute an X piece corresponding to* (a,b,c,d,e,x,z) *with* $a,b,c,d,e,x,z>1$, *iff*

$$\frac{\tilde{M}}{\tilde{L}} = -\frac{ae+bd+(ad+be)(c-\sqrt{c^2-1})+\sqrt{c^2-1}((c-\sqrt{c^2-1})z-x)}{a^2+b^2+c^2+2abc-1}.$$

Proof. Let us call the righthand expression of the equation p. We must show that $(a+A)$, $(b+B)$ and $(d+D)$ corresponding to $(a,b,c,d,e,x,z) \in \mathcal{P}_{(0,4)}$ are such that

$$(A,D) = -ad-x \quad \text{and} \quad (B,D) = -bd-z. \qquad (*)$$

Furthermore we have to prove that $p>0$, which implies a situation as in Figure 3.1 (fixed points of α and ε of different sign).

With the coordinates of Proposition 3.1 the equations $(*)$ yield a non degenerated system of two equations linear in $\tilde{p} = \frac{\tilde{M}}{\tilde{L}}$ and $\tilde{q} = \frac{\tilde{L}}{\tilde{M}}$. The unique solution for $\tilde{p}$ is p but the solution for $\tilde{q}$ is not its inverse at first sight. However, we can extract a factor $Q_{(0,4)}$ from $\tilde{p}\tilde{q}-1$ which is therefore zero because of Corollary 3.5.

The condition $p>0$ is clearly equivalent to

$$ae+bd+(ad+be)(c-\sqrt{c^2-1})+\sqrt{c^2-1}((c-\sqrt{c^2-1})z-x)<0.$$

If we solve this condition for z, we get $z < \frac{\sqrt{c^2-1}x-(ae+bd)-(ad+be)(c-\sqrt{c^2-1})}{\sqrt{c^2-1}(c-\sqrt{c^2-1})}$. Let us call this last expression z_{asym}.

Let us now consider the polynomial of the Corollary 3.5 as a function $Q_{(0,4)}(x,z)$ whose zero-set is a hyperbola. The branch of this hyperbola with x and z positive has an upper asymptote (with respect to z) given by the equation $z = z_{\text{asym}}$. Thus $z < z_{\text{asym}}$ for all (x,z) on the branch of the hyperbola. This means that $p>0$ for any set of parameters satisfying $\det(H)=0$. $\square$

Remark 3.7 If we want to fix the fundamental domain of the X piece (i.e. not only up to an isometry leaving the imaginary axis invariant) we can for instance fix $\tilde{M} = p$ and $\tilde{L} = 1$ in the formula of Proposition 3.1.

Theorem 3.8 *The subgroup of $SL(2,\mathbb{R})$ acting on the upper half plane $\mathbb{H}$ by Möbius transformations such that the quotient surface has signature $(0,4)$, is freely generated by α, β and δ or a simultaneous conjugation of these three; where $\alpha = (a + A), \beta = (b + B)$ and $\delta = (d + D)$ are the ones given in Proposition 3.1 with*

$$\tilde{M} = -\frac{ae + bd + (ad + be)(c - \sqrt{c^2 - 1}) + \sqrt{c^2 - 1}((c - \sqrt{c^2 - 1})z - x)}{a^2 + b^2 + c^2 + 2abc - 1} \quad \textit{and} \quad \tilde{L} = 1.$$

Here (a, b, c, d, e, x, z) is an element of the Teichmüller space $\mathcal{T}_{(0,4)}$ of surfaces of signature $(0,4)$ in trace coordinates, i.e.

$$\mathcal{T}_{(0,4)} := \left\{(a, b, c, d, e, x, z) \in \mathcal{P}_{(0,4)} \;\middle|\; Q_{(0,4)} = 0\right\}$$

with the parameter space

$$\mathcal{P}_{(0,4)} := \left\{(a, b, c, d, e, x, z) \in \mathbb{R}^7 \;\middle|\; a, b, c, d, e, x, z > 1\right\}$$

and the polynomial

$$\begin{aligned} Q_{(0,4)} \;:=\; & a^2 + b^2 + c^2 + d^2 + e^2 + x^2 + z^2 + 4abde - 1 \\ & + 2c(ab + de) + 2x(ad + be) + 2z(ae + bd) - 2cxz. \end{aligned}$$

Proof. Direct deduction from Propositions 3.1 and 3.6 together with Poincaré's Polyhedron Theorem showing that Figure 3.1 is the fundamental domain for the X piece. □

Remark 3.9 If we know the generating elements α, β and δ of an X piece in terms of matrices (with positive traces), we can easily extract the trace coordinates of the Teichmüller space:

$$(a, b, c, d, e, x, z) = (tr(\alpha), tr(\beta), -tr(\alpha * \beta), tr(\delta), tr(\alpha * \beta * \delta), -tr(\alpha * \delta), -tr(\beta * \delta))$$

On the other hand, given a point in the Teichmüller space, we can calculate the matrices (up to conjugation) using Proposition 3.1.

3.2 The “Fish” Piece

In this section, we will study a surface which we will call the “Fish” piece. It's a Riemann surface of signature (1,2) obtained by joining a Y piece (generated by h_α, h_β) and a Q piece (generated by h_ϱ, h_σ) along $\gamma = -(\alpha * \beta)^{-1} = -[\varrho, \sigma]$. As for the X piece, Poincaré's Polyhedron Theorem shows that the fundamental domain of the “Fish” piece

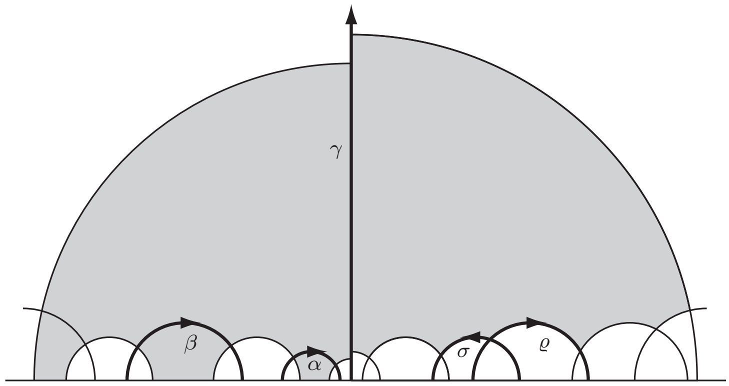

Figure 3.2

can be obtained by joining the domains of the Y piece and the Q piece along γ as in Figure 3.2.

The proofs of the propositions and the corollary in this section run along exactly the same lines as the corresponding ones of the previous section. For ease of reading, they are postponed to Appendix 3.A.

3.2.1 Coordinates in the Quaternion Basis

As before, if we take the hyperbolic elements α, β, ϱ, σ such that $\alpha * \beta * [\varrho, \sigma] = 1$, there is no guarantee that the geodesic corresponding to $\gamma = -[\varrho, \sigma]$ is dividing.

If we conjugate all the elements such that $\gamma = (c + \sqrt{c^2-1}\ \mathbf{I})$, the condition for γ corresponding to a dividing geodesic is equivalent to the condition that the attracting fixed points of α and σ are of different signs; which leads us to the Proposition 3.10.

Proposition 3.10 *Let α, β, γ, ϱ and σ be hyperbolic elements such that h_α and h_β constitute a Y piece, h_ϱ and h_σ a Q piece and $\gamma = -(\alpha * \beta)^{-1} = -[\varrho, \sigma] = (c + \sqrt{c^2-1}\ \mathbf{I})$. Suppose that the attracting fixed points of α and σ are $-L \in \mathbb{R}$ and $M \in \mathbb{R}$, then*

$$
\begin{aligned}
(a+A) &= a - \tfrac{ac+b}{\sqrt{c^2-1}}\mathbf{I} - \tfrac{\tilde{L}}{2(c^2-1)}(\mathbf{J}+\mathbf{K}) + \tfrac{h(a,b,-c)}{2\tilde{L}}(\mathbf{J}-\mathbf{K}),\\
(b+B) &= b - \tfrac{bc+a}{\sqrt{c^2-1}}\mathbf{I} - \tfrac{\tilde{L}(c+\sqrt{c^2-1})}{2(c^2-1)}(\mathbf{J}+\mathbf{K}) + \tfrac{h(a,b,-c)(c-\sqrt{c^2-1})}{2\tilde{L}}(\mathbf{J}-\mathbf{K}),\\
(r+R) &= r + \tfrac{r(c+1)}{\sqrt{c^2-1}}\mathbf{I} - \tfrac{\tilde{M}(2rs-t+t(c+\sqrt{c^2-1}))}{2(c-1+2s^2)}(\mathbf{J}+\mathbf{K}) + \tfrac{2rs-t+t(c-\sqrt{c^2-1})}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K}),\\
(s+S) &= s - \tfrac{s(c+1)}{\sqrt{c^2-1}}\mathbf{I} + \tfrac{\tilde{M}}{2}(\mathbf{J}+\mathbf{K}) - \tfrac{c-1+2s^2}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K}),
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{L} &= L\sqrt{c^2-1}(b + ac + \sqrt{a^2-1}\sqrt{c^2-1}),\\
\tilde{M} &= M\tfrac{s\sqrt{c^2-1}+(c-1)\sqrt{s^2-1}}{c-1},\\
h(u,v,w) &= u^2 + v^2 + w^2 - 2uvw - 1.
\end{aligned}
$$

Remark 3.11 This proposition, together with Proposition 2.7 proves also that every Q piece can be constructed by a quotient $\mathbb{H}/_{\langle h_\varrho, h_\sigma \rangle}$, where ϱ and σ satisfy H_Q because we can choose $\tilde{M} = 1$ by a simultaneous conjugation of ϱ, σ and γ. And the Teichmüller space of Q pieces can therefore be given as

$$\mathcal{T}_{(1,1)} := \Big\{ (c,r,s,t) \in \mathbb{R}^4 \;\Big|\; c,r,s,t > 1, \; -(c+1) = 2(r^2+s^2+t^2-2rst-1) \Big\}$$

3.2.2 Teichmüller Space

As in the last section, $\tilde{M}$ and $\tilde{L}$ are not invariant under conjugation. We therefore introduce the elements $\xi = (x+X) = -(\alpha * \varrho)^{-1}$ and $\zeta = (z+Z) = -(\beta * \varrho)^{-1}$ in order to express $\tilde{M}$ and $\tilde{L}$ in terms of these traces (using another corollary of Lemma 3.3). This leads to the parameter space of the "Fish" piece in terms of traces[1] only:

$$\mathcal{P}_{(1,2)} := \Big\{ (a,b,c,r,s,t,x,z) \in \mathbb{R}^8 \;\Big|\; a,b,c,r,s,t,x,z > 1, \; -\tfrac{c+1}{2} = r^2+s^2+t^2-2rst-1 \Big\}$$

Corollary 3.12 (of Lemma 3.3)

$$Q_{(1,2)} := a^2+b^2+c^2+2abc-1+x^2+z^2-2cxz+2r(a+b)(x+z)+2r^2(1+c+2ab) = 0.$$

Proposition 3.13 *The hyperbolic elements $(a+A)$, $(b+B)$, $(r+R)$ and $(s+S)$ with the coordinates of Proposition 3.10 constitute a "Fish" piece corresponding to the point $(a,b,c,r,s,t,x,z) \in \mathcal{P}_{(1,2)}$, iff*

$$\frac{\tilde{M}}{\tilde{L}} = -\frac{(c-1+2s^2)((a+b)r(c-1+\sqrt{c^2-1}) - (c-1)((c+\sqrt{c^2-1})x-z))}{(c-1)(a^2+b^2+c^2+2abc-1)(2rs-t+(c+\sqrt{c^2-1})t)}\,.$$

[1] We could have eliminated the trace c in $\mathcal{P}_{(1,2)}$ but it is a convenient abbreviation as we will see.

Theorem 3.14 *The subgroup of* $SL(2,\mathbb{R})$ *acting on the upper half plane* $\mathbb{H}$ *by Möbius transformations such that the quotient surface has signature* $(1,2)$*, is generated by* α,β,ϱ *and* σ *(with the relation* $\alpha * \beta * [\varrho,\sigma] = 1$*) or a simultaneous conjugation of these four; where* $\alpha = (a+A)$*,* $\beta = (b+B)$*,* $\varrho = (r+R)$ *and* $\sigma = (s+S)$ *are the ones given in Proposition 3.10 with*

$$\tilde{M} = -\frac{(c-1+2s^2)((a+b)r(c-1+\sqrt{c^2-1}) - (c-1)((c+\sqrt{c^2-1})x - z))}{(c-1)(a^2+b^2+c^2+2abc-1)(2rs-t+(c+\sqrt{c^2-1})t)}$$

$$\text{and} \qquad \tilde{L} = 1.$$

Here (a,b,c,r,s,t,x,z) *is an element of the Teichmüller space* $\mathcal{T}_{(1,2)}$ *of surfaces of signature* $(1,2)$ *in trace coordinates, i.e.*

$$\mathcal{T}_{(1,2)} := \left\{(a,b,c,r,s,t,x,z) \in \mathcal{P}_{(1,2)} \;\middle|\; Q_{(1,2)} = 0\right\}$$

with the parameter space

$$\mathcal{P}_{(1,2)} := \left\{(a,b,c,r,s,t,x,z) \in \mathbb{R}^8 \;\middle|\; \begin{array}{l} a,b,c,r,s,t,x,z > 1, \\ -\frac{c+1}{2} = r^2+s^2+t^2-2rst-1 \end{array}\right\}$$

and the polynomial

$$Q_{(1,2)} := a^2+b^2+c^2+2abc-1+x^2+z^2-2cxz+2r(a+b)(x+z)+2r^2(1+c+2ab).$$

Proof. Direct deduction from Propositions 3.10 and 3.13 together with Poincaré's Polyhedron Theorem showing that Figure 3.2 is the fundamental domain for the "Fish" piece. □

Remark 3.15 As in the case of an X piece, we can easily calculate the trace coordinates of the Teichmüller space for the "Fish" piece given the generating elements in terms of matrices and vice-versa.

3.3 The Genus 2 Surface

A surface of genus 2 can be obtained by joining two Q pieces generated by h_{ϱ_l}, h_{σ_l} and h_{ϱ_m}, h_{σ_m} along $\gamma = -[\varrho_l,\sigma_l] = -[\varrho_m,\sigma_m]$. Using again Poincaré's Polyhedron Theorem, it is easy to see that the fundamental domain of a genus 2 surface can be obtained by joining the domains of two Q pieces along γ as in Figure 3.3.

As in the last section, the proofs of the propositions and the corollary in this section are postponed to Appendix 3.B.

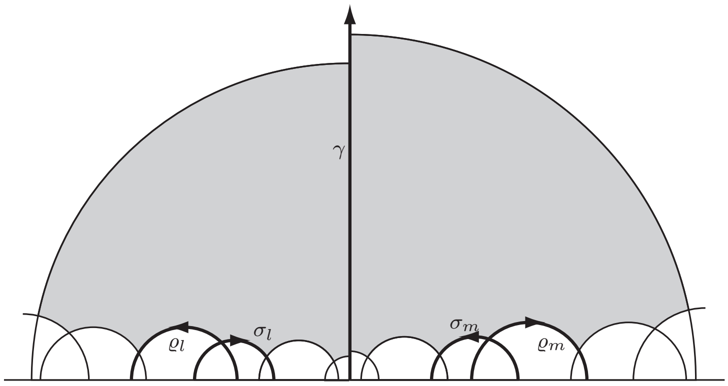

Figure 3.3

3.3.1 Coordinates in the Quaternion Basis

Again, we conjugate the hyperbolic elements ϱ_l, σ_l and ϱ_m, σ_m (each pair satisfying H_Q as well as $[\sigma_l, \varrho_l] * [\varrho_m, \sigma_m] = 1$) such that $\gamma = (c + \sqrt{c^2-1}\ \mathbf{I})$ and use the attracting fixed points of σ_l and σ_m in order to get the coordinates of ϱ_l, σ_l, ϱ_m and σ_m in the quaternion basis:

Proposition 3.16 *Let ϱ_l, σ_l, ϱ_m and σ_m be hyperbolic elements such that ϱ_l, σ_l and ϱ_m, σ_m constitute Q pieces and $\gamma = -[\varrho_l, \sigma_l] = -[\varrho_m, \sigma_m] = (c + \sqrt{c^2-1}\ \mathbf{I})$. Suppose that the attracting fixed points of ϱ_l and σ_m are $-L \in \mathbb{R}$ and $M \in \mathbb{R}$, then*

$$\begin{aligned}
(r_l + R_l) &= r_l + \tfrac{r_l(c+1)}{\sqrt{c^2-1}}\mathbf{I} + \tfrac{\tilde{L}}{2}(\mathbf{J}+\mathbf{K}) - \tfrac{c-1+2r_l^2}{2\tilde{L}(c-1)}(\mathbf{J}-\mathbf{K}),\\
(s_l + S_l) &= s_l - \tfrac{s_l(c+1)}{\sqrt{c^2-1}}\mathbf{I} - \tfrac{\tilde{L}(2r_l s_l - t_l + (c-\sqrt{c^2-1})t_l)}{2(c-1+2r_l^2)}(\mathbf{J}+\mathbf{K})\\
&\qquad + \tfrac{2r_l s_l - t_l + (c+\sqrt{c^2-1})t_l}{2\tilde{L}(c-1)}(\mathbf{J}-\mathbf{K}),\\
(r_m + R_m) &= r_m + \tfrac{r_m(c+1)}{\sqrt{c^2-1}}\mathbf{I} - \tfrac{\tilde{M}(2r_m s_m - t_m + (c+\sqrt{c^2-1})t_m)}{2(c-1+2s_m^2)}(\mathbf{J}+\mathbf{K})\\
&\qquad + \tfrac{2r_m s_m - t_m + (c-\sqrt{c^2-1})t_m}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K}),\\
(s_m + S_m) &= s_m - \tfrac{s_m(c+1)}{\sqrt{c^2-1}}\mathbf{I} + \tfrac{\tilde{M}}{2}(\mathbf{J}+\mathbf{K}) - \tfrac{c-1+2s_m^2}{2\tilde{M}(c-1)}(\mathbf{J}-\mathbf{K}),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{L} &= L\tfrac{r_l\sqrt{c^2-1}+(c-1)\sqrt{r_l^2-1}}{c-1},\\
\tilde{M} &= M\tfrac{s_m\sqrt{c^2-1}+(c-1)\sqrt{s_m^2-1}}{c-1},\\
h(u,v,w) &= u^2+v^2+w^2-2uvw-1.
\end{aligned}$$

3.3.2 Teichmüller Space

As before, $\tilde{M}$ and $\tilde{L}$ are not invariant under conjugation. We therefore introduce the elements
$\xi = (x + X) = -(\sigma_l * \varrho_m)^{-1}$ and $\zeta = (z + Z) = -(\varrho_l * \sigma_l^{-1} * \varrho_l^{-1} * \varrho_m)^{-1}$ and use a third corollary of Lemma 3.3 to get the Teichmüller space:

Corollary 3.17 (of Lemma 3.3)

$$Q_{(2,0)} := 4s_l^2 r_m^2 + 4s_l r_m(x + y) + 2(c + 1)(s_l^2 + r_m^2) + c^2 + x^2 + z^2 - 2cxz - 1 = 0.$$

Proposition 3.18 *The hyperbolic elements ϱ_l, σ_l, ϱ_m and σ_m with the coordinates of Proposition 3.16 constitute a genus 2 surface corresponding to $(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z)$ in $\mathcal{P}_{(2,0)}$, iff*

$$\frac{\tilde{M}}{\tilde{L}} = -\frac{(c-1+2s_m^2)\left(2r_l s_l - t_l + \left(c-\sqrt{c^2-1}\right)t_l\right)\left(2r_m s_l(c-1+\sqrt{c^2-1}) - (c-1)\left(\left(c+\sqrt{c^2-1}\right)x - z\right)\right)}{(c-1+2r_l^2)(c-1+2s_l^2)\sqrt{c^2-1}\left(2r_m s_m - t_m + \left(c+\sqrt{c^2-1}\right)t_m\right)}.$$

Here $$\mathcal{P}_{(2,0)} := \left\{(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \in \mathbb{R}^9 \;\middle|\; \begin{array}{l} r_l, s_l, t_l, c, r_m, s_m, t_m, x, z > 1, \\ -\frac{c+1}{2} = h(r_l, s_l, t_l) = h(r_m, s_m, t_m) \end{array}\right\}$$

and $$h(u, v, w) := u^2 + v^2 + w^2 - 2uvw - 1.$$

Theorem 3.19 *The subgroup of $SL(2, \mathbb{R})$ acting on the upper half plane $\mathbb{H}$ by Möbius transformations such that the quotient surface is compact and has genus 2, is generated by $\varrho_l, \sigma_l, \varrho_m$ and σ_m (with the relation $[\sigma_l, \varrho_l] * [\varrho_m, \sigma_m] = 1$) or a simultaneous conjugation of these four; where $\varrho_l = (r_l + R_l), \sigma_l = (s_l + S_l), \varrho_m = (r_m + R_m)$ and $\sigma_m = (s_m + S_m)$ are the ones given in Proposition 3.16 with*

$$\tilde{M} = -\frac{(c-1+2s_m^2)\left(2r_l s_l - t_l + \left(c-\sqrt{c^2-1}\right)t_l\right)\left(2r_m s_l(c-1+\sqrt{c^2-1}) - (c-1)\left(\left(c+\sqrt{c^2-1}\right)x - z\right)\right)}{(c-1+2r_l^2)(c-1+2s_l^2)\sqrt{c^2-1}\left(2r_m s_m - t_m + \left(c+\sqrt{c^2-1}\right)t_m\right)} \quad \textit{and} \quad \tilde{L} = 1.$$

Here $(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z)$ is an element of the Teichmüller space $\mathcal{T}_{(2,0)}$ of surfaces of genus 2 in trace coordinates, i.e.

$$\mathcal{T}_{(2,0)} := \left\{(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \in \mathcal{P}_{(2,0)} \;\middle|\; Q_{(2,0)} = 0\right\}$$

with the parameter space

$$\mathcal{P}_{(2,0)} := \left\{(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \in \mathbb{R}^9 \;\middle|\; \begin{array}{l} r_l, s_l, t_l, c, r_m, s_m, t_m, x, z > 1, \\ -\frac{c+1}{2} = h(r_l, s_l, t_l) = h(r_m, s_m, t_m) \end{array}\right\},$$

and the polynomials

$$h(u, v, w) := u^2 + v^2 + w^2 - 2uvw - 1 \quad \textit{and}$$

$$Q_{(2,0)} := 4s_l^2 r_m^2 + 4s_l r_m(x + y) + 2(c + 1)(s_l^2 + r_m^2) + c^2 + x^2 + z^2 - 2cxz - 1.$$

Proof. The theorem follows directly from Propositions 3.16 and 3.18 together with Poincaré's Polyhedron Theorem showing that Figure 3.3 is the fundamental domain for the "Fish" Piece. □

Remark 3.20 Again, we can easily calculate the trace coordinates of the Teichmüller space for a genus 2 surface, given the generating elements in terms of matrices and vice-versa.

3.4 The Polynomial Q

As we have seen, the polynomial Q is quite similar in the three cases we studied. Indeed,

$$\begin{aligned} Q_{(1,2)}(a,b,c,r,s,t,x,z) &= Q_{(0,4)}(a,b,c,r,r,x,z) \\ \text{and} \quad Q_{(2,0)}(r_l,s_l,t_l,c,r_m,s_m,t_m,x,z) &= Q_{(0,4)}(s_l,s_l,c,r_m,r_m,x,z). \end{aligned}$$

But the polynomial Q is not only very similar in the three cases, the zero-set of Q determines the Teichmüller spaces. This leads to the question how to obtain the Teichmüller spaces of new surfaces from the ones of known surfaces, which is the subject of this section.

3.4.1 Gluing a Y piece along a boundary geodesic

We have seen that the Teichmüller space of X pieces can be parameterized using the traces of the boundary geodesics of the Y pieces that are glued together, the traces of two more geodesics intersecting the glued geodesic and an equation between these traces. This opens the question how to obtain the Teichmüller space of a surface S of signature (g,n) given the Teichmüller space of a surface S' of signature $(g,n-1)$ by gluing a Y piece along one boundary geodesic to the surface S'.

Let $\mathcal{T}_{(g,n-1)}$ be given by the traces $a_1,\ldots,a_u$ and a set of equalities $Q_1 = Q_2 = \cdots = Q_v = 0$ (polynomial in the trace coordinates), a_1 the trace of the boundary geodesic α_1 along which we want to glue and a_2, a_3 the traces of the geodesics α_2, α_3 such that α_1, α_2 and α_3 form a Y piece embedded in S'. Then, $\mathcal{T}_{(g,n)}$ is obtained by adding the trace coordinates d,e,x and z and the equation $Q_{(0,4)}(a_2,a_3,a_1,d,e,x,z)=0$. Here, d and e are the traces of the boundary geodesics δ and ε such that α_1, δ and ε form the Y piece that is glued to S' and $x=-tr(\alpha_2 * \delta)$, $z=-tr(\alpha_3*\delta)$.

Given the matrices for $\alpha_1,\ldots,\alpha_u$, the matrices for δ and ε can be obtained as follows:

- Conjugate α_1 and α_2 by the appropriate matrix φ such that $\varphi*\alpha_1*\varphi^{-1} = (a_1+\sqrt{a_1^2-1}\,\mathbf{I})$ and $\varphi*\alpha_2*\varphi^{-1} = a_2 - \frac{a_2a_1+a_3}{\sqrt{a_1^2-1}}\mathbf{I} - \frac{1}{2(a_1^2-1)}(\mathbf{J}+\mathbf{K}) + \frac{h(a_2,a_3,-a_1)}{2}(\mathbf{J}-\mathbf{K})$.
- Define the matrices

$$\begin{aligned} \tilde{\delta} &= d + \frac{a_1d+e}{\sqrt{a_1^2-1}}\mathbf{I} - \frac{\tilde{M}(a_1+\sqrt{a_1^2-1})}{2(a_1^2-1)}(\mathbf{J}+\mathbf{K}) + \frac{h(a_1,d,-e)(a_1-\sqrt{a_1^2-1})}{2\tilde{M}}(\mathbf{J}-\mathbf{K}) \quad \text{and} \\ \tilde{\varepsilon} &= e + \frac{a_1e+d}{\sqrt{a_1^2-1}}\mathbf{I} - \frac{\tilde{M}}{2(a_1^2-1)}(\mathbf{J}+\mathbf{K}) + \frac{h(a_1,d,-e)}{2\tilde{M}}(\mathbf{J}-\mathbf{K}), \end{aligned}$$

$$\begin{array}{rcl} \text{where} \quad \tilde{M} & := & -\frac{a_2e+a_3d+(a_2d+a_3e)(a_1-\sqrt{a_1^2-1})+\sqrt{a_1^2-1}((a_1-\sqrt{a_1^2-1})z-x)}{a_2^2+a_3^2+a_1^2+2a_2a_3a_1-1} \quad \text{and} \\ h(a,b,c) & := & a^2+b^2+c^2-2abc-1. \end{array}$$

- Conjugating $\tilde{\delta}$ and $\tilde{\varepsilon}$ by φ^{-1}we get the matrices $\delta = \varphi^{-1} * \tilde{\delta} * \varphi$ and $\varepsilon = \varphi^{-1} * \tilde{\varepsilon} * \varphi$.

