On the Asymptotic Distortion Behavior of the Distributed Karhunen-Loève Transform*

Olivier Roy and Martin Vetterli†

School of Computer and Communication Sciences (I&C)

Ecole Polytechnique Fédérale de Lausanne (EPFL)

CH-1015 Lausanne, Switzerland

{olivier.roy, martin.vetterli}@epfl.ch

Abstract

The ability to design efficient distributed communications systems heavily depends on a proper understanding of the distortion incurred by their constituting elements. To this end, we study the mean squared reconstruction error behavior of the distributed Karhunen-Loève transform. We provide a general formula to compute this distortion under asymptotic considerations in a pure approximation viewpoint. Using a simple illustrative example, we show how this approach allows to obtain closed-form formulas that permit to efficiently assess the performance of the distributed scheme.

1 Introduction

The Karhunen-Loève Transform (KLT) has received considerable attention over the last decades owing to its significance in a plethora of signal processing tasks including image, audio and video coding. With the emergence of distributed infrastructures like sensor networks, there is a growing need for schemes that exhibit efficient decentralized processing capabilities. To this end, the KLT was recently reconsidered in its distributed form in [1] under the name of distributed KLT (dKLT). Similarly to its centralized counterpart, the dKLT is expected to play an important role in tasks involving distributed approximation, classification and compression.

In this context, a thorough understanding of the distortion incurred by such transforms is needed in order to properly address miscellaneous questions of both theoretical and practical interest. Among the challenges that arise in the design, use and maintenance of sensor networks, the judicious processing of the data provided by the sensing devices is crucial. In fact, the inherent low-power constraints require the development of efficient communication protocols. A typical example is the data gathering of physical measurements to a central base station. In this monitoring scenario, the amount

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†Also with the Department of Electrical Engineering and Computer Sciences, University of California at Berkeley, Berkeley, CA 94720, USA.

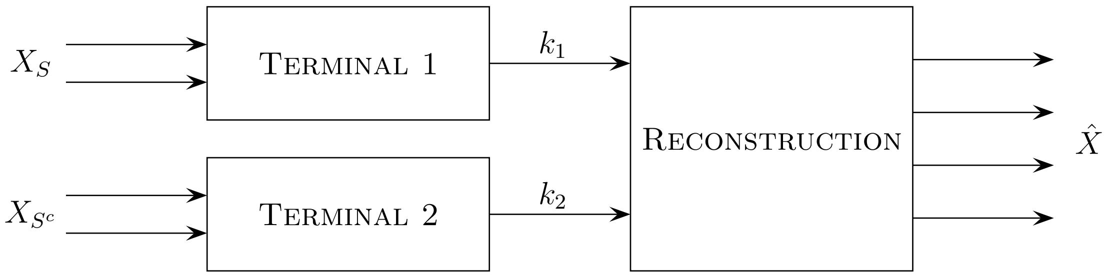


Figure 1: Block diagram of the dKLT with two terminals.

of information that needs to be conveyed in order to fulfill a given fidelity criterion is of fundamental interest. Careful design of such networks is needed in order to account for sensor breakdowns and unreliable communication links. The aforementioned problems require a precise knowledge of the distortion that can be expected considering the specificities of the communication system at hand. Unfortunately, it is often analytically difficult to gain such knowledge and the analysis of optimized transmission strategies becomes untractable.

In this paper, we wish to give a precise characterization of the mean-squared reconstruction error incurred by the dKLT for the various scenarios considered in [1]. To this end, we study the asymptotic distortion of jointly Gaussian stationary sources[1] from a pure approximation point of view and extend some results about large Toeplitz matrices to support the theoretical foundation of our analysis. This allows us to provide a general formula to compute the distortion under these assumptions. We then illustrate our results with a first-order Gauss-Markov correlation model and come up with closed-form formulas of the reconstruction error. A crucial observation is that, in this example, the asymptotic analysis carried in this work furnishes an excellent approximation of the distortion in the finite dimensional regime. This highlights the practical applicability of our approach. The formulas obtained also enable us to precisely compare the gain that can be achieved by the presence of correlated side information at the receiver or the loss incurred by sensors that fail to provide any approximation of their local observation.

The outline of the paper is as follows. In Section 2, we present the distributed approximation problem. Section 3 provides our theoretical analysis, illustrated in Section 4 with a simple correlation model. We finally offer some conclusions in Section 5.

