ANALYSIS AND NUMERICAL SIMULATION OF VISCOELASTIC FLOWS: Deterministic and Stochastic Models

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À mes parents.

Abstract

Mathematical and numerical aspects of viscoelastic flows are investigated here.

Two simplified mathematical models are considered. They are motivated by a splitting algorithm for solving viscoelastic flows with free surfaces. The first model is a simplified Oldroyd-B model. Existence on a fixed time interval is proved in several Banach spaces provided the data are small enough. Short time existence is also proved for arbitrarily large data in Hölder spaces for the time variable. These results are based on the maximal regularity property of the Stokes operator and on the analycity behavior of the corresponding semi-group. A finite element discretization in space is then proposed. Existence of the numerical solution is proved for small data, as well as a priori error estimates, using an implicit function theorem framework. Then, the extension of these results to a stochastic simplified Hookean dumbbells model is discussed. Because of the presence of the Brownian motion, existence in a fixed time interval, provided the data are small enough, is proved only in some of the Banach spaces considered previously. The dumbbells' elongation is split in two parts, one satisfying a standart stochastic differential equation, the other satisfying a partial differential equation with a stochastic source term. A finite element discretization in space is also proposed. Existence of the numerical solution is proved for small data, as well as a priori error estimates.

A numerical algorithm for solving viscoelastic flows with free surfaces is also described. This algorithm is based on a splitting method in time and two different meshes are used for the space discretization. Convergence of the numerical model is checked for the pure extensional flow and the filling of a pipe. Then, numerical results are reported for the stretching of a filament and for jet buckling.

Keywords: viscoelastic fluid, non-Newtonian fluid, Oldroyd-B, Hookkean dumbbells, finite elements, free surface, semi-group, Stokes problem, time splitting method.

Version abrégée

Nous nous intéressons à quelques aspects mathématiques et numériques liés à des écoulements de fluides viscoélastiques.

Dans la première partie de cette thèse, nous traitons quelques problèmes mathématiques simplifiés issus d'un algorithme numérique proposé pour résoudre numériquement des écoulements de fluides viscoélastiques avec surfaces libres. Le modèle d'Oldroyd-B est le premier considéré. L'existence dans un intervalle de temps fixé est prouvée dans plusieurs espaces de Banach en supposant que les données soient suffisamment petites. L'existence pour des données arbitrairement grandes est aussi prouvée mais cette fois uniquement dans l'espace de fonctions Hölderiennes en temps et pour un intervalle de temps suffisamment petit. Ces résultats sont basés sur la propriété de régularité maximale du problème de Stokes et sur le fait qu'il soit générateur d'un semi-groupe analytique. Une méthode d'éléments finis est ensuite proposée. L'existence de la solution numérique ainsi que sa convergence est prouvée pour des données suffisament petites en utilisant le cadre du théorème des fonctions implicites. Ces résultats sont ensuite étendus à un problème stochastique, le modèle des "dumbbells" (haltères). La présence du mouvement Brownien nous conduit à considérer des espaces de Banach spécifiques dans lesquels l'existence est prouvée pour des données suffisamment petites. Pour cela, nous découplons la partie stochastique du reste du problème afin d'invoquer des résultats classiques sur les équations stochastiques et de mettre en place un cadre similaire à celui utilisé pour le problème déterministe. Comme pour le modèle d'Oldroyd-B, une approximation par éléments finis est proposée. L'existence de la solution numérique ainsi que sa convergence est prouvée pour des données suffisament petites.

Dans la deuxième partie de la thèse, nous décrivons l'algorithme numérique proposé pour résoudre l'écoulement de fluides viscoélastiques avec surfaces libres. Cet algorithme est basé sur une méthode de pas fractionnaires en temps et deux maillages différents sont utilisés pour l'approximation en espace. Nous vérifions la convergence du modèle numérique dans les cas d'élongation pure de fluides et de remplissage de tubes. Pour finir, des simulations numériques du flambage d'un jet viscoélastique et d'étirement d'un filament viscoélastique sont présentées.

Mots clés: fluide viscoélastique, fluide non Newtonien, Oldroyd-B, Hookkean dumbbells, éléments finis, surface libre, semi-groupe, problème de Stokes, méthode de pas fractionnaires.

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Lausanne, le 13 mars 2006 Andrea Bonito

Contents

Introduction

Numerical modeling of viscoelastic flows is of great importance for complex engineering applications involving blood, paints and adhesives. Many manufacturers use viscoelastic (or more generally non-Newtonian) fluids in order to benefit from their rheological properties; for instance to thicken soup one can add inelastic polymers, while multi-grade oils have polymer additives [**KDI$^+$05, KAHC05, GOS05, OSG02**]. This thesis investigates simplified deterministic and stochastic models for viscoelastic flows. These simplifications are motivated by the numerical algorithm proposed for simulating viscoelastic flows with complex free surface, the latter is presented in the second part of this thesis.

Newtonian fluids enjoy the property that the relaxation time - the time for the stress to return to zero under constant-strain condition - is immediate. In practice such behavior is never realized and is a mathematical idealization. Indeed, stress-relaxation after the imposition of a constant-strain condition takes place over some finite non-zero time interval which is the defining characteristic of viscoelastic fluids. For instance, time relaxation of water is about 10^{-12} s, that of a low density polyethylene about 10 s and in excess of 28 hours for a glass, see [**OP02**]. The non-dimensional parameter which characterizes this property is the *Deborah*[1] number

$$De := \frac{\text{relaxation time} \times \text{characteristic velocity}}{\text{characteristic length}}$$

of flow process. Viscoelastic fluids may also differ from purely Newtonian fluids by the presence of normal stress differences which causes, in particular, unusual behavior of the liquid shape. When a rod is rotated inside a fluid, inertia in Newtonian case would dominate and the fluid would flow away from the center, whilst the viscoelastic fluid would "climb" along the rod as shown in Fig. 1. Another example is the so-called fingering instabilities. The flow of a thin layer of viscoelastic fluid leads to complex instabilities when the top end-plate is moved vertically as shown in Fig. 1.

In the traditional macroscopic approach for modelling viscoelastic flows, the unknowns are the velocity, the pressure and the extra-stress (the non-Newtonian component of the stress) satisfying the mass and momentum equations supplemented with a so-called constitutive or closure equation. This constitutive equation relating the extra-stress to the velocity can be either differential or integral, refer to [**BCAH87, OP02**].

The mass and momentum conservation laws lead to the following partial differential equations for the velocity u, the pressure p and the extra-stress σ (the non-Newtonian part of the stress)

$$\rho\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) - \nabla \cdot (2\eta_s \epsilon(u) + \sigma) + \nabla p = f, \tag{0.1}$$

$$\nabla \cdot u = 0. \tag{0.2}$$

[1] In report with the prophetess Deborah, who stated "The mountains flowed before the Lord", Judges 5:5 in the Old Testament, see [**OP02**].

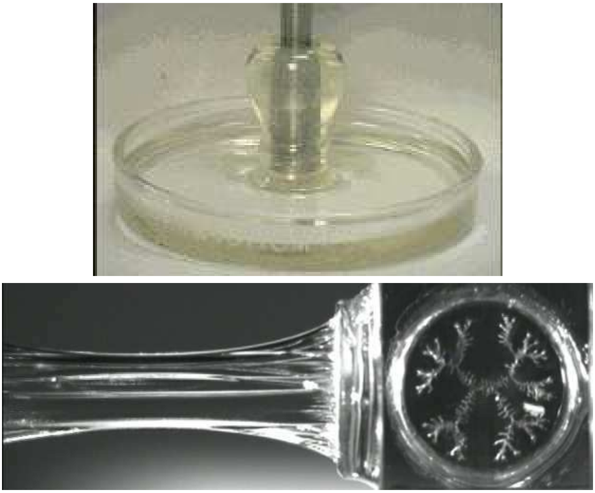

FIGURE 1. **Top:** Rod climbing effect or "Weissenberg effect". A Rod is rotated with its end immersed in a viscoelastic fluid. **Bottom:** The separation of twoplates with a thin layer of a viscoelastic liquid between them can lead to complex instabilities. For more details and original pictures see the web site of the group of Prof. Gareth McKinley at MIT *HTTP://web.mit.edu/nnf/research/*. Reprinted with permission.

Here ρ is the density, f a force term, η_s is the solvent viscosity and

$$\epsilon(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

the strain rate tensor. Oldroyd-B fluids are those where equations (0.1) and (0.2) are supplemented with the following constitutive or closure equation

$$\sigma + \lambda\Big(\frac{\partial \sigma}{\partial t} + (u \cdot \nabla)\sigma - (\nabla u)\sigma - \sigma(\nabla u)^T\Big) - 2\eta_p \epsilon(u) = 0, \tag{0.3}$$

where η_p is the polymer viscosity, λ the relaxation time and $(\nabla u)\sigma$ denotes the matrix-matrix product between ∇u and σ. Obviously, when $\lambda = 0$, the Oldroyd-B model reduces to the incompressible Navier-Stokes equations, refer to [**vW85, Soh01, Tem84, Glo03**].

Although the Oldroyd-B model is too simple to describe complex experiments such as shear thinning, it already contains some mathematical and numerical difficulties. Indeed, when solving numerically Oldroyd-B fluids, one is faced by the "high Deborah (or Weissenberg) number problem", i.e. the breakdown in convergence of algorithms when the Deborah number increases. The sources of this problem are due to :

i) the presence of the quadratic term $(\nabla u)\sigma + \sigma(\nabla u)^T$ which prevents a priori estimates and therefore existence to be proved for large data;

ii) the presence of a convective term $(u \cdot \nabla)\sigma$ which needs to use numerical schemes more suited to transport dominated problems;

iii) when using finite element methods, the case $\eta_s = 0$ requires either a compatibility condition between the finite element spaces for u and σ or the use of adequate stabilization procedures such as EVSS.

The Oldroyd-B model is the simplest macroscopic example but several other models have been introduced such as the Giesekus [**Gie82**] and Phan-Thien Tanner [**PTT77**] models.

Concerning the mathematical analysis of system (0.1)-(0.3), the existence of slow steady viscoelastic flow has been proved in [**Ren88, AS03**]. For the time-dependent case, existence of solutions locally in time and, for small data, globally in time has been proved in [**GS90**] in Hilbert spaces. Extensions to Banach spaces and a review can be found in [**FCGO02**]. Finally, existence for any data has been proved in [**LM00**] for a corotational Oldroyd model only.

For a description of numerical procedures used for solving viscoelastic flows in the engineering community, refer to [**Baa98, OP02**]. Convergence of finite element methods for the linear three fields Stokes problem have been studied in [**FP89, San93, FGP00, BPS01**]. Convergence of continuous and discontinuous finite element methods for steady state viscoelastic fluids have been presented in [**BS92, San94, NS95, FZ03**], provided that the solution of the continuous problem is smooth and small enough. Extensions to time-dependent problems have been proposed in [**BW95, EH04, EM03, ME01**].

The Oldroyd-B model can be derived by a kinetic theory [**BCAH87, Ött96**] from the mesoscopic Hookean dumbbells model. The stochastic dumbbells model is defined as a dilute solution of a liquid polymer, i.e. a Newtonian solvent with non interacting polymer chains. The polymer chains are modelled by dumbbells, two beads connected with elastic springs, see Fig. 2. In this case, the Newton law for the beads leads to the following stochastic differential equation

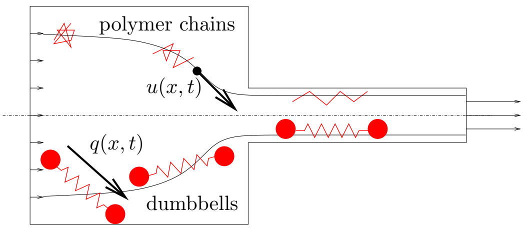

FIGURE 2. The mesoscopic dumbbells model for a dilute solution of liquid polymer.

for the dimensionless spring elongation q

$$dq = \left(-(u\cdot\nabla)q) + (\nabla u)q - \frac{1}{2\lambda}\mathsf{F}(q)\right)dt + \frac{1}{\sqrt{\lambda}}\,dB, \tag{0.4}$$

where λ is the relaxation time, F is the force due to the elastic spring and B is a vector of independent Wiener processes modelling the thermal agitation and collisions with the solvent molecules. The transport term $(u\cdot\nabla)q$ in (0.4) reflects that the trajectories of the center of mass of the dumbbells are those of the liquid particles. The term $(\nabla u)q$ takes into account the drag force due to the beads. The extra-stress σ is then obtained by

$$\sigma = \frac{\eta_p}{\lambda}(\mathbb{E}(q\otimes\mathsf{F}(q)) - I), \tag{0.5}$$

with η_p being the polymer viscosity. Equations (0.1) and (0.2) coupled with (0.4) and (0.5) yield the dumbbells model. To the major difficulties already present in the determinist model, those resulting from the stochastic model must be added:

iv) the vector field F may be non-linear;
v) the presence of the quadratic term $(\nabla u)q$ prevents a priori estimates and therefore existence to be proved for large data;
vi) the presence of the convective term $(u \cdot \nabla)q$ which requires an adequate mathematical analysis [**LBL04**] and the use of numerical schemes suited to transport dominated problems;
vii) the Wiener process in (0.4) which requires a specific mathematical treatment.

The case $\mathsf{F}(q) = q$, namely Hookean dumbbells leads, using formal stochastic calculus, to the Oldroyd-B model (0.3). The FENE dumbbells model (see [**BCAH87, Ött96**] for a detailed description) is a more realistic model corresponding to $\mathsf{F}(q) = \frac{q}{1-q^2/b}$, where $b > 0$ depends on the number of monomer units of a polymer chain. In that case, there is no equivalent constitutive relation for the extra-stress, but closure approximations (such as FENE-P, see [**BCAH87, Ött96**]) have been derived. These approximations can have a significant impact on rheological predictions (see [**AZ03, BHH98, Keu97**]). Recently, thanks to increasing computational resources, the equations (0.1), (0.2),(0.4) and (0.5) have been solved numerically to obtain more realistic results [**BHH97, BP01, BHÖ97, Keu97, LÖP97, LÖ93**]. For a review of numerical methods used for kinetic models see [**OP02, Keu04**].

Only a few mathematical papers pertaining to the kinetic theory have been published. For one dimensional FENE shear flows, a complete mathematical and numerical analysis can be found in [**JLLB02, JL03, JLLB04, Lel04**]. Well posedness of the dumbbell model in three dimensions has been proved for nonlinear elastic dumbbells in [**ELZ04**].

The kinetic theory can also be formulated by introducing the probability density $f(x, q, t)$ of the spring elongation which must satisfy a Fokker-Planck equation

$$\frac{\partial}{\partial t} f(x,q,t) + (u \cdot \nabla) f(x,q,t) = -\nabla_q \cdot \left[\left((\nabla u)q - \frac{1}{2\lambda}\mathsf{F}(q) \right) f(x,q,t) \right] + \frac{1}{2\lambda} \Delta_q f(x,q,t).$$

Refer to [**Fan89a, Fan89b, CL03**] for numerical experiments and [**BSS05, Ren91**] for a mathematical analysis. This deterministic approach seems to be inappropriate when considering more complex kinetic models involving chains [**Keu04**], although recent advances are encouraging [**vPS04**].

Several experiments, such as jet swell, jet buckling, mould filling and impacting drops involve free surfaces with complex topological changes. Therefore, Lagrangian or ALE (Arbitrary Lagrangian Eulerian) methods can not be used in this context. An alternative is to use Eulerian methods and to solve an additional advection equation, namely

$$\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi = 0.$$

In the level-set [**Set96, CHMO96, SS03**] or pseudo-concentration [**Tho86**] approach, the function φ is smooth and the free surface is defined to be the zero level-set. In the VOF (Volume of Fluid) [**HN81**] or the volume tracking method [**RK98, SZ99**] approach, the function φ denotes the volume fraction of liquid and is a step function having a value of one in the liquid and zero in the surrounding vacuum.

Level-set and VOF methods have produced an enormous amount of literature, both methods having their advantages and disadvantages. Roughly speaking, VOF-like methods suffer from the lack of regularity of the volume fraction of liquid on the interface. For instance, post-processing algorithms such as SLIC or PLIC have to be used in order to reduce numerical diffusion, see [**SZ99**] for a review. Moreover, special care is needed when computing geometrical properties of the interface (such as normal and curvature) which are relevant when surface tension effects have to be accounted for, see [**SZ99, RR02**]. Since the level-set method involves a continuous function, better accuracy should be expected. However, mass conservation is difficult to obtain so special procedures must be added, see [**vdPSVW05, Par04, Smo01**].

Although the VOF model was initially solved using finite volumes, finite element implementations have recently been proposed in three space dimensions [**PC98, MPR03**]. In [**MPR99, MPR03, CPR05, Cab05**], the VOF formulation was solved using a first order implicit splitting algorithm and two different grids, see Fig. 3. In the prediction step, the advection equations

$$\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi = 0,$$
$$\frac{\partial u}{\partial t} + (u \cdot \nabla)\, u = 0,$$
$$\frac{\partial \sigma}{\partial t} + (u \cdot \nabla)\, \sigma = 0,$$

were solved on a structured grid of small cubic cells. In the correction step, the equations

$$\rho \frac{\partial u}{\partial t} - \nabla \cdot (2\eta_s \epsilon(u) + \sigma) + \nabla p = f, \tag{0.6}$$
$$\nabla \cdot u = 0, \tag{0.7}$$
$$\sigma + \lambda\Big(\frac{\partial \sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\Big) - 2\eta_p \epsilon(u) = 0, \tag{0.8}$$

were solved on an unstructured finite element tetrahedral mesh. Refer to [**Glo03, Mar90**] for a general description of splitting methods and to [**Glo03**] for their applications to the Navier-Stokes equations. The reasons for using two different grids are the following: Firstly, the advection step is much easier to implement and requires less CPU time on a structured grid rather than on a general unstructured finite element mesh. Secondly, the size of the structured cells should be small compared to the size of the unstructured finite element mesh (typically three to five times smaller), so that numerical diffusion of the volume fraction of liquid is as small as possible. Thirdly, the use of unstructured finite elements during the diffusion step of the algorithm allows computational domains with complex shapes to be considered. This approach has been successfully applied to Newtonian flows with complex free surfaces, see [**MPR99, MPR03, CPR05, Cab05**]. The goal is to extend it to non-Newtonian (viscoelastic) simulations.

Recently, numerical simulation of viscoelastic flows with free surfaces have received much attention, see [**TMC$^+$02, YM98, SL97, RH99, BRLH02, LOQ02**]. In a number of papers, the filament stretching rheometer was considered, this experiment being well suited to Lagrangian computations in two [**YM98, SL97**] and three dimensions [**RH99, BRLH02**]. Two dimensional computations of viscoelastic flows using the VOF method have already been presented. For instance in [**TMC$^+$02, LR00**], the capabilities of the VOF method has been demonstrated for a number of problems such as jet swell, jet buckling and impacting drops. Moreover, two dimensional computations using the VOF method and CONNFFESSIT like models have been presented in [**GLP03**]. It should be mentioned that the level set method has also been successfully used for viscoelastic fluids with free surfaces in [**GBO04**] using the hybrid particle level set method of [**EFFM02**].

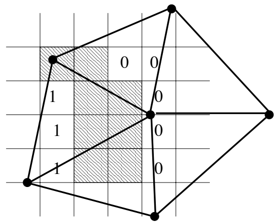

FIGURE 3. Two grids are used for the computations. In order to reduce numerical diffusion and to simplify the implementation, the volume fraction of liquid is computed on a structured grid of small cells. The velocity, pressure and extra-stress are computed on an unstructured finite element mesh with larger size. The symbol 1 (resp. 0) denotes a cell completely filled (resp. empty). The partially filled cells are shaded.

The theoretical study of simplified problems will require concepts and results which will be provided in the first chapter. Functional spaces, semi-group results and maximal regularity property are introduced. Then, existence and uniqueness on the perturbed abstract Cauchy problem:

$$\frac{\partial u}{\partial t} = -Au - k * Au + f, \qquad u(0) = u_0,$$

are stated, where $k * Au$ denotes the convolution in time of the kernel $k(t) = e^{-t/\lambda}$ with Au. In this thesis, only the case when A is the Stokes operator will be considered, consequently the first part ends with deeper investigations about the Stokes operator.

The second part (chapters 2 and 3) is devoted to the mathematical and numerical analysis of simplified models. In chapter 2, the simplified time-dependent Oldroyd-B problem (0.7)-(0.8) is considered. The implicit function theorem can be used to prove an existence result, whenever the data are small enough, in accordance with the results of [**FCGO02**]. The regularity of the solution is sufficient to prove convergence of a finite element discretization in space by extending the techniques presented in [**PR01**] in a time-dependent framework. It should be noted that the analysis remains valid for more realistic fluids such as Giesekus or Phan-Thien-Tanner. In the third chapter, these ideas are extended to the stochastic description (0.2), (0.1), (0.4) and (0.5). The points $v)$ and $vii)$ will be investigated and, as in the deterministic case, assume $\eta_s > 0$, remove the convective terms and consider $\mathsf{F}(q) = q$, namely the simplified Hookean dumbbells model.

Finally, in the last chapter, the numerical algorithm proposed for solving viscoelastic flows with complex free surfaces is described. Convergence of the numerical model is checked for the pure extensional flow and the filling of a pipe. Then, numerical results are reported for the stretching of a filament and for jet buckling.

CHAPTER 1

Preliminary materials

In chapter 2 and 3, existence and uniqueness of solutions to simplified Oldroyd-B and Hookean dumbbells problems will be based on properties of the initial value problem:

$$\dot{u} = -A_r u + f, \qquad u(0) = 0,$$

where A_r is the Stokes operator defined in the last part of this chapter. This preliminary chapter is devoted to the introduction of some definitions and results used throughout this thesis. Function spaces framework and the semi-group theory will be detailed in the first two sections. Then results on the abstract Cauchy problem are provided and extended when the perturbation $-k * A_r u$ is added on the right hand side, where $k * A_r u$ denotes the convolution in time of the kernel $k(t) = e^{-t/\lambda}$ with $A_r u$. Finally, this chapter ends with the description of the Stokes operator and some of its properties.

1.1. Function spaces

Let $(M, \Sigma, \boldsymbol{\mu})$ be a measurable space and let $(E, \|.\|_E)$ be a Banach space. The spaces $L^r(M, \Sigma, \boldsymbol{\mu}; E)$ (in short $L^r(M; E)$), $1 \leq r < \infty$ are the spaces of (equivalence classes of) all Bochner-measurable functions $f : M \to E$ such that $\|f(.)\|_E^r$ is integrable, see [**Yos80**, section V.5]. These spaces are Banach spaces when normed by

$$\|f\|_{L^r(M;E)} := \left(\int_M \|f(x)\|_E^r \, d\boldsymbol{\mu}(x) \right)^{1/r}.$$

When $\boldsymbol{\mu}(M) < \infty$, the spaces of (equivalence classes of) functions $f \in L^r(M, \Sigma, \boldsymbol{\mu}; E)$ such that

$$\int_M f(x) \, d\boldsymbol{\mu}(x) = 0$$

are denoted by $L_0^r(M, \Sigma, \boldsymbol{\mu}; E)$. When $r = \infty$, the space of (equivalence classes of) functions $L^\infty(M, \Sigma, \boldsymbol{\mu}; E)$ (in short $L^\infty(M; E)$) denotes the Banach space of all Bochner-measurable functions $f : M \to E$ such that

$$\inf \{C \geq 0; \|f(.)\|_E \leq C \text{ almost everywhere in } M\} < \infty$$

and is a Banach space with the norm

$$\|f\|_{L^\infty(M;E)} := \inf \{C \geq 0; \|f(.)\|_E \leq C \text{ almost everywhere in } M\}.$$

From now on let D be an open set of $\mathbb{R}^d$, $d \geq 1$ with the *Lebesgue* measure. The Sobolev spaces $W^{m,r}(D; E)$, m a non-negative integer and $1 \leq r < \infty$, are the spaces of functions $f \in L^r(D; E)$ such that all the distributional derivatives of f of order up to m belong to $L^r(D; E)$.

Other classes of functions will be useful when considering a Brownian motion. Let $\mathcal{C}^0(\overline{D}; E)$ be the class of uniformly continuous functions belonging to $L^\infty(D; E)$. The spaces $\mathcal{C}^0(\overline{D}; E)$ is a Banach space endowed with the norm defined for $f \in \mathcal{C}^0(\overline{D}; E)$ by

$$\|f\|_{\mathcal{C}^0(\overline{D};E)} := \sup_{x \in D} \|f\|_E.$$

The Hölder continuous function spaces $\mathcal{C}^\mu(\overline{D};E)$, $0<\mu<1$ are the spaces of all functions belonging to $\mathcal{C}^0(\overline{D};E)$ such that

$$\sup_{\substack{x,y\in D\\ x\neq y}}\frac{\|f(x)-f(y)\|_E}{|x-y\,|^\mu}<\infty.$$

The spaces $\mathcal{C}^\mu(\overline{D};E)$ endowed with the norm

$$\|f\|_{\mathcal{C}^\mu(\overline{D};E)}:=\|f\|_{\mathcal{C}^0(\overline{D};E)}+\sup_{\substack{x,y\in D\\ x\neq y}}\frac{\|f(x)-f(y)\|_E}{|x-y\,|^\mu},$$

are Banach spaces. Higher order spaces can also be considered in this context. Let $0<\mu<1$ and m be a non negative integer. The spaces $\mathcal{C}^{m+\mu}(\overline{D};E)$ are all m times continuously differentiable functions such that all the derivatives up to order m, namely $f^{(k)}$ with $k=0,\dots,m$, are bounded and such that $f^{(m)}\in\mathcal{C}^\mu(\overline{D};E)$. The spaces $\mathcal{C}^{m+\mu}(\overline{D};E)$ provided with the norms

$$\|f\|_{\mathcal{C}^{m+\mu}(\overline{D};E)}:=\sum_{k=0}^m\|f^{(k)}\|_{\mathcal{C}^0(\overline{D};E)}+\sup_{\substack{x,y\in D\\ x\neq y}}\frac{\|f^{(m)}(x)-f^{(m)}(y)\|_E}{|x-y\,|^\mu},$$

are Banach spaces. The spaces $\mathcal{C}^{m+\mu}(\overline{D};E)$ are not separable. So, instead, consider the little Hölder spaces

$$h^{m+\mu}(\overline{D};E):=\Big\{f\in\mathcal{C}^{m+\mu}(\overline{D};E);\lim_{\delta\to0}\sup_{\substack{x,y\in D\\ |x-y\,|<\delta}}\frac{\|f^{(m)}(x)-f^{(m)}(y)\|_E}{|x-y\,|^\mu}=0\Big\},$$

provided with the norm of $\mathcal{C}^{m+\mu}(\overline{D};E)$. The spaces $h^\mu(\overline{D};E)$ are the closures of $\mathcal{C}^\theta(\overline{D};E)$ in $\mathcal{C}^\mu(\overline{D};E)$ for $\theta>\mu$. Moreover, assuming E is a separable Banach space, $h^\mu(\overline{D};E)$ with $0<\mu<1$ are separable Banach spaces. When considering $D=(0,T)$, $0<T<\infty$, the restriction of functions of $h^\mu([0,T];E)$ vanishing at the origin of the interval $[0,T]$ is denoted by $h^\mu_0([0,T];E)$.

Assume now that D has boundary ∂D of class $\mathcal{C}^2$. An important property of the spaces $W^{1,r}(D;\mathbb{R})$, is the continuous embedings [**Ada70**, chapter V]:

$$\begin{aligned}
&W^{1,r}(D;\mathbb{R})\subset_> \mathcal{C}^\mu(\overline{D};\mathbb{R}) &&\text{for } r>d,\ 0<\mu\le 1-r/d &&\text{if } d>1,\\
&W^{1,r}(D;\mathbb{R})\subset_> \mathcal{C}^0(\overline{D};\mathbb{R}) &&\text{for } r\ge 1 &&\text{if } d=1.
\end{aligned}$$

In particular, $W^{1,r}(D;\mathbb{R})$ with $r>d$ is a Banach algebra (again see [**Ada70**, chapter V]). Moreover, using the same arguments as in [**Ada70**] ($d=1$), it holds for $0<T<\infty$

$$W^{1,r}(0,T;E)\subset_>\mathcal{C}^0([0,T];E)\qquad\forall r\ge1.$$

The following spaces will be considered in this thesis. Let $D\subset\mathbb{R}^d$, $d\ge2$ be a bounded domain considered with the *Lebesgue* measure, $[0,T]$ be a time interval also considered with the *Lebesgue* measure and $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space. For simplicity, the notation will be abridged as follows whenever there is no possible confusion. For $1<r,q<\infty$ and $0<\mu<1$ the space L^r denotes $L^r(D;\mathbb{R})$ or $L^r(D;\mathbb{R}^d)$. Also, $L^q(L^r)$ stands for $L^q(0,T;L^r(D;\mathbb{R}))$ or $L^q(0,T;L^r(D;\mathbb{R}^d))$ and $h^\mu(L^r)$ stands for $h^\mu([0,T];L^r(D;\mathbb{R}))$ or $h^\mu([0,T];L^r(D;\mathbb{R}^d))$. For $1<r<\infty$, $0<\mu<1$ and $1<\gamma<\infty$, $L^\gamma(h^\mu(L^r))$ stands for $L^\gamma(\Omega;h^\mu([0,T];L^r(D;\mathbb{R})))$ or $L^\gamma(\Omega;h^\mu([0,T];L^r(D;\mathbb{R}^d)))$. The same notation applies for higher order spaces such as $W^{1,r}$, $h^{1+\mu}(W^{1,r})$ and $L^\gamma(h^{1+\mu}(W^{1,r}))$.

1.2. Semi-group theory

This section contains basics definitions and classical results for the semi-group theory. References will be constantly made to [**Yos80, HP74, Lun95, BB67, CHA$^+$87, DL88**]. Let $(E, \|.\|_E)$ be a Banach space (restricting considerations to that case, see [**Yos80**] for a more general approach).

Start with the definition of a $(\mathcal{C}_0)$ semi-group.

Definition 1.1. If $\{T_t; t \geq 0\} \subset \mathcal{L}(E)$ satisfy the conditions

$$\begin{aligned} T_t T_s &= T_{t+s} \quad (\text{for } t, s \geq 0), \\ T_0 &= I, \\ \lim_{t \to t_0} \|T_t x - T_{t_0} x\|_E &= 0 \qquad \text{for each } t_0 \geq 0 \text{ and each } x \in E, \end{aligned}$$

then $\{T_t\}$ is called a semi-group of class $(\mathcal{C}_0)$.

Given a $(\mathcal{C}_0)$ semi-group, one can define its infinitesimal generator.

Definition 1.2. The infinitesimal generator $A : \mathcal{D}_A \subset E \to E$ of T_t is defined by

$$\mathcal{D}_A := \left\{ x \in E; \lim_{t \to 0^+} t^{-1}(T_t - I)x \in E \right\} \quad \text{and} \quad Ax := \lim_{t \to 0^+} t^{-1}(T_t - I)x, \quad \forall x \in \mathcal{D}_A.$$

The well known Hille-Yosida Theorem provides a necessary and sufficient condition for an operator A with dense domain to be the (infinitesimal) generator of a $(\mathcal{C}_0)$ semi-group. In order to state this theorem, notion of the resolvent of an operator is needed. Let A_λ be defined by $A_\lambda := \lambda I - A$. The resolvent set $\rho(A)$ is defined by the set of all $\lambda \in \mathbb{C}$ such that A_λ^{-1} exists with domain $\mathcal{D}_{A_\lambda^{-1}}$ dense in E and such that A_λ^{-1} is bounded. The inverse operator A_λ^{-1} will be denoted $R(\lambda; A)$ and is called the resolvent of A. Lemma is the following:

Lemma 1.3. *(Hille-Yosida) A necessary and sufficient condition for a closed linear operator A with dense domain $\mathcal{D}_A$ and range in E to be the generator of a semi-group $\{T_t; 0 \leq t < \infty\}$ of class $(\mathcal{C}_0)$ is that there exist real numbers C and w such that for every real $\lambda > w$, λ belongs to $\rho(A)$ and*

$$\|R(\lambda; A)^r\|_{\mathcal{L}(E)} \leq C(\lambda - w)^{-r} \quad (r = 1, 2, \dots).$$

In this case $\|T_t\|_{\mathcal{L}(E)} \leq Ce^{wt}$ for $t \geq 0$.

A important particular subset of $(\mathcal{C}_0)$ semi-groups are analytic (or holomorphic) ones. They are characterized by the fact they can, as functions of parameter t, be continued holomorphically into a sector of the complex plane containing the positive $t-$axis. From now on and until the end of this section, assume E is a complex Banach space.

Lemma 1.4. *If $\{T_t; 0 \geq t < \infty\}$ is of class $(\mathcal{C}_0)$ in $\mathcal{L}(E)$ (E a complex Banach space) such that for each $t > 0$, the range of T_t in E belongs to $\mathcal{D}_A$ and if there is a constant $N > 0$ with $t\|AT_t\|_{\mathcal{L}(E)} \leq N$ $(0 < t \leq 1)$, then this semi-group has a holomorphic extension $\{T_\zeta; \zeta \in \Delta\}$, where*

$$\Delta := \{\zeta \in \mathbb{C}\,; \Re(\zeta) > 0, |\arg \zeta\,| < 1/(eN)\}\,.$$

Remark 1.5. Other sufficient conditions for a $(\mathcal{C}_0)$ semi-group to be continued holomorphically are provided in [**Yos80**].

A class of operator being the generator of analytic semi-groups will be now determined.

Definition 1.6. $A : \mathcal{D}_A \subset E \to E$ is said to be sectorial if there are constants $w \in R$, $\theta \in]\pi/2, \pi[$, $C > 0$ such that

$$\begin{cases} \rho(A) \supset S_{\theta,w} := \{\lambda \in \mathbb{C} : \lambda \neq w, |\arg(\lambda - w)| < \theta\}, \\ \qquad\qquad \|R(\lambda, A)\|_{\mathcal{L}(E)} \leq \dfrac{C}{|\lambda - w|} \quad \forall \lambda \in S_{\theta,w}. \end{cases}$$

In the case of a sectorial operator, it is possible to define for every $t > 0$ a linear bounded operator e^{tA} in E, by the mean of the Dunford integral

$$e^{tA} := \frac{1}{2\pi i} \int_{w+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda \quad t > 0, \qquad e^{0A} := I, \tag{1.1}$$

where $r > 0$, $\eta \in]\pi/2, 0[$ are properly chosen, and $\gamma_{r,\eta}$ is the curve

$$\{\lambda \in \mathbb{C} : |\arg \lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \eta, |\lambda| = r\},$$

oriented counterclockwise, for more precisions see [**Yos80, Lun95, DHP03**].

Definition 1.7. Let $A : \mathcal{D}_A \subset E \to E$ be a sectorial operator. The family $\{e^{tA} : t \geq 0\}$ defined by (1.1) is said to be the analytic semi-group generated by A in E.

In [**Lun95**], the validity of the previous definition is ensured. More precisely, it is proved that the definition of e^{tA} does not depends on r, η and the family semi-group $\{e^{tA} : t \geq 0\}$ satisfies the hypothesis of Proposition 1.4.

Remark 1.8. Reciprocally, the infinitesimal generator of an analytic semi-group is sectorial, see [**ABHN01**, Proposition 3.7.4 and Corollary 3.7.12].

From now on and until the end of this section, assume A is a sectorial operator. Intermediate spaces between E and $\mathcal{D}_A$ will be useful when considering initial value problems as in sections 1.3 and 1.4. General definitions of those spaces will not be provided but will be restricted to those needed in this thesis. Denote $E_{1-1/q,q} := (E, \mathcal{D}_A)_{1-1/q,q}$ a real interpolation space which can be defined as

$$E_{1-1/q,q} := \left\{ v \in E; \int_0^\infty \|Ae^{tA}v\|_E^q < \infty \right\},$$

with $1 < q < \infty$. The space $E_{1-1/q,q}$ is a Banach space with norm

$$\|v\|_{E_{1-1/q,q}} := \|v\|_E + \left(\int_0^\infty \|Ae^{tA}v\|_E^q \right)^{1/q}.$$

Moreover, the spaces $E_{\mu,\infty}$ can be defined as

$$E_{\mu,\infty} := \left\{ v \in E; \sup_{t>0} \|t^{1-\mu} Ae^{tA}v\|_E < \infty \right\},$$

with $0 < \mu < 1$ and is a Banach space endowed with the norm

$$\|v\|_{E_{\mu,\infty}} := \|v\|_E + \sup_{t>0} \|t^{1-\mu} Ae^{tA}v\|_E.$$

Refer to [**DB84, Sin85, DPS87**] for more details on intermediate spaces.

1.3. The abstract Cauchy problem

Let $(E, \|.\|)$ be a complex or real Banach space. The aim of this section is to recall some existing results concerning the abstract Cauchy problem value problem in E

$$\dot{u}(t) = Au(t) + f(t), \quad t > 0; \qquad u(0) = u_0, \tag{1.2}$$

where $A : \mathcal{D}_A \subset E \to E$ is a linear, closed, densely defined operator, where $f : [0, T] \to E$ is a given source term and where $u_0 \in E$. Then, in next section, these results will be extended when a perturbation of lower order in time is added to (1.2). Constant reference to the works of Da Prato and Sinestrari [**Sin85, DPS87**], Butzer and Berens [**BB67**] and Lunardi [**Lun95**] will be made.

The notion of integral solution of (1.2) suggested by its formal integration is now introduced.

DEFINITION 1.9. Let $u_0 \in E$ and let $f \in L^1(0, T; E)$. A function $u \in \mathcal{C}^0([0, T]; E)$ is said to be an integral solution of the Cauchy problem (1.2) if

$$\int_0^t u(s)ds \in \mathcal{D}_A, \quad \forall t \in [0, T]$$

and if

$$u(t) = u_0 + A\int_0^t u(s)ds + \int_0^t f(s)ds, \quad t \in [0, T]. \tag{1.3}$$

REMARK 1.10. Let u be an integral solution of (1.2) with $f \in \mathcal{C}^0([0, T]; E)$. Assume $u \in \mathcal{C}^1([0, T]; E)$ or $u \in \mathcal{C}^0([0, T]; \mathcal{D}_A)$ then $u \in \mathcal{C}^1([0, T]; E) \cap \mathcal{C}^0([0, T]; \mathcal{D}_A)$ and satisfies (1.2).

LEMMA 1.11. *Let $(E, \|.\|)$ be a Banach space. Let $A : \mathcal{D}_A \subset E \to E$ be a linear, closed, densely defined operator, the generator of a $(\mathcal{C}_0)$ semi-group on E. Let $f \in L^1(0, T; E)$ and $u_0 \in E$. Then there exists an unique integral solution u of (1.2) and there exist two constants $C \geq 1$, $w \in \mathbb{R}$ independent of u, f, u_0 such that*

$$\|u(t)\|_E \leq Ce^{wt}\|u_0\|_E + C\int_0^t e^{w(t-s)}\|f(s)\|_E ds, \quad t \in [0, T].$$

LEMMA 1.12. *Let $(E, \|.\|_E)$ be a Banach space. Let $A : \mathcal{D}_A \subset E \to E$ be a linear, closed, densely defined operator, the generator of a $(\mathcal{C}_0)$ semi-group on E. Let $f \in W^{1,1}(0, T; E)$ and $u_0 \in \mathcal{D}_A$, then the unique integral solution u of (1.2) satisfies $u \in \mathcal{C}^1([0, T]; E) \cap \mathcal{C}([0, T]; \mathcal{D}_A)$. Moreover, there exists two constants $C \geq 1$ and $w \in \mathbb{R}$ independent of u, f, u_0 such that*

$$\|\dot{u}(t)\|_E + \|Au(t)\|_E \leq Ce^{wt}\|Au_0 + f(0)\|_E + C\int_0^t e^{w(t-s)}\|\dot{f}(s)\|_E ds.$$

LEMMA 1.13. *Let $(E, \|.\|)$ be a Banach space. Let $A : \mathcal{D}_A \subset E \to E$ be a linear, closed, densely defined operator, the generator of an analytic semi-group e^{tA} on E. Let $f = 0$ and $u_0 \in E$. Let $1 < q < \infty$. Then the integral solution u of (1.2) satisfies $u \in W^{1,q}(0, T; E) \cap L^q(0, T; \mathcal{D}_A)$ if and only if $u_0 \in E_{1-1/q,q} := (E, \mathcal{D}_A)_{1-\frac{1}{q},q}$. Moreover $u \in \mathcal{C}^0([0, T]; E_{1-1/q,q})$ and there exists a constant C independent of u, u_0 such that*

$$\|\dot{u}\|_{L^q(0,T;E)} + \|Au\|_{L^q(0,T;E)} + \|u\|_{L^\infty(0,T;E_{1-1/q,q})} \leq C\|u_0\|_{E_{1-1/q,q}}.$$

When $f \neq 0$, the previous Lemma can not be generalized in Sobolev spaces, a maximal regularity result has to be assumed.

