

INFILTRATION SUBJECT TO TIME-DEPENDENT SURFACE PONDING: EXACT RESULTS AND CORRESPONDENCE TO SOLUTE TRANSPORT WITH NONLINEAR REACTION

D. A. Barry
Department of Environmental Engineering
Centre for Water Research
University of Western Australia
Nedlands 6009 Western Australia

J.-Y. Parlange
Department of Agricultural and Biological Engineering
Cornell University
Ithaca NY 14853-5701 USA

ABSTRACT. Richards' equation, the governing equation for single phase unsaturated flow in soil, is highly nonlinear and, consequently, is difficult to solve either analytically or numerically. The nonlinearity enters through the soil hydraulic functions, viz., the soil moisture characteristic curve and the unsaturated hydraulic conductivity. We present an exact quasi-analytical solution for the one-dimensional Richards' equation subject to an arbitrary, time-dependent ponding depth. The solution depends on solving a first-order ordinary differential equation. For certain special cases, such as when the surface ponding depth is constant, fully analytical results can be derived. The theory is used in a number of applications. For example, an existing analytical approximation for cumulative infiltration subject to a ponded, time-dependent surface head is checked. As well, a drainage formula is derived. New results for solute transport coupled with a nonlinear adsorption isotherm are derived by making use of an exact mapping between Richards' equation and the governing solute transport equation.

1. Notation

<i>arg</i>	argument
<i>A</i>	function of time
<i>B</i>	constant defined by (54)
<i>c</i>	normalised solute concentration
<i>c_B</i>	adsorption isotherm fitting parameter
<i>c₀</i>	normalised solute concentration at soil surface
<i>c₁, c₂</i>	constants, L
<i>C</i>	function defined by (7), T ⁻¹
<i>C_c</i>	value of <i>C</i> when ψ_s is constant, T ⁻¹
<i>D</i>	solute diffusion/dispersion coefficient, L ² T ⁻¹
<i>D_r</i>	cumulative drainage, L
<i>Ei(.)</i>	exponential integral with argument .
<i>g</i>	magnitude of the gravitational acceleration, LT ⁻²
<i>H(.)</i>	Heaviside function with argument .
<i>I</i>	cumulative infiltration, L
<i>I_c</i>	quantity of solute in the soil profile, L
<i>K</i>	hydraulic conductivity, LT ⁻¹
<i>K_s</i>	saturated hydraulic conductivity, LT ⁻¹
<i>n</i>	time step number
<i>p</i>	fluid pressure offset such that $p = 0$ is 1 atmosphere, ML ⁻¹ T ⁻²
<i>s</i>	normalised solid phase concentration
<i>S</i>	sorptivity, LT ^{1/2}
<i>S₀</i>	sorptivity for zero surface head, LT ^{1/2}

t	time, T
v	pore fluid velocity, LT^{-1}
$W(.)$	function, with argument ., defined by (19)
z	position, L
z_f	solute front position, L
z_i	initial height of reservoir, L
β	constant, L
γ	constant, L
δ	parameter defined by (27)
ϵ	constant, T^{-1}
Δt	time increment in numerical scheme to compute A, T
$\Delta\theta$	$\theta_s - \theta_r$
$\Delta\psi$	$\psi_s(0) - \psi_a$, L
θ	volumetric water content
θ_r	residual volumetric water content
θ_s	saturated volumetric water content
μ	parameter defined by (26)
ρ	density of soil water, ML^{-3}
ψ	pressure head, L
ψ_a	air-entry head, L
ψ_s	surface pressure head, L

2. Introduction

The prediction of infiltration into soil is a key element of hydrologic catchment modelling. Green and Ampt (1911) presented a remarkably simple infiltration formula for the cumulative amount of water to enter a dry soil profile due to a ponded surface condition. Their formula is still used widely because it captures the essential short and long time behaviour of ponded water infiltration. Thus, it is of sufficient accuracy for many applications (e.g., Onstad et al., 1973).

The Green-Ampt formula relies on the assumption of a sharp front of infiltrating water. Recently, Barry et al. (1993b) showed that the Green-Ampt formula can be derived without this assumption. Instead, the soil moisture characteristic curve is a particular function of the hydraulic conductivity. It is this latter, less prescriptive, relationship which leads to the Green-Ampt infiltration formula.

The Green-Ampt approach is easily extended to other, more complicated scenarios. For example, a depth-dependence of the hydraulic conductivity or the initial water content in the soil profile can be introduced. A common extension is to consider the ponding depth at the soil surface to be time dependent as it would be, for instance, if runoff or evaporation occurred. In the following, the exact approach of Barry et al. (1993b) is extended to derive an exact solution for the case of a time-dependent surface condition. In addition, a corresponding exact solution is presented for solute transport subject to a nonlinear sorption isotherm.