We will use the following notation throughout this paper: A^T, $\mathrm{tr}(A)$, $\|A\|$ and $\mathrm{E}[A]$ denote the transpose, trace, Frobenius norm and expectation of the matrix A, respectively. The notation A_n designates a matrix A of size $n \times n$. In particular, I_n stands for the identity matrix and O_n for the all-zero matrix. Finally, $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ designates a diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$.

2 Problem Statement

The problem that we consider in this paper is that of the distributed Karhunen-Loève transform introduced in [1]. The benefits, tradeoffs and complexity issues that are inherent to the dKLT problem are fully captured by a two-terminal scenario as illustrated in Figure 1. Therefore, we will concentrate on this setup for the rest of the discussion.

The source X consists in a zero mean jointly Gaussian random vector of size $2n$. It is

[1]For the scope of the present paper, we only consider jointly Gaussian random sources, even though some of our results are more general.

divided into the part X_S sampled by terminal 1 and the part X_{S^c} observed by terminal 2, both of size n.[2] The corresponding covariance and cross-covariance matrices are denoted Σ_X, Σ_S, Σ_{S^c} and Σ_{SS^c}, respectively. Each terminal l provides a k_l-dimensional approximation of its observation by mean of a suitable $k_l \times n$ transform C_l $(0 \leq k_l \leq n)$. The ultimate goal is to find C_1 and C_2 such as to minimize the mean-squared reconstruction error between the source X and the reconstruction $\hat{X}$. We will denote this distortion by $D_n(k_1, k_2)$, which is easily shown to be expressible as

$$D_n(k_1, k_2) = \mathrm{E}[\|X - \hat{X}\|_F^2] = \mathrm{tr}\left(\Sigma_X - \Sigma_X C^T (C \Sigma_X C^T)^{-1} C \Sigma_X\right) \tag{1}$$

where C is a 2×2 block-diagonal matrix with diagonal blocks C_1 and C_2. Unfortunately, a simple solution to the above optimization problem does not seem to exist in general. The authors of [1] provide an iterative procedure, referred to as the dKLT algorithm, that is proved to converge to locally optimal solutions. The algorithm works in a round-robin fashion by fixing one transform while optimizing the other with respect to the fidelity criterion (1). This local optimization step is referred to as the local KLT (lKLT). The following cases are of particular interest in this discussion:

1. The conditional KLT (cKLT) is obtained by setting $k_2 = n$ and $C_2 = I_n$, i.e. X_{S^c} is perfectly conveyed to the receiver (side information). The distortion $D_n(k_1, n)$, denoted $D_n^c(k_1)$, corresponds to the distortion incurred by the KLT of a zero mean jointly Gaussian random vector with covariance matrix [1]

$$\Sigma_W = \Sigma_S - \Sigma_{SS^c} \Sigma_{S^c}^{-1} \Sigma_{SS^c}^T. \tag{2}$$

2. The partial KLT (pKLT) is obtained by setting $k_2 = 0$, i.e. X_{S^c} is completely discarded (hidden part). The distortion $D_n(k_1, 0)$, denoted $D_n^p(k_1)$, can be shown to correspond to the distortion incurred by the KLT of a zero mean jointly Gaussian random vector with covariance matrix

$$\Sigma_W = \Sigma_S + \Sigma_{SS^c} \Sigma_{SS^c}^T \Sigma_S^{-1} \tag{3}$$

   plus an additional distortion $\mathrm{tr}(\Sigma_{S^c} - \Sigma_{SS^c}^T \Sigma_S^{-1} \Sigma_{SS^c})$. Note that the latter expression corresponds to the maximal distortion ($k_1 = 0$) incurred by the cKLT of the source X_{S^c} with X_S as side information.

3. The joint KLT (jKLT) corresponds to the centralized scenario, i.e. where the two terminals are merged. The distortion, denoted $D_n^j(k)$ or simply $D_n(k)$, is that of the KLT of a zero mean jointly Gaussian random vector with covariance matrix Σ_X. In the sequel, the denomination KLT and jKLT will be used alternatively depending on the context.

We observe that the computation of the distortion, for the cases at hand, relies exclusively on the eigenvalues of some carefully chosen covariance matrices. Such eigenvalues are generally not expressible analytically when n is finite. In the next section, we alleviate this problem by considering the infinite dimensional regime with additional spatial stationarity assumptions.