DEFINITION 1.14. Let $1 < q < \infty$. The operator A possesses the maximal L^q-regularity property (MRp) if for any $u_0 = 0$ and any $f \in L^q(0, T; E)$, the unique integral solution u of

(1.2) is in $W^{1,q}(0,T;E)\cap L^q(0,T;\mathcal{D}_A)$ and there exists a constant C independent of u, f such that

$$\|\dot{u}\|_{L^q(0,T;E)}+\|Au\|_{L^q(0,T;E)}\leq C\|f\|_{L^q(0,T;E)}.$$

REMARK 1.15. Let $1<q_0<\infty$. If A possesses the maximal L^{q_0}-regularity property, then $\beta A+\omega I$ possesses the maximal L^q-regularity property for $1<q<\infty$, for $\beta>0$ and for $w\in\mathbb{R}$ (see [**Sin85, DPS87**]).

LEMMA 1.16. *Let $(E,\|.\|_E)$ be a Banach space. Let $A:\mathcal{D}_A\subset E\to E$ be a linear, closed, densely defined operator, the generator of an analytic semi-group e^{tA} on E. Assume A satisfies MRp, let $1<q<\infty$, $f\in L^q(0,T;E)$ and $u_0\in E_{1-1/q,q}:=(E,\mathcal{D}_A)_{1-\frac{1}{q},q}$. Then the solution u of (1.2) satisfies $u\in W^{1,q}(0,T;E)\cap L^q(0,T;\mathcal{D}_A)$ and*

$$u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s)ds,\quad t\in[0,T].$$

Moreover, there exists a constant C independent of u, f, u_0 such that

$$\|\dot{u}\|_{L^q(0,T;E)}+\|Au\|_{L^q(0,T;E)}\leq C\left(\|u_0\|_{E_{1-1/q,q}}+\|f\|_{L^q(0,T;E)}\right).$$

COROLLARY 1.17. *Let $(E,\|.\|_E)$ be a Banach space. Let $A:\mathcal{D}_A\subset E\to E$ be a linear, closed, densely defined operator, the generator of an analytic semi-group e^{tA} on E. Suppose A satisfies MRp, let $1<q<\infty$. Let $f\in W^{1,q}(0,T;E)$ and $u_0\in E$. Assume that the compatibility conditions $u_0\in\mathcal{D}_A$ and $Au_0+f(0)\in E_{1-1/q,q}:=(E,\mathcal{D}_A)_{1-\frac{1}{q},q}$ hold. Then the solution u of (1.2) satisfies $\dot{u}\in W^{1,q}(0,T;E)\cap L^q(0,T;\mathcal{D}_A)$ and*

$$u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s)ds,\quad t\in[0,T].$$

Moreover, there exists a constant C independent of u, f, u_0 such that

$$\|\ddot{u}\|_{L^q(0,T;E)}+\|A\dot{u}\|_{L^q(0,T;E)}\leq C\left(\|Au_0+f(0)\|_{E_{1-1/q,q}}+\|\dot{f}\|_{L^q(0,T;E)}\right).$$

The maximal regularity property holds in Hölder spaces and Lemma 1.16 can be stated in this context.

LEMMA 1.18. *Let $(E,\|.\|)$ be a Banach space. Let $A:\mathcal{D}_A\subset E\to E$ be a linear, closed, densely defined operator, the generator of an analytic semi-group e^{tA} on E. Let $0<\mu<1$ and let $f\in\mathcal{C}^\mu([0,T];E)$. Assume that the compatibility conditions $u_0\in\mathcal{D}_A$ and $Au_0+f(0)\in E_{\mu,\infty}:=(E,\mathcal{D}_A)_{\mu,\infty}$ hold. Then there exists a unique solution u of problem (1.2) in $\mathcal{C}^{1+\mu}([0,T];E)\cap\mathcal{C}^\mu([0,T];\mathcal{D}_A)$ and u satisfies*

$$u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s)ds,\quad t\in[0,T].$$

Moreover there exists a constant C independent of u, f, u_0 such that

$$\|\dot{u}\|_{C^\mu([0,T];E)}+\|Au\|_{C^\mu([0,T];E)}\leq C\left(\|f-f(0)\|_{C^\mu([0,T];E)}+\|Au_0+f(0)\|_{E_{\mu,\infty}}\right).$$

The same result is also true in little Hölder space $h^\mu(0,T;E)$ with a slight modification for the initial conditions.

LEMMA 1.19. *Let $(E,\|.\|_E)$ be a Banach space. Let $A:\mathcal{D}_A\subset E\to E$ be a linear, closed, densely defined operator, the generator of an analytic semi-group e^{tA} on E. Let $0<\mu<1$ and let $f\in h^\mu(0,T;E)$. Assume the compatibility conditions $u_0\in\mathcal{D}_A$ and $Au_0+f(0)\in$*

$\overline{\mathcal{D}_A}^{E_{\mu,\infty}}$ *where* $E_{\mu,\infty} := (E, \mathcal{D}_A)_{\mu,\infty}$. *Then there exists a unique solution* u *of problem (1.2) in* $h^{1+\mu}([0,T];E) \cap h^{\mu}([0,T];\mathcal{D}_A)$ *and* u *satisfies*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)ds, \quad t \in [0,T].$$

Moreover there exists a constant C *independent of* u, f, u_0 *such that*

$$\|\dot{u}\|_{h^{\mu}([0,T];E)} + \|Au\|_{h^{\mu}([0,T];E)} \leq C \left(\|f - f(0)\|_{h^{\mu}([0,T];E)} + \|Au_0 + f(0)\|_{E_{\mu,\infty}} \right).$$

1.4. A perturbed abstract Cauchy problem

In this section, it is proved that the results presented in the previous section also hold when a convolution $k * Au$ is added to (1.2). Such results can be found in [**Prü93**] in a more general framework. For the convenience of the reader, simpler proofs are provided for the special case considered.

The convolution product $*$ between $f, g \in L^1(0,T)$ is defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

REMARK 1.20. From [**ABHN01**, Proposition 1.3.2]: if $f \in L^q(0,T;E)$, $g \in L^1(0,T)$ then $f * g \in L^q(0,T;E)$ and

$$\|f * g\|_{L^q(0,T;E)} \leq \|g\|_{L^1(0,T)} \|f\|_{L^q(0,T;E)}.$$

Let us start with a technical Lemma.

LEMMA 1.21. *Given* $\beta \neq 0$, $m \geq 1$ *and* $k \in W^{m,1}(0,T)$, *there exists an unique* $b \in W^{m+1,1}(0,T)$ *such that*

$$\beta b + k * b = 1.$$

PROOF. We recall a result given in Prüss [**Prü93**, Theorem 1.4 p.46]: for $p \geq 1$, there exists an unique $r : W^{m,p}(0,T) \to W^{m,p}(0,T)$ such that for $a \in W^{m,p}(0,T)$

$$r(a) + a * r(a) = a.$$

Then, taking $a = \beta^{-1}k$ in the equation above, the unique solution $S(b_0) \in W^{m,1}(0,T)$ such that $S(b_0) + \beta^{-1}k * S(b_0) = 1$ is given by

$$S(b_0) = 1 - r(\beta^{-1}k) * 1.$$

Thus

$$b(t) = \beta^{-1}S(b_0) = \beta^{-1} - \beta^{-1}r(\beta^{-1}k) * 1$$

and

$$\dot{b}(t) = -\beta^{-1}r(\beta^{-1}k) \quad \text{in } L^1(0,T).$$

Since $r(\beta^{-1}k) \in W^{m,1}(0,T)$, it follows $b \in W^{m+1,1}(0,T)$. □

LEMMA 1.22. *Let* $(E, \|.\|_E)$ *be a Banach space. Let* $A : \mathcal{D}_A \subset E \to E$ *be a linear, closed, densely defined operator, the generator of an analytic semi-group on* E. *Let* $1 < q < \infty$, $\beta > 0$, $\gamma \in \mathbb{R}$ *and* $a \in L^1(0,T)$. *Assume* A *satisfies MRp and let* $f \in L^q(0,T;E)$, $u_0 \in E_q := (E, \mathcal{D}_{A_r})_{1-\frac{1}{q},q}$, *then there exists a unique* $u \in W^{1,q}(0,T;E) \cap L^q(0,T;\mathcal{D}_{A_r})$ *satisfying*

$$\dot{u} = \beta Au + \gamma u + a * u + f, \qquad u(0) = u_0.$$

Moreover, there exists a constant C *independent of* u, f, u_0 *such that*

$$\|\dot{u}\|_{L^q(0,T;E)} + \|Au\|_{L^q(0,T;E)} \leq C \left(\|f\|_{L^q(0,T;E)} + \|u_0\|_{E_q} \right). \tag{1.4}$$

PROOF. Let $B := \beta A+\gamma$, since A satisfies the MRp using Remark 1.15 it follows B satisfies the MRp and there exists a constant C independent of u, f, u_0 such that

$$\|Au\|_{L^q(E)} + \|u\|_{L^q(E)} \leq C\left(\|Bu\|_{L^q(E)} + \|u\|_{L^q(E)}\right), \tag{1.5}$$

for $u \in \mathcal{D}_A$. Therefore, it remains to prove for given $f \in L^q(E)$ and $u_0 \in E_{1-1/q,q}$ there exists an unique $z \in W^{1,q}(E) \cap L^q(\mathcal{D}_A)$ such that

$$z = e^{tB}u_0 + \int_0^t e^{(t-s)B}f(s)ds + \int_0^t e^{(t-s)B}a * z(s)ds. \tag{1.6}$$

Lemma 1.13 ensures $z_0 := e^{tB}u_0 + \int_0^t e^{(t-s)B}f(s)ds \in W^{1,q}(E) \cap L^q(\mathcal{D}_A)$. Then rewrite (1.6) as a fixed point problem. Given $z_0 \in Z := L^q(E)$ and let $F : Z \to Z$ defined for $z \in Z$ by

$$F(z) := v,$$

where, $v \in Z$ satisfies

$$\dot{v} = Bv + a * z, \quad v(0) = 0.$$

Notice F is well defined using Remark 1.20 and Lemma 1.13. Then (1.6) becomes

$$z = z_0 + F(z).$$

It will be shown that there exists $n > 0$ such that

$$\|F^n\|_{\mathcal{L}(Z)} < 1. \tag{1.7}$$

Lemma 1.13 again ensures that there exists a constant C independent of u, f, u_0 such that

$$\|v\|_{L^q(E)} \leq C\|a * u\|_{L^q(E)} \leq C\|a\|_{L^1(0,T)}\|u\|_{L^q(E)}.$$

Denoting by $c^{(n)} := \underbrace{c * \ldots * c}_{n \text{ times}}$ for $c \in L^1(0,T)$, it follows

$$\|F^n\|_{\mathcal{L}(Z)} \leq C^n\|a^{(n)}\|_{L^1(0,T)}. \tag{1.8}$$

Since

$$\|c * c\|_{L^1(0,T)} \leq \|c\|_{L^\infty(0,T)} * \|c\|_{L^\infty(0,T)} = \|c\|^2_{L^\infty(0,T)} 1 * 1 = \|c\|^2_{L^\infty(0,T)}T,$$

for $c \in L^\infty(0,T)$ it follows

$$\|c^{(n)}\|_{L^1(0,T)} \leq \|c\|^n_{L^\infty(0,T)}\frac{T^{n-1}}{(n-1)!}, \quad \forall c \in L^\infty(0,T).$$

Using the above inequality in (1.8), it follows

$$\|F^n\|_{\mathcal{L}(Z)} \leq \frac{T^{n-1}}{(n-1)!}C^n\|a\|^n_{L^\infty(0,T)}$$

which tends to 0 when n goes to infinity. Thus (1.7) is proved and a fixed point theorem (see [**Car67**, Theorem 4.4.1]) ensures the existence of an unique $z \in Z$ satisfying (1.6) and there exists a constant C independent of u, f, u_0 such that

$$\|z\|_{L^q(E)} \leq C\|z_0\|_{L^q(E)}. \tag{1.9}$$

The fact that $z \in W^{1,q}(E) \cap L^q(E)$ is a direct consequence of (1.6) since z_0 and $a * z \in W^{1,q}(E) \cap L^q(E)$. It remains to prove the estimation (1.4). Going back to (1.6) and using Lemma 1.13 again, there exists a constant C independent of u, f, u_0 such that

$$\|\dot{z}\|_{L^q(E)} + \|Bu\|_{L^q(E)} \leq C\left(\|u_0\|_{E_{1-1/q,q}} + \|f\|_{L^q(E)} + \|a * u\|_{L^q(E)}\right). \tag{1.10}$$

Using (1.9) we obtain

$$\|a * z\|_{L^q(E)} \leq \|a\|_{L^1(0,T)}\|z\|_{L^q(E)} \leq C\|a\|_{L^1(0,T)}\|z_0\|_{L^q(E)},$$

which coupled with (1.5) and (1.10) proves (1.18). □

COROLLARY 1.23. *Let $(E, \|.\|_E)$ be a Banach space. Let $A : \mathcal{D}_A \subset E \to E$ be a linear, closed, densely defined operator, the generator of an analytic semi-group on E. Let $1 < q < \infty$, $k \in W^{1,1}(0,T)$ and $\beta > 0$. Assume A satisfies MRp and let $f \in L^q(0,T;E)$, $u_0 \in E_q := (E, \mathcal{D}_{A_r})_{1-\frac{1}{q},q}$, then there exists a unique $u \in W^{1,q}(0,T;E) \cap L^q(0,T;\mathcal{D}_{A_r})$ satisfying*

$$\dot{u} = \beta A u + k * Au + f, \qquad u(0) = u_0. \tag{1.11}$$

Moreover, there exists a constant C independent of u, f, u_0 such that

$$\|\dot{u}\|_{L^q(0,T;E)} + \|Au\|_{L^q(0,T;E)} \leq C\left(\|f\|_{L^q(0,T;E)} + \|u_0\|_E\right). \tag{1.12}$$

PROOF. Since $k \in W^{1,1}(0,T)$, Lemma 1.21 ensure the existence of a $b \in W^{2,1}(0,T)$ such that

$$\beta b + k * b = 1. \tag{1.13}$$

Moreover $b(0) = \beta^{-1}$. Convolving the equation for u in (1.11) and using the equation above, we have

$$b * \dot{u} = 1 * Au + b * f.$$

Differentiating with respect to time the equation above, using $b(0) = \beta^{-1}$, we obtain

$$\beta^{-1}\dot{u} + \dot{b} * \dot{u} = Au + \beta^{-1} f + \dot{b} * f. \tag{1.14}$$

Noticing that

$$\dot{b} * \dot{u} + \dot{b} u_0 = \frac{d}{dt}\left(\dot{b} * u\right) = \dot{b}(0) u + \ddot{b} * u,$$

the equation (1.14) becomes

$$\dot{u} = \beta A u - \beta \dot{b}(0) u + f + \beta \dot{b} * f + \beta \dot{b} u_0 - \beta \ddot{b} * u.$$

Differentiating equation (1.13) and since $\dot{b} \in \mathcal{C}^0([0,T])$, $k \in W^{1,1}(0,T)$ we find $\dot{b}(0) = -\beta^{-2} k(0)$. Finally, the system (1.11) reduces to

$$\dot{u} = \beta A u + f + \beta \dot{b} * f + \beta \dot{b} u_0 - \beta \ddot{b} * u + \beta^{-1} k(0) u, \qquad u(0) = u_0.$$

The Lemma 1.22 completes the proof. □

The previous Corollary also holds in little Hölder spaces.

LEMMA 1.24. *Let $(E, \|.\|_E)$ be a Banach space. Let $A : \mathcal{D}_A \subset E \to E$ be a linear, closed, densely defined operator, the generator of an analytic semi-group on E. Let $0 < \mu < 1$. Let $k \in W^{1,\frac{1}{1-\mu}}(0,T)$, $f \in h^\mu([0,T];E)$ and $u_0 \in \mathcal{D}_A$ satisfying $Au_0 + f(0) \in \overline{\mathcal{D}_A}^{(E,\mathcal{D}_A)_{\mu,\infty}}$, then there exists a unique $u \in h^{1+\mu}([0,T];E) \cap h^\mu([0,T];\mathcal{D}_A)$ satisfying*

$$\dot{u} = \beta A u + k * Au + f, \qquad u(0) = u_0. \tag{1.15}$$

Moreover, there exists a constant C independent of u, f, u_0 such that

$$\|\dot{u}\|_{h^\mu([0,T];E)} + \|Au\|_{h^\mu([0,T];E)} \leq C\left(\|f - f(0)\|_{h^\mu([0,T];E)} + \|Au_0 + f(0)\|_{E_{\mu,\infty}}\right). \tag{1.16}$$

PROOF. The proof uses the same arguments for the proof for Corollary 1.23. It has to be slightly modified in the two following senses. Remark 1.20, has to be replaced by the affirmation: let $0 < \mu < 1$, for $g \in L^{\frac{1}{1-\mu}}(0,T)$, for $f \in h^\mu([0,T];E)$ there exists a constant C independent of $f$and g such that

$$\|f * g\|_{h^\mu(E)} \leq C \|g\|_{L^{\frac{1}{1-\mu}}(0,T)} \|f\|_{h^\mu(E)}. \tag{1.17}$$

In the proof for Lemma 1.22, relation (1.8) does not holds in $h^{\mu}(E)$ but the same conclusion follows by using a fixed point theorem in $h^{\mu}(E)$ and (1.17). $\square$

LEMMA 1.25. *Let $(E, \|.\|_E)$ be a Banach space. Let $A : \mathcal{D}_A \subset E \to E$ be a linear, closed, densely defined operator, the generator of an analytic semi-group on E. Let $1 < q < \infty$, $\beta > 0$ and $k \in W^{1,q}(0,T)$. Assume A satisfies MRp and let $f \in W^{1,q}(0,T;E)$, $u_0 \in \mathcal{D}_{A_r}$ such that $\beta A u_0 + f(0) \in E_q := (E, \mathcal{D}_{A_r})_{1-\frac{1}{q},q}$, then there exists a unique $u \in W^{2,q}(0,T;E) \cap W^{1,q}(0,T;\mathcal{D}_{A_r})$ satisfying*

$$\dot{u} = \beta A u + k * A u + f, \qquad u(0) = u_0.$$

Moreover, there exists a constant C independent of u, f, u_0 such that

$$\|\ddot{u}\|_{L^q(0,T;E)} + \|A\dot{u}\|_{L^q(0,T;E)} \leq C\left(\|\dot{f}\|_{L^q(0,T;E)} + \|Au_0\|_{L^q(0,T;E)} + \|Au_0 + f(0)\|_{E_q}\right). \tag{1.18}$$

PROOF. Let u be the unique solution in $W^{1,q}(0,T;E) \cap L^q(0,T;\mathcal{D}_{A_r})$ satisfying

$$\dot{u} = \beta A u + k * A u + f, \qquad u(0) = u_0.$$

Define $z \in W^{1,q}(0,T;E) \cap L^q(0,T;\mathcal{D}_{A_r})$ such that

$$\dot{z} = \beta A z + k A u_0 + k * A z + \dot{f}, \qquad z(0) = \beta A u_0 + f(0). \tag{1.19}$$

Corollary 1.23 ensures z is well defined since $\dot{f} \in L^q(0,T;E)$ and since $\beta A u_0 + f(0) \in E_q$. Moreover there exists a constant C independent of u, f, u_0 such that

$$\|\dot{z}\|_{L^q(0,T;E)} + \|Az\|_{L^q(0,T;E)} \leq C\left(\|Au_0 + f(0)\|_{E_q} + \|\dot{f}\|_{L^q(0,T;E)} + \|kAu_0\|_{L^q(0,T;E)}\right). \tag{1.20}$$

Let $v \in W^{2,q}(0,T;E) \cap W^{1,q}(0,T;\mathcal{D}_{A_r})$ defined by

$$v(t) := u_0 + \int_0^t z(s)ds. \tag{1.21}$$

It will be shown that $v = u$. Recalling that since A is closed, it holds

$$\int_0^t Ay(s)ds = A\int_0^t y(s)ds, \quad \forall y \in L^1(0,t';\mathcal{D}_{A_r}),\ \forall t' \in [0,t],$$

(see [**ABHN01**, Proposition 1.1.7]) from (1.19) it follows

$$z(t) = \beta A u_0 + f(0) + \beta A\int_0^t z(s)ds + f(t) - f(0) + 1 * kAu_0 + k * A\int_0^T z(s)ds.$$

Then

$$\dot{v} = \beta A(v - u_0) + \beta A u_0 + f(t) + (1 * k)Au_0 + k * A(v - u_0), \qquad v_0 = u_0$$

is obtained using the definition of v (1.21). Thus the uniqueness of the solution ensured by Corollary 1.23 proves $v = u$ or $z = \dot{u}$. The estimate (1.18) is a direct consequence of (1.20). $\square$

1.5. The Stokes operator

Let D be a bounded, connected open set of $\mathbb{R}^d$, $d \geq 2$ with boundary ∂D of class $\mathcal{C}^2$, and let $T > 0$. We introduce the Helmholtz-Weyl projector [**FM77, Gal94, GSS05**] defined by

$$P_r : L^r(D;\mathbb{R}^d) \to \mathcal{H}_r \qquad 1 < r < \infty,$$

where $\mathcal{H}_r$ is the completion of the divergence free $C_0^\infty(D)$ vector fields with respect to the L^r norm. The space $\mathcal{H}_r$ can be characterized as follows (again see [**Gal94**])

$$\mathcal{H}_r := \left\{v \in L^r(D;\mathbb{R}^d)\ ;\ \nabla\cdot v = 0,\ v\cdot n = 0 \text{ on } \partial D, \text{ hold weakly}\right\}.$$

Since D is of class $\mathcal{C}^2$, there exists a constant C such that for $f \in L^r$

$$\|P_r f\|_{L^r} \le C\|f\|_{L^r}.$$

Define $A_r := -P_r\Delta : \mathcal{D}_{A_r} \to \mathcal{H}_r$ the Stokes operator, where

$$\mathcal{D}_{A_r} := \left\{ v \in W^{2,r}(D;\mathbb{R}^d) \cap W_0^{1,r}(D;\mathbb{R}^d) \mid \nabla \cdot v = 0 \right\}.$$

It is well known (see [**Gig81**] for instance) that, for D of class $\mathcal{C}^2$, the operator A_r equipped with the usual norm of $L^r(D;\mathbb{R}^d)$ is closed and densely defined in $\mathcal{H}_r$. Moreover, the graph norm of A_r is equivalent to the $W^{2,r}$ norm.

Since D is of class $\mathcal{C}^2$, the operator $-A_r$ is the generator of an analytic semi-group [**Gig81**] and satisfies the maximal regularity property, see [**Sol68**, Theorem 15, section 17] for the $L^r(L^r)$ estimate and Remark 1.15 for the $L^q(L^r)$ estimate.

Remark 1.26. It will be needed (see sections 2.2 and 2.3 and sections 3.2 and 3.3) that the operator $-A_r$ still satisfies the maximal regularity property when considering a polygonal domain. We did not find such result in the literature, therefore this assumption will be made. It should be noted that the corresponding property is true in the stationary case for some $r > 2$ depending on the angles of the polygon, see [**PR01**].

In what follows, the intermediate spaces between $\mathcal{H}_r$ and $\mathcal{D}_{A_r}$ defined in section 1.2, will only be considered with the Stokes operator. So, the notations

$$E_{1-1/q,q} := (\mathcal{H}_r, \mathcal{D}_{A_r})_{1-1/q,q} \quad \text{and} \quad E_{\mu,\infty} := \left\{ v \in \mathcal{H}_r; \sup_{t>0} \|t^{1-\mu} A_r e^{-tA_r} v\|_{L^r(D)} < \infty \right\},$$

will be used. The following Lemma (personal communication of Prof. Philippe Clément) will be useful thereafter.

Lemma 1.27. *Let $2 \le d < r < \infty$ and $2 \le q$ the following embbedings hold*

$$E_{1-1/q,q} \subset_{>} W_0^{1,r} \cap \mathcal{H}_r \subset_{>} H_0^1 \cap \mathcal{H}_2.$$

CHAPTER 2

Mathematical and numerical analysis of the Oldroyd-B model

In this chapter, a time-dependent model corresponding to an Oldroyd-B or convected Jeffreys fluid in a finite domain is considered, the convective terms being disregarded. The reason for considering this model without the convective terms and in a non-moving domain is motivated by the second step (0.6)-(0.8) of the splitting algorithm presented in the introduction. Existence in a finite time interval is proved in Banach spaces provided the data are small enough, using the implicit function theorem and a maximum regularity property for a three fields Stokes problem. On the other hand, short time existence for arbitrarily large data is proved in Hölder spaces for the time variable using the analyticity behavior of the semi-group generated by the Stokes operator. A finite element discretization in space is then proposed. Existence of the numerical solution is proved for small data, as well as a priori error estimates, using again an implicit function theorem.

2.1. The simplified Oldroyd-B problem and its finite element approximation in space

Let D be a bounded, connected open set of $\mathbb{R}^d$, $d \geq 2$ with boundary ∂D of class $\mathcal{C}^2$, and let $T > 0$. Consider the following problem. Given initial conditions $u_0 : D \to \mathbb{R}^d$, $\sigma_0 : D \to \mathbb{R}^{d\times d}_{sym}$, a force term f, constant solvent and polymer viscosities $\eta_s > 0$, $\eta_p > 0$, a constant relaxation time $\lambda > 0$, find the velocity $u : D \times (0,T) \to \mathbb{R}^d$, pressure $p : D \times (0,T) \to \mathbb{R}$ and extra-stress $\sigma : D \times (0,T) \to \mathbb{R}^{d\times d}_{sym}$ such that

$$\rho \frac{\partial u}{\partial t} - 2\eta_s \, \nabla \cdot \epsilon(u) + \nabla p - \nabla \cdot \sigma = f \qquad \text{in } D \times (0,T), \tag{2.1}$$

$$\nabla \cdot u = 0 \qquad \text{in } D \times (0,T), \tag{2.2}$$

$$\frac{1}{2\eta_p}\sigma + \frac{\lambda}{2\eta_p}\Big(\frac{\partial \sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\Big) - \epsilon(u) = 0 \qquad \text{in } D \times (0,T), \tag{2.3}$$

$$u(\cdot,0) = u_0 \qquad \text{in } D, \tag{2.4}$$

$$\sigma(\cdot,0) = \sigma_0 \qquad \text{in } D, \tag{2.5}$$

$$u = 0 \qquad \text{on } \partial D \times (0,T). \tag{2.6}$$

The implicit function theorem will be used to prove that (2.1)-(2.6) admits a unique solution

$$u \in W^{1,q}(L^r) \cap L^q(W^{2,r} \cap H^1_0), \qquad p \in L^q(W^{1,r} \cap L^2_0), \qquad \sigma \in W^{1,q}(W^{1,r}), \tag{2.7}$$

or

$$u \in h^{1+\mu}(L^r) \cap h^{\mu}(W^{2,r} \cap H^1_0), \qquad p \in h^{\mu}(W^{1,r} \cap L^2_0), \qquad \sigma \in h^{1+\mu}(W^{1,r}), \tag{2.8}$$

with $1 < q < \infty$, $d < r < \infty$ and $0 < \mu < 1$ for any data f, u_0, σ_0 small enough in appropriate spaces. Moreover, assuming more regularity on the data, it will also be proved that

$$u \in W^{2,q}(L^r) \cap W^{1,q}(W^{2,r} \cap H^1_0), \qquad p \in W^{1,q}(W^{1,r} \cap L^2_0), \qquad \sigma \in W^{2,q}(W^{1,r}), \tag{2.9}$$

or

$$u \in h^{2+\mu}(L^r) \cap h^{1+\mu}(W^{2,r} \cap H_0^1), \qquad p \in h^{1+\mu}(W^{1,r} \cap L_0^2), \qquad \sigma \in h^{2+\mu}(W^{1,r}), \tag{2.10}$$

for any data f, u_0, σ_0 again small enough in appropriate spaces. The regularity (2.7) is sufficient to prove convergence of a finite element discretization in space, see section 2.3. On the other hand, the regularity (2.9) will be needed to prove convergence of a space and time discretization. Finally, the regularities (2.8) and (2.10) are related to considerations of chapter 3 when considering a Brownian motion.

Alternatively, local existence in time is proved for arbitrarily large data, using an abstract theorem for fully nonlinear parabolic equations, namely Theorem 8.1.1 of [**Lun95**]. More precisely, proof will be provided that there exists $0 < T_* \leq T$ such that (2.1)-(2.6) admits a solution

$$u \in \mathcal{C}^1([0,T_*], L^r) \cap \mathcal{C}^0([0,T_*], W^{2,r} \cap H_0^1), \qquad p \in \mathcal{C}^0([0,T_*]; W^{1,r} \cap L_0^2), \qquad \sigma \in \mathcal{C}^1([0,T_*], W^{1,r}),$$

with $d < r < \infty$ and for any data f, u_0 and σ_0 in appropriate spaces.

The finite element approximation in space is now introduced. For any $h > 0$, let $\mathcal{T}_h$ be a decomposition of D into triangles K with diameter h_K less than h, regular in the sense of [**CL91**]. Consider V_h, M_h and Q_h the finite element spaces for the velocity, extra-stress and pressure, respectively defined by :

$$\begin{aligned}
V_h &:= \{v_h \in \mathcal{C}^0(\overline{D}; \mathbb{R}^d); v_h\,|_K \in (\mathbb{P}_1)^d, \forall K \in \mathcal{T}_h\} \cap H_0^1(D; \mathbb{R}^d),\\
M_h &:= \{\tau_h \in \mathcal{C}^0(\overline{D}; \mathbb{R}^{d\times d}_{sym}); \tau_h\,|_K \in (\mathbb{P}_1)^{d\times d}_{sym}, \forall K \in \mathcal{T}_h\},\\
Q_h &:= \{q_h \in \mathcal{C}^0(\overline{D}; \mathbb{R}); q_h\,|_K \in \mathbb{P}_1, \forall K \in \mathcal{T}_h\} \cap L_0^2(D; \mathbb{R}).
\end{aligned}$$

Denote i_h the $L^2(D)$ projection onto V_h, M_h or Q_h and consider the following stabilized finite element discretization in space of (2.1)-(2.6). Given f, u_0, σ_0 find

$$(u_h, \sigma_h, p_h) : t \to (u_h(t), \sigma_h(t), p_h(t)) \in V_h \times M_h \times Q_h$$

such that $u_h(0) = i_h u_0$, $\sigma_h(0) = i_h \sigma_0$ and such that the following weak formulation holds in $(0,T)$:

$$\begin{aligned}
&\rho\Big(\frac{\partial u_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(u_h), \epsilon(v_h)\Big) - \Big(p_h, \nabla\cdot v_h\Big) + \Big(\sigma_h, \epsilon(v_h)\Big) - \Big(f, v_h\Big) + \Big(\nabla\cdot u_h, q_h\Big)\\
&+\sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla p_h, \nabla q_h\Big)_K + \frac{1}{2\eta_p}\Big(\sigma_h, \tau_h\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial \sigma_h}{\partial t} - (\nabla u_h)\sigma_h - \sigma_h(\nabla u_h)^T, \tau_h\Big) - \Big(\epsilon(u_h), \tau_h\Big) = 0,
\end{aligned} \tag{2.11}$$

for all $(v_h, \tau_h, q_h) \in V_h \times M_h \times Q_h$. Here $\alpha > 0$ is a dimensionless stabilization parameter and $(\cdot,\cdot)$ (respectively $(\cdot,\cdot)_K$) denotes the $L^2(D)$ (resp. $L^2(K)$) scalar product for scalars, vectors and tensors.

The above nonlinear finite element scheme has already being studied in the stationary case [**PR01**]. Indeed, using the convergence result of [**BPS01**] for the linear three fields Stokes problem and an implicit function theorem taken from [**BRR81, BRR81, CR97**], existence and convergence could be proved for small λ, the difficulty being again due to the fact that no a priori estimates can be obtained because of the presence of the quadratic terms $(\nabla u_h)\sigma_h + \sigma_h(\nabla u_h)^T$.

Proceeding in an analogous manner for the time dependent case, existence and convergence of a solution to (2.11) will be proven for a given λ but for small data f, u_0, σ_0. It should be noted that the case $\eta_s = 0$ is not considered, therefore some of the stabilization terms present in [**BPS01, PR01**] are not included in the finite element formulation (2.11).

2.2. Existence of a solution to the simplified Oldroyd-B problem

This section starts with the definition of a solution. Using the Stokes operator A_r and the notations introduced in section 1.5, (u,σ) is said to be a solution of (2.1)-(2.6) if

$$u \in W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r}), \qquad \sigma \in W^{1,q}(W^{1,r}),$$

with $1 < q < \infty$, $d < r < \infty$ and satisfies

$$\rho \frac{\partial u}{\partial t} + \eta_s A_r u - P_r \nabla \cdot \sigma = P_r f, \tag{2.12}$$

$$\frac{1}{2\eta_p}\sigma + \frac{\lambda}{2\eta_p}\Big(\frac{\partial \sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\Big) - \epsilon(u) = 0, \tag{2.13}$$

$$u(\cdot,0) = u_0, \tag{2.14}$$

$$\sigma(\cdot,0) = \sigma_0. \tag{2.15}$$

Assume that the source term $f \in L^q(L^r)$, the initial data $u_0 \in E_{1-1/q,q}$ and $\sigma_0 \in W^{1,r}$ (see chapter 1 for the definition of $E_{1-1/q,q}$).

Remark 2.1. For $1 < q < \infty$, $d < r < \infty$, we have

$$W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r}) \subset L^2(W_0^{1,r}),$$

(personal communication with Prof. Philippe Clément). Thus a solution of (2.12)-(2.15) satisfies

$$\|u(T)\|_{L^2} + \|\nabla u\|_{L^2(L^2)} < \infty.$$

Uniqueness of a solution to problem (2.12)-(2.15) can be obtained proceeding as in [**FCGO02**], i.e. by proving an a priori estimate for the difference of two solutions when $q \geq 2$.

Lemma 2.2. *Let $d \geq 2$, let $D \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$, let $T > 0$ and assume $2 \leq q < \infty$, $d < r < \infty$. Then, for any $f \in L^q(L^r)$, $u_0 \in E_{1-1/q,q}$, $\sigma_0 \in W^{1,r}$, there exists at most one solution (u,σ) of problem (2.12)-(2.15).*

Proof. Start by noticing that for $2 \leq q < \infty$, $d < r < \infty$, $v \in L^q(W^{2,r})$ and $\tau \in W^{1,q}(W^{1,r})$ the nonlinearity $(\nabla v)\tau + \tau(\nabla v)^T \in L^q(W^{1,r})$. Indeed, $W^{1,r}$ is a Banach algebra for $d < r$ (see section 1.1) and there exists $C > 0$ independent of v and τ such that

$$\|(\nabla v)\tau + \tau(\nabla v)^T\|_{W^{1,r}} \leq C\|\tau\|_{W^{1,r}}\|v\|_{W^{1,r}}.$$

Moreover, since $W^{1,q}(E) \subset_> \mathcal{C}^0(E)$ for all Banach space E (see section 1.1), it holds

$$\begin{aligned}
\|(\nabla v)\tau\|^q_{L^q(W^{1,r})} = \int_0^T \|(\nabla v)\tau\|^q_{W^{1,r}} &\leq C\int_0^T \|\nabla v\|^q_{W^{1,r}}\|\tau\|^q_{W^{1,r}} \\
&\leq C\|\tau\|^q_{L^\infty(W^{1,r})}\int_0^T \|\nabla v\|^q_{W^{1,r}} \\
&\leq \tilde{C}\|\tau\|^q_{W^{1,q}(W^{1,r})}\|v\|^q_{L^q(W^{2,r})},
\end{aligned}$$

where $\tilde{C}$ is independent of v and τ. Define the mapping S by

$$\begin{aligned}
S : L^q(W^{2,r}) \times W^{1,q}(W^{1,r}) &\longrightarrow L^q(W^{1,r}) \\
(v,\tau) &\longmapsto S(v,\tau) := \frac{\lambda}{2\eta_p}\Big((\nabla v)\tau + \tau(\nabla v)^T\Big).
\end{aligned}$$

Now let $(u_i,\sigma_i) \in W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r}) \times W^{1,q}(W^{1,r})$ $i = 1,2$, be two solutions of problem (2.12)-(2.15) and let $u := u_1 - u_2$, $\sigma := \sigma_1 - \sigma_2$. Using the well known properties of the Helmholtz-Weyl projector (see section 1.5), there exists a unique pressure $p_i \in L^q(W^{1,r} \cap L_0^2)$,

$i = 1, 2$, corresponding to each pair (u_i, σ_i) such that (u_i, σ_i, p_i) satisfies (2.1)-(2.6). When $2 \le q < \infty$, take the weak formulation to obtain

$$\begin{aligned} \rho \int_0^t \left(\frac{\partial u_i}{\partial t}, u \right) + \frac{\lambda}{2\eta_p} \int_0^t \left(\frac{\partial \sigma_i}{\partial t}, \sigma \right) + \eta_s \int_0^t (\nabla u_i, \nabla u) \\ + \int_0^t (\sigma_i, \epsilon(u)) + \frac{1}{2\eta_p} \int_0^t (\sigma_i, \sigma) - \int_0^t (\epsilon(u_i), \sigma) = \int_0^t (S(u_i, \sigma_i), \sigma) , \end{aligned} \tag{2.16}$$

for $i = 1, 2$. Hereabove the fact has been used that, since $\nabla \cdot u_i = 0$, then

$$2\nabla \cdot \epsilon(u_i) = \Delta u_i.$$

All the terms in the previous equation are well defined because of the regularity of u_i and σ_i and since $u(0) = \sigma(0) = 0$ we obtain

$$\rho \int_0^t \left(\frac{\partial u_i}{\partial t}, u \right) + \frac{\lambda}{2\eta_p} \int_0^t \left(\frac{\partial \sigma_i}{\partial t}, \sigma \right) = \frac{\rho}{2} \|u(t)\|_{L^2}^2 + \frac{\lambda}{4\eta_p} \|\sigma(t)\|_{L^2}^2,$$

for $i = 1, 2$ and $t \in (0, T)$. Subtracting the two equalities (2.16), it follows that

$$\begin{aligned} \left(\frac{\rho}{2} \|u(t)\|_{L^2}^2 + \frac{\lambda}{4\eta_p} \|\sigma(t)\|_{L^2}^2 \right) + \eta_s \|\nabla u\|_{L^2(L^2)}^2 + \frac{1}{2\eta_p} \|\sigma\|_{L^2(L^2)}^2 \\ = \int_0^t (S(u, \sigma_1), \sigma) + \int_0^t (S(u_2, \sigma), \sigma) . \end{aligned} \tag{2.17}$$

Then, using Cauchy-Schwarz and Young inequalities, it follows that for $t \in (0, T)$

$$\int_0^t (S(u, \sigma_1), \sigma) \le \frac{2\lambda}{2\eta_p} \int_0^t \|\sigma_1\|_{L^\infty} \|\nabla u\|_{L^2} \|\sigma\|_{L^2} \le \frac{\lambda^2}{2\eta_s \eta_p^2} \int_0^t \|\sigma_1\|_{L^\infty}^2 \|\sigma\|_{L^2}^2 + \frac{\eta_s}{2} \int_0^t \|\nabla u\|_{L^2}^2$$

and

$$\int_0^t (S(u_2, \sigma), \sigma) \le \frac{\lambda}{2\eta_p} \int_0^t \|\nabla u_2\|_{L^\infty} \|\sigma\|_{L^2}^2.$$

Hence, with (2.17) and the continuous injection $W^{1,r} \subset_{>} \mathcal{C}^0$ it follows that

$$\frac{\rho}{2} \|u(t)\|_{L^2}^2 + \frac{\lambda}{4\eta_p} \|\sigma(t)\|_{L^2}^2 \le C \int_0^t \left(\|u_2\|_{W^{2,r}} + \|\sigma_1\|_{W^{1,r}}^2 \right) \|\sigma\|_{L^2}^2,$$

for $t \in (0, T)$. Here C is a constant independent of u_1, u_2, σ_1 and σ_2. Since $u(0) = 0$ and $\sigma(0) = 0$, Gronwall's Lemma is used to obtain for $t \in (0, T)$

$$\rho \|u(t)\|_{L^2}^2 + \frac{\lambda}{2\eta_p} \|\sigma(t)\|_{L^2}^2 = 0,$$

so that $u \in W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r})$ and $\sigma \in W^{1,q}(W^{1,r})$ vanish. $\square$

The main results of this section are in the Theorems 2.3 and 2.5.