3.4.2 Genus from Funnels

Looking at the cases of signature $(0,4)$, $(1,2)$ and $(2,0)$, we can derive a rule how to get the Teichmüller space $\mathcal{T}_{(g,n)}$ of surfaces of signature (g,n), given the Teichmüller space $\mathcal{T}_{(g-1,n+2)}$ of surfaces of signature $(g-1,n+2)$.

Let $\mathcal{T}_{(g-1,n+2)}$ be given by the traces $a_1,\ldots,a_u$ and a set of equalities $Q_1 = Q_2 = \cdots = Q_v = 0$, with a_1, a_2 traces of boundary geodesics α_1, α_2 and a_3 the trace of a geodesic α_3 forming a Y piece together with α_1 and α_2. Then, $\mathcal{T}_{(g,n)}$ can be constructed adding the trace coordinates r, s and t, as well as the equations $a_1 = a_2 = r$ and $-\frac{a_3+1}{2} = r^2+s^2+t^2-2rst-1$.

We have shown this statement for the cases of signatures $(0,4)$, $(1,2)$ and $(2,0)$, but it remains true in the general case as every geodesic that does not cross the Y piece $\alpha_1, \alpha_2, \alpha_3$ is left untouched by the exchange of this building block with a Q piece and every geodesic that does cross the Y piece still exists after the transformation.

As in before, we can get the explicit matrices ϱ and σ by conjugating, replacing α_1 and α_2 with the matrices $\tilde{\varrho}$ and $\tilde{\sigma}$ according to Proposition 3.10 and Theorem 3.14 and re-conjugating the matrices $\tilde{\varrho}$ and $\tilde{\sigma}$.

Remark 3.21 Together with the previous subsection, we can thus construct the Teichmüller spaces and even the explicit matrices for surfaces of signature (g,n) in an inductive way, following any given tree-like decomposition of the surface into Y pieces.

3.4.3 Limit Cases

Following a path in the Teichmüller space of X pieces, we can continuously deform an X piece such that the four geodesics corresponding to β, γ, δ and ε keep their lengths and such that α gets smaller and smaller. In the limit case (the length of α goes to 0), the formula of the generating elements (Proposition 3.1) are still valid (with $a = 1$), with the difference that the isometry h_α is now parabolic (it fixes an infinite point). We can therefore use the same arguments as before and get the Teichmüller space of a Y piece with a cusp:

$$\mathcal{T}_{(0;-;1;3)} := \left\{(b,c,d,e,x,z) \in \mathcal{P}_{(0;-;1;3)} \;\middle|\; Q_{(0;-;1;3)} = 0\right\}$$

with the parameter space

$$\mathcal{P}_{(0;-;1;3)} := \left\{(b,c,d,e,x,z) \in \mathbb{R}^6 \;\middle|\; b,c,d,e,x,z > 1\right\}$$

and the polynomial

$$Q_{(0;-;1;3)}(b,c,d,e,x,z) := Q_{(0,4)}(1,b,c,d,e,x,z).$$

Obviously, we can proceed in the same way with the geodesics corresponding to β, δ and ε and get the Teichmüller spaces of a double-funnel with two cusps, a sphere with a funnel and three cusps and a sphere with four cusps.

If we apply this procedure to the Fish piece, we can get for instance the Teichmüller space of tori with two cusps:

$$\mathcal{T}_{(1;-;2;0)} := \left\{(c,r,s,t,x,z) \in \mathcal{P}_{(1;-;2;0)} \;\middle|\; Q_{(1;-;2;0)} = 0\right\}$$

with the parameter space

$$\mathcal{P}_{(1;-;2;0)} := \left\{(c,r,s,t,x,z) \in \mathbb{R}^6 \;\middle|\; \begin{array}{l} c,r,s,t,x,z > 1, \\ -\frac{c+1}{2} = r^2+s^2+t^2-2rst-1 \end{array}\right\}$$

and the polynomial

$$Q_{(1;-;2;0)}(c,r,s,t,x,z) := Q_{(1,2)}(1,1,c,r,s,t,x,z) = Q_{(0,4)}(1,1,c,r,r,x,z).$$

However, we cannot proceed in the same way with a dividing geodesic like γ. Indeed, if its length goes to 0, the polynomial Q is strictly positive:

$$\begin{aligned} Q_{(0,4)}(a,b,1,d,e,x,z) \;\; := \;\; & a^2+b^2+d^2+e^2+4abde+(x-z)^2 \\ & +2(ab+de)+2x(ad+be)+2z(ae+bd) \;\; > \;\; 0. \end{aligned}$$

3.4.4 From Cusps to Cones

In the previous subsection, we considered the limit of the Teichmüller space of X pieces when a goes to 1 and we saw that this corresponds to Y pieces with a cusp. We will now look at what happens if $0 \leq a < 1$.

For such an a, let $\alpha = (a+A)$ and $\beta = (b+B) \in SL(2,\mathbb{R})$ be such that $b > 1$ and such that there is $\gamma = (c+C) \in SL(2,\mathbb{R})$ with $c > 1$ and $\alpha * \beta * \gamma = -1$. Then, α is elliptic and corresponds to the fixed point of m_α in $\mathbb{H}$. By exactly the same proof as for Lemma 2.2 in Appendix 2.A.1, we show that α is in the domain whose boundaries are β and γ. Taking the perpendiculars to β and γ through the fixed point α and their images by $h_\beta^{-1}(\alpha)$ and h_γ, we get the situation of Figure 3.4 in the upper half plane (up to simultaneous conjugation).

Unfortunately, the gray domain D in Figure 3.4 is not always a fundamental domain for $\Gamma = \langle h_\alpha, h_\beta \rangle$. Indeed, if we want Γ to be discrete and D to be a fundamental domain, we must verify the conditions of Poincaré's Polyhedron Theorem (theorem 1.13). Condition (f) says in our case, that the sum of the interior angles at the points corresponding to α and $h_\beta^{-1}(\alpha) = \beta^{-1} * \alpha * \beta$ should be $\frac{2\pi}{t}$ for some $t \in \mathbb{N}$. Cutting D along β, γ and their common perpendicular, we get two pentagons with four right angles P_1 and P_2 containing respectively θ_1 and θ_2. Using the fact that a hyperbolic pentagon with four right angles is uniquely determined by the lengths of three of its sides (see e.g. [Bus92, p.36-37]), we derive that $\theta_1 = \theta_2$. Calculating the cosine of this angle, we get

$$\cos(\theta_1) \;=\; \frac{|(A\wedge B, A\wedge C)|}{\sqrt{(A\wedge B, A\wedge B)(A\wedge C, A\wedge C)}} \;=\; a.$$

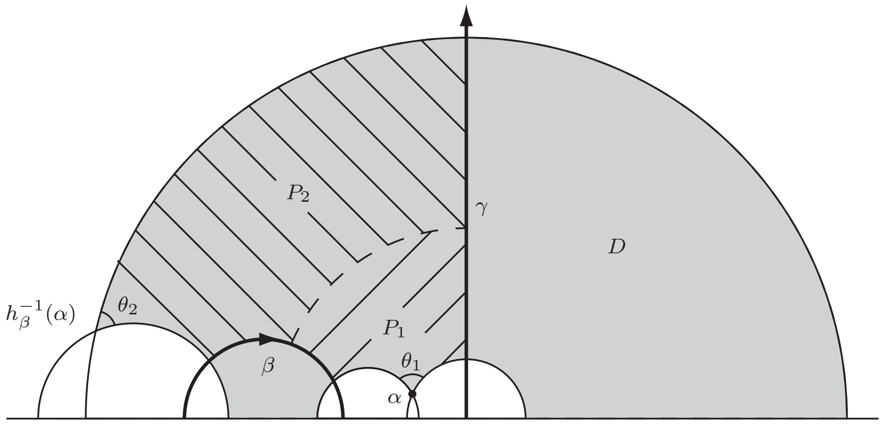

Figure 3.4

Therefore, $\Gamma = \langle h_\alpha, h_\beta \rangle$ is a surface group with fundamental domain D if $a = \cos\left(\frac{\pi}{t}\right)$ with $t \in \mathbb{N}$, $t \geq 2$ and the quotient surface $\mathbb{H}/\Gamma$ is a cylinder with a cone-like singularity (of order t).

We can now glue such a degenerated Y piece to other building blocks (cutting off the half-cylinders) and use the matrices given in Propositions 3.1 and 3.10. If we want to get surfaces with more than one cone-like singularity, we can apply the procedure to other generators but have to take care about the cases of trace 0. Indeed, if $a = b = 0$ and $\alpha = (a + A)$ and $\beta = (b + B) \in SL(2, \mathbb{R})$ are such that $b > 1$ and such that there is $\gamma = (c + C) \in SL(2, \mathbb{R})$ with $c > 1$ and $\alpha * \beta * \gamma = -1$, then α and β are elliptic (they are half-turns) and correspond to points on the geodesic corresponding to γ. In this case, a fundamental domain would have γ and its perpendiculars through α and $h_\beta^{-1}(\alpha)$ as borders, i.e. only the funnel we cut off in order to glue the degenerated Y piece to other building blocks. Nevertheless, a degenerated X piece having two cone-like singularities of order 2 still makes sense and we can use the same formula as for the non-degenerated X piece (with $a = b = 0$).

An example for this is the sphere with four cone-like singularities of order 2, 2, 3 and 3. This is the first example of a spectrally rigid surface obtained by a 3-generator group as has recently been shown in [BFS05] (announced in [BS05]). Its Teichmüller space can be expressed as

$$\mathcal{T}_{(0;2,2,3,3;0;0)} := \left\{ (c, x, z) \in \mathcal{P}_{(0;2,2,3,3;0;0)} \;\middle|\; Q_{(0;2,2,3,3;0;0)} = 0 \right\}$$

with the parameter space

$$\mathcal{P}_{(0;2,2,3,3;0;0)} := \left\{ (c, x, z) \in \mathbb{R}^3 \;\middle|\; c, x, z > 1 \right\}$$

and the polynomial

$$Q_{(0;2,2,3,3;0;0)}(c, x, z) := Q_{(0,4)}(0, 0, c, \tfrac{1}{2}, \tfrac{1}{2}, x, z).$$

3.A Appendix: Proofs for the “Fish” Piece

3.A.1 Proof of Proposition 3.10

Using the part of the proof of Proposition 3.1 that concerns the “left” Y piece, we get the coordinates of α and β.

As we have the special form of $\gamma = (c + C) = \left(c + \sqrt{c^2-1}\,\mathbf{I}\right)$ and the fact that (R, C) and (S, C) are known in terms of traces, we can calculate the components r_1 and s_1.

We know that the attracting fixed point of σ is $M = \frac{s_1+\sqrt{s^2-1}}{s_2-s_3}$. Thus we can express $(s_2 - s_3)$ in terms of r, s, t and M. We know also that $\sigma \in SL(2, \mathbb{R})$, i.e. $s^2 - 1 = s_1^2 + (s_2 - s_3)(s_2 + s_3)$. Therefore, $(s_2 + s_3)$ can also be expressed in terms of r, s, t and M. The solution of this set of linear equations is (after simplification) s_2 and s_3.

We know that $(R, S) = t - rs = r_1 s_1 + r_2 s_2 - r_3 s_3$. Thus we get r_2 as a linear function of r_3. As ϱ is normalized $(r^2 - 1 = r_1^2 + r_2^2 - r_3^2)$ we can replace r_2 by its function of r_3 in the normalizing condition. We get a quadratic equation for r_3 and we can exclude one of the solution for r_3 because because the equation $\gamma = -[\varrho, \sigma]$ must be satisfied.

After simplification, we get the result. □

3.A.2 Proof of Corollary 3.12

Replacing D with R in Lemma 3.3 (where $\varrho = (r + R)$) and using the fact that α, ϱ, ξ and β, ϱ, ζ constitute Y pieces, we get $\det(H) = (a^2 + b^2 + c^2 + 2abc - 1)Q_{(1,2)}$. But we know that $a^2 + b^2 + c^2 + 2abc - 1 > 0$, which leads to the result. □

3.A.3 Proof of Proposition 3.13

We call the righthand expression of the equation p, as before. We must show that $(a + A)$, $(b + B)$, $(r + R)$ and $(s + S)$ corresponding to $(a, b, c, r, s, t, x, z) \in \mathcal{P}_{(1,2)}$ are such that

$$(A, R) = -ar - x \quad \text{and} \quad (B, R) = -br - z. \qquad (**)$$

Furthermore, we have to prove that $p > 0$, which implies a situation where the fixed points of α and σ are of opposite sign.

With the coordinates of Proposition 3.10 the equations $(**)$ yield a non degenerated system of two equations linear in $\tilde{p} = \frac{\tilde{M}}{\tilde{L}}$ and $\tilde{q} = \frac{\tilde{L}}{\tilde{M}}$. The unique solution for $\tilde{p}$ is p but the solution for $\tilde{q}$ is not its inverse at first sight. However, we can extract a factor $Q_{(1,2)}$ from $\tilde{p}\tilde{q} - 1$ which is therefore zero because of Corollary 3.12.

As $2rs - t$ is positive[2], the condition $p > 0$ is equivalent to

$$(a + b)r(c - 1 + \sqrt{c^2-1}) - (c - 1)((c + \sqrt{c^2-1})x - z) < 0.$$

[2]because $0 < \frac{c-1}{2} = (2rs - t)t - (r^2 + s^2)$

If we solve this condition for z, we get $z < \frac{x(c-1)(c+\sqrt{c^2-1})-r(a+b)(c-1+\sqrt{c^2-1})}{c-1}$. Let us call this last expression z_{asym}.

Let us now consider the polynomial of Corollary 3.12 as a function $Q_{(1,2)}(x,z)$ whose zero-set is a hyperbola. The branch of this hyperbola with x and z positive has an upper asymptote (with respect to z) given by the equation $z = z_{\text{asym}}$. Thus $z < z_{\text{asym}}$ for all (x,z) on the branch of the hyperbola. This means $p > 0$ for any set of parameters satisfying $\det(H) = 0$. □

3.B Appendix: Proofs for Genus 2

3.B.1 Proof of Proposition 3.16

Using the part of the proof of Proposition 3.10 that concerns the Q piece (see previous appendix), we get the coordinates of ϱ_m and σ_m.

As we have the special form of $\gamma = (c+C) = \left(c + \sqrt{c^2-1}\,\mathbf{I}\right)$ and the fact that (R_l, C) and (S_l, C) are known in terms of traces, we can calculate the components r_{l1} and s_{l1}.

We know that the attracting fixed point of σ_l is $-L = \frac{r_{l1}+\sqrt{{r_l}^2-1}}{r_{l2}-r_{l3}}$. Thus we can express $(r_{l2} - r_{l3})$ in terms of r_l, s_l, t_l and L. We know also that $\varrho_l \in SL(2,\mathbb{R})$, i.e. ${r_l}^2 - 1 = r_{l1}^2 + (r_{l2} - r_{l3})(r_{l2} + r_{l3})$. Therefore, $(r_{l2} + r_{l3})$ can also be expressed in terms of r_l, s_l, t_l and L. The solution of this set of linear equations is (after simplification) r_{l2} and r_{l3}.

We know that $(R_l, S_l) = t_l - r_l s_l = r_{l1}s_{l1} + r_{l2}s_{l2} - r_{l3}s_{l3}$. Thus we get s_{l2} as a linear function of s_{l3}. As σ is normalized $({s_l}^2 - 1 = s_{l1}^2 + s_{l2}^2 - s_{l3}^2)$ we can replace s_{l2} by its function of s_{l3} in the normalizing condition. We get a quadratic equation for s_{l3} and we can exclude one of the solution for s_{l3} because the equation $\gamma = -[\varrho_l, \sigma_l]$ must be satisfied.

After simplification, we get the result. □

3.B.2 Proof of Corollary 3.17

Replacing A, B and D with S_l, $\tilde{S}_l$ and R_m in Lemma 3.3 (where $(s_l + S_l) = \sigma_l$, $(s_l + \tilde{S}_l) = \varrho_l * \sigma_l^{-1} * \varrho_l^{-1}$ and $(r_m + R_m) = \varrho_m$) and using the fact that σ_l, ϱ_m, ξ and $\varrho_l * \sigma_l^{-1} * \varrho_l^{-1}, \varrho_m, \zeta$ constitute Y pieces, we get $\det(H) = (c+1)(c-1+2s_l^2)Q_{(2,0)}$. But we know that $(c+1)(c-1+2s_l^2) > 0$, which leads to the result. □

3.B.3 Proof of Proposition 3.18

We call again the righthand expression of the equation p. We must show that $(s_l + S_l) = \varrho_l$, $(s_l + \tilde{S}_l) = \varrho_l * \sigma_l^{-1} * \varrho_l^{-1}$ and $(r_m + R_m) = \varrho_m$ corresponding to $(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z)$ in $\mathcal{P}_{(2,0)}$ are such that

$$(S_l, R_m) = -s_l r_m - x \quad \text{and} \quad (\tilde{S}_l, R_l) = -s_l r_m - z. \qquad (***)$$

Furthermore, we have to prove that $p > 0$, which implies a situation where the fixed points of α and ε are of opposite sign.

With the coordinates of Proposition 3.16 the equations $(***)$ yield a non degenerated system of two equations linear in $\tilde{p} = \frac{\tilde{M}}{\tilde{L}}$ and $\tilde{q} = \frac{\tilde{L}}{\tilde{M}}$. The unique solution for $\tilde{p}$ is p but the solution for $\tilde{q}$ is not its inverse at first sight. However, we can extract a factor $Q_{(2,0)}$ from $\tilde{p}\tilde{q} - 1$ which is therefore zero because of Corollary 3.17.

As $2r_m s_m - t_m$ and $2r_l s_l - t_l$ are positive, the condition $p > 0$ is equivalent to

$$2r_m s_l(c - 1 + \sqrt{c^2 - 1}) - (c - 1)\left((c + \sqrt{c^2 - 1})x - z\right).$$

If we solve this condition for z, we get $z < \frac{x(c+\sqrt{c^2-1})-2r_m s_l(c-1+\sqrt{c^2-1})}{c-1}$. Let us call this last expression z_{asym}.

Let us now consider the polynomial of Corollary 3.12 as a function $Q_{(2,0)}(x, z)$ whose zero-set is a hyperbola. The branch of this hyperbola with x and z positive has an upper asymptote (with respect to z) given by the equation $z = z_{\text{asym}}$. Thus $z < z_{\text{asym}}$ for all (x, z) on the branch of the hyperbola. This means $p > 0$ for any set of parameters satisfying $\det(H) = 0$. □

Chapter 4

The Modular Group

As each hyperbolic structure on a surface is represented by infinitely many different points in Teichmüller space[1], it is possible to represent isometric surfaces by infinitely many different points. The transformations mapping one of these points to another form a group, the modular group (or extended mapping class group).

The aim of this chapter is to explicitly give the modular group in terms of generating elements and to prove that it acts in a particularly nice manner on the Teichmüller space in trace coordinates:

Theorem 4.1 *The modular group of the Teichmüller space $\mathcal{T}_{(0,4)}$ of surfaces of signature $(0,4)$ acts polynomially on $\mathcal{T}_{(0,4)}$ in these trace coordinates.*

Theorem 4.2 *The modular group of the Teichmüller space $\mathcal{T}_{(1,2)}$ of surfaces of signature $(1,2)$ acts rationally on $\mathcal{T}_{(1,2)}$ in these trace coordinates.*

Theorem 4.3 *The modular group of the Teichmüller space $\mathcal{T}_{(2,0)}$ of surfaces of signature $(2,0)$ acts rationally on $\mathcal{T}_{(2,0)}$ in these trace coordinates.*

But before giving the generating elements and proving these theorems, we state a well known fact concerning quadratic equations which we will use quite often:

Lemma 4.4 *Let $P(x) \in \mathbb{R}[x]$ be a quadratic expression with leading coefficient 1 such that $P(1) > 0$. If the equation $P(x) = 0$ has a real solution $\lambda_1 > 1$, then its other solution λ_2 is also greater than 1.*

Proof. We know $P(x) = (x - \lambda_1)(x - \lambda_2)$, $P(1) > 0$ and $1 - \lambda_1 < 0$. Thus $1 - \lambda_2 < 0$. □

In order to prove Theorems 4.1, 4.2 and 4.3, we give in each case some elements of the modular group and show that they generate the whole group. The fact that these elements and their inverses are polynomial for X pieces and rational in the other two cases, implies that the modular group contains only polynomial or rational elements.

[1]corresponding to the distinct markings of the surface

4.1 Some Elements

4.1.1 Signature $(0,4)$

Proposition 4.5 *The following transformations are elements of the modular group of the Teichmüller space of X pieces:*

$$\begin{array}{ll}\varphi_{Y_{ab}} & : \mathcal{T}_{(0,4)} \longrightarrow \mathcal{T}_{(0,4)} : (a,b,c,d,e,x,z) \longmapsto (b,a,c,d,e,2(cx-ae-bd)-z,x)\\ \varphi_{Y_{de}} & : \mathcal{T}_{(0,4)} \longrightarrow \mathcal{T}_{(0,4)} : (a,b,c,d,e,x,z) \longmapsto (a,b,c,e,d,2(cx-ae-bd)-z,x)\\ \varphi_{\text{turn}} & : \mathcal{T}_{(0,4)} \longrightarrow \mathcal{T}_{(0,4)} : (a,b,c,d,e,x,z) \longmapsto (b,d,z,e,a,2(cz-ad-be)-x,c)\end{array}$$

Proof. In each case we we look for elements $\alpha', \beta', \delta', \varepsilon' \in SL(2,\mathbb{R})$ such that the group of isometries $\langle h_\alpha, h_\beta, h_\delta, h_\varepsilon\rangle$ is the same as $\langle h'_\alpha, h'_\beta, h'_\delta, h'_\varepsilon\rangle$ and such that $\alpha' * \beta' * \delta' * \varepsilon' = 1$. Then we must show that $\varphi(a,b,c,d,e,x,z) = (tr(\alpha'), tr(\beta'), -tr(\alpha' * \beta'), tr(\delta'), tr(\varepsilon'), -tr(\alpha' * \delta'), -tr(\beta' * \delta'))$ and that these coordinates are all greater than 1. Thus they necessarily satisfy the condition $\det(H') = 0$ because that is a consequence of $\dim(H_0) = 3$.

In the first case we choose $\alpha' = \alpha * \beta * \alpha^{-1}$, $\beta' = \alpha$, $\delta' = \delta$ and $\varepsilon' = \varepsilon$. They clearly generate the same group and $\alpha' * \beta' * \delta' * \varepsilon' = \alpha * \beta * \alpha^{-1} * \alpha * \delta * \varepsilon = 1$ and calculating the traces of the new elements we get the coordinates of $\varphi_{Y_{ab}}(a,b,c,d,e,x,z)$. Considering the polynomial of Corollary 3.5 as a function in z and using Lemma 4.4, it is easy to see that the two solutions z and $2(cx-ae-bd)-z$ to the equation $Q_{(0,4)}(z) = 0$ are both greater than 1. Thus the coordinates of $\varphi_{Y_{ab}}(a,b,c,d,e,x,z)$ are all greater than 1.

In the second case we choose $\alpha' = \alpha$, $\beta' = \beta$, $\delta' = \delta * \varepsilon * \delta^{-1}$ and $\varepsilon' = \delta$.

In the third case we choose $\alpha' = \beta$, $\beta' = \delta$, $\delta' = \varepsilon$ and $\varepsilon' = \alpha$. Using Lemma 4.4 for the polynomial equation $Q_{(0,4)}(x) = 0$, we prove that the two solutions x and $2(cz-ad-be)-x$ are both greater than 1. Thus the coordinates of $\varphi_{\text{turn}}(a,b,c,d,e,x,z)$ are greater than 1 as well. □

Remark 4.6 The inverses of $\varphi_{Y_{ab}}$, $\varphi_{Y_{de}}$ and φ_{turn} are also polynomial:

$$\begin{array}{ll}\varphi^{-1}_{Y_{ab}} & : \mathcal{T}_{(0,4)} \longrightarrow \mathcal{T}_{(0,4)} : (a,b,c,d,e,x,z) \longmapsto (b,a,c,d,e,z,2(cz-ad-be)-x)\\ \varphi^{-1}_{Y_{de}} & : \mathcal{T}_{(0,4)} \longrightarrow \mathcal{T}_{(0,4)} : (a,b,c,d,e,x,z) \longmapsto (a,b,c,e,d,z,2(cz-ad-be)-x)\\ \varphi^{-1}_{\text{turn}} & : \mathcal{T}_{(0,4)} \longrightarrow \mathcal{T}_{(0,4)} : (a,b,c,d,e,x,z) \longmapsto (e,a,z,b,d,2(cz-ad-be)-x,c)\end{array}$$

Remark 4.7 Geometrically, an element of the modular group can be interpreted as a choice of other geodesics on the same surface. Figure 4.1 shows this for the transformations of Proposition 4.5.

Remark 4.8 As the Teichmüller space can be defined as the space of isotopy classes of marked complex structures of the Riemann surface (where two structures define the same point in the Teichmüller space if there exists a holomorphic homeomorphism homotopic to the identity leading from one to the other), an element of the modular group is such a class and we can represent it as one homeomorphism between marked X pieces. Thus,

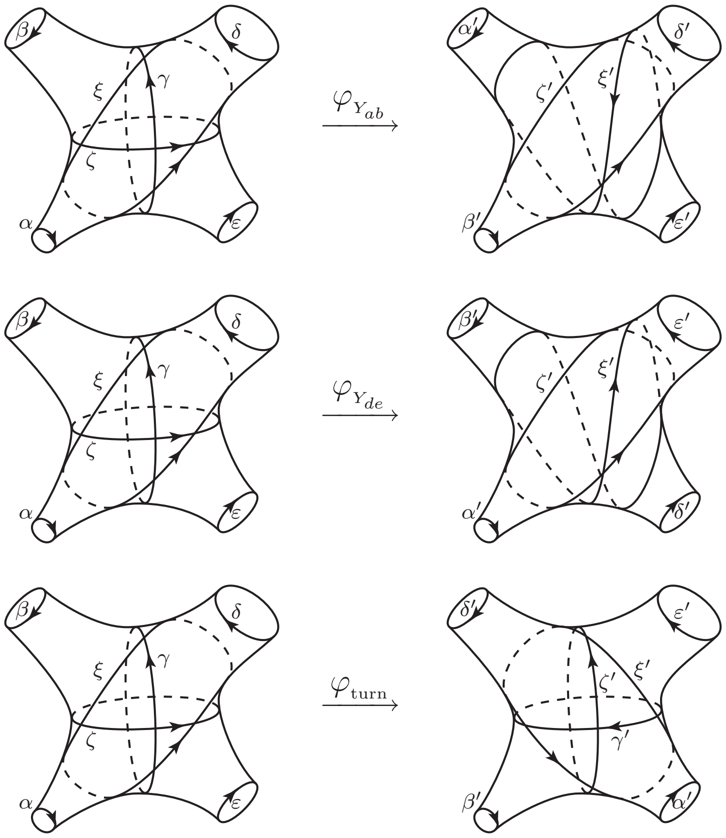

Figure 4.1

$\varphi_{Y_{ab}}$ can be interpreted as the following transformation of an X piece: We (pointwise) fix the Y piece with border geodesics (corresponding to) δ, ε and γ. We deform the Y piece with border α, β and γ such that γ is fixed pointwise and such that the lengths of α and β are exchanged ($tr(\alpha') = tr(\beta)$, $tr(\beta') = tr(\alpha)$). Then we twist this Y piece along γ^{-1} *half* way round, fixing γ pointwise.