[2] This restriction allows to work with square cross-covariance matrices, which greatly simplify the derivations that will be presented in the sequel.

3 Asymptotic Normalized Distortion

Let us study the setup considered in the finite case but letting n grow to infinity. In this context, terminal l provides the reconstruction point with a description of size $\lfloor \alpha_l n \rfloor$ with $\alpha_l \in [0,1]$, i.e. where only a fraction α_l of transformed coefficients is kept. It is worth noting that both the size of the source and the transformed vector go to infinity whereas the ratio remains constant and is given by $\alpha_l \sim k_l/n$.[3] The study of the mean squared error under these asymptotic considerations is achieved by means of the following definition.

Definition 3.1 (Asymptotic Normalized Distortion)**.** *Let $\alpha_l \sim k_l/n$ for $l = 1, 2$. The asymptotic normalized distortion $D(\alpha_1, \alpha_2)$ is defined as*[4]

$$D(\alpha_1, \alpha_2) = \lim_{n\to\infty} \frac{D_n(k_1, k_2)}{n}$$

if the limit exists.

The terminology adopted in this section follows from that of Section 2, simply replacing k_l by α_l. In particular, $D(\alpha_1)$, $D^c(\alpha_1)$ and $D^p(\alpha_1)$ denote respectively the asymptotic normalized distortion incurred by the KLT, cKLT and pKLT of the source X_S. Similarly to [2], we denote by $T_n(f)$ the Toeplitz matrix generated by the sequence $\{t_k\}$ with corresponding Fourier series f. For the rest of the discussion, we will assume that $\{t_k\} \in \ell^1(\mathbb{Z})$, i.e. that f exists, is continuous and bounded.[5] If f is real, we will denote its greatest lower bound and least upper bound by m_f and M_f, respectively. The following lemma is central to the computation of the asymptotic normalized distortion.

Lemma 3.2. *Let $T_n(f)$ be a sequence of Hermitian Toeplitz matrices with (real) eigenvalues $\lambda_m^{(n)}$, such that*[6] *$\int_{\lambda : f(\lambda)=x} d\lambda = 0$ for all x. Then, for any non-negative integer s,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \left(\lambda_m^{(n)}\right)^s 1_x\left(\lambda_m^{(n)}\right) = \frac{1}{2\pi} \int_{\lambda : f(\lambda) \leq x} f^s(\lambda)\, d\lambda\,.$$

Proof: Let us proceed by induction. For $s = 0$, we know that the assertion holds true by [2, Corollary 4.1]. Assume it has been proved for $s-1$, we now prove it for s. We first note that the LHS of the assertion can be expressed as

$$\frac{1}{n} \sum_{m=0}^{n-1} \left(\lambda_m^{(n)}\right)^{s-1} \min\left(x, \lambda_m^{(n)}\right) - \frac{x}{n} \sum_{m=0}^{n-1} \left(\lambda_m^{(n)}\right)^{s-1} + \frac{x}{n} \sum_{m=0}^{n-1} \left(\lambda_m^{(n)}\right)^{s-1} 1_x\left(\lambda_m^{(n)}\right).$$

Since min is a continuous function, we can apply Szegö's theorem [2, Theorem 4.2] to the first and second summations and our induction assumption to the third one. Passing to

[3] The notation $\alpha_l \sim k_l/n$ means that $\lim_{n\to\infty} k_l/n = \alpha_l$.

[4] The normalization factor n adopted here corresponds to the size of the source vectors X_S and X_{S^c}. In the rest of the discussion however, this factor will always be chosen such as to have a maximal distortion equal to 1.

[5] All the results presented here hold for the more general case where $\{t_k\} \in \ell^2(\mathbb{Z})$, but the generalization involves mathematical sophistication which is beyond the scope of this work.

[6] This technical condition ensures the continuity of the limiting eigenvalue distribution. Unless otherwise stated, λ implicitly ranges from 0 to 2π in all the subsequent integrals involving Fourier series.