THEOREM 2.3. *Let $d \ge 2$, let $D \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$, let $T > 0$ and assume $d < r < \infty$, $1 < q < \infty$, $0 < \mu < 1$. Then, there exists $\delta_0 > 0$ such that the following holds.*
i) If $f \in L^q(L^r)$, $u_0 \in E_{1-1/q,q}$, $\sigma_0 \in W^{1,r}$ satisfy

$$\|P_r f\|_{L^q(L^r)} + \|u_0\|_{E_{1-1/q,q}} + \|\sigma_0\|_{W^{1,r}} \le \delta_0,$$

then there exists a solution of (2.12)-(2.15).
ii) If $f \in W^{1,q}(L^r)$, $u_0 \in \mathcal{D}_{A_r}$, $\sigma_0 \in W^{1,r}$ satisfy the compatibility condition

$$-\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0 \in E_{1-1/q,q}$$

and are such that

$$\|P_r f\|_{W^{1,q}(L^r)} + \|u_0\|_{W^{2,r}} + \|\sigma_0\|_{W^{1,r}} + \| -\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0\|_{E_{1-1/q,q}} \leq \delta_0,$$

then there exists a solution of (2.12)-(2.15) with

$$u \in W^{2,q}(\mathcal{H}_r) \cap W^{1,q}(\mathcal{D}_{A_r}), \qquad \sigma \in W^{2,q}(W^{1,r}).$$

iii) If $f \in h^\mu(L^r)$, $u_0 \in \mathcal{D}_{A_r}$, $\sigma_0 \in W^{1,r}$ *satisf the compatibility condition*

$$-\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0 \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$$

and are such that

$$\|P_r(f - f(0))\|_{h^\mu(L^r)} + \|u_0\|_{W^{2,r}} + \|\sigma_0\|_{W^{1,r}} + \| -\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}} \leq \delta_0,$$

then there exists a solution of (2.12)-(2.15) with

$$u \in h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r}), \qquad \sigma \in h^{1+\mu}(W^{1,r}).$$

iv) If $f \in h^{1+\mu}(L^r)$, $u_0 \in \mathcal{D}_{A_r}$, $\sigma_0 \in W^{1,r}$ *satisfy the compatibility conditions*

$$-\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0 \in \mathcal{D}(A_r),$$
$$-\eta_s A_r(-\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0) + \frac{\partial f}{\partial t}(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}},$$

and are such that

$$\|P_r(f - f(0))\|_{h^\mu(L^r)} + \|P_r(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial t}(0))\|_{h^\mu(L^r)} + \| -\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0\|_{W^{2,r}}$$
$$+ \|u_0\|_{W^{2,r}} + \|\sigma_0\|_{W^{1,r}} + \| -\eta_s A_r(-\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0) + \frac{\partial f}{\partial t}(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}} \leq \delta_0,$$

then there exists a solution of (2.12)-(2.15) with

$$u \in h^{2+\mu}(\mathcal{H}_r) \cap h^{1+\mu}(\mathcal{D}_{A_r}), \qquad \sigma \in h^{2+\mu}(W^{1,r}).$$

Moreover, in all cases, the mappings

$$(P_r f, u_0, \sigma_0) \mapsto \Big(u(P_r f, u_0, \sigma_0), \sigma(P_r f, u_0, \sigma_0)\Big)$$

are analytic in their respective spaces.

Using the well known properties of the Helmholtz-Weyl projector (again see section 1.5), the following result is obtained.

COROLLARY 2.4. *Under the assumptions of the above Theorem, for each* (u, σ) *solution of (2.12)-(2.15) there exists a unique* p *satisfying*

i) $p \in L^q(W^{1,r} \cap L^2_0)$,
ii) $p \in W^{1,q}(W^{1,r} \cap L^2_0)$,
iii) $p \in h^\mu(W^{1,r} \cap L^2_0)$,
iv) $p \in h^{1+\mu}(W^{1,r} \cap L^2_0)$,

such that (u, σ, p) *is a solution of problem (2.1)-(2.6). Moreover, the mappings*

$$(f, u_0, \sigma_0) \mapsto \Big(u(f, u_0, \sigma_0), \sigma(f, u_0, \sigma_0), p(f, u_0, \sigma_0)\Big)$$

are analytic in their respective spaces.

Local existence in time can be proved for arbitrarily large data, using an abstract theorem for fully nonlinear parabolic equations, namely Theorem 8.1.1 of [**Lun95**].

THEOREM 2.5. *Let $d \geq 2$, let $D \subset \mathbb{R}^d$ be a bounded, connected open set with a boundary of class $\mathcal{C}^2$ and assume $d < r < \infty$, $0 < \mu < 1$, $T > 0$. If*

$$f \in \mathcal{C}^\mu(L^r), \qquad u_0 \in \mathcal{D}_{A_r}, \qquad \sigma_0 \in W^{1,r},$$

then there exists $T_* \in (0,T]$ such that problem (2.12)-(2.15) possesses a solution

$$u \in \mathcal{C}^1([0,T_*],\mathcal{H}_r) \cap \mathcal{C}^0([0,T_*],\mathcal{D}_{A_r}), \qquad \sigma \in \mathcal{C}^1([0,T_*],W^{1,r}).$$

As for Corollary 2.4 the following result can be deduced.

COROLLARY 2.6. *Under the assumptions of the above Theorem, for each (u,σ) solution of (2.12)-(2.15) there exists a unique p satisfying*

$$p \in \mathcal{C}^1([0,T_*],W^{1,r} \cap L^2_0)$$

such that (u,σ,p) is a solution of problem (2.1)-(2.6).

REMARK 2.7. Parts iii) and iv) of Theorem 2.3 still hold when replacing little Hölder spaces by the classical Hölder spaces. Indeed, the only difference is that the trace space is not $\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$ anymore but $E_{\mu,\infty}$.

REMARK 2.8. The trace spaces $E_{1-1/q,q}$ or $\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$ are abstract spaces but they both contain $\mathcal{D}_{A_r}$. For instance in part i), if $u_0 \in W^{2,r} \cap W^{1,r}_0$ then $u_0 \in E_{1-1/q,q}$. Also, the condition "$\|P_r f\|_{L^q(L^r)}$ small" is satisfied whenever $\|f\|_{L^q(L^r)}$ is small.

REMARK 2.9. Assuming that the operator $-A_r$ satisfies the maximal regularity property when D is a convex polygon, see Remark 1.26, then Theorem 2.3 still holds.

2.2.1. Proof of Theorem 2.3. The proof is detailed for part ii) only, which contains the major mathematical difficulties. Then a brief explaination of how the same arguments can be used to prove parts i), iii) and iv) will be provided.

In order to prove part ii) of Theorem 2.3, the mapping $F : Y \times X \to Z$ will be introduced, where

$$\begin{aligned} Y &:= \big\{(P_r f, u_0, \sigma_0), \text{ such that } (f,u_0,\sigma_0) \in W^{1,q}(L^r) \times \mathcal{D}_{A_r} \times W^{1,r} \\ &\qquad \text{and } -\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0 \in E_{1-1/q,q}\big\}, \\ X &:= W^{2,q}(\mathcal{H}_r) \cap W^{1,q}(\mathcal{D}_{A_r}) \times W^{2,q}(W^{1,r}), \\ Z &:= W^{1,q}(W^{1,r}) \times Y. \end{aligned}$$

The mapping F is defined for $y = (P_r f, u_0, \sigma_0) \in Y$ and $x = (u,\sigma) \in X$ by

$$F(y,x) := \begin{pmatrix} \dfrac{\lambda}{2\eta_p}\dfrac{\partial \sigma}{\partial t} + \dfrac{1}{2\eta_p}\sigma - \epsilon(u) - S(u,\sigma) \\ \rho\dfrac{\partial u}{\partial t} + \eta_s A_r u - P_r \nabla \cdot \sigma - P_r f \\ u(.,0) - u_0 \\ \sigma(.,0) - \sigma_0 \end{pmatrix},$$

with

$$S(u,\sigma) := \frac{\lambda}{2\eta_p}\Big((\nabla u)\sigma + \sigma(\nabla u)^T\Big). \tag{2.18}$$

Then problem (2.12)-(2.15) can be reformulated as follows. Given $y \in Y$, find $x \in X$ such that

$$F(y,x) = 0 \qquad \text{in } Z. \tag{2.19}$$

The aim is to use the implicit function theorem, hence noticing that $F(0,0) = 0$, it will be proved that

- the spaces X, Y and Z equipped with appropriate norms are Banach spaces,
- F is a well defined, real analytic mapping,
- the Fréchet derivative $D_xF(0,0)$ is an isomorphism from X to Z.

This will establish existence for part ii) of Theorem 2.3. Uniqueness follows from Lemma 2.2 for $2 \le q < \infty$.

The space X is equipped with the norm $\|\cdot\|_X$ defined for $x := (u,\sigma) \in X$ by

$$\|x\|_X = \|u,\sigma\|_X := \|u\|_{W^{2,q}(L^r)} + \|u\|_{W^{1,q}(W^{2,r})} + \|\sigma\|_{W^{2,q}(W^{1,r})}.$$

Clearly, $(X, \|.\|_X)$ becomes a Banach space. The space Y is equipped with the norm $\|\cdot\|_Y$ defined for $y := (P_rf, u_0, \sigma_0) \in Y$ by

$$\begin{aligned}\|y\|_Y &= \|P_rf, u_0, \sigma_0\|_Y \\ &:= \|P_rf\|_{W^{1,q}(L^r)} + \|u_0\|_{W^{2,r}} + \|\sigma_0\|_{W^{1,r}} + \|-\eta_sA_ru_0 + P_rf(0) + P_r\nabla\cdot\sigma_0\|_{E_{1-1/q,q}}.\end{aligned}$$

As a consequence of the continuity of the linear mapping

$$(P_rf, u_0, \sigma_0) \longmapsto -\eta_sA_ru_0 + P_rf(0) + P_r\nabla\cdot\sigma_0$$

from $W^{1,q}(\mathcal{H}_r)\times\mathcal{D}_{A_r}\times W^{1,r}$ (equipped with the product norm) to L^r and due to the completeness of $E_{1-1/q,q}$, the space $(Y, \|.\|_Y)$ is a closed subspace of $W^{1,q}(\mathcal{H}_r)\times\mathcal{D}_{A_r}\times W^{1,r}$ and thus a Banach space. The space Z is equipped with the product norm and becomes a Banach space .

In order to prove that F is well defined and analytic it is needed to prove that $S : X \to W^{1,q}(W^{1,r})$ is well defined and analytic. For this purpose, will use the following Lemma.

LEMMA 2.10. *For every pair $x_1 := (u_1,\sigma_1)$, $x_2 := (u_2,\sigma_2) \in X$,*

$$b(x_1,x_2) := \nabla u_1\sigma_2 + \sigma_1(\nabla u_2)^T \in W^{1,q}(W^{1,r}).$$

Moreover, the corresponding bilinear mapping $b : X\times X \to W^{1,q}(W^{1,r})$ is continuous, that is, there exists a constant C such that for all $x_1,\ x_2 \in X$ it follows that

$$\|b(x_1,x_2)\|_{W^{1,q}(W^{1,r})} \le C\|x_1\|_X\|x_2\|_X. \tag{2.20}$$

PROOF. Let $x_1 := (u_1,\sigma_1)$, $x_2 := (u_2,\sigma_2) \in X$. Since $r > d$, $W^{1,r}(D) \subset_> \mathcal{C}^0(D)$ so that $W^{1,r}(D)$ is a Banach algebra (see section 1.1) and there exists a constant C depending only on D such that

$$\|b(x_1,x_2)\|_{W^{1,r}} \le C\|u_1\|_{W^{2,r}}\|\sigma_2\|_{W^{1,r}}.$$

Then

$$\begin{aligned}\|b(x_1,x_2)\|^q_{L^q(W^{1,r})} &= \int_0^T \|b(x_1,x_2)\|^q_{W^{1,r}} \\ &\le C\int_0^T \|u_1\|^q_{W^{2,r}}\|\sigma_2\|^q_{W^{1,r}} \\ &\le C\|u_1\|^q_{L^\infty(W^{2,r})}\int_0^T \|\sigma_2\|^q_{W^{1,r}}.\end{aligned}$$

Since $q > 1$, $W^{1,q}(0,T) \subset_> \mathcal{C}^0([0,T])$ and also that $W^{1,q}(E) \subset_> \mathcal{C}^0(E)$ for any Banach space E, thus

$$\|b(x_1,x_2)\|^q_{L^q(W^{1,r})} \le C\|u_1\|^q_{W^{1,q}(W^{2,r})}\|\sigma_2\|^q_{L^q(W^{1,r})},$$

which proves that

$$\|b(x_1, x_2)\|_{L^q(W^{1,r})} \leq C\|u_1, \sigma_1\|_X \|u_2, \sigma_2\|_X. \tag{2.21}$$

Similarly, there exists a constant C depending only on D, λ and η_p such that

$$\begin{aligned}
\|\frac{\partial b}{\partial t}(x_1, x_2)\|^q_{L^q(W^{1,r})} &= \int_0^T \|\frac{\partial b}{\partial t}(x_1, x_2)\|^q_{W^{1,r}} \\
&\leq C \int_0^T \Big(\|\frac{\partial u_1}{\partial t}\|^q_{W^{2,r}} \|\sigma_2\|^q_{W^{1,r}} + \|u_1\|^q_{W^{2,r}} \|\frac{\partial \sigma_2}{\partial t}\|^q_{W^{1,r}}\Big) \\
&\leq C \left(\|\sigma_2\|^q_{L^\infty(W^{1,r})} \int_0^T \|\frac{\partial u_1}{\partial t}\|^q_{W^{2,r}} + \|u_1\|^q_{L^\infty(W^{2,r})} \int_0^T \|\frac{\partial \sigma_2}{\partial t}\|^q_{W^{1,r}} \right) \\
&\leq C \left(\|\sigma_2\|^q_{W^{1,q}(W^{1,r})} \int_0^T \|\frac{\partial u_1}{\partial t}\|^q_{W^{2,r}} + \|u_1\|^q_{W^{1,q}(W^{2,r})} \int_0^T \|\frac{\partial \sigma_2}{\partial t}\|^q_{W^{1,r}} \right),
\end{aligned}$$

which proves that

$$\|\frac{\partial b}{\partial t}(x_1, x_2)\|_{L^q(W^{1,r})} \leq C\|u_1, \sigma_1\|_X \|u_2, \sigma_2\|_X. \tag{2.22}$$

The estimates (2.21) and (2.22) prove that $b(x_1, x_2) \in W^{1,q}(W^{1,r})$ and (2.20). □

REMARK 2.11. In fact it also has been proved that $W^{1,q}(0, T; W^{1,r}(D; \mathbb{R}))$ is a Banach algebra for $1 < q < \infty$ and $d < r < \infty$.

REMARK 2.12. For $x \in X$, we have $S(x) = \dfrac{\lambda}{2\eta_p} b(x, x)$, where $S : X \to W^{1,q}(W^{1,r})$ is introduced in (2.18). Thus, in virtue of Proposition 5.4.1 in [**Car67**], S is well defined and analytic.

COROLLARY 2.13. *The mapping $F : Y \times X \to Z$ is well defined and analytic. Moreover, for $x := (v, \tau) \in X$ its Fréchet derivative $D_x F(0, 0)x$ is given by*

$$D_x F(0, 0)x = \begin{pmatrix} \dfrac{\lambda}{2\eta_p} \dfrac{\partial \tau}{\partial t} + \dfrac{1}{2\eta_p} \tau - \epsilon(v) \\ \rho \dfrac{\partial v}{\partial t} + \eta_s A_r v - P_r \nabla \cdot \tau \\ v\,(.,0) \\ \tau\,(.,0) \end{pmatrix}.$$

PROOF. In order to study the property of the mapping $F : Y \times X \to Z$ rewrite it as follows

$$F(y, x) = L_1 y + L_2 x + \begin{pmatrix} \dfrac{\lambda}{2\eta_p} b(x, x) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{2.23}$$

where $L_1 : Y \to Z$, $L_2 : X \to Z$ are bounded linear operator defined for $y := (P_r f, u_0, \sigma_0) \in Y$ and $x := (u, \sigma) \in X$ by

$$L_1 y := \begin{pmatrix} 0 \\ -y \end{pmatrix}, \qquad L_2 x := \begin{pmatrix} \dfrac{\lambda}{2\eta_p} \dfrac{\partial \sigma}{\partial t} + \dfrac{1}{2\eta_p} \sigma - \epsilon(u) \\ \rho \dfrac{\partial u}{\partial t} + \eta_s A_r u - P_r \nabla \cdot \sigma \\ u\,(.,0) \\ \sigma\,(.,0) \end{pmatrix}$$

and $b : X \times X \to W^{1,q}(W^{1,r})$ is defined in Lemma 2.10. Clearly, the first two terms in (2.23) are analytic. The last term is also analytic in virtue of Proposition 5.4.1 in [**Car67**], which proves that F is analytic. Moreover $D_xF(0,0) = L_2$ which completes the proof. $\square$

In order to use the implicit function theorem, it remains to check that $D_xF(0,0)$ is an isomorphism from X to Z. Therefore, checking for $g \in W^{1,q}(W^{1,r})$ and $(h, v_0, \tau_0) \in Y$ there exists a unique $(v, \tau) \in X$ such that

$$\begin{cases} \dfrac{\lambda}{2\eta_p}\dfrac{\partial \tau}{\partial t} + \dfrac{1}{2\eta_p}\tau - \epsilon(v) = g, \\ \rho\dfrac{\partial v}{\partial t} + \eta_s A_r v - P_r \nabla \cdot \tau = h, \\ \qquad\qquad v\,(.,0) = v_0, \\ \qquad\qquad \tau\,(.,0) = \tau_0. \end{cases} \tag{2.24}$$

LEMMA 2.14. *Let $d \geq 2$, let $D \subset \mathbb{R}^d$ be a bounded,connected open set with boundary of class $\mathcal{C}^2$, let $T > 0$ and assume $d < r < \infty$, $1 < q < \infty$. Given $g \in W^{1,q}(W^{1,r})$ and $(h, v_0, \tau_0) \in Y$, there exists a unique $(v, \tau) \in X$ solution of (2.24). Moreover, there exists a constant C such that for $g \in W^{1,q}(W^{1,r})$ and $(h, v_0, \tau_0) \in Y$*

$$\|v, \tau\|_X \leq C\left(\|g\|_{W^{1,q}(W^{1,r})} + \|h, v_0, \tau_0\|_Y\right). \tag{2.25}$$

PROOF. Solving the first equation of (2.24) the extra-stress is

$$\tau = k\tau_0 + \frac{2\eta_p}{\lambda} k * (\epsilon(v) + g)\,, \tag{2.26}$$

with $k \in C^\infty([0,T])$ defined by $k(t) := e^{-\frac{t}{\lambda}}$ and the convolution operator $*$ by

$$(f * g)(t) := \int_0^t f(t-s)g(s)\, ds \qquad \forall t \in [0,T], \forall f, g \in L^1(0,T).$$

Introducing (2.26) in the second equation of (2.24), yields

$$\begin{cases} \rho\dfrac{\partial v}{\partial t} + \eta_s A_r v + \dfrac{\eta_p}{\lambda} k * A_r v = \tilde{h}, \\ \qquad\qquad v(.,0) = v_0. \end{cases} \tag{2.27}$$

where $\tilde{h} := h + P_r\nabla \cdot (k\tau_0) + \frac{2\eta_p}{\lambda} P_r \nabla \cdot (k * g) \in W^{1,q}(\mathcal{H}_r)$. Since D is of class $\mathcal{C}^2$, $-A_r$ satisfies the maximal regularity property (see section 1.3). Moreover, $\overline{\mathcal{D}_{A_r}} = L^r(D)$ and since $v_0 \in \mathcal{D}_{A_r}$, $-A_r v_0 + \tilde{h}(0) \in E_{1-1/q,q}$, Corollary 1.23 and Lemma 1.25 prove the existence and uniqueness of the solution $v \in W^{2,q}(\mathcal{H}_r) \cap W^{1,q}(\mathcal{D}_{A_r})$. The estimates of Corollary 1.23, Lemma 1.25 and Remark 1.20 ensure the existence of a constant C such that for $(h, v_0, \tau_0) \in Y$, $g \in W^{1,q}(W^{1,r})$

$$\|v\|_{W^{2,q}(L^r)} + \|v\|_{W^{1,q}(\mathcal{D}_{A_r})} \leq C\left(\|h, v_0, \tau_0\|_Y + \|g\|_{W^{1,q}(W^{1,r})}\right).$$

Because of the regularity of D, the graph norm $\|.\|_{\mathcal{D}_{A_r}}$ is equivalent to the whole norm $\|.\|_{W^{2,r}}$, thus there exists a constant C such that

$$\|v\|_{W^{2,q}(L^r)} + \|v\|_{W^{1,q}(W^{2,r})} \leq C\left(\|h, v_0, \tau_0\|_Y + \|g\|_{W^{1,q}(W^{1,r})}\right). \tag{2.28}$$

Going back to the extra-stress, equation (2.26), since $g + \epsilon(v) \in W^{1,q}(W^{1,r})$, Remark 1.20 ensures that $k * (g + \epsilon(v)) \in W^{2,q}(W^{1,r})$ and there exists a constant C such that

$$\|k * (g + \epsilon(v))\|_{W^{2,q}(W^{1,r})} \leq C\left(\|g\|_{W^{1,q}(W^{1,r})} + \|v\|_{W^{1,q}(W^{2,r})}\right).$$

It remains to use (2.26) to obtain the existence and uniqueness of $\tau \in W^{2,q}(W^{1,r})$. Moreover there exists a constant C such that

$$\|\tau\|_{W^{2,q}(W^{1,r})} \leq C\left(\|h, v_0, \tau_0\|_Y + \|g\|_{W^{1,q}(W^{1,r})}\right). \tag{2.29}$$

Collecting the estimations (2.28) and (2.29) we obtain (2.25). □

PROOF. (of Theorem 2.3, part ii)) The implicit function theorem is applied to (2.19). From Corollary 2.13, F is well defined and analytic, $F(0,0) = 0$. Moreover, from Lemma 2.14 $D_x F(0,0)$ is an isomorphism from X to Z. Therefore, the implicit function theorem (see [**BT03**, Theorem 4.5.4 chapter 4 p. 56]) can be applied. Thus there exists $\delta_0 > 0$ and $\varphi : Y \to X$ analytic such that for $y := (P_r f, u_0, \sigma_0) \in Y$ with $\|y\|_Y < \delta_0$, we have $F(y, \varphi(y)) = 0$. □

A brief explanation follows of how the same arguments can be used to prove parts i), iii) and iv) of Theorem 2.3.

The proof of part i) is very similar to the one presented hereabove. Indeed, it suffices to use the spaces

$$\begin{aligned}
&Y := \{(P_r f, u_0, \sigma_0), \text{ such that } (f, u_0, \sigma_0) \in L^q(L^r) \times E_{1-1/q,q} \times W^{1,r}\},\\
&X := W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r}) \times W^{1,q}(W^{1,r}),\\
&Z := L^q(W^{1,r}) \times Y
\end{aligned}$$

and to use Corollary 1.23 in order to prove the existence and uniqueness of the function v solution of (2.27). Concerning part iii), the spaces

$$\begin{aligned}
Y := \{&(P_r f, u_0, \sigma_0), \text{ such that } (f, u_0, \sigma_0) \in h^\mu(L^r) \times \mathcal{D}_{A_r} \times W^{1,r}\\
&\text{and } -\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0 \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}\},\\
X := {}&h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r}) \times h^{1+\mu}(W^{1,r}),\\
Z := {}&h^\mu(W^{1,r}) \times Y
\end{aligned}$$

will be used. Moreover, in order to prove the existence and uniqueness of the function v solution of (2.27), Lemma 1.24 will be used. Finally, the link between parts i) and ii) is the same as between parts iii) and iv). So the arguments presented in part ii) for little Hölder spaces can be extended in order to obtain more regularity in time.

REMARK 2.15. Part i) of Theorem 2.3 is compatible with Theorem 9.2 of [**FCGO02**], in which the convective terms have been taken into account. Moreover, if $2/q + d/r < 1$, it follows that $L^q(\mathcal{D}_{A_r}) \cap W^{1,q}(\mathcal{H}_r) \subset_> \mathcal{C}^0(\mathcal{C}^1)$ and thus $\nabla u \in \mathcal{C}^0([0,T] \times \overline{D})$ which implies $(u \cdot \nabla)u \in L^q(L^r)$. Therefore, Theorem 2.3 part i) still holds when the convective term $(u \cdot \nabla)u$ is added to the momentum equation (2.1) or (2.12). However, since $(u \cdot \nabla)\sigma \notin W^{1,q}(W^{1,r})$, the convective term $(u \cdot \nabla)\sigma$ can not be added to (2.13) in the present analysis.

REMARK 2.16. Since

$$\|(u \cdot \nabla)u\|_{W^{1,q}(L^r)} \leq C\Big(\|u\|_{W^{1,q}(\mathcal{D}_{A_r})} + \|u\|_{W^{2,q}(\mathcal{H}_r)}\Big),$$

then Theorem 2.3 part ii) still holds if the convective term $(u \cdot \nabla)u$ is added to (2.12). However, since $(u \cdot \nabla)\sigma \notin W^{2,q}(W^{1,r})$, the convective term $(u \cdot \nabla)\sigma$ can not be added to (2.13) in the present analysis.

2.2.2. Proof of Theorem 2.5. This result is obtained using the fully nonlinear theory for parabolic problems which can be found in [**Lun95**]. More precisely, Theorem 8.1.1 page 290 will be used for problem (2.12)-(2.15) that can be rewritten as follows

$$\dot{x}(t) = G(t, x(t)), \quad t > 0, \quad x(0) = x_0,$$

where $x := (u, \sigma)$, $x_0 := (u_0, \sigma_0)$ and $G : [0, T] \times \mathcal{D}_{A_r} \times W^{1,r} \to \mathcal{H}_r \times W^{1,r}$ is defined by

$$G(t, x) := \begin{pmatrix} -\frac{\eta_s}{\rho} A_r & G_1 \\ G_2 & -\frac{1}{\lambda} I_d \end{pmatrix} x + \begin{pmatrix} \frac{1}{\rho} P_r f(t) \\ \hat{S}(x) \end{pmatrix}.$$

Hereabove, $G_1 \in \mathcal{L}(W^{1,r}; \mathcal{H}_r)$ and $G_2 \in \mathcal{L}(\mathcal{D}_{A_r}, W^{1,r})$ are defined by

$$G_1 \sigma := \frac{1}{\rho} P_r \nabla \cdot \sigma, \quad G_2 u := \frac{2\eta_p}{\lambda} \epsilon(u)$$

whilst $\hat{S} : \mathcal{D}_{A_r} \times W^{1,r} \to W^{1,r}$ is defined by

$$\hat{S}(u, \sigma) := (\nabla u)\sigma + \sigma(\nabla u)^T.$$

LEMMA 2.17. *The application $\hat{S} : \mathcal{D}_{A_r} \times W^{1,r} \to W^{1,r}$ is well defined and analytic. Moreover, $G : [0, T] \times \mathcal{D}_{A_r} \times W^{1,r} \to \mathcal{H}_r \times W^{1,r}$ is continuous with respect to (t, x).*

PROOF. The same arguments as provided in Lemma 2.10 and Remark 2.12 in the previous subsection can be used to ensure $\hat{S} : \mathcal{D}_{A_r} \times W^{1,r} \to W^{1,r}$ is well defined and analytic. The continuity of G needs to be proved. In order to simplify the notations, the linear part of G is introduced, namely $L \in \mathcal{L}(\mathcal{D}_{A_r} \times W^{1,r}, \mathcal{H}_r \times W^{1,r})$ defined by

$$L := \begin{pmatrix} -\frac{\eta_s}{\rho} A_r & G_1 \\ G_2 & -\frac{1}{\lambda} I_d \end{pmatrix}.$$

Fix $(t, x) \in (0, T) \times \mathcal{D}_{A_r} \times W^{1,r}$ and let $\{t_n\}_{n \geq 0} \subset (0, T)$, $\{x_n\}_{n \geq 0} \subset \mathcal{D}_{A_r} \times W^{1,r}$ such that $t_n \to t$ and $x_n \to x$ when t goes to infinity. Therefore,

$$\begin{aligned} &\|G(t, x) - G(t_n, x_n)\|_{L^r \times W^{1,r}} \\ &\qquad \leq \|L(x - x_n)\|_{L^r \times W^{1,r}} + \|\frac{1}{\rho}(P_r f(t) - P_r f(t_n))\|_{L^r} + \|\hat{S}(x) - \hat{S}(x_n)\|_{W^{1,r}}. \end{aligned}$$

Thus, since $f \in \mathcal{C}^\mu(L^r)$ and $\hat{S}$ is continuous from $\mathcal{D}_{A_r} \times W^{1,r}$ to $W^{1,r}$, it follows

$$\begin{aligned} &\|G(t, x) - G(t_n, x_n)\|_{L^r \times W^{1,r}} \\ &\leq \|L\|_{\mathcal{L}(\mathcal{D}_{A_r} \times W^{1,r}, \mathcal{H}_r \times W^{1,r})} \|x - x_n\|_{W^{2,r} \times W^{1,r}} + C\left(\|P_r f\|_{\mathcal{C}^\mu(L^r)} |t - t_n|^\mu + \|x - x_n\|_{W^{2,r} \times W^{1,r}}\right). \end{aligned}$$

Hence

$$\|G(t, x) - G(t_n, x_n)\|_{L^r \times W^{1,r}} \to 0 \quad \text{when} \quad n \to \infty.$$

$\square$

The crucial point in order to prove Theorem 2.5 is

$$\left\{ \begin{array}{l} \text{for } t \in [0, T] \text{ and } x \in \mathcal{D}_{A_r} \times W^{1,r} \text{ the Fréchet derivative } D_x G(t, x) \\ \text{is the generator of an analytic semi-group.} \end{array} \right. \tag{2.30}$$

The above property will be a consequence of a result by S. B. Angenent [**Ang90**].

LEMMA 2.18. *For $t \in [0, T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$ the Fréchet derivative $D_x G(t, x)$ is the generator of an analytic semi-group.*

PROOF. Let $x := (u,\sigma) \in \mathcal{D}_{A_r} \times W^{1,r}$. In order to characterize the Fréchet derivative $D_x G(t,x)$, $t \in [0,T]$, define the operators $S_u \in \mathcal{L}(W^{1,r}, W^{1,r})$ and $S_\sigma \in \mathcal{L}(\mathcal{D}_{A_r}, W^{1,r})$ by

$$S_u \tau := (\nabla u)\tau + \tau \nabla u, \quad \forall \tau \in W^{1,r}$$

and

$$S_\sigma v := (\nabla v)\sigma + \sigma \nabla v, \quad \forall v \in \mathcal{D}_{A_r}.$$

Lemma 2.10 ensures $S_\sigma \in \mathcal{L}(\mathcal{D}_{A_r}, W^{1,r})$ and $S_u \in \mathcal{L}(W^{1,r}, W^{1,r})$. Using these notations, we obtain for $t \in [0,T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$

$$D_x G(t,x) = \begin{pmatrix} -\dfrac{\eta_s}{\rho} A_r & G_1 \\ G_2 + S_\sigma & -\dfrac{1}{\lambda} I_d + S_u \end{pmatrix}, \tag{2.31}$$

and since $G_1 \in \mathcal{L}(W^{1,r}, \mathcal{H}_r)$, $G_2 \in \mathcal{L}(\mathcal{D}_{A_r}, W^{1,r})$, then

$$D_x G(t,x) \in \mathcal{L}(\mathcal{D}_{A_r} \otimes W^{1,r}, \mathcal{H}_r \otimes W^{1,r}).$$

Finally, since $-A_r : \mathcal{D}_{A_r} \to \mathcal{H}_r$ is the generator of an analytic semi-group (see chapter 1), Lemma 2.6 p. 98 (part (a)) of [**Ang90**] concludes the proof. $\square$

Theorem 2.5 can now be proven.

PROOF. (of Theorem 2.5) Apply Theorem 8.1.1 p. 290 of [**Lun95**] with $\overline{u} := x_0 := (u_0, \sigma_0)$, $t_0 = 0$, $\bar{t} = 0$ and $\mathcal{O} = \mathcal{D}_{A_r} \times W^{1,r}$. Since $\overline{\mathcal{D}_{A_r} \times W^{1,r}} = \mathcal{H}_r \times W^{1,r}$, for $x_0 \in \mathcal{D}_{A_r} \times W^{1,r}$ we have $G(0,x_0) \in \overline{\mathcal{D}_{A_r} \times W^{1,r}}$. Thus it remains to check

- $\imath$) property (2.30) is satisfied,
- $\imath\imath$) for $t \in [0,T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$, the graph norm of the operator $D_x G(t,x)$ is equivalent to the norm $\|.\|_{W^{2,r} \times W^{1,r}}$,
- $\imath\imath\imath$) $(t,x) \mapsto G(t,x)$ is continuous with respect to (t,x), and it is Fréchet differentiable with respect to x,
- $\imath v$) for $\overline{x} := (\overline{u}, \overline{\sigma}) \in \mathcal{D}_{A_r} \times W^{1,r}$ there are $R = R(\overline{x})$, $L = L(\overline{x})$, $K = K(\overline{x}) > 0$ verifying

$$\|D_x G(t,x) - D_x G(t,z)\|_{\mathcal{L}(\mathcal{D}_{A_r} \times W^{1,r}, \mathcal{H}_r \times W^{1,r})} \leq L \|x - z\|_{W^{2,r} \times W^{1,r}},$$

$$\|G(t,x) - G(s,x)\|_{L^r \times W^{1,r}} + \|D_x G(t,x) - D_x G(s,x)\|_{\mathcal{L}(\mathcal{D}_{A_r} \times W^{1,r}, \mathcal{H}_r \times W^{1,r})} \leq K \left| t - s \right|^\mu,$$

for $t, s \in [0,T]$, $x, z \in B(\overline{x}, R) \subset \mathcal{D}_{A_r} \times W^{1,r}$.

Relation $\imath$) is satisfied by using Lemma 2.18. Property $\imath\imath$) is satisfied since $W^{1,r} \subset_> \mathcal{C}^0$ (see section 1.1). The application G is continuous by Lemma 2.17. The Fréchet derivative is given by (2.31) and is well defined. Finally, iv) may be proved as follow. Let $x := (u,\sigma)$, $z := (v,\tau)$ and $\tilde{z} := (w,\xi)$ all belonging to $\mathcal{D}_{A_r} \times W^{1,r}$, again using the continuous embedding $W^{1,r} \subset_> \mathcal{C}^0$ it follows that

$$\begin{aligned}\|D_x G(t,x)\tilde{z} - D_x G(t,z)\tilde{z}\|_{L^r \times W^{1,r}} &= \|D_x \hat{S}(x)\tilde{z} - D_x \hat{S}(z)\tilde{z}\|_{L^r \times W^{1,r}} \\ &= \|\nabla(u-v)\xi + \xi(\nabla(u-v))^T + \nabla w(\sigma - \tau) + (\sigma - \tau)(\nabla w)^T\|_{W^{1,r}} \\ &\leq C \|\tilde{z}\|_{W^{2,r} \times W^{1,r}} \|x - z\|_{W^{2,r} \times W^{1,r}},\end{aligned}$$

where C is independent of u and σ. Moreover, for $t, s \in [0,T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$

$$D_x G(t,x) = D_x G(s,x).$$

Hence, since $f \in \mathcal{C}^\mu(L^r)$, we have for $t, s \in [0,T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$

$$\|G(t,x) - G(s,x)\|_{L^r \times W^{1,r}} = \|\frac{1}{\rho}(P_r f(t) - P_r f(s))\|_{L^r} \leq C \left| t - s \right|^\mu,$$

where C is independent of t, s and x. Relations $\imath) - \imath v)$ ensure the existence of $0 < T_* < T$ such that there exists a solution

$$x \in \mathcal{C}^0([0,T_*], \mathcal{D}_{A_r} \times W^{1,r}) \cap \mathcal{C}^1([0,T_*], \mathcal{H}_r \times W^{1,r})$$

of (2.12)-(2.15). $\square$

2.2.3. Other deterministic models. The existence results presented in this section still hold when considering more realistic constitutive equations for the extra-stress tensor σ. This is the case of the simplified Giesekus [**Gie82**] and Phan-Thien Tanner [**PTT77**] models, respectively defined by

$$\sigma + \lambda\left(\frac{\partial\sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\right) + \alpha\frac{\lambda}{\eta_p}\sigma\sigma = 2\eta_p\epsilon(u),$$
$$\sigma + \lambda\left(\frac{\partial\sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\right) + \epsilon\frac{\lambda}{\eta_p}tr(\sigma)\sigma = 2\eta_p\epsilon(u),$$

where α and ϵ are given positive parameters.

2.3. Existence of the finite element approximation and a priori error estimates

In this section it will be assumed that D is a convex polygon, that

$$2 \leq q < \infty, \qquad 2 = d < r < \infty$$

and that the results presented in the previous section still hold when D is a convex polygon (see Remark 2.9). Let

$$Y := L^q(L^r) \times E_{1-1/q,q} \times W^{1,r},$$
$$X := W^{1,q}(L^r) \cap L^q(W^{2,r}) \times W^{1,q}(W^{1,r}),$$

be the data and solution spaces, respectively. According to Theorem 2.3 part i), Corollary 2.4 and Remark 2.9, it follows that if $y := (f, u_0, \sigma_0) \in Y$ is sufficiently small, then there exists a unique solution $(u(y), \sigma(y), p(y))$ of (2.1)-(2.6), the mapping

$$y \mapsto \Big(u(y), \sigma(y), p(y)\Big)$$

being analytic (therefore continuous).

