Percolation in the signal to interference ratio graph

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Abstract

Continuum percolation models in which pairs of points of a two-dimensional Poisson point process are connected if they are within some range to each other have been extensively studied. This paper considers a variation in which a connection between two points depends not only on their Euclidean distance, but also on the positions of *all* other points of the point process. This model has been recently proposed to model interference in radio communication networks. Our main result shows that, despite the infinite range dependencies, percolation occurs in the model when the density λ of the Poisson point process is greater than the critical density value λ_c of the independent model, provided that interference from other nodes can be sufficiently reduced (without vanishing).

1 Introduction

Continuum percolation models originated with a paper of Gilbert [3], who considered the following construction of a random graph: each pair of points of a two-dimensional Poisson point process of density λ is joined by an edge if the points are within distance $2r$ of each other. The motivation to introduce such a construction was to model networks of broadcasting stations that can exchange messages if they are within a certain range of each other. Gilbert proved a phase transition behavior, namely the existence of a critical value $\lambda_c = \lambda_c(r)$ for the density of the Poisson point process, such that, for $\lambda > \lambda_c$, an unbounded connected subgraph a.s. forms (i.e., the model *percolates*), and so the network can provide long distance communication by multi-hopping messages along a path of connected stations. On the other hand, for $\lambda < \lambda_c$ any connected component is bounded. His results sparked a wide range of interest and were extended considerably by many mathematicians. We refer to [4] for a survey of the literature.

Gilbert's model applies to multi-hop wireless networks, when the circular discs centered at the Poisson points are considered as the radiation patterns of signals transmitted by the broadcasting stations. Pick two points of the Poisson process and label them a transmitter x_i and a receiver x_j. The transmitter x_i radiates a signal with intensity proportional to the power P spent to generate the transmission. The signal diffuses isotropically in the environment and is then received by x_j with intensity P times a loss factor $\ell(x_i, x_j) \leq 1$, due to isotropic dispersion and absorption in the environment. Furthermore, the reception mechanism is affected by noise, which means that x_j is able to

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detect the signal only if its intensity is sufficiently high compared to the environmental noise $N > 0$. Assuming that $\ell(x_i, x_j) = \ell(x_j, x_i)$, we conclude that x_i and x_j are able to establish a communication link if the signal to noise ratio (SNR) is above a given threshold T. That is, if

$$SNR = \frac{P\ell(x_i, x_j)}{N} > T. \tag{1}$$

It is reasonable to assume the loss factor $\ell(x, y)$ to be a decreasing function of the Euclidean distance between x and y. It follows that fixing the threshold T is equivalent to fixing the radius r of the discs in Gilbert's model.

From a practical viewpoint, however, this simple model does not account for interference effects that arise when all nodes transmit at the same time. In this case, all nodes can contribute to the amount of noise present at the receiver and increasing the density of the transmitters may not always be beneficial for connectivity. These observations motivated Dousse, Baccelli and Thiran [2] to introduce a dependent percolation model that can be described as follows.

Consider two points of a planar Poisson point process x_i and x_j, and assume x_i wants to communicate a message to x_j. At the same time, however, all other nodes x_k, $k \neq i, j$, also transmit an interfering signal that reaches x_j. We write the total interference term at x_j as $\gamma \sum_{k \neq i,j} P\ell(x_k, x_j)$, where P is the transmitted power, and $\gamma > 0$ is a factor that depends on the technology adopted in the system. Node x_j can then successfully receive the message from x_i if the signal to noise plus interference ratio (SINR) is greater than a given threshold, that is

$$SINR = \frac{P\ell(x_i, x_j)}{N + \gamma \sum_{k \neq i,j} P\ell(x_k, x_j)} > T. \tag{2}$$

A random graph is now constructed as follows. For each pair of Poisson points, the $SINR$ level at both ends is computed and an undirected edge between the two is drawn if this exceeds the threshold T in both cases. In this way, the presence of an edge indicates the possibility of direct bidirectional communication between the two nodes, while the presence of a path between two nodes in the graph indicates the possibility of multi-hop bidirectional communication. Note that the constructed random graph does not have the independence structure of Gilbert's model, because the presence of an edge between any pair of nodes now depends on the random positions of all other nodes in the plane that are causing interference, and not only on the two end-nodes of the link. Such dependencies, as we shall see, make the mathematical analysis of this kind of graph considerably more challenging. We call this model the SINR-model.