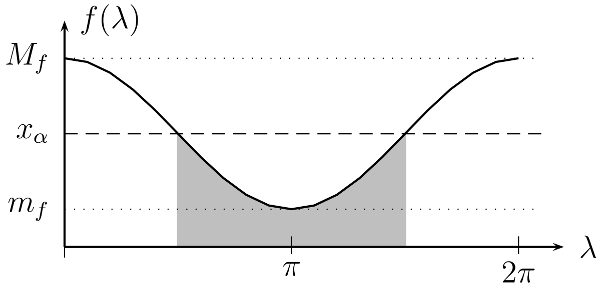


Figure 2: Computation of the asymptotic normalized distortion in a "water-filling" fashion. The shaded area corresponds to the distortion (up to a scaling factor 2π).

the limit, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \left(\lambda_m^{(n)}\right)^s 1_x\left(\lambda_m^{(n)}\right) \\
&= \frac{1}{2\pi} \int f^{s-1}(\lambda) \min\left(x, f(\lambda)\right) d\lambda - \frac{x}{2\pi} \int f^{s-1}(\lambda)\, d\lambda + \frac{x}{2\pi} \int_{\lambda: f(\lambda)\le x} f^{s-1}(\lambda)\, d\lambda \\
&= \frac{1}{2\pi} \int_{\lambda: f(\lambda)\le x} f^s(\lambda)\, d\lambda
\end{aligned}$$

and the proof follows. □

The LHS expression of Lemma 3.2, with $s = 1$, corresponds to the asymptotic normalized distortion incurred by the KLT of a zero mean jointly Gaussian random vector with covariance matrix $T_n(f)$, where only the eigenvalues greater than x are preserved. Keeping a fraction α of transformed coefficients amounts to evaluate the above expression at the value x_α satisfying

$$F(x_\alpha) = P(\lambda \le x_\alpha) = 1 - \alpha \tag{4}$$

where $F(x)$ denotes the limiting eigenvalue distribution [2, Corollary 4.1] of $T_n(f)$. This is summarized in the following theorem.

Theorem 3.3 (Asymptotic Normalized Distortion)**.** *Consider a zero mean jointly Gaussian random vector X of size n whose covariance matrix is given by the Hermitian Toeplitz matrix $T_n(f)$ with $\int_{\lambda: f(\lambda)=x} d\lambda = 0$ for all x. The asymptotic normalized distortion incurred by the KLT of the vector X when only a fraction α of transformed coefficients is kept is given by*

$$D(\alpha) = \frac{1}{2\pi} \int_{\lambda: f(\lambda)\le x_\alpha} f(\lambda)\, d\lambda$$

where x_α satisfies $F(x_\alpha) = 1 - \alpha$.

Theorem 3.3 illustrates the fact that the computation of the asymptotic normalized distortion amounts to integrate f in a "water-filling" fashion as shown in Figure 2. As α ranges from 0 to 1, x_α scopes from M_f to m_f as dictated by equation (4). A closed-form computation thus heavily depends on the ability to express x_α as a function of α. In particular, when f is symmetric (i.e. $T_n(f)$ is a real Hermitian matrix) and strictly decreasing in $[0, \pi]$, it can be seen that the above formula reduces to

$$D(\alpha) = \frac{1}{\pi} \int_{\pi\alpha}^{\pi} f(\lambda)\, d\lambda \tag{5}$$

i.e. the knowledge of the limiting eigenvalue distribution is not required. Note that Theorem 3.3 holds in the more general setting where the covariance matrix is asymptotically Toeplitz with real eigenvalues.

In the dKLT context, the asymptotic normalized distortion, for the cases at hand, can be computed by considering the covariance matrices given by equations (2) and (3). The derivation relies on the assumption that these matrices are asymptotically Toeplitz, i.e. that the underlying processes are asymptotically stationary. This is the case for the example provided in the next section.

4 Example: First-Order Gauss-Markov

In this section, we apply the results obtained in Section 3 to a first-order Gauss-Markov process. Owning to its simplicity, it is particulary suited for the analytical development that will be presented in the sequel. We come up with closed-form formulas for the asymptotic normalized distortion that allow us to assess the performance of the dKLT and to precisely compare the different scenarios considered previously. We also relate our results to the general two-terminal solution obtained numerically using the dKLT algorithm.