In order to prove that the solution of the nonlinear finite element discretization (2.11) exists and converges to that of (2.1)-(2.6), $X_h \subset X$ is introduced and defined by

$$X_h := L^2(V_h) \times L^\infty(M_h)$$

equipped with the norm $||\cdot||_{X_h}$ defined for $x_h = (u_h, \sigma_h) \in X_h$ by

$$\|x_h\|^2_{X_h} := 2\eta_s \int_0^T \|\epsilon(u_h(t))\|^2_{L^2(D)} dt + \frac{\lambda}{4\eta_p} \sup_{t\in[0,T]} \|\sigma_h(t)\|^2_{L^2(D)}.$$

Then, rewrite the solution of (2.11) as the following fixed point problem. Given $y := (f, u_0, \sigma_0) \in Y$, find $x_h := (u_h, \sigma_h) \in X_h$ such that

$$x_h = \mathsf{T}_h\Big(y, S(x_h)\Big), \tag{2.32}$$

where S is still defined as in (2.18) but has been extended to the larger space

$$S : L^2(H^1) \times L^\infty(L^2) \to L^2(L^2).$$

The operator T_h is the semi-discrete time-dependent three fields Stokes problem defined by

$$\begin{aligned}\mathsf{T}_h : Y \times L^2(L^2) &\to X_h\\ (f, u_0, \sigma_0, g) &\to \mathsf{T}_h(f, u_0, \sigma_0, g) := (\tilde{u}_h, \tilde{\sigma}_h)\end{aligned}$$

where for $t \in (0, T)$

$$(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) : t \longmapsto (\tilde{u}_h(t), \tilde{\sigma}_h(t), \tilde{p}_h(t)) \in V_h \times M_h \times Q_h$$

satisfies $\tilde{u}_h(0) = i_h u_0$, $\tilde{\sigma}_h(0) = i_h \sigma_0$ and

$$\begin{aligned}(2.33)\quad &\rho\Big(\frac{\partial \tilde{u}_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \Big(\tilde{p}_h, \nabla\cdot v_h\Big) + \Big(\tilde{\sigma}_h, \epsilon(v_h)\Big) - \Big(f, v_h\Big)\\ &+\Big(\nabla\cdot\tilde{u}_h, q_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}_h, \nabla q_h\Big)_K + \frac{1}{2\eta_p}\Big(\tilde{\sigma}_h, \tau_h\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial\tilde{\sigma}_h}{\partial t}, \tau_h\Big) - \Big(\epsilon(\tilde{u}_h), \tau_h\Big) - \Big(g, \tau_h\Big) = 0\end{aligned}$$

for all $(v_h, \tau_h, q_h) \in V_h \times M_h \times Q_h$, a.e in $(0, T)$.

Note that, given $y := (f, u_0, \sigma_0) \in Y$ sufficiently small, the solution $x(y) := (u(y), \sigma(y)) \in X$ of the continuous Oldroyd-B problem (2.1)-(2.6) also satisfies a fixed point problem, namely

$$x(y) = \mathsf{T}\Big(y, S(x(y))\Big). \tag{2.34}$$

Here the operator T is the time-dependent three fields Stokes problem defined by

$$\begin{aligned}\mathsf{T} : Y \times L^q(W^{1,r}) &\to X\\ (f, u_0, \sigma_0, g) &\to \mathsf{T}(f, u_0, \sigma_0, g) := (\tilde{u}, \tilde{\sigma}),\end{aligned}$$

where $(\tilde{u}, \tilde{\sigma}, \tilde{p})$ satisfy

$$\rho\frac{\partial\tilde{u}}{\partial t} - 2\eta_s\ \nabla\cdot\epsilon(\tilde{u}) + \nabla\tilde{p} - \nabla\cdot\tilde{\sigma} = f \qquad \text{in } D \times (0, T), \tag{2.35}$$

$$\nabla\cdot\tilde{u} = 0 \qquad \text{in } D \times (0, T), \tag{2.36}$$

$$\frac{1}{2\eta_p}\tilde{\sigma} + \frac{\lambda}{2\eta_p}\frac{\partial\tilde{\sigma}}{\partial t} - \epsilon(\tilde{u}) = g \qquad \text{in } D \times (0, T), \tag{2.37}$$

$$\tilde{u}(\cdot, 0) = u_0 \qquad \text{in } D, \tag{2.38}$$

$$\tilde{\sigma}(\cdot, 0) = \sigma_0 \qquad \text{in } D, \tag{2.39}$$

$$\tilde{u} = 0 \qquad \text{on } \partial D \times (0, T). \tag{2.40}$$

The following stability and convergence result holds.

LEMMA 2.19. *The operator T_h is well defined and uniformly bounded with respect to h : there exists $C_1 > 0$ such that for all $h > 0$ and for all $(f, u_0, \sigma_0, g) \in Y \times L^2(L^2)$ it holds*

$$||\mathsf{T}_h(f, u_0, \sigma_0, g)||_{X_h} \leq C_1\Big(||f, u_0, \sigma_0||_Y + ||g||_{L^2(L^2)}\Big). \tag{2.41}$$

Moreover, there exists $C_2 > 0$ such that for all $h > 0$ and for all $(f, u_0, \sigma_0, g) \in Y \times L^q(W^{1,r})$ it holds

$$||(\mathsf{T} - \mathsf{T}_h)(f, u_0, \sigma_0, g)||_{X_h} \leq C_2 h\Big(||f, u_0, \sigma_0||_Y + ||g||_{L^q(W^{1,r})}\Big). \tag{2.42}$$

PROOF. A priori, (2.33) has to be understood in a weak sense with respect to the time variable: find $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{q}_h) \in X_h$ such that

$$
\begin{gathered}
-\rho \int_0^T \Big(\tilde{u}_h, \frac{\partial v_h}{\partial t}\Big) + \rho\Big(u_0, v_h(0)\Big) + 2\eta_s \int_0^T \Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \int_0^T \Big(\tilde{p}_h, \nabla \cdot v_h\Big) + \int_0^T \Big(\tilde{\sigma}_h, \epsilon(v_h)\Big) \\
- \int_0^T \Big(f, v_h\Big) + \int_0^T \Big(\nabla \cdot \tilde{u}_h, q_h\Big) + \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \int_0^T \Big(\nabla \tilde{p}_h, \nabla q_h\Big)_K \\
+ \frac{1}{2\eta_p} \int_0^T \Big(\tilde{\sigma}_h, \tau_h\Big) - \frac{\lambda}{2\eta_p} \int_0^T \Big(\tilde{\sigma}_h, \frac{\partial \tau_h}{\partial t}\Big) + \frac{\lambda}{2\eta_p}\Big(\sigma_0, \tau_h(0)\Big) - \Big(\epsilon(\tilde{u}_h), \tau_h\Big) - \frac{\lambda}{2\eta_p} \int_0^T \Big(g, \tau_h\Big) = 0,
\end{gathered}
$$

for all $v_h \in H^1(V_h)$ such that $v_h(T) = 0$, for all $\tau_h \in H^1(M_h)$ such that $\tau_h(T) = 0$ and for all $q_h \in L^2(Q_h)$. Problem (2.33) has a unique solution $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) \in X_h$. Indeed, when writing $(\tilde{u}_h, \tilde{p}_h, \tilde{\sigma}_h)$ with respect to a finite basis of $V_h \times Q_h \times M_h$, problem (2.33) can be expressed as a linear differential system. The degrees of freedom corresponding to the pressure can be eliminated. By a classical result of ODE, the resulting differential system has a unique solution, each components being in $H^1(0,T)$.

In order to prove (2.41), choose $v_h = \tilde{u}_h(t)$, $\tau_h = \tilde{\sigma}_h(t)$, $q_h = \tilde{p}_h(t)$ in (2.33), integrate from $t = 0$ to s with $0 \le s \le T$ and obtains

$$
\begin{aligned}
\frac{\rho}{2}\|\tilde{u}_h(s)\|^2_{L^2(D)} &+ \frac{\lambda}{4\eta_p}\|\tilde{\sigma}_h(s)\|^2_{L^2(D)} + 2\eta_s \int_0^s \|\epsilon(\tilde{u}_h)\|^2_{L^2(D)} \\
&+ \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \int_0^s \|\nabla \tilde{p}_h\|^2_{L^2(K)} + \frac{1}{2\eta_p} \int_0^s \|\tilde{\sigma}_h\|^2_{L^2(D)} \\
&= \frac{\rho}{2}\|\tilde{u}_h(0)\|^2_{L^2(D)} + \frac{\lambda}{4\eta_p}\|\tilde{\sigma}_h(0)\|^2_{L^2(D)} + \int_0^s (f, \tilde{u}_h) + \int_0^s (g, \tilde{\sigma}_h).
\end{aligned}
$$

Using Young and Poincaré inequalities, there exists a constant C such that

$$
\begin{aligned}
\frac{\rho}{2}\|\tilde{u}_h(s)\|^2_{L^2(D)} &+ \frac{\lambda}{4\eta_p}\|\tilde{\sigma}_h(s)\|^2_{L^2(D)} + \eta_s \int_0^s \|\epsilon(\tilde{u}_h)\|^2_{L^2(D)} \\
&+ \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \int_0^s \|\nabla \tilde{p}_h\|^2_{L^2(K)} + \frac{\lambda}{2\eta_p} \int_0^s \|\tilde{\sigma}_h\|^2_{L^2(D)} \\
&\le \frac{\rho}{2}\|\tilde{u}_h(0)\|^2_{L^2(D)} + \frac{\lambda}{4\eta_p}\|\tilde{\sigma}_h(0)\|^2_{L^2(D)} + C\left(\int_0^s \|f\|^2_{L^2(D)} + \int_0^s \|g\|^2_{L^2(D)}\right).
\end{aligned}
$$

It suffices to note that Lemma 1.27 implies

$$
\begin{aligned}
\|\tilde{u}_h(0)\|_{L^2(D)} &= \|i_h u_0\|_{L^2(D)} \le \|u_0\|_{L^2(D)} \le C\|u_0\|_{E_{1-1/q,q}}, \\
\|\tilde{\sigma}_h(0)\|_{L^2(D)} &= \|i_h \sigma_0\|_{L^2(D)} \le \|\sigma_0\|_{L^2(D)} \le C\|\sigma_0\|_{W^{1,r}},
\end{aligned}
$$

to obtain (2.41).

The convergence result (2.42) is now proved. Let

$$
\begin{aligned}
e_u &:= \tilde{u} - \tilde{u}_h = \Pi_u + C_u, \qquad & \Pi_u &:= \tilde{u} - i_h \tilde{u}, \qquad & C_u &:= i_h \tilde{u} - \tilde{u}_h, \\
e_\sigma &:= \tilde{\sigma} - \tilde{\sigma}_h = \Pi_\sigma + C_\sigma, \qquad & \Pi_\sigma &:= \tilde{\sigma} - i_h \tilde{\sigma}, \qquad & C_\sigma &:= i_h \tilde{\sigma} - \tilde{\sigma}_h, \\
e_p &:= \tilde{p} - \tilde{p}_h = \Pi_p + C_p, \qquad & \Pi_p &:= \tilde{p} - i_h \tilde{p}, \qquad & C_p &:= i_h \tilde{p} - \tilde{p}_h,
\end{aligned}
$$

where $(\tilde{u}_h, \tilde{p}_h, \tilde{\sigma}_h)$ solve (2.33) and $(\tilde{u}, \tilde{p}, \tilde{\sigma})$ solve (2.35)-(2.40). The triangle inequality leads to

$$
\|e_u, e_\sigma\|_{X_h} \le \|\Pi_u, \Pi_\sigma\|_{X_h} + \|C_u, C_\sigma\|_{X_h}.
$$

Using classical interpolation results, we have

$$\|\Pi_u, \Pi_\sigma\|_{X_h} \leq Ch\|u, \sigma\|_X.$$

The norm $\|C_u, C_\sigma\|_{X_h}$ is now estimated. The solution of (2.35)-(2.40) satisfies

$$\rho\Big(\frac{\partial \tilde{u}}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}), \epsilon(v_h)\Big) - \Big(\tilde{p}, \nabla\cdot v_h\Big) + \Big(\tilde{\sigma}, \epsilon(v_h)\Big) - \Big(f, v_h\Big)$$
$$+ \Big(\nabla\cdot\tilde{u}, q_h\Big) + \frac{1}{2\eta_p}\Big(\tilde{\sigma}, \tau_h\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial\tilde{\sigma}}{\partial t}, \tau_h\Big) - \Big(\epsilon(\tilde{u}), \tau_h\Big) - \Big(g, \tau_h\Big) = 0$$

for all $(v_h, \tau_h, q_h) \in V_h \times M_h \times Q_h$. Subtracting (2.33) from the above equation, it follows that

$$\rho\Big(\frac{\partial e_u}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(e_u), \epsilon(v_h)\Big) - \Big(e_p, \nabla\cdot v_h\Big) + \Big(e_\sigma, \epsilon(v_h)\Big)$$
$$+ \Big(\nabla\cdot e_u, q_h\Big) + \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla e_p - \nabla\tilde{p}, \nabla q_h\Big)_K$$
$$+ \frac{1}{2\eta_p}\Big(e_\sigma, \tau_h\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial e_\sigma}{\partial t}, \tau_h\Big) - \Big(\epsilon(e_u), \tau_h\Big) = 0, \tag{2.43}$$

for all $(v_h, \tau_h, q_h) \in V_h \times M_h \times Q_h$. On the other hand, from the definition of C_u, C_σ and C_p, we have

$$\rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) - \Big(C_p, \nabla\cdot C_u\Big) + \Big(C_\sigma, \epsilon(C_u)\Big) + \Big(\nabla\cdot C_u, C_p\Big)$$
$$+ \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K + \frac{1}{2\eta_p}\Big(C_\sigma, C_\sigma\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial C_\sigma}{\partial t}, C_\sigma\Big) - \Big(\epsilon(C_u), C_\sigma\Big)$$
$$= \rho\Big(\frac{\partial(e_u - \Pi_u)}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(e_u - \Pi_u), \epsilon(C_u)\Big) - \Big(e_p - \Pi_p, \nabla\cdot C_u\Big)$$
$$+ \Big(e_\sigma - \Pi_\sigma, \epsilon(C_u)\Big) + \Big(\nabla\cdot(e_u - \Pi_u), C_p\Big) + \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla(e_p - \Pi_p), \nabla C_p\Big)_K$$
$$+ \frac{1}{2\eta_p}\Big((e_\sigma - \Pi_\sigma), C_\sigma\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial(e_\sigma - \Pi_\sigma)}{\partial t}, C_\sigma\Big) - \Big(\epsilon(e_u - \Pi_u), C_\sigma\Big). \tag{2.44}$$

From the definition of i_h (the L^2 projection onto the finite element spaces), it is clear that

$$\Big(\frac{\partial\Pi_u}{\partial t}, C_u\Big) = 0, \qquad \Big(\Pi_\sigma, C_\sigma\Big) = 0, \qquad \Big(\frac{\partial\Pi_\sigma}{\partial t}, C_\sigma\Big) = 0,$$

so that, using (2.43), (2.44) yields

$$\rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) + \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K$$
$$+ \frac{1}{2\eta_p}\Big(C_\sigma, C_\sigma\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial C_\sigma}{\partial t}, C_\sigma\Big)$$
$$= 2\eta_s\Big(\epsilon(\Pi_u), \epsilon(C_u)\Big) + \Big(\Pi_p, \nabla\cdot C_u\Big) - \Big(\Pi_\sigma, \epsilon(C_u)\Big)$$
$$- \Big(\nabla\cdot(\Pi_u), C_p\Big) - \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\Pi_p, \nabla C_p\Big)_K + \Big(\epsilon(\Pi_u), C_\sigma\Big) + \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}, \nabla C_p\Big)_K$$
$$= I_1 + \cdots + I_7. \tag{2.45}$$

It now remains to bound the terms $I_1, ..., I_7$ in the above equality. Cauchy-Schwarz and Young's inequalities lead to

$$\begin{aligned} I_1 &= 2\eta_s\Big(\epsilon(\Pi_u), \epsilon(C_u)\Big) \\ &\leq 2\eta_s||\epsilon(\Pi_u)||_{L^2(D)}||\epsilon(C_u)||_{L^2(D)} \\ &\leq 3\eta_s||\epsilon(\Pi_u)||^2_{L^2(D)} + \frac{\eta_s}{3}||\epsilon(C_u)||^2_{L^2(D)}. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 = \Big(\Pi_p, \nabla\cdot C_u\Big) &\leq \frac{3}{4\eta_s}||\Pi_p||^2_{L^2(D)} + \frac{\eta_s}{3}||\nabla\cdot C_u||^2_{L^2(D)} \\ &\leq \frac{3}{4\eta_s}||\Pi_p||^2_{L^2(D)} + \frac{\eta_s}{3}||\epsilon(C_u)||^2_{L^2(D)}, \end{aligned}$$

and

$$I_3 = -\Big(\Pi_\sigma, \epsilon(C_u)\Big) \leq \frac{3}{4\eta_s}||\Pi_\sigma||^2_{L^2(D)} + \frac{\eta_s}{2}||\epsilon(C_u)||^2_{L^2(D)}.$$

An integration by parts yields, since $\Pi_u = 0$ on ∂D,

$$\begin{aligned} I_4 &= \Big(\nabla\cdot(\Pi_u), C_p\Big) = -\Big(\Pi_u, \nabla C_p\Big) = -\sum_{K\in\mathcal{T}_h}\Big(\Pi_u, \nabla C_p\Big)_K \\ &\leq \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \sum_{K\in\mathcal{T}_h}\frac{3\eta_p}{\alpha h_K^2}||\Pi_u||^2_{L^2(K)}. \end{aligned}$$

Again, Cauchy-Schwarz and Young's inequalities yield

$$\begin{aligned} I_5 &= -\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\Pi_p, \nabla C_p\Big)_K \\ &\leq \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \frac{3\alpha h^2}{4\eta_p}||\nabla\Pi_p||^2_{L^2(D)}, \end{aligned}$$

and

$$I_6 = \Big(\epsilon(\Pi_u), C_\sigma\Big) \leq \eta_p||\epsilon(\Pi_u)||^2_{L^2(D)} + \frac{1}{4\eta_p}||C_\sigma||^2_{L^2(D)}.$$

Finally, we obtain

$$\begin{aligned} I_8 &= \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}, \nabla C_p\Big)_K \\ &\leq \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \frac{3\alpha h^2}{4\eta_p}\left\|\nabla\tilde{p}\right\|^2_{L^2(D)}. \end{aligned}$$

The above estimates of $I_1, ..., I_7$ in (2.45) yield

$$\begin{aligned} \rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + \frac{1}{2}2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K + \frac{1}{4\eta_p}\Big(C_\sigma, C_\sigma\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial C_\sigma}{\partial t}, C_\sigma\Big) \\ \leq C\Big(||\epsilon(\Pi_u)||^2_{L^2(D)} + ||\Pi_p||^2_{L^2(D)} + ||\Pi_\sigma||^2_{L^2(D)} \\ + \sum_{K\in\mathcal{T}_h}\frac{1}{h_K^2}\|\Pi_u\|^2_{L^2(K)} + h^2||\nabla\Pi_p||^2_{L^2(D)} + h^2||\nabla\tilde{p}||^2_{L^2(D)}\Big) \end{aligned}$$

where C depends only on ρ, η_s, η_p and α. Time integration for $0 \le s \le T$ yields

$$\begin{aligned}\frac{\rho}{2}\|C_u(s)\|^2_{L^2(D)} + \eta_s \int_0^s \|\epsilon(C_u)\|^2_{L^2(D)} + \frac{\lambda}{4\eta_p}\|C_\sigma(s)\|^2_{L^2(D)} \\ \le \frac{\rho}{2}\|C_u(0)\|^2_{L^2(D)} + \frac{\lambda}{4\eta_p}\|C_\sigma(0)\|^2_{L^2(D)} \\ + C\int_0^s \Big(||\epsilon(\Pi_u)||^2_{L^2(D)} + ||\Pi_p||^2_{L^2(D)} + ||\Pi_\sigma||^2_{L^2(D)} \\ + \sum_{K\in\mathcal{T}_h} \frac{1}{h_K^2}\|\Pi_u\|^2_{L^2(K)} + h^2||\nabla\Pi_p||^2_{L^2(D)} + h^2||\nabla\tilde{p}||^2_{L^2(D)}.\end{aligned}$$

Using standard interpolation results, we obtain

$$\|C_u, C_\sigma\|^2_{X_h} \le Ch^2\Big(\|\tilde{u},\tilde{\sigma}\|^2_X + \|\tilde{p}\|^2_{L^q(W^{1,r})} + \|\nabla u_0\|^2_{L^2(D)} + \|\nabla\sigma_0\|^2_{L^2(D)}\Big),$$

where C does not depend on h, f, u_0, σ_0 and g. Then, using Lemma 1.27, we have

$$\|\nabla u_0\|^2_{L^2(D)} \le C\|u_0\|^2_{E_{1-1/q,q}},$$

where C does not depend on h, f, u_0, σ_0 and g. Moreover, using the fact that the mapping

$$(f, u_0, \sigma_0, g) \mapsto (\tilde{u}, \tilde{\sigma}, \tilde{p})$$

is continuous from $Y \times L^q(W^{1,r})$ to $L^q(W^{2,r}) \times L^q(W^{1,r}) \times L^q(W^{1,r})$, one obtains

$$||C_u, C_\sigma||_{X_h} \le Ch\Big(||f, u_0, \sigma_0||_Y + ||g||_{L^q(W^{1,r})}\Big),$$

which concludes the proof. □

The goal is now to prove that (2.32) has a unique solution converging to that of (2.34). For this purpose, use, as in [**PR01**], an abstract framework and write (2.32) as the following problem : given $y := (f, u_0, \sigma_0) \in Y$, find $x_h := (u_h, \sigma_h) \in X_h$ such that

$$F_h(y, x_h) = 0, \tag{2.46}$$

where $F_h : Y \times X_h \to X_h$ is defined by

$$F_h(y, x_h) := x_h - \mathsf{T}_h\Big(y, S(x_h)\Big). \tag{2.47}$$

In order to prove existence and convergence of a solution to (2.46), use Theorem 2.1 of [**CR97**]. The mapping $F_h : Y \times X_h \to X_h$ is $\mathcal{C}^1$. Moreover, it is necessary to prove that the scheme is consistent, that D_xF_h has bounded inverse at i_hx - recall that i_h is the $L^2(D)$ projection onto the finite element space, x is the solution of (2.34) - and that D_xF_h is locally Lipschitz at i_hx.

LEMMA 2.20. *Let δ_0 be as in Theorem 2.3 part i) and assume Remark 2.9. Let $y := (f, u_0, \sigma_0) \in Y$ with $\|y\|_Y \le \delta_0$ and let $x(y) = (u(y), \sigma(y)) \in X$ be the solution of (2.34). Then, there exists a constant C_1 such that for $y \in Y$ with $\|y\|_Y \le \delta_0$, for $0 < h \le 1$, it holds*

$$\|F_h(y, i_hx(y))\|_{X_h} \le C_1h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|^2_X\Big). \tag{2.48}$$

Moreover, there exists a constant C_2 such that for $y \in Y$ with $\|y\|_Y \le \delta_0$, for $0 < h \le 1$, for $z \in X_h$ it holds

$$\|D_xF_h(y, i_hx(y)) - D_xF_h(y, z)\|_{\mathcal{L}(X_h)} \le \frac{C_2}{h}\|i_hx(y) - z\|_{X_h}. \tag{2.49}$$

PROOF. Using (2.34) and (2.47), it follows that

$$\begin{aligned}F_h(y, i_h x) &= i_h x - x - \mathsf{T}_h(y, S(i_h x)) + \mathsf{T}(y, S(x))\\ &= i_h x - x + \mathsf{T}_h(0, S(x) - S(i_h x)) + (\mathsf{T} - \mathsf{T}_h)(y, S(x)),\end{aligned}$$

so that,

$$\frac{1}{3}\|F_h(y, i_h x)\|^2_{X_h} \leq \|i_h x - x\|^2_{X_h} + \|\mathsf{T}_h(0, S(x) - S(i_h x))\|^2_{X_h} + \|(\mathsf{T} - \mathsf{T}_h)(y, S(x))\|^2_{X_h}.$$

Using standard interpolation results for the first term of the right hand side, Lemma 2.19 for the second and third terms, it follows that

$$\|F_h(y, i_h x)\|^2_{X_h} \leq C\Big(h^2\|x\|^2_X + \|S(x) - S(i_h x)\|^2_{L^2(L^2)} + h^2\|y\|^2_Y + h^2\|S(x)\|^2_{L^q(W^{1,r})}\Big), \tag{2.50}$$

C being independent of h and y. Proceeding as in Lemma 2.2, it follows that

$$\|S(x)\|^2_{L^q(W^{1,r})} \leq C\|x\|^4_X, \tag{2.51}$$

C being independent of h and y. On the other hand, we also have

$$\begin{aligned}\frac{2\eta_p}{\lambda}\Big(S(x) - S(i_h x)\Big) &= \nabla u\ \sigma + \sigma(\nabla u)^T - (\nabla i_h u) i_h\sigma - i_h\sigma\ (\nabla i_h u)^T\\ &= \nabla(u - i_h u)\sigma + (\nabla i_h u)(\sigma - i_h\sigma) + \sigma\ (\nabla(u - i_h u))^T + (\sigma - i_h\sigma)(\nabla i_h u)^T,\end{aligned}$$

so that, using a Cauchy-Schwarz inequality

$$\|S(x) - S(i_h x)\|^2_{L^2(L^2)} \leq C\|x - i_h x\|^2_{X_h}\left(\|\sigma\|^2_{L^\infty(L^\infty)} + \|\nabla i_h u\|^2_{L^2(L^\infty)}\right),$$

C being independent of h and y. Standard interpolation results lead to

$$\|\nabla i_h u\|_{L^\infty} \leq \|\nabla u\|_{L^\infty} + \|\nabla(i_h u - u)\|_{L^\infty} \leq C\|u\|_{W^{2,r}},$$

C being independent of h and y. Thus, using again standard interpolation results, we have

$$\|S(x) - S(i_h x)\|^2_{L^2(L^2)} \leq Ch^2\|x\|^4_X, \tag{2.52}$$

C being independent of h and y. Finally, (2.51) and (2.52) in (2.50) yields (2.48).

Relation (2.49) is now proved. Let $z := (v, \tau) \in X_h$, let $\tilde{z} := (\tilde{v}, \tilde{\tau}) \in X_h$, we have

$$\Big(D_x F_h(y, i_h x) - D_x F_h(y, z)\Big)\tilde{z} = -\mathsf{T}_h\Big(0, (DS(i_h x) - DS(z))\tilde{z}\Big).$$

Using Lemma 2.19 we obtain

$$\|\left(D_x F_h(y, i_h x) - D_x F_h(y, z)\right)\tilde{z}\|_{X_h} \leq C\|(DS(i_h x) - DS(z))\tilde{z}\|_{L^2(L^2)}, \tag{2.53}$$

C being independent of h and y. It follows that

$$\begin{aligned}\frac{2\eta_p}{\lambda}\Big(DS(i_h x) - DS(z)\Big)\tilde{z} &= (\nabla(i_h u - v))\tilde{\tau} + \tilde{\tau}(\nabla(i_h u - v))^T\\ &\quad + \nabla\tilde{v}(i_h\sigma - \tau) + (i_h\sigma - \tau)(\nabla\tilde{v})^T.\end{aligned}$$

Then, using Cauchy-Schwarz inequality, there exists a constant C independent of h and y such that

$$\|(DS(i_h x) - DS(z))\tilde{z}\|_{L^2(L^2)} \leq C\Big(\|\nabla(i_h u - v)\|_{L^2(L^\infty)}\|\tilde{\tau}\|_{L^\infty(L^2)} + \|\nabla\tilde{v}\|_{L^2(L^\infty)}\|i_h\sigma - \tau\|_{L^\infty(L^2)}\Big).$$

A classical inverse inequality yields

$$\|(DS(i_h x) - DS(z))\tilde{z}\|_{L^2(L^2)} \leq \frac{C}{h}\Big(\|\nabla(i_h u - v)\|_{L^2(L^2)}\|\tilde{\tau}\|_{L^\infty(L^2)} + \|\nabla\tilde{v}\|_{L^2(L^2)}\|i_h\sigma - \tau\|_{L^\infty(L^2)}\Big),$$

so that we finally obtain

$$\|(DS(i_h x) - DS(z))\tilde{z}\|_{L^2(L^2)} \leq \frac{C}{h}\|i_h x - z\|_{X_h}\|\tilde{z}\|_{X_h}.$$

This last inequality in (2.53) yields (2.49). □

Before proving existence of a solution to (2.46) it is necessary to check that $D_x F_h(y, i_h x)$ is invertible.

LEMMA 2.21. *Let δ_0 be as in Theorem 2.3 part i) and assume Remark 2.9. Let $y := (f, u_0, \sigma_0) \in Y$ with $\|y\|_Y \leq \delta_0$ and let $x(y) := (u(y), \sigma(y)) \in X$ be the solution of (2.34). Then, there exists $0 < \delta_1 \leq \delta_0$ such that for $y \in Y$ with $\|y\|_Y \leq \delta_1$, for $0 < h \leq 1$, it holds*

$$\|D_x F_h(y, i_h x(y))^{-1}\|_{\mathcal{L}(X_h)} \leq 2.$$

PROOF. By definition of F_h, we have

$$D_x F_h(y, i_h x) = I - \mathsf{T}_h(0, DS(i_h x)),$$

so that it follows that

$$D_x F_h(y, i_h x) = I - G_h \quad \text{with} \quad G_h := \mathsf{T}_h(0, DS(i_h x)).$$

Proving that $\|G_h\|_{\mathcal{L}(X_h)} \leq 1/2$ for y sufficiently small will ensure that $D_x F_h(y, i_h x)$ is invertible and

$$\|D_x F_h(y, i_h x)^{-1}\|_{\mathcal{L}(X_h)} \leq 2.$$

Let $z := (v, \tau) \in X_h$. Using Lemma 2.19 we have

$$\|G_h(z)\|_{X_h} \leq C_1\|DS(i_h x)z\|_{L^2(L^2)},$$

C_1 being independent of y and h. Using the same arguments as in the proof of Lemma 2.20, we obtain

$$\begin{aligned}
\frac{2\eta_p}{\lambda}\|DS(i_h x)z\|_{L^2(L^2)} &= \|(\nabla i_h u)\tau + \tau(\nabla i_h u)^T + (\nabla v)i_h\sigma + i_h\sigma(\nabla v)^T\|_{L^2(L^2)} \\
&\leq 2\Big(\|\nabla i_h u\|_{L^2(L^\infty)}\|\tau\|_{L^\infty(L^2)} + \|\nabla i_h\sigma\|_{L^\infty(L^\infty)}\|\nabla v\|_{L^2(L^2)}\Big) \\
&\leq C_2\Big(\|u\|_{L^2(W^{2,r})}\|\tau\|_{L^\infty(L^2)} + \|\nabla v\|_{L^2(L^2)}\|\sigma\|_{W^{1,q}(W^{1,r})}\Big),
\end{aligned}$$

C_2 being independent of y and h. Hence,

$$\|G_h(z)\|_{X_h} \leq C_3\|x\|_X\|z\|_{X_h},$$

where C_3 is independent of y and h. From Corollary 2.4, the mapping $y \to x(y)$ is continuous, thus if $||y||_Y$ is sufficiently small it follows that $\|x\|_X \leq 1/(2C_3)$ so that

$$\|G_h(z)\|_{X_h} \leq \frac{1}{2}\|z\|_{X_h}.$$

□

The existence of a solution to the finite element scheme (2.11) and convergence to the solution of (2.1)-(2.6) can now be proved.

THEOREM 2.22. *Let δ_0 be as in Theorem 2.3 part i) and assume Remark 2.9. Let $y := (f, u_0, \sigma_0) \in Y$ with $\|y\|_Y \leq \delta_0$ and let $x(y) := (u(y), \sigma(y)) \in X$ be the solution of (2.34). Then, there exists $0 < \delta_2 \leq \delta_0$ and $\zeta > 0$ such that for $y \in Y$ with $\|y\|_Y \leq \delta_2$, for $0 < h \leq 1$, there exists a unique $x_h(y) := (u_h(y), \sigma_h(y))$ in the ball of X_h centered at $i_h x(y)$ with radius ζh, satisfying*

$$F_h(y, x_h(y)) = 0.$$

Moreover, the mapping $y \mapsto x_h(y)$ is continuous and there exists $C > 0$ independent of h and y such that the following a priori error estimate holds

$$\|x(y) - x_h(y)\|_{X_h} \leq Ch. \tag{2.54}$$

REMARK 2.23. The above Theorem still holds when the stabilization term in (2.11) is replaced by

$$\sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \Big(-\nabla \cdot (2\eta_s \epsilon(u_h) + \sigma_h) + \nabla p_h - f, \nabla q_h \Big)_K,$$

provided $0 < \alpha \leq C_I$. Here C_I is the largest constant satisfying the following inverse estimate

$$C_I \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla \cdot \tau_h\|^2_{L^2(K)} \leq \|\tau_h\|^2_{L^2(D)} \qquad \forall \tau_h \in M_h.$$

REMARK 2.24. Theorem 2.22 also holds when $y := (f, u_0, \sigma_0) \in Y$ is sufficiently small, with Y corresponding to Theorem 2.3 part iii) thus defined by

$$Y = \{(f, u_0, \sigma_0) \in h^\mu(L^r) \times \mathcal{D}_{A_r} \times W^{1,r} \text{ such that } -\eta_s A_r u_0 + P_r f(0) + P_r \nabla \cdot \sigma_0 \in E_{1-1/q,q}\}.$$

In order to prove the above Theorem, the following abstract result will be used.

LEMMA 2.25 (Theorem 2.1 in [**CR97**]). *Let Y and Z be two real Banach spaces with norms $\|.\|_Y$ and $\|.\|_Z$ respectively. Let $G : Y \to Z$ be a $\mathcal{C}^1$ mapping and $v \in Y$ be such that $DG(v) \in \mathcal{L}(Y; Z)$ is an isomorphism. We introduce the notations*

$$\epsilon = \|G(v)\|_Z,$$
$$\gamma = \|DG(v)^{-1}\|_{\mathcal{L}(Y;Z)},$$
$$L(\alpha) = \sup_{x \in \overline{B}(v,\alpha)} \|DG(v) - DG(x)\|_{\mathcal{L}(Y;Z)},$$

with $\overline{B}(v, \alpha) = \{y \in Y; \|v - y\|_Y \leq \alpha\}$, and we are interested in finding $u \in Y$ such that

$$G(u) = 0. \tag{2.55}$$

We assume that $2\gamma L(2\gamma\epsilon) \leq 1$. Then Problem (2.55) has a unique solution u in the ball $\overline{B}(v, 2\gamma\epsilon)$ and, for $x \in \overline{B}(v, 2\gamma\epsilon)$, we have

$$\|x - u\|_Y \leq 2\gamma \|G(x)\|_Z.$$

PROOF OF THEOREM 2.22. Apply Lemma 2.25 with $Y := X_h$, $Z := X_h$, $G := F_h$ and $v := i_h x(y)$. According to Lemma 2.20 there exists a constant C_1 independent of y and h such that

$$\epsilon = \|F_h(y, i_h x(y))\|_{X_h} \leq C_1 h \Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \Big).$$

According to Lemma 2.21, for $\|y\|_Y$ sufficiently small

$$\gamma = \|D_x F_h(y, i_h x(y))\|_{\mathcal{L}(X_h)} \leq 2.$$

According to Lemma 2.20, there is a constant C_2 independent of y and h such that

$$L(\beta) = \sup_{z \in \overline{B}(i_h x(y), \beta)} \|DF_h(i_h x(y)) - DF_h(z)\|_{\mathcal{L}(X_h)} \leq \frac{C_2}{h} \beta.$$

Hence, we have

$$\begin{aligned} 2\gamma L(2\gamma\epsilon) &\leq 2.2 \frac{C_2}{h} \Big(2.2 C_1 h \Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \Big) \Big) \\ &= 16 C_1 C_2 \Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \Big). \end{aligned}$$

Using the continuity of the mapping $y \mapsto x(y)$, there exists $0 < \delta_2 \leq \delta_0$ such that for $y \in Y$ with $\|y\|_Y \leq \delta_2$, then

$$\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \leq \frac{1}{32C_1C_2}$$

so that $2\gamma L(2\gamma\epsilon) \leq 1/2 < 1$ and Lemma 2.25 applies. There exists a unique $x_h(y)$ in the ball $\overline{B}(i_h x(y), 2\gamma\epsilon)$ such that

$$F_h(y, x_h(y)) = 0$$

and we obtain

$$\|i_h x(y) - x_h(y)\|_{X_h} \leq 4C_1 h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big) \leq \frac{4C_1 h}{32C_1C_2} = \frac{1}{8C_2}h.$$

It suffices to use the triangle inequality

$$\|x(y) - x_h(y)\|_{X_h} \leq \|x(y) - i_h x(y)\|_{X_h} + \|i_h x(y) - x_h(y)\|_{X_h},$$

and standard interpolation results to obtain (2.54). The fact that the mapping $y \mapsto x_h(y)$ is continuous is as a direct consequence of the implicit function theorem used to prove Lemma 2.25. $\square$

CHAPTER 3

Mathematical and numerical analysis of the Hookean dumbbells model

The ideas of the previous chapter are now extended to a stochastic model. Limitations on the regularity of the Brownian motion involve the use of particular function spaces. As in the previous chapter, the convective terms are disregarded and the focus will be on a non-moving domain. Existence on a fixed time interval is proved provided the data are small enough, using the implicit function theorem and a maximum regularity property of type h^μ for a three fields Stokes problem. A finite element discretization in space is then proposed. Existence of the numerical solution is proved for small data, as well as a priori error estimates.

3.1. The simplified Hookean dumbbells model and its finite elements approximation in space

As in the previous chapter, D denotes a bounded, connected open set of $\mathbb{R}^d$, $d = 2$ or 3 with boundary ∂D of class $\mathcal{C}^2$, and let $T > 0$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete filtered probability space. The filtration $\mathcal{F}_t$ upon which the Brownian process B is defined is completed with respect to $\mathcal{P}$ and is assumed to be right continuous on $[0, T]$. Assume also that the space Ω is rich enough to accommodate a random vector $q_0 : \Omega \to \mathbb{R}^d$ such that

$$\left\{\begin{array}{l} q_0 \text{ is independent of } B \text{ and } (q_0)_i \text{ is independent of } (q_0)_j, 1 \leq i \neq j \leq d, \\ \text{and } \mathbb{E}(q_0) = 0,\ \mathbb{E}(q_0 \otimes q_0) = I. \end{array}\right. \tag{3.1}$$

In fact, q_0 is an initial condition for the dumbbells elongation q which corresponds to the equilibrium state since the conditions $\mathbb{E}(q_0) = 0$ and $\mathbb{E}(q_0 \otimes q_0) = I$ lead to a vanishing initial extra-stress. These conditions could be relaxed to yield constant initial stresses with respect to the space variable $x \in D$. Refer to [**RY94**] for all notions related to stochastic processes.

Consider the following problem. Given initial conditions $u_0 : D \to \mathbb{R}^d$, $q_0 : \Omega \to \mathbb{R}^d$ satisfying (3.1), a force term $f : D \times [0, T] \to \mathbb{R}^d$, constant solvent and polymer viscosities $\eta_s > 0$, $\eta_p > 0$, a constant relaxation time $\lambda > 0$, find the velocity $u : D \times [0, T] \to \mathbb{R}^d$, the pressure $p : D \times [0, T] \to \mathbb{R}$ and the dumbbells elongation vector $q : D \times [0, T] \times \Omega \to \mathbb{R}^d$ such that

$$dq - \left((\nabla u)q - \frac{1}{2\lambda}q\right) dt - \frac{1}{\sqrt{\lambda}}\, dB = 0 \qquad \text{in } D \times (0, T) \times \Omega, \tag{3.2}$$

$$\rho\frac{\partial u}{\partial t} - \nabla \cdot \left(2\eta_s\epsilon(u) + \frac{\eta_p}{\lambda}\left(\mathbb{E}(q \otimes q) - I\right)\right) + \nabla p = f \qquad \text{in } D \times (0, T), \tag{3.3}$$

$$\nabla \cdot u = 0 \qquad \text{in } D \times (0, T), \tag{3.4}$$

$$u\,(., 0) = u_0 \qquad \text{in } D, \tag{3.5}$$

$$q(., 0, .) = q_0 \qquad \text{in } D \times \Omega, \tag{3.6}$$

$$u = 0 \qquad \text{on } \partial D \times (0, T). \tag{3.7}$$

REMARK 3.1. Equations (3.2) and (3.6) are notations for

$$q(x,t,\omega) - q_0(t,\omega) - \int_0^t \left((\nabla u(x,s))q(x,s,\omega) - \frac{1}{2\lambda}q(x,s,\omega) \right) ds - \frac{1}{\sqrt{\lambda}}B(t,\omega) = 0,$$

where $(x,t,\omega) \in D \times [0,T] \times \Omega$.