3. Exact Solution for Infiltration with a Time-Dependent Surface Head

The governing equation for one-dimensional, unsaturated fluid flow is (Richards, 1931)

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z} \left[K(\psi) \frac{\partial\psi}{\partial z} - K(\psi) \right], \quad (1)$$

where K is the hydraulic conductivity and $\psi (= p/\rho g)$ is the pressure head. The z -axis is assumed to be pointed downwards. Recalling that if only infiltration or drainage is considered, then the soil moisture characteristic curve, $\psi(\theta)$, is single-valued, and (1) can be written as

$$\frac{d\theta}{d\psi} \frac{\partial\psi}{\partial t} = \frac{dK}{d\psi} \frac{\partial\psi}{\partial z} \left(\frac{\partial\psi}{\partial z} - 1 \right) + K \frac{d^2\psi}{dz^2}. \quad (2)$$

Following Barry et al. (1993b), a solution for (2) is sought in the form

$$\psi = \psi_s(t) + \frac{z}{A(t)}, \quad (3)$$

where ψ_s is the surface pressure head and A is an unknown function of time. Further, define the soil moisture capacity as

$$C(t) \frac{d\theta}{d\psi} = \frac{dK}{d\psi} \frac{\frac{dA}{dt}}{\frac{d}{dt} [A(\psi_s - \psi)]} \quad (4)$$

where C is an arbitrary function of time. Note that ψ and t are independent variables in (4). With (4), substitution of (3) into (2) yields

$$A \frac{dA}{dt} = (1 - A) C(t) \quad (5)$$

which has the solution

$$0 = A(t) - A(0) + \ln \left[\frac{1 - A(t)}{1 - A(0)} \right] + \int_0^t C(i) di. \quad (6)$$

We consider soils which have a well defined air-entry value, given by ψ_a . Then, using (4), C is defined as

$$C(t) = \frac{1}{\Delta\theta} \int_{-\infty}^{\psi_a} \frac{\frac{dK}{d\psi} \frac{dA}{dt}}{\frac{d}{dt} [A(\psi_s - \psi)]} d\psi. \quad (7)$$

A slightly different definition of C is appropriate if the surface pressure head becomes lower than ψ_a . We focus here on infiltration under ponded conditions ($\psi_s > 0$), in which case (7) should be used (Barry et al., 1993b). With this definition, the soil moisture characteristic curve is

$$\frac{\theta - \theta_r}{\Delta\theta} = \begin{cases} \frac{\int_{-\infty}^{\psi} \frac{\frac{dK}{d\bar{\psi}}}{\frac{d}{dt} [A(\psi_s - \bar{\psi})]} d\bar{\psi}}{\int_{-\infty}^{\psi_a} \frac{\frac{dK}{d\psi}}{\frac{d}{dt} [A(\psi_s - \psi)]} d\psi}, & \psi < \psi_a \\ 1, & \psi \geq \psi_a \end{cases} \quad (8)$$

This theoretical development is similar to that presented by Barry et al. (1993b), except that here the surface head is permitted to be a function of time. The right side of (8) is unrealistic because of the presence of the time-dependent surface head, whereas the soil moisture characteristic curve is usually taken to be independent of these quantities. However, it will be shown below that useful results can be obtained, particularly for soils which are of the Green and Ampt (1911) type. Before proceeding to the Green-Ampt case, we determine when the soil moisture characteristic curve is independent of time. Any surface pressure head that satisfies

$$\psi_s(t) = c_1 + c_2/A(t) \quad (9)$$

will reduce (8) to

$$\frac{\theta - \theta_r}{\Delta\theta} = \begin{cases} \frac{\int_{-\infty}^{\psi} \frac{\frac{dK}{d\bar{\psi}}}{c_1 - \bar{\psi}} d\bar{\psi}}{\int_{-\infty}^{\psi_a} \frac{\frac{dK}{d\bar{\psi}}}{c_1 - \bar{\psi}} d\bar{\psi}}, & \psi < \psi_a \\ 1, & \psi \geq \psi_a \end{cases} \quad (10)$$

The solution presented by Barry et al. (1993b) is for soil moisture characteristic curves defined by (10), with c_1 representing the (constant) surface pressure head. Further, (7) then shows that C is constant, and so (6) identical to (11) of Barry et al. (1993b). Thus, their results appear to apply not only to a fixed head surface pressure head, but also to the time dependent surface head given by (9). However, for this latter case, Barry et al. (1993b) showed that the solution (3) reduces to the case of a constant pressure head condition at the surface under a translation of distance c_2 in the profile.

Regardless of the functional form of $\psi_s(t)$, integration of Darcy's law gives the cumulative infiltration at the soil surface as

$$I = K_s \int_0^t \left[1 - \frac{1}{A(\bar{t})} \right] d\bar{t}, \quad (11)$$

assuming that $\psi_s > \psi_a$.