It is obvious that transformations $\varphi_{Y_{ab}}$, $\varphi_{Y_{de}}$ and φ_{turn} cannot generate the modular group because they do not permit a change of orientation for γ without an exchange of the sets $\{a, b\}$ and $\{d, e\}$. We thus introduce φ_{inv} in Proposition 4.9.

Proposition 4.9 *The following transformation is an element of the modular group of the Teichmüller space:*

$$\varphi_{\text{inv}} \quad : \mathcal{T}_{(0,4)} \longrightarrow \mathcal{T}_{(0,4)} : (a, b, c, d, e, x, z) \longmapsto (b, a, c, e, d, 2(cz - ad - be) - x, z)$$

It is an involution.

Proof. We choose $\alpha' = \beta^{-1}$, $\beta' = \alpha^{-1}$, $\delta' = \varepsilon^{-1}$ and $\varepsilon' = \delta^{-1}$. Again, they generate the same group and $\alpha' * \beta' * \delta' * \varepsilon' = \beta^{-1} * \alpha^{-1} * \varepsilon^{-1} * \delta^{-1} = (-\gamma) * (-\gamma^{-1}) = 1$.

Calculating the traces of the new elements we get the coordinates of $\varphi_{\text{inv}}(a,b,c,d,e,x,z)$ (after some simplifications using Corollary 3.5). The traces are greater than 1 analogously to the proof of Proposition 4.5.

Direct calculation leads to $\varphi_{\text{inv}} \circ \varphi_{\text{inv}}(a,b,c,d,e,x,z) = (a,b,c,d,e,x,z)$. □

4.1.2 Signature $(1,2)$

Proposition 4.10 *The following transformations are elements of the modular group of the Teichmüller space of "Fish" pieces:*

$$\begin{array}{lcl} \varphi_Y & : & \mathcal{T}_{(1,2)} \longrightarrow \mathcal{T}_{(1,2)} : (a,b,c,r,s,t,x,z) \longmapsto (b,a,c,r,s,t,2(cx-r(a+b))-z,x) \\ \varphi_{Q_1} & : & \mathcal{T}_{(1,2)} \longrightarrow \mathcal{T}_{(1,2)} : (a,b,c,r,s,t,x,z) \longmapsto (a,b,c,r,t,2rt-s,x,z) \\ \varphi_{Q_2} & : & \mathcal{T}_{(1,2)} \longrightarrow \mathcal{T}_{(1,2)} : (a,b,c,r,s,t,x,z) \longmapsto (a,b,c,t,s,2st-r,x_{Q_2},z_{Q_2}) \\ \varphi_{\text{inv}} & : & \mathcal{T}_{(1,2)} \longrightarrow \mathcal{T}_{(1,2)} : (a,b,c,r,s,t,x,z) \longmapsto (b,a,c,r,s,2rs-t,2(cz-r(a+b))-x,z) \end{array}$$

$$\textit{where} \quad x_{Q_2} = \frac{(a+b)t+(2rt-s)x+s(2(cx-r(a+b))-z)}{c-1+2r^2} \quad \textit{and} \quad z_{Q_2} = \frac{(a+b)t+sx+(2rt-s)z}{c-1+2r^2}.$$

Proof. As before, we look for the elements $\alpha', \beta', \varrho', \sigma' \in SL(2,\mathbb{R})$ such that $\langle \alpha, \beta, \varrho, \sigma \rangle = \langle \alpha', \beta', \varrho', \sigma' \rangle$ and $\alpha' * \beta' * [\varrho', \sigma'] = 1$. Then we calculate the traces of the new elements that are the coordinates of $\varphi(a,b,c,r,s,t,x,z)$ (after some simplifications using Corollary 3.12). The proof that the traces are greater than 1 is analogous to the proof of Proposition 4.5 using Lemma 4.4 applied to the polynomial $Q_{(1,2)}$ of Corollary 3.12 and the fact that $-(c+1) = 2(r^2+s^2+t^2-2rst-1) < -2$ which implies that $2rs-t$, $2rt-s$ and $2st-r$ are all positive and thus greater than 1 (they are traces of hyperbolic elements).

In the first case we choose $\alpha' = \alpha * \beta * \alpha^{-1}$, $\beta' = \alpha$, $\varrho' = \varrho$ and $\sigma' = \sigma$.

In the second case we choose $\alpha' = \alpha$, $\beta' = \beta$, $\varrho' = \varrho$ and $\sigma' = \sigma * \varrho$.

In the third case we choose $\alpha' = \alpha$, $\beta' = \beta$, $\varrho' = \varrho * \sigma$ and $\sigma' = \sigma$.

In the last case, we finally choose $\alpha' = \beta^{-1}$, $\beta' = \alpha^{-1}$, $\varrho' = \sigma * \varrho * \sigma^{-1}$ and $\sigma' = \sigma^{-1}$. □

Remark 4.11 φ_{inv} is an involution and the inverses of φ_Y, φ_{Q_1} and φ_{Q_2} are rational:

$$\begin{array}{lcl} \varphi_Y^{-1} & : & \mathcal{T}_{(1,2)} \longrightarrow \mathcal{T}_{(1,2)} : (a,b,c,r,s,t,x,z) \longmapsto (b,a,c,r,s,t,2(cx-r(a+b))-z,x) \\ \varphi_{Q_1}^{-1} & : & \mathcal{T}_{(1,2)} \longrightarrow \mathcal{T}_{(1,2)} : (a,b,c,r,s,t,x,z) \longmapsto (a,b,c,r,2rs-t,s,x,z) \\ \varphi_{Q_2}^{-1} & : & \mathcal{T}_{(1,2)} \longrightarrow \mathcal{T}_{(1,2)} : (a,b,c,r,s,t,x,z) \longmapsto (a,b,c,2rs-t,s,r,x_{Q_2^{-1}},z_{Q_2^{-1}}) \end{array}$$

$$\text{where} \quad x_{Q_2^{-1}} = \frac{(a+b)(2rs-t)+(2r(2rs-t)-s)x+sz}{c-1+2r^2} \quad \text{and}$$
$$z_{Q_2^{-1}} = \frac{(a+b)(2rs-t)+s(2(cz-r(a+b))-x)+(2r(2rs-t)-s)z}{c-1+2r^2}.$$

Remark 4.12 Geometrically, we can again interpret these elements of the modular group as other choices of geodesics on the same surface as shown in Figure 4.2.

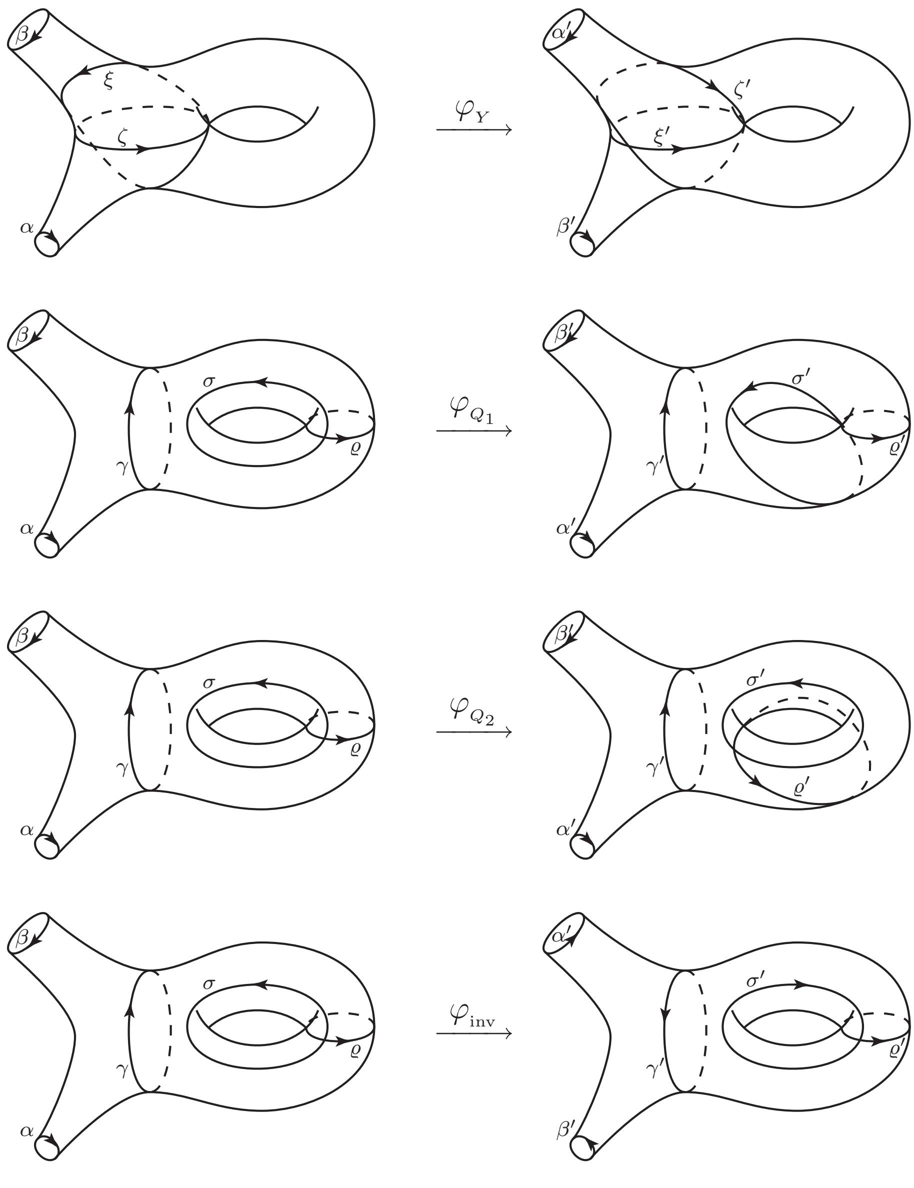

Figure 4.2

For signature $(1, 2)$, we have seen so far only transformations that leave the geodesic γ invariant. At first glance, one might think that the natural choice of the dividing geodesic γ is unique but that is not the case as Figure 4.3 and Proposition 4.13 show.

Proposition 4.13 *The following transformation is an element of the modular group of the Teichmüller space of "Fish" pieces:*

$$\varphi_{\infty} \;:\; \mathcal{T}_{(1,2)} \;\longrightarrow\; \mathcal{T}_{(1,2)} \;:\; (a, b, c, r, s, t, x, z) \;\longmapsto\; (a', b', c', r', s', t', x', z')$$

where

$$\begin{array}{rclcrcl} a' &=& b & \quad & s' &=& \frac{(a+b)s+(2rs-t)x+tz}{c-1+2r^2} \\ b' &=& a & & t' &=& \frac{(a+b)s+tx+(2rt-s)z}{c-1+2r^2} \\ c' &=& 2(xz-ab-r^2)-c & & x' &=& z \\ r' &=& r & & z' &=& x \end{array}$$

Proof. We choose $\alpha' = \beta$, $\beta' = \varrho * \alpha * \varrho^{-1}$, $\varrho' = \varrho$ and $\sigma' = -\alpha^{-1} * \sigma$. They clearly generate the same group and $\alpha' * \beta' * [\varrho', \sigma'] = \beta * \varrho * \alpha * \varrho^{-1} * \varrho * \alpha^{-1} * \sigma * \varrho^{-1} * \sigma^{-1} * \alpha = \alpha^{-1} * \alpha * \beta * [\varrho, \sigma] * \alpha = 1$.

Calculating the traces of the new elements we get the ones of the proposition after some simplifications using Corollary 3.12 and $-(c+1) = 2(r^2 + s^2 + t^2 - 2rst - 1)$.

Applying Lemma 4.4 to the polynomial $Q_{(1,2)}$ of Corollary 3.12, we proof that the two solutions c and $2(xz - ab - r^2) - c$ to the equation $Q_{(1,2)}(c) = 0$ are both greater than 1. Thus $c' > 1$. Clearly $s' > 1$ and $t' > 1$ because $(2rs - t) > 1$ and $(2rt - s) > 1$ analogous to the proof of Proposition 4.10. ☐

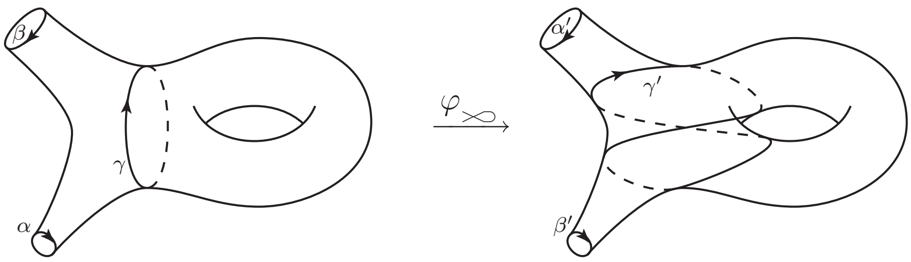

Figure 4.3

Remark 4.14 φ_{∞} is an involution.

4.1.3 Signature $(2,0)$

Proposition 4.15 *The following transformations are elements of the modular group of the Teichmüller space of compact surfaces of genus 2:*

$$\begin{array}{rcl} \varphi_{Q_{1_l}} &:& \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} : (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, t_l, \overline{s_l}, c, r_m, s_m, t_m, x_{Q_{1_l}}, z_{Q_{1_l}}) \\ \varphi_{Q_{2_l}} &:& \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} : (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (t_l, s_l, \overline{r_l}, c, r_m, s_m, t_m, x, z) \\ \varphi_{Q_{1_m}} &:& \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} : (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, s_l, t_l, c, r_m, t_m, \overline{s_m}, x, z) \\ \varphi_{Q_{2_m}} &:& \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} : (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, s_l, t_l, c, t_m, s_m, \overline{r_m}, x_{Q_{2_m}}, z_{Q_{2_m}}) \\ \varphi_{\text{inv}} &:& \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} : (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, s_l, \overline{t_l}, c, r_m, s_m, \overline{t_m}, z, x) \\ \varphi_{l \leftrightarrow m} &:& \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} : (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (s_m, r_m, \overline{t_m}, c, s_l, r_l, \overline{t_l}, x, \overline{z}) \\ \varphi_{\infty} &:& \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} : (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_{\infty_l}, s_l, t_{\infty_l}, \overline{c}, r_m, s_{\infty m}, t_{\infty m}, z, x) \end{array}$$

where

$$\begin{array}{lll}
\overline{r_l} = 2s_lt_l - r_l & \qquad & \overline{r_m} = 2s_mt_m - r_m \\
\overline{s_l} = 2r_lt_l - s_l & & \overline{s_m} = 2r_mt_m - s_m \\
\overline{t_l} = 2r_ls_l - t_l & & \overline{t_m} = 2r_ms_m - t_m \\
\overline{x} = 2(cz - 2s_lr_m) - x & & \overline{z} = 2(cx - 2s_lr_m) - z \\
x_{Q_{1_l}} = \frac{2r_mt_l+\overline{r_l}x+r_l\overline{z}}{c-1+2s_l^2} & & z_{Q_{1_l}} = \frac{2r_mt_l+r_lx+\overline{r_l}z}{c-1+2s_l^2} \\
x_{Q_{2m}} = \frac{2s_lt_m+\overline{s_m}x+s_m\overline{z}}{c-1+2r_m^2} & & z_{Q_{2m}} = \frac{2s_lt_m+s_m\overline{x}+\overline{s_m}z}{c-1+2r_m^2} \\
r_{\infty_l} = \frac{2r_mr_l+\overline{t_l}x+t_lz}{c-1+2s_l^2} & & s_{\infty_m} = \frac{2s_ls_m+\overline{t_m}x+t_mz}{c-1+2r_m^2} \\
t_{\infty_l} = z_{Q_{1_l}} & & t_{\infty_m} = z_{Q_{2m}} \\
\overline{c} = 2(xz - s_l^2 - r_m^2) - c & &
\end{array}$$

Proof. Again, we look for the elements $\varrho'_l, \sigma'_l, \varrho'_m, \sigma'_m \in SL(2,\mathbb{R})$ such that $\langle \varrho_l, \sigma_l, \varrho_m, \sigma_m\rangle = \langle \varrho'_l, \sigma'_l, \varrho'_m, \sigma'_m\rangle$ and $[\sigma'_l, \varrho'_l] * [\varrho'_m, \sigma'_m] = 1$. Then we calculate the traces of the new elements and show that they are the coordinates of $\varphi(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z)$ (after some simplifications using Corollary 3.17). The proof that the traces are greater than 1 is analogous to the proof of Proposition 4.5 using Lemma 4.4 applied to the polynomial $Q_{(2,0)}$ of Corollary 3.17 and the fact that

$$-(c+1) = 2(r_l^2 + s_l^2 + t_l^2 - 2r_ls_lt_l - 1) = 2(r_m^2 + s_m^2 + t_m^2 - 2r_ms_mt_m - 1) < -2.$$

For each transformation, the choices for $\varrho'_l, \sigma'_l, \varrho'_m, \sigma'_m \in SL(2,\mathbb{R})$ are the following:

$$\begin{array}{rcl}
\varphi_{Q_{1_l}} & : & \varrho'_l = \varrho_l,\ \sigma'_l = \sigma_l * \varrho_l,\ \varrho'_m = \varrho_m \text{ and } \sigma'_m = \sigma_m; \\
\varphi_{Q_{2_l}} & : & \varrho'_l = \varrho_l * \sigma_l,\ \sigma'_l = \sigma_l,\ \varrho'_m = \varrho_m \text{ and } \sigma'_m = \sigma_m; \\
\varphi_{Q_{1m}} & : & \varrho'_l = \varrho_l,\ \sigma'_l = \sigma_l,\ \varrho'_m = \varrho_m \text{ and } \sigma'_m = \sigma_m * \varrho_m; \\
\varphi_{Q_{2m}} & : & \varrho'_l = \varrho_l,\ \sigma'_l = \sigma_l,\ \varrho'_m = \varrho_m * \sigma_m \text{ and } \sigma'_m = \sigma_m; \\
\varphi_{\text{inv}} & : & \varrho'_l = \sigma_l * \varrho_l * \sigma_l^{-1},\ \sigma'_l = \sigma_l^{-1},\ \varrho'_m = \sigma_m * \varrho_m * \sigma_m^{-1} \text{ and } \sigma'_m = \sigma_m^{-1}; \\
\varphi_{l\leftrightarrow m} & : & \varrho'_l = \varrho_m * \sigma_m * \varrho_m^{-1},\ \sigma'_l = \varrho_m^{-1},\ \varrho'_m = \sigma_l^{-1} \text{ and } \sigma'_m = \sigma_l * \varrho_l * \sigma_l^{-1}; \\
\varphi_{\infty} & : & \varrho'_l = -\varrho_m * \varrho_l^{-1},\ \sigma'_l = \varrho_l * \sigma_l^{-1} * \varrho_l^{-1},\ \varrho'_m = \varrho_m \text{ and } \sigma'_m = -\sigma_l^{-1} * \sigma_m.
\end{array}$$

□

Remark 4.16 $\varphi_{\text{inv}}, \varphi_{l\leftrightarrow m}$ and φ_∞ are involutions and the inverses of $\varphi_{Q_{1_l}}, \varphi_{Q_{2_l}}, \varphi_{Q_{1m}}$ and $\varphi_{Q_{2m}}$ are rational:

$$\begin{array}{rcl}
\varphi^{-1}_{Q_{1_l}} & : & \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} \ :\ (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, \overline{t_l}, s_l, c, r_m, s_m, t_m, x_{Q^{-1}_{1_l}}, z_{Q^{-1}_{1_l}}) \\
\varphi^{-1}_{Q_{2_l}} & : & \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} \ :\ (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (\overline{t_l}, s_l, r_l, c, r_m, s_m, t_m, x, z) \\
\varphi^{-1}_{Q_{1m}} & : & \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} \ :\ (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, s_l, t_l, c, r_m, \overline{t_m}, s_m, x, z) \\
\varphi^{-1}_{Q_{2m}} & : & \mathcal{T}_{(2,0)} \longrightarrow \mathcal{T}_{(2,0)} \ :\ (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, s_l, t_l, c, \overline{t_m}, s_m, r_m, x_{Q^{-1}_{2m}}, z_{Q^{-1}_{2m}})
\end{array}$$

where

$$\begin{array}{lll}
x_{Q^{-1}_{1_l}} = \frac{2r_m\overline{t_l}+(2s_l\overline{t_l}-r_l)x+r_lz}{c-1+2s_l^2} & \qquad & z_{Q^{-1}_{1_l}} = \frac{2r_m\overline{t_l}+r_l\overline{x}+(2s_l\overline{t_l}-r_l)z}{c-1+2s_l^2} \\
x_{Q^{-1}_{2m}} = \frac{2s_l\overline{t_m}+(2r_m\overline{t_m}-s_m)x+s_mz}{c-1+2r_m^2} & & z_{Q^{-1}_{2m}} = \frac{2s_l\overline{t_m}+s_m\overline{x}+(2r_m\overline{t_m}-s_m)z}{c-1+2r_m^2}
\end{array}$$

Remark 4.17 As before, we can geometrically interpret these elements of the modular group as other choices of geodesics on the same surface as shown in Figure 4.4.

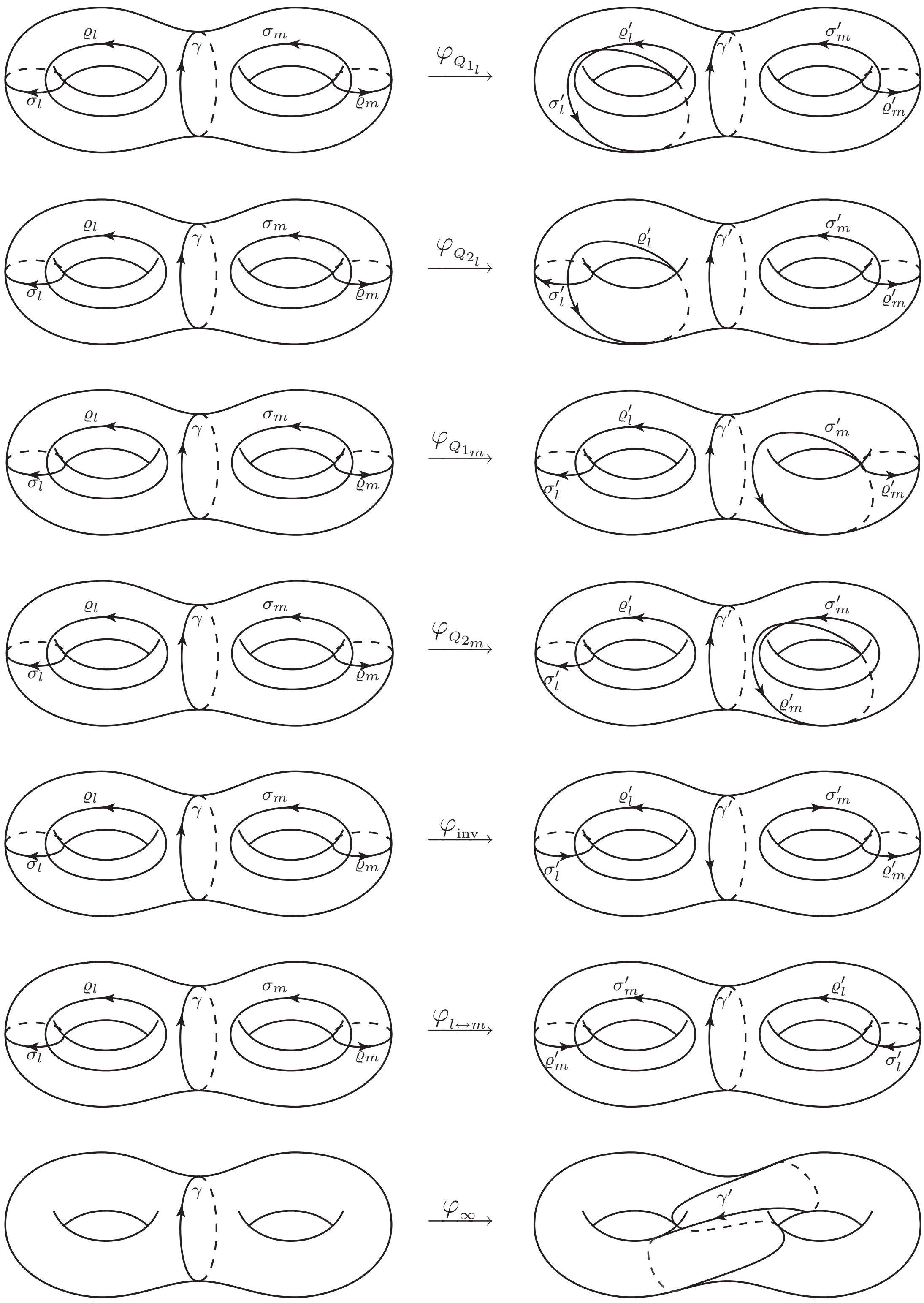
ϱ_l
σ_m
γ
σ_l
ϱ_m
$\varphi_{Q_{1_l}}$
$\varphi_{Q_{2_l}}$
$\varphi_{Q_{1_m}}$
$\varphi_{Q_{2_m}}$
φ_{inv}
$\varphi_{l\leftrightarrow m}$
φ_{∞}
ϱ'_l
σ'_m
γ'
σ'_l
ϱ'_m

Figure 4.4

4.2 Generating the Modular Groups

In order to prove Theorems 4.1, 4.2 and 4.3, it remains to show in each case that the elements of the modular group we exposed in the previous section generate the whole group. We will proceed in two steps; in this section we prove that the distinguished elements generate the subgroups of the modular groups that leave the dividing geodesic invariant, in the next section we prove that we can generate all dividing geodesics and thus accomplish the proof.