It is shown in [2] that by taking λ large enough, there exists a $\overline{\gamma}(\lambda) > 0$ such that for $\gamma < \overline{\gamma}$ the network percolates. In order to deal with the dependency structure of the model, however, they assumed that the function ℓ has bounded support. This allows to considerably simplify the mathematical analysis and to use a standard coupling argument with a finite range dependent percolation model to immediately obtain the result. In fact, the main focus of their paper was not mathematical, but rather to present a novel model of engineering interest.

Nevertheless, one reasonably expects stronger results. If we denote by λ_c the critical density of the model when $\gamma = 0$, then one expects percolation for all $\lambda > \lambda_c$, by taking $\gamma > 0$ sufficiently small. Furthermore, this should be the case also when the function ℓ has unbounded support.

Our contribution in this paper is to show that this is correct under the most general class of loss functions ℓ. We remark that most of the difficulties that we need to overcome deal with the long range dependencies introduced by the unbounded support of ℓ. It shall be clear from the proof that a tight density threshold for percolation at λ_c is easy to obtain when bounded support is assumed. However, from physics we know that in reality attenuation of a signal does not have bounded support, which gives some applied motivation to our additional mathematical efforts.

2 The main result

In this paper, the underlying point process will always be a Poisson process with density $\lambda > 0$. Further parameters of the model are N, γ, P and T. In the sequel, we consider $N > 0, P > 0$ and $T > 0$ fixed, and study the existence of a percolation phenomenon for varying values of λ and γ. The function ℓ is called the *attenuation function* and we assume it to have the following properties:

1. $\ell(x, y)$ only depends on $|x - y|$, that is, $\ell(x, y) = l(|x - y|)$ for some function $l : \mathbb{R}^+ \to \mathbb{R}^+$;
2. $l(x) \leq 1$;
3. l is continuous and as long as it does not vanish, it is strictly decreasing.

These three properties characterize our physical model for wave propagation. In order to ensure that the model is not degenerated, we must impose two more conditions on l:

4. $l(0) > TN/P$;
5. $\int_0^\infty x l(x) dx < \infty$.

Some comments on these assumptions are perhaps necessary. If Condition 4 is not verified, Equation (2) never holds. Furthermore, the sum in the denominator of (2) is almost surely finite if and only if Condition 5 is satisfied (see e.g. [1]).

We remark at this point that the length of the edges are uniformly bounded: when $\ell(x, y) \leq TN/P$, no edge can form between x and y. In [2] it is also shown that the degree of any vertex is bounded above uniformly by $1 + \frac{1}{T\gamma}$; their proof is also valid in the case of unbounded support of ℓ.

We write $x \leftrightarrow y$ if there exists a sequence $x_1, x_2, \ldots, x_k$ of Poisson points such that $x_1 = x$, $x_k = y$, and x_l is connected by an edge to x_{l+1} for $1 \leq l < k$. A *(connected) component* or *cluster* is a set $\{x_i : i \in J\}$ of points which is maximal with the property that $x_i \leftrightarrow x_j$ for all $i, j \in J$.

As mentioned before, we denote by λ_c the critical density of the model when $\gamma = 0$ (and all other parameters N, P and T fixed). It is known that for $\lambda \leq \lambda_c$ we have no infinite cluster a.s., while for $\lambda > \lambda_c$ there is an infinite cluster with probability 1. If there is an unbounded component of points with positive probability, we say that the signal to interference ratio graph *percolates*. (In fact, standard results from ergodic theory say that the existence of an unbounded component with positive probability implies that this latter probability is equal to one. For us, this is not immediately relevant.) Here is our main result.

Theorem 1 *Let λ_c be the critical node density when $\gamma = 0$ and assume that the attenuation function ℓ satisfies the assumptions 1-5 stated above. Then for any node density $\lambda > \lambda_c$, there exists $\gamma^*(\lambda) > 0$ such that for $\gamma \leq \gamma^*(\lambda)$, the SINR-model percolates.*

3 Proof of Theorem 1

The main strategy of the proof is by coupling the model to a discrete edge percolation model on the grid. By doing so, we end up with a dependent discrete model, such that the existence of an infinite connected component in the edge percolation model implies the existence of an infinite connected component in the original graph. Although the edges of the discrete model are not finite-range dependent, we show that the probability of having a collection of n closed edges in the discrete model decreases exponentially as q^n, where q can be made arbitrarily small by appropriate choice of the parameters, and therefore the existence of an infinite connected component follows from a Peierls argument.