Let us consider a first order Gauss-Markov source X with correlation parameter ρ, i.e. a random process that satisfies

$$X_n = \rho X_{n-1} + Z_n \tag{6}$$

for $\rho \in (0,1)$ [7] and where Z_n are i.i.d. zero mean Gaussian random variables with variance $1-\rho^2$. We will consider the case where terminal 1 samples the odd coefficients and terminal 2 observes the even ones, i.e.

$$X_S = (X_1, X_3, \ldots, X_{2n-1})^T \quad \text{and} \quad X_{S^c} = (X_2, X_4, \ldots, X_{2n})^T. \tag{7}$$

The covariance matrix Σ_X is thus given by the 2×2 block Toeplitz matrix with Toeplitz blocks

$$\Sigma_S = \Sigma_{S^c} = T_n(f) \quad \text{and} \quad \Sigma_{SS^c} = T_n(g) \tag{8}$$

where f and g are the Fourier series given by

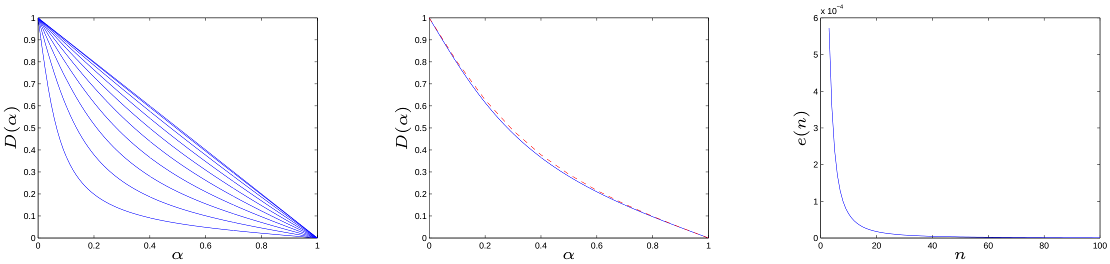
$$f(\lambda) = \frac{1-\rho^4}{1+\rho^4 - 2\rho^2\cos\lambda} \quad \text{and} \quad g(\lambda) = \frac{\rho(1-\rho^2)(1+e^{-i\lambda})}{1+\rho^4 - 2\rho^2\cos\lambda}. \tag{9}$$

This approach is motivated, for example, by super-resolution imaging problems, where two subsampled versions of the same signal are processed in order to build a higher resolution image. The analysis of the distortion in this scenario allows to precisely quantify the gain achievable by the use of a low-resolution image as side information. It also gives a useful characterization of the loss incurred due to the interpolation of missing samples. The next theorem provides a closed-form formula for the asymptotic normalized distortion incurred by the KLT of the source X_S.

Theorem 4.1 (Asymptotic Normalized Distortion - KLT)**.** *The asymptotic normalized distortion incurred by the KLT of the source X_S where only a fraction α of the coefficients is kept is given by*

$$D(\alpha) = 1 - \frac{2}{\pi}\arctan\left(\frac{1+\rho^2}{1-\rho^2}\tan\left(\frac{\pi\alpha}{2}\right)\right).$$

[7] The case $\rho \in (-1,0)$ follows immediately by considering $|\rho|$.



(a) (b) (c)

Figure 3: Asymptotic normalized distortion of the KLT. (a) Asymptotic normalized distortion $D(\alpha)$ of the source X_S for $\rho = 0, 0.1, \ldots, 0.9$ (from top to bottom). (b) Asymptotic normalized distortion $D(\alpha)$ (plain) of the source X_S, and its approximation (dashed) for $\rho = 0.6$ and $n = 10$. (c) Approximation error $e(n)$ as a function of the size of the source vector for $\rho = 0.6$.

Proof: Let $\Sigma_S = T_n(f)$. We can readily check that $f > 0$, f is symmetric and strictly decreasing in $[0, \pi]$. The asymptotic normalized distortion can thus be computed using (5) as [3, p. 181]

$$D(\alpha) = \frac{1-\rho^4}{\pi} \int_{\pi\alpha}^{\pi} \frac{1}{1+\rho^4 - 2\rho^2 \cos\lambda} \, d\lambda = 1 - \frac{2}{\pi} \arctan\left(\frac{1+\rho^2}{1-\rho^2} \tan\left(\frac{\pi\alpha}{2}\right)\right)$$

and the proof follows. $\square$

We provide in Figure 3 the asymptotic normalized distortion $D(\alpha)$ of the KLT of the source X_S as function of α. We also show how the asymptotic normalized distortion is approximated for $\rho = 0.6$ and $n = 10$. We observe that even for small values of n, the asymptotic analysis presented here offers a very good approximation of the distortion in the finite dimensional regime. This highlights the practical applicability of our results. We also compute the approximation error $e(n) = \frac{1}{n} \sum_{k=0}^{n} |D_n(k) - D(k/n)|^2$ to quantify the quality of the estimate with respect to the size of the source vector. The observed exponential decay suggests, once again, that the results obtained by our asymptotical analysis approximate accurately the distortion we would compute with a finite number of measurements.