3.1. Green-Ampt Soil

A Green-Ampt soil is one which has a hydraulic conduction function defined by

$$K(\psi) = K_s H(\psi - \psi_a), \quad (12)$$

where H is the Heaviside step function. Using (12), C is found from (7) to be

$$\frac{dA}{dt} C^{-1} = \frac{\Delta\theta}{K_s} \frac{d}{dt} [A(\psi_s - \psi_a)]. \quad (13)$$

Then, from (5), A is defined by the solution of

$$\frac{\Delta\theta}{K_s} \frac{d}{dt} [A(\psi_s - \psi_a)] A = 1 - A, \quad (14)$$

with $A(0) = 0$ (Barry et al., 1993b). For a Green-Ampt soil, (11) becomes

$$I = -\Delta\theta A(t) (\psi_s - \psi_a). \quad (15)$$

Now, using (15), (14) can be written as the familiar Green-Ampt infiltration equation

$$I \frac{dI}{dt} = K_s \Delta\theta (\psi_s - \psi_a) + K_s I. \quad (16)$$

Thus, the theory has reduced to the Green-Ampt model as it should. Exact infiltration formulas can be derived from (16) upon specification of suitable functional forms of ψ_s . For example, if a fixed head of

water exists at the soil surface at $t = 0$ then, if the fluid is not replenished, the falling head Green-Ampt model results. Equation (16) is solved by setting $\psi_s = \psi_s(0) - I$. Since $I(0) = 0$, the solution to (16) can be written as (Philip, 1992a)