We describe the construction of the discrete model first, then we prove percolation of this model, and finally we show the final result by coupling it with the SINR model.

3.1 Mapping on a lattice

If we set $\gamma = 0$, we obtain a fixed radius Poisson boolean model with radius r_b given by

$$2r_b = l^{-1}\left(\frac{TN}{P}\right).$$

Since l is continuous, strictly monotone and larger than TN/P at the origin, we have that $l^{-1}(TN/P)$ exists.

We consider next a supercritical boolean model $\mathcal{B}(\lambda, r_b)$ with radius r_b where the node density λ is higher than the critical value λ_c. By rescaling the model, we can establish that the critical radius for a fixed density λ is

$$r^*(\lambda) = \sqrt{\frac{\lambda_c}{\lambda}} r_b < r_b.$$

Therefore, a boolean model $\mathcal{B}(\lambda, r)$ with density λ and radius r satisfying $r^*(\lambda) < r < r_b$, is still supercritical.

We map this latter model on a discrete percolation model as follows. For $d > 0$, we denote by $\mathcal{L}_d$ the two-dimensional square lattice whose vertices are located at all points of the form (dx, dy) with $(x, y) \in \mathbb{Z}^2$. For each horizontal edge a of $\mathcal{L}_d$, we denote by $\boldsymbol{z}_a = (x_a, y_a)$ the point in the middle of the edge, and introduce the random field A_a, indexed by the edges of $\mathcal{L}_d$, that takes the value 1 if the following two events (illustrated in Figure 1) occur, and 0 otherwise:

1. the rectangle $[x_a - 3d/4, x_a + 3d/4] \times [y_a - d/4, y_a + d/4]$ is crossed from left to right by an occupied component in $\mathcal{B}(\lambda, r)$, and
2. both squares $[x_a - 3d/4, x_a - d/4] \times [y_a - d/4, y_a + d/4]$; $[x_a + d/4, x_a + 3d/4] \times [y_a - d/4, y_a + d/4]$ are crossed from top to bottom by an occupied component in $\mathcal{B}(\lambda, r)$.

We define A_a similarly for vertical edges, by rotating the above conditions by 90°.

According [4, Corollary 4.1], the probability that $A_a = 1$ can be made as large as we like by choosing d large. The variables A_a are not independent in general. However, if a and b are not adjacent, then A_a and A_b are independent: these variables thus define a 1-dependent edge percolation process.

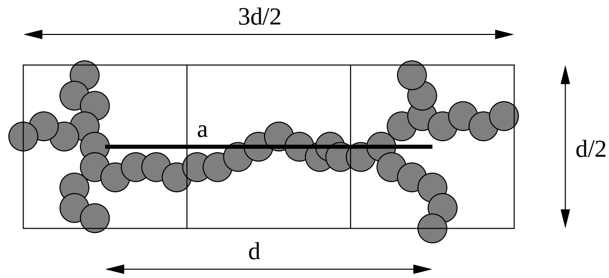

Figure 1: A horizontal edge a that fulfills the two conditions for having $A_a = 1$.

We next define a second random field, B_a, indexed again by the edges in $\mathcal{L}_d$, as follows. We first define $\tilde{l}$, a shifted version of l, as follows:

$$\tilde{l}(x) = \begin{cases} l(0) & x \leq \frac{\sqrt{10}d}{4}, \\ l(x - \frac{\sqrt{10}d}{4}) & x > \frac{\sqrt{10}d}{4}. \end{cases}$$

We define the *shot-noise* processes I and $\tilde{I}$ as follows:

$$I(\boldsymbol{z}) = \sum_k l(|\boldsymbol{z} - x_k|)$$

and

$$\tilde{I}(\boldsymbol{z}) = \sum_k \tilde{l}(|\boldsymbol{z} - x_k|),$$

where $\boldsymbol{z} \in \mathbb{R}^2$ is an arbitrary point, and where the sum is over all points of the Poisson process X. Note that the shot-noises are random variables, since they depend on the random position of the points of X.