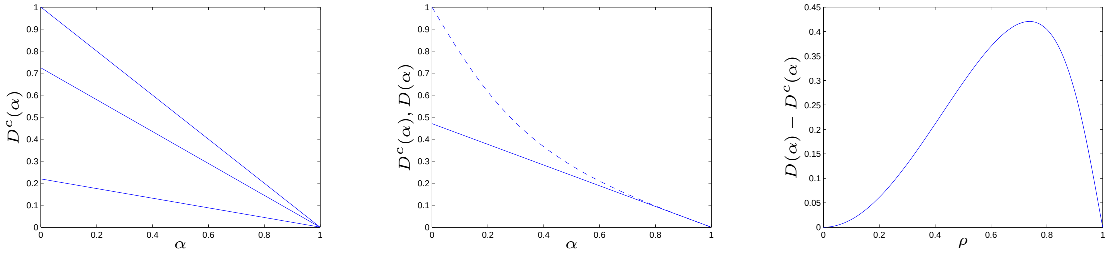
Let us now consider the case where the side information X_{S^c} is available at the receiver. The next theorem provides a closed-form formula for the asymptotic normalized distortion in this scenario.

Theorem 4.2 (Asymptotic Normalized Distortion - cKLT)**.** *Consider the source X_S and the receiver side information X_{S^c}. The asymptotic normalized distortions incurred by the cKLT of the source where only a fraction α of the coefficients is kept is given by*

$$D^c(\alpha) = \frac{1-\rho^2}{1+\rho^2}(1-\alpha).$$

Proof: It can be easily shown that Σ_W defined in (2) is asymptotically equivalent to the Toeplitz matrix $T_n(f - |g|^2/f)$ where $s = f - |g|^2/f$ can be computed as

$$s(\lambda) = \frac{1-\rho^4}{1+\rho^4 - 2\rho^2\cos\lambda} - \frac{2\rho^2(1-\rho^2)(1+\cos\lambda)}{(1+\rho^2)(1+\rho^4-2\rho^2\cos\lambda)} = \frac{1-\rho^2}{1+\rho^2}.$$



(a) (b) (c)

Figure 4: Asymptotic normalized distortion of the cKLT. (a) Asymptotic normalized distortion $D^c(\alpha)$ of the source X_S with receiver side information X_{S^c} for $\rho = 0, 0.4, 0.8$ (from top to bottom). (b) Asymptotic normalized distortion of the source X_S with ($D^c(\alpha)$, plain) and without ($D(\alpha)$, dashed) receiver side information X_{S^c} for $\rho = 0.6$. (c) Gain due to receiver side information as a function of ρ for $\alpha = 0.1$.

Since $\int_{\lambda:s(\lambda)=x} d\lambda \neq 0$ for all x, Theorem 3.3 cannot be applied. Furthermore, Σ_W is only asymptotically Toeplitz thus [2, Lemma 4.1] does not hold. Nevertheless, using the inverse formula of a Kac-Murdoch-Szegö matrix, we can easily show that

$$\Sigma_W = \Sigma_S - \Sigma_{SS^c}\Sigma_{S^c}^{-1}\Sigma_{SS^c}^T = \mathrm{diag}(1-\rho^2, \frac{1-\rho^2}{1+\rho^2}, \ldots, \frac{1-\rho^2}{1+\rho^2})$$

i.e. all the eigenvalues $\lambda_m^{(n)}$ are given by $(1-\rho^2)/(1+\rho^2)$ except the maximum one $\lambda_n^{(n)}$ which is equal to $1-\rho^2$. Since the asymptotical normalized distortion is not affected by the change of a finite number of eigenvalues, it can be computed as

$$D^c(\alpha) = D(\alpha, 1) = \lim_{n\to\infty} \frac{\sum_{m=1}^{n-k} \lambda_m^{(n)}}{n} = \lim_{n\to\infty} \frac{1-\rho^2}{1+\rho^2}\left(1 - \frac{k}{n}\right) = \frac{1-\rho^2}{1+\rho^2}(1-\alpha)$$