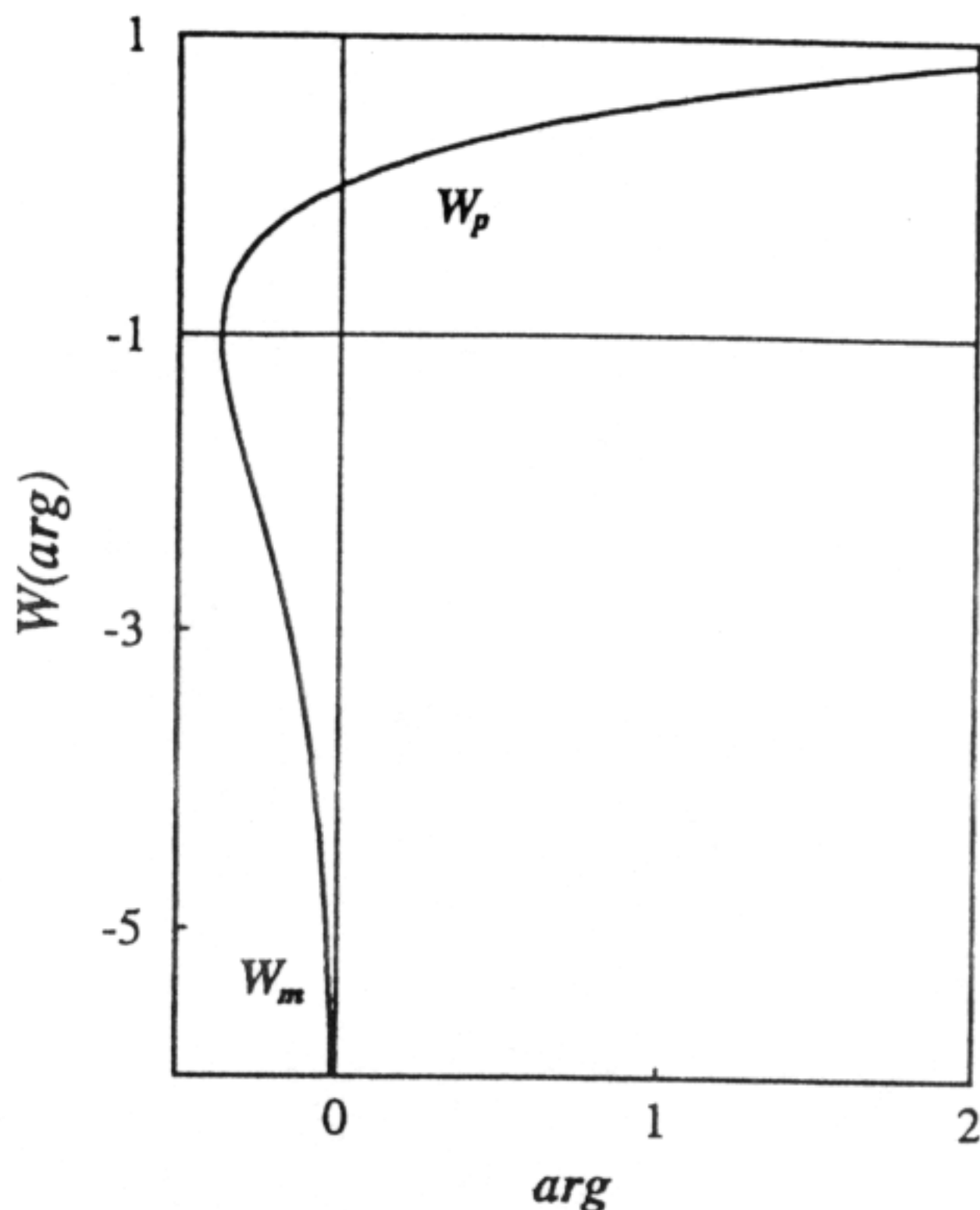


Figure 1. The function $W(arg)$. Above and below the line $W = -1$ lie the two branches W used here, i.e., W_p and W_m , respectively.

$$(1 - \Delta\theta)^2 tK_s = I(1 - \Delta\theta) - \Delta\psi\Delta\theta \ln \left[1 + \frac{I(1 - \Delta\theta)}{\Delta\psi\Delta\theta} \right]. \quad (17)$$

which can be expressed explicitly in the equivalent form

$$\frac{I(1 - \Delta\theta)}{\Delta\psi\Delta\theta} = -1 - W_m \left\{ -\exp \left[-1 - \frac{tK_s(1 - \Delta\theta)^2}{\Delta\psi\Delta\theta} \right] \right\}. \quad (18)$$

The function $W(arg)$ is defined by solutions of (Fritsch et al., 1974)

$$W \exp(W) = arg. \quad (19)$$

The behaviour of the function W is displayed in Fig. 1. Note that Barry et al. (1993b) give simple approximations for portion of this function in the region $arg < 1$. Barry et al. (1993a) have derived an efficient scheme to compute W for all arg . Obviously, other solutions can be determined for (16) upon specification of ψ_s (e.g., Philip, 1992b). However, we wish to consider more general soils than those defined by (12).

3.2. Solution for a General Soil

We consider soils having the property that their soil moisture characteristic curve satisfies (8). As mentioned already, (8) is not considered as a realistic model. Rather, it defines a class of soils for which exact results can be derived. Subsequently, these results will be used to test a general infiltration formula.

Equation (11) gives an exact infiltration formula in terms of the unknown function A , which is defined in turn by (5). A short time expansion of (5) in powers of $t^{1/2}$ can easily be calculated. It is not difficult to show as $t \rightarrow 0$ that

$$A = - \sqrt{\frac{2t}{\Delta\theta} \int_{-\infty}^{\psi_s} \frac{\frac{dK}{d\psi}}{\psi_s(0) - \psi} d\psi} + O(t). \quad (20)$$

Thus, from (11), the sorptivity for this class of soils is

$$S = K_s \sqrt{\frac{2\Delta\theta}{\int_{-\infty}^{\psi_s} \frac{\frac{dK}{d\psi}}{\psi_s(0) - \psi} d\psi}}. \quad (21)$$

An approximate solution for $A(t)$ can now be developed. First, we define X as

$$X(t) = \psi_s + A \frac{\frac{d\psi_s}{dt}}{\frac{dA}{dt}} = \psi_s + \frac{A^2 \frac{d\psi_s}{dt}}{C(1-A)}. \quad (22)$$

With $A(0) = 0$, the solution to (5) is, in terms of the W function,

$$A = 1 + W_m \left\{ -\exp \left[-1 - \int_0^t C(i) di \right] \right\}. \quad (23)$$

For a given K , (7) and (22) define C implicitly. In (23), however, the integral of C is needed. This can be achieved using a simple numerical approximation. Upon setting $t = n\Delta t$, and using the trapezoidal rule (Davis and Polonsky, 1964), (23) becomes

$$A(n\Delta t) = 1 + W_m \left\{ -\exp \left[-1 - \frac{\Delta t}{2} C(0) - \frac{\Delta t}{2} C(n\Delta t) - \Delta t \sum_{i=1}^{n-1} C(i\Delta t) \right] \right\}. \quad (24)$$

Thus, (7) and (22) give a single equation with $A(n\Delta t)$ and $C(n\Delta t)$ as the unknowns, as does (24). However, (24) gives $A(n\Delta t)$ explicitly, so that the combination of (7) and (22) reduces to a single nonlinear equation with $C(n\Delta t)$ as the only unknown. These equations are solved stepwise in time starting at Δt using (20) to give $A(\Delta t)$. It is easily shown that, to the same order of accuracy, $C(\Delta t) = A(\Delta t)^2 / 2\Delta t$. With C , and hence A known, the cumulative infiltration can be determined from (11) using a convenient quadrature formula, for example, the trapezoidal rule. Accuracy is specified by suitable choice of Δt . The results, however, are limited to the class of soils defined by (8). One use of the exact solution is to check more general analytical approximations for infiltration. A general infiltration formula for a time-dependent head condition has been proposed (Parlange et al., 1985). The exact results presented here provide the opportunity to check directly the accuracy of this formula.

