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Explicit infiltration equations and the Lambert *W*-function

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Abstract

The Green and Ampt infiltration formula, as well as the Talsma and Parlange formula, are two-parameter equations that are both expressible in terms of Lambert *W*-functions. These representations are used to derive explicit, simple and accurate approximations for each case. The two infiltration formulas are limiting cases that can be deduced from an existing three-parameter infiltration equation, the third parameter allowing for interpolation between the limiting cases. Besides the limiting cases, there is another case for which the three-parameter infiltration equation yields an exact solution. The three-parameter equation can be solved by fixed-point iteration, a scheme which can be exploited to obtain a sequence of increasingly complex explicit infiltration equations. For routine use, a simple, explicit approximation to the three-parameter infiltration equation is derived. This approximation eliminates the need to iterate for most practical circumstances. © 2002 Published by Elsevier Science Ltd.

1. Introduction

Due to the many circumstances where infiltration into porous media plays a role, theoretical equations for predicting quantities such as infiltration flux and cumulative infiltration are in widespread use. A subset of these circumstances involves one-dimensional vertical infiltration, a branch of vadose-zone hydrology that has a rich history stretching back to the early part of last century. For a given soil type, the formulas aim to estimate $I(t)$, the cumulative infiltration, I, that enters the soil as a function of time, t. The archetype problem to which infiltration laws apply is infiltration into an initially dry, homogeneous soil where the surface of the soil is saturated, but not ponded. It is this situation that is considered below.

In practice, it is useful to have infiltration laws that are both physically based and easy to implement. The latter feature is inherent in explicit expressions for $I(t)$, whereas the former is a feature of laws that are based on standard soil properties such as the soil-water diffusivity, D, and hydraulic conductivity, K. Physically based infiltration laws for one-dimensional infiltration typically use the sorptivity, S, and particular values of the hydraulic conductivity, e.g., the hydraulic conductivity at saturation, K_s, or at the surface moisture content. The sorptivity, we recall, is derived from D and the boundary and initial conditions that pertain [1–4].

As demonstrated elsewhere [5–8], infiltration laws have two "limiting" behaviors. We remark that they are limits in that they appear to cover the possible range of infiltration behaviour; they are not formal mathematical limits. One limit is represented by the Green and Ampt formula [9], which relies on a soil having a rapidly varying diffusivity and a near-constant hydraulic conductivity. The other is represented by Talsma and Parlange [10] result relying on proportionality between D and $\mathrm{d}K/\mathrm{d}\theta$ (θ being the volumetric moisture content), a relationship that was first proposed in [11]. These limiting cases are both easily derived from Richards' equation [8].

The difference in these two formulas fundamentally relates to different assumptions concerning the behaviour of K. The Green and Ampt result assumes that $K \sim \int \psi(\theta)\,\mathrm{d}\theta$, where ψ is the soil-water pressure. On the other hand, the Talsma and Parlange limit assumes K

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varies exponentially with ψ, a functional behaviour which is known to be satisfactory for many soils [1]. It has been shown that the Green and Ampt assumption means that the curvature of K has the wrong sign as it varies with the moisture content [12] and, thus, it comes closest to reality when K varies as a step function of the soil-water pressure head. As a result of the curvature of the Green and Ampt case, water moves more rapidly into a Green and Ampt soil than into a Talsma and Parlange soil (for the same S and K_s).

While the limiting cases are useful for bracketing infiltration behaviour, it is not surprising that the behaviour of natural soils lie somewhere between them. An infiltration law that interpolates between the two limits has been provided [13]. Apart from S and K_s, it includes an additional interpolation parameter, α. For $\alpha = 0$, it reduces to the Green and Ampt formula, whereas for $\alpha = 1$ the Talsma and Parlange formula results. It has been suggested that most natural soils typically are represented by taking $\alpha \approx 0.85$ [13]. This interpolation applies to the situation where there is no ponding at the soil surface. Other interpolations are available that account for ponded infiltration [14]. Here, however, we consider only the non-ponded case.