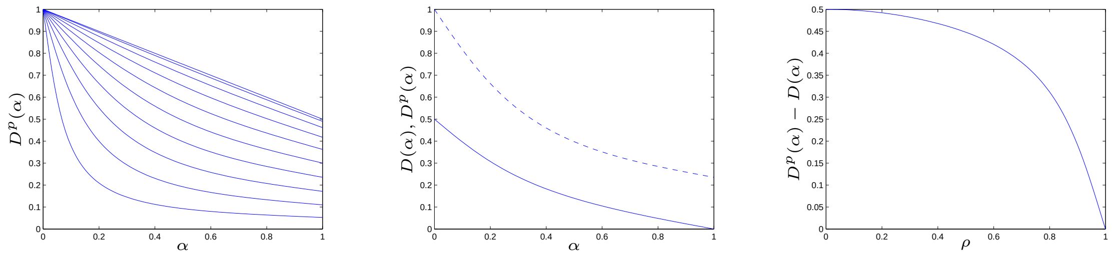
where $\alpha \sim k/n$. □

Theorem 4.2 shows that the prediction error of the odd coefficients by the even ones has uncorrelated components, i.e. the error process is white. This is due to the first-order property of the Gauss-Markov process and it yields the above linear decrease in distortion. We show in Figure 4 the asymptotic normalized distortion $D^c(\alpha)$ of the cKLT of the source X_S with side information X_{S^c} as function of α. We also compare the asymptotic normalized distortion of the source X_S with (cKLT) and without (KLT) receiver side information for different values of ρ. We clearly see the gain achieved by providing the central decoder with some correlated side information. The exact gain can be expressed using Theorems 4.1 and 4.2. As $\rho \to 0$, X_S and X_{S^c} become uncorrelated, i.e. the presence of side information does not provide any gain. When $\rho \to 1$, the correlation among the components of X_S allows to perfectly recover the discarded coefficients without the need for X_{S^c}.

We treat now the source vector X_{S^c} as hidden part and derive the asymptotic normalized distortion of the pKLT.

Theorem 4.3 (Asymptotic Normalized Distortion - pKLT)**.** *Consider the source X_S and the hidden part X_{S^c}. The asymptotic normalized distortion incurred by a pKLT of the source where only a fraction α of the coefficients is kept is given by*

$$D^p(\alpha) = 1 + \frac{\alpha(1-\rho^2)}{2(1+\rho^2)} - \frac{2}{\pi}\arctan\left(\frac{1+\rho^2}{1-\rho^2}\tan\left(\frac{\pi\alpha}{2}\right)\right).$$



(a) (b) (c)

Figure 5: Asymptotic normalized distortion of the pKLT. (a) Asymptotic normalized distortion $D^p(\alpha)$ of the source X_S with hidden part X_{S^c} for $\rho = 0, 0.1, \ldots, 0.9$ (from top to bottom). (b) Asymptotic normalized distortion of the source X_S with ($D^p(\alpha)$, dashed) and without ($D(\alpha)$, plain) hidden part X_{S^c} for $\rho = 0.6$. (c) Loss due to the hidden part as a function of ρ for $\alpha = 0.1$.

Proof: Let us denote by $D_1^p(\alpha)$ the asymptotic normalized distortion that corresponds to Σ_W defined in (3). It can be easily shown that Σ_W is asymptotically equivalent to the Toeplitz matrix $T_n(f + |g|^2/f)$ where $s = f + |g|^2/f$ can be computed as

$$s(\lambda) = \frac{(1-\rho^2)(1+4\rho^2+\rho^4+2\rho^2\cos\lambda)}{(1+\rho^2)(1+\rho^4-2\rho^2\cos\lambda)}.$$

We can readily check that $s > 0$, s is symmetric and strictly decreasing in $[0, \pi]$. The asymptotic normalized distortion can thus be computed using (5) as [3, p. 181]

$$\begin{aligned} D_1^p(\alpha) &= \frac{1-\rho^2}{\pi(1+\rho^2)} \int_{\pi\alpha}^{\pi} \frac{1+4\rho^2+\rho^4+2\rho^2\cos\lambda}{1+\rho^4-2\rho^2\cos\lambda}\, d\lambda \\ &= 2 - \frac{(1-\alpha)(1-\rho^2)}{(1+\rho^2)} - \frac{4}{\pi}\arctan\left(\frac{1+\rho^2}{1-\rho^2}\tan\left(\frac{\pi\alpha}{2}\right)\right). \end{aligned}$$