Before proceeding, we note that when the surface pressure head is constant, an infiltration equation of the form of (16) can always be derived following the approach used here, as shown by Barry et al. (1993b). For a time-dependent surface pressure head, such a simple formula is possible only in the case of the step function hydraulic conductivity, (12). Thus, the Green-Ampt assumption is extremely useful in simplifying the governing equation.

3.3. Infiltration Formula

Parlange et al. (1985) proposed an approximate infiltration formula for an arbitrary ponded, time-dependent surface condition, extending the earlier, less general, infiltration formula of Parlange et al. (1982). For infiltration into a dry soil, the Parlange et al. (1985) formula is:

$$I - \frac{K_s \psi_s(t) \Delta\theta (1 + \mu)}{\frac{dI}{dt} - K_s} = \frac{S_0^2}{2\delta K_s} \ln \left(1 + \frac{\delta K_s}{\frac{dI}{dt} - K_s} \right), \quad (25)$$

where S_0 is the sorptivity for a zero surface head, and μ and δ are defined, respectively, by

$$[S_0^2 + 2K_s \psi_s(0) \Delta\theta] \mu = \int_{-\infty}^{\psi_s} [\theta_s - \theta(\psi)] K(\psi) d\psi \quad (26)$$

and

$$\delta = 1 - \frac{1}{\Delta\theta K_s} \int_{-\infty}^{\psi_s} \frac{d\theta}{d\psi} K(\psi) d\psi. \quad (27)$$

The parameters μ and δ are constants for any particular soil, and so (25) is a first-order differential equation for I which is solved numerically. The most convenient solution method results from rewriting (25) as

$$\frac{dI}{dt} - K_s = \frac{\delta K_s}{\frac{B}{H} - 1}, \quad (28)$$

where

$$B = W_p \left[H \exp \left(H + \frac{2\delta K_s I}{S_0^2} \right) \right] \quad (29)$$

and

$$H = \frac{2\psi_s(t) K_s \Delta\theta (1 + \mu)}{S_0^2}. \quad (30)$$

The advantage of (28) is that it isolates the derivative term. Standard integration methods, such as the Runge-Kutta technique (e.g., Davis and Polonsky, 1964), can then be employed to solve (28) very easily and efficiently. The only difficulty with numerical integration of (28) is that $dI/dt \rightarrow \infty$ as $t \rightarrow 0$. However, in this limit, I is given by $S t^{1/2}$. In general, S will be unknown, although it is assumed that S_0 is known. Then, S can be approximated by (Green and Ampt, 1911)

$$S^2 = S_0^2 + 2K_s \psi_s(0) \Delta\theta. \quad (31)$$

Clearly, the approximation (31) has been used to derive (26). More accurate sorptivity approximations are available and have been extensively tested (Parlange, 1975; Barry et al., 1992; Parlange et al.,

1992, 1993), should they be required.

For a constant ψ_s , (28) can be integrated explicitly. The solution is

$$\frac{2t\delta K_s^2(1-\delta)}{S_0^2} = \delta \ln\left(1 + \frac{B-H}{\delta H}\right) (H\delta - H - 1) + \ln\left(\frac{B}{H}\right) - (H-B)(1-\delta) \quad (32)$$

Equation (32) provides a straightforward formula to calculate $I(t)$ for constant head infiltration.

3.3.1. Application to falling head infiltration. A simple calculation shows that (25) reduces to (16) in the limit $\delta \rightarrow 0$ and $\mu \rightarrow 0$ (Parlange et al., 1985). In other words, (25) is exact for the case of a Green-Ampt soil. As shown by (12), such soils are characterised by rapid changes in hydraulic conductivity around the air-entry value. To test (25) then, we choose a conductivity that is slowly varying, e.g.,

$$K = K_s \left(\frac{\psi_a}{\psi}\right)^2, \quad \psi < \psi_a \quad (33)$$

There is nothing particularly significant about this hydraulic conductivity, other than its slow variation. Indeed, it is unrealistic since the conductivity in most soils would vary much more rapidly than predicted by (33) (Reichardt et al., 1972).