We define now the second random field B_a as taking the value 1 if the value of the shot-noise $\tilde{I}(\boldsymbol{z}_a)$ does not exceed a certain threshold M, and 0 otherwise. As the distance between any point $\boldsymbol{z}$ inside the rectangle $R(\boldsymbol{z}_a) = [x_a - 3d/4, x_a + 3d/4] \times [y_a - d/4, y_a + d/4]$ and its center $\boldsymbol{z}_a$ is at most $\sqrt{10}d/4$, the triangle inequality implies that $|\boldsymbol{z}_a - x_k| \leq \sqrt{10}d/4 + |\boldsymbol{z} - x_k|$, and thus that $I(\boldsymbol{z}) \leq \tilde{I}(\boldsymbol{z}_a)$ for all $\boldsymbol{z} \in R(\boldsymbol{z}_a)$. Therefore, $B_a = 1$ implies that $I(\boldsymbol{z}) \leq M$ for all $\boldsymbol{z} \in R(\boldsymbol{z}_a)$. Later, we will make an appropriate choice for first d and then M.

3.2 Percolation in the lattice

For any edge a of $\mathcal{L}_d$, we call the edge *open* if the product $C_a = A_a B_a = 1$, that is, if both of the following events occur: there exist crossings in the rectangle $R(z_a)$ as described above, *and* the shot noise is bounded by M for all points inside $R(z_a)$. An edge a that is not open is *closed*. We want to show that for appropriate choice of the parameters M and d, there exists an infinite connected component of open edges at the origin, with positive probability.

To do this, we need an exponential bound on the probability of a collection of n closed edges. Most of the difficulty of obtaining this resides in the infinite range dependencies introduced by the random variables B_i's. A careful application of Campbell's theorem will take care of this.

Consider any collection of n edges $a_1, \ldots, a_n$. To keep the notation simple, we write $A_{a_i} = A_i$, $B_{a_i} = B_i$ and $C_{a_i} = C_i$, $i = 1, \ldots, n$. In Proposition 3, we will prove that the probability that all these edges are closed simultaneously, decreases exponentially with n. To do this, we first prove this for the fields A and B.

Proposition 1 *Let $\{a_i\}_{i=1}^n$ be a collection of n distinct edges, and let $\{A_i\}_{i=1}^n$ be the random variables of the field A associated with them. Then there exists a constant $q_A < 1$, independent of the particular collection, such that*

$$\mathbb{P}(A_1 = 0, A_2 = 0, \ldots, A_n = 0) \leq q_A^n.$$

Furthermore, for any $\varepsilon > 0$, one can choose d large enough so that $q_A \leq \varepsilon$.

Proof: This propositions follows directly from the observation that it is always possible to find a subset of indices $\{k_j\}_{j=1}^m$ with $1 \leq k_j \leq n$ for each j, such that the variables $\{A_{k_j}\}_{j=1}^m$ are independent, and such that $m \geq n/4$. Therefore we have

$$\begin{aligned} \mathbb{P}(A_1 = 0, A_2 = 0, \ldots, A_n = 0) &\leq \mathbb{P}(A_{k_1} = 0, A_{k_2} = 0, \ldots, A_{k_m} = 0) \\ &= \mathbb{P}(A_1 = 0)^m \\ &\leq \mathbb{P}(A_1 = 0)^{\frac{n}{4}} \\ &\equiv q_A^n. \end{aligned}$$

Furthermore, since $q_A = \mathbb{P}(A_1 = 0)^{1/4}$, it follows from [4, Corollary 4.1] that q_A tends to zero when d tends to infinity. $\square$

Proposition 2 *Let $\{a_i\}_{i=1}^n$ be a collection of n distinct edges, and let $\{B_i\}_{i=1}^n$ be the random variables of the field B associated with them. Then there exists a constant $q_B < 1$, independent of the particular collection, such that*

$$\mathbb{P}(B_1 = 0, B_2 = 0, \ldots, B_n = 0) \leq q_B^n.$$

Furthermore, for any $\varepsilon > 0$ and fixed d, one can choose M large enough so that $q_B \leq \varepsilon$.