System (3.2)-(3.7) formally contains the simplified Oldroyd-B problem studied in the previous chapter. Indeed, using Îto's formula, one obtains the variance V defined by

$$V := \mathbb{E}(q \otimes q)$$

satisfies the deterministic equation

$$\frac{\partial V}{\partial t} = \left(\nabla u - \frac{I}{2\lambda}\right) V + V \left(\nabla u - \frac{I}{2\lambda}\right)^T + \frac{1}{\lambda} I \qquad \text{in } D \times [0,T], \tag{3.8}$$

see Problem 6.1 p.355 in [**KS91**]. Thus setting

$$\sigma := \frac{\eta_p}{\lambda}(V - I), \tag{3.9}$$

equation (3.8) corresponds to the constitutive equation of the simplified Oldroyd-B model without convective terms

$$\sigma + \lambda \left(\frac{\partial \sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T \right) = 2\eta_p \epsilon(u), \quad \sigma(0) = 0. \tag{3.10}$$

In Remark 3.14 the link between the solution of the Oldroyd-B problem seen in the previous chapter and that of system (3.2)-(3.7) will be made precise from the mathematical viewpoint.

One of the difficulties of problem (3.2)-(3.7) is to deal with stochastic processes with value in Banach spaces. Indeed in classical textbooks, Îto formula (see [**RY94**, Theorem 3.3, chapter IV]), relation (3.8) as well as classical existence and uniqueness results for linear stochastic differential equation (Theorem 2.1, chapter IX, in [**RY94**]) are not presented in this context. Hence, we split the dumbbells elongation into two components

$$q = q^S + q^D, \tag{3.11}$$

where $q^S : \Omega \times [0,T] \to \mathbb{R}^d$ is the solution at equilibrium (that is to say when $u = 0$) and obviously $q^D : \Omega \times [0,T] \times D \to \mathbb{R}^d$ is the discrepancy with respect to the equilibrium. The stochastic process q^S is an *Ornstein-Uhlenbeck* process (see [**RY94**]) satisfying

$$dq^S = -\frac{1}{2\lambda}q^S dt + \frac{1}{\sqrt{\lambda}}dB, \qquad q^S(0) = q_0, \tag{3.12}$$

while the equation satisfied by q^D

$$\frac{\partial q^D}{\partial t} = \nabla u \ (q^D + q^S) - \frac{1}{2\lambda}q^D, \qquad q^D(0) = 0, \tag{3.13}$$

is a differential equation with a stochastic forcing term $(\nabla u)q^S$.

In the previous chapter, it was proved that the extra-stress σ solution of the simplified Oldroyd-B problem (2.1)-(2.6) was in spaces $W^{1,q}(0,T; W^{1,r}(D; \mathbb{R}^{d\times d}_{sym}))$ or in spaces $h^{1+\mu}([0,T]; W^{1,r}(D; \mathbb{R}^{d\times d}_{sym}))$. Since, according to [**RY94**] (Theorem 2.2 p. 26, Corollary 2.6 p 28 and Theorem 2.7 p. 29), a Brownian motion can not be expected in a more regular space than $L^\gamma(\Omega; \mathcal{C}^{\mu'}([0,T]))$, $\mu' < \frac{1}{2}$ and $2 \le \gamma < \infty$, the use of Sobolev spaces in time is not appropriate. Moreover, the reason for using little Hölder spaces is that in a stochastic context, it is more convenient to deal with separable spaces and the spaces $h^k([0,T]; E)$ provided with the norm of $C^\mu([0,T]; E)$ are separable Banach spaces if E is separable.

REMARK 3.2. In order to avoid confusion, one must recall that for $2 \leq \gamma < \infty$, $0 < \mu < 1/2$ and $d < r < \infty$, $L^\gamma(h^\mu(W^{1,r}))$ stands for

$$L^\gamma(\Omega; h^\mu([0,T]; W^{1,r}(D;\mathbb{R}^d))),$$

where D and $[0,T]$ are provided with the Lebesgue measure and Ω with the probability measure $\mathcal{P}$.

The implicit function theorem will be used to prove that (3.2)-(3.7) admits a unique solution

$$u \in h^{1+\mu}(L^r) \cap h^\mu(W^{2,r} \cap H_0^1), \quad q \in L^\gamma(h^\mu(W^{1,r})), \quad p \in h^\mu(W^{1,r} \cap L_0^2), \tag{3.14}$$

with $2 \leq \gamma < \infty$, $0 < \mu < 1/2$ and $d < r < \infty$ whenever the data f, u_0 are small enough in appropriate spaces. It will be shown that $h^\mu([0,T]; W^{1,r}(D)) \subset \mathcal{C}([0,T] \times \overline{D})$, which implies, in particular, that a process $q \in L^\gamma(h^\mu(W^{1,r}))$ has a continuous sample path for almost each realization.

The finite element approximation in space for D a convex polygon in $\mathbb{R}^2$ is now introduced. For any $h > 0$, let $\mathcal{T}_h$ be a decomposition of D into triangles K with diameter h_K less than h, regular in the sense of [**CL91**]. Let V_h, R_h and Q_h be the finite element spaces for the velocity, dumbbells elongation and pressure, respectively defined by :

$$\begin{aligned}
V_h &:= \{v_h \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R}^d); v_h \,|_K \in (\mathbb{P}_1)^d \quad \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega;\mathbb{R}^d),\\
R_h &:= \{r_h \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R}^d); r_h \,|_K \in (\mathbb{P}_1)^d \quad \forall K \in \mathcal{T}_h\},\\
Q_h &:= \{s_h \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R}); s_h \,|_K \in \mathbb{P}_1 \quad \forall K \in \mathcal{T}_h\} \cap L_0^2(\Omega;\mathbb{R}).
\end{aligned}$$

Denote i_h the $L^2(D)$ projection onto V_h, R_h or Q_h and consider the following stabilized finite element discretization in space of (3.2)-(3.7). Given f, u_0, q_0 find

$$(t,\omega) \in [0,T] \times \Omega \mapsto (u_h(t), q_h(\omega,t), p_h(t)) \in V_h \times R_h \times Q_h$$

such that $u_h(0) = i_h u_0$, $q_h(0) = q_0$ and such that the following weak formulation holds in $(0,T) \times \Omega$:

$$\begin{aligned}
\rho\Big(\frac{\partial u_h}{\partial t}, v_h\Big) &+ 2\eta_s\Big(\epsilon(u_h), \epsilon(v_h)\Big) - \Big(p_h, \nabla \cdot v_h\Big) + \frac{\eta_p}{\lambda}\Big(\mathbb{E}(q_h \otimes q_h) - I, \epsilon(v_h)\Big) - \Big(f, v_h\Big)\\
&+ \Big(\nabla \cdot u_h, s_h\Big) + \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} (\nabla p_h, \nabla s_h)_K + (q_h(t), r_h) - (1, r_h) q_0\\
&+ \Big(\int_0^t \Big(\frac{1}{2\lambda} q_h(k) - (\nabla u_h(k)) q_h(k)\Big)\, dk, r_h\Big) - \frac{1}{\sqrt{\lambda}}(1, r_h) B = 0,
\end{aligned} \tag{3.15}$$

for all $(v_h, r_h, s_h) \in V_h \times R_h \times Q_h$. Here $\alpha > 0$ is a dimensionless stabilization parameter and $(\cdot,\cdot)$ (respectively $(\cdot,\cdot)_K$) denotes the $L^2(D)$ (resp. $L^2(K)$) scalar product for scalars, vectors and tensors.

The above nonlinear finite element scheme is closely linked to the Oldroyd-B scheme studied in the previous chapter. Using an implicit function theorem taken from [**CR97**], existence and convergence will be proved for small data f and u_0, again the difficulty being due to the fact that no a priori estimates are available due to the nonlinear term $(\nabla u_h) q_h$. The proof will be as in the continuous problem. More precisely, it will be proven that the linearized problem in the neighborhood of the equilibrium state $u_h = 0$, $q_h = q^S$ is well posed.

It should be noted that the case $\eta_s = 0$ is not considered, therefore some of the stabilization terms present in [**BP01**] are not included in the finite element formulation (3.15).

3.2. Existence of the simplified Hookean dumbbells model

This section starts with the definition of a solution. Using the Stokes operator A_r and the notations introduced in section 1.5, (u,q) is said to be a solution of (3.2)-(3.7) if

$$u \in h^{1+\mu}(\mathcal{H}_r) \cap h^{\mu}(\mathcal{D}_{A_r}), \quad q \in L^{\gamma}(h^{\mu}(W^{1,r})),$$

q adapted to $(\mathcal{F}_t)$ (refer to [**RY94**, Definition 4.2, chapter I]), with $0 < \mu < 1/2$, $d < r < \infty$, $2 \le \gamma < \infty$ and satisfies

$$dq - \left(\nabla u q - \frac{1}{2\lambda} q\right) dt - \frac{1}{\sqrt{\lambda}}\, dB = 0 \qquad \text{in } D \times (0,T) \times \Omega, \tag{3.16}$$

$$\rho \frac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda} P_r \nabla \cdot (\mathbb{E}(q \otimes q) - I) = P_r f \qquad \text{in } D \times (0,T), \tag{3.17}$$

$$u\,(.,0) = u_0 \qquad \text{in } D, \tag{3.18}$$

$$q\,(.,0) = q_0 \qquad \text{in } \Omega. \tag{3.19}$$

The main result of this paper can now be stated.

THEOREM 3.3. *Let $d \ge 2$, let $D \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$ and let $T > 0$. Assume $0 < \mu < \frac{1}{2}$, $2 \le \gamma < \infty$ and $d < r < \infty$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ a complete filtered probability space with $\mathcal{F}_t$ right continuous for $t \in [0,T]$ and upon which the Brownian process $B \in L^{\gamma}(h^{\mu})$ and the initial condition $q_0 \in L^{\gamma}$ are defined. Moreover assume that q_0 satisfies (3.1). Then there exists $\delta_0 > 0$ such that for every $f \in h^{\mu}(L^r)$, $u_0 \in \mathcal{D}_{A_r}$ satisfying*

$$-\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}},$$

and

$$\|P_r f - P_r f(0)\|_{h^{\mu}(L^r)} + \|u_0\|_{W^{2,r}} + \| - \eta_s A_r + P_r f(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}} \le \delta_0,$$

there exists exactly one solution of (3.16)-(3.19) (q is unique up to indistinguishability, see [**RY94**, Definition 1.7, chapter I]*). Moreover, the mapping*

$$(P_r f, u_0) \mapsto (u(f,u_0), q(f,u_0))$$

is analytic.

Using the well known properties of the Helmholtz-Weyl projector (see section 1.5), the following result is obtained.

COROLLARY 3.4. *Under the assumptions of the above theorem, there exists a unique solution (u,q,p) of (3.2)-(3.7) with the regularity (3.14).*

The vector field $q^S : \Omega \times [0,T] \to \mathbb{R}^d$ solution of (3.12) is now more precisely defined. Given $q_0 \in L^{\gamma}(\Omega)$ satisfying (3.1), $q^S \in L^{\gamma}(\Omega; h^{\mu}([0,T]; \mathbb{R}^d))$ is the unique solution (up to indistinguishability and ensured by Theorem 2.1, chapter IX, in [**RY94**]) of (3.12). Moreover, (using equation (6.8) section 5.6 of [**KS91**]) a relation for the covariance of q^S is obtained:

$$\mathbb{E}(q^S(s) \otimes q^S(t)) = e^{-\frac{|t-s|}{2\lambda}} I, \quad \forall s,t \in [0,T]. \tag{3.20}$$

Even though q^S does not depend on $x \in D$, when needed, q^S will be considered as an element of $L^{\gamma}(h^{\mu}(W^{1,r}))$.

Using (3.11) and (3.13) problem (3.16)-(3.19) can be rewritten as, find

$$u \in h^{1+\mu}(\mathcal{H}_r) \cap h^{\mu}(\mathcal{D}_{A_r}), \quad q^D \in L^{\gamma}(h^{1+\mu}(W^{1,r}) \cap h_0^{\mu}(W^{1,r})),$$

with $2 \le \gamma < \infty,\ 0 < \mu < 1/2,\ d < r < \infty$, such that

$$\frac{\partial q^D}{\partial t} + \frac{1}{2\lambda} q^D - S_d(u, q^D) = (\nabla u) q^S \qquad \text{in } D \times (0,T) \times \Omega, \tag{3.21}$$

$$\rho \frac{\partial u}{\partial t} + \eta_s A_r u$$
$$- \frac{\eta_p}{\lambda} P_r \nabla \cdot \Big(\mathbb{E}(q^D \otimes q^S + q^S \otimes q^D) + S_c(u, q^D) \Big) = P_r f \qquad \text{in } D \times (0,T), \tag{3.22}$$

$$u(.,0) = u_0 \qquad \text{in } D, \tag{3.23}$$

with

$$S_d(u, q^D) := (\nabla u) q^D, \tag{3.24}$$

$$S_c(u, q^D) := \mathbb{E}(q^D \otimes q^D). \tag{3.25}$$

It will be shown S_d and S_c are well defined in appropriate spaces.

In order to prove Theorem 3.3, the mapping $F : Y \times X \to Z$ is introduced, where

$$Y := \{(P_r f, u_0), \text{ such that } (f, u_0) \in h^\mu(L^r) \times \mathcal{D}_{A_r} \text{ and } -\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}\},$$
$$W := \Big\{ w \in L^\gamma(h^\mu(W^{1,r})); w \text{ adapted to } (\mathcal{F}_t)_{t\in[0,T]} \Big\}, \quad Z := W \times Y,$$
$$X := \{(u,q) \in h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r}) \times L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r})); q \text{ adapted to } (\mathcal{F}_t)_{t\in[0,T]}\},$$

and for $y := (P_r f, u_0) \in Y$ and $x := (u, q^D) \in X$

$$F(y,x) := \begin{pmatrix} \dfrac{\partial q^D}{\partial t} + \dfrac{1}{2\lambda} q^D - S_d(u, q^D) - (\nabla u) q^S \\ \rho \dfrac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda} P_r \nabla \cdot \Big(\mathbb{E}(q^D \otimes q^S + q^S \otimes q^D) + S_c(u, q^D) \Big) - P_r f \\ u(.,0) - u_0 \end{pmatrix}, \tag{3.26}$$

with $q^S \in L^\gamma(h^\mu(W^{1,r}))$ defined by (3.12). Then problem (3.21)-(3.23) can be reformulated as: given $q_0 \in L^\gamma(\Omega)$ satisfying (3.1), $q^S \in L^\gamma(h^\mu(W^{1,r}))$ defined by (3.12) and $y = (P_r f, u_0) \in Y$, find $x = (u, q^D) \in X$ such that

$$F(y,x) = 0 \text{ in } Z. \tag{3.27}$$

The aim is to use the implicit function theorem by proving

- the spaces $X,\ Y,\ W$ and Z equipped with appropriate norms are Banach spaces,
- F is a well defined, real analytic mapping,
- $F(0,0) = 0$ and the Fréchet derivative $D_x F(0,0)$ is an isomorphism from X to Z.

This establishes the existence part in the conclusion of Theorem 3.3. The uniqueness part is treated separately.

The space X is equipped with the norm $\|\cdot\|_X$ defined for $x := (u, q^D) \in X$ by

$$\|x\|_X = \|u, q^D\|_X := \|u\|_{h^{1+\mu}(L^r)} + \|u\|_{h^\mu(W^{2,r})} + \|q^D\|_{L^\gamma(h^{1+\mu}(W^{1,r}))}.$$

Since X is a closed subspace of $h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r}) \times L^\gamma(h^\mu(W^{1,r}))$, it becomes a Banach space. The space Y is equipped with the norm $\|\cdot\|_Y$ defined for $y := (P_r f, u_0) \in Y$ by

$$\|y\|_Y = \|P_r f, u_0\|_Y$$
$$:= \|P_r f - P_r f(0)\|_{h^\mu(L^r)} + \|u_0\|_{W^{2,r}} + \| - \eta_s A_r u_0 + P_r f(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}}.$$

As a consequence of the continuity of the linear mapping

$$(P_r f, u_0) \longmapsto -\eta_s A_r u_0 + P_r f(0)$$

from $h^\mu(\mathcal{H}_r)\times\mathcal{D}_{A_r}$ (equipped with the product norm) to $\mathcal{H}_r$ and of the completeness of $\overline{\mathcal{D}_{A_r}}^{E_\mu,\infty}$, the space $(Y,\|.\|_Y)$ is a closed subspace of $h^\mu(\mathcal{H}_r)\times\mathcal{D}_{A_r}\times W^{1,r}$ and thus a Banach space. Similarly, using [**Mun53**, Corollary 31.5.1 and Theorem 32.1] and since the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is complete, W is a Banach space equipped with the induced norm of $L^\gamma(h^\mu(W^{1,r}))$. The space Z is equipped with the product norm and becomes a Banach space .

The following Lemma ensures the space $h^\mu(W^{1,r})$ is a Banach algebra and as a Corollary, it will be deduced that the function $F:Y\times X\to Z$ is well defined and analytic.

LEMMA 3.5. *For $0<\mu<1$ and $r>d$, the space $h^\mu(W^{1,r})\subset\mathcal{C}^0([0,T]\times\overline{D})$ is a Banach algebra. Moreover, there exists a constant C such that for all $u,v\in h^\mu(W^{1,r})$ the following inequality holds*
$$\|uv\|_{h^\mu(W^{1,r})}\le C\|u\|_{h^\mu(W^{1,r})}\|v\|_{h^\mu(W^{1,r})},$$
where $(uv)(x,t):=u(x,t)v(x,t)$ for $(x,t)\in D\times[0,T]$.

PROOF. Let $u,v\in h^\mu(W^{1,r})$. Let $0\le s<t\le T$, using the triangle inequality it follows,
$$\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}\le\|u(t)(v(t)-v(s))\|_{W^{1,r}}+\|(u(t)-u(s))v(s)\|_{W^{1,r}}.$$
For all Banach space E we have $h^\mu(E)\subset_> \mathcal{C}^0(E)$. Moreover, since $W^{1,r}$ is a Banach algebra for $r>d$ (see section 1.1), we obtain
(3.28)
$$\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}\le C(\|u\|_{h^\mu(W^{1,r})}\|v(t)-v(s)\|_{W^{1,r}}+\|u(t)-u(s)\|_{W^{1,r}}\|v\|_{h^\mu(W^{1,r})}),$$
where C_1 is a constant independent of u and v. Thus, we find
$$\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}\le C_2(\|u\|_{h^\mu(W^{1,r})}\|v\|_{h^\mu(W^{1,r})}\,|t-s\,|^\mu+\|u\|_{h^\mu(W^{1,r})}\,|t-s\,|^\mu\,\|v\|_{h^\mu(W^{1,r})}),$$
where C_2 is a constant independent of u and v. Hence, $u\cdot v\in\mathcal{C}^0(W^{1,r})$ and
$$\|uv\|_{h^\mu(W^{1,r})}:=\sup_{t\in(0,T)}\|u(t)v(t)\|_{W^{1,r}}+\sup_{\substack{t,s\in(0,T)\\ t\neq s}}\frac{\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}}{|t-s\,|^\mu}\le C\|u\|_{h^\mu(W^{1,r})}\|v\|_{h^\mu(W^{1,r})}.$$
Moreover, from (3.28) we also deduce
$$\lim_{\delta\to0}\sup_{|t-s\,|<\delta}\frac{\|u(t)v(t)-u(s)v(s)\|_{W^{1,r}}}{|t-s\,|^\mu}=0,$$
which ensures $u\cdot v\in h^\mu(W^{1,r})$ and ends the proof of the Lemma. □

The same arguments can be used to prove

COROLLARY 3.6. *Let $u,\ v\in h^{1+\mu}(W^{1,r})$, $0<\mu<1$, $d<r<\infty$, then the product $u\cdot v$ belongs to $h^{1+\mu}(W^{1,r})$ and there exists a constant C such that*
$$\|uv\|_{h^{1+\mu}(W^{1,r})}\le C\|u\|_{h^{1+\mu}(W^{1,r})}\|v\|_{h^{1+\mu}(W^{1,r})}.$$

COROLLARY 3.7. *Let $x_1:=(u^1,q^1)$, $x_2:=(u^2,q^2)\in X$, then*
$$b_d(x_1,x_2):=(\nabla u^1)q^2\in W,$$
$$b_c(x_1,x_2):=P_r\nabla\cdot\mathbb{E}(q^1\otimes q^2)\in h^\mu(\mathcal{H}_r).$$
Moreover, the corresponding bilinear mappings $b_d:X\times X\to W$ and $b_c:X\times X\to h^\mu(\mathcal{H}_r)$ are continuous, i.e. there exist two constants C_1,C_2 such that for $x_1,x_2\in X$ it holds
$$\|b_d(x_1,x_2)\|_W\le C_1\|x_1\|_X\|x_2\|_X,$$
$$\|b_c(x_1,x_2)\|_{h^\mu(\mathcal{H}_r)}\le C_2\|x_1\|_X\|x_2\|_X.$$

Proof. Let (u^1, q^1), $(u^2, q^2) \in X$. Using Lemma 3.5 it follows for almost all $\omega \in \Omega$

$$(\nabla u^1)q^2(\omega) \in h^\mu(W^{1,r})$$

and

$$\|(\nabla u^1)q^2(\omega)\|_{h^\mu(W^{1,r})} \leq C\|u^1\|_{h^\mu(W^{1,r})}\|q^2(\omega)\|_{h^\mu(W^{1,r})},$$

where C is a constant independent of (u^1, q^1) and (u^2, q^2). Hence,

$$\|b_d((u^1, q^1), (u^2, q^2))\|_W \leq C\|(u^1, q^1)\|_X\|(u^2, q^2)\|_X.$$

This ensures $b_d((u^1, q^1), (u^2, q^2)) \in L^\gamma(h^\mu(W^{1,r}))$. Similarly, using Lemma 3.5 it follows that for almost all $w \in \Omega$

$$q^1(\omega) \otimes q^2(\omega) \in h^\mu(W^{1,r})$$

and there exists a constant C independent of (u^1, q^1) and (u^2, q^2) such that for almost all $\omega \in \Omega$

$$\|q^1(\omega) \otimes q^2(\omega)\|_{h^\mu(W^{1,r})} \leq C\|q^1(\omega)\|_{h^\mu(W^{1,r})}\|q^2(\omega)\|_{h^\mu(W^{1,r})}.$$

Using Bochner's Theorem [**Yos80**, chapter V], $\omega \in \Omega \mapsto q^1(\omega) \otimes q^2(\omega) \in h^\mu(W^{1,r})$ is Bochner integrable and $b_c((u^1, q^1), (u^2, q^2)) \in h^\mu(\mathcal{H}_r)$. Moreover, the Cauchy-Schwarz inequality implies

$$\|b_c((u^1, q^1), (u^2, q^2))\|_{h^\mu(L^r)} \leq C\|u^1, q^1\|_X\|u^2, q^2\|_X.$$

□

Remark 3.8. The mappings $S_d : X \to W$ and $S_c : X \to h^\mu(W^{1,r})$ can be characterized for $x \in X$ by $S_d(x) = b_d(x, x)$ and $S_c(x) = b_c(x, x)$. Thus, in virtue of Proposition 5.4.1 in [**Car67**], the mappings S_d and S_c are well defined and even analytic in their respective spaces.

Remark 3.9. Using similar arguments, we also have for $(u, q^D) \in X$

$$\int_0^\cdot (\nabla u(s))q^D(s)ds \in L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r})).$$

Lemma 3.10. *The mapping $F : Y \times X \to Z$ is well defined and analytic. Moreover, for all $x := (u, q^D) \in X$ its Fréchet derivative in $(0,0)$, $D_xF(0,0)$ is given by*

$$D_xF(0,0)x = \begin{pmatrix} \dfrac{\partial q^D}{\partial t} + \dfrac{1}{2\lambda}q^D - (\nabla u)q^S \\ \rho\dfrac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda}P_r\nabla\cdot\mathbb{E}(q^D \otimes q^S + q^S \otimes q^D) \\ u(0) \end{pmatrix},$$

where $q^S \in L^\gamma(h^\mu(W^{1,r}))$ is defined by (3.12).

Proof. In order to study the property of the mapping $F : Y \times X \to Z$ we rewrite it as follows

$$F(y, x) = L_1y + L_2x - \begin{pmatrix} S_d(x) \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{\eta_p}{\lambda}P_r\nabla\cdot S_c(x) \\ 0 \end{pmatrix}, \tag{3.29}$$

where $L_1 : Y \to Z$, $L_2 : X \to Z$ are bounded linear operator defined for $y := (P_rf, u_0) \in Y$ and $x := (u, q^D) \in X$ by

$$L_1y := \begin{pmatrix} 0 \\ -P_rf \\ u_0 \end{pmatrix}, \; L_2x := \begin{pmatrix} \dfrac{\partial q^D}{\partial t} + \dfrac{1}{2\lambda}q^D - (\nabla u)q^S \\ \rho\dfrac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda}P_r\nabla\cdot\mathbb{E}\left(q^D \otimes q^S + q^S \otimes q^D\right) \\ u(0) \end{pmatrix}$$

and $S_d : X \to W$, $S_c : X \to h^\mu(\mathcal{H}_r)$ are well defined and analytic (see Remark 3.8). Clearly, the first two terms in (3.29) are also analytic. Thus, $F : Y \times X \to Z$ is analytic.

Moreover for $x \in X$

$$D_x F(0,0)x = L_2 x,$$

which completes the proof. □

In order to use the implicit function theorem, it remains to check that $D_x F(0,0)$ is an isomorphism from X to Z. Therefore, it is necessary to check that, for $w \in W$ and $(f, u_0) \in Y$ there exists a unique $(u, q^D) \in X$ such that

$$\left\{\begin{aligned} \frac{\partial q^D}{\partial t} + \frac{1}{2\lambda} q^D - (\nabla u) q^S = w,\\ \rho \frac{\partial u}{\partial t} + \eta_s A_r u - \frac{\eta_p}{\lambda} P_r \nabla \cdot \mathbb{E}(q^D \otimes q^S + q^S \otimes q^D) = f,\\ u(0) = u_0. \end{aligned}\right. \tag{3.30}$$

LEMMA 3.11. *Given $w \in W$, $(f, u_0) \in Y$, there exists a unique $(u, q^D) \in X$ solution of (3.30).*

PROOF. Solving the first equation of the above system, it follows for $t \in [0, T]$

$$q^D(t) = \int_0^t e^{-\frac{t-s}{2\lambda}} (\nabla u(s) q^S(s) + w(s))\, ds, \text{ a.e in } \Omega. \tag{3.31}$$

The aim is to use equation (3.31) in the third equation of (3.30) in order to obtain a relation for u. For $t \in [0, T]$ we have

$$\mathbb{E}(q^D(t) \otimes q^S(t)) = \int_0^t e^{-\frac{t-s}{2\lambda}} \Big(\mathbb{E}(\nabla u(s) q^S(s) \otimes q^S(t)) + \mathbb{E}(w(s) \otimes q^S(s))\Big)\, ds.$$

Using (3.20) we obtain for the first term on the right hand side of the above equation

$$\int_0^t e^{-\frac{t-s}{2\lambda}} \mathbb{E}(\nabla u(s) q^S(s) \otimes q^S(t)) ds = \int_0^t e^{-\frac{t-s}{\lambda}} \nabla u(s) ds.$$

Using the same arguments for the term $\mathbb{E}(q^S(t) \otimes q^D(t))$, we obtain

$$\begin{aligned} \mathbb{E}\left(q^D(t) \otimes q^S(t) + q^S(t) \otimes q^D(t)\right) &= \int_0^t e^{-\frac{t-s}{\lambda}} (\nabla u(s) + (\nabla u(s))^T) ds \\ &+ \int_0^t e^{-\frac{t-s}{2\lambda}} \mathbb{E}(w(s) \otimes q^S(t) + q^S(t) \otimes w(s)) ds. \end{aligned} \tag{3.32}$$

Going back to (3.30) it follows that u satisfies

$$\rho \frac{\partial u}{\partial t} + \eta_s A_r u + k * A_r u = P_r f + P_r \nabla \cdot g, \qquad u(0) = u_0, \tag{3.33}$$

where $k \in \mathcal{C}^\infty([0, T])$ is defined for $t \in [0, T]$ by $k(t) := \frac{\eta_p}{\lambda} e^{-\frac{t}{\lambda}}$, $g \in h^\mu(W^{1,r})$ is defined for $t \in [0, T]$ by

$$g(t) := \frac{\eta_p}{\lambda} \int_0^t e^{-\frac{t-s}{2\lambda}} \mathbb{E}(w(s) \otimes q^S(t) + q^S(t) \otimes w(s)) ds, \tag{3.34}$$

and $k * A_r u$ denotes the convolution in time of the kernel k with $A_r u$. The right hand side of (3.33) belongs to $h^\mu(\mathcal{H}_r)$ using Corollary 3.7. Moreover, $g(0) = 0$ and since $(f, u_0) \in Y$, the compatibility condition

$$-\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$$

is satisfied. It suffices now to apply Lemma 1.24, which ensures the existence and uniqueness of $u \in h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r})$. Going back to (3.31) with a given $u \in h^\mu(W^{1,r})$, using the regularity of w and Remark 3.9, there exists an unique $q^D \in L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r}))$ adapted to the filtration (up to indistinguishability). ☐

We are now in a position to prove the next Lemma.

LEMMA 3.12. *Let $d \geq 2$, let $D \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$ and let $T > 0$. Assume $0 < \mu < \frac{1}{2}$, $2 \leq \gamma < \infty$ and $d < r < \infty$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ a complete filtered probability space with $\mathcal{F}_t$ right continuous for $t \in [0,T]$ and upon which the Brownian process $B \in L^\gamma(h^\mu)$ and the initial condition $q_0 \in L^\gamma$ are defined. Moreover assuming q_0 satisfies (3.1) and let $q^S \in L^\gamma(h^\mu(W^{1,r}))$ satisfying (3.12). Then there exists $\delta_0 > 0$ such that for every $f \in h^\mu(L^r)$, $u_0 \in \mathcal{D}_{A_r}$ satisfying*

$$-\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_\mu,\infty},$$

and

$$\|P_r f - P_r f(0)\|_{h^\mu(L^r)} + \|u_0\|_{W^{2,r}} + \| - \eta_s A_r + P_r f(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_\mu,\infty}} \leq \delta_0,$$

there exists exactly one solution of (3.21)-(3.23). Moreover, the mapping

$$(P_r f, u_0) \mapsto (u(f, u_0), q^D(f, u_0))$$

is analytic.

PROOF. Apply the implicit function theorem to (3.27). From Lemma 3.10, F is well defined and analytic, $F(0,0) = 0$ and from Lemma 3.11 $D_x F(0,0)$ is an isomorphism from X to Z. Therefore, the implicit function theorem in the analytic case (see Theorem 4.5.4 chapter 4 p. 56 of [**BT03**]) implies the existence of $\delta_0 > 0$ and $\varphi : Y \to X$ analytic such that for $y := (P_r f, u_0) \in Y$ with $\|y\|_Y < \delta_0$ we have $F(y, \varphi(y)) = 0$.

Now check the uniqueness. Assume $(u, q^D) \in X$ satisfying (3.21)-(3.23). Using a standard result on a system of ordinary differential equation we obtain that q^D satisfies for $t \in [0,T]$

$$q^D(t) = \Phi(t) \int_0^t \Phi(s)^{-1} (\nabla u(s)) q^S(s) ds, \tag{3.35}$$

where $\Phi : D \times [0,T] \to \mathbb{R}^{d\times d}$ is the fundamental matrix satisfying

$$\Phi(t,x) = I + \int_0^t \left(\nabla u(s) - \frac{I}{2\lambda}\right) \Phi(s,x) ds.$$

Using a fixed point theorem in $h^\mu(W^{1,r})$ (see [**Car67**]), we obtain $\Phi \in h^{1+\mu}([0,T]; W^{1,r}(D; \mathbb{R}^{d\times d}))$. Then by reversing time, it is possible to show Φ^{-1} also belongs to $h^{1+\mu}([0,T]; W^{1,r}(D; \mathbb{R}^{d\times d}))$. Moreover, let σ_1, σ_2, $\sigma_3 : D \times [0,T] \to \mathbb{R}^{d\times d}$ defined for $t \in [0,T]$ by

$$\sigma_1(t) := \mathbb{E}(q^D(t) \otimes q^S(t)) = \Phi \int_0^t e^{-\frac{t-s}{2\lambda}} \Phi^{-1}(s,.) \nabla u(s,.) ds,$$

$$\sigma_2(t) := \mathbb{E}(q^S(t) \otimes q^D(t)) = \int_0^t e^{-\frac{t-s}{2\lambda}} (\nabla u(s,.))^T (\Phi^{-1}(s,.))^T ds \ \Phi^T$$

and

$$\begin{aligned}\sigma_3(t) &:= \mathbb{E}(q^D(t) \otimes q^D(t)) \\ &= \Phi \int_0^t \int_0^t \Phi(s,.)^{-1} (\nabla u(s,.)) e^{-\frac{|s-k|}{2\lambda}} (\nabla u(k,.))^T (\Phi^{-1}(k,.))^T ds \ dk \ \Phi^T.\end{aligned}$$

Using representation (3.35), relation (3.20), Corollary 3.7, Remarks 3.9 and Corollary 3.6, it follows

$$\sigma_1,\ \sigma_2,\ \sigma_3 \in h^{1+\mu}(W^{1,r}).$$

Thus, we have

$$\frac{\partial \sigma_1}{\partial t} = -\frac{1}{\lambda}\sigma_1 + (\nabla u)\sigma_1 + \nabla u,$$
$$\frac{\partial \sigma_2}{\partial t} = -\frac{1}{\lambda}\sigma_2 + \sigma_2(\nabla u)^T + (\nabla u)^T$$

and

$$\frac{\partial \sigma_3}{\partial t} = -\frac{1}{\lambda}\sigma_3 + (\nabla u)\sigma_3 + \sigma_3(\nabla u)^T + (\nabla u)\sigma_2 + \sigma_1(\nabla u)^T.$$

Finally, setting $\sigma := \frac{\eta_p}{\lambda}(\sigma_1+\sigma_2+\sigma_3) \in h^{1+\mu(W^{1,r})}$ we obtain

$$\frac{\lambda}{\eta_p}\left(\frac{\partial \sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\right) + \frac{1}{\eta_p}\sigma = \left(\nabla u + (\nabla u)^T\right),$$

which corresponds to (3.8) with definition (3.9). Then, using Theorem 2.3, one obtains the existence of exactly one solution $u \in h^{1+\mu}(\mathcal{H}_r) \cap h^{\mu}(\mathcal{D}_{A_r})$. Going back to (3.35), we obtain the uniqueness (up to indistinguishability) of $q^D \in L^{\gamma}(h^{1+\mu}(W^{1,r}) \cap h_0^{\mu}(W^{1,r}))$ adapted to the filtration. $\square$

Corollary 3.13. *Under the assumptions of the above Lemma, there exists a unique $(u, q^D, p) \in X \times h^{\mu}(W^{1,r} \cap L_0^2)$, satisfying*

$$\frac{\partial q^D}{\partial t} + \frac{1}{2\lambda}q^D - (\nabla u)q^D = (\nabla u)q^S \qquad \textit{in } D \times (0,T)\times\Omega, \tag{3.36}$$
$$\rho\frac{\partial u}{\partial t} - 2\eta_s\nabla\cdot\epsilon(u) - \frac{\eta_p}{\lambda}\nabla\cdot\mathbb{E}((q^S+q^D)\otimes(q^S+q^D)) + \nabla p = f \qquad \textit{in } D\times(0,T), \tag{3.37}$$
$$u\,(.,0) = u_0 \qquad \textit{in } D,, \tag{3.38}$$

with q^D adapted to $(\mathcal{F}_t)_{t\geq 0}$.

Let us go back to the proof of Theorem 3.3.

Proof. (of Theorem 3.3) Let $y := (P_r f, u_0) \in Y$. It was proved in Lemma 3.12 that $x := (u(y), q^D(y))$ is unique in X but $q(y) := q^S + q^D(y)$ is only in $L^{\gamma}(h^{\mu}(W^{1,r}))$ because of the regularity imposed by q^S. Obviously, q is unique (up to indistinguishability) and the mapping $y \in Y \mapsto x(y) \in X$ is analytic. $\square$

Remark 3.14. In the proof of uniqueness (in Lemma 3.12), it was proved that $V := \mathbb{E}(q\otimes q)$ satisfies (3.8) with $V \in h^{1+\mu}(W^{1,r})$. Setting $\sigma := \dfrac{\eta_p}{\lambda}(V - I)$, then $\sigma \in h^{1+\mu}(W^{1,r})$ and satisfies (2.3). Lemma 2.2 and Theorem 2.3 ensure $(u,\sigma) \in h^{1+\mu}(\mathcal{H}_r) \cap h^{\mu}(\mathcal{D}_{A_r}) \times h^{1+\mu}(W^{1,r})$ coincides with the unique solution of the Oldroyd-B problem. This fact having been established, the existence and uniqueness could also be directly ensured by Lemma 2.2 and Theorem 2.3, but this approach is more general. Indeed, it is only necessary that the linearized problem has a unique solution so that the original problem does not need to have a deterministic equivalent. In that case, existence still holds but uniqueness is only ensured by the implicit function theorem. More precisely, there exists a neighborhood $\mathcal{V}\times\mathcal{U} \subset Y \times X$ of $(u, q^D) = (0,0)$ and an analytic mapping $\varphi : Y \to \mathcal{U}$ such that $(y,x) \in \mathcal{V}\times\mathcal{U}$ and $F(y,x) = 0$ is equivalent to $y \in \mathcal{V}$ and $x = \varphi(y)$.

REMARK 3.15. Since, having proved $\mathbb{E}(q \otimes q) \in h^{1+\mu}(W^{1,r})$, and assuming $f \in h^{1+\mu}(L^r)$, some compatibility conditions and using the same arguments as in the previous chapter, it is possible to prove the existence of a solution for (3.2)-(3.7) satisfying $u \in h^{2+\mu}(\mathcal{H}_r) \cap h^{1+\mu}(\mathcal{D}_{A_r})$ and $q \in L^\gamma(h^\mu(W^{1,r}))$ (q still remains in $L^\gamma(h^\mu(W^{1,r}))$ because of the Brownian motion).

REMARK 3.16. As in the previous chapter, the key point to prove this result when D is a convex polygon is to prove that the negative Stokes operator $-A_r$ is still the generator of an analytic semi-group, see Remark 1.26. This assumption will be made and convergence of the finite element scheme will be proven.

3.3. Existence of the finite element approximation and a priori error estimates

In this section it is assumed that D is a convex polygon, that

$$2 \leq \gamma < \infty, \qquad 0 < \mu < 1/2, \qquad 2 = d < r < \infty$$

and that the results of the previous section still hold when D is a convex polygon (see Remark 3.16). Let

$$\begin{aligned}
Y &:= \{(f, u_0) \in h^\mu(L^r) \times \mathcal{D}_{A_r} \text{ such that } -\eta_s A_r u_0 + P_r f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}\},\\
W &:= \Big\{w \in L^\gamma(h^\mu(W^{1,r})); w \text{ adapted to } (\mathcal{F}_t)_{t\in[0,T]}\Big\}, \quad Z := W \times Y,\\
X &:= \{(u, q) \in h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r}) \times L^\gamma(h^{1+\mu}(W^{1,r}) \cap h_0^\mu(W^{1,r})); q \text{ adapted to } (\mathcal{F}_t)_{t\in[0,T]}\},
\end{aligned}$$

be the data and solution spaces provided with the norms

$$\begin{aligned}
\|f, u_0\|_Y &:= \|f\|_{h^\mu(L^r)} + \|u_0\|_{W^{2,r}} + \| - \eta_s A_r u_0 + P_r f(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}},\\
\|w\|_W &:= \|w\|_{L^\gamma(h^{1+\mu}(W^{1,r}))}, \qquad \|w, f, u_0\|_Z := \|w\|_W + \|f, u_0\|_Y,\\
\|u, q\|_X &:= \|u\|_{h^{1+\mu}(L^r)} + \|u\|_{h^\mu(W^{2,r})} + \|q\|_{L^\gamma(h^{1+\mu}(W^{1,r}))}.
\end{aligned}$$

According to Theorem 3.3, Corollary 3.4 and Remark 3.16 it is known that if $y := (f, u_0) \in Y$ is sufficiently small, then there exists a unique solution

$$(u(y), q(y), p(y))$$

of (3.2)-(3.7), the mapping

$$y \longmapsto (u(y), q(y), p(y))$$

being analytic (therefore continuous).