Before solving (28), μ and δ must be calculated. Upon substituting (8) and (33), as necessary, into (26) and (27), we find, respectively,

$$\frac{-[S_0^2 + 2K_s\psi_s(0)\Delta\theta]\mu}{2K_s\psi_a\Delta\theta} = \frac{\psi_a^3 \ln\left(1 - \frac{X}{\psi_a}\right) + \psi_a^2 X + \frac{X^2\psi_a}{2} + \frac{X^3}{3}}{X\left[2\psi_a X + X^2 + 2\psi_a^2 \ln\left(1 - \frac{X}{\psi_a}\right)\right]} \quad (34)$$

and

$$\frac{(1-\delta)X^2}{2} = \frac{\psi_a^4 \ln\left(1 - \frac{X}{\psi_a}\right) + X\left(\frac{\psi_a X^2}{3} + \frac{X^3}{4} + \frac{X\psi_a^2}{2} + \psi_a^3\right)}{2\psi_a X + X^2 + 2\psi_a^2 \ln\left(1 - \frac{X}{\psi_a}\right)} \quad (35)$$

Equations (34) and (35) are functions of time. This is due to the time-dependence of the soil moisture characteristic curve, (8). Of course, for real soils, both μ and δ are constants. Also, the functional form of X means that (28) is no longer an explicit function of dI/dt , since (11) can be used to eliminate dA/dt and A from X . We found, however, that μ and δ are only slowly varying functions of time, and can be replaced by average values. This point will be addressed below. However, we note that using average values for these parameters is clearly a more stringent test of (28).

For the purpose of illustration, we used the following parameters to carry out the calculations: $\psi_s(0) = 10$ cm, $\psi_a = -1$ cm, $K_s = 1$ cm d⁻¹ and $\Delta\theta = 0.5$. With these values we first computed the exact cumulative infiltration using (7), (11), and (24). The exact results were used to compute μ and δ from (34) and (35). The behaviour of these parameters, along with that of X , is shown in Fig. 2. Note that in this example the surface ceases to be ponded at $t = 5.56$ d. Figure 2a shows that the μ varies only marginally, while the variation of δ is more marked (Fig. 2b). The variation of X is in the range $\psi_s(0) \geq X > \psi_a$ (Fig. 2c). A suitable average X for the calculation of the average μ and δ is clearly $X(2.28$ d). However, there is no reason that this time would be known *a priori*. A more reasonable choice is to calculate μ and δ using a fraction of $\psi_s(0)$, rather than X . For the sake of simplicity we selected $\psi_s(0)/2 = 5$ cm. Equations (34) and (35) then give $\mu(5$ cm) = 0.03 and $\delta(5$ cm) = 0.47. As can be seen from Fig. 2b, this