Proof: To simplify notation, we denote by $\boldsymbol{z}_i$ the center $\boldsymbol{z}_{a_i}$ of the edge a_i. By Markov's inequality, we have for any $s \geq 0$,

$$\begin{aligned} \mathbb{P}(B_1 = 0, B_2 = 0, \ldots, B_n = 0) &\leq \mathbb{P}\Big(\tilde{I}(\boldsymbol{z}_1) > M, \tilde{I}(\boldsymbol{z}_2) > M, \ldots, \tilde{I}(\boldsymbol{z}_n) > M\Big) \\ &\leq \mathbb{P}\left(\sum_{i=1}^n \tilde{I}(\boldsymbol{z}_i) > nM\right) \\ &\leq e^{-snM}\mathbb{E}\Big(e^{s\sum_{i=1}^n \tilde{I}(\boldsymbol{z}_i)}\Big). \end{aligned}$$

Using Campbell's theorem (see e.g. [5]) applied to the function

$$f(x) = \sum_{i=1}^n \tilde{l}(|\boldsymbol{z}_i - x|)$$

we obtain

$$\mathbb{E}\Big(e^{s\sum_{i=1}^n \tilde{I}(\boldsymbol{z}_i)}\Big) = \exp\left(\lambda \int_{\mathbb{R}^2} (e^{s\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|)} - 1)\boldsymbol{dx}\right). \tag{3}$$

We need to estimate the exponent $s\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|)$. As $\{\boldsymbol{z}_i\}$ are centers of edges, they are located on a square lattice with edge length $d/\sqrt{2}$. So, if we consider the square in which $\boldsymbol{x}$ is located, the contribution to $\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|)$ coming from the four corners of this square is at most equal to 4, since $\tilde{l}(x)\leq 1$. Around this square, there are 12 nodes, each located at distance at least $d/\sqrt{2}$ from $\boldsymbol{x}$. Further away, there are 20 other nodes at distance at least $2d/\sqrt{2}$, and so on. Consequently,

$$\begin{aligned}\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|) &\leq \sum_{i=1}^\infty \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|)\\ &\leq 4+\sum_{k=1}^\infty (4+8k)\tilde{l}\left(\frac{kd}{\sqrt{2}}\right)\equiv K.\end{aligned}$$

Now Assumption 5 above on l can easily be extended to $\tilde{l}$, and we clearly have

$$\int_y^\infty x\tilde{l}(x)dx<\infty \text{ for some } y>0. \tag{4}$$

Using the integral criterion and (4), we conclude that the sum converges and thus $K<\infty$.

The computation made above holds for any $s\geq 0$. We now take $s=1/K$, so that $s\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|)\leq 1$, for all $\boldsymbol{x}$. Furthermore, since $e^x-1<2x$ for all $x\leq 1$ we have

$$e^{s\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|)}-1<2s\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|)=\frac{2}{K}\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|).$$

Substituting this in (3), we obtain

$$\begin{aligned}\mathbb{E}\left(e^{\sum_{i=1}^n \tilde{I}(\boldsymbol{z}_i)/K}\right) &\leq \exp\left(\lambda\int_{\mathbb{R}^2}\frac{2}{K}\sum_{i=1}^n \tilde{l}(|\boldsymbol{x}-\boldsymbol{z}_i|)\boldsymbol{dx}\right)\\ &= \exp\left(\frac{2n\lambda}{K}\int_{\mathbb{R}^2}\tilde{l}(|\boldsymbol{x}|)\boldsymbol{dx}\right)\\ &= \left[\exp\left(\frac{2\lambda}{K}\int_{\mathbb{R}^2}\tilde{l}(|\boldsymbol{x}|)\boldsymbol{dx}\right)\right]^n.\end{aligned}$$

Putting things together, we have that

$$\begin{aligned}&\mathbb{P}\left(\tilde{I}(\boldsymbol{z}_1)>M,\tilde{I}(\boldsymbol{z}_2)>M,\ldots,\tilde{I}(\boldsymbol{z}_n)>M\right)\\ &\quad\leq e^{-snM}\mathbb{E}\left(e^{s\sum_{i=1}^n \tilde{I}(\boldsymbol{z}_i)}\right)\\ &\quad\leq e^{-nM/K}\left[\exp\left(\frac{2\lambda}{K}\int_{\mathbb{R}^2}\tilde{l}(|\boldsymbol{x}|)\boldsymbol{dx}\right)\right]^n\\ &= q_B^n,\end{aligned}$$

where q_B is defined as

$$q_B\equiv\exp\left(\frac{2\lambda}{K}\int\tilde{l}(|\boldsymbol{x}|)\boldsymbol{dx}-\frac{M}{K}\right). \tag{5}$$

Finally, it is easy to observe that this expression tends to zero when M tends to infinity (for fixed d and hence, since K depends only on d, for fixed K). $\square$

We next combine the two propositions, in order to obtain a similar result for the field C.