In order to prove that the solution of the nonlinear finite element discretization (3.15) exists and converges to that of (3.2)-(3.7), let $X_h \subset X$ be defined by

$$X_h := L^2(V_h) \times L^2(L^\infty(R_h)),$$

provided with the norm $\|.\|_{X_h}$ defined for $x_h := (u_h, q_h) \in X_h$ by

$$\|x_h\|_{X_h}^2 := 2\eta_s \int_0^T \|\epsilon(u_h(t))\|_{L^2(D)}^2 dt + \int_\Omega \sup_{t\in[0,T]} \|q_h(\omega, t)\|_{L^2(D)}^2 \, d\mathcal{P}(\omega).$$

The splitting $q_h = q^S + q_h^D$ will also be used for space discretization (remember q^S does not depend on the space variable and satisfies (3.12)) where $q_h^D \in L^2(L^\infty(R_h))$ satisfies

$$(q_h^D(t), r_h) + \left(\int_0^t \left(\frac{1}{2\lambda} q_h^D(k) - (\nabla u_h(k))(q^S(k) + q_h^D(k))\right) dk, r_h\right) = 0, \tag{3.39}$$

for all $r_h \in R_h$, a.e in $(0, T)$ and a.e in Ω.

It will be shown that there exists a unique $(u_h, q_h^D) \in X_h$ converging to $(u, q^D) \in X$ and thus a unique (u_h, q_h) converging to (u, q). For this purpose, the discrete problem corresponding to the unknowns (u_h, q_h^D, p_h) will be written in the abstract framework of the previous chapter and [**CR97**]. Using the splitting $q_h = q^S + q_h^D$, we rewrite the solution of (3.15) as the following fixed point problem. Given $y := (f, u_0) \in Y$, find $x_h := (u_h, q_h^D) \in X_h$ such that

$$x_h = \mathsf{T}_h\Big(y, S_c(x_h), S_d(x_h)\Big), \tag{3.40}$$

where S_c and S_d have to be extended to the larger spaces

$$\begin{aligned} S_c : L^2(H^1) \times L^2(L^\infty(L^2)) &\longrightarrow L^2(L^2) \\ x_h := (u_h, q_h^D) &\longmapsto S_c(x_h) := \mathbb{E}(q_h^D \otimes q_h^D), \end{aligned}$$

$$\begin{aligned} S_d : L^2(H^1) \times L^2(L^\infty(L^2)) &\longrightarrow L^2(L^2(L^2)) \\ x_h := (u_h, q_h^D) &\longmapsto S_d(x_h) := (\nabla u_h) q_h^D. \end{aligned}$$

The linear operator T_h is defined as follows

$$\begin{aligned} \mathsf{T}_h : Y \times L^2(L^2) \times L^2(L^2(L^2)) &\longrightarrow X_h \\ (f_1, u_0, f_2, w) &\longmapsto \mathsf{T}_h(f_1, u_0, f_2, w) := (\tilde{u}_h, \tilde{q}_h^D), \end{aligned} \tag{3.41}$$

where for almost all $t \in (0, T)$ and almost all $\omega \in \Omega$

$$(\tilde{u}_h, \tilde{q}_h^D, \tilde{p}_h) : (\omega, t) \longmapsto (\tilde{u}_h(t), \tilde{q}_h^D(\omega, t), \tilde{p}_h(t)) \in V_h \times R_h \times Q_h$$

satisfies $\tilde{u}_h(0) = i_h u_0$ and

$$\begin{aligned} \rho\Big(\frac{\partial \tilde{u}_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \Big(\tilde{p}_h, \nabla\cdot v_h\Big) + \frac{\eta_p}{\lambda}\Big(\mathbb{E}(\tilde{q}_h^D \otimes q^S + q^S \otimes \tilde{q}_h^D) + f_2, \epsilon(v_h)\Big) \\ - \Big(f_1, v_h\Big) + \Big(\nabla\cdot\tilde{u}_h, s_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}_h, \nabla s_h\Big)_K \\ (\tilde{q}_h^D(t), r_h) + \left(\int_0^t \left(\frac{1}{2\lambda}\tilde{q}_h^D(k) - (\nabla\tilde{u}_h(k)) q^S(k)\right) dk, r_h\right) - (w, r_h) = 0, \end{aligned} \tag{3.42}$$

for all $(v_h, r_h, s_h) \in V_h \times R_h \times Q_h$, a.e. in $(0, T)$ and a.e. in Ω.

It should be noted that, given $y := (f, u_0) \in Y$ sufficiently small, the solution $x(y) := (u(y), q^D(y)) \in X$ of the continuous Hookean dumbbells problem (3.21)-(3.23) also satisfies a fixed point problem, namely

$$x(y) = \mathsf{T}\Big(y, S_c(x(y)), S_d(x(y))\Big). \tag{3.43}$$

Here the operator T is defined by

$$\begin{aligned} \mathsf{T} : Y \times U \times W &\longrightarrow X \\ (f_1, u_0, f_2, w) &\longmapsto \mathsf{T}(f_1, u_0, f_2, w) := (\tilde{u}, \tilde{q}^D), \end{aligned}$$

where $(\tilde{u}, \tilde{q}^D, \tilde{p}) \in X \times h^\mu(W^{1,r} \cap L_0^2)$ satisfy

$$\left\{\begin{aligned} \frac{\partial \tilde{q}^D}{\partial t} + \frac{1}{2\lambda}\tilde{q}^D - (\nabla\tilde{u}) q^S &= w, \\ \rho\frac{\partial\tilde{u}}{\partial t} - \nabla\cdot(2\eta_s\epsilon(\tilde{u})) - \frac{\eta_p}{\lambda}\nabla\cdot\Big(\mathbb{E}(\tilde{q}^D \otimes q^S + q^S \otimes \tilde{q}^D) + f_2\Big) + \nabla\tilde{p} &= f_1, \\ u\,(0) &= u_0. \end{aligned}\right. \tag{3.44}$$

Note that it has been proved that for $x := (u, q^D) \in X$, then $S_c(x) \in h^\mu(W^{1,r})$ and $S_d(x) \in W$ (see Remark 3.8). Moreover, since $q^D(0) = 0$, it follows that $S_c(x) = \mathbb{E}(q^D \otimes q^D)$ vanishes at time $t = 0$ and thus $S_c(x) \in U$ for $x \in X$. Therefore, Lemma 3.11 and the well known properties of the Helmholtz-Weyl projector (see section 1.5) ensure that problem (3.43) is well defined.

The elongation vector $\tilde{q}^D$ can be eliminated from (3.42) and the next Lemma provides the equation satisfied by $\tilde{u}$. This equation is a discrete approximation of (3.33).

LEMMA 3.17. *Let $\gamma \geq 2$, $0 < \mu < 1/2$ and $r > 2$. Let $(f_1, u_0) \in Y$, $f_2 \in L^2(L^2)$, $w \in L^2(L^2(L^2))$ and let $q^S \in L^\gamma(h^\mu(W^{1,r}))$ be defined by (3.12). Then problem (3.42) admits a unique solution $(\tilde{u}_h, \tilde{q}_h^D) \in X_h$. Moreover, $(\tilde{u}_h, \tilde{p}_h)$ satisfies*

$$\begin{aligned}(3.45)\quad \rho\Big(\frac{\partial \tilde{u}_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \Big(\tilde{p}_h, \nabla \cdot v_h\Big) + 2\Big(k * i_h\epsilon(u_h), i_h\epsilon(v_h)\Big)\\ + \Big(\nabla \cdot \tilde{u}_h, s_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}_h, \nabla s_h\Big)_K = \Big(f_1, v_h\Big) + \Big(-\frac{\eta_p}{\lambda} f_2 + i_h g, \epsilon(v_h)\Big),\end{aligned}$$

where $k \in \mathcal{C}^\infty([0,T])$ is defined by $k(t) := \frac{\eta_p}{\lambda} e^{-t/\lambda}$, where $g \in L^\gamma(h^\mu(W^{1,r}))$ is defined by (3.34) and where i_h is the $L^2(D)$ projection onto $R_h \otimes R_h$.

PROOF. In order to prove the existence (and uniqueness) of a solution $(\tilde{u}_h, \tilde{q}_h^D)$, (3.42) will be written using a basis of R_h. Hence, we introduce φ_i^n, $n = 1, 2$, $i = 1, \dots, P$ an orthonormal basis of R_h where P is the number of nodes of the mesh. Let $\tilde{q}_{h,i}^{D,n}$ be the components of q_h^D and $\tilde{u}_{h,i}^n$ be those of u_h with respect to the given basis φ_i^n. Then

$$\tilde{q}_h^D(\omega, t, x) = \sum_{n=1}^{2}\sum_{i=1}^{P} q_{h,i}^{D,n}(\omega, t)\varphi_i^n(x), \quad \tilde{u}_h(t, x) = \sum_{n=1}^{2}\sum_{i=1}^{P} \tilde{u}_{h,i}^n(t)\varphi_i^n(x).$$

Choosing $v_h = 0$, $s_h = 0$, $r_h = \varphi_i^n$ in (3.42) we have

$$\tilde{q}_{h,i}^{D,n}(t) = \int_0^t e^{-\frac{t-s}{2\lambda}} \left(\sum_{k=1}^{2} \frac{\partial \tilde{u}_h^n}{\partial x_k}(s) q^{S,k}(s) + w^n, \varphi_i^n\right) ds, \quad n = 1, 2, \ i = 1, \dots, P,$$

a.e in $(0,T)$, a.e. in Ω and with $w = (w^1, w^2)^T$. The definition of the $L^2(D)$ projection i_h implies for $t \in [0,T]$

$$(3.46)\qquad \tilde{q}_h^D(t) = \int_0^t e^{-\frac{t-s}{2\lambda}} \Big(i_h(\nabla\tilde{u}_h)q^S + i_h w\Big)\, ds,$$

a.e. in Ω. Owning (3.20) and going back to (3.42), $(\tilde{u}_h, \tilde{p}_h)$ satisfies

$$\begin{aligned}\rho\Big(\frac{\partial \tilde{u}_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \Big(\tilde{p}_h, \nabla \cdot v_h\Big) + 2\Big(k * i_h\epsilon(u_h), \epsilon(v_h)\Big)\\ + \Big(\nabla \cdot \tilde{u}_h, s_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}_h, \nabla s_h\Big)_K = \Big(f_1, v_h\Big) + \Big(-\frac{\eta_p}{\lambda} f_2 + i_h g, \epsilon(v_h)\Big).\end{aligned}$$

Using the property of the L^2-projection

$$(i_h\epsilon(\tilde{u}_h), \epsilon(v_h) - i_h\epsilon(v_h)) = 0 \qquad \forall v_h \in V_h,$$

relation (3.45) is obtained. Thus, problem (3.42) is equivalent to (3.45) and (3.46). Existence (and uniqueness) of $\tilde{u}_h \in \mathcal{C}^1([0,T]; V_h)$ satisfying (3.45) is ensured by a standard argument on Stokes system (see [**GR86, QV91**],) and a contraction mapping theorem (see [**LTW98**] or sections 1.3 and 1.4). Finally, since $q^S \in L^\gamma(h^\mu(W^{1,r}))$, equation (3.46) ensures the existence (and uniqueness) of $\tilde{q}_h^D \in L^\gamma(\mathcal{C}^1(R_h))$ thus in $L^2(L^\infty(R_h))$. $\square$

REMARK 3.18. Having proved in the previous Lemma that $\tilde{q}_h^D$ belongs to $L^\gamma(\mathcal{C}^1(R_h))$, (3.39) can be rewritten

$$(\frac{\partial q_h^D}{\partial t}, r_h) + (\frac{1}{2\lambda} q_h^D, r_h) - ((\nabla u_h)(q^S + q_h^D), r_h) = 0 \quad \forall r_h \in R_h, \qquad \text{a.e. in } \Omega. \tag{3.47}$$

The following stability and convergence result holds.

LEMMA 3.19. *The operator T_h is well defined and uniformly bounded with respect to h: there exists $C_1 > 0$ such that for all $h > 0$ and for all $(f_1, u_0) \in Y$, $f_2 \in L^2(L^2)$, $w \in L^2(L^2(L^2))$ it holds*

$$||\mathsf{T}_h(f_1, u_0, f_2, w)||_{X_h} \leq C_1\Big(||f_1, u_0||_Y + ||f_2||_{L^2(L^2)} + ||w||_{L^2(L^2(L^2))}\Big). \tag{3.48}$$

Moreover, there exists $C_2 > 0$ such that for all $h > 0$ and for all $(f_1, u_0, f_2, w) \in Y \times U \times W$, it holds

$$||(\mathsf{T} - \mathsf{T}_h)(f_1, u_0, f_2, w)||_{X_h} \leq C_2 h\Big(||f_1, u_0||_Y + ||f_2||_U + ||w||_W\Big). \tag{3.49}$$

PROOF. Let $(\tilde{u}_h, \tilde{q}_h^D) := \mathsf{T}_h(f_1, u_0, f_2, w)$, where $(\tilde{u}_h, \tilde{q}_h^D) \in X_h$ satisfies (3.45). From Lemma 1 in [**Sob64**], we have

$$\int_0^T \int_0^T e^{-\frac{t-s}{\lambda}} \, (i_h \epsilon(\tilde{u}_h(s)), i_h \epsilon(\tilde{u}_h(t))) \, ds \; dt \geq 0. \tag{3.50}$$

Therefore, choosing $v_h = u_h(t)$ in (3.45), there exists a constant C independent of f_1, f_2, g such that

$$||\tilde{u}_h||_{L^2(H^1)} \leq C\Big(||f_1, u_0||_Y + ||f_2||_{L^2(L^2)} + ||i_h g||_{L^2(L^2)}\Big), \tag{3.51}$$

where $g \in h_0^\mu(W^{1,r})$ is defined by (3.34). Moreover since i_h is bounded in $L^2(D)$, using the continuous embedding $h^\mu([0,T]) \subset_> \mathcal{C}^0([0,T])$ and a Cauchy-Schwarz inequality, we have

$$\|i_h g\|_{L^2(L^2)} \leq C \|q^S\|_{L^2(h^\mu)} \|w\|_{L^2(L^2(L^2))} \tag{3.52}$$

where C is a constant independent of h, w, q^S and g.

On the other hand from (3.46) we have

$$\|\tilde{q}_h^D\|_{L^2(L^\infty(L^2))} \leq C\left(\|\tilde{u}_h\|_{L^2(H^1)} + \|w\|_{L^2(L^2(L^2))}\right), \tag{3.53}$$

where C is a constant independent of h, f_1, u_0, f_2 and w. Thus (3.52) in (3.51) and (3.53) leads to (3.48). Let us now prove (3.49). Let

$$\begin{aligned} e_{\dot{:}}u = \tilde{u} - \tilde{u}_h = \Pi_u + C_u, \qquad & \Pi_u := \tilde{u} - i_h\tilde{u}, \qquad & C_u := i_h\tilde{u} - \tilde{u}_h, \\ e_p := \tilde{p} - \tilde{p}_h = \Pi_p + C_p, \qquad & \Pi_p := \tilde{p} - i_h\tilde{p}, \qquad & C_p := i_h\tilde{p} - \tilde{p}_h, \end{aligned}$$

where $(\tilde{u}_h, \tilde{p}_h)$ solves (3.45) and $(\tilde{u}, \tilde{p})$ solves

$$\rho\frac{\partial \tilde{u}}{\partial t} - \nabla\cdot(2\eta_s\epsilon(\tilde{u}) - 2k * \epsilon(\tilde{u})) + \nabla\tilde{p} = f_1 + \frac{\eta_p}{\lambda}\nabla\cdot f_2 + \nabla\cdot g, \quad \nabla\tilde{u} = 0, \quad \tilde{u}(.,0) = u_0. \tag{3.54}$$

Using the triangle inequality we have

$$\|e_u\|_{L^2(H^1)} \leq \|\Pi_u\|_{L^2(H^1)} + \|C_u\|_{L^2(H^1)}$$

and classical interpolation results lead to

$$\|\Pi_u\|_{L^2(H^1)} \leq Ch\|\tilde{u}\|_{L^2(H^2)}.$$

We now estimate $\|C_u\|_{L^2(H^1)}$. The solution of (3.54) satisfies

$$\rho\Big(\frac{\partial \tilde{u}}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}), \epsilon(v_h)\Big) - \Big(\tilde{p}, \nabla\cdot v_h\Big) + 2\Big(k * \epsilon(\tilde{u}), \epsilon(v_h)\Big)$$
$$-\Big(f_1, v_h\Big) + \Big(\nabla\cdot\tilde{u}, s_h\Big) + \Big(\frac{\eta_p}{\lambda} f_2 - g, \epsilon(v_h)\Big) = 0$$

for all $(v_h, s_h) \in V_h \times Q_h$. Subtracting (3.45) from the above equation, it follows that

$$\rho\Big(\frac{\partial e_u}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(e_u), \epsilon(v_h)\Big) - \Big(e_p, \nabla\cdot v_h\Big) + 2\Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u}_h)), \epsilon(v_h)\Big)$$
$$+\Big(\nabla\cdot e_u, r_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla e_p - \nabla\tilde{p}, \nabla r_h\Big)_K - \Big(g - i_h g, \epsilon(v_h)\Big) = 0, \tag{3.55}$$

for all $(v_h, r_h) \in V_h \times Q_h$. On the other hand, from the definition of i_h (the L^2 projection onto the finite element spaces), C_u and Π_u, we have

$$\Big(k * i_h\epsilon(C_u), i_h\epsilon(C_u)\Big) = \Big(k * i_h\epsilon(C_u), \epsilon(C_u)\Big)$$
$$= \Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u}_h)) + k * (i_h\epsilon(\tilde{u}) - \epsilon(\tilde{u})) - k * i_h\epsilon(\Pi_u), \epsilon(C_u)\Big).$$

Hence, we obtain

$$\rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) - \Big(C_p, \nabla\cdot C_u\Big) + 2\Big(k * i_h\epsilon(C_u), i_h\epsilon(C_u)\Big)$$
$$+\Big(\nabla\cdot C_u, C_p\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K$$
$$= \rho\Big(\frac{\partial(e_u - \Pi_u)}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(e_u - \Pi_u), \epsilon(C_u)\Big) - \Big(e_p - \Pi_p, \nabla\cdot C_u\Big)$$
$$+2\Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u}_h)), \epsilon(C_u)\Big) - 2\Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u})), \epsilon(C_u)\Big) - 2\Big(k * i_h\epsilon(\Pi_u), \epsilon(C_u)\Big)$$
$$+\Big(\nabla\cdot(e_u - \Pi_u), C_p\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla(e_p - \Pi_p), \nabla C_p\Big)_K \tag{3.56}$$

From the definition of i_h again, it is clear that

$$\Big(\frac{\partial \Pi_u}{\partial t}, C_u\Big) = 0,$$

so that, using (3.55), (3.56) yields

$$\rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + 2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K + 2\Big(k * i_h\epsilon(C_u), i_h\epsilon(C_u)\Big)$$
$$= -2\eta_s\Big(\epsilon(\Pi_u), \epsilon(C_u)\Big) + \Big(\Pi_p, \nabla\cdot C_u\Big) - \Big(\nabla\cdot(\Pi_u), C_p\Big) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\Pi_p, \nabla C_p\Big)_K$$
$$+\sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}, \nabla C_p\Big)_K + 2\Big(k * (\epsilon(\tilde{u}) - i_h\epsilon(\tilde{u})), \epsilon(C_u)\Big) \tag{3.57}$$
$$+2\Big(k * i_h\epsilon(\Pi_u), \epsilon(C_u)\Big) + (g - i_h g, \epsilon(C_u))$$
$$= I_1 + \cdots + I_8.$$

It now remains to bound the terms $I_1, ..., I_8$ in the above equality. Using Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned} I_1 &= -2\eta_s\Big(\epsilon(\Pi_u), \epsilon(C_u)\Big) \\ &\leq 2\eta_s||\epsilon(\Pi_u)||_{L^2(D)}||\epsilon(C_u)||_{L^2(D)} \\ &\leq 5\eta_s||\epsilon(\Pi_u)||^2_{L^2(D)} + \frac{\eta_s}{5}||\epsilon(C_u)||^2_{L^2(D)}. \end{aligned}$$

Similarly

$$\begin{aligned} I_2 = \Big(\Pi_p, \nabla\cdot C_u\Big) &\leq \frac{5}{4\eta_s}||\Pi_p||^2_{L^2(D)} + \frac{\eta_s}{5}||\nabla\cdot C_u||^2_{L^2(D)} \\ &\leq \frac{5}{4\eta_s}||\Pi_p||^2_{L^2(D)} + \frac{\eta_s}{5}||\epsilon(C_u)||^2_{L^2(D)}. \end{aligned}$$

An integration by parts yields, since $\Pi_u = 0$ on ∂D

$$I_3 = \Big(\nabla\cdot(\Pi_u), C_p\Big) = -\Big(\Pi_u, \nabla C_p\Big) = -\sum_{K\in\mathcal{T}_h}\Big(\Pi_u, \nabla C_p\Big)_K$$
$$\leq \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \sum_{K\in\mathcal{T}_h}\frac{3\eta_p}{\alpha h_K^2}||\Pi_u||^2_{L^2(K)}.$$

Again, Cauchy-Schwarz and Young's inequalities yield

$$I_4 = -\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\Pi_p, \nabla C_p\Big)_K \leq \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \frac{3\alpha h^2}{4\eta_p}||\nabla\Pi_p||^2_{L^2(D)}$$

and

$$I_5 = \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla\tilde{p}, \nabla C_p\Big)_K \leq \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{12\eta_p}||\nabla C_p||^2_{L^2(K)} + \frac{3\alpha h^2}{4\eta_p}\left\|\nabla\tilde{p}\right\|^2_{L^2(D)}.$$

Since $k \in \mathcal{C}^\infty([0,T]) \subset_{>} \mathcal{C}^0([0,T])$, using Cauchy-Schwarz and Young's inequalities yield

$$I_6 = -2\Big(k*(i_h\epsilon(\tilde{u}) - \epsilon(\tilde{u})), \epsilon(C_u)\Big) \leq \frac{5}{\eta_s}\|k\|^2_{L^\infty(0,T)}\|i_h\epsilon(\tilde{u}) - \epsilon(\tilde{u})\|^2_{L^2(D)} + \frac{\eta_s}{5}\|\epsilon(C_u)\|^2_{L^2(D)}$$

and

$$I_7 = -2\Big(k*i_h\epsilon(\Pi_u), \epsilon(C_u)\Big) \leq \frac{5}{\eta_s}\|k\|^2_{L^\infty([0,T])}\|\epsilon(\Pi_u)\|^2_{L^2(D)} + \frac{\eta_s}{5}\|\epsilon(C_u)\|^2_{L^2(D)},$$

where in the last inequality the stability of $i_h : L^2(D) \to L^2(D)$ was used. Finally, a Cauchy-Schwarz and Young's inequalities again yields

$$I_8 = (g - i_h g, \epsilon(C_u)) \leq \frac{5}{4\eta_s}\|g - i_h g\|^2_{L^2(D)} + \frac{\eta_s}{5}\|\epsilon(C_u)\|^2_{L^2(D)}.$$

The above estimates of $I_1, ..., I_8$ in (3.57) yield

$$\rho\Big(\frac{\partial C_u}{\partial t}, C_u\Big) + \frac{1}{2}2\eta_s\Big(\epsilon(C_u), \epsilon(C_u)\Big) + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\Big(\nabla C_p, \nabla C_p\Big)_K + 2\Big(k*i_h\epsilon(C_u), i_h\epsilon(C_u)\Big)$$
$$\leq C\Big(||\epsilon(\Pi_u)||^2_{L^2(D)} + ||\Pi_p||^2_{L^2(D)} + \sum_{K\in\mathcal{T}_h}\frac{1}{h_K^2}\|\Pi_u\|^2_{L^2(K)} + h^2||\nabla\Pi_p||^2_{L^2(D)} + h^2||\nabla\tilde{p}||^2_{L^2(D)}\Big)$$

where C depends only on ρ, η_s, η_p, k and α. Time integration for $0 \le s \le T$ yields

$$\begin{aligned}\frac{\rho}{2}\|C_u(s)\|^2_{L^2(D)} + \eta_s \int_0^s \|\epsilon(C_u)\|^2_{L^2(D)} + 2\int_0^s \Big((k * i_h\epsilon(C_u))(s), i_h\epsilon(C_u(s))\Big)ds \\ \le \frac{\rho}{2}\|C_u(0)\|^2_{L^2(D)} + C\int_0^s \Big(||\epsilon(\Pi_u)||^2_{L^2(D)} + ||\Pi_p||^2_{L^2(D)} + ||\Pi_\sigma||^2_{L^2(D)} \\ + \sum_{K\in\mathcal{T}_h} \frac{1}{h_K^2}\|\Pi_u\|^2_{L^2(K)} + h^2||\nabla\Pi_p||^2_{L^2(D)} + h^2||\nabla\tilde{p}||^2_{L^2(D)}.\end{aligned}$$

Using standard interpolation results and (3.50), we obtain

$$\|\epsilon(C_u)\|^2_{L^2(L^2)} \le Ch^2\Big(\|\tilde{u}\|^2_{h^\mu(W^{2,r})} + \|\tilde{p}\|^2_{h^\mu(W^{1,r})} + \|\nabla u_0\|^2_{L^2(D)} + \|g\|^2_{L^2(W^{1,r}(D))}\Big),$$

where C does not depend on h, f_1, f_2, u_0 and g. Then, using the well known properties of the Helmholtz-Weyl projector (see section 1.5) and estimation (1.16) there exists a constant C independent of h, f_1, f_2, u_0 and g such that

$$\|\frac{\partial\tilde{u}}{\partial t}\|_{h^\mu(L^r)} + \|\tilde{u}\|_{h^\mu(W^{2,r})} + \|\tilde{p}\|_{h^\mu(W^{1,r})} \le C\left(\|f_1,u_0\|_Y + \|f_2\|_{h^\mu(W^{1,r})} + \|w\|_{L^\gamma(h^\mu(W^{1,r}))}\right).$$

In addition, since

$$\|g\|_{L^2(W^{1,r})} \le C\|q^S\|_{L^2(\Omega;L^\infty([0,T]))}\|w\|_{L^2(L^2(W^{1,r}))} \quad \text{and} \quad \|\nabla u_0\|_{L^2(D)} \le C\|u_0\|_{\mathcal{D}_{A_r}}$$

we obtain

$$||\epsilon(C_u)||_{L^2(L^2)} \le Ch\Big(||f_1,u_0||_Y + ||f_2||_U + ||w||_W\Big)$$

where C does not depend on h, f_1, f_2, u_0 and w. Thus

$$||\epsilon(e_u)||_{L^2(L^2)} \le Ch\Big(||f_1,u_0||_Y + ||f_2||_U + ||w||_W\Big). \tag{3.58}$$

It remains to prove that

$$\|\tilde{q}^D - \tilde{q}_h^D\|_{L^2(L^\infty(L^2))} \le Ch\Big(||y||_Y + ||f_2||_U + ||w||_W\Big), \tag{3.59}$$

where C does not depend on h, f_1, f_2, u_0 and w. The solution $(\tilde{u}, \tilde{q}^D) := \mathsf{T}(y, f_2, w)$ satisfies for $t \in [0,T]$

$$\tilde{q}^D(t) = \int_0^t e^{-\frac{t-s}{2\lambda}}\left((\nabla\tilde{u})q^S + w\right)ds.$$

Hence, (3.59) follows by subtracting (3.46) from the above equation. □

The goal is now to prove that (3.15) has a unique solution (u_h, q_h) converging to that of (3.21)-(3.21). Since q^S does not depend on $x \in D$, it suffices to show that (3.40) has a unique solution (u_h, q_h^D) converging to (u, q^D) solution of (3.2)-(3.7). For this purpose, use, as in the previous chapter, an abstract framework and write (3.15) as follows: given $y := (f, u_0) \in Y$, find $x_h := (u_h, q_h^D) \in X_h$ such that

$$F_h(y, x_h) = 0, \tag{3.60}$$

where F_h is defined by

$$\begin{aligned}F_h : Y \times X_h &\longrightarrow X_h \\ (y, x_h) &\longmapsto F_h(y, x_h) := x_h - \mathsf{T}_h\Big(y, S_c(x_h), S_d(i_h x)\Big).\end{aligned}$$

In order to prove existence and convergence, use Lemma 2.25. The mapping $F_h : Y \times X_h \to X_h$ is $\mathcal{C}^1$. First prove that the scheme is consistent and that $D_x F$ is locally Lipschitz.

LEMMA 3.20. *Let $y := (f, u_0) \in Y$ be sufficiently small, assume Remark 3.16 and let $x(y) := (u(y), q^D(y)) \in X$ be the solution of (3.21)-(3.23). Then, there exists a constant C_1 such that for all $0 < h < 1$, for all $y \in Y$ it holds*

$$\|F_h(y, i_h x(y))\|_{X_h} \leq C_1 h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big). \tag{3.61}$$

Moreover, there exists a constant C_2 such that for all $h > 0$, for all $y \in Y$, for all $z \in X_h$ we have

$$\|D_x F_h(y, i_h x(y)) - D_x F_h(y, z)\|_{\mathcal{L}(X_h)} \leq \frac{C_2}{h}\|i_h x(y) - z\|_{X_h}. \tag{3.62}$$

PROOF. From the definition of F_h we have

$$\begin{aligned}F_h(y, i_h x) &= i_h x - x - \mathsf{T}_h\Big(y, S_c(i_h x), S_d(i_h x)\Big)\\ &= (i_h x - x) + \mathsf{T}_h(0, 0, S_c(i_h x) - S_c(x), S_d(x) - S_d(i_h x)) + (\mathsf{T} - \mathsf{T}_h)(y, S_c(x), S_d(x))\end{aligned}$$

so that,

$$\begin{aligned}\frac{1}{3}\|F_h(y, i_h x)\|_{X_h}^2 \leq& \|i_h x - x\|_{X_h}^2 + \|\mathsf{T}_h(0, 0, S_c(i_h x) - S_c(x), S_d(x) - S_d(i_h x))\|_{X_h}^2\\ &+ \|(\mathsf{T} - \mathsf{T}_h)(y, S_c(x), S_d(x))\|_{X_h}^2.\end{aligned}$$

Using standard interpolation results for the first term of the right hand side and Lemma 3.19 for the second and third terms, it follows that

$$\begin{aligned}\|F_h(y, i_h x)\|_{X_h}^2 \leq C\Big(&h^2\|x\|_X^2 + \|S_c(x) - S_c(i_h x)\|_{L^2(L^2)}^2 + \|S_d(x) - S_d(i_h x)\|_{L^2(L^2(L^2))}^2\\ &+ h^2\|y\|_Y^2 + h^2\|S_c(x)\|_{h^\mu(W^{1,r})} + h^2\|S_d(x)\|_{L^2(h^\mu(W^{1,r}))}^2\Big),\end{aligned} \tag{3.63}$$

C being independent of h and y. Proceeding as in Corollary 3.7, we have

$$\|S_c(x)\|_{h^\mu(W^{1,r})} + \|S_d(x)\|_{L^2(h^\mu(W^{1,r}))} = \|\frac{\lambda}{\eta_p}\mathbb{E}(q^D \otimes q^D)\|_{h^\mu(W^{1,r})} + \|(\nabla u)q^D\|_{L^2(h^\mu(W^{1,r}))} \leq C\|x\|_X^2, \tag{3.64}$$

C being independent of h and y. On the other hand, we also have

$$\begin{aligned}S_d(x) - S_d(i_h x) &= (\nabla u)q^D - (\nabla i_h u) i_h q^D\\ &= (\nabla(u - i_h u))q^D + (\nabla i_h u)(q^D - i_h q^D)\end{aligned}$$

so that, using a Cauchy-Schwarz inequality

$$\|S_d(x) - S_d(i_h x)\|_{L^2(L^2(L^2))}^2 \leq C\|x - i_h x\|_{X_h}\Big(\|q^D\|_{L^2(L^\infty(L^\infty))} + \|\nabla i_h u\|_{L^2(L^\infty)}\Big)$$

C being independent of h and y. Standard interpolation results lead to

$$\|x - i_h x\|_{X_h} \leq C_1 h\|x\|_X$$

and

$$\|\nabla i_h u\|_{L^2(L^\infty)} \leq \|\nabla u\|_{L^2(L^\infty)} + \|\nabla(u - i_h u\|_{L^2(L^\infty)} \leq C_2\|u\|_{h^\mu(W^{2,r})},$$

for $0 < h < 1$ and where C_1, C_2 are constants independent of h and x. Thus

$$\|S_d(x) - S_d(i_h x)\|_{L^2(L^2(L^2))} \leq Ch\|x\|_X, \tag{3.65}$$

C being independent of h and y. Similarly

$$\|S_c(x) - S_c(i_h x)\|_{L^2(L^2)} \leq Ch\|x\|_X. \tag{3.66}$$

Finally, (3.64), (3.65) and (3.66) in (3.63) yields (3.61).

Now prove (3.62). Let $z := (v, r) \in X_h$, let $\tilde{z} := (\tilde{v}, \tilde{r}) \in X_h$, we have

$$\Big(D_x F_h(y, i_h x) - D_x F_h(y, z)\Big)\tilde{z} = -\mathsf{T}_h\Big(0, 0, (DS_c(i_h x) - DS_c(z))\tilde{z}, (DS_d(i_h x) - DS_d(z))\tilde{z}\Big).$$

Using Lemma 3.19 we obtain

$$\begin{aligned} (3.67)\quad & \| \left(D_x F_h(y, i_h x) - D_x F_h(y, z)\right) \tilde{z}\|_{X_h} \\ & \qquad \leq C\left(\|(DS_c(i_h x) - DS_c(z))\tilde{z}\|_{L^2(L^2)} + \|(DS_d(i_h x) - DS_d(z))\tilde{z}\|_{L^2(L^2(L^2))}\right), \end{aligned}$$

C being independent of h and y. Using Cauchy-Schwarz inequality, there exists a constant C independent of h and y such that

$$\begin{aligned} & \|(DS_d(i_h x) - DS_d(z))\tilde{z}\|_{L^2(L^2(L^2))} \\ & \qquad \leq C\Big(\|\nabla(i_h u - v)\|_{L^2(L^\infty)}\|\tilde{r}\|_{L^2(L^\infty(L^2))} + \|\nabla\tilde{v}\|_{L^2(L^\infty)}\|i_h q^D - r\|_{L^2(L^\infty(L^2))}\Big). \end{aligned}$$

A classical inverse inequality yields

$$\begin{aligned} & \|(DS_d(i_h x) - DS_d(z))\tilde{z}\|_{L^2(L^2(L^2))} \\ & \qquad \leq \frac{\tilde{C}}{h}\Big(\|\nabla(i_h u - v)\|_{L^2(L^2)}\|\tilde{r}\|_{L^2(L^\infty(L^2))} + \|\nabla\tilde{v}\|_{L^2(L^2)}\|i_h q^D - r\|_{L^2(L^\infty(L^2))}\Big), \end{aligned}$$

$\tilde{C}$ being independent of h and y, so that we finally have

$$\|(DS_d(i_h x) - DS_d(z))\tilde{z}\|_{L^2(L^2(L^2))} \leq \frac{\tilde{C}}{h}\|i_h x - z\|_{X_h}\|\tilde{z}\|_{X_h}. \tag{3.68}$$

Similarly

$$\begin{aligned} \|(DS_c(i_h x) - DS_c(z))\tilde{z}\|_{L^2(0,T;L^2(D))} & \leq C\|i_h q - r\|_{L^2(L^\infty(L^\infty))}\|\tilde{r}\|_{L^2(L^\infty(L^2))} \\ & \leq \frac{\tilde{C}}{h}\|i_h x - z\|_{X_h}\|\tilde{z}\|_{X_h}. \qquad (3.69) \end{aligned}$$

Inequalities (3.68) and (3.69) in (3.67) yields (3.62). □

Before proving existence of a solution to (3.60) it is still necessary to check that $D_x F_h(y, i_h x)$ is invertible.

LEMMA 3.21. *Let $y := (f, u_0) \in Y$ be sufficiently small, assume Remark 3.16 and let $x(y) := (u(y), q^D(y)) \in X$ be the solution of (3.21)-(3.23). Then, for y sufficiently small, for $h \leq 1$ it holds*

$$\|D_x F_h(y, i_h x(y))^{-1}\|_{\mathcal{L}(X_h)} \leq 2.$$

PROOF. By definition of F_h, we have

$$D_x F_h(y, i_h x) = I - \mathsf{T}_h(0, 0, DS_c(i_h x), DS_d(i_h x)),$$

so that

$$D_x F_h(y, i_h x) = I - G_h \quad \text{with} \quad G_h := \mathsf{T}_h(0, 0, DS_c(i_h x), DS(i_h x)).$$

Proving that $\|G_h\|_{\mathcal{L}(X_h)} \leq 1/2$ for y sufficiently small will imply that $D_x F_h(y, i_h x)$ is invertible and

$$\|D_x F_h(y, i_h x)^{-1}\|_{\mathcal{L}(X_h)} \leq 2.$$

Let $z := (v, \tau) \in X_h$. Using Lemma 3.19 we have

$$\|G_h(z)\|_{X_h} \leq C_1\left(\|DS_c(i_h x)z\|_{L^2(L^2)} + \|DS_d(i_h x)z\|_{L^2(L^2(L^2))}\right),$$

C_1 being independent of y and h. Proceeding as in the proof of Lemma 3.20, we have

$$\|DS_d(i_h x)z\|_{L^2(L^2(L^2))} \leq C_2\Big(\|\nabla u\|_{L^2(W^{1,r})}\|\tau\|_{L^2(L^\infty(L^2))} + \|\nabla v\|_{L^2(L^2)}\|q^D\|_{L^2(L^\infty(W^{1,r}))}\Big),$$

C_2 being independent of y, h and z. Hence,

$$\|G_h(z)\|_{X_h} \leq C_3\|x\|_X\|z\|_{X_h},$$

where C_3 is independent of y, h and z. From Lemma 3.12, the mapping $y \mapsto x(y)$ is continuous, thus if $||y||_Y$ is sufficiently small we have $\|x\|_X \leq 1/(2C_3)$ so that

$$\|G_h(z)\|_{X_h} \leq \frac{1}{2}\|z\|_{X_h}.$$

□

Existence of a solution to the finite element scheme (3.15) and convergence to the solution of (3.2)-(3.7) can now be proved.

THEOREM 3.22. *Let $y := (f, u_0) \in Y$ be sufficiently small, assume Remark 3.16 and let $x(y) := (u(y), q^D(y)) \in X$ be the solution of (3.21)-(3.21). Then, there exists $\zeta > 0$ such that for y and h sufficiently small, there exists a unique $x_h(y) = (u_h(y), q_h^D(y))$ in the ball of X_h centered at $i_h x(y)$ with radius ζh, satisfying*

$$F_h(y, x_h(y)) = 0.$$

Moreover, the mapping $y \mapsto x_h(y)$ is continuous and there exists $C > 0$ independent of h and y such that the following a priori error estimate holds

$$\|x(y) - x_h(y)\|_{X_h} \leq Ch. \tag{3.70}$$

The proof of Theorem 3.22 can now be provided.