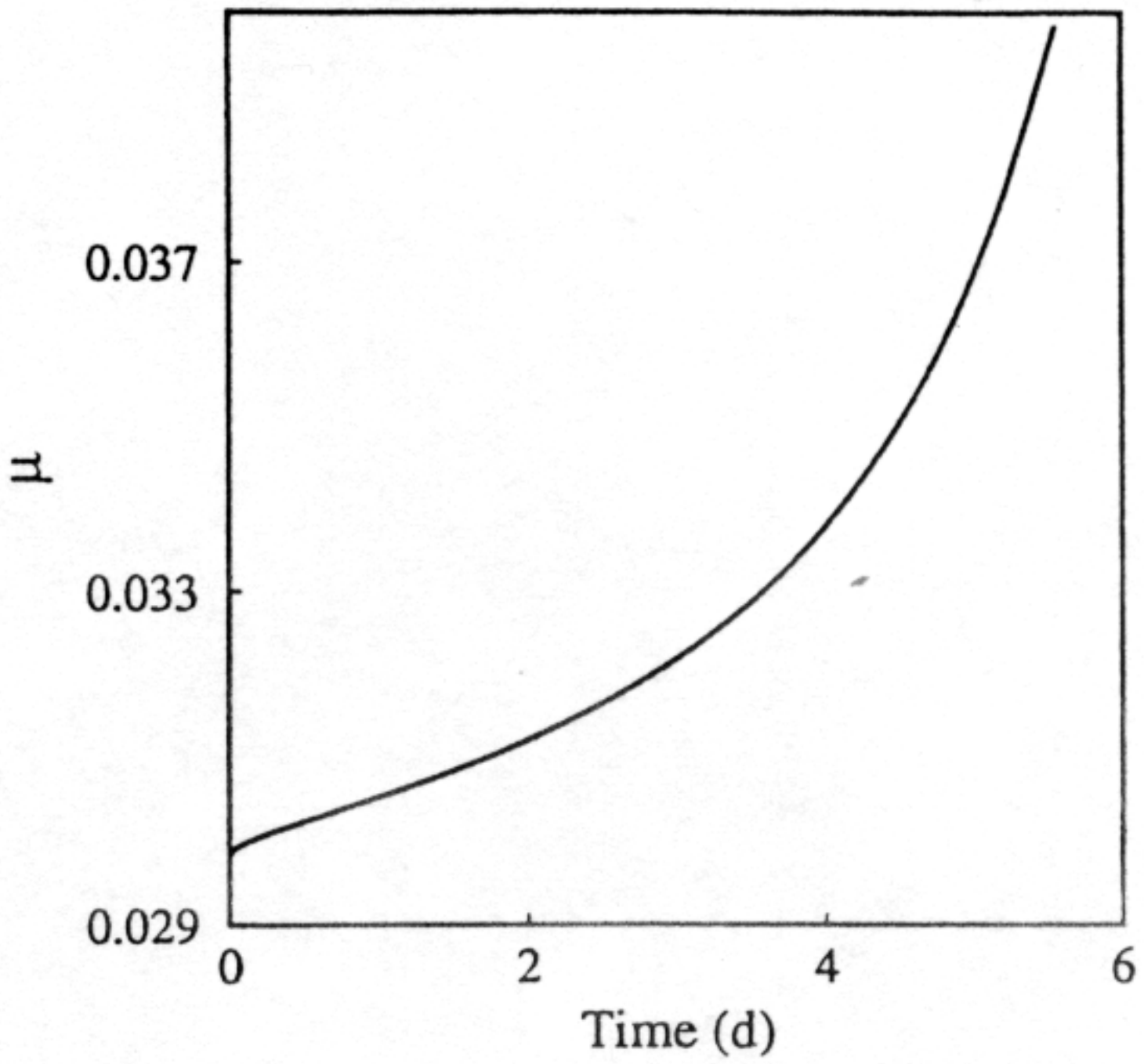


Figure 2a. Variation of μ with time for the falling head infiltration example.