We first consider the subgroups of the modular groups of X and "Fish" pieces that have representatives that twist one Y piece along the dividing geodesic, fixing the other building block:

Lemma 4.18

(i) *Let $\mathcal{Y}_{ab}$ be the set of elements of the modular group of X pieces that have representatives that (pointwise) fix the Y piece whose border geodesics correspond to δ, ε and γ. Then, $\mathcal{Y}_{ab}$ is a cyclic subgroup of the modular group generated by $\varphi_{Y_{ab}}$.*

(ii) *Let $\mathcal{Y}_{de}$ be the set of elements of the modular group of X pieces that have representatives that (pointwise) fix the Y piece whose border geodesics correspond to α, β and γ. Then, $\mathcal{Y}_{de}$ is a cyclic subgroup of the modular group generated by $\varphi_{Y_{de}}$.*

(iii) *Let $\mathcal{Y}$ be the set of elements of the modular group of "Fish" pieces that have representatives that (pointwise) fix the Q piece whose border geodesic corresponds to γ. Then, $\mathcal{Y}$ is a cyclic subgroup of the modular group generated by φ_{Y}.*

Proof. We only prove (i); the proofs (ii) and (iii) are essentially the same (after renaming) because the nature of the fixed building block does not matter:

Let ψ be a generic element of the set and consider $\tilde{\psi}$, its action on the generators $(\alpha, \beta, \delta, \varepsilon) \in SL(2, \mathbb{R})^4$. As the "right" Y piece (whose border geodesics correspond to δ, ε and γ) is fixed, we can consider its action on $(\alpha, \beta) \in SL(2, \mathbb{R})^2$ only (δ and ε are fixed), i.e.

$$\tilde{\psi} : SL(2, \mathbb{R})^2 \longrightarrow SL(2, \mathbb{R})^2 : (\alpha, \beta) \longmapsto (\alpha', \beta'), \quad \text{where} \quad \alpha * \beta = \alpha' * \beta'.$$

Thus, these elements are stabilizers of γ and form therefore a subgroup of $Aut(\langle \alpha, \beta \rangle)$ and it only remains to show that the action of $\varphi_{Y_{ab}}$ defined by $\tilde{\varphi}_{Y_{ab}} : (\alpha, \beta) \longmapsto (\alpha * \beta * \alpha^{-1}, \alpha)$ generates the whole group.

As $\alpha * \beta = \tilde{\psi}(\alpha) * \tilde{\psi}(\beta)$, the action $\tilde{\psi}$ maps (α, β) to either one of the following:

(1) $(\alpha * \beta * g, g^{-1})$,

(2) $(\alpha * g, g^{-1} * \beta)$ or

(3) $(g, g^{-1} * \alpha * \beta)$,

where g is a length-reduced word in α and β.

Note that $\alpha' = \tilde{\psi}(\alpha)$ and $\beta' = \tilde{\psi}(\beta)$ must be conjugates of α and β because their corresponding geodesics together with the geodesic corresponding to γ must form the "left" Y piece[2] and thus $\{tr(\alpha), tr(\beta)\} = \{tr(\alpha'), tr(\beta')\}$, and the geodesics corresponding to α' and β' cannot change orientation (see Proposition 2.3, the orientation of the geodesic corresponding to γ is fixed).

Let us first consider the second case: either g starts with α^{-1} and we are in case (3) or it ends with α^{-1} (because $\alpha * g$ must be a conjugate to α or β). But in that case, g^{-1} starts with α and therefore ends with β^{-1} (because $g^{-1} * \beta$ is a conjugate to α or β) and we are in the case (1).

Let us now consider case (1) and prove the hypothesis by induction on the length of g: if $g = \beta^{-1}$, (α, β) maps to (α, β) and $\tilde{\psi} = id$, if $g = \alpha^{-1}$, (α, β) maps to $(\alpha * \beta * \alpha^{-1}, \alpha)$ and $\tilde{\psi} = \tilde{\varphi}_{Y_{ab}}$. Other one-letter words are not possible because $\alpha * \beta * g$ as well as g^{-1} must be conjugates to α or β. Suppose now that the length of g is $n > 1$ and that the hypothesis is true for any word strictly shorter than n. If g does not begin with β^{-1}, then $\tilde{\varphi}_{Y_{ab}}^{-1}(\alpha*\beta*g, g^{-1}) = (g^{-1}, g*\alpha*\beta*g*g^{-1}) = (\alpha*\beta*g', g'^{-1})$ with length of g' smaller than n because g must end with $\beta^{-1}*\alpha^{-1}$ and is a conjugate to α or β. If g does begin with β^{-1}, then it must begin with $\beta^{-1}*\alpha^{-1}$ because $\alpha*\beta*g$ and g^{-1} must both be conjugates to α or β. But in that case $\tilde{\varphi}_{Y_{ab}}(\alpha*\beta*g, g^{-1}) = (\alpha*\beta*g*g^{-1}*g^{-1}*\beta^{-1}*\alpha^{-1}) = (\alpha*\beta*g', g'^{-1})$ with length of g' smaller than n.

The proof of case (3) is analogous to the one of case (1). □

In order to show the analogous lemma for the subgroups that twist one Q piece along the dividing geodesic, (pointwise) fixing the other building block, we use the following proposition and a lemma based on it:

Proposition 4.19 *Let $\Gamma = \langle \varrho, \sigma \rangle$ be a free group on two generators. Then $Aut(\Gamma)$ is generated by the following Nielsen-transformations:*

$$\begin{array}{rcccc} k & : & (\varrho, \sigma) & \longmapsto & (\varrho * \sigma, \sigma) \\ l & : & (\varrho, \sigma) & \longmapsto & (\varrho^{-1}, \sigma) \\ m & : & (\varrho, \sigma) & \longmapsto & (\sigma, \varrho) \end{array}$$

Proof. See for instance [LS77, p.4-6]. □

Lemma 4.20 *Let $\Gamma = \langle \varrho, \sigma \rangle$ be a free group on two generators. Then*

$$F_{[\varrho,\sigma]} := \{g \in Aut(\Gamma) \mid g([\varrho, \sigma]) \textit{is a conjugate of } [\varrho, \sigma]\}$$

is a subgroup of $Aut(\Gamma)$ generated by k, mkm and ml, where $mkm = m \circ k \circ m$ and $ml = m \circ l$.

[2] If α' is a conjugate of α, β' must be a conjugate of β. If α' is a conjugate of β, β' must be a conjugate of α.

Proof. It is easy to see that $F_{[\varrho,\sigma]}$ is a subgroup of $Aut(\Gamma)$. We will prove that k, mkm and ml generate this subgroup by induction on the length of a word in $F_{[\delta,\epsilon]}$:

- We have $k([\varrho,\sigma]) = mkm([\varrho,\sigma]) = [\varrho,\sigma]$, $ml([\varrho,\sigma]) = \varrho^{-1} * [\varrho,\sigma] * \varrho$, $l([\varrho,\sigma]) = \varrho^{-1} * ([\varrho,\sigma])^{-1} * \varrho$ and $m([\varrho,\sigma]) = ([\varrho,\sigma])^{-1}$. Thus k, mkm and ml are elements of $F_{[\varrho,\sigma]}$ but m and l are not, as they map the commutator to its inverse.
- Assume now the hypothesis to be true for any length-reduced word in $F_{[\varrho,\sigma]}$ of length strictly smaller than n ($n > 1$). Let w be a length-reduced word in $F_{[\varrho,\sigma]}$ of length n. It starts with either $k^{\pm 1}$, $lk^{\pm 1}$, $mk^{\pm 1}$, lm or ml because $l^2 = m^2 = 1$. Thus, one of the words $k^{\mp 1}w$, $mk^{\mp 1}m \cdot ml \cdot w$, $mk^{\mp 1}m \cdot w$, $ml \cdot w$ or $lm \cdot w$ can be reduced to a word of length strictly smaller than n because $mk^{-1}m = (mkm)^{-1}$ and $lm = (ml)^{-1}$. Therefore we can generate w.

□

Lemma 4.21

(*i*) *Let $\mathcal{Q}$ be the set of elements of the modular group of "Fish" pieces that have representatives that (pointwise) fix the Y piece whose border geodesics correspond to α, β and γ. Then, $\mathcal{Q}$ is a subgroup of the modular group generated by φ_{Q_1} and φ_{Q_2}.*

(*ii*) *Let $\mathcal{Q}_m$ be the set of elements of the modular group of genus 2 surfaces that have representatives that (pointwise) fix the Q piece containing the geodesics corresponding to ϱ_l and σ_l and whose border geodesic corresponds to γ. Then, $\mathcal{Q}_m$ is a subgroup of the modular group generated generated by $\varphi_{Q_{1_m}}$ and $\varphi_{Q_{2_m}}$.*

(*iii*) *Let $\mathcal{Q}_l$ be the set of elements of the modular group of genus 2 surfaces that have representatives that (pointwise) fix the Q piece containing the geodesics corresponding to ϱ_m and σ_m and whose border geodesic corresponds to γ. Then, $\mathcal{Q}_l$ is a subgroup of the modular group generated generated by $\varphi_{Q_{1_l}}$ and $\varphi_{Q_{2_l}}$.*

Proof. Again, we only prove (i); the proofs (ii) and (iii) are essentially the same (after renaming):

Let ψ be a generic element of the set and consider $\tilde{\psi}$, its action on the generators $(\alpha,\beta,\varrho,\sigma) \in SL(2,\mathbb{R})^4$. As the Y piece (whose border geodesics correspond to α, β and γ) is fixed, we can consider its action on $(\varrho,\sigma) \in SL(2,\mathbb{R})^2$ only, i.e.

$$\tilde{\psi} : SL(2,\mathbb{R})^2 \longrightarrow SL(2,\mathbb{R})^2 : (\varrho,\sigma) \longmapsto (\varrho',\sigma'), \quad \text{where} \quad [\varrho,\sigma] = [\varrho',\sigma'].$$

Thus, these elements are stabilizers of γ and form therefore a subgroup of $Aut(\langle\varrho,\sigma\rangle)$. We also know that $\tilde{\psi} \in F_{[\varrho,\sigma]}$, which is generated by k, mkm and ml (see Proposition 4.19 and Lemma 4.20). Clearly, k and mkm correspond to φ_{Q_1} and φ_{Q_2}. Unfortunately $ml(\gamma) = \varrho^{-1}\gamma\varrho \neq \gamma$ and we have to consider its action on the generators as

$$(\alpha,\beta,\varrho,\sigma) \longmapsto (\varrho^{-1} * \alpha * \varrho, \varrho^{-1} * \beta * \varrho, \sigma, \varrho^{-1}),$$

if we want to fix the Y piece. This gives an element of the modular group acting on the Teichmüller space as follows:

$$(a,b,c,r,s,t,x,z) \longmapsto (a,b,c,s,r,2rs-t,-tr(\varrho^{-1}*\alpha*\varrho*\sigma),-tr(\varrho^{-1}*\beta*\varrho*\sigma)).$$

But this is the same transformation as $\varphi_{Q_1} \circ \varphi_{Q_2} \circ \varphi_{Q_1}^{-1} \circ \varphi_{Q_2}^2$.

Therefore φ_{Q_1} and φ_{Q_2} generate the subgroup of the modular group of the "Fish" piece that has representatives that fix the Y piece whose border geodesics correspond to α, β and γ. □

Proposition 4.22

(*i*) *Any element of the modular group of the X piece that has a representative leaving the dividing geodesic* γ *invariant can be generated by* $\varphi_{Y_{ab}}$, $\varphi_{Y_{de}}$, φ_{turn} *and* φ_{inv}.

(*ii*) *Any element of the modular group of the "Fish" piece that has a representative leaving the dividing geodesic* γ *invariant can be generated by* φ_{Y}, φ_{Q_1}, φ_{Q_2} *and* φ_{inv}.

(*iii*) *Any element of the modular group of the surface of genus* 2 *that has a representative leaving the dividing geodesic* γ *invariant can be generated by* $\varphi_{Q_{1_l}}$, $\varphi_{Q_{2_l}}$, $\varphi_{Q_{1m}}$, $\varphi_{Q_{2m}}$, $\varphi_{l\leftrightarrow m}$ *and* φ_{inv}.

Proof. We use Lemmas 4.18 and 4.21 to show that we can generate any element of the modular group that has a representative that leaves γ invariant but not necessarily its direction:

(*i*) By Lemma 4.18, we know that $\varphi_{Y_{ab}}$ and $\varphi_{Y_{de}}$ generate any element that fixes the dividing geodesic pointwise without exchanging sides (α stays to the left of γ). Using $\varphi_{\text{inv}} \circ \varphi_{\text{turn}}^2$, we can exchange the left and the right sides of the dividing geodesic that stays fixed. Finally, using φ_{inv} we can invert the direction of γ.

(*ii*) By Lemmas 4.18 and 4.21, we know that φ_{Y}, φ_{Q_1} and φ_{Q_2} generate any element that fixes the dividing geodesic pointwise. Using φ_{inv} we can reverse the direction of γ.

(*iii*) By Lemma 4.21, we know that $\varphi_{Q_{1_l}}$, $\varphi_{Q_{2_l}}$, $\varphi_{Q_{1m}}$ and $\varphi_{Q_{2m}}$ generate any element that fixes the dividing geodesic pointwise without exchanging sides (σ_l stays to the left of γ). Using $\varphi_{l\leftrightarrow m}$, we can exchange the left and the right sides of the dividing geodesic that stays fixed. Finally, using φ_{inv} we can reverse the direction of γ.

□

4.3 Dividing Geodesics

In this section we develop an algorithm reducing any dividing geodesic to a neighbor (see the following definition) of a smallest dividing geodesic through a series of shorter and shorter dividing geodesics, using the transformations of section 4.1. We then show that we can generate any neighbor of the dividing geodesic and thus finish the proof of Theorems 4.1, 4.2 and 4.3.

We will need some notions introduced in [Sem88] as well as some new ones concerning dividing geodesics:

Definition 4.23 *A geodesic segment without self-intersection on a building block (a Y piece or a Q piece) with endpoints on the same boundary component will be called an* arc.

Two non-intersecting arcs are said to be parallel *or* homotopic *if there exists a homeomorphism homotopic to the identity of the building block transforming one arc into the other.*

A road *is a simply connected domain on a building block between two adjacent parallel arcs.*

A square *is a connected domain on the building block that is not a road (if it is delimited by one geodesic arc and completely contains a border geodesic, it will be called a roundabout).*

A long-road *is a simply connected domain that is a series of roads (alternating on one of the building blocks) leading from a square to another.*

A neighbor *of a dividing geodesic μ is a dividing geodesic ν intersecting μ exactly four times ($|\mu \cap \nu| = 4$).*

In the following figure we illustrate these notions on an easy example:

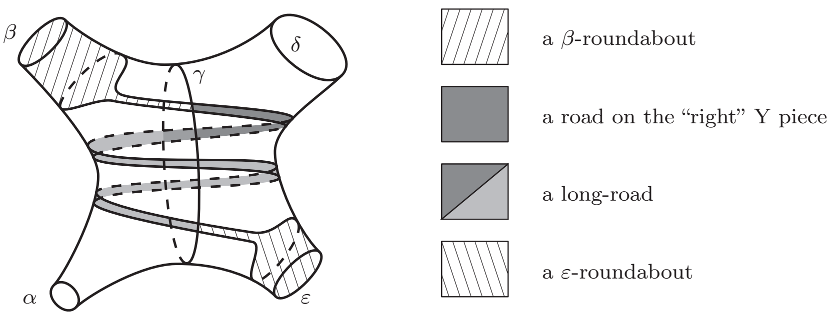

Figure 4.5

In order to prove that we can generate any dividing geodesic, we will show the following statements:

1. Any dividing geodesic η can be transformed by an element of the groups generated by the transformations of section 4.1 into a dividing geodesic that intersects the original dividing geodesic γ in a multiple of 4 points (Proposition 4.24).

2. If the surface is not an X piece, then there are geodesics that intersect neither η nor γ such that the surface obtained by cutting along these geodesics is an X piece (Proposition 4.25).

3. If η and γ intersect in a multiple of four points on an X piece, then there is a series of dividing geodesics $\eta = \eta_0, \eta_1, \ldots, \eta_{n+1} = \gamma$ such that η_i and η_{i+1} are neighbors (algorithm 4.27).

4. We can generate any neighbors of γ using an element of the groups generated by the transformations of section 4.1 (Proposition 4.29).

Proposition 4.24 *Let γ and η be two distinct dividing geodesics of the surface S obtained by joining two building blocks along γ. Then*

(i) $|\eta \cap \gamma| \mod 4 = 0$*, if S is a "Fish" piece or a genus 2 surface;*

(ii) either $|\eta \cap \gamma| \mod 4 = 0$*, or* $|\eta \cap \gamma| \mod 4 = 2$ *and we can generate φ such that* $|\varphi(\eta) \cap \gamma| \mod 4 = 0$*, if S is an X piece.*

Proof. As η is a dividing geodesic, we can color S in red and white such that the colors change along η. Each color defines a building block that is topologically equivalent to one of the original building blocks.

In case (i), this means that the red domain (a Q piece) is a square followed by a long-road leading to the same square. In case (ii), it means that the red domain (a Y piece) is an α-roundabout followed by a long-road leading to another roundabout.

A long road always has an even number of intersections with γ. Furthermore, in the case of a "Fish" piece or a genus 2 surface, $|\eta \cap \gamma| \mod 4 = 0$, as the long road returns to the same square on the same side of γ. In the case of an X piece, the β-roundabout is either red or white. If it is red, then $|\eta \cap \gamma| \mod 4 = 0$ (the long road returns to the same side of γ). If the δ-roundabout is red (this means that the β-roundabout is white), we apply $\varphi = \varphi_{\text{turn}}^{-1} \circ \varphi_{Y_{ab}}^{-1} \circ \varphi_{\text{turn}}$. If the ε-roundabout is red, we apply $\varphi = \varphi_{\text{turn}}^{-1}$. In the new configuration, the red roundabouts are on the same side of γ and thus $|\varphi(\eta) \cap \gamma| \mod 4 = 0$. □

Proposition 4.25 *Let γ and η be two distinct dividing geodesics of the surface S obtained by joining two building blocks along γ, where at least one building block is a Q piece. Then,*

(i) there is a simple closed geodesic δ that intersects neither γ nor η, if S is a "Fish" piece;

(ii) there are two simple closed geodesics α and δ (one on each building block) that intersect neither γ nor η, if S is a genus 2 surface.

Proof. As η is a dividing geodesic, we can color S in red and white such that the colors change along η and such that the red domain defines a Q piece. This domain is a square followed by a long-road leading to the same square. As any long-road is simply connected and as a Q piece is topologically equivalent to a wedge of two circles, the square cannot be simply connected and there has to be a simple closed geodesic δ that lies entirely in the square. But the square is part of the intersection of a building block with the red domain. Thus δ intersects neither γ nor η.

Obviously, this argument holds as well for the white domain if S is a genus 2 surface. Therefore, there is another simple closed geodesic α that intersects neither γ nor η. The geodesics α and δ do not intersect because α is part of the white domain and δ is part of the red domain. They are separated by γ (and therefore on distinct building blocks) because two simple closed geodesics on a Q piece different from the border geodesic intersect at least once. □

We have to show that on an X piece, any geodesic η intersecting γ in a multiple of four points is an iterated neighbor of γ. To do this, we introduce diagrams for long-roads leading from the α- to the β-roundabout corresponding to the geodesic η:

The border of the β-roundabout is an arc $\hat{\eta}$ of η and an arc $\hat{\gamma}$ of γ. Following γ (in its direction), we choose a point P that is at a small distance from $\hat{\gamma}$ such that there is no intersection of γ and η between $\hat{\gamma}$ and P. Now we punch a hole into the surface in P and deform it into a disc in $\mathbb{R}^2$ (whose border corresponds to P) with four holes (that correspond to α, β, δ and ε). We contract α, β, δ and ε to four points and deform the disc such that γ is a vertical segment oriented upward and such that the points corresponding to α, β, δ and ε form a parallelogram whose side (α, β) is vertical. Not tracing the border of the disc corresponding to P, we get the diagram (see Figure 4.6).

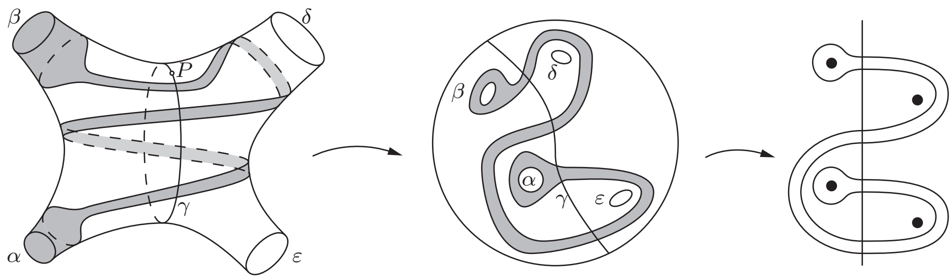

Figure 4.6

Remark 4.26 The choice of P induces that there are no roads enclosing the β-roundabout in the diagram (such that an arc of the border of the road that is an arc of η followed by an arc of $\gamma \setminus \{P\}$ is freely homotopic to β). But there may be several roads enclosing the α-, δ- and ε-roundabouts.

If there are no roads enclosing the α-roundabout, then η is a neighbor of γ.

Two diagrams are equal if they there is a homeomorphism leading from one to the other. This means that a diagram of a dividing geodesic η separating α and β from δ and ε is the same as the diagram of any multiple Dehn twist of η along γ or γ^{-1}.

The following algorithm acting on diagrams will show that any long-road η leading from the α- to the β-roundabout is an iterated neighbor of γ. In fact, at each step, it constructs a new long-road η' that is a neighbor of η such that $|\eta \cap \gamma| > |\eta' \cap \gamma|$; it stops when η has been reduced to a neighbor of γ:

Algorithm 4.27
While the current diagram is not a neighbor of γ, do the α-simplification step:

- *orient the long-road from the α- towards the β-roundabout,*
- *with the road just beside the one leaving the α-roundabout that is oriented in the same direction, do the following transformation:*

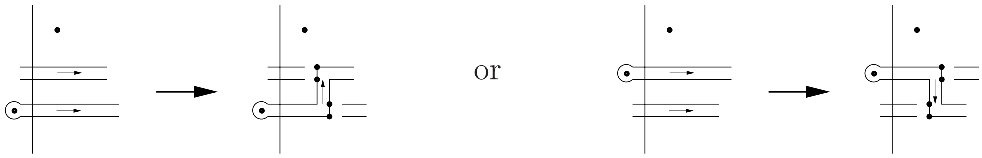

Figure 4.7

- *remove the roads that are no longer part of the new long-road from the α- to the β-roundabout;*

Proof. At each simplification step we pass to a neighbor of the current diagram (the four intersection points are emphasized), thus the final diagram is an iterated neighbor of the original one. At each simplification step the number of parallel roads decreases strictly because some are left out.

The α-simplification step is possible if there is a road that encloses the α-roundabout as in Figure 4.8 because in that case, this road can be oriented in either direction without prohibiting the simplification step.

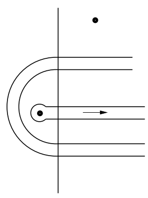

Figure 4.8

Thus, the only situations where the α-simplification step is not possible are the ones in Figure 4.9 where η is not a neighbor of γ and the road leaving the α-roundabout is either adjacent to the δ- or the ε-roundabout and the red road next to it has opposite direction, or there is no road enclosing the α-roundabout.

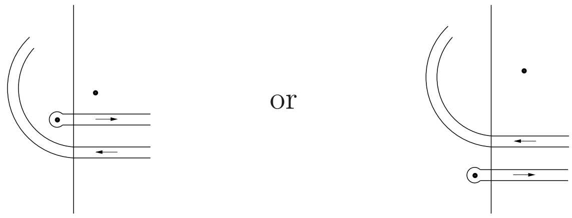

Figure 4.9

We will prove that these situations are impossible, finishing by that the proof of Proposition 4.22: If the road leaving the α-roundabout is either adjacent to the δ- or the ε-roundabout, then we must be in one of the situations of Figure 4.10 and the α-simplification is possible.

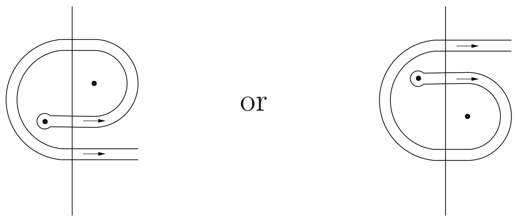

Figure 4.10

If there is no road enclosing the α-roundabout, η must be a neighbor because of the choice of the diagrams. □

This algorithm is not only an important step for the proof of Theorems 4.1, 4.2 and 4.3, it is also very useful to prove the following proposition:

Proposition 4.28 *If the geodesic γ divides a surface S into two building blocks and is shorter than its neighbors, then we have the following:*

1. *If S is an X piece, γ is a shortest geodesic dividing S into two Y pieces whose boundaries contain the border geodesics α, β and δ, ε respectively.*

2. *γ is a shortest dividing geodesic if S is a "Fish" piece or a genus 2 surface.*

Proof. Let η be any such dividing geodesic and γ shorter than its neighbors. Without loss of generality we can assume that S is an X piece, because otherwise there are geodesics that intersect neither η nor γ such that the surface obtained by cutting along these geodesics is an X piece (c.f. Proposition 4.25).

We have to prove that there is a length decreasing sequence of iterated neighbors leading from η to γ. In fact, using the algorithm, we have already constructed this sequence. It remains to prove that the α-simplification step is length decreasing.

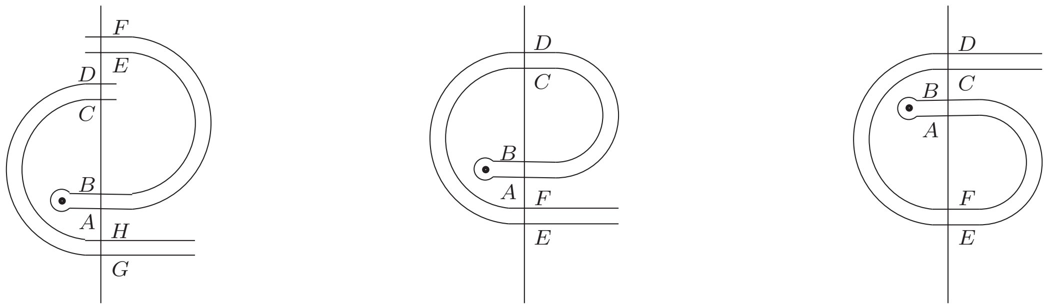

Figure 4.11

Take the arc of η that is part of the boundary of the α-roundabout and give its intersection with γ the names A and B. Following γ we get to the next intersections with η and name them C and D. We are thus in one of the three situations in the following figure:

In the first situation the names $ABCDEFGH$ come in that order on γ and in the order $ABECHGDF$ on η. We can now construct the curves γ' and η', longer[3] and freely homotopic to neighbors of γ and η:

- For γ', we will first stay on η from A to E, then follow γ^{-1} to C, η to H, γ^{-1} to F and finish on η to close the curve in A.
- For η', we follow γ from A to C, η^{-1} to E, γ to F, η^{-1} to H and finish on γ to close the curve in A. Note that this curve is freely homotopic to the geodesic obtained by the α-simplification step.