The main drawback of all those infiltration laws is that the cumulative infiltration is not obtained explicitly in terms of the time, t, making their practical application somewhat inconvenient. Even the Green and Ampt law, for example, is given implicitly as $t(I)$—meaning that I must be determined numerically for a given t—rather than the more useful $I(t)$. In previous investigations then, we have provided explicit approximations to implicit infiltration formulas. An accurate approximation (within 1% relative error) to the result of [14] yielding $I(t)$ explicitly is available [15]. Elsewhere [12], we showed that an explicit solution to the Green and Ampt infiltration equation was available in terms of the Lambert W-function [16], and furthermore provided some accurate approximations for evaluating W. In addition to its role in Green and Ampt infiltration, we will show below that this function is intimately connected to the three-parameter infiltration equation.

The purpose of this paper is to re-examine the limiting cases of the Green and Ampt and Talsma and Parlange infiltration laws making use of the Lambert W-function, showing the exact results that are available when this function is used. Next, we show that, based on approximations to the various branches of the Lambert W-function, new approximations to the limiting cases can be deduced from simple analytical iteration schemes. This approach is then applied to the three-parameter infiltration equation [13], resulting in a new, very accurate, explicit approximation to that formula. We begin, however, by providing some background information on the Lambert W-function.

2. Lambert W-function

Following previously used notation [16], we consider real values of the function $W(x)$ defined by

$$W\exp(W) = x, \quad x \geqslant -\exp(-1), \tag{1}$$

which has two branches $W_0(x) \geqslant -1$ and $W_{-1}(x) \leqslant -1$. These names follow established usage [16]. The branches are shown in Fig. 1. The range of the lower branch is $-1 \geqslant W_{-1}$, while the upper branch W_0 is divided into $-1 \leqslant W_0^- \leqslant 0$ and $0 \leqslant W_0^+$. The latter portion of the upper branch is not used below, although it has been shown to be a solution for soil profile drainage [17].

In applications, using W to obtain formal solutions to problems is useful because it means immediately that a considerable body of W-related work can be drawn upon. On the other hand, in practical situations where formulas need to be evaluated W is not directly useful as it must be computed numerically. Thus, analytical approximations to W are useful for providing rapid estimates.

3. Limiting cases

3.1. Green and Ampt

The Green and Ampt infiltration law is given by

$$I = t + \ln(1 + I), \tag{2}$$

where, as usual [12], I is made dimensionless with $S^2/2K_s$ and t with S^2/K_s^2. Apparently, Barry et al. [12] were the first to notice the relationship between W_{-1} and the Green and Ampt [9] infiltration law into a dry soil. The relationship is more easily discerned by comparing (1) and an equivalent form of (2):

$$(1 + I)\exp[-(1 + I)] = \exp[-(1 + t)]. \tag{3}$$

Hence,

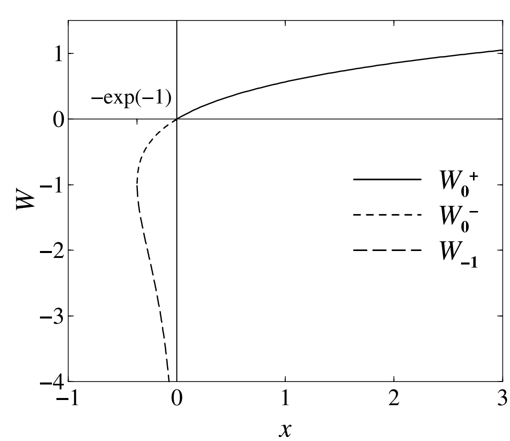

Fig. 1. Branches of the W-function, showing the division into W_{-1}, W_0^- and W_0^+.

$$I = -1 - W_{-1}[-\exp(-1-t)]. \tag{4}$$

Barry et al. [12,18–20] used this relationship to obtain an estimate of $W_{-1}(x)$ by extending an earlier approximation to $I(t)$—here denoted I_{B}—provided by Brutsaert [21]:

$$I_{\mathrm{B}} = t + \frac{(2t)^{1/2}}{1 + (2t)^{1/2}/6}. \tag{5}$$

Remarkably, (5) agrees with the short-time expansion of (3) up to $\mathrm{O}(t^{3/2})$ and differs by a $\ln(t)$ term in the asymptotic (long-time) expansion. To correct for this shortcoming, it is convenient to use an iterative scheme such that the nth approximation is related to the previous one by