PROOF OF THEOREM 3.22. Apply Lemma 2.25 with $Y := X_h$, $Z := X_h$, $G := F_h$ and $v := i_h x(y)$. According to Lemma 3.20 there exists a constant C_1 independent of y and h such that

$$\epsilon = \|F_h(y, i_h x(y))\|_{X_h} \leq C_1 h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big).$$

According to Lemma 3.21, for $\|y\|_Y$ sufficiently small

$$\gamma = D_x F_h(y, i_h x(y))_{\mathcal{L}(X_h)} \leq 2.$$

According to Lemma 3.20, there is a constant C_2 independent of y and h such that

$$L(\alpha) = \sup_{x \in \overline{B}(i_h x(y), \alpha)} \|DF_h(i_h x(y)) - DF_h(x)\|_{\mathcal{L}(X_h)} \leq \frac{C_2}{h}\alpha.$$

Hence, we have

$$\begin{aligned} 2\gamma L(2\gamma\epsilon) &\leq 2.2\frac{C_2}{h}\Big(2.2C_1 h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big)\Big) \\ &= 16C_1C_2\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big). \end{aligned}$$

Using the continuity of the mapping $y \to x(y)$ it follows that, for sufficiently small $\|y\|_Y$ then $2\gamma L(2\gamma\epsilon) < 1$ and Lemma 2.25 applies. There exists a unique $x_h(y)$ in the ball $\overline{B}(i_h x(y), 2\gamma\epsilon)$ such that

$$F_h(y, x_h(y)) = 0$$

and we have

$$\|i_h x(y) - x_h(y)\|_{X_h} \leq 4C_1 h.$$

It suffices to use the triangle inequality

$$\|x(y) - x_h(y)\|_{X_h} \leq \|x(y) - i_h x(y)\|_{X_h} + \|i_h x(y) - x_h(y)\|_{X_h},$$

and standard interpolation results to obtain (3.70). The fact that the mapping $y \mapsto (x_h(y))$ is continuous is as a direct consequence of the implicit function theorem. $\square$

CHAPTER 4

Numerical simulation of 3D viscoelastic flows with free surfaces

This chapter is mainly extracted from [**BPL05**] and involves the simulation of viscoelastic flows with complex free surfaces in three space dimensions. The mathematical formulation of the model is similar to that of the volume of fluid (VOF) method, but the numerical procedures are different.

A splitting method is used for the time discretization. The prediction step consists of solving three advection problems, one for the volume fraction of liquid (which allows the new liquid domain to be obtained), one for the velocity field and one for the extra-stress. The correction step corresponds to solving an Oldroyd-B fluid flow problem without advection in the new liquid domain.

Two different grids are used for the space discretization. The three advection problems are solved on a fixed, structured grid made up of small cubic cells, using a forward characteristics method. The Oldroyd-B problem without advection is solved using continuous, piecewise linear stabilized finite elements on a fixed, unstructured mesh of tetrahedrons.

Efficient post-processing algorithms enhance the quality of the numerical solution. A hierarchical data structure reduces the memory requirements.

Convergence of the numerical method is checked for the pure extensional flow and the filling of a pipe. Numerical results are presented for the stretching of a filament. Fingering instabilities are obtained when the aspect ratio is large. Results pertaining to jet buckling are also shown.

4.1. The model

Let Λ be a cavity of $\mathbb{R}^3$ in which an Oldroyd-B fluid is confined, and let $T > 0$ be the final time of the simulation. At time t, the liquid region is denoted $D(t)$. Finally, let Q_T be the space-time domain containing the liquid

$$Q_T := \{(x,t) \in \Lambda \times (0,T); x \in D(t), 0 < t < T\},$$

and let Σ_T be the space-time free surface between the liquid and the surrounding air. The notation is shown in two space dimensions in Fig. 1.

In the liquid region, the velocity field $u : Q_T \to \mathbb{R}^3$, the pressure field $p : Q_T \to \mathbb{R}$ and the symmetric extra-stress tensor field $\sigma : Q_T \to \mathbb{R}^{3\times 3}$ must satisfy :

$$\begin{aligned} \rho\frac{\partial u}{\partial t} + \rho(u\cdot\nabla)u - 2\eta_s\ \nabla\cdot\epsilon(u) + \nabla p - \nabla\cdot\sigma &= \rho g, \\ \nabla\cdot u &= 0, \\ \sigma + \lambda\Big(\frac{\partial\sigma}{\partial t} + (u\cdot\nabla)\sigma - \nabla u\,\sigma - \sigma\nabla u^T\Big) - 2\eta_p\epsilon(u) &= 0. \end{aligned}$$

Here ρ is the density, g the gravity, $\eta_s \geq 0$ and $\eta_p > 0$ are the solvent and polymer viscosities and λ the relaxation time.

Let $\varphi : \Lambda \times [0,T] \to \mathbb{R}$ be the volume fraction of liquid. The function φ is a step function, which equals one if liquid is present and zero if it is not; thus φ is the characteristic function of

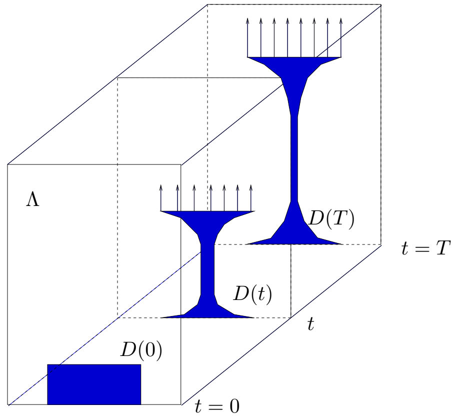

FIGURE 1. Calculation domain for the stretching of a filament in two space dimensions. At the initial time, the viscoelastic fluid is at rest and occupies the domain $D(0)$. At $t > 0$, the upper plate moves at a given velocity.

the liquid region

$$D(t) := \{x \in \Lambda; \varphi(x,t) = 1\}.$$

Since the interface moves with the liquid, the function φ must satisfy (in a weak sense)

$$\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi = 0 \qquad \text{in } Q_T. \tag{4.1}$$

From a Lagrangian point of view, the function φ is constant along the trajectories of the fluid particles. More precisely, $\varphi(X(t),t) = \varphi(X(0),0)$, where $X(t)$ is the trajectory of a fluid particle, thus $X'(t) = u(X(t),t)$.

Initial and boundary conditions are as follows: At the initial time, the volume fraction of liquid $\varphi(\cdot,0)$ is given, which defines the liquid region,

$$D(0) := \{x \in \Lambda; \varphi(x,0) = 1\},$$

(see Fig. 1 for the notations in two space dimensions). The initial velocity field u and extra-stress tensor σ are then prescribed in $D(0)$. Now turn to the boundary conditions for the velocity field. It is assumed that no external forces act on the free surface Σ_T (effects of surface tension are neglected) :

$$-pn + \Big(2\eta_s \epsilon(u) + \sigma\Big)n = 0 \qquad \text{on } \Sigma_T, \tag{4.2}$$

where n is the unit outer normal of Σ_T. Neglecting surface tension effects may not be correct in many applications, however, surface tension was not implemented for the following reasons: Firstly, realistic results can be obtained when including viscoelastic effects and without considering surface tension forces (see the numerical results of section 4.4). Secondly, even though accurate procedures are available in VOF-like methods to compute surface tension [**GLN+99, LF04, SZ99, Ren88**], the mesh size required to obtain a good approximation of curvature would be too small to allow viscoelastic computations in three space dimensions.

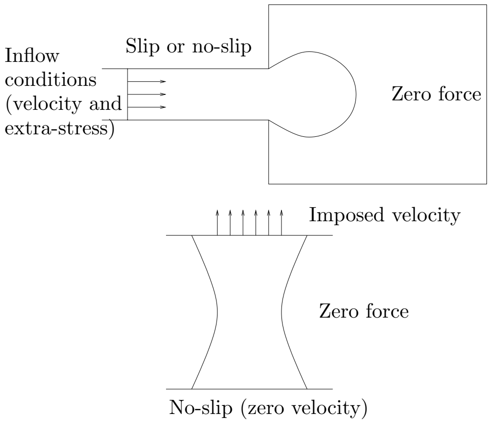

FIGURE 2. Boundary conditions. Top : jet emerging from a die. Bottom : stretching of a filament.

On the boundary of the liquid region being in contact with the walls (i.e. the boundary of the cavity Λ, see Fig. 1), essential boundary conditions (i.e. imposed velocity components) or natural boundary conditions (i.e. boundary conditions which are not explicitly enforced but which are implicitly included in the weak formulation) can be imposed for the velocity and the extra-stress. Consider the two following situations : $\imath$) a jet emerging from a die $\imath\imath$) the stretching of a filament, see Fig. 2. In case $\imath$), both the velocity and extra-stress are imposed at the inflow boundary. Either slip or no-slip boundary conditions apply on the other boundaries of the cavity Λ. If no-slip conditions are enforced for the velocity, then no other conditions apply. If slip boundary conditions are enforced i.e. $u \cdot n = 0$, then the fluid tangent force should also be set to zero, namely

$$\Big(-pn + \Big(2\eta_s \epsilon(u) + \sigma\Big) n\Big) \cdot t_i = 0 \quad i = 1, 2, \tag{4.3}$$

where t_1 and t_2 are two unit vectors tangent to the boundary of the cavity. In case $\imath\imath$), the velocity is imposed on the top and bottom sides of the cavity whereas condition (4.2) applies on the lateral side. Boundary conditions (4.2) and (4.3) are straightforward to implement in the framework of finite element methods since the corresponding terms vanish after integration by parts in the variational formulation, see section 3.2 hereafter.

4.2. Time discretization: an implicit splitting algorithm

The implicit, order one, splitting algorithm described in [**MPR99, MPR03, CPR05**] for Newtonian flows is extended here to viscoelastic situations. This splitting algorithm allows advection phenomena to be decoupled from other phenomena, see [**Glo03**]. The reader should note that a similar algorithm has already been presented for viscoelastic flow computations in fixed domains [**PW99**].

Let $0 = t^0 < t^1 < t^2 < \ldots < t^N = T$ be a subdivision of the time interval $[0, T]$, define $\Delta t^n = t^n - t^{n-1}$ as the n-th time step, $n = 1, 2, \ldots, N$, Δt the largest time step. At time t^{n-1},

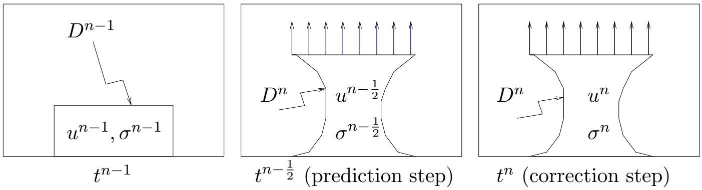

FIGURE 3. The splitting algorithm.

assume that an approximation $\varphi^{n-1} : \Lambda \to \mathbb{R}$ of the volume fraction of liquid is known, which defines the liquid region :

$$D^{n-1} := \{x \in \Lambda; \varphi^{n-1}(x) = 1\}.$$

Also assume that approximations of the velocity $u^{n-1} : D^{n-1} \to \mathbb{R}^3$ and the extra stress $\sigma^{n-1} : D^{n-1} \to \mathbb{R}^{3\times 3}$ are available. Then φ^n, D^n, u^n, σ^n are computed by means of a splitting algorithm as illustrated in Fig. 3. The prediction step consists of solving three advection problems, which yields the new volume fraction of liquid φ^n, the new liquid region D^n, the predicted velocity $u^{n-\frac{1}{2}} : D^n \to \mathbb{R}^3$ and the predicted extra-stress $\sigma^{n-\frac{1}{2}} : D^n \to \mathbb{R}^{3\times 3}$. Then, the correction step is performed, a generalized Stokes problem is solved, which yields the new velocity $u^n : D^n \to \mathbb{R}^3$ and pressure $p^n : D^n \to \mathbb{R}$. The new extra-stress $\sigma^n : D^n \to \mathbb{R}^{3\times 3}$ is then updated from the Oldroyd-B constitutive equation.

4.2.1. Prediction step : advection. The prediction step consists of solving between time t^{n-1} and t^n the three advection problems :

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = 0, \tag{4.4}$$

$$\frac{\partial \tau}{\partial t} + (v \cdot \nabla)\tau = 0, \tag{4.5}$$

$$\frac{\partial \psi}{\partial t} + v \cdot \nabla \psi = 0, \tag{4.6}$$

with initial conditions

$$\begin{aligned} v(t^{n-1}) &= u^{n-1}, \\ \tau(t^{n-1}) &= \sigma^{n-1}, \\ \psi(t^{n-1}) &= \varphi^{n-1}. \end{aligned}$$

These three problems can be solved exactly using the method of characteristics [**Pir89, PLT92, QV91**], the trajectories of the velocity field being straight lines. Indeed, the trajectories are given by $X'(t) = v(X(t), t)$, but since v is constant along the trajectories, it follows that $X'(t) = v(X(t^{n-1}), t^{n-1}) = u^{n-1}(X(t^{n-1}))$. Let $u^{n-\frac{1}{2}}$, $\sigma^{n-\frac{1}{2}}$, φ^n denote the solution at time t^n of (4.4), (4.5), (4.6), respectively. Thus

$$u^{n-\frac{1}{2}}(x + \Delta t^n u^{n-1}(x)) = u^{n-1}(x), \tag{4.7}$$

$$\sigma^{n-\frac{1}{2}}(x + \Delta t^n u^{n-1}(x)) = \sigma^{n-1}(x), \tag{4.8}$$

$$\varphi^n(x + \Delta t^n u^{n-1}(x)) = \varphi^{n-1}(x), \tag{4.9}$$

hold for x belonging to D^{n-1}. Once φ^n is known in the cavity Λ, then the liquid region at time t^n is defined by :

$$D^n := \{y \in \Lambda; \varphi^n(y) = 1\}.$$

At this point it should be stressed that after the prediction step, the obtained velocity $u^{n-\frac{1}{2}}$ is not divergence free. The divergence free property is obtained after the correction step, see eq. (4.10) and (4.11) below.

4.2.2. Correction step : Stokes and Oldroyd-B. The new liquid region D^n being known, the predicted velocity $u^{n-\frac{1}{2}} : D^n \to \mathbb{R}^3$ and the extra-stress $\sigma^{n-\frac{1}{2}} : D^n \to \mathbb{R}^{3\times 3}$ being also known, the new velocity u^n is obtained by solving a generalized Stokes problem :

$$\rho \frac{u^n - u^{n-\frac{1}{2}}}{\Delta t^n} - 2\eta_s \nabla \cdot \epsilon(u^n) + \nabla p^n - \nabla \cdot \sigma^{n-\frac{1}{2}} = \rho g \qquad \text{in } D^n, \tag{4.10}$$

$$\nabla \cdot u^n = 0 \qquad \text{in } D^n, \tag{4.11}$$

then, the new extra-stress σ^n is obtained from Oldroyd-B constitutive equation :

$$\sigma^n + \lambda\Big(\frac{\sigma^n - \sigma^{n-\frac{1}{2}}}{\Delta t^n} - \nabla u^n \sigma^{n-\frac{1}{2}} - \sigma^{n-\frac{1}{2}}(\nabla u^n)^T\Big) - 2\eta_p \epsilon(u^n) = 0. \tag{4.12}$$

4.3. Space discretization and implementation

Two distinct grids are used to solve the prediction and correction steps. Since the shape of the cavity Λ can be complex (for instance the case in mould filling or extrusion processes), finite element techniques are well suited for solving (4.10)-(4.12) using an unstructured mesh. On the other hand, a structured grid of cubic cells is used to implement (4.7)-(4.9). The reasons for using a structured grid are the following: Firstly, the method of characteristics can be easily implemented on structured grids. Secondly, the size of the cells can be tuned in order to control numerical diffusion when projecting (4.7)-(4.9) on the structured grid. Numerical experiments reported in [**MPR99, MPR03, CPR05**] have shown that choosing the cell spacing three to five times smaller than the mesh spacing is a good trade-off between numerical diffusion and computational costs or memory storage.

4.3.1. Advection step : structured grid of cubic cells. The implementation of (4.7)-(4.9) is now discussed. Assume that the grid is made out of cubic cells C_{ijk} of size h. Let φ_{ijk}^{n-1}, u_{ijk}^{n-1} and σ_{ijk}^{n-1} be the approximate value of φ, u and σ at center of cell number (ijk) and time t^{n-1}. According to (4.7)-(4.9), the advection step on cell number (ijk) consists of advecting φ_{ijk}^{n-1}, u_{ijk}^{n-1} and σ_{ijk}^{n-1} by $\Delta t^n u_{ijk}^{n-1}$ and then projecting the values onto the structured grid. An example of cell advection and projection is presented in Fig. 4 in two space dimensions.

This advection algorithm is unconditionally stable with respect to the CFL condition - velocity multiplied by the time step divided by the cells spacing h - and $O(\Delta t + h^2/\Delta t)$ convergent, according to the theoretical results available for the characteristics-Galerkin method [**Pir89, PLT92, QV91**]. However, this algorithm has two drawbacks. The first is that numerical diffusion is introduced when projecting the values of the advected cells on the grid (remember that the volume fraction of liquid is discontinuous across the interface) and the second is if the time step is too large, two cells may arrive at the same place, producing numerical (artificial) compression.

In order to enhance the quality of the volume fraction of liquid, two post-processing procedures have been implemented. Refer to [**MPR99, MPR03, CPR05**] for a description in two and three space dimensions. The first procedure reduces numerical diffusion and is a simplified

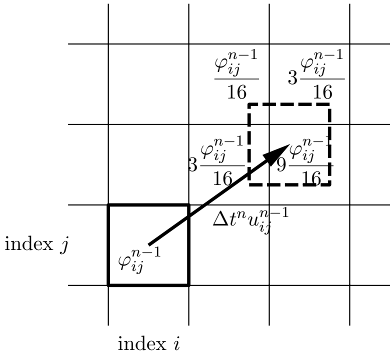

FIGURE 4. An example of two dimensional advection of φ_{ij}^{n-1} by $\Delta t^n u_{ij}^{n-1}$, and projection on the grid. The advected cell is represented by the dashed lines. The four cells containing the advected cell receive a fraction of φ_{ij}^{n-1}, according to the position of the advected cell. In this example, the new values of the volume fraction of liquid φ^n are updated as follows : $\varphi_{i+1,j+1}^n = \varphi_{i+1,j+1}^n + 3/16\varphi_{ij}^{n-1}$; $\varphi_{i+2,j+1}^n = \varphi_{i+2,j+1}^n + 9/16\varphi_{ij}^{n-1}$; $\varphi_{i+1,j+2}^n = \varphi_{i+1,j+2}^n + 1/16\varphi_{ij}^{n-1}$; $\varphi_{i+2,j+2}^n = \varphi_{i+2,j+2}^n + 3/16\varphi_{ij}^{n-1}$;

implementation of the SLIC (Simple Linear Interface Calculation) algorithm [**Cho80, NW76, SZ99**], see Figures 5 and 6 for a simple example. In the SLIC procedure, if a cell is partially filled with liquid, then the volume fraction of liquid is condensed either along the cells faces, edges or corners (see Fig. 7), according to the volume fraction of liquid of the neighbouring cells (see Fig. 8).

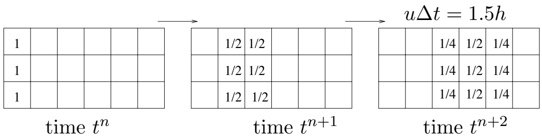

FIGURE 5. Numerical diffusion during the advection step. At time t^n, the cells have a volume fraction of liquid one or zero. The velocity u is horizontal, the time step Δt is chosen so that $u\Delta t = 1.5h$ where h is the cells spacing.

The second procedure removes artificial compression (that is values of the volume fraction of liquid greater than one), which may happen when the volume fraction of liquid advected in two cells arrive at the same place, see Fig. 9. The aim of this procedure is to produce new values φ_{ijk}^n which are between zero and one and is as follows: At each time step, all the cells having values φ_{ijk}^n greater than one (strictly) or between zero and one (strictly) are sorted according to their values φ_{ijk}^n. This can be done in an efficient way using quick sort algorithms. The cells having values φ_{ijk}^n greater than one are called the dealer cells, whereas the cells having values φ_{ijk}^n between zero and one are called the receiver cells. The second procedure then consists of moving the excess fraction of liquid in the dealer cells to the receiver cells, see [**MPR99, Mar00**] for details.

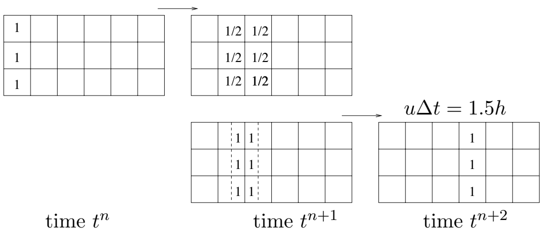

FIGURE 6. Reducing numerical diffusion using the SLIC algorithm. Before advecting a cell partially filled with liquid, the volume fraction of liquid is condensed along the cells boundaries, according to the neighbouring cells.

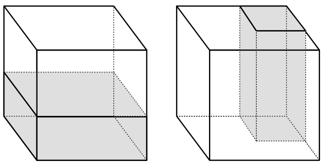

FIGURE 7. SLIC algorithm. If the cell is partially filled with liquid, the liquid is pushed along a face, an edge, or a vertex of the cell, according to the neighbours volume fraction of liquid.

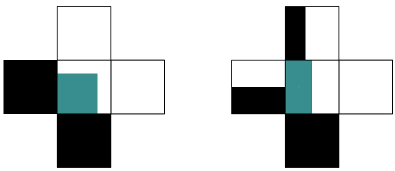

FIGURE 8. SLIC algorithm. The volume fraction of liquid in a cell partially filled with liquid is pushed according to the volume fraction of liquid of the neighbouring cells. Two examples are proposed. Left: the left and bottom neighbouring cells are full of liquid, the right and top neighbouring cells are empty, the liquid is pushed at the bottom left corner of the cell. Right: the bottom neighbouring cell is full of liquid, the right neighbouring cell is empty, the other two neighbouring cells are partially filled with liquid, the volume fraction of liquid is pushed along the left side of the cell.

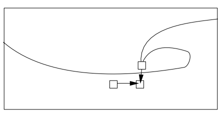

FIGURE 9. An example of numerical (artificial) compression.

Validation of these procedures using standard two dimensional test cases taken from [**AMS03, RK98**] have been performed in [**CPR05**]. Translation, rotation and stretching of a circular region of fluid are shown in Fig. 10. For more details refer to section 5.1 of [**CPR05**].

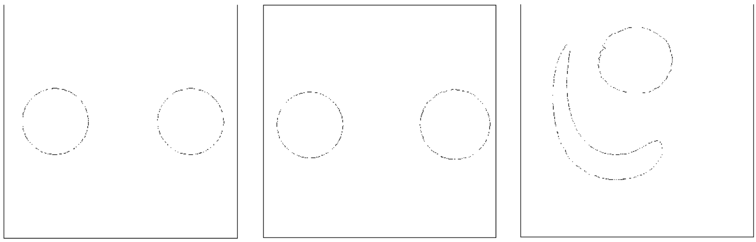

FIGURE 10. Validation of the advection step. Left: Translation of a circular region of liquid, the interface is shown at time $t = 0$ and $t = 0.06$ s. Middle: Rotation of a circular region of liquid, the interface is shown at time $t = 0$ and $t = 0.126$ s. Right: Single vortex test case, the interface is shown at time $t = 1$ (maximal deformation) and $t = 2$ s (return to initial circular shape).

In a number of industrial applications, the shape of the cavity containing the liquid is complex. Therefore, a special data structure has been implemented in order to reduce the memory requirements used to store the cell data. An example is proposed in Fig. 11. The cavity containing the liquid is meshed into tetrahedrons. Without any particular cells data structure, a great number of cells would be stored in the memory without ever being used. The data structure makes use of three hierarchical levels to define the cells. At the coarsest level, the cavity is meshed into windows which can be glued together. Each window is then subdivided into blocks. Finally, a block is cut into smaller cubes, namely the cells (ijk). When a block is free of liquid ($\varphi = 0$), it is switched off, i.e. the memory corresponding to the cells is not allocated. When liquid enters a block, the block is switched on, i.e. the memory corresponding to the cells is allocated.

Once values φ^n_{ijk}, $u^{n-\frac{1}{2}}_{ijk}$ and $\sigma^{n-\frac{1}{2}}_{ijk}$ have been computed on the cells (ijk), values are interpolated at the vertices P of the finite element mesh. More precisely, the volume fraction of liquid at vertex P is computed by considering all the cells (ijk) contained in the triangles K containing the vertex P, see Fig. 12, using the following formula :

$$\varphi^n_h(P) = \frac{\displaystyle\sum_{\substack{K\\P\in K}} \sum_{(ijk)\subset K} \phi_P(x_{ijk})\varphi^n_{ijk}}{\displaystyle\sum_{\substack{K\\P\in K}} \sum_{(ijk)\subset K} \phi_P(x_{ijk})}, \tag{4.13}$$

where x_{ijk} denotes the center of cell (ijk) and ϕ_P is the finite element basis function attached to vertex P. Similar formulae hold for the velocity and extra-stress. Thus, the liquid region is defined as follows: An element (tetrahedron) of the mesh is said to be liquid if, at least one of its vertices has a volume fraction of liquid $\varphi^n_h > 0.5$, see Fig. 13. The computational domain D^n_h used for solving (4.10)-(4.12) is then defined to be the union of all liquid elements. At this point, it is necessary to stress that the values of the volume fraction of liquid on the unstructured finite element mesh are only used in order to define the liquid region. Again, advection of the volume

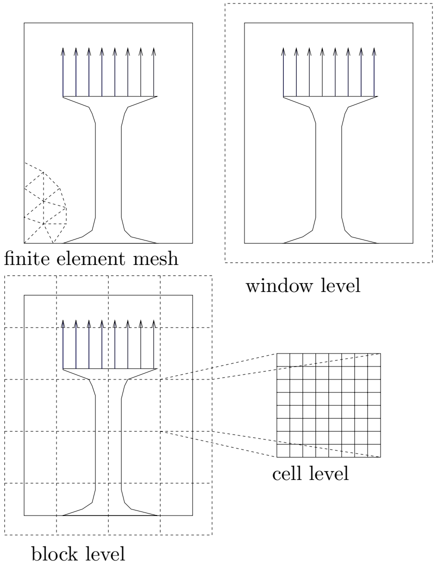

FIGURE 11. The hierarchical window-block-cell data structure used to implement cells advection in the framework of the 2D filament stretching.

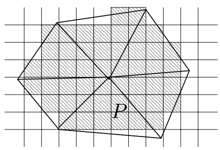

FIGURE 12. Interpolation of the volume fraction of liquid from the structured cells to the unstructured finite element mesh. The volume fraction of liquid at vertex P depends on the volume fraction of liquid in the shaded cells.

fraction of liquid only occurs on the structured cells, and not on the unstructured finite element mesh. Also, the volume constraint is not directly enforced in the numerical model. However, if numerical diffusion of the volume fraction of liquid is small, then the volume constraint will be satisfied. This is precisely the goal of the two post-processing procedures that have been added. In all the computations, it has been observed that the (numerical) diffusion layer of the volume fraction of liquid ($0 < \varphi < 1$) is of the order of one or two cells and that the volume constraint is satisfied up to 1%. In order to achieve this goal, the two post-processing procedures must be switched on and the cells spacing must be three to five times smaller than the mesh spacing.

4.3.2. Correction step : Stokes and Oldroyd-B with finite elements. Turn to the finite element techniques used for solving (4.10)-(4.12). Following [**BPS01, PR01**], an EVSS

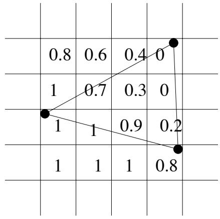

FIGURE 13. A two dimensional example of a liquid element. The values of the volume fraction of liquid φ at the center of the cells are known. A value φ is then interpolated at the vertices of the finite element mesh. The displayed triangle has at least one vertex with value φ greater than 0.5. Therefore, the triangle is liquid and the velocity, the pressure and the extra-stress will be computed at the three vertices of the triangle.

(Elastic Viscous Split Stress) formulation with continuous, piecewise linear stabilized finite elements has been used. More precisely, given the predicted velocity $u_h^{n-\frac{1}{2}}$, an extra-variable $B_h^{n-\frac{1}{2}}$ defined by

$$\int_{D^n} B_h^{n-\frac{1}{2}} : E_h dx = \int_{D^n} \epsilon(u_h^{n-\frac{1}{2}}) : E_h dx \qquad \forall E_h \in M_h,$$

(see section 2.1 for the definition of M_h) is introduced for stability purposes. No boundary conditions apply to $B_h^{n-\frac{1}{2}}$. This equation results in solving a diagonal linear system provided a mass lumping quadrature formula is used. Since the mass lumping quadrature formula is order two accurate in space, the global accuracy of the method is not affected. Once $B_h^{n-\frac{1}{2}}$ is computed, the new velocity u_h^n and pressure p_h^n are obtained by solving the following Stokes problem :

(4.14)

$$\begin{aligned}
&\int_{D^n} \rho \frac{u_h^n - u_h^{n-\frac{1}{2}}}{\Delta t^n} \cdot v_h dx + 2(\eta_s + \eta_p) \int_{D^n} \epsilon(u_h^n) : \epsilon(v_h) dx - \int_{D^n} p_h^n \nabla \cdot v_h dx \\
&\quad = \int_{D^n} \left(2\eta_p B_h^{n-\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}\right) : \epsilon(v_h) dx + \int_{D^n} \rho g \cdot v_h dx \qquad \forall v_h \in V_h, \\
&\int_{D^n} \nabla \cdot u_h^n q_h dx + \sum_{K \subset D^n} \alpha_K \int_K \left(\rho \frac{u_h^n - u_h^{n-\frac{1}{2}}}{\Delta t^n} + \nabla p_h^n - \nabla \cdot \sigma_h^{n-\frac{1}{2}} - \rho g \right) \cdot \nabla q_h \, dx = 0 \ \ \forall q_h \in Q_h,
\end{aligned}$$

see the definitions of V_h and Q_h in section 2.1. Here K denotes a tetrahedron, α_K is the local stabilization coefficient defined by

$$\alpha_K := \begin{cases} \dfrac{|K|^{\frac{2}{3}}}{12(\eta_s + \eta_p)} & \text{if } Re_K \leq 3, \\ \dfrac{|K|^{\frac{2}{3}}}{4Re_K(\eta_s + \eta_p)} & \text{else,} \end{cases}$$

where the local Reynolds number Re_K is defined by

$$Re_K := \frac{\rho |K|^{\frac{1}{3}} \max_{x\in K} |u^{n-\frac{1}{2}}(x)|}{2(\eta_s + \eta_p)}.$$

Note that in (4.14) the corrected velocity u_h^n can be prescribed on the boundary of the cavity Λ whenever needed, see Fig. 2 for an illustration of the boundary conditions. Also note that the boundary condition (4.2) is implicitly contained in the above variational formulation. Indeed, (4.14) has been obtained by multiplying the momentum equation with a test function $v_h \in V_h$, integrating by parts, and using the boundary condition (4.2). Thus, from the implementation point of view, no additional work is required to enforce (4.2). All the degrees of freedom corresponding to velocity and pressure are stored in a single matrix and the linear system is solved using the GMRES algorithm with a classical incomplete LU preconditioner and no restart.

It then remains to update the extra-stress σ_h^n from Oldroyd-B constitutive equation :

$$\left(1 + \frac{\lambda}{\Delta t^n}\right) \int_{D^n} \sigma_h^n : \tau_h dx = \lambda \int_{D^n} \left(\frac{1}{\Delta t^n} \sigma_h^{n-\frac{1}{2}} + \nabla u_h^n \sigma_h^{n-\frac{1}{2}} + \sigma_h^{n-\frac{1}{2}} (\nabla u^n)^T \right) : \tau_h dx$$
$$+ 2\eta_p \int_{D^n} \epsilon(u_h^n) : \tau_h dx \qquad \forall \tau_h \in M_h.$$

Here σ_h^n must be prescribed at the inflow boundary, if there is one, see Fig. 2. Again, this equation results in solving a diagonal linear system whenever a mass lumping quadrature formula is used. In [**BPS01, PR01**] it is proved that this finite element scheme is convergent for stationary problems in fixed computational domains. More precisely, it is proved that the approximate velocity gradient, the approximate pressure and the approximate extra-stress converge with order one in space in the L^2 norm, even when the solvent viscosity is small compared to the polymer viscosity.

Finally, once the new velocity u_h^n and extra-stress σ_h^n are computed at the vertices of the finite element mesh, values are interpolated at the center of the cells (ijk):

$$u_{ijk}^n = \sum_P \phi_P(x_{ijk}) u_h^n(P), \tag{4.15}$$

where P denotes a mesh vertex, x_{ijk} denotes the center of cell (ijk), ϕ_P denotes the finite element basis function corresponding to vertex P and $u_h^n(P)$ is the approximated velocity at vertex P. A similar formula is used for the extra-stress σ_{ijk}^n. Please note that the volume fraction of liquid is not interpolated from the finite element mesh to the cells. Indeed, the volume fraction of liquid is only computed on the structured cells. It is interpolated on the unstructured finite element mesh only in order to define the liquid region after the prediction step, see the end of section 3.1 above.

4.3.3. Implementations details. The memory storage is as follows: For each cubic cell, the volume fraction of liquid, the velocity and the extra-stress must be stored in order to implement (4.7)-(4.9), therefore $1 + 3 + 6 = 10$ values. For each vertex of the finite element mesh, the velocity, the pressure, the extra-stress and the EVSS field $B_h^{n-1/2} = i_h \epsilon(u_h^{n-1/2})$ (where i_h is the L^2 projection onto the finite element space M_h) must be stored, therefore $3 + 1 + 6 + 6 = 16$ values. The code is written in the C++ programming language and the finite element data structure is classical. The data structure of the cells is as follows. Each cell is labeled by indices (ijk) within a block. Also, each block is labeled by indices (ijk) within a window, see Fig. 11.

Mesh	Subdivisions (radius × height)	Vertices	Tetrahedrons
coarse	10×80	29565	160320
intermediate	15×120	115155	648024
fine	23×180	397377	2284800

TABLE 1. Elongational flow; the three meshes used to check convergence.

Efficient interpolation between the two grids (structured cells/unstructured finite elements) has been performed by using the following data structure. In order to implement interpolation from the finite element mesh to the cells, eq. (4.15), the index of the finite element (tetrahedron) containing each cell is needed. Alternatively, in order to implement interpolation from the cells to the finite element mesh, eq. (4.13), the list of the cells contained in each finite element (tetrahedron) is required. This additional data structure is built at the beginning of each computation. It can be stored in case several computations are performed using the same grids. The additional CPU time required to build this data structure is small (less than 1%) compared to the total CPU time.

4.4. Numerical results

Several tests are presented in this section. Firstly, our numerical model is validated for two simple flows, namely an elongational flow and the filling of a pipe. Then, numerical experiments corresponding to the stretching of a filament and jet buckling are considered.

4.4.1. Elongational flow. At the initial time, liquid at rest occupies a cylinder with radius $R_0 = 0.0034\ m$ and height $L_0 = 0.0019\ m$. Then, the velocity field on the top and bottom sides of the cylinder is imposed to be

$$u(x,y,z,t) = \begin{pmatrix} -\frac{1}{2}\dot{\epsilon_0}x \\ -\frac{1}{2}\dot{\epsilon_0}y \\ \dot{\epsilon_0}z \end{pmatrix},$$

with $\dot{\epsilon_0} = 4.68\ s^{-1}$, whereas (4.2) applies to the lateral sides. Since there is no inflow velocity, no boundary conditions have to be enforced for the extra-stress. A simple calculation shows that, for all time t, the above velocity field satisfies the momentum equations, that the extra-stress tensor is homogeneous, for instance

$$\begin{aligned} \sigma_{xx}(x,y,z,t) &= -\frac{\eta_p\dot{\epsilon_0}}{1+\dot{\epsilon_0}\lambda}\left(1-e^{-\left(\frac{1}{\lambda}+\dot{\epsilon_0}\right)t}\right), \\ \sigma_{xz}(x,y,z,t) &= 0, \\ \sigma_{zz}(x,y,z,t) &= \frac{2\eta_p\dot{\epsilon_0}}{1-2\dot{\epsilon_0}\lambda}\left(1-e^{-\left(\frac{1}{\lambda}-2\dot{\epsilon_0}\right)t}\right), \end{aligned}$$

and that the liquid region remains a cylinder with radius $R(t) = R_0 e^{-\frac{1}{2}\dot{\epsilon_0}t}$. Indeed, the trajectories of the fluid particles are defined by $X'(t) = u(X(t),t)$ which yields

$$\begin{cases} X(t) = X(0)e^{-\frac{1}{2}\dot{\epsilon_0}t}, \\ Y(t) = Y(0)e^{-\frac{1}{2}\dot{\epsilon_0}t}, \\ Z(t) = Z(0)e^{\dot{\epsilon_0}t}. \end{cases}$$

The computational domain is the cylinder with radius of $0.004\ m$ and high of $0.02\ m$. Three meshes are used for the computations, the finer mesh being obtained by dividing the mesh size by 1.5 see Table 1. The cells' spacing is four times smaller than the mesh size and When using the coarse (resp. fine) mesh, the cell size is 0.0001 m (resp. 0.00005 m). The time steps where chosen so that the CFL number of the cells - velocity multiplied by the time step divided by the cells spacing - equals 0.9 at time $t = 0$ and 3.7 at time $t = 0.3$.

Numerical results corresponding to 0.05 % by weight Polystyrene (the parameter values are taken from [**CLLD01**], $\rho = 1030\ kg/m^3$, $\eta_s = 9.15\ Pa.s$, $\eta_p = 25.8\ Pa.s$, $\lambda = 0.421\ s$, thus $De = \lambda\dot{\epsilon_0} = 1.97$) are illustrated in Fig. 14 and 15. Clearly the computed velocity agrees perfectly with the exact velocity whereas the error for the extra-stress is within 10% on the fine grid. The fact that the velocity is more precise than the extra-stress is not surprising since the finite element method is expected to be of order two (in the L^2 norm and in a fixed domain) for the velocity but only of order one for the extra-stress. Convergence rates are shown in Figure 16 and it can be seen that each component is of the order of one.

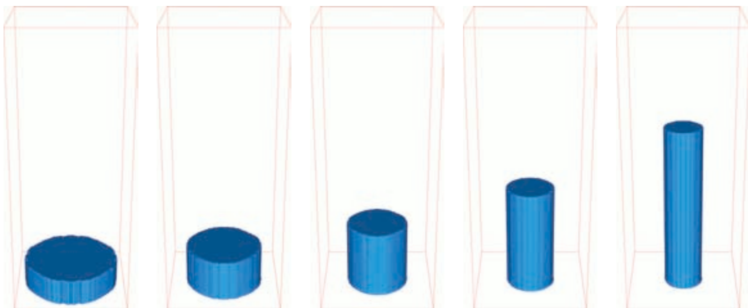

FIGURE 14. Elongational flow : shape of the liquid region (the volume corresponding to volume fraction of liquid $\varphi > 0.5$ is shown); from left to right: $t = 0\ s$, $t = 0.1\ s$, $t = 0.2\ s$, $t = 0.3\ s$, $t = 0.4\ s$.

4.4.2. Filling of a pipe. Consider a rectangular pipe of dimensions $[0, L_1]\times[0, L_2]\times[0, L_3]$ in the xyz directions, where $L_1 = 4\ m$, $L_2 = 1\ m$, $L_3 = 0.3\ m$. At the initial time, the pipe is empty. Then, fluid enters from the left side ($x = 0$) with the velocity and extra-stress given by

$$u(x,y,z,t) = \begin{pmatrix} u_x \\ 0 \\ 0 \end{pmatrix}, \qquad \sigma(x,y,z,t) = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.16}$$

with $u_x(y) = 6y(L_2 - y)$, $\sigma_{xx}(y) = 72\eta_s\lambda(2y - L_2)^2$ and $\sigma_{xy}(y) = -6\eta_p(2y - L_2)$. The boundary conditions are detailed in Fig. 17 and as follows: On the top and bottom sides ($y = 0$ and $y = L_2$), no-slip boundary conditions apply. On the front and rear sides ($z = 0$ and $z = L_3$), slip boundary conditions apply. On the right side ($x = L_1$) the fluid is free to exit the pipe with zero vertical velocity. The parameter values are taken from [**TMC$^+$02**] subsection 6.1 and are the following : $\rho = 1\ kg/m^3$, $\eta_s = \eta_p = 0.5\ Pa.s$ and $\lambda = 5\ s$. Three finite element meshes are used in this subsection, see Table 2 and Fig. 18 for details. The cells spacing is five times smaller than the finite element mesh spacing.