Proposition 3 *Let $\{a_i\}_{i=1}^n$ be a collection of n distinct edges, and let $\{C_i\}_{i=1}^n$ be the random variables of the field C associated to them. Then there exists a constant $q_C < 1$, independent of the particular collection, such that*

$$\mathbb{P}(C_1 = 0, C_2 = 0, \ldots, C_n = 0) \leq q_C^n.$$

Furthermore, for any $\varepsilon > 0$, one can choose d and M so that $q_C \leq \varepsilon$.

Proof: For convenience in the following calculations, we introduce the notation $\bar{A}_i = 1 - A_i$ and $\bar{B}_i = 1 - B_i$. First observe that

$$1 - C_i = 1 - A_i B_i \leq (1 - A_i) + (1 - B_i) = \bar{A}_i + \bar{B}_i.$$

Let us denote by $p(n)$ the probability that we want to bound, and let $(k_i)_{i=1}^n$ be a binary sequence (i.e. $k_i = 0$ or 1) of length n. We denote by $\mathcal{K}$ the set of the 2^n such sequences. Then we can write

$$\begin{aligned}
p(n) &= \mathbb{P}(C_1 = 0, C_2 = 0, \ldots, C_n = 0) \\
&= \mathbb{E}((1 - C_1)(1 - C_2) \ldots (1 - C_n)) \\
&\leq \mathbb{E}\big((\bar{A}_1 + \bar{B}_1)(\bar{A}_2 + \bar{B}_2) \ldots (\bar{A}_n + \bar{B}_n)\big) \\
&= \sum_{(k_i) \in \mathcal{K}} \mathbb{E}\left(\prod_{i:k_i=0} \bar{A}_i \prod_{i:k_i=1} \bar{B}_i \right) \\
&\leq \sum_{(k_i) \in \mathcal{K}} \sqrt{\mathbb{E}\left(\prod_{i:k_i=0} \bar{A}_i^2 \right) \mathbb{E}\left(\prod_{i:k_i=1} \bar{B}_i^2 \right)} \\
&= \sum_{(k_i) \in \mathcal{K}} \sqrt{\mathbb{E}\left(\prod_{i:k_i=0} \bar{A}_i \right) \mathbb{E}\left(\prod_{i:k_i=1} \bar{B}_i \right)},
\end{aligned}$$

where the two last inequalities follow respectively from Schwartz's inequality and from the observation that $\bar{A}_i^2 = \bar{A}_i$ and $\bar{B}_i^2 = \bar{B}_i$. Applying Propositions 1 and 2, we can bound each expectation in the sum. We have thus

$$\begin{aligned}
p(n) &\leq \sum_{(k_i) \in \mathcal{K}} \sqrt{\prod_{i:k_i=0} q_A \prod_{i:k_i=1} q_B} \\
&= \sum_{(k_i) \in \mathcal{K}} \prod_{i:k_i=0} \sqrt{q_A} \prod_{i:k_i=1} \sqrt{q_B} \\
&= \left(\sqrt{q_A} + \sqrt{q_B}\right)^n \\
&\equiv q_C^n.
\end{aligned}$$

Choosing first d large, and then M appropriately, we can make q_C is smaller than any given ε. □

With Proposition 3, the existence of percolation in our dependent bond percolation model follows from standard arguments. Indeed, with our exponential bound in Proposition 3, we can apply the usual Peierls argument to establish the existence of percolation for appropriate M and d.

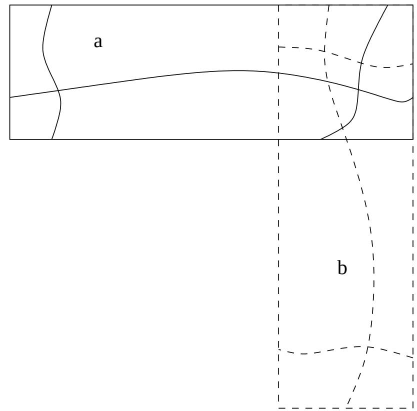

Figure 2: Two adjacent edges a (plain) and b (dashed) with $A_a = 1$ and $A_b = 1$. The crossings overlap, and form a connected component.