$$I_n = t + \ln(I_{n-1} + 1), \quad n = 1, 2, \ldots, \tag{6}$$

which is just a fixed-point iteration scheme to solve (2). It is easy to show that the iteration (6) maintains the short-time expansion of the initial estimate, I_0, whatever estimate is used. Here, we take $I_0 = I_{\mathrm{B}}$. Furthermore, the iteration improves the long-time estimate dramatically. For instance, I_{B} has already a small maximum relative error (defined as $\max|1 - \text{approximation/exact}|$, $\forall t \geqslant 0$ [22]) of 0.36%, but the next approximation:

$$I_1 = t + \ln\left[t + 1 + \frac{(2t)^{1/2}}{1 + (2t)^{1/2}/6}\right], \quad t \geqslant 0, \tag{7}$$

has a maximum relative error of 0.036% , comparable to the 0.025% of Barry et al. [12], who used a slightly more complex expression. Subsequent iterations reduce the error further, roughly by a factor 5 for the first few steps.

Eqs. (4) and (7) also provide a new approximation for W_{-1}:

$$W_{-1}(x) \approx \ln(-x) - \ln\left\{-\ln(-x) + \frac{[-2 - 2\ln(-x)]^{1/2}}{1 + [-2 - 2\ln(-x)]^{1/2}/6}\right\}, \tag{8}$$

valid for $0 \geqslant x \geqslant -\exp(-1)$, which has a maximum relative error of 0.03%.

3.2. Talsma and Parlange

The infiltration law of Talsma and Parlange [10] is

$$(I - t - 1)\exp(I - t - 1) = -\exp(-t-1). \tag{9}$$

As for the Green and Ampt case, an explicit expression for I is available in terms of W_0^-:

$$I = 1 + t + W_0^-[-\exp(-1-t)]. \tag{10}$$

This relationship between the Talsma and Parlange infiltration law and the Lambert W-function has apparently not been recognised before. Following the Green and Ampt case, it is tempting to use the iteration:

$$I_n = t + 1 - \exp(-I_{n-1}), \quad n = 1, 2, \ldots, \tag{11}$$

with the first guess written by analogy with (5) such that the iteration produces infiltration formulas that have short-time expansions that are exact to $\mathrm{O}(t^{3/2})$, or

$$I_0 = t + \frac{(2t)^{1/2}}{1 + (2t)^{1/2}/3 + t/6}. \tag{12}$$

However, the relative error for I_1 is almost 0.2%, which is significantly larger than in the Green and Ampt case. The following study of the general case suggests more appropriate approximations.

4. General case

Between the two limiting cases, Parlange et al. [13] obtained the infiltration formula:

$$I - t = (1-\alpha)^{-1}\ln\left[\frac{1 + (\alpha - 1)\exp(-\alpha I)}{\alpha}\right], \tag{13}$$

where as already mentioned, α is a curve fitting parameter, varying between 0 for the Green and Ampt case and 1 for the Talsma and Parlange case. As for the limiting cases, the inversion of (13) to obtain $I(t)$ is based on the iteration:

$$I_n - t = (1-\alpha)^{-1}\ln\left[\frac{1 + (\alpha - 1)\exp(-\alpha I_{n-1})}{\alpha}\right], \quad n = 1, 2, \ldots, \tag{14}$$

which converges for all t [23]. By taking the difference between two successive approximations, (14) yields, for $t \to 0$,

$$I_n - I_{n-1} = I_{n-1} - I_{n-2} + \mathrm{O}[I_{n-1} - I_{n-2}]. \tag{15}$$

Because the first two approximations differ by a term of $\mathrm{O}(t^{5/2})$, that term remains the same between two consecutive iterations and after n iterations the nth approximation will differ from the first by n-times that term. However, this also means that the next larger term, here of order t^2, remains unchanged after each iteration. Hence, if that term is incorrect, which is the case in our scheme, it will remain so at each iteration.

Taking I_{B} as the first approximation for $\alpha = 0$ ensures that all subsequent approximations are correct to $\mathrm{O}(t^{3/2})$. I_1 in (7) is simple enough to be amenable to analytical manipulations while being very accurate. However, an obvious extension of the procedure to the other limit $\alpha = 1$, starting with (12), was not very accurate, as indicated already above in Section 3.2. Thus, for $\alpha > 0$ we shall use a different approach.

We use the interesting result that, for $\alpha = 1/2$, (13) can be inverted:

$$I_{\alpha=1/2} = t + 2\ln\left\{1 + [1 - \exp(-t/2)]^{1/2}\right\}. \tag{16}$$

This is the only value of α that allowed us to obtain $I(t)$ exactly in terms of elementary functions (the cases of $\alpha = 0$ and 1 are also exact of course but involve the transcendental Lambert W-function).