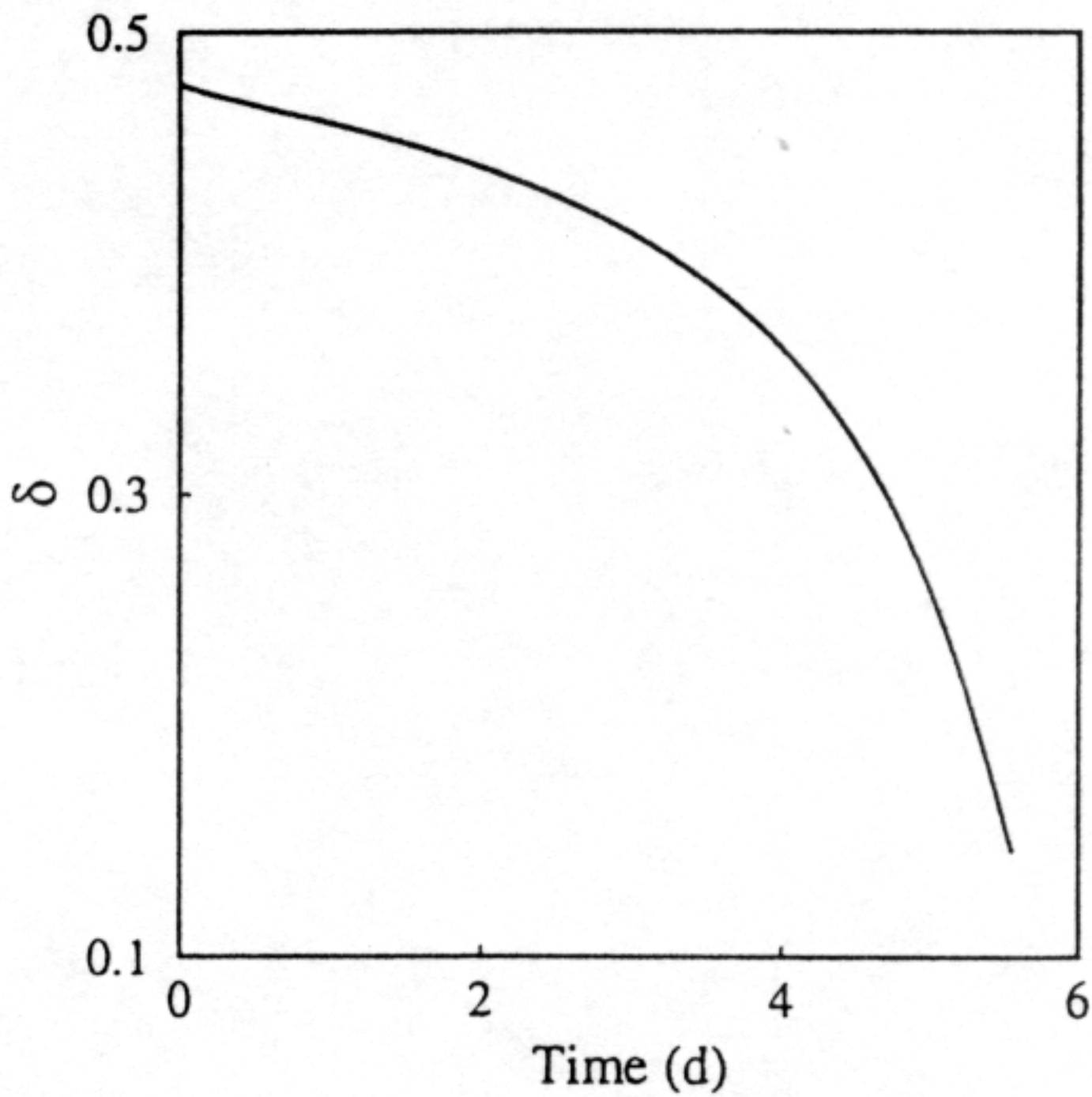


Figure 2b. Variation of δ with time for the falling head infiltration example.

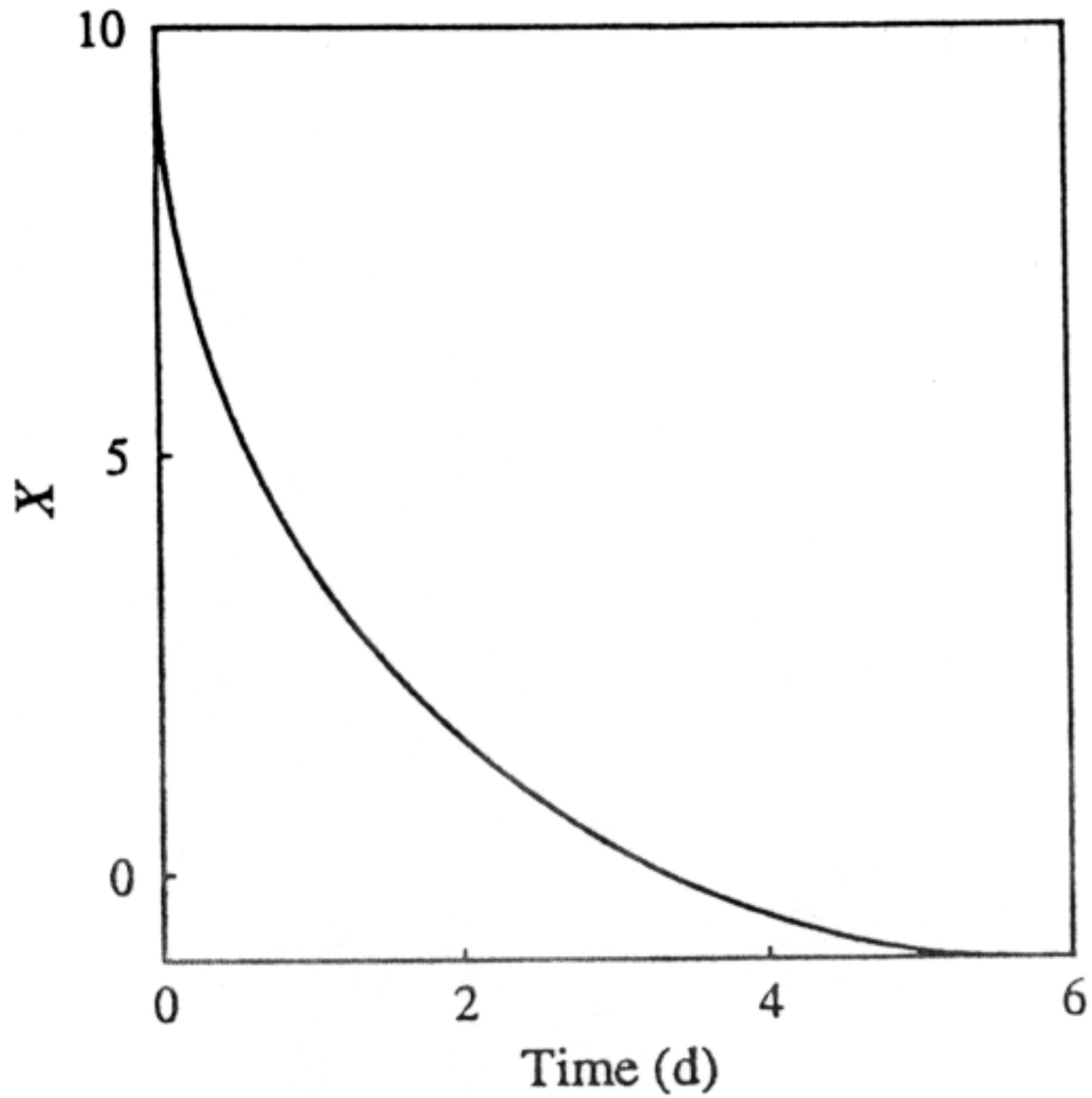


Figure 2c. Variation of X with time for the falling head infiltration example.