It remains to prove that the length of E to F plus the length of H to C on γ is strictly smaller than the length of F to E plus the length of C to H on η; i.e.

$$l(E \xrightarrow{\gamma} F) + l(H \xrightarrow{\gamma} C) < l(F \xrightarrow{\eta} E) + l(C \xrightarrow{\eta} H).$$

This implies that the length of η' is smaller than the length of η (and the α-simplification step is thus length decreasing).

Suppose now that $l(E \xrightarrow{\gamma} F) + l(H \xrightarrow{\gamma} C) \geq l(F \xrightarrow{\eta} E) + l(C \xrightarrow{\eta} H)$. This implies $l(\gamma') - l(\eta) \leq l(\gamma) - l(\eta)$ and thus $l(\gamma') \leq l(\gamma)$ which is in contradiction to the hypothesis that γ is shorter than its neighbors.

In the second situation the names $ABCDEF$ come in that order on γ and in the order $ABCFED$ on η. We can now construct the curves γ' and η':

- For γ', we follow η from A to F, then follow γ^{-1} to D and finish on η to close the curve in A.
- For η', we follow γ from A to D, η^{-1} to F and finish on γ to close the curve in A. Note that this curve is freely homotopic to the geodesic obtained by the α-simplification step.

[3] A closed non-geodesic curve on any Riemann surface is strictly longer than its freely homotopic geodesic (see e.g. [Bus92, p.18-23]).

Suppose that $l(F \xrightarrow{\gamma} D) \geq l(D \xrightarrow{\eta} F)$. This implies $l(\gamma') - l(\eta) \leq l(\gamma) - l(\eta)$ and thus $l(\gamma') \leq l(\gamma)$ which is in contradiction to the hypothesis that γ is shorter than its neighbors. Thus $l(F \xrightarrow{\gamma} D) < l(D \xrightarrow{\eta} F)$ and therefore $l(\eta') < l(\eta)$.

The proof for the third situation is exactly analogous to the one for the second (mirror situation). □

It remains to prove that we can generate any neighbor of γ:

Proposition 4.29 *If γ is a geodesic that divides a surface S into two building blocks, we can generate any neighbor of γ using the elements of the modular group given in section 4.1.*

Proof. We first prove that on an X piece, the neighbors of the dividing geodesic γ are of the form $\varphi_X \circ \varphi^n_{Y_{de}}(\gamma)$ where $n \in \mathbb{Z}$ and $\varphi_X := \varphi^{-1}_{\text{turn}} \circ \varphi_{Y_{de}} \circ \varphi_{\text{turn}} \circ \varphi^{-1}_{Y_{ab}} \circ \varphi_{\text{turn}}$.

There are only two possible diagrams for neighbors on an X piece:

Figure 4.12

These two diagrams can be obtained by

$$\varphi_X(\gamma) = -\delta^{-1} * \beta^{-1} * \delta * \alpha^{-1} \quad \text{and} \quad \varphi_X \circ \varphi^{-1}_{Y_{de}}(\gamma) = -\varepsilon^{-1} * \beta^{-1} * \varepsilon * \alpha^{-1},$$

but each diagram corresponds to more than only one geodesic. Indeed, if two geodesics correspond to the same diagram, there is a (multiple) Dehn twist along γ that brings one onto the other. Therefore, all neighbors are of the form $-\gamma^m*\delta^{-1}*\gamma^{-m}*\beta^{-1}*\gamma^m*\delta*\gamma^{-m}*\alpha^{-1}$ or $-\gamma^m * \varepsilon^{-1} * \gamma^{-m} * \beta^{-1} * \gamma^m * \varepsilon * \gamma^{-m} * \alpha^{-1}$ for some $m \in \mathbb{Z}$. But

$$\begin{aligned}\varphi_X \circ \varphi^{2m}_{Y_{de}}(\gamma) &= -\gamma^m * \varepsilon^{-1} * \gamma^{-m} * \beta^{-1} * \gamma^m * \varepsilon * \gamma^{-m} * \alpha^{-1} \quad \text{and} \\ \varphi_X \circ \varphi^{2m-1}_{Y_{de}}(\gamma) &= -\gamma^m * \varepsilon^{-1} * \gamma^{-m} * \beta^{-1} * \gamma^m * \varepsilon * \gamma^{-m} * \alpha^{-1}.\end{aligned}$$

It remains to prove the proposition in case S is a “Fish” piece or a genus 2 surface.

For any neighbor η, there are geodesics that intersect neither η nor γ such that the surface obtained by cutting along these geodesics is an X piece (Proposition 4.25). But we can generate any element of the modular group of the “Fish” piece or the surface of genus 2 that has a representative which leaves the dividing geodesic γ invariant (c.f. Proposition 4.22). Therefore we can assume that the geodesics along which we have to cut to obtain an X piece are ϱ in case of a “Fish” piece and σ_l, ϱ_m in case of a genus 2 surface. To finish the proof, it is thus enough to show that in both cases, we can generate elements that act on S in the same way as φ_X and $\varphi_{Y_{de}}$ act on the corresponding X pieces (fixing the geodesics along which we have to cut):

1. The case of a "Fish" piece:
   The geodesics α, β, δ and ε on the X piece correspond to the geodesics α, β, ϱ and $\sigma * \varrho^{-1} * \sigma^{-1}$ on the "Fish" piece. It is thus easy to verify that the action of φ_X and $\varphi_{Y_{de}}$ on α, β, δ and ε corresponds to the action of φ_{∞} and $\varphi_{Q_2} \circ \varphi_{Q_1}^{-1} \circ \varphi_{Q_2} \circ \varphi_{Q_1}^{-1} \circ \varphi_{Q_2}$ on α, β, ϱ and $\sigma * \varrho^{-1} * \sigma^{-1}$.

2. The case of a surface of genus 2:
   The geodesics α, β, δ and ε on the X piece correspond to the geodesics σ_l, $\varrho_l * \sigma_l^{-1} * \varrho_l^{-1}$, ϱ_m and $\sigma_m * \varrho_m^{-1} * \sigma_m^{-1}$ on the genus 2 surface. It is thus easy to verify that the action of φ_X and $\varphi_{Y_{de}}$ on α, β, δ and ε corresponds to the action of φ_{∞} and $\varphi_{Q_{2m}} \circ \varphi_{Q_{1m}}^{-1} \circ \varphi_{Q_{2m}} \circ \varphi_{Q_{1m}}^{-1} \circ \varphi_{Q_{2m}}$ on σ_l, $\varrho_l * \sigma_l^{-1} * \varrho_l^{-1}$, ϱ_m and $\sigma_m * \varrho_m^{-1} * \sigma_m^{-1}$.

□

Chapter 5

The Riemann Moduli Problem

The Riemann moduli problem is to describe the space of isomorphism classes of Riemann surfaces of a given signature which is known as the *moduli space*. In this chapter, we solve this problem for surfaces of signature $(0, 4)$, $(1, 2)$ and $(2, 0)$ giving precise fundamental domains for the action of the modular groups in the Teichmüller spaces.

It would also be interesting to investigate the boundary set of this fundamental domain and the elements of the modular group ("side pairing transformations") that map parts ("sides ") of it to other parts in order to get a manifold using Poincaré's Polyhedron Theorem. However, there is no immediate reason to believe that the fundamental domain is a polyhedron for the Teichmüller metric (or any other metric on Teichmüller space) and we will not treat this question any further.

Probably more important than knowing moduli space is answering the question whether two given points in the Teichmüller space correspond to isometric surfaces. To answer this question, we develop in this chapter three algorithms (one for each signature) leading to the point in moduli space corresponding to a given surface. Two surfaces will then be isometric if and only if the algorithm gives the same point in moduli space for both of them.

For each signature we proceed as follows:

First, we get a pre-fundamental domain for the action of the modular group on $\mathcal{T}$ using the trace of the dividing geodesic γ: For every element φ of the modular group we want $tr(\gamma) \leq tr(\varphi(\gamma))$ (this means that γ is a shortest dividing geodesic).

Then we use the elements that fix the trace of γ to get a standard situation for the other traces.

This gives a *infinite* but redundant set of inequalities between the coordinates of the Teichmüller space. Omitting redundant inequalities, we get a *finite* set of inequalities between the coordinates of the Teichmüller space that describe the moduli space.

The algorithm is then obtained by checking each inequality and, if one is violated, applying the right element of the modular group to the point in the Teichmüller space.

5.1 The X Piece

On a X piece, let γ be a shortest dividing geodesic and orient it arbitrarily. It cuts the X piece into two Y pieces. Take the Y piece that contains the shortest boundary geodesic of the X piece, name this geodesic α and its third boundary geodesic β and orient them such that $\alpha * \beta * \gamma = -1$. Let δ and ε be the boundary geodesics of the X piece we haven't yet named and orient them such that $\alpha * \beta * \delta * \varepsilon = 1$ (there are two possibilities to do this). Now we use $\varphi_{Y_{de}}$ or $\varphi_{Y_{de}}^{-1}$ repeatedly to "unwind" the X piece such that $x \leq cz - ad - be$ and $z \leq cx - ae - bd$. By Lemma 5.1, this situation gives a global minimum for $x + z$. To finish, we want to be in a unique situation and have $z \leq x$ which can be achieved by applying $\varphi_{\text{inv}} \circ \varphi_{Y_{ab}} : \mathcal{T}_{(0,4)} \longrightarrow \mathcal{T}_{(0,4)} : (a, b, c, d, e, x, z) \longmapsto (a, b, c, e, d, z, x)$ if needed.

Lemma 5.3 proves that $z \leq cx - ae - bd$ for any $z \leq x \leq cz - ad - be$. Thus it only remains to translate "shortest dividing geodesic" into terms of traces in order to get a fundamental domain.

Lemma 5.1 *Let $f : \mathcal{T}_{(0,4)} \to \mathbb{R} : (a, b, c, d, e, x, z) \mapsto x + z$ and $(a, b, c, d, e, x, z) \in \mathcal{T}_{(0,4)}$ such that $x \leq cz - ad - be$ and $z \leq cx - ae - bd$.*

Then $f(a, b, c, d, e, x, z) \leq f(\varphi_{Y_{de}}^n(a, b, c, d, e, x, z))$ for all $n \in \mathbb{Z}$.

Proof. $x \leq cz - ad - be$ and $z \leq cx - ae - bd$ imply that (a, b, c, d, e, x, z) is a local minimum of f under the action of $\varphi_{Y_{de}}$. It remains to prove that it is also a global minimum, i.e. that $\hat{g} : \mathbb{Z} \to \mathbb{R} : n \mapsto f(\varphi_{Y_{de}}^n(a, b, c, d, e, x, z))$ is a restriction of a convex function $g : \mathbb{R} \to \mathbb{R}$.

This is indeed the case as $\varphi_{Y_{de}}^n(a, b, c, d, e, x, z) = (a, b, c, d_n, e_n, x_n, z_n)$ can be written as

$$\begin{aligned}
x_n &= \tfrac{1}{c^2-1}\left(\sqrt{h(a,b,-c)h(c,d,-e)}\cosh(A(n+1)+B) + k_{n+1}\right)\\
z_n &= \tfrac{1}{c^2-1}\left(\sqrt{h(a,b,-c)h(c,d,-e)}\cosh(An+B) + k_n\right)\\
k_n &= c(ad_n + be_n) + (ae_n + bd_n)\\
(d_n, e_n) &= \begin{cases} (d,e) & \text{if } n \text{ even}\\ (e,d) & \text{if } n \text{ odd}\end{cases}
\end{aligned}$$

$$\begin{aligned}
\text{where } \cosh(A) &= c,\\
\cosh(B) &= \tfrac{z(c^2-1)-(c(ad+be)+(ae+bd))}{\sqrt{h(a,b,-c)h(c,d,-e)}}\\
\text{and } h(u,v,w) &= u^2 + v^2 + w^2 - 2uvw - 1.
\end{aligned}$$

This implies that g can be given as

$$t \mapsto \frac{\sqrt{h(a,b,-c)h(c,d,-e)}\,(\cosh(A(t+1)+B) + \cosh(At+B))}{c^2-1} + \frac{(a+b)(d+e)}{c-1},$$

which is a convex function in t. □

Remark 5.2 In [KR78] and [FR95], the authors investigate similar cases for two generator Fuchsian groups, showing the existence of a global minimum without explicitly giving a convex function.

Note also that the function g is the sum of two geodesic length functions (i.e. functions that associate the length to a geodesic according to its homotopy class under the action of a continuous transformation of the surface) along an earthquake path (i.e. continuous twisting along a simple closed geodesic). In [Ker83], Kerckhoff proves that the geodesic length function of a simple closed geodesic is convex along earthquake paths. Thus g is convex.

Lemma 5.3 *Let $(a,b,c,d,e,x,z) \in \mathcal{T}_{(0,4)}$ such that $z \leq x \leq cz - ad - be$.*
Then $z \leq cx - ae - bd$.

Proof. Fixing a, b, c, d and e, the Teichmüller space corresponds to a branch of a hyperbola (a connected component of the zero-set of the polynomial $Q_{(0,4)}$). The part of this hyperbola between the points of horizontal and vertical tangents corresponds to the set

$$\{(x,z) \in \mathbb{R} \mid x, z > 1,\ Q_{(0,4)} = 0,\ x \leq cz - ad - be,\ z \leq cx - ae - bd\}.$$

We show that the point of vertical tangent (x_v, z_v) is such that $z_v \geq x_v$, which proves the lemma: Substituting $z_v = cx_v - ae - bd$ and $x = x_v$ in the equation $Q_{(0,4)} = 0$ we get

$$\begin{aligned} z_v - x_v &= (c-1)x_v - ae - bd \\ &= \frac{(b-a)(e-d) + \sqrt{(a^2+b^2+c^2+2abc-1)(c^2+d^2+e^2+2cde-1)}}{c+1} \\ &> \frac{(b-a)(e-d) + \sqrt{(a^2+2ab+b^2)(d^2+2de+e^2)}}{c+1} = \frac{2(ad+be)}{c+1} > 0\,. \end{aligned}$$

□

Proposition 5.4 *Let $\alpha = (a+A), \beta = (b+B), \delta = (d+D)$ and $\varepsilon = (e+E)$ be the generators of an X piece, such that $a = \min\{a,b,d,e\}$ and $c \leq z \leq x \leq cz - ad - be$ where $c = -tr(\alpha * \beta)$, $x = -tr(\alpha * \delta)$ and $z = -tr(\beta * \delta)$.*
Then $\gamma = -\delta * \varepsilon$ is a shortest dividing geodesic.

Proof. We must show that γ is shorter than its iterated neighbors, ξ, ζ and their iterated neighbors (see section 4.3). We know that a dividing geodesic is shorter than its *iterated* neighbors if it is shorter than its neighbors (see Proposition 4.28).

We first prove that γ is shorter than its neighbor $\bar{\gamma} := \varphi_X(\gamma)$. For this we construct two curves ζ_1 and ζ_2 both homotopic to ζ using only arcs of γ and $\bar{\gamma}$:

- for ζ_1, we follow γ from A to B, then take $\bar{\gamma}$ to C, continue on γ back to B and finish on $\bar{\gamma}$ from B to A;

- for ζ_2, we follow γ from C to D, then take $\bar{\gamma}$ to A, continue on γ back to D and finish on $\bar{\gamma}$ from D to C.

Here, A, B, C, D are the intersections of γ and $\bar{\gamma}$ as in the following figure:

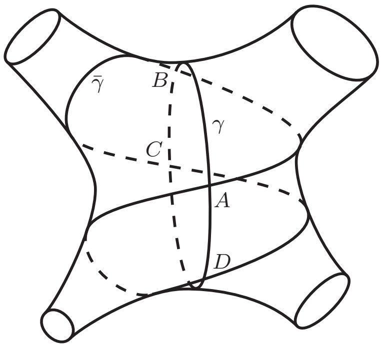

Figure 5.1

Using the argument that the geodesic is always the shortest curve in its homotopy class (see e.g. [Bus92, p.18-23]), we show that ζ is shorter than $\bar{\gamma}$ which implies that γ is shorter than $\bar{\gamma}$: $l(\gamma)+l(\bar{\gamma}) = l(\zeta_1)+l(\zeta_2) > 2l(\zeta)$. But $c \leq z$, therefore $l(\gamma) \leq l(\zeta)$, thus $l(\zeta) < l(\bar{\gamma})$ and finally $l(\gamma) < l(\bar{\gamma})$.

By Lemma 5.3 we know that $z \leq cx - ae - bd$ i.e. the X piece is unwound.

We now show that $\bar{\gamma}$ is the shortest neighbor of γ: Its trace is $2(xz - ab - de) - c$. All other neighbors are of the form $\varphi_X \circ \varphi^n_{Y_{de}}(\gamma)$ (see Proposition 4.29) and thus have traces $2(tr(\varphi^n_{Y_{de}}(\xi))tr(\varphi^n_{Y_{de}}(\zeta)) - ab - de) - c$. But as the X piece is unwound, $tr(\varphi^n_{Y_{de}}(\xi)) \geq x$ and $tr(\varphi^n_{Y_{de}}(\zeta)) \geq z$ for n even and $tr(\varphi^n_{Y_{de}}(\xi)) \geq z$ and $tr(\varphi^n_{Y_{de}}(\zeta)) \geq x$ for n odd.

We prove that ζ is shorter than its neighbors: as in Proposition 4.29, we can write the neighbors of ζ as $\varphi_{\text{turn}} \circ \varphi_X \circ \varphi^n_{Y_{de}} \circ \varphi^{-1}_{\text{turn}}(\zeta)$ and show that the geodesic $\varphi_{\text{turn}} \circ \varphi_X \circ \varphi^{-1}_{Y_{de}} \circ \varphi^{-1}_{\text{turn}}(\zeta)$ is a shortest neighbor of ζ if $c \leq xz - ab - de$ and $x \leq cz - ad - be$. But $tr(\varphi_{\text{turn}} \circ \varphi_X \circ \varphi^{-1}_{Y_{de}} \circ \varphi^{-1}_{\text{turn}}(\zeta)) = 2(cx - ae - bd) - z \geq z$.

The proof that ξ is shorter than its neighbors if $c \leq xz - ab - de$ and $z \leq cx - ae - bd$ is analogous, using the following shortest neighbor of ξ: $\varphi^{-1}_{Y_{ab}} \circ \varphi_{\text{inv}} \circ \varphi_{\text{turn}} \circ \varphi_X \circ \varphi^{-1}_{Y_{de}} \circ \varphi^{-1}_{\text{turn}} \circ \varphi_{\text{inv}} \circ \varphi_{Y_{ab}}(\xi)$ with trace $2(cz - ad - be) - x \geq x$. $\square$

This leads us to the fundamental domain $\mathcal{F}_{(0,4)}$ for the action of the modular group on the Teichmüller space $\mathcal{T}_{(0,4)}$ that can be formulated as follows:

Theorem 5.5 *The set of isometry classes of surfaces of signature* $(0,4)$ *is in a 1-1-correspondence with the set*

$$\mathcal{F}_{(0,4)} := \Big\{(a,b,c,d,e,x,z) \in \mathbb{R}^7 \;\Big|\; \begin{array}{l} 1 < a \leq \min\{b,d,e\}, \\ 1 < c \leq z \leq x \leq cz - ad - be \\ \text{and } Q_{(0,4)} = 0 \end{array}\Big\},$$

where $Q_{(0,4)}$ *is the following polynomial:*

$$\begin{aligned} Q_{(0,4)} \;\; := \;\; & a^2 + b^2 + c^2 + d^2 + e^2 + x^2 + z^2 + 4abde - 1 \\ & +2c(ab+de) + 2x(ad+be) + 2z(ae+bd) - 2cxz. \end{aligned}$$

Proof. Direct deduction from Propositions 4.22 and 5.4 together with Theorem 3.8. □

Knowing the moduli space $\mathcal{F}_{(0,4)}$, we can now give an algorithm leading from a point in the Teichmüller space to a point in $\mathcal{F}_{(0,4)}$ corresponding to the same surface:

Algorithm 5.6

1. *If* $\min\{a,b,d,e\} \notin \{a,b\}$, *apply* φ^2_{turn} *to the current point.*
   Go to step 2.

2. *If* $b > a$, *apply* $\varphi_{Y_{ab}}$ *to the current point.*
   Go to step 3.

3. *Unwind the X piece, using* $\varphi_{Y_{de}}$ *or* $\varphi^{-1}_{Y_{de}}$ *repeatedly until* $x \leq cz - ad - be$ *and* $z \leq cx - ae - bd$.
   Go to step 4.

4. *If* $z > x$, *apply* $\varphi_{\text{inv}} \circ \varphi_{Y_{ab}}$ *to the current point.*
   Go to step 5.

5. *If* $c > z$, *apply* $\varphi_{Y_{ab}} \circ \varphi^{-1}_{\text{turn}}$ *to the current point, then go to step 3.*
   Else STOP.

Proof. It is easy to see that if this algorithm stops, it will lead to a point in the moduli space. It remains to show that it does indeed stop.

In step 5, the algorithm either stops or we exchange the distinguished dividing geodesic γ with one that is strictly shorter. In the other steps, the dividing geodesic stays the same. As the number of closed geodesics of lengths smaller than $l(\gamma)$ is finite (see e.g. [Bus92, p.162]), the algorithm has to stop. □

5.2 The "Fish" Piece

As before, let γ be a shortest dividing geodesic and orient it arbitrarily. It cuts the "Fish" piece into a Y piece and a Q piece. Let $\varrho = (r + R)$ be the shortest simple (without self-intersections) closed geodesic on the Q piece, $\sigma = (s + S)$ the second to shortest, oriented such that $\gamma = -[\varrho, \sigma]$ and $\tau = (\varrho * \sigma)^{-1} = (t + T)$, which is equivalent to $1 < r \leq s \leq t \leq 2rs - s$ (cf. [BS88, Sem88]). Let us now consider the geodesics ξ and ζ: without changing the generators of the Q piece we can "unwind" ξ and ζ by twisting along γ using φ_Y or φ_Y^{-1} repeatedly until $x \leq cz - r(a+b)$ and $z \leq cx - r(a+b)$. By Lemma 5.7, this situation gives a global minimum for $x + z$. Using $\varphi_{\text{inv}} \circ \varphi_Y : (a, b, c, r, s, t, x, z) \longmapsto (a, b, c, r, s, 2rs - t, z, x)$ (if needed) the inequality $z \leq x$ is obtained[1] which leads to the inequality $1 < z \leq x \leq cz - r(a + b)$.

Combining $\varphi_{\text{wind}} := \varphi_{Q_1}^{-1} \circ \varphi_{Q_2}^2 \circ \varphi_{Q_1}^{-1} \circ \varphi_{Q_2}^2$ with φ_Y^{-1}, we can demand that $a \leq b$ because $\varphi_Y^{-1} \circ \varphi_{\text{wind}}(a, b, c, r, s, t, x, z) = (b, a, c, r, s, t, x, z)$. We end up in a unique situation which corresponds to a point in the fundamental domain.

It remains to compare γ to its neighbors and to exclude some of the resulting inequalities in order to get a finite set.

Lemma 5.7 *Let* $f : \mathcal{T}_{(1,2)} \to \mathbb{R} : (a, b, c, r, s, t, x, z) \mapsto x + z$. *Then* (a, b, c, r, s, t, x, z) *in* $\mathcal{T}_{(1,2)}$ *is a global minimum of* f *under the action of* φ_Y *iff* $x \leq cz - r(a + b)$ *and* $z \leq cx - r(a + b)$.

Proof. As f is the sum of two geodesic length functions, it is convex along earthquake paths (see [Ker83]). The action of φ_Y is the restriction of a continuous twist along γ (an earthquake path) to half twists. $x \leq cz - d(a + b)$ and $z \leq cx - d(a + b)$ imply that (a, b, c, d, e, f, x, z) is a local minimum of f under the action of φ_Y; it is therefore also a global minimum. □

In order to compare γ only to a finite number of neighbors, we introduce border-to-border paths associated to dividing geodesics:

Definition 5.8 *A geodesic segment without self-intersections on a "Fish" piece with one endpoint on the border geodesic* α *and the other endpoint on the border geodesic* β *perpendicular to both border geodesics is called* the border-to-border path associated to the dividing geodesic γ *if the segment does not intersect* γ.

Remark 5.9 Using hyperbolic trigonometry, it is easy to show that the border-to-border path π associated to a dividing geodesic γ is unique and that its length $l(\pi)$ has the property $p = \cosh(l(\pi)) = \frac{c+ab}{\sqrt{(a^2-1)(b^2-1)}}$ (see e.g. [Bus92, p.454]). Therefore, the shortest border-to-border path is associated to the shortest dividing geodesic and a border-to-border path π

[1]Instead of this inequality which is consistent with the fundamental domain for X pieces, we could have chosen $t \leq 2rs - t$ which leads to $1 < r \leq s \leq t \leq rs$. Parameters r, s and t satisfying this relation would then be in a 1-1-correspondence to the set of geometrically distinct Q pieces (cf. [BS88, Sem88]).

is shortest if it is shorter than any other border-to-border path (associated to a neighbor of γ) not intersecting π.

Lemma 5.10 *On a "Fish" piece let ϱ be a non-dividing simple closed geodesic and π a border-to-border path intersecting ϱ at least twice and name two consecutive (following π) intersections A and B.*

Let π' and ϱ' be the piecewise geodesic curves obtained by interchanging the segments from A to B on π and on ϱ (for π' follow π up to A then follow ϱ up to B and finish on π; for ϱ' follow ϱ from B to A then π up to B).

If the geodesic $\tilde{\varrho}$ homotopic to ϱ' is longer than ϱ, then the border-to-border path $\tilde{\pi}$ homotopic to π' is shorter than π and $|\tilde{\pi} \cap \varrho| = |\pi \cap \varrho| - 1$.

Proof. Note that given π' and ϱ', $\tilde{\pi}$ and $\tilde{\varrho}$ are unique and strictly shorter than π' and ϱ' (see e.g. [Bus92, p.18-23]).

As $\tilde{\varrho}$ is longer than ϱ, ϱ' must also be longer than ϱ and thus the segment from A to B on π is longer than the one on ϱ (i.e. $l(A \xrightarrow{\pi} B) > l(A \xrightarrow{\varrho} B)$). Therefore π' and thus also $\tilde{\pi}$ are shorter than π.