We try an approximation to (13) which will reduce automatically to (16) for $\alpha = 1/2$ and to (7) for $\alpha = 0$:

$$I = t + (1-\alpha)^{-1} \ln\left[\frac{1+(1-\alpha)}{\alpha}(1-f)^{1/2}\right], \tag{17}$$

where

$$f = \exp\left\{-2\alpha^2 t\left[\frac{1 + A(2t)^{1/2} + 2Bt}{1 + C(2t)^{1/2} + 2Bt(2\alpha)^{1/2}}\right]^2\right\}. \tag{18}$$

We note that to obtain the $\alpha = 0$ limit from (17), the right-hand side must be expanded in a Taylor series for α small, following which the limit as $\alpha \to 0$ is taken. Equation (17) reduces to (7) when the proper values of A, B and C are taken; see (19)–(21) below.

In general, the structure of this approximation is chosen to match the exact behaviour of the three-parameter equation in the short-and-long time limits. The term in brackets on the right-hand side of (18) has the form of a continued fraction, a standard approach to generating approximations designed to produce series expansions [24]. The two B terms are chosen so that I behaves like $t - (1-\alpha)^{-1}\ln(\alpha) - \mathrm{O}[\exp(-\alpha t)]$ when $t \to \infty$, in agreement with (13). The parameters in (18) are chosen so that I is correct to $\mathrm{O}(t^{3/2})$ for small t, or

$$A = \frac{1}{2} + \frac{\lambda - 2\alpha}{3}, \tag{19}$$

$$B = \frac{1+(2\alpha)^{1/2}}{12}\left(\frac{4\lambda - 11\alpha}{3} + 1\right) \tag{20}$$

and

$$C = \frac{1}{6} + \frac{\lambda}{3}. \tag{21}$$

We observe that, whatever the value of λ, (17) reduces to (16) for $\alpha = 1/2$ and (7) for $\alpha = 0$.

Eq. (17) is in a form to approximate I as given by (13). Because λ is arbitrary, it can be used to minimise the error of this approximation. The optimal value of λ was determined as a function of α by minimising the maximum relative error. Then, we fitted an approximation to this numerically determined $\lambda(\alpha)$ and found a good fit using:

$$\lambda = \frac{35}{17}\alpha - \frac{3}{2}\alpha^{1/4}\exp\left(-\frac{15}{4}\alpha^{1/2}\right). \tag{22}$$

A plot of (22) is shown in Fig. 2. The rapid variation in λ evident near $\alpha = 0$ is due to the change in behaviour of I at large t; it changes from being dominated by $\ln(t)$ to $\mathrm{O}[\exp(-\alpha t)]$. Note that this fit was determined by best-fitting of (17) to (13), as shown in Fig. 3. As shown in this figure, the maximum relative error of the approximation is 0.048%.

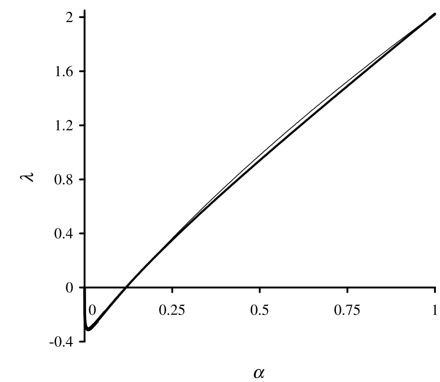

Fig. 2. Numerically determined $\lambda(\alpha)$—thin line and (22)—thick line.

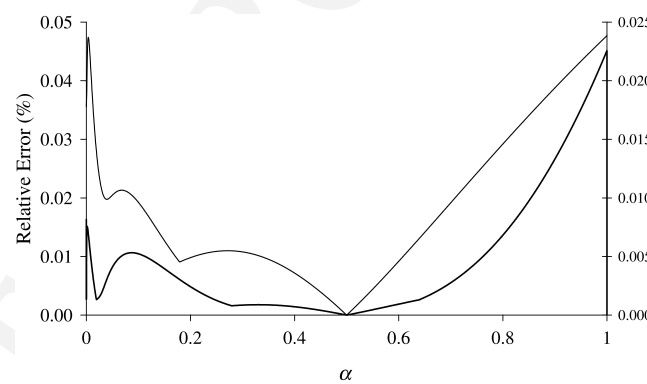

Fig. 3. Thin line (uses left ordinate axis): relative error of the approximation (17)–(22) for the three-parameter infiltration equation (13). Thick line (uses right ordinate axis): relative error of a single iteration using (14).