3.3 Percolation of the SINR model

To conclude the proof we need to show that percolation of C_a implies percolation in the SINR model, with appropriate γ. If $B_a = 1$, the interference level in the rectangle $R(z_a)$ is at most equal to M. Therefore, for two nodes x_i and x_j in $R(z_a)$ such that $|x_i - x_j| \leq 2r$, we have

$$\begin{aligned} \frac{Pl(|x_i - x_j|)}{N + \gamma \sum_{k \neq i,j} Pl(|x_k - x_j|)} &\geq \frac{Pl(|x_i - x_j|)}{N + \gamma PM} \\ &\geq \frac{Pl(2r)}{N + \gamma PM}. \end{aligned} \tag{6}$$

As $r < r_b$ and as l is strictly decreasing, we choose

$$\gamma = \frac{N}{PM}\left(\frac{l(2r)}{l(2r_b)} - 1\right) > 0, \tag{7}$$

yielding

$$\frac{Pl(2r)}{N + \gamma PM} = \frac{Pl(2r_b)}{N} = T. \tag{8}$$

Therefore, there exists a positive value of γ such that any two nodes separated by a distance less than r are connected in the SINR model. This means that in the rectangle $R(z_a)$ all connections of $\mathcal{B}(\lambda, r)$ also exist in the SINR model.

Finally, if $A_a = 1$, there exist crossings along edge a, as shown in Figure 1. These crossings are designed such that if for two adjacent edges a and b, $A_a = 1$ and $A_b = 1$, the crossings overlap, and they all belong to the same connected component (see Figure 2). Thus, an infinite cluster of such edges implies an infinite cluster in the boolean model of radius r and density λ. Since all edges a of the infinite cluster of the discrete model are such that $A_a = 1$ and $B_a = 1$, this means that the crossings also exist in the SINR model, and thus form an infinite connected component.

4 Conclusion

In this paper, we proved that a percolation phenomenon occurs for some values of the parameters λ (node density) and γ (weight of the interference term) in the SINR-model.

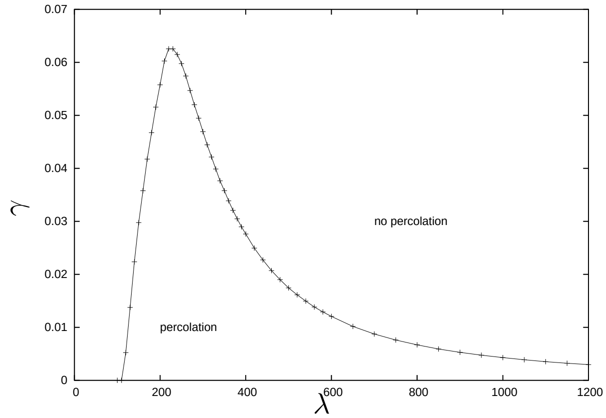

Figure 3: The percolation domain for $l(x) = \min(1, x^{-3})$ computed by simulation.

When $\gamma = 0$, the model boils down to a standard boolean model, and it is known that there exists λ_c such that percolation occurs whenever $\lambda > \lambda_c$. We showed that for any density $\lambda > \lambda_c$, one can pick γ small enough but non zero, so that percolation still occurs.

We thus improved the results in [2] in two ways: first we extended the range of node densities where a percolation phenomenon is proved to exist to the actual range where percolation can occur, and second we established the result for a large class of attenuation functions (in particular with unbounded support), that includes all isotropic, continuous and strictly decreasing functions bounded from above by 1.

We conclude this paper with a summary of what is known about the set of couples (λ, γ) for which percolation occurs:

- no percolation occurs when $\lambda < \lambda_c$,
- no percolation occurs when $\gamma > 1/T$,
- when $\lambda > \lambda_c$, there exists $\gamma^*(\lambda) > 0$ such that percolation occurs whenever $\gamma < \gamma^*(\lambda)$, and
- there exists $c_1 < \infty$ and $\lambda' < \infty$ such that $\gamma^*(\lambda) \le c_1/\lambda$ for all $\lambda > \lambda'$.

The last property follows from [2]. Figure 3 shows a simulation of the percolation domain for an attenuation function of the form $l(x) = \min(1, x^{-3})$.

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