$|\tilde{\pi} \cap \varrho| = |\pi \cap \varrho| - 1$ is obvious from the construction. □

Proposition 5.11 *$(a, b, c, r, s, t, x, z) \in \mathcal{T}_{(1,2)}$ with $r \le s \le t \le 2rs - s$, $c \le xz - ab - r^2$, $x \le cz - r(a + b)$ and $z \le cx - r(a + b)$ corresponds to a "Fish" piece where a shortest border-to-border path associated to a neighbor of γ or associated to γ itself is either the border-to-border path π associated to γ or a border-to-border path intersecting ϱ once and not intersecting σ more than once.*

Proof. We know that $r \le s \le t \le 2rs - s$ means that ϱ and σ are the shortest and second to shortest geodesics on the Q piece of border γ (cf. [BS88, Sem88]).

We first consider border-to-border paths that do not intersect ϱ associated to neighbors of γ, i.e. not intersecting π. They are half twists of π along ζ and ζ^{-1} and therefore associated to $\varphi_{\infty}(\varphi_Y^{-1}(\gamma)) = -(\beta^{-1} * \alpha * \beta * \varrho * \beta * \varrho^{-1})^{-1}$ and $\varphi_{\infty}(\gamma) = -(\beta * \varrho * \alpha * \varrho^{-1})^{-1}$. But we know that $tr(\varphi_{\infty}(\varphi_Y^{-1}(\gamma))) = 2((2(cz - r(a + b)) - x)z - ab - r^2) - c \ge \varphi_{\infty}(\gamma) = 2(xz - ab - r^2) - c \ge tr(\gamma) = c$; thus the associated paths are longer than π.

Consider now the shortest border-to-border path ψ associated to a neighbor of γ intersecting ϱ more than once and name two consecutive intersections A and B. As this path is associated to a neighbor of γ, its segment between A and B will not intersect γ and is thus completely on the Q piece. We can now apply the previous lemma and construct the border-to-border path and the closed geodesic $\tilde{\varrho}$. $\tilde{\psi}$ is shorter than ψ because we know that ϱ' and thus $\tilde{\varrho}$ are on the Q piece and therefore are longer than ϱ. Furthermore, $|\tilde{\psi} \cap \varrho| = |\psi \cap \varrho| - 1$ and thus the shortest border-to-border path does not intersect ϱ more than once.

Let now ψ be a border-to-border path intersecting ϱ once and intersecting σ consecutively in A and B. The closed geodesic $\tilde{\sigma}$ constructed by the previous lemma is entirely on the Q piece and intersects ϱ once (because both σ and ψ intersect ϱ exactly once).

$\tilde{\sigma}$ is thus neither equal to ϱ nor equal to σ (nor their inverses) and thus longer than σ. By the previous lemma, there is a border-to-border path $\tilde{\psi}$ shorter than ψ such that $|\tilde{\psi} \cap \varrho| = |\psi \cap \varrho| - 1$.

The shortest border-to-border path is therefore either π or a border-to-border path intersecting ϱ once and not intersecting σ more than once. □

This means that in order to prove that γ is the shortest dividing geodesic, it is enough to require that $r \leq s \leq t \leq 2rs - s$, $c \leq xz - ab - r^2$, $x \leq cz - r(a+b)$ and $z \leq cx - r(a+b)$ and that γ is shorter than any neighbor intersecting ϱ twice and intersecting σ at most twice (as it is dividing, it cannot intersect σ only once).

Lemma 5.12 *Let* $(a, b, c, r, s, t, x, z) \in \mathcal{T}_{(1,2)}$ *with* $r \leq s \leq t \leq 2rs - s$, $c \leq xz - ab - r^2$, $x \leq cz - r(a+b)$ *and* $z \leq cx - r(a+b)$.

Then, a shortest neighbor of γ *that intersects* ϱ *twice and does not intersect* σ *is*

$$\begin{array}{lll} \textit{either} & \varphi_{\infty} \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma) & \textit{if } \frac{t}{2rs-t} \leq \frac{cx-z-r(a+b)}{cz-x-r(a+b)} \\ \textit{or} & \varphi_{\infty} \circ \varphi_Y \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma) & \textit{otherwise.} \end{array}$$

Proof. Obviously, $\varphi_{\infty} \circ \varphi_Y \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma) = -(\alpha * \sigma^{-1} * \alpha * \beta * \alpha^{-1} * \sigma)^{-1}$ and $\varphi_{\infty} \circ \varphi_Y \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma) = -(\beta * \sigma^{-1} * \alpha * \sigma)^{-1}$ are neighbors that intersect the geodesic ϱ twice and do not intersect the geodesic σ. All other such geodesics are multiple Dehn twists of these two and thus of the form $-(\gamma^{-m} * \alpha * \gamma^m * \sigma^{-1} * \gamma^{-m} * \alpha * \beta * \alpha^{-1} * \gamma^{-m}\sigma)^{-1}$ and $-(\gamma^{-m} * \beta * \gamma^m * \sigma^{-1} * \gamma^{-m} * \alpha * \gamma^{-m}\sigma)^{-1}$ for some $m \in \mathbb{Z}$.

But $\varphi_{\infty} \circ \varphi_Y^{2m+1} \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma) = -(\gamma^{-m} * \alpha * \gamma^m * \sigma^{-1} * \gamma^{-m} * \alpha * \beta * \alpha^{-1} * \gamma^{-m}\sigma)^{-1}$ and $\varphi_{\infty} \circ \varphi_Y^{2m} \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma) = -(\gamma^{-m} * \beta * \gamma^m * \sigma^{-1} * \gamma^{-m} * \alpha * \gamma^{-m}\sigma)^{-1}$. Therefore, it is enough to show that among the geodesics of the form $\varphi_{\infty} \circ \varphi_Y^n \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma)$ with $n \in \mathbb{Z}$, the shortest is $\varphi_{\infty} \circ \varphi_Y \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma)$ or $\varphi_{\infty} \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma)$. As $tr(\varphi_{\infty} \circ \varphi_Y^n \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(\gamma)) = 2(\bar{x}\bar{z} - ab - s^2) - c$ for $\varphi_Y^n \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(a, b, c, r, s, t, x, z) = (\bar{a}, \bar{b}, c, s, t, r, \bar{x}, \bar{z})$ with $\{\bar{a}, \bar{b}\} = \{a, b\}$, it is enough to show that the corresponding "Fish" piece is unwound (i.e. that $\bar{x}\bar{z}$ is globally minimal) for either $n = 0$ or $n = 1$. But using the argument of Kerckhoff (cf. Lemma 5.7) and the fact that $\bar{x} > 0$, $\bar{z} > 0$, we know that the function $\bar{x}\bar{z}$ is convex along earthquake paths; thus we only have to prove that there is a local minimum for $n = 0$ or $n = 1$:

Case $\frac{t}{2rs-t} \leq \frac{cx-z-r(a+b)}{cz-x-r(a+b)}$:

$$\begin{aligned} c\bar{x}_0 - \bar{z}_0 - s(a+b) &= \tfrac{1}{2r^2+c-1}\big((cx - z - r(a+b))(2rs - t) - (cz - x - r(a+b))t\big) \geq 0 \\ c\bar{z}_0 - \bar{x}_0 - s(a+b) &= \tfrac{1}{2r^2+c-1}\big((cz - x - r(a+b))(2rs - t) \\ &\quad + (c\underbrace{(2cz - x - 2r(a+b))}_{\geq x} - z - r(a+b))t\big) \geq 0 \end{aligned}$$

$$
\begin{array}{rcl}
\text{Case } \frac{t}{2rs-t} > \frac{cx-z-r(a+b)}{cz-x-r(a+b)} : & & \\
c\bar{x}_1 - \bar{z}_1 - s(a+b) & = & c\bar{x}_0 - (2c\bar{x}_0 - \bar{z}_0 - 2s(a+b)) - s(a+b) \\
& = & -(c\bar{x}_0 - \bar{z}_0 - s(a+b)) \ > \ 0 \\
c\bar{z}_1 - \bar{x}_1 - s(a+b) & = & \frac{1}{2r^2+c-1}\big((cx-z-r(a+b))t \\
& & \quad + (c\underbrace{(2cx-z-2r(a+b))}_{\geq z} - x - r(a+b))(2rs-t)\big) \ \geq \ 0
\end{array}
$$

$$
\begin{array}{ll}
\text{where} & \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(a,b,c,r,s,t,x,z) = (a,b,c,s,t,r,\bar{x}_0,\bar{z}_0) \\
\text{and} & \varphi_Y \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(a,b,c,r,s,t,x,z) = (b,a,c,s,t,r,\bar{x}_1,\bar{z}_1).
\end{array}
$$

□

Lemma 5.13 $(a,b,c,r,s,t,x,z) \in \mathcal{T}_{(1,2)}$ *with* $r \leq s \leq t \leq 2rs - s$, $c \leq xz - ab - r^2$, $x \leq cz - r(a+b)$ *and* $z \leq cx - r(a+b)$ *corresponds to a “Fish” piece where the shortest neighbor of* γ *that intersects exactly twice each one of the geodesics* ϱ *and* σ *is one of the two following:*

$$
\textit{not intersecting } \tau = (\varrho * \sigma)^{-1} : \quad \begin{cases} \varphi_{\infty} \circ \varphi_Y^{-1} \circ \varphi_{Q_2}(\gamma) & \textit{if } \frac{2rt-s}{s} \leq \frac{cx-z-r(a+b)}{cz-x-r(a+b)} \\ \varphi_{\infty} \circ \varphi_{Q_2}(\gamma) & \textit{otherwise.} \end{cases}
$$

$$
\textit{intersecting } \tau : \quad \begin{cases} \varphi_{\infty} \circ \varphi_{Q_2}^{-1}(\gamma) & \textit{if } \frac{s}{2r(2rs-t)-s} \leq \frac{cx-z-r(a+b)}{cz-x-r(a+b)} \\ \varphi_{\infty} \circ \varphi_Y \circ \varphi_{Q_2}^{-1}(\gamma) & \textit{otherwise.} \end{cases}
$$

Proof. Analogous to proof of Lemma 5.12. □

This now leads to the following theorem, giving the fundamental domain:

Theorem 5.14 *The set of isometry classes of surfaces of signature* $(1,2)$ *is in a 1-1-correspondence with the set*

$$
\mathcal{F}_{(1,2)} := \left\{ (a,b,c,r,s,t,x,z) \in \mathbb{R}^8 \;\middle|\; \begin{array}{l} -(c+1) = 2(r^2+s^2+t^2-2rst-1), Q_{(1,2)} = 0, \\ 1 < a \leq b, \ 1 < r \leq s \leq t \leq 2rs - s, \\ 1 < z \leq x \leq cz - r(a+b), \\ 1 < c \leq xz - ab - r^2, \\ c \leq x_s z_s - ab - s^2, \\ c \leq x_t z_t - ab - t^2, \\ c \leq x_{\bar{t}} z_{\bar{t}} - ab - \bar{t}^2 \end{array} \right\}.
$$

Here,

$$
Q_{(1,2)} := a^2 + b^2 + c^2 + 2abc - 1 + x^2 + z^2 - 2cxz + 2r(a+b)(x+z) + 2r^2(1+c+2ab),
$$

$$
\bar{t} := 2rs - t
$$

and the transformations

$$\varphi_v(a,b,c,r,s,t,x,z) = (a,b,c,v,*,*,x_v,z_v)$$

are given by

$$\varphi_s := \begin{cases} \varphi_{Q_2}^{-1} \circ \varphi_{Q_1} & \text{if } \frac{t}{2rs-t} \leq \frac{cx-z-r(a+b)}{cz-x-r(a+b)}, \\ \varphi_Y \circ \varphi_{Q_2}^{-1} \circ \varphi_{Q_1} & \text{if not;} \end{cases}$$

$$\varphi_t := \begin{cases} \varphi_Y^{-1} \circ \varphi_{Q_2} & \text{if } \frac{2rt-s}{s} \leq \frac{cx-z-r(a+b)}{cz-x-r(a+b)}, \\ \varphi_{Q_2} & \text{otherwise;} \end{cases}$$

$$\varphi_{\bar{t}} := \varphi_{Q_2}^{-1}.$$

Proof. Using Theorem 3.14, Proposition 4.22 and the inequalities $1 < a \leq b$, $1 < r \leq s \leq t \leq 2rs - s$ and $1 < z \leq x \leq cz - r(a+b)$, it remains to prove that γ is shorter than its neighbors for any surface in $\mathcal{F}_{(1,2)}$.

As $c \leq xz - ab - r^2$, we know that γ is shorter than its neighbors that do not intersect ϱ (see Proposition 5.11).

As $\frac{s}{2r(2rs-t)-s} \leq 1 \leq \frac{cx-z-r(a+b)}{cz-x-r(a+b)}$, we conclude using Lemmas 5.12 and 5.13. □

As before, knowing the moduli space $\mathcal{F}_{(1,2)}$, we can give an algorithm (using the same notations as in Theorem 5.14) leading from a point in the Teichmüller space to a point in $\mathcal{F}_{(1,2)}$ corresponding to the same surface:

Algorithm 5.15

1. (a) *Obtain* $r = \min\{r,s,t\}$*, using either* $\varphi_{Q_2}^{-1} \circ \varphi_{Q_1}$ *or* $\varphi_{Q_1}^{-1} \circ \varphi_{Q_2}$ *if needed*[2].
   Go to (b).

   (b) *If* $t > 2rs - t$*, apply* $\varphi_{Q_1}^{-1}$ *to the current point and return to (a).*
   Else go to (c).

   (c) *If* $s > 2rt - s$*, apply* φ_{Q_1} *to the current point and return to (a).*
   Else go to (d).

   (d) *If* $s > t$*, apply* φ_{Q_1} *to the current point.*

   Go to step 2.

2. *Unwind the "Fish" piece, using* φ_Y *or* φ_Y^{-1} *repeatedly until* $x \leq cz - r(a+b)$ *and* $z \leq cx - r(a+b)$*.*
   Go to step 3.

3. *If* $z > x$*, apply* $\varphi_{\text{inv}} \circ \varphi_Y$ *to the current point.*
   Go to step 4.

[2] $\varphi_{Q_2}^{-1} \circ \varphi_{Q_1}(a,b,c,r,s,t,x,z) = (a,b,c,s,t,r,*,*)$ and $\varphi_{Q_1}^{-1} \circ \varphi_{Q_2}(a,b,c,r,s,t,x,z) = (a,b,c,t,r,s,*,*)$.

4. *If $a > b$, apply $\varphi_Y^{-1} \circ \varphi_{\text{wind}}$ to the current point.*
   Go to step 5.

5. (a) *If $c > xz - ab - r^2$, apply φ_{∞} to the current point and go to step 1.*
   Else go to (b).

   (b) *If $c > x_s z_s - ab - s^2$, apply $\varphi_{\infty} \circ \varphi_s$ to the current point and go to step 1.*
   Else go to (c).

   (c) *If $c > x_t z_t - ab - t^2$, apply $\varphi_{\infty} \circ \varphi_t$ to the current point and go to step 1.*
   Else go to (d).

   (d) *If $c > x_{\bar{t}} z_{\bar{t}} - ab - \bar{t}^2$, apply $\varphi_{\infty} \circ \varphi_{\bar{t}}$ to the current point and go to step 1.*
   Else STOP.

Proof. We have to show that the algorithm is finite and that the point obtained is really in $\mathcal{F}_{(1,2)}$.

We first show that the point obtained is in $\mathcal{F}_{(1,2)}$ if the algorithm stops:

- Step 1 gives the inequalities $1 < r \leq s \leq t \leq 2rs - s$.
  Indeed, when the algorithm gets to (d), we have the inequalities $1 < r \leq \min\{s, t\}$, $s \leq 2rt - s$ and $t \leq 2rs - t$. If $s \leq t$ we are done, otherwise applying φ_{Q_1} means setting $s' = t$ and $t' = 2rt - s$. But in this case, s' is certainly smaller than t' and $t' = 2rt - s \leq 2rt - t = 2rs' - s'$.

- It is easy to see that steps 2 and 3 lead to a "Fish" piece where the inequalities $1 < z \leq x \leq cz - r(a + b)$ hold.

- By step 4 we get $1 < a \leq b$.

- If the algorithm comes to a stop in step 5, we know that the inequalities $1 < c \leq xz - ab - r^2$, $c \leq x_s z_s - ab - s^2$, $c \leq x_t z_t - ab - t^2$ and $c \leq x_{\bar{t}} z_{\bar{t}} - ab - \bar{t}^2$ are satisfied and the point obtained is in $\mathcal{F}_{(1,2)}$.

We now prove that the algorithm is finite:

- It is not possible that the algorithm loops inside step 1 because every time we return to (a), $r + s + t$ is strictly smaller than before and there are only finitely many geodesics shorter than the maximum of the lengths of the initial ϱ, σ and τ.

- For the same reason, steps 2 to 5 will only be used finitely many times because in step 5 the algorithm either stops or strictly diminishes the length of the dividing geodesic γ before returning to step 1.

□

5.3 The Genus 2 Surface

Again, let γ be a shortest dividing geodesic and orient it arbitrarily. It cuts the surface into two Q pieces. Let $\varrho_l = (r_l + R_l)$ be the shortest simple closed geodesic on one of the Q pieces, $\sigma_l = (s_l + S_l)$ the second to shortest on the same Q piece, oriented such that $\gamma = -[\varrho_l, \sigma_l]$ and $\tau_l = (\varrho_l * \sigma_l)^{-1} = (t_l + T_l)$, which is equivalent to $1 < r_l \le s_l \le t_l \le 2r_l s_l - s_l$ (see [BS88, Sem88]). Do the same on the other Q piece; i.e. let $\varrho_m = (r_m + R_m)$ and $\sigma_m = (s_m + S_m)$ be the shortest and second to shortest simple closed geodesics, oriented such that $\gamma = -[\varrho_m, \sigma_m]$ and $\tau_m = (\varrho_m * \sigma_m)^{-1} = (t_m + T_m)$, which is equivalent to $1 < r_m \le s_m \le t_m \le 2r_m s_m - s_m$ (cf. [BS88, Sem88]).

Let us now consider the geodesics ξ and ζ: without changing the traces of the generators of the Q pieces we can "unwind" ξ and ζ by twisting along γ using $\varphi_{\text{wind}} := \varphi_{Q_{1m}}^{-1} \circ \varphi_{Q_{2m}}^{2} \circ \varphi_{Q_{1m}}^{-1} \circ \varphi_{Q_{2m}}^{2}$ or $\varphi_{\text{wind}}^{-1}$ repeatedly[3] until $x \le cz - 2s_l r_m$ and $z \le cx - 2s_l r_m$. Again, as for Lemma 5.1 or using the argument of Kerckhoff, this situation gives a global minimum for $x + z$.

Using φ_{inv} (if needed), we can demand $z \le x$ (i.e. $1 < z \le x \le cx - 2s_l r_m$) and we get a unique situation corresponding to a point in the fundamental domain.

It remains to compare γ to its neighbors in terms of traces and prove that we only have to compare it to finitely many.

Definition 5.16 *A geodesic η on a surface has the* property $\binom{n}{k}^{\{\alpha_1,\dots,\alpha_n\}}$ *if η intersects k of the n geodesics $\alpha_1, \dots, \alpha_n$ exactly twice and does not intersect the other $n-k$ geodesics.*

Proposition 5.17 *Let γ be the dividing geodesic on a genus 2 surface whose coordinates in the Teichmüller space are $(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z)$ with $r_l \le r_m$, $1 < r_l \le s_l \le t_l \le 2r_l s_l - s_l$, $1 < r_m \le s_m \le t_m \le 2r_m s_m - s_m$ and $1 < z \le x \le cz - 2s_l r_m$. Then γ is a shortest dividing geodesic iff it is shorter than neighbors that have property $\binom{4}{3}^{\{\varrho_l, \sigma_l, \varrho_l*\sigma_l, \varrho_l*\sigma_l^{-1}\}}$ as well as property $\binom{4}{3}^{\{\varrho_m, \sigma_m, \varrho_m*\sigma_m, \varrho_m*\sigma_m^{-1}\}}$.*

Proof. We have to prove that a shortest neighbor η of a shortest dividing geodesic γ has the two properties.

Due to Proposition 4.25, we know that there is a geodesic α on the "left" Q piece (containing ϱ_l and σ_l) that does not intersect η. Cutting the surface along α we get a "Fish" piece. Proposition 5.11 shows that η has property $\binom{4}{3}^{\{\varrho_m, \sigma_m, \varrho_m*\sigma_m, \varrho_m*\sigma_m^{-1}\}}$.

We now re-glue the "Fish" piece along α and cut the original surface along the one geodesic in $\{\varrho_m, \sigma_m, \varrho_m * \sigma_m, \varrho_m * \sigma_m^{-1}\}$ that does not intersect η. The arguments of Proposition 5.11 show again that η has property $\binom{4}{3}^{\{\varrho_l, \sigma_l, \varrho_l*\sigma_l, \varrho_l*\sigma_l^{-1}\}}$. □

[3] φ_{wind} and $\varphi_{\text{wind}}^{-1}$ act as follows on $\mathcal{T}_{(2,0)}$:

$$\begin{aligned}
\varphi_{\text{wind}} &: (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, s_l, t_l, c, r_m, s_m, t_m, 2(cx - 2s_l r_m) - z, x), \\
\varphi_{\text{wind}}^{-1} &: (r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \longmapsto (r_l, s_l, t_l, c, r_m, s_m, t_m, z, 2(cz - 2s_l r_m) - x).
\end{aligned}$$

Remark 5.18 Knowing that any shortest neighbor η of the dividing geodesic γ has both properties $\binom{4}{3}^{\{\varrho_l,\sigma_l,\varrho_l*\sigma_l,\varrho_l*\sigma_l^{-1}\}}$ and $\binom{4}{3}^{\{\varrho_m,\sigma_m,\varrho_m*\sigma_m,\varrho_m*\sigma_m^{-1}\}}$, gives a method to construct a set of 16 inequalities that force γ to be a shortest dividing geodesic:

For every pair (α_l, α_m) in $\{\varrho_l, \sigma_l, \varrho_l*\sigma_l, \varrho_l*\sigma_l^{-1}\}\times\{\varrho_l, \sigma_l, \varrho_l*\sigma_l, \varrho_l*\sigma_l^{-1}\}$ with $tr(\alpha_l) = a_l$ and $tr(\alpha_m) = a_m$ do the following:

- Find an element $\tilde{\varphi}_{a_la_m}$ of the modular group such that
$$\tilde{\varphi}_{a_la_m}(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) = (*, a_l, *, c, a_m, *, *, \tilde{x}_{a_la_m}, \tilde{z}_{a_la_m})$$
(using $\varphi^{-1}_{Q_{1_l}} \circ \varphi_{Q_{2_l}}$, $\varphi^{-1}_{Q_{2_m}} \circ \varphi_{Q_{1_m}}$, $\varphi^{-1}_{Q_{1_l}}$ and $\varphi^{-1}_{Q_{2_m}}$ [4]).
- Unwind this surface using φ_{wind} or $\varphi^{-1}_{\text{wind}}$ repeatedly until
$$x_{a_la_m} \leq cz_{a_la_m} - 2a_la_m \quad \text{and} \quad z_{a_la_m} \leq cx_{a_la_m} - 2a_la_m,$$
where
$$\varphi_{a_la_m}(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) = (*, a_l, *, c, a_m, *, *, x_{a_la_m}, z_{a_la_m})$$
and $\varphi_{a_la_m} = \varphi^k_{\text{wind}} \circ \tilde{\varphi}_{a_la_m}$ for some $k \in \mathbb{Z}$.
- Therefore, the inequality corresponding to a neighbor intersecting neither α_l nor α_m is
$$c \leq x_{a_la_m} z_{a_la_m} - a_l^2 - a_m^2.$$

This now leads to the following theorem, giving the fundamental domain:

Theorem 5.19 *The set of isometry classes of surfaces of signature* $(2, 0)$ *is in a 1-1-correspondence with the set*

$$\mathcal{F}_{(2,0)} := \Big\{(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) \in \mathbb{R}^9 \;\Big|\; \begin{array}{l} -(c+1) = 2(r_l^2 + s_l^2 + t_l^2 - 2r_ls_lt_l - 1) \\ \qquad = 2(r_m^2 + s_m^2 + t_m^2 - 2r_ms_mt_m - 1), \\ Q_{(2,0)} = 0, \\ 1 < r_l \leq r_m, \\ r_l \leq s_l \leq t_l \leq 2r_ls_l - s_l, \\ r_m \leq s_m \leq t_m \leq 2r_ms_m - s_m, \\ 1 < z \leq x \leq cz - 2s_lr_m, \\ 1 < c \leq x_{a_la_m}z_{a_la_m} - a_l^2 - a_m^2 \\ \quad \forall (a_l, a_m) \in \{r_l, s_l, t_l, \overline{t_l}\} \times \{r_m, s_m, t_m, \overline{t_m}\} \end{array}\Big\}.$$

Here,

$$Q_{(2,0)} := 4s_l^2r_m^2 + 4s_lr_m(x+y) + 2(c+1)(s_l^2 + r_m^2) + c^2 + x^2 + z^2 - 2cxz - 1,$$

[4] $\varphi^{-1}_{Q_{1_l}} \circ \varphi_{Q_{2_l}}$ and $\varphi^{-1}_{Q_{2_m}} \circ \varphi_{Q_{1_m}}$ act as cyclic rotations on (r_l, s_l, t_l) and (r_m, s_m, t_m). $\varphi^{-1}_{Q_{1_l}}$ and $\varphi^{-1}_{Q_{2_m}}$ change t_l and t_m into $2r_ls_l - t_l$ and $2r_ms_m - t_m$, fixing r_l, s_l and r_m, s_m.

$$\overline{t_l} := 2r_l s_l - t_l, \qquad \overline{t_m} := 2r_m s_m - t_m,$$

and the transformations

$$\varphi_{a_l a_m}(r_l, s_l, t_l, c, r_m, s_m, t_m, x, z) = (*, a_l, *, c, a_m, *, *, x_{a_l a_m}, z_{a_l a_m})$$

are given by Remark 5.18.

Proof. Direct deduction from Proposition 5.17 and Remark 5.18. □

Thus, we can again give an algorithm (using the notations of theorem 5.19) leading from a point in the Teichmüller space to a point in $\mathcal{F}_{(2,0)}$ corresponding to the same surface:

Algorithm 5.20

1. (a) *Obtain* $r_l = \min\{r_l, s_l, t_l\}$, *using either* $\varphi_{Q_{2_l}}^{-1} \circ \varphi_{Q_{1_l}}$ *or* $\varphi_{Q_{1_l}}^{-1} \circ \varphi_{Q_{2_l}}$ *if needed. Go to (b).*

   (b) *If* $t_l > 2r_l s_l - t_l$, *apply* $\varphi_{Q_{1_l}}^{-1}$ *to the current point and return to (a). Else go to (c).*

   (c) *If* $s_l > 2r_l t_l - s_l$, *apply* $\varphi_{Q_{1_l}}$ *to the current point and return to (a). Else go to (d).*

   (d) *If* $s_l > t_l$, *apply* $\varphi_{Q_{1_l}}$ *to the current point.*

   Go to step 2.