Even though the relative error shown in Fig. 3 would be satisfactory for most applications, this error can be further reduced by iteration using (14). This error is also plotted in Fig. 3, where we have used (17)–(22) in the right-hand side of (14) and iterated once. In both the curves in Fig. 3, there are several discontinuities in slope. These occur because, for any given α, there can be more than one peak (when the relative error is plotted with t or I). As α changes, different peaks dominate. The slope changes, then, signify when the largest peak in the relative error plot changes.

5. Discussion and concluding remarks

We have obtained relatively simple but very accurate approximations to estimate the solution of the three-parameter infiltration equation, $I(t)$, as defined by (13). Our main analytical result is summarised in (17)–(22), which has a maximum relative error of less than 0.05% as shown in Fig. 3. This simple result will be sufficient for most practical purposes. If, however, greater precision is required, (14) can be used for iterative processes.

We observe that, since any value of λ can be used in the approximation (17)–(21), other simple, yet potentially useful approximations can be deduced. For example, the case of $\lambda = 2$ has the virtue of simplifying the approximation considerably. Taking this value, and iterating using (14) for the case of $\alpha = 1$ gives the approximation:

$$I_1 = t + 1 - \exp[-t - (1 - f)^{1/2}] \tag{23}$$

with

$$f = \exp\left\{ -2t\left[\frac{1 + (2t)^{1/2}/2}{1 + 5(2t)^{1/2}/6}\right]^2\right\}, \tag{24}$$

which is a quite simple expression, yet has a maximum relative error of only 0.02%. As may be noted from Fig. 3, this relative error is less than the relative error of the iterated version of (17)–(22), even though before iteration it has a relative error of only 0.048% (compared with an error of about 0.058% for $\lambda = 2$). The reason for this is that in each case the maximum relative error occurs at different times (or, equivalently, values of I), and the convergence rate of the fixed-point iteration is not uniform over t (or, indeed, α).

Other related results might not be of sufficient accuracy, however. For instance, for $t \to \infty$, the one-dimensional intercept is defined by $I - t$ [25,26], a concept of practical use when it is finite. This is the case when $\alpha > 0$; for the Green and Ampt case of $\alpha = 0$, $I - t$ behaves like $\ln(t)$ and the one-dimensional intercept does not exist. It is indeed that difference in behaviour for $\alpha = 0$ and $\alpha > 0$ (no matter how small), which is responsible for the rapid variation of λ near $\alpha = 0$ shown in Fig. 2.

Since I behaves like t in the long-time limit, $I - t$ will have a larger relative error in that limit than I by itself, since in the latter case t will dominate. Here, we find that using (23) to estimate $I - t$ for $\alpha = 1$, when the one-dimensional intercept exists, gives a maximum relative error of 0.03%, which can be compared with the 0.02% error of (23). Even worse would be to estimate $I - t - 1$ for $\alpha = 1$, i.e., W_0^-, see (10), since as $t \to \infty$, $I \to t + 1$. Here, the maximum error obtained using (23) increases to 0.2%. Thus, the present expression, largely obtained from the short time behaviour of I, is excellent to obtain I, and still quite good for $I - t$, but some care should be taken in its use. Even in such cases, however, the iteration (14) could be applied to improve predictions, as we have already indicated in Fig. 3.

Finally, we have already mentioned that for $\alpha = 0$ and 1, i.e., the limiting cases, the branches of the Lambert W-function are related to I, see (4) and (10). Thus, for $0 < \alpha < 1$, I provides an interpolation between W_0^- and W_{-1} which can be used to define a generalised W function, W_g, by

$$W_g(x) = (2\alpha - 1)I[-1 - \ln(-x)] + [1 + \ln(-x)]\alpha - 1, \tag{25}$$

valid for $-\exp(-1) \leqslant x < 0$. Approximating I using (17) provides a convenient estimate of W_g, which is accurate, except for α very close to 1, when one or more iterations should be used, particularly as $x \to 0$. The reason for this is that the denominator in the relative error vanishes in this limit, so any imprecision in the numerator is exacerbated.

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