First consider the filling of the pipe, starting from empty. This experiment has been considered in [**PC98, TMC$^+$02**] and is sometimes called fountain flow. The imposed velocity and extra-stress profile at the inlet are those corresponding to Poiseuille flow, see (4.16). Following [**GVV04**], after some time the shape of free surface should be close to a half circle. In Fig. 17,

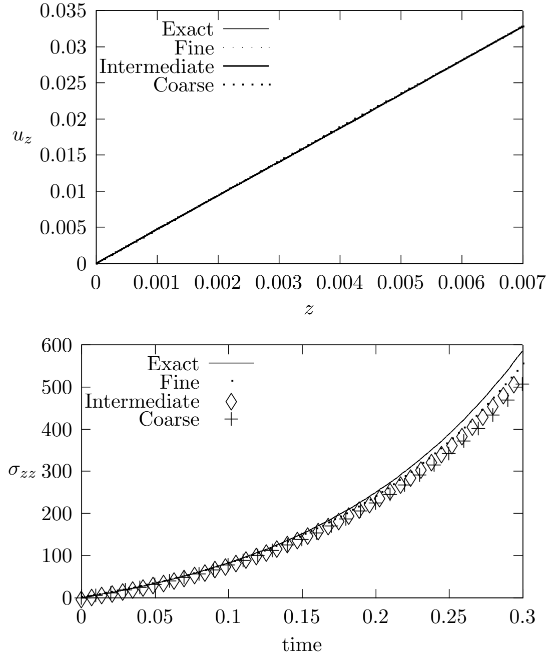

FIGURE 15. Elongational flow; top : vertical velocity u_z along the vertical axis Oz at final time $t = 0.3$ s; bottom : extra-stress σ_{zz} at $z = 0.0006$ m as a function of time.

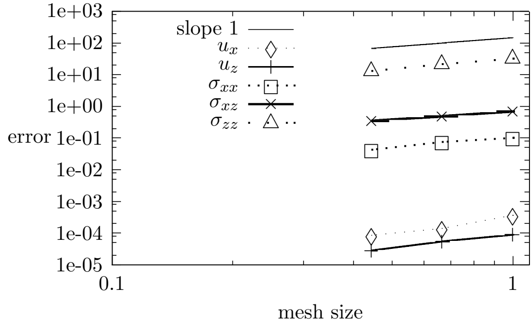

FIGURE 16. Elongational flow; error in the L^2 norm with respect to the mesh size.

Mesh	Subdivisions	Vertices	Tetrahedrons
coarse	$40 \times 10 \times 3$	1804	7200
intermediate	$80 \times 20 \times 6$	11900	57600
fine	$160 \times 40 \times 12$	85813	460800

TABLE 2. Filling of a pipe; the three meshes used to check convergence.

the velocity and the shape of the free surface is shown at several times. The mesh is the finest one and the time step is $\Delta t = 0.03\ s$, so that the CFL number of the cells - velocity multiplied by the time step divided by the cells spacing - equals 4.5. Away from the inlet, the position of the free surface is the same for both Newtonian and viscoelastic flows, see Fig. 19. As predicted theoretically [**GVV04**], the shape is almost circular. Details of the fountain flow at the free surface are provided in Fig. 19.

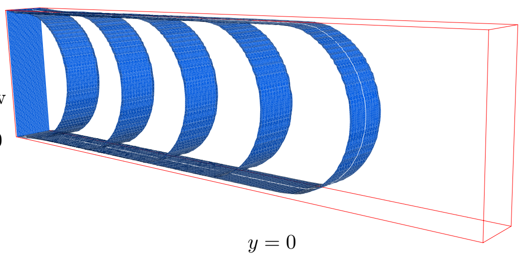

FIGURE 17. Filling of a pipe; notations and isovalue $\varphi = 0.5$ for a Newtonian fluid at times $t = 0,\ 0.6,\ 1.2,\ 1.8,\ 2.4,\ 3.0\ s$.

Once totally filled with liquid, the velocity and extra-stress must satisfy (4.16) in the whole pipe. Convergence of the stationary solution is checked with $\lambda = 1\ s$, thus $De = \lambda U/L_2 = 1$, where $U = 1\ m/s$ is the average velocity. In Fig. 20, σ_{xx}, σ_{xy} and u_x are plotted along the vertical line $x = L_1/2$, $0 \leq y \leq L_2$, $z = L_3/2$. Convergence can be observed even though boundary layer effects are present, this being classical with low order finite elements. In Fig. 21, the error in the L^2 norm of σ_{xx}, σ_{xy} and u_x is plotted versus the mesh size. Clearly order one convergence rate is observed for the extra-stress, order two for the velocity, this being consistent with theoretical predictions for simpler problems [**BPS01**].

4.4.3. Stretching of a filament. The flow of an Oldroyd-B fluid contained between two parallel coaxial circular disks with radius $R_0 = 0.003\ m$ is considered. At the initial time, the

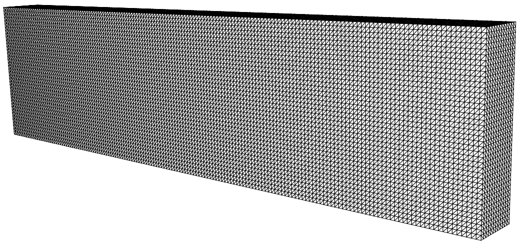

FIGURE 18. Filling of a pipe; fine mesh

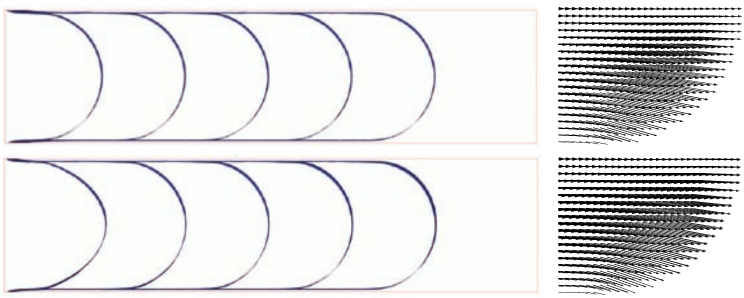

FIGURE 19. Filling of a pipe. Left : position of the free surface at time $t = 0,\ 0.6,\ 1.2,\ 1.8,\ 2.4,\ 3.0\ s$. Right : Velocity field close to the free surface at time $t = 1.8\ s$. Top : Newtonian flow. Bottom : viscoelastic flow ($\lambda = 5\ s$ thus $De = 5$).

distance between the two end-plates is $L_0 = 0.0019\ m$ and the liquid is at rest. Then, the top end-plate is moved vertically with velocity $L_0 \dot{\epsilon_0} e^{\dot{\epsilon_0} t}$. The model data ($\rho$, η_s, η_p, λ, $\dot{\epsilon_0}$) are those of subsection 4.4.1. The fine mesh of subsection 4.4.1 was used with an initial time step $\Delta t^0 = 0.005\ s$, thus the initial CFL number of the cells - velocity multiplied by the time step divided by the cells spacing - is close to one, the time step at time t^n being such, that the distance of the moving end-plate between two time steps is constant, i.e.

$$\Delta t^n = \Delta t^{n-1} e^{-\dot{\epsilon_0} \Delta t^{n-1}}.$$

Therefore, the CFL number remains constant throughout the simulation. The shape of the liquid region at several times is represented in Fig. 22, for both Newtonian and non-Newtonian

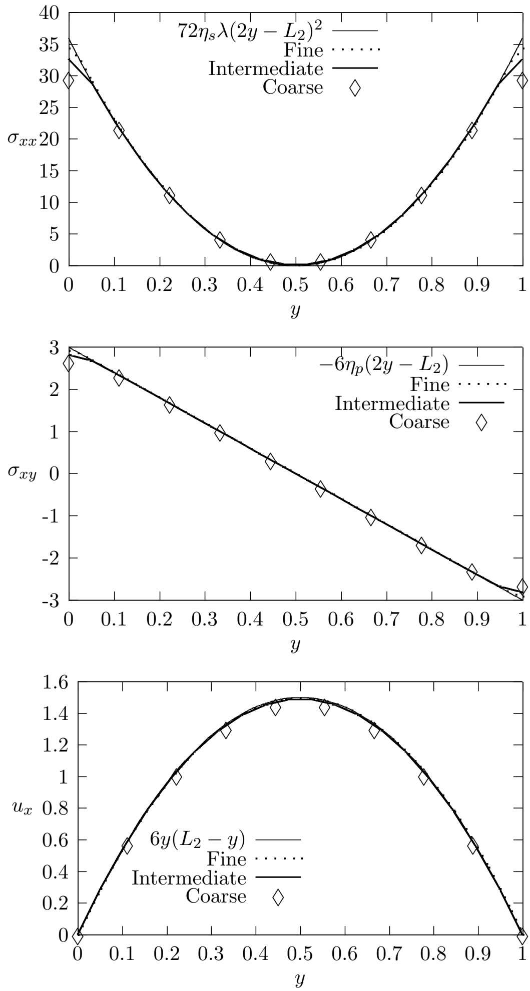

FIGURE 20. Filling of a pipe; top : σ_{xx} along the vertical line $x = L_1/2$, $0 \leq y \leq L_2$, $z = L_3/2$, middle : σ_{xy}, bottom : horizontal velocity u_x.

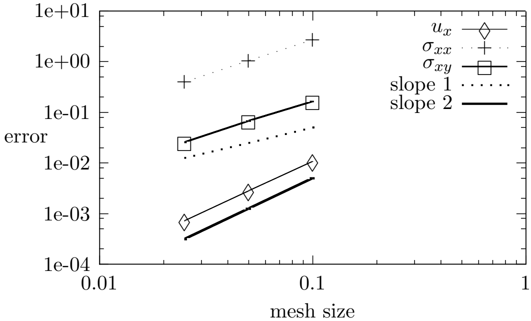

FIGURE 21. Filling of a pipe; error in the L^2 norm with respect to the mesh size.

computations. 2D cuts along plane $y = 0$ are show in Fig. 23. As reported in [**YM98**], the 'necking' phenomena occurring in the central part of the liquid for Newtonian fluids is not observed for viscoelastic fluids, due to strain hardening. This calculation requires 2 hours (resp. 24 hours) on the coarse mesh (resp. fine mesh) using a single user Pentium 4 CPU 2.8 Ghz, with 2Gb memory, under the Linux operating system. Most of the time is spent in solving the associated Stokes problem. The memory usage is 200 Mb for the coarse mesh, resp. 1.6 Gb for the fine mesh.

Unfortunately, the high Hencky strains as reported in [**YM98, CLLD01**] for 2D axisymetric computations has not been reached. The reasons may be the following: $\imath$) When the Hencky strain is large, the filament is highly stretched and the number of vertices in the thinnest region of the filament decreases, and so does the accuracy. Therefore, Lagrangian numerical models should be more accurate than Eulerian models provided the mesh is not too distorted. $\imath\imath$) Surface tension, which should stabilize the filament shape, is not include in the model. $\imath\imath\imath$) Since the filament breaking is due to 3D instabilities, 2D axisymetric computations should be more stable than 3D computations. At this point, it should be mentioned that in [**RH99**], the authors have also reported the same discrepancies when comparing results achieved by their 3D Lagrangian model with experiments [**SM96**]. Moreover, from Fig. 11 of [**RH99**], it is predicted that the onset of instability when $De = 2$ is obtained for Hencky strains $\epsilon \simeq 2$. This is in accordance with the results of Fig. 23.

It will now be shown that the numerical model is capable of reproducing fingering instabilities reported in [**RH99, BRLH02, MS02, DLCB03**] for non-Newtonian flows. Following section 4.4 in [**MS02**], take an aspect ratio $L_0/R_0 = 1/20$ ($R_0 = 0.003\ m$, $L_0 = 0.00015\ m$), so that the Weissenberg number $We = DeR_0^2/L_0^2$ is large. The finite element mesh has 50 vertices along the radius and 25 vertices along the height, thus the mesh size is 0.00006 m. The cells size is 0.00001 m and the initial time step is $\Delta t^0 = 0.01\ s$ thus the CFL number of the cells - velocity multiplied by the time step divided by the cells spacing - is close to one. The shape of the filament is shown in Fig. 24 and fingering instabilities can be observed from the very beginning of the stretching, leading to branched structures, as described in [**MS02, BRLH02, DLCB03**]. Clearly, such complex shapes cannot be obtained using Lagrangian models, the mesh distortion would be too large. In Fig. 25, the mesh is changed in order to check for the dependence of the results on the mesh topology. In Fig. 26, the same simulation is performed for a Newtonian

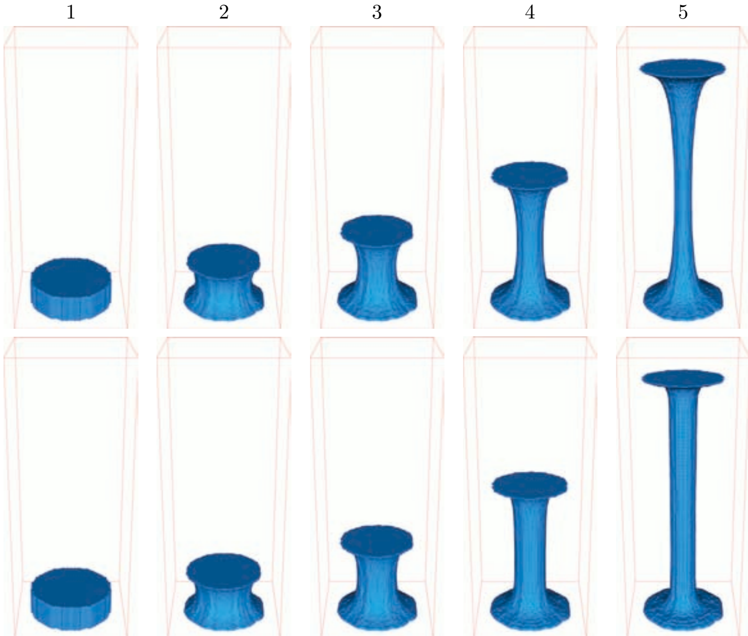

FIGURE 22. Filament stretching. Aspect ratio $L_0/R_0 = 19/30$. Shape of the liquid region in the $y = 0$ plane (the isovalues of φ are shown); column 1 : $\epsilon = \dot{\epsilon_0}t = 0$; column 2 : $\epsilon = \dot{\epsilon_0}t = 0.57$; column 3 : $\epsilon = \dot{\epsilon_0}t = 1.12$; column 4 : $\epsilon = \dot{\epsilon_0}t = 2.25$; column 5 : $\epsilon = \dot{\epsilon_0}t = 4.49$; top row : Newtonian fluid; bottom row : Viscoelastic fluid with $\lambda = 0.421$ s ($De = 1.97$).

fluid and no fingering instabilities can be seen. It is possible therefore to conclude that these instabilities are essentially elastic, as reported in [**RH99**]. However, it should be noted that fingering instabilities can also be obtained for Newtonian flows, see [**STW97**].

4.4.4. Jet buckling. The transient flow of a jet injected into a parallelepiped cavity is reproduced. First, the 3D computations are compared to the 2D results reported in [**TMC$^+$02**], section 7.3. Then, the 3D computations are shown.

In order to compare the 3D computations to those of [**TM99, TMC$^+$02**], consider a thin cavity of width 0.05 m, variable height H and depth 0.004 m, the width of the jet being $D = 0.005$ m and the vertical gravity $g = 9.81$ m/s^2. Slip boundary conditions apply whenever the jet hits the boundary of the cavity. The finite element mesh can be seen in Fig. 27; the mesh size is 0.00125 m, it has 16605 vertices and 76800 tetrahedrons; it is obtained by generating $40 \times 80 \times 4$ hexahedrons, each hexahedron being cut into 6 tetrahedrons. The cells size is 0.0002 m and the time step is 0.004 s thus the CFL number of the cells - velocity multiplied by the time step divided by the cells spacing - is ten.

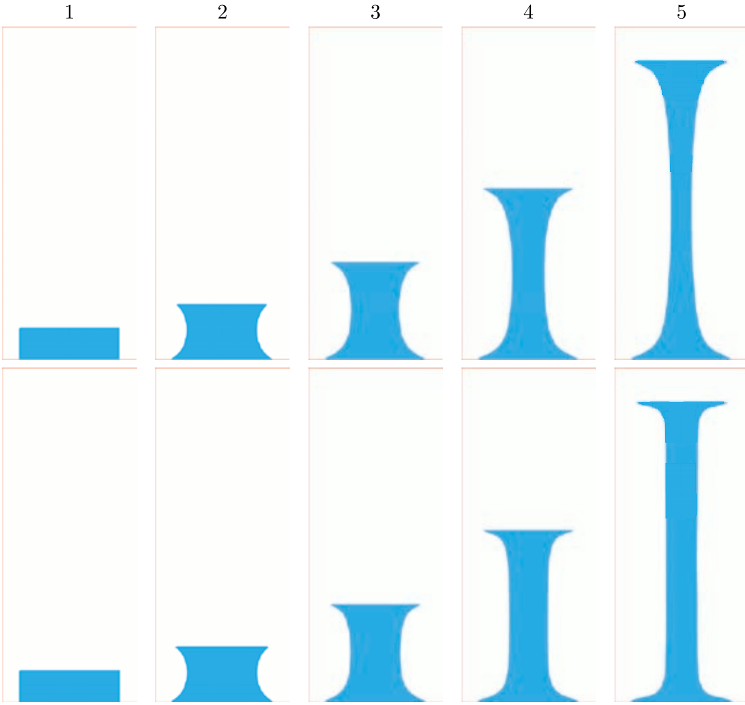

FIGURE 23. Filament stretching. Aspect ratio $L_0/R_0 = 19/30$. Shape of the liquid region in the $y = 0$ plane (the isovalues of φ are shown); column 1 : $\epsilon = \dot{\epsilon}_0 t = 0$; column 2 : $\epsilon = \dot{\epsilon}_0 t = 0.57$; column 3 : $\epsilon = \dot{\epsilon}_0 t = 1.12$; column 4 : $\epsilon = \dot{\epsilon}_0 t = 2.25$; column 5 : $\epsilon = \dot{\epsilon}_0 t = 4.49$; top row : Newtonian fluid; bottom row : Viscoelastic fluid with $\lambda = 0.421\ s$ ($De = 1.97$).

From [**TM99**], the condition for a Newtonian jet to buckle is

$$Re^2 \le \frac{1}{\pi}\frac{(H/D)^{2.6} - 8.8^{2.6}}{(H/D)^{2.6}}, \tag{4.17}$$

where $Re = \rho U D/(\eta_s + \eta_p)$ is the Reynolds number, U being the inflow jet velocity at the top of the cavity. The aim is to check that the numerical model is consistent with such a condition. The fluid parameters for the Newtonian case are $\rho = 1030\ kg/m^3$, $\eta_s + \eta_p = 10.3\ Pa.s$ and $\lambda = 0\ s$. Firstly, we set $U = 1\ m/s$, so that $Re = 0.5$ and find the critical cavity height H in order to obtain buckling. When $H/D = 14$ no buckling is observed whereas buckling occurs when $H/D = 16$. This is consistent with relation (4.17) which predicts buckling when $H/D > 15.9$. Secondly, choose a constant ratio $H/D = 20$ and vary the jet velocity U in order to determine the maximum Reynolds number for which buckling occurs. The results are shown in Fig. 28.

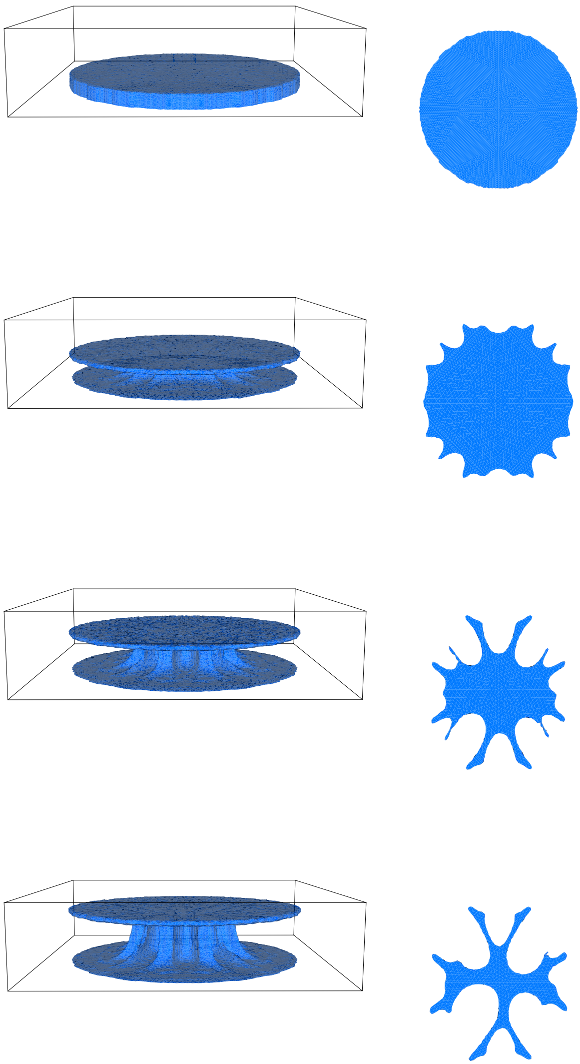

FIGURE 24. Filament stretching, $\lambda = 0.421\ s$ ($De = 1.97$), aspect ratio $L_0/R_0 = 1/20$. Left: shape of the liquid region at time $t = 0\ s$, $t = 0.33\ s$, $t = 0.66\ s$ and $t = 1\ s$. Right: horizontal cut through the middle of the liquid region.

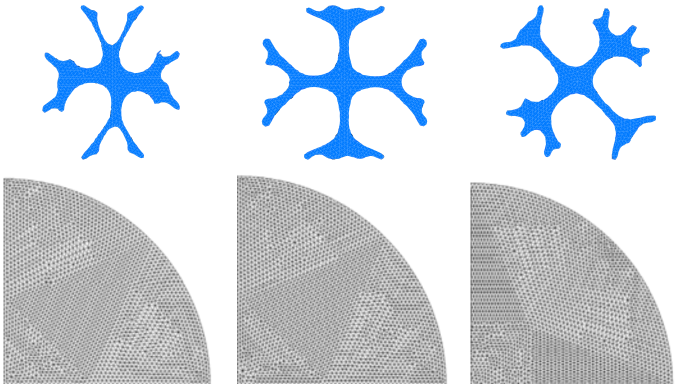

FIGURE 25. Filament stretching, $\lambda = 0.421\ s$ ($De = 1.97$), aspect ratio $L_0/R_0 = 1/20$, time $t = 1\ s$. Horizontal cut through the middle of the liquid region. Left: the mesh has 200 vertices along the diameter. Middle : the mesh has 204 vertices along the diameter. Right: the middle mesh is rotated by $\pi/4$.

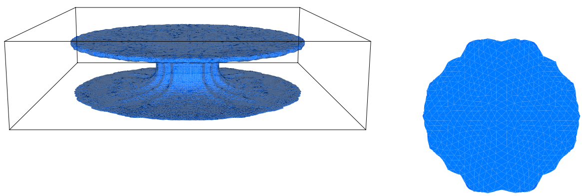

FIGURE 26. Filament stretching, Newtonian fluid, aspect ratio $L_0/R_0 = 1/20$. Left: shape of the liquid region at time $t = 1\ s$. Right: horizontal cut at the middle of the liquid region. Compare with Fig. 24.

The jet buckles when $Re = 0.2$ and $Re = 0.5$ but does not buckle when $Re = 0.7$. This is again consistent with condition (4.17) which yields buckling whenever $Re \leq 0.53$. Finally, set $Re = 1$, $H/D = 20$ so that no buckling occurs in the Newtonian case and perform a viscoelastic computation with $\eta_s = 1.03$, $\eta_p = 9.27\ Pa.s$, $\lambda = 0.1\ s$. The result is shown in Fig. 28, and obviously buckling occurs. Other computations show that the jet buckles whenever $\lambda < 0.005\ s$. Therefore it is possible to conclude that the numerical model yields results which agree with condition (4.17) in the Newtonian case and that this condition depends on λ for viscoelastic flows.

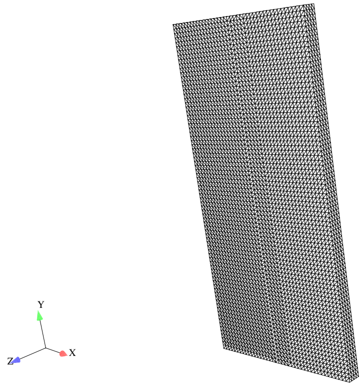

FIGURE 27. Jet buckling in a thin cavity. The mesh.

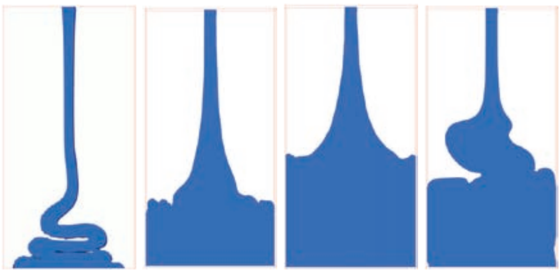

FIGURE 28. Jet buckling of a fluid in a thin cavity, $H/D = 20$. The first three figures correspond to Newtonian flows at time $t = 0.408$ s ($Re = 0.2,\ 0.5,\ 0.7$), the fourth figure corresponds to a viscoelastic flow at time $t = 0.264$ s ($\lambda = 0.1$ s).

Then compare the 3D viscoelastic computations to those 2D of [**TMC+02**], with $H = 0.1$ m ($H/D = 20$) and $U = 0.5$ m/s ($Re = 0.25$), λ ranges from 0 to 1 s so that $De = \lambda U/D$ ranges from 0 to 100. The shape of the jet is shown in Fig. 29 and 30. As in [**TMC+02**], when the Newtonian jet starts to buckle, the non-Newtonian jet has already produced many folds. However, a thin viscoelastic jet as in [**TMC+02**] (time $t = 0.3$ s) was not observed at all during the whole experiment.

Finally, simulations for a thick cavity of width 0.05 m, depth 0.05 m and height 0.1 m were undertaken, the diameter of the jet being $D = 0.005$ m. Liquid enters from the top of the cavity with vertical velocity $U = 0.5$ m/s. The relaxation time $\lambda = 1$ s so that $De = 100$. The finite element mesh has 503171 vertices and 2918760 tetrahedrons. The cells size is 0.0002 m and the time step is 0.001 s thus the CFL number of the cells - velocity multiplied by the time step divided by the cells spacing - is 2.5. The shape of the jet is shown in Fig. 31 and 32 for

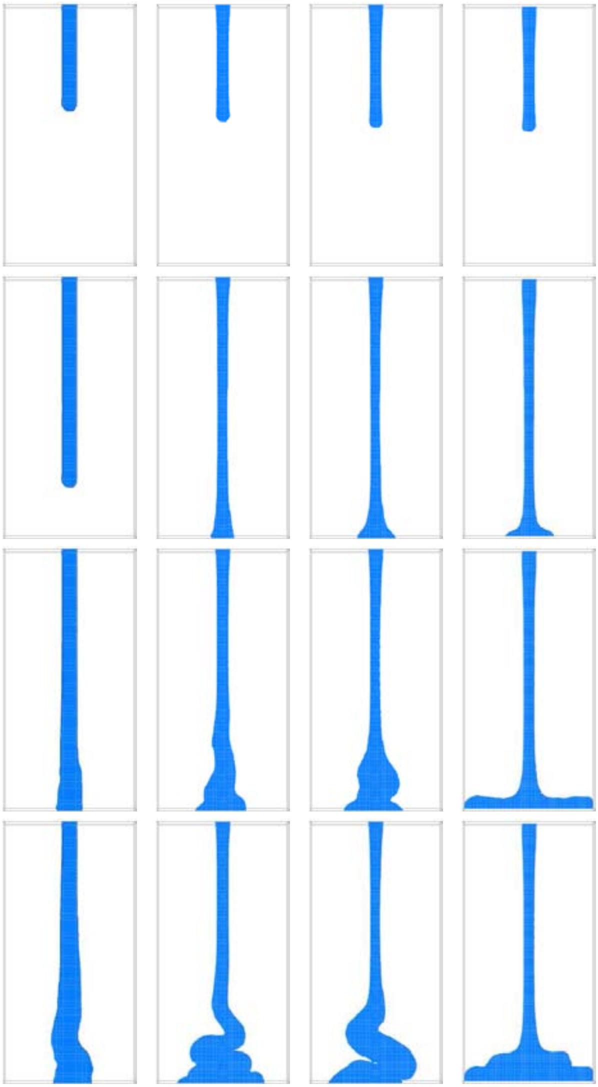

FIGURE 29. Jet buckling in a thin cavity. Shape of the jet at time $t = 0.1\ s$ (first row), $t = 0.2\ s$ (second row), $t = 0.3\ s$ (third row), $t = 0.4\ s$ (last row), Newtonian flow (first column), $\lambda = 0.01\ s$ (second column), $\lambda = 0.1\ s$ (third column), $\lambda = 1\ s$ (last column).

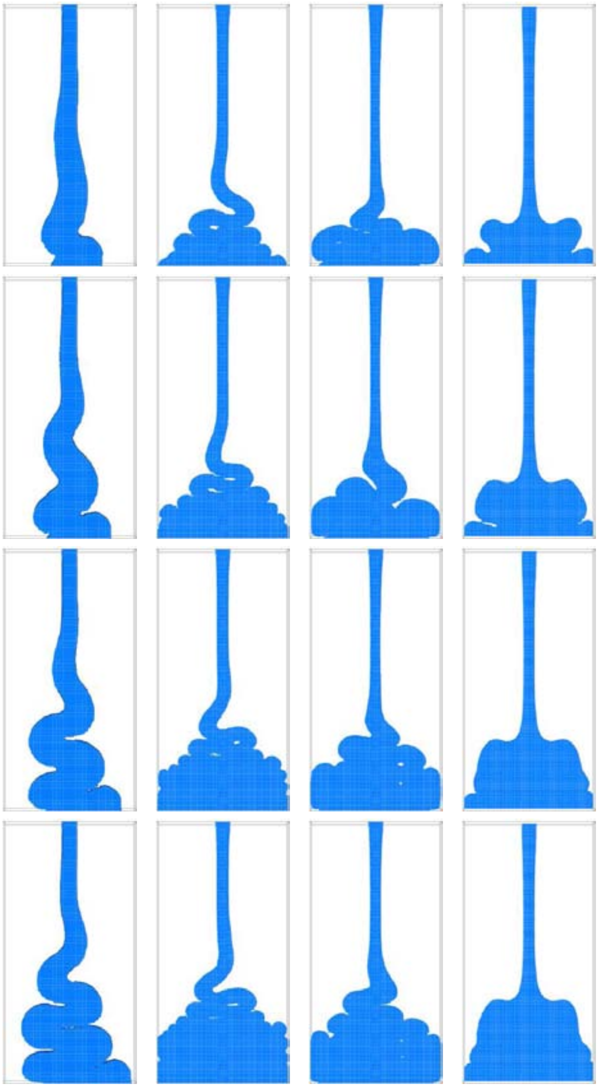

FIGURE 30. Jet buckling in a thin cavity. Shape of the jet at time $t = 0.5$ s (first row), $t = 0.6$ s (second row), $t = 0.7$ s (third row), $t = 0.8$ s (last row), Newtonian flow (first column), $\lambda = 0.01$ s (second column), $\lambda = 0.1$ s (third column), $\lambda = 1$ s (last column).

Newtonian and viscoelastic flows. This computation took 64 hours on a AMD opteron CPU with 8Gb memory.

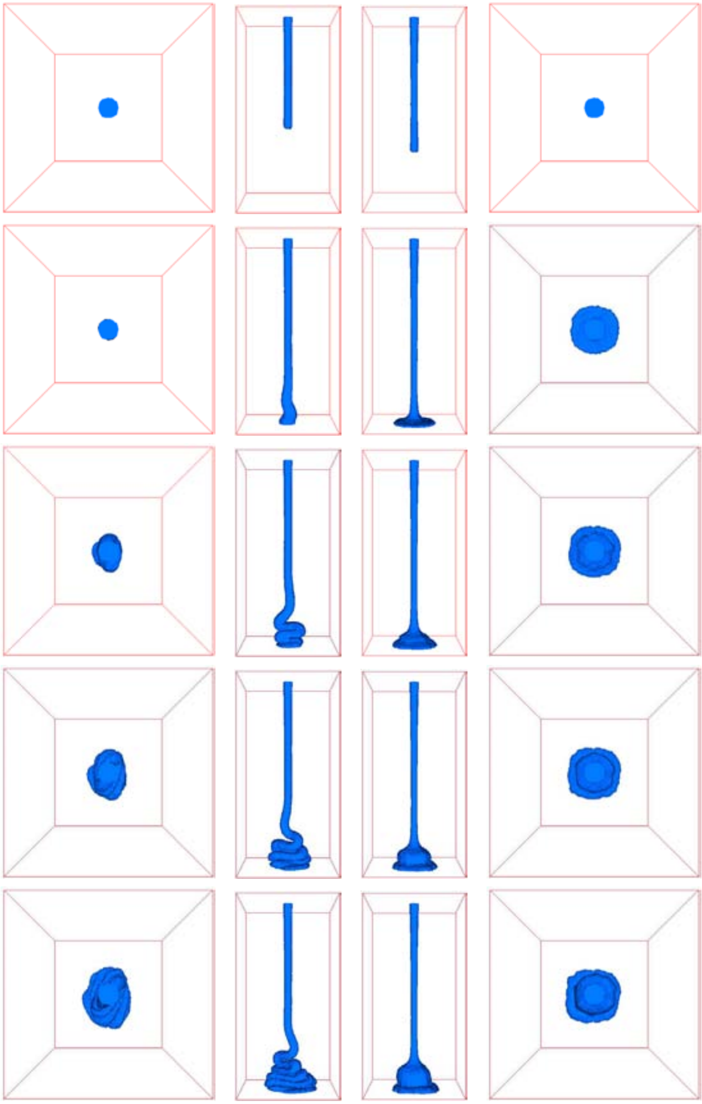

FIGURE 31. Jet buckling in a thick cavity. Shape of the jet at time $t = 0.1$ s (row 1), $t = 0.2$ s (row 2), $t = 0.3$ s (row 3), $t = 0.4$ s (row 4), $t = 0.5$ s (row 5), Newtonian fluid (col. 1 and 2), $\lambda = 1$ s (col. 3 and 4).

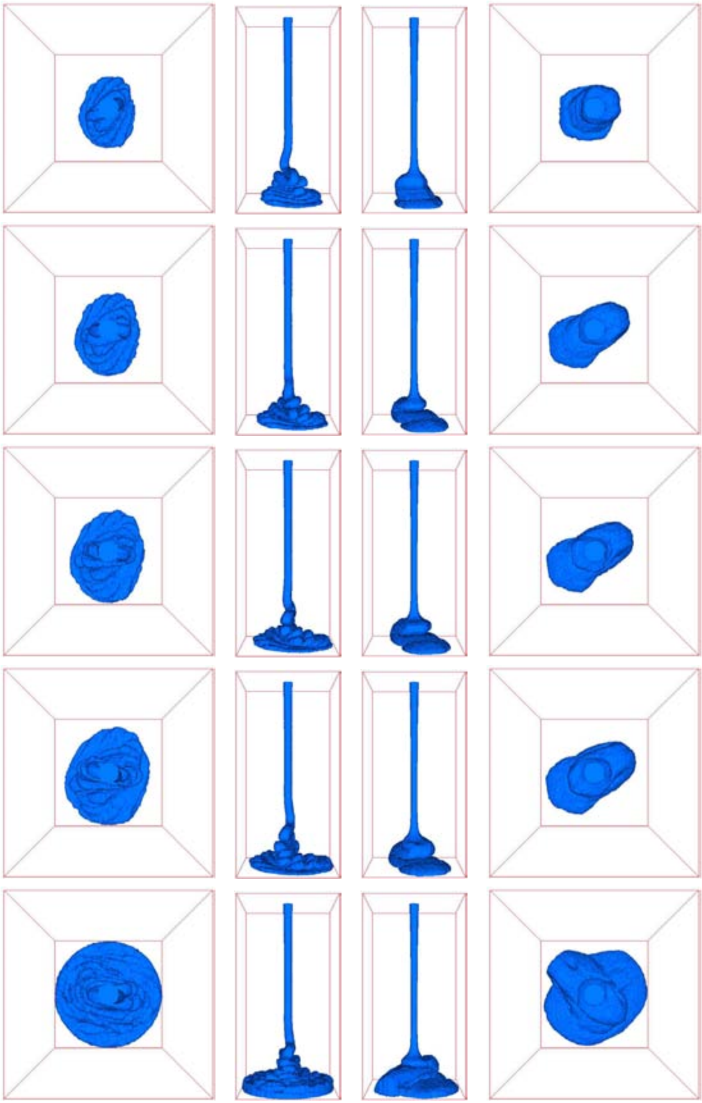

FIGURE 32. Jet buckling in a thick cavity. Shape of the jet at time $t = 0.6$ s (row 1), $t = 0.7$ s (row 2), $t = 0.8$ s (row 3), $t = 0.9$ s (row 4), $t = 1.6$ s (row 5), Newtonian fluid (col. 1 and 2), $\lambda = 1$ s (col. 3 and 4).

Conclusions

Viscoelastic flows have been investigated from both mathematical and numerical points of view.

Initially the mathematical study of two simplified models was considered. The first was the time dependent Oldroyd-B model without convective terms. Existence on a fixed time interval has been proved in several Banach spaces provided the data are small enough, and short time existence for arbitrarily large data was proved in Hölder continuous spaces for the time variable. These results are based on properties of the Stokes operator: the maximal regularity property and the analycity behavior of the corresponding semi-group. A finite element discretization in space was proposed. Existence of the numerical solution was proved for small data, as well as a priori error estimates, using an abstract framework closely related to the one used for the continuous problem.

Then, the extension of these results to the stochastic Hookean dumbbells model was discussed. Due to the presence of Brownian motion, existence on a fixed time interval, provided the data are small enough, was proved only in some Banach spaces considered for the previous deterministic model. A splitting was used in order to avoid complications due to the stochastic behavior of the solution. Classical results were used for the stochastic component and the same framework as for the Oldroyd-B model was set up for the other component. A finite element discretization in space was also proposed. Existence of the numerical solution was proved for small data, as well as a priori error estimates.

It is anticipated that the existence as well as the convergence of a space-time discretization using the implicit Euler scheme will be proven.

Secondly, a numerical algorithm for solving viscoelastic flows was described. An Eulerian model based on the VOF formulation was presented for the simulation of viscoelastic flows with complex free surfaces in three space dimensions.

A splitting method was used for time discretization. The prediction step consists of solving three advection problems, one for the volume fraction of liquid, one for the velocity field, one for the extra-stress. The correction step corresponds to solving an Oldroyd-B flow problem without advection. Two different grids were used for space discretization. The three advection problems were solved on a fixed, structured grid made out of small cubic cells, using a forward characteristics method. The viscoelastic flow problem, without advection, was solved using continuous, piecewise linear stabilized finite elements on a fixed, unstructured mesh of tetrahedrons.

Convergence of the numerical method was checked for two test cases, namely an elongational flow and the filling of a pipe. Numerical results were then presented and typical viscoealstic behavior of the fluid was computed. In the framework of the stretching of a filament, when the aspect ratio is large, fingering instabilities were obtained. Jet buckling was also presented and a comparison with two dimentional results present in the literature was succesfull.

It is anticipated that the numerical simulations will be improved in several different ways. The Stokes solver requires much attention and could be optimized by using, for instance, a decoupling velocity-pressure algorithm, see [**FG83, GLT76a, GLT76b, GLT81, Glo03**]. The stabilizing effect of the surface tension should be taken into account. Moreover the FENE dumbbells model remains to be implemented in three dimensional spaces in order to reproduce physical behavior of which the Oldroyd-B model is unable.

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Curriculum vitae

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