2. (a) *Obtain* $r_m = \min\{r_m, s_m, t_m\}$, *using either* $\varphi_{Q_{2m}}^{-1} \circ \varphi_{Q_{1m}}$ *or* $\varphi_{Q_{1m}}^{-1} \circ \varphi_{Q_{2m}}$ *if needed. Go to (b).*

   (b) *If* $t_m > 2r_m s_m - t_m$, *apply* $\varphi_{Q_{1m}}^{-1}$ *to the current point and return to (a). Else go to (c).*

   (c) *If* $s_m > 2r_m t_m - s_m$, *apply* $\varphi_{Q_{1m}}$ *to the current point and return to (a). Else go to (d).*

   (d) *If* $s_m > t_m$, *apply* $\varphi_{Q_{1m}}$ *to the current point.*

   Go to step 3.

3. *If* $r_m > r_l$, *apply* $\varphi_{l \leftrightarrow m} \circ \varphi_{Q_{1_l}} \circ \varphi_{Q_{1m}}$ *to the current point. Go to step 4.*

4. *Unwind the surface, using* φ_{wind} *or* $\varphi_{\text{wind}}^{-1}$ *repeatedly until* $x \le cz - 2s_l r_m$ *and* $z \le cx - 2s_l r_m$. *Go to step 5.*

5. *If* $z > x$, *apply* φ_{inv} *to the current point. Go to step 6.*

6. *For all 16 cases of* $(a_l, a_m) \in \{r_l, s_l, t_l, \overline{t_l}\} \times \{r_m, s_m, t_m, \overline{t_m}\}$ *do :*
   If $c > x_{a_l a_m} z_{a_l a_m} - a_l^2 - a_m^2$*, apply* $\varphi_{\infty} \circ \varphi_{a_l a_m}$ *and go to step 1.*
   STOP.

Proof. As in the proof of algorithm 5.15, it is easy to show that the algorithm ends in $\mathcal{F}_{(2,0)}$ if it does not loop and that it is finite because lengths of some marked geodesics get smaller and smaller every time the algorithm gets to step 1 or 2. □

Chapter 6

Simple Closed Geodesics

The fact that our parameterization of the Teichmüller space depends only on traces (or equivalently on lengths of closed geodesics) gives a powerful tool using simple mathematics to state general results on geodesic lengths on Riemann surfaces. In this chapter we give evidence for this by investigating the following questions concerning simple closed geodesics:

1. What is the connection between the intersection number and the lengths of two simple closed geodesics ?

2. Are X pieces spectrally rigid with respect to the length spectrum of simple closed geodesics ?

6.1 Intersecting Geodesics

The connection between the intersection number and the lengths of two simple closed geodesics can be specified as follows:

If two simple closed geodesics on any purely hyperbolic[1] Riemann surface intersect one another n times, can we be sure that at least one of them is longer than a positive constant l_n independent of the surface ? Furthermore, is this constant sharp ? I.e. for $\varepsilon > 0$, is there a purely hyperbolic surface containing two simple closed geodesics intersecting one another n times whose lengths are both smaller or equal to $l_n + \varepsilon$?

In this section, we show the existence of positive sharp constants, prove that they are unbounded when n goes to infinity, and explicitly give l_n for $n = 1, 2, 3$. We also conjecture that l_n is strictly increasing and that for every $n \in \mathbb{N}$, there is a degenerated Q piece (a torus with a cusp) containing two geodesics of lengths l_n intersecting one another n times.

6.1.1 Existence and Behavior of a Sharp Solution l_n

Buser proves the existence of l_n for $n = 1$ in [Bus92, p.95-96] giving the following result using collars:

[1] No elliptic or parabolic elements in its fundamental group.

If a simple closed geodesic γ and a closed geodesic δ on a purely hyperbolic Riemann surface intersect each other, then $\sinh \frac{1}{2}l(\gamma) \sinh \frac{1}{2}l(\delta) > 1$ (this is equivalent to the statement that on a purely hyperbolic Riemann surface, two simple closed geodesics shorter than $2\,\text{arccosh}(\sqrt{2})$ are pairwise disjoint). He then shows the sharpness of this constant constructing a family of Q pieces converging to a torus with one cusp such that each Q piece contains two simple geodesic intersecting perpendicularly, one of length $2\,\text{arccosh}(\sqrt{2})$ and one whose length tends to the same value.

Thus, a (non-sharp) solution to the problem would be $l_n = 2\,\text{arccosh}(\sqrt{2})$ for any $n \in \mathbb{N}$, but we can do better:

Proposition 6.1 *There is a positive sharp constant l_n such that at least one of two simple closed geodesics on any purely hyperbolic Riemann surface intersecting one another n times is longer than l_n independent of the surface.*

Furthermore, l_n is unbounded for n to infinity, i.e. $\lim_{n\to\infty} l_n = \infty$.

Proof. Let $l_n = \inf\ l(\beta_n)$ be the infimum over all possible surfaces S_n and all pairs of simple closed geodesics (α_n, β_n) intersecting one another exactly n times such that $l(\alpha) \leq l(\beta_n)$. It is obvious that l_n is sharp as it is an infimum. As two simple closed geodesics shorter than $2\,\text{arccosh}(\sqrt{2})$ are pairwise disjoint, this constant l_n is also strictly positive.

Suppose now that there is $L > 0$ such that $l_n < L\ \forall\ n \in \mathbb{N}$. This means that for any natural n, there exist two simple closed geodesics α_n and β_n on some surface S_n that intersect one another n times with $l(\alpha_n) \leq l(\beta_n) \leq L$. By the Collar Theorem (see [Bus92, p.94-95]), this means that there is a collar containing α_n, isometric to a cylinder of height $2\,\text{arcsinh}\left(\frac{1}{\sinh(L/2)}\right) \leq 2\,\text{arcsinh}\left(\frac{1}{\sinh(l(\alpha_n)/2)}\right)$. As β_n intersects α_n exactly n times, its length must be at least n times the height of this cylinder, i.e. $l(\beta_n) \geq 2n\,\text{arcsinh}(\frac{1}{\sinh(L/2)})$. Thus, for any L, n can be chosen so that $l(\beta_n) > L$, which leads to a contradiction. □

We have thus shown that l_n tends to infinity for $n \to \infty$. It remains the question how fast (what is its asymptotic behaviour).

In the proof of Proposition 6.1, we used the fact that l_n is longer than n times the height of the collar of some geodesic. Consider now L_n, the length of a geodesic equal to n times the height of its collar, i.e. let L_n be the positive solution of the equation $L_n = 2n\,\text{arcsinh}(\frac{1}{\sinh(L_n/2)})$.

Proposition 6.2 *For $n \in \mathbb{N}$, let L_n be the positive solution of the equation $L_n = 2n\,\text{arcsinh}(\frac{1}{\sinh(L_n/2)})$. Then L_n is strictly increasing in n.*

Proof. The equation $L_n = 2n\,\text{arcsinh}(\frac{1}{\sinh(L_n/2)})$ is equivalent to $\sinh(\frac{L_n}{2n})\sinh(\frac{L_n}{2}) = 1$. Suppose now that there is an $n \in \mathbb{N}$ such that $L_n \geq L_{n+1}$. Then $\sinh(\frac{L_n}{2}) \geq \sinh(\frac{L_{n+1}}{2})$ which implies $\sinh(\frac{L_n}{2n}) \leq \sinh(\frac{L_{n+1}}{2n+2})$. But $\frac{L_n}{2n} \leq \frac{L_{n+1}}{2n+2}$ implies $L_n < L_{n+1}$ which leads to a contradiction. □

In the following proposition, we show that $L_n \leq l_n < 2L_n$. This proves that the behavior of l_n is similar to the one of L_n.

Proposition 6.3 *Let l_n be the positive sharp constant such that at least one of two simple closed geodesics on any purely hyperbolic Riemann surface intersecting one another n times is longer than l_n independent of the surface. Let L_n be the positive solution of the equation $L_n = 2n \operatorname{arcsinh}(\frac{1}{\sinh(L_n/2)})$.*

Then

$$L_n \leq l_n < 2L_n.$$

Proof. It is easy to see that $L_n \leq l_n$:

If a simple closed geodesic α of length L_n intersects a simple closed geodesic β n times, then β is at least as long as n times the height of the collar of α. Thus $l(\beta) \geq L_n$. As the height of the collar of α gets bigger when α gets smaller, $L_n \leq l_n$.

It remains to show that $l_n < 2L_n$:

For $n \in \mathbb{N}$, let $\mathcal{Y}$ be the degenerated Y piece that has a cusp and two boundary geodesics of length L_n. Glue these two geodesics together (and name the resulting geodesic α) such that the smallest boundary to boundary geodesic in $\mathcal{Y}$ becomes a simple closed geodesic (twist zero). Then, this simple closed geodesic δ is obviously of length L_n/n. We now build the curve $\bar{\beta}$ following δ n times and then α once. Therefore $l(\bar{\beta}) = 2L_n > l(\beta)$, where β is the simple closed geodesic in the homotopy class of $\bar{\beta}$. We have thus constructed a (degenerated) Q piece containing two simple closed geodesics intersecting one another n times and that are both shorter than $2L_n$. As there is a family of non-degenerated Q pieces that tend to this surface, $l_n < 2L_n$. □

6.1.2 Explicit Sharp Solutions

In order to find explicit sharp constants l_n for $n = 1, 2, 3$ we proceed as follows:

- First, we list all possible situations (up to homeomorphism) for two simple closed geodesics α_n and β_n on a surface S_n intersecting one another n times and such that all other interior simple closed geodesics intersect either α_n or β_n.
- Then, we find an explicit sharp constant l_{S_n} for every situation or show that $l_{S_n} \geq l_{S'_n}$ for a situation S'_n for which we have already found $l_{S'_n}$.
- Finally, we define l_n to be the minimum of all such l_{S_n} which is thus an explicit sharp constant for all possible situations.

Remark 6.4 If there is a surface containing two simple closed geodesics α_n and β_n intersecting one another n times as well as some simple closed internal geodesics that intersect neither α_n nor β_n, we can cut the surface along these geodesic and get at least one connected component containing α_n and β_n that is a hyperbolic Riemann surface S_n (half-cylinders dropped) containing no geodesics that intersect neither α_n nor β_n except for boundary geodesics. Thus, we can restrict the considered situations to those surfaces only.

The Case $n = 1$

Let ϱ and σ be two simple closed geodesics intersecting one another once on a hyperbolic Riemann surface S. Then we can build the geodesic corresponding to $[\varrho, \sigma]$ (as ϱ and σ are not trivial and intersect once, this curve is not trivial either). Cutting S along $[\varrho, \sigma]$, we get a Q piece containing ϱ and σ. Thus, the only situation to consider is a Q piece and we can give the following proof for the explicit sharp constant $l_1 = 2\operatorname{arccosh}(\sqrt{2})$:

Proposition 6.5 *If, on a purely hyperbolic Riemann surface, there are two simple closed geodesics ϱ and σ that intersect one another once, then* $\max\{l(\varrho), l(\sigma)\} > 2\operatorname{arccosh}(\sqrt{2})$. *This bound is sharp.*

Proof. We know already that it is enough to prove the proposition for Q pieces. As ϱ and σ are simple closed geodesics on a Q piece intersecting once, there is an element of the Teichmüller space of Q pieces $(c, r = \cosh(l(\varrho)/2), s = \cosh(l(\sigma)/2), t) \in \mathcal{T}_{(1,1)}$ corresponding to this particular Q piece.

Thus, $Q_{(1,1)} = r^2 + s^2 + t^2 - 2rst + \frac{c-1}{2} = 0$. The discriminant $\operatorname{disc}(Q_{(1,1)}, t)$ of $Q_{(1,1)}$ with respect to t must therefore be positive. But $\operatorname{disc}(Q_{(1,1)}, t) \geq 0$ is equivalent to $r^2s^2 - (r^2 + s^2) \geq \frac{c-1}{2}$. Without loss of generality, we may assume that $s \geq r$; thus $s^2(s^2 - 2) \geq \frac{c-1}{2} > 0$ and therefore $s > \sqrt{2}$.

For any $\varepsilon > 0$, the element $\left(1 + 2(\sqrt{2}+\varepsilon)^2\left((\sqrt{2}+\varepsilon)^2 - 2\right), \sqrt{2}+\varepsilon, \sqrt{2}+\varepsilon, (\sqrt{2}+\varepsilon)^2\right)$ in the Teichmüller space of Q pieces corresponds to a Riemann surface that contains two simple closed geodesics of lengths $2\operatorname{arccosh}(\sqrt{2}+\varepsilon)$ that intersect once. Thus the bound is sharp. ☐

The Case $n = 2$

Let γ and ζ be two simple closed geodesics on a purely hyperbolic Riemann surface that intersect one another exactly twice (in A and in B). Arbitrarily orient ζ and orient γ such that ζ intersects γ "positively" [2] in A. Then there are only two possible situations for the intersection B, namely ζ intersects γ "negatively" or "positively" in B as in the following figure:

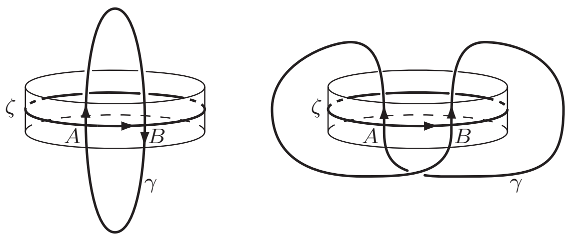

Figure 6.1

[2] If we turn the "arrow" ζ on the surface counter-clockwise onto the arrow γ, the turning angle is less than π.

1. If ζ intersects γ "negatively" in B, we build the oriented closed curves $\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}$ and $\tilde{\varepsilon}$ as follows:

    - for $\tilde{\alpha}$, we follow γ^{-1} from A to B, then back to A on ζ;
    - for $\tilde{\beta}$, we follow ζ^{-1} from A to B, then back to A on γ^{-1};
    - for $\tilde{\delta}$, we follow γ from A to B, then back to A on ζ^{-1};
    - for $\tilde{\delta}$, we follow ζ from A to B, then back to A on γ.

    None of these curves are null-homotopic because otherwise γ and ζ would be homotopic to two other geodesics intersecting at most once, and in each homotopy class there is only one geodesic (see e.g. [Bus92, p.18-23]). Some of the geodesics α, β, δ and ε homotopic to $\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}$ and $\tilde{\varepsilon}$ may be the same but do not intersect transversely because we can build the homotopic curves $\bar{\alpha}, \bar{\beta}, \bar{\delta}$ and $\bar{\varepsilon}$ that do not intersect, as shown in the following figure:

    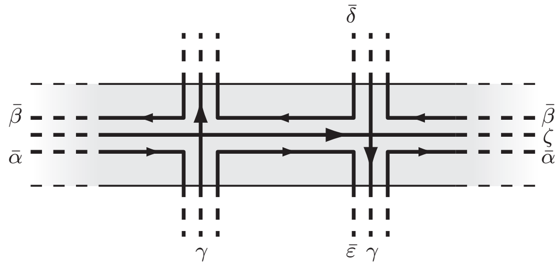

    Figure 6.2

    We may now cut the surface along the geodesics α, β, δ and ε and get a surface that is an X piece containing γ and ζ.

2. If ζ intersects γ "positively" in B, we build the oriented curves $\tilde{\alpha}$ and $\tilde{\beta}$ as follows:

    - for $\tilde{\alpha}$, we follow γ^{-1} from A to B, then ζ^{-1} back to A, then γ to B and finish on ζ back to A;
    - for $\tilde{\beta}$, we follow ζ^{-1} from A to B, then γ^{-1} back to A, then ζ to B and finish on γ back to A.

    One of these curves may be null-homotopic, but not both because otherwise the surface would be a torus as can be seen by cutting the surface along ζ and from A to B along γ and tracing the curves $\bar{\alpha}$ and $\bar{\beta}$ homotopic to $\tilde{\alpha}$ and $\tilde{\beta}$ and (if they are not null-homotopic) to the geodesics α and β as in the following figure:

    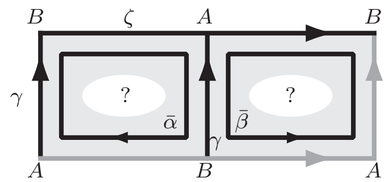

    Figure 6.3

If one of $\bar{\alpha}$ and $\bar{\beta}$ is null-homotopic (say $\bar{\beta} \sim 0$), we cut along the other geodesic (in our case α) and get a surface that is a Q piece and contains both γ and ζ (see Figure 6.4).

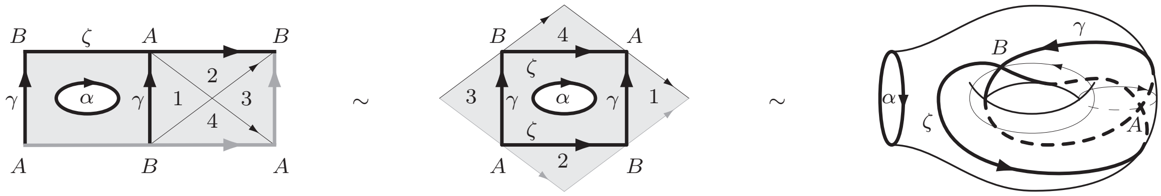

Figure 6.4

If none of $\bar{\alpha}$ and $\bar{\beta}$ is null-homotopic, we cut the original surface along the geodesics α and β and get a surface that is a "Fish" piece and contains both γ and ζ (see Figure 6.5).

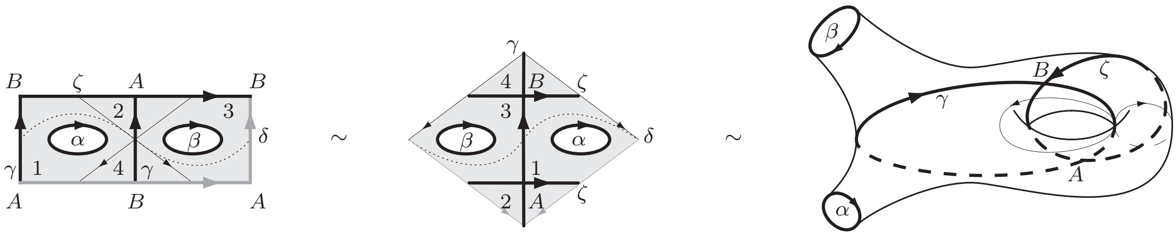

Figure 6.5

The situations that have to be considered are therefore the X piece (signature $(0,4)$), the Q piece (signature $(1,1)$) and the "Fish" piece (signature $(1,2)$).

Lemma 6.6 *Let (a,b,c,d,e,x,z) be an element of the Teichmüller space of Riemann surfaces of signature $(0,4)$. Then $(z-1)(c-1) \geq 4a_m^2$, where $a_m = \min\{a,b,d,e\}$.*

Proof. By Corollary 3.5 we know that $Q_{(0,4)} = 0$. The discriminant $\mathrm{disc}(Q_{(0,4)}, x)$ of $Q_{(0,4)}$ with respect to x must therefore be positive. But $\mathrm{disc}(Q_{(0,4)}, x)(z)$ is a quadratic polynomial in z that is negative if evaluated in 1 ($\mathrm{disc}(Q_{(0,4)}, x)(1) < 0$). Thus z must be greater than its right root, i.e.

$$\begin{aligned} z &\geq \tfrac{ae+bd+c(ad+be)+\sqrt{h(a,b,-c)h(c,d,-e)}}{c^2-1} \\ &\geq \tfrac{2(c+1)a_m^2+h(a_m,a_m,-c)}{c^2-1} = \tfrac{4a_m^2}{c-1} + 1. \end{aligned}$$

But this is equivalent to $(z-1)(c-1) \geq 4a_m^2$. □

Corollary 6.7 *If, on an X piece, there are two simple closed geodesics γ and ζ that intersect twice, then $\max\{l(\gamma), l(\zeta)\} > 2\operatorname{arccosh}(3)$. This bound is sharp.*

Proof. As any simple closed geodesic (other than the boundary geodesics) on an X piece is dividing, there is a point $(a, b, c = \cosh(l(\gamma)/2), d, e, x, z = \cosh(l(\zeta)/2))$ in the Teichmüller space of Riemann surfaces of signature $(0, 4)$ corresponding to the X piece.

Lemma 6.6 implies that $(z-1)(c-1) > 4$ and thus $\max\{z, c\} > 3$ which is equivalent to $\max\{l(\gamma), l(\zeta)\} > 2\operatorname{arccosh}(3)$.

It is easy to see that for any $\varepsilon > 0$, $(1+\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}, 3+\varepsilon, 1+\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}, 7+4\varepsilon+\frac{\varepsilon^2}{2}, 3+\varepsilon)$ is an element of moduli space of surfaces of signature $(0, 4)$ corresponding to a Riemann surface that has two geodesics of lengths $2\operatorname{arccosh}(3+\varepsilon)$ that intersect twice; hence the sharpness of the bound. □

Lemma 6.8 *If, on a Q piece, there are two simple closed geodesics τ and $\bar{\tau}$ that intersect twice, then* $\max\{l(\tau), l(\bar{\tau})\} > 2\operatorname{arccosh}(2)$. *This bound is sharp.*

Proof. We will use the following result of [BS88], investigating the length spectrum of the one holed torus: the geodesics corresponding to the first three lengths of the spectrum (of simple non-boundary geodesics) intersect only once and are such that their traces r, s and t satisfy $1 < r \leq s \leq t \leq rs$ and $\frac{c-1}{2} = t(2rs-t) - r^2 - s^2$ where c is the trace of the boundary geodesic γ and $\bar{t} = 2rs - t$ is the trace of the forth to shortest simple non-boundary geodesic.

As τ and $\bar{\tau}$ intersect twice, the minimum of $\max\{l(\tau), l(\bar{\tau})\}$ is obtained when they are third and forth shortest and have equal length, i.e. $t = rs = \bar{t}$. But in that case, $0 < \frac{c-1}{2} = r^2s^2 - r^2 - s^2 \leq r^2s^2 - 2rs$ and thus $t = \bar{t} > 2$, which implies

$$\max\{l(\tau), l(\bar{\tau})\} > 2\operatorname{arccosh}(2).$$

For any $\varepsilon > 0$, $(1+2\varepsilon(2+\varepsilon), \sqrt{2+\varepsilon}, \sqrt{2+\varepsilon}, 2+\varepsilon)$ is an element of the Teichmüller space of Q pieces corresponding to a Riemann surface that has two geodesics of lengths $2\operatorname{arccosh}(2+\varepsilon)$ that intersect twice. Thus the bound is sharp. □

Lemma 6.9 *If, on a "Fish" piece, there are two simple closed geodesics γ and ζ that intersect twice, such that there is no simple closed geodesic (other than a boundary geodesic) that intersects neither γ nor ζ, then* $\max\{l(\gamma), l(\zeta)\} > 2\operatorname{arccosh}(2)$.

Proof. Without loss of generality, we can assume that $l(\gamma) \leq l(\zeta)$ and that we are in the situation of Figure 6.5, where α and β are the two boundary geodesics. Let δ be a geodesic intersecting γ twice, such that α, δ and ζ are the boundary geodesics of a Y piece (dotted in Figure 6.5). If we cut along δ, we get an X piece with boundary geodesics α, β, δ and a copy of δ that has ζ as a dividing geodesic.

Let now η be another dividing geodesic of this X piece that does not intersect the geodesic arcs constituting γ. Let $z = \cosh(l(\zeta)/2)$, $a = \cosh(l(\alpha)/2)$, $b = \cosh(l(\beta)/2)$, $d = \cosh(l(\delta)/2)$ and $e = \cosh(l(\eta)/2)$. By the proof of Lemma 6.6, we know that

$$z \geq \frac{d(a+b)(e+1)+\sqrt{h(a,b,e)(e+1)(e-1+2d^2)}}{e^2-1} > \frac{2d+\sqrt{(e+1)(e-1+2d^2)}}{e^2-1}.$$

If we cut the original "Fish" piece along η, we get a Y piece and a Q piece that contains γ and δ intersecting one another twice. If γ and δ are not the third and forth shortest simple closed geodesics of this Q piece, then $c = \cosh(l(\gamma)/2) > 2$ due to the proof of Lemma 6.8. If, on the other hand, γ and δ are the third and forth shortest simple closed geodesics of this Q piece, then $\frac{e-1}{2} = cd - r^2 - e^2 < d(c-1)$ because $c = 2rs - d$.

Thus, $c < 2$ implies

$$z > \frac{2d+\sqrt{(e+1)(e-1+2d^2)}}{e^2-1} > \frac{d}{\frac{e-1}{2}} + \sqrt{\frac{e+1}{e-1}\left(1+\frac{d^2}{\frac{e-1}{2}}\right)} > \frac{1}{c-1} + \sqrt{1+\frac{d}{c-1}} > 2,$$

which proves the lemma. □

Proposition 6.10 *If, on a purely hyperbolic Riemann surface, there are two simple closed geodesics intersecting one another twice, then at least one of them is longer than* $2\,\mathrm{arccosh}(2)$. *This bound is sharp.*

Proof. As the only situations we have to consider are the X piece, the Q piece and the "Fish" piece, the proposition follows directly from Corollary 6.7, Lemma 6.8 and Lemma 6.9. □

Remark 6.11 As the shortest self-intersecting geodesic on any purely hyperbolic Riemann surface is a figure eight geodesic and is strictly longer than $4\,\mathrm{arcsinh}(1) = 2\,\mathrm{arccosh}(3)$ (cf. [Bus92, p.99]), the statements of corollary 6.7, Lemma 6.8, Lemma 6.9 and Proposition 6.10 are true even for non-simple geodesics.

The Case $n = 3$

Let α and β be two simple closed geodesics on a purely hyperbolic Riemann surface that intersect three times. Name the intersections A, B and C and orient α and β such that A, B and C come in that order on α and on β. This leads to two possible situations:

1. There is a change of orientation and/or renaming of intersections such that α intersects β "positively" in A and "negatively" in B or α intersects β "negatively" in A and "positively" in B.

2. There is no change of orientation and/or renaming of intersections that makes this possible, i.e. α intersects β "positively" in every intersection or α intersects β "negatively" in every intersection.

In the first case, Lemma 6.12 gives a lower bound for $\max\{l(\alpha), l(\beta)\}$.

In the second case, we cut the surface along α and from A to B along β as in Figure 6.6 and use a series of lemmas to show that in this configuration, we can restrict us to the situation on a Q piece.