Let us denote by $D_2^p(\alpha)$ the asymptotic normalized distortion that corresponds to $\mathrm{tr}(\Sigma_{S^c} - \Sigma_{SS^c}^T \Sigma_S^{-1} \Sigma_{SS^c})$. Since $\Sigma_S = \Sigma_{S^c}$, we can directly compute $D_2^p(\alpha)$ by evaluating at $\alpha = 0$ the asymptotic normalized distortion of the cKLT obtained in Theorem 4.2. Thus,

$$D_2^p(\alpha) = D^c(0) = \frac{1-\rho^2}{1+\rho^2}.$$

Using a normalization factor $2n$, the result follows from adding the two above contributions pondered by a factor 1/2. $\square$

We show in Figure 5 the asymptotic normalized distortion $D^p(\alpha)$ of the pKLT of the source X_S with hidden part X_{S^c} as function of α. We also compare the asymptotic normalized distortion of the source X_S with (pKLT) and without (KLT) hidden part. We clearly see the loss incurred by having to reconstruct the missing information at the central decoder. Furthermore, increasing ρ allows to estimate more and more accurately the missing data, hence reducing the gap between the two distortions. The exact loss can be expressed using Theorems 4.1 and 4.3 with appropriate normalization.

Let us now consider again the general two terminal setup introduced in Section 2. Assume that terminal 1 (resp. terminal 2) only keeps a fraction α_1 (resp. α_2) of transformed coefficients. We can then represent the asymptotic normalized distortion surface

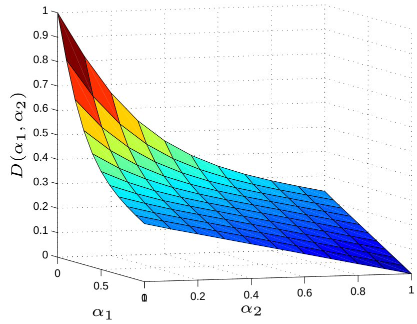


Figure 6: Asymptotic normalized distortion $D(\alpha_1, \alpha_2)$ for $\rho = 0.6$. The inside of the distortion surface is obtained using the dKLT algorithm.

as a function of α_1 and α_2. This is shown in Figure 6 for $\rho = 0.6$. The inside of the distortion surface is obtained numerically using the dKLT algorithm whereas the analytical expression of the border is perfectly known. In fact, it corresponds to the asymptotic normalized distortion obtained for the pKLT ($\alpha_1 = 0$ or $\alpha_2 = 0$) and the cKLT ($\alpha_1 = 1$ or $\alpha_2 = 1$). The inherent symmetry is due to the fact that $\Sigma_S = \Sigma_{S^c}$. Note that the dKLT algorithm only provides a locally optimal solution which here is conjectured to be global owing to the regularity of the covariance matrix. This perspective shows that even if the overall distortion surface is not known, it is somehow constrained by its borders. Furthermore, the approximation problem allows a convenient representation of the asymptotic normalized distortion as a function of parameters that range over a finite interval.

5 Conclusions

In this work, we have studied the mean squared distortion behavior of the dKLT under asymptotic considerations. A "water-filling" type formula has been provided in order to compute the asymptotic normalized distortion of stationary processes. Our findings have been illustrated with a simple correlation model that has permitted us to obtain closed-form formulas for the distortion incurred by different scenarios of both theoretical and practical interest.

The analysis provided in this paper has been carried from a pure approximation point of view. However, the optimality of transform coding, for jointly Gaussian random sources, allows us to relate this approach to the compression (rate-distortion) framework. In particular, we can precisely compare the distortion introduced by the approximation and compression stages in an optimal transform coder.

References

[1] M. Gastpar, P. L. Dragotti, and M. Vetterli, "The distributed Karhunen-Loève transform," *submitted to IEEE Transactions on Information Theory*, November 2004.

[2] R. M. Gray, *Toeplitz and Circulant Matrices: a Review.* Available online at http://ee.standford.edu/~gray/toeplitz.pdf.

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products.* Academic Press, London, 5th ed., 1994.