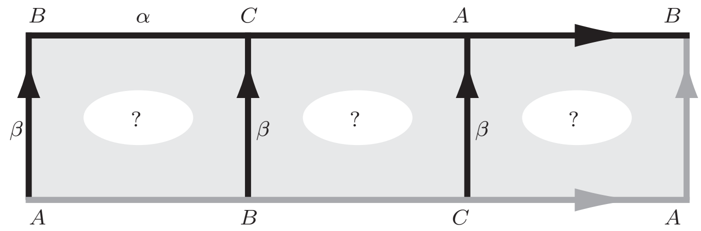

Figure 6.6

Lemma 6.12 *Let α and β be two simple closed oriented geodesics on a purely hyperbolic Riemann surface that intersect three times in A, B and C, such that A, B, C are consecutive on α and also on β.*

If α intersects β "positively" in A and "negatively" in B or if α intersects β "negatively" in A and "positively" in B, then $\max\{l(\alpha), l(\beta)\} > 2\,\text{arccosh}(3)$.

Proof. Comparing the lengths of the arcs between B and C, there are two possible situations:

1. $l(B \xrightarrow{\alpha} C) \leq l(B \xrightarrow{\beta} C)$,

2. $l(B \xrightarrow{\alpha} C) > l(B \xrightarrow{\beta} C)$.

We now build the oriented closed curves $\bar{\gamma}$ and $\bar{\zeta}$:

- In situation 1, we set $\bar{\gamma} = \alpha$; in situation 2, $\bar{\gamma}$ is obtained following α from A to B, then β from B to C and again α from C to A.

- In situation 1, $\bar{\zeta}$ is obtained following β from A to B, then α from B to C and again β from C to A; in situation 2, we set $\bar{\zeta} = \beta$.

These two curves $\bar{\gamma}$ and $\bar{\zeta}$ are thus homotopic to two shorter simple closed geodesics γ and ζ intersecting one another twice and that lie on an imbedded X piece (because γ intersects ζ once "positively" and once "negatively"). This means that we can conclude applying Corollary 6.7. □

We now prove in Lemma 6.14, that in the other cases, we have only to consider Q pieces and "Fish" pieces. The proof uses the following theorem of [Par05] (first published in slightly different form in [Par03]):

Theorem 6.13 (H. Parlier) *Let S be a surface of signature (g, n) with $n > 0$. Let $\gamma_1, \ldots, \gamma_n$ be the boundary geodesics of S. For $(\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{R}^-)^n$ with at least one $\varepsilon_i \neq 0$, there exists a surface $\tilde{S}$ with boundary geodesics of length $\gamma_1 + \varepsilon_1, \ldots, \gamma_n + \varepsilon_n$ such that all the corresponding simple closed geodesics in $\tilde{S}$ are of length strictly less than those of S.*

Lemma 6.14 *Let S be a purely hyperbolic Riemann surface and α and β two oriented simple closed geodesics on S intersecting one another three times such that α intersects β "positively" in every intersection or α intersects β "negatively" in every intersection. Name the intersections A, B, C such that they are consecutive on α. If A, B, C are also consecutive on β, then there is a "Fish" piece with two boundary geodesics of equal lengths or a Q piece containing two simple closed geodesics α' and β' intersecting one another three times such that* $\max\{l(\alpha), l(\beta)\} \geq \max\{l(\alpha'), l(\beta')\}$.

Proof. If one or two of the domains with a question mark in Figure 6.6 are simply connected, we cut along the geodesics homotopic to the boundaries of the not simply connected domains with a question mark and get a Q piece or a "Fish" piece. If we get a "Fish" piece with boundary geodesics of different lengths, we use Theorem 6.13 to diminish the length of the longer boundary geodesic such that the lengths of the new boundary geodesics are equal.

If none of the domains with a question mark in Figure 6.6 are simply connected, we cut along the geodesics homotopic to the boundaries of these domains and get a surface of signature $(1, 3)$. On this surface, there is a simple closed geodesic γ not intersecting α and intersecting β exactly four times such that it cuts the surface into a Q piece containing α and an X piece containing two geodesic arcs of β ($T \xrightarrow{\beta} W$ and $U \xrightarrow{\beta} V$) as in the following figure:

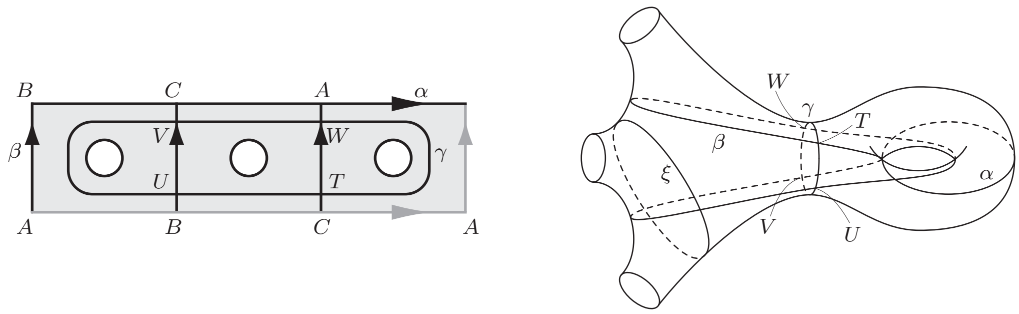

Figure 6.7

Consider first the geodesic arc of β joining T and W: There is a dividing geodesic ξ on the X piece, that does not intersect this arc. Cutting along ξ, we get a Y piece. We can now diminish lengths of the boundary geodesics other than γ and get a Y piece with one boundary geodesic of length $l(\gamma)$ and two of lengths σ, where σ is the systole (length of the shortest closed geodesic) of the original surface S. This Y piece contains a geodesic arc joining T and D of length smaller than $l(T \xrightarrow{\beta} W)$ (this is part of the statement of the technical lemma used in [Par05] in order to show Theorem 6.13).

Obviously, we can do the same for the geodesic arc joining U and V. Thus we can replace the X piece by a Y piece with one boundary geodesic of length $l(\gamma)$ and two of lengths σ. If we glue this Y piece to the Q piece containing α, we get a "Fish" piece with boundary geodesics of lengths σ that contains a geodesic α and a curve $\tilde{\beta}$ that intersect three times and such that $l(\beta) \geq l(\tilde{\beta})$. Therefore, the geodesic β' that is homotopic to $\tilde{\beta}$ intersects α three times and $\max\{l(\alpha), l(\beta)\} \geq \max\{l(\alpha), l(\beta')\}$. □

Lemma 6.15 *If, on a Q piece, there are two simple closed geodesics $\bar{\sigma}$ and $\bar{\tau}$ that intersect three times, then* $\max\{l(\bar{\sigma}), l(\bar{\tau})\} > l_3 := 2\,\mathrm{arccosh}\left(\sqrt{\frac{1}{2}\left(7+\frac{11}{3}\sqrt{\frac{11}{3}}\right)}\right) \approx 2\,\mathrm{arccosh}(2.648)$. *This bound is sharp.*

Proof. Let (c, r, s, t) be an element of the Teichmüller space of surfaces of signature $(1, 1)$ such that $1 < r \leq s \leq t \leq rs$, where r, s and t are the traces of the shortest three geodesics ϱ, σ and $\tau = (\varrho * \sigma)^{-1}$ of the Q piece corresponding to (c, r, s, t). Then, the geodesics $\bar{\tau} = \varrho * \sigma^{-1}$ and $\bar{\sigma} = \tau * \varrho^{-1}$ intersect three times and $\bar{\tau}$ is the forth shortest internal simple closed geodesic (cf. [BS88]). The traces of $\bar{\tau}$ and $\bar{\sigma}$ are $\bar{t} = 2rs - t$ and $\bar{s} = 2rt - s$.

For a fixed c and r, $\max\{\bar{s}, \bar{t}\} = \bar{s} = 2rt - s$ is therefore minimal if $s = t = e$. In this case $\frac{c-1}{2} = 2rst - r^2 - s^2 - t^2 = 2e^2(r-1) - r^2$ and therefore $\bar{s}^2 = e^2(2r-1)^2 = \frac{(c-1+2r^2)(2r-1)^2}{4(r-1)} > \frac{r^2(2r-1)^2}{2(r-1)}$. But for $r > 1$, this last quantity is minimal for

$$\frac{d}{dr}\frac{r^2(2r-1)^2}{2(r-1)} = 0 \iff \frac{r(2r-1)(6r^2-9r+2)}{2(r-1)^2} = 0 \iff r = \frac{1}{4}\left(3+\sqrt{\frac{11}{3}}\right).$$

Therefore $\bar{s}^2 = e^2(2r-1)^2 > \frac{1}{2}\left(7+\frac{11}{3}\sqrt{\frac{11}{3}}\right)$.

There is a degenerated Q piece[3] on which there are two geodesics of lengths l_3 intersecting one another three times. Therefore there is a family of Q pieces whose border geodesic tends to length zero containing two geodesics tending to length l_3 intersecting one another three times and thus the bound is sharp. □

It remains to show $\max\{l(\alpha), l(\beta)\} \geq l_3$ for any two geodesics α and β intersecting one another three times on a "Fish" piece. But before that let us state some facts concerning hyperellipticity:

1. A "Fish" piece with two boundary geodesics of equal lengths is hyperelliptic.
2. The hyperelliptic involution φ_{hyp} of such a "Fish" piece is unique.
3. If γ is a simple closed geodesic on such a "Fish" piece, then $\varphi_{\mathrm{hyp}}(\gamma) = \gamma$.

[3] A torus with a cusp whose trace coordinates are

$$(r, s, t) = \left(\frac{1}{4}\left(3+\sqrt{\frac{11}{3}}\right), \sqrt{\frac{13+7\sqrt{\frac{11}{3}}}{8}}, \sqrt{\frac{13+7\sqrt{\frac{11}{3}}}{8}}\right).$$

Remark 6.16 These facts follow directly from the corresponding ones on closed Riemann surfaces of genus 2 (see e.g. [Mas00]).

Lemma 6.17 *Let S be a "Fish" piece with two boundary geodesics of equal lengths and α and β two oriented simple closed geodesics on S intersecting one another three times such that α intersects β "positively" in every intersection or α intersects β "negatively" in every intersection. Name the intersections A, B, C such that they are consecutive on α. If A, B, C are also consecutive on β, then there is a Q piece containing two simple closed geodesics α' and β' intersecting one another three times such that* $\max\{l(\alpha), l(\beta)\} \geq \max\{l(\alpha'), l(\beta')\}$.

Proof. The statement is obviously true if there is a dividing geodesic intersecting neither α nor β. We may thus assume that there is no such dividing geodesic.

Without loss of generality, we may assume that S is in a minimal configuration for a given length $2\operatorname{arccosh}(e)$ of the boundary geodesics; i.e. there is no other "Fish" piece with boundary geodesics of lengths $2\operatorname{arccosh}(e)$ containing two simple closed geodesics α' and β' intersecting one another three times such that $\max\{l(\alpha), l(\beta)\} > \max\{l(\alpha'), l(\beta')\}$.

By the convexity of the length function (see [Ker83]), we may assume that $l(\alpha) = l(\beta)$ (otherwise we may slightly lengthen the shorter geodesic and slightly shorten the longer geodesic and thus are not in a minimal configuration).

As the situation is symmetric in α and β, we may assume that $l(\gamma) < 2l(\beta)$. Indeed, we show that if $\bar{\gamma}$ is a dividing geodesic of the "Fish" piece intersecting α twice and not intersecting β (see Figure 6.8), then $l(\gamma) + l(\bar{\gamma}) < 4l(\beta)$. Indeed, as S is hyperelliptic with the hyperelliptic involution φ_{hyp} that leaves α, β and γ invariant, we get

$$l(A \xrightarrow{\alpha} B) = l(C \xrightarrow{\alpha} A) \quad \text{and} \quad l(A \xrightarrow{\beta} B) = l(C \xrightarrow{\beta} A).$$

Thus

$$l(\alpha) = 2l(A \xrightarrow{\alpha} B) + l(B \xrightarrow{\alpha} C) = l(\beta) = 2l(A \xrightarrow{\beta} B) + l(B \xrightarrow{\beta} C).$$

But

$$\begin{aligned} l(\gamma) + l(\bar{\gamma}) &< \Big(2l(A \xrightarrow{\alpha} B) + 2l(B \xrightarrow{\alpha} C) + 2l(A \xrightarrow{\beta} B)\Big) \\ &\qquad + \Big(2l(A \xrightarrow{\beta} B) + 2l(B \xrightarrow{\beta} C) + 2l(A \xrightarrow{\alpha} B)\Big) \\ &= 2l(\alpha) + 2l(\beta) = 4l(\beta). \end{aligned}$$

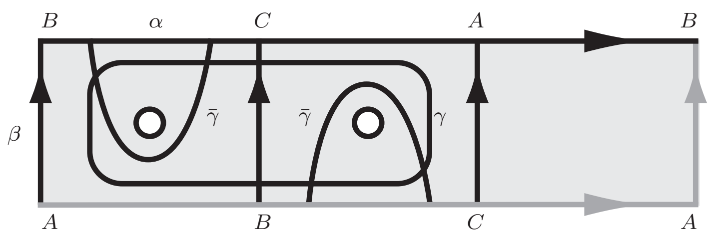

Figure 6.8

If ϱ is a simple closed geodesic on the "Fish" piece intersecting neither β nor γ, then $r = \cosh(l(\varrho)/2) < \sqrt{(b^2-1)\frac{c+1}{2}} - b$ where $b = \cosh(l(\beta)/2)$ and $c = \cosh(l(\gamma)/2)$. Indeed, the "Fish" piece has the trace coordinates (e, e, c, r, s, t, x, b) as can be seen in Figure 6.9 and thus $Q_{(1,2)}(e, e, c, r, s, t, x, b) = 0$. As in the proof of Lemma 6.6 we obtain therefore

$$b \geq \frac{2re + \sqrt{(c-1+2r^2)(c-1+2e^2)}}{c-1} > \frac{2r + \sqrt{(c-1+2r^2)(c+1)}}{c-1}.$$

But this implies $r < \sqrt{(b^2-1)\frac{c+1}{2}} - b$.

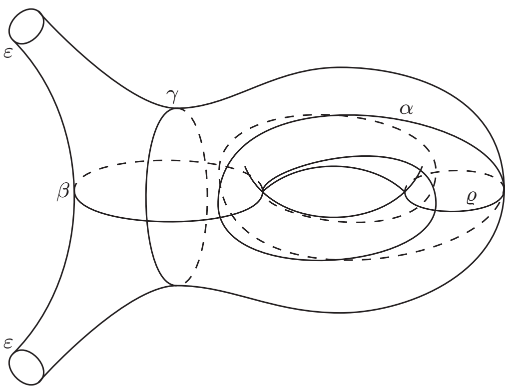

Figure 6.9

If we cut the "Fish" piece along ϱ, we get an X piece. Let π be the shortest border-to-border path on the "right" Y piece whose boundary geodesics are γ and two copies of ϱ. Then $p = \cosh(l(\pi)) = \frac{c+r^2}{r^2-1}$, as may easily be obtained by hyperbolic trigonometry (see e.g. [Bus92, p.454]). But, as $\frac{c+r^2}{r^2-1}$ is a decreasing function in r and as $r < \sqrt{(b^2-1)\frac{c+1}{2}} - b$, we get

$$p > \frac{c + \left(\sqrt{(b^2-1)\frac{c+1}{2}} - b\right)^2}{\left(\sqrt{(b^2-1)\frac{c+1}{2}} - b\right)^2 - 1}.$$

This last term is again decreasing in c and $l(\gamma) < 2l(\beta)$ which is equivalent to $c < 2b^2 - 1$. Thus

$$p > \frac{1 - b^2(b^2 - 2\sqrt{b^2-1} + 2)}{1 - b^2(b^2 - 2\sqrt{b^2-1})}.$$

Suppose now that $l_2 \leq l(\beta) \leq l_3$ (otherwise the lemma is trivial because l_3 is sharp for Q pieces). But $l_2 \leq l(\beta) \leq l_3$ is equivalent to $2 \leq b \leq \sqrt{\frac{1}{2}\left(7 + \frac{11}{3}\sqrt{\frac{11}{3}}\right)} \approx 2.648$ which implies that $\frac{1-b^2(b^2-2\sqrt{b^2-1}+2)}{1-b^2(b^2-2\sqrt{b^2-1})}$ is decreasing and thus $p > 2.017$ which means that

$l(\pi) > 1.326$. But, obviously $l(\alpha) > 3l(\pi) > 3.98 > l_3 \geq l(\beta)$ which is in contradiction with $l(\alpha) = l(\beta)$. Thus $l(\beta) > l_3$ and we finish the proof using Lemma 6.15.

□

Proposition 6.18 *If, on a purely hyperbolic Riemann surface, there are two simple closed geodesics intersecting one another three times, then at least one of them is longer than*

$$l_3 := 2\,\mathrm{arccosh}\left(\sqrt{\tfrac{1}{2}\left(7 + \tfrac{11}{3}\sqrt{\tfrac{11}{3}}\right)}\right).$$

Proof. The only situations we have to consider are the situations when the two geodesics intersect at least once "positively" and at least once "negatively" as well as the situation on a "Fish" piece with two boundary geodesics of equal lengths or a Q piece (due to Lemma 6.14). Thus the proposition follows directly from Lemmas 6.12, 6.15 and 6.17. □

6.1.3 Conjectures

For $n = 1, 2, 3$, there are tori with one cusp containing two simple closed geodesics of lengths l_1, l_2 and l_3, intersecting one another n times. This fact, together with Proposition 6.2 and Proposition 6.3 make the following two conjectures very plausible:

Conjecture 6.19 *Let l_n be the positive sharp constant such that if two simple closed geodesics on any purely hyperbolic Riemann surface intersect one another n times, at least one of them is longer than l_n independent of the surface.*

Then l_n is strictly increasing in n.

Conjecture 6.20 *For every n there is degenerated Q piece (a torus with a cusp) containing two simple closed geodesics of lengths l_n intersecting one another n times.*

Note that the conjecture 6.19 implies the following reformulation of the connection between the intersection number and the lengths of simple closed geodesics:

Proposition 6.21 *Conjecture 6.19 implies the following:*

If, on a purely hyperbolic Riemann surface, there are two simple closed geodesics both shorter than (or equal to) l_n, then they intersect at most $n-1$ times.

6.2 Spectral Questions

Questions concerning the eigenvalues of the Laplacian have first arisen in mathematical physics in the context of the wave and the heat equations. Quite soon though, interest shifted to a more geometric point of view, giving birth to spectral geometry, i.e. trying to answer questions concerning the geometry of a manifold given its spectrum or part of it.

The most popular question in spectral geometry is probably the following one asked by Kac (cf. [Kac66]): "Can one hear the shape of a drum?" i.e. the question about the (non-)existence of domains of the Euclidian plane that are isospectral but not isometric[4]. Obviously, this question can also be asked for non-Euclidian manifolds and especially for Riemann surfaces (with a metric of constant curvature -1).

In a series of papers ([Hub59, Hub61a, Hub61b]), Huber introduced the length spectrum for Riemann surfaces (i.e. the ordered set [with multiplicity] of the lengths of all closed geodesics) and showed that for compact Riemann surfaces the length spectrum and the spectrum of the Laplacian are geometrically equivalent. As the length $l(\gamma)$ and the trace c of a closed geodesic γ are equivalent ($c = \cosh(l(\gamma)/2)$), the trace spectrum and the spectrum of the Laplacian are also geometrically equivalent.

It is well known that there are isospectral non-isometric Riemann surfaces for genus $g \geq 4$ (see e.g. [BT87, BT90, Bus86, Vig80]) but the question is still open for instance for Riemann surfaces of genus 2 and 3 of constant curvature[5]. On the other hand, it has recently been shown in [BFS05] that surfaces of signature $(0; 2, 2, 3, 3; 0; 0)$ are spectrally rigid (i.e. there are no isospectral non-isometric surfaces of signature $(0; 2, 2, 3, 3; 0; 0)$) and there is some hope to generalize some of the methods used in that paper to other three-generator surfaces such as signature $(0; 2, 2, n, n; 0; 0)$, $(0; m, m, n, n; 0; 0)$, $(0; k, l, m, n; 0; 0)$ or even non-degenerated X pieces.

In this section, we will only look at part of the trace spectrum, i.e. only at the spectrum of simple closed geodesics. In this context, we show the following proposition:

Proposition 6.22 *An X piece is uniquely determined by the set (with multiplicity) of traces of its boundary geodesics and the trace spectrum of its internal simple closed geodesics.*

Proof. We show that there is only one point in the moduli space $\mathcal{F}_{(0,4)}$ of X pieces (see 5.5) with the given set of traces of boundary geodesics and the given trace spectrum of internal simple closed geodesics:

As every simple closed geodesic of an X piece is dividing, it is obvious that the three smallest traces in the spectrum are $c \leq z \leq x$.

It is also obvious that a is the minimum of the set of traces of the boundary geodesics.

Consider now the polynomial $Q_{(0,4)}$ as a function $Q_{(0,4)}(b, d, e)$ in b, d and e. Let b, d and e be the remaining boundary traces such that $Q_{(0,4)}(b, d, e) = 0$. If there is only one way to do this, then the X piece is uniquely determined. If there is a permutation $\varsigma \neq id$ such that $Q_{(0,4)}(b', d', e') = 0$ for $(b', d', e') = \varsigma(b, d, e)$, then it remains to prove that there is an element φ of the modular group such that $\varphi(a, b, c, d, e, x, z) = (a, b', c, d', e', x, z)$ (i.e. (a, b, c, d, e, x, z) is on the boundary of $\mathcal{F}_{(0,4)}$) or that the spectra of the two corresponding surfaces are different. Here we show this in the cases of $\varsigma_1(b, d, e) = (b, e, d)$ and $\varsigma_2(b, d, e) = (d, e, b)$; the other cases are analogous:

[4]Gordon, Webb and Wolpert have shown (see [GWW92a, GWW92b]), that there are such domains; in [BCDS94], Buser, Conway, Doyle and Semmler have given a particularly nice way to construct examples that are not only isospectral without being isometric but also homophonic, i.e. that have each a distinguished point such that corresponding normalized Dirichlet eigenfunctions take equal values at the distinguished points.

[5]It has recently been shown that there are isospectral non-isometric surfaces of genus 2 and 3 of non-constant curvature (see [Kan05]).

1. $Q_{(0,4)}(b,d,e) = 0 = Q_{(0,4)}(b,e,d)$ implies
$$Q_{(0,4)}(b,d,e) - Q_{(0,4)}(b,e,d) = 2(b-a)(d-e)(x-z) = 0.$$
   - If $a = b$ then $\varphi^{-1}_{Y_{ab}} \circ \varphi_{Y_{de}}(a,b,c,d,e,x,z) = (b,a,c,e,d,x,z) = (a,b,c,e,d,x,z)$.
   - If $d = e$ then $(a,b,c,d,e,x,z) = (a,b,c,e,d,x,z)$.
   - If $x = z$ then $\varphi^{-1}_{Y_{ab}} \circ \varphi_{\text{inv}}(a,b,c,d,e,x,z) = (a,b,c,e,d,z,x) = (a,b,c,e,d,x,z)$.
2. $Q_{(0,4)}(b,d,e) = 0 = Q_{(0,4)}(d,e,b)$ implies
$$Q_{(0,4)}(b,d,e) - Q_{(0,4)}(d,e,b) = 2(b-a)(d-e)(x-z) + 2(e-a)(d-b)(z-c) = 0.$$

We may assume that a, b, d and e have distinct values and that $c < x < z$ (otherwise we are in a case analogous to the previous). Therefore $Q_{(0,4)}(b,d,e) = 0 = Q_{(0,4)}(d,e,b)$ implies $\frac{d-e}{b-d} = \frac{(e-a)(z-c)}{(b-a)(x-z)} > 0$ and thus either $b > d > e$ or $b < d < e$.

Due to the arguments of the proof of Proposition 5.4, we know that the forth shortest simple closed geodesic on an X piece corresponding to $(a,b,c,d,e,x,z) \in \mathcal{F}_{(0,4)}$ has the trace
$$y = \min\{2(xz - ab - de) - c, 2(cx - ae - bd) - z, 2(cz - ad - be) - x\}$$
and the forth shortest simple closed geodesic on an X piece corresponding to the point $(a,d,c,e,b,x,z) \in \mathcal{F}_{(0,4)}$ has the trace
$$y' = \min\{2(xz - ad - be) - c, 2(cx - ab - de) - z, 2(cz - ae - bd) - x\}.$$

- If $b > d > e$, then
  - $(b-d)(e-a) > 0$ which implies $ad + be > ab + de$,
  - $(d-e)(b-a) > 0$ which implies $ae + bd > ad + be$ and
  - $(b-e)(d-a) > 0$ which implies $ae + bd > ab + de$.

  Therefore
$$\begin{aligned} y &= \min\{2(cx - ae - bd) - z, 2(cz - ad - be) - x\}\} \quad \text{and} \\ y' &= 2(cz - ae - bd) - x. \end{aligned}$$
  But, as
$$\begin{aligned} (2(cz - ad - be) - x) - (2(cz - ae - bd) - x) &= (d-e)(b-a) > 0 \qquad \text{and} \\ (2(cx - ae - bd) - z) - (2(cz - ae - bd) - x) &= (2c-1)(x-z) > 0, \end{aligned}$$
  we get $y > y'$.

  This means that the spectra of the X pieces corresponding to (a,b,c,d,e,x,z) and (a,d,c,e,b,x,z) are not the same.

- If $b < d < e$, then
  - $(d-b)(e-a) > 0$ which implies $ad + be > ab + de$ and
  - $(e-b)(d-a) > 0$ which implies $ab + de > ae + bd$.

  Therefore

$$\begin{aligned} y &= \min\{2(xz - ab - de) - c, 2(cz - ad - be) - x\} \quad \text{and} \\ y' &= \min\{2(cx - ab - de) - z, 2(cz - ae - bd) - x\}. \end{aligned}$$

  But, as

$$\begin{aligned} (2(cz - ae - bd) - x) - (2(cz - ad - be) - x) &= (e-d)(b-a) > 0 \qquad \text{and} \\ (2(cx - ab - de) - z) - (2(cz - ad - be) - x) &= (2c-1)(x-z) \\ &\quad + (d-b)(e-a) \; > \; 0, \end{aligned}$$

  we get $y < y'$.

  This means again that the spectra of the X pieces corresponding to the points (a, b, c, d, e, x, z) and (a, d, c, e, b, x, z) are not the same.

□

Remark 6.23 Note that this proof uses the fact that we deal only with the simple closed spectrum. For the entire spectrum, it is not so easy to determine c, x and z, nor can we easily compare the minimum of the spectrum without c, x and z for the different cases.

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List of Symbols and Notations

Curriculum Vitæ

Originaire de Laufen-Uhwiesen ZH, je suis né le 3 janvier 1977 à St. Gallen. Après avoir effectué ma scolarité obligatoire, j'ai fréquenté la Kantonsschule am Burggraben St. Gallen où j'ai obtenu une maturité de type C (scientifique) en 1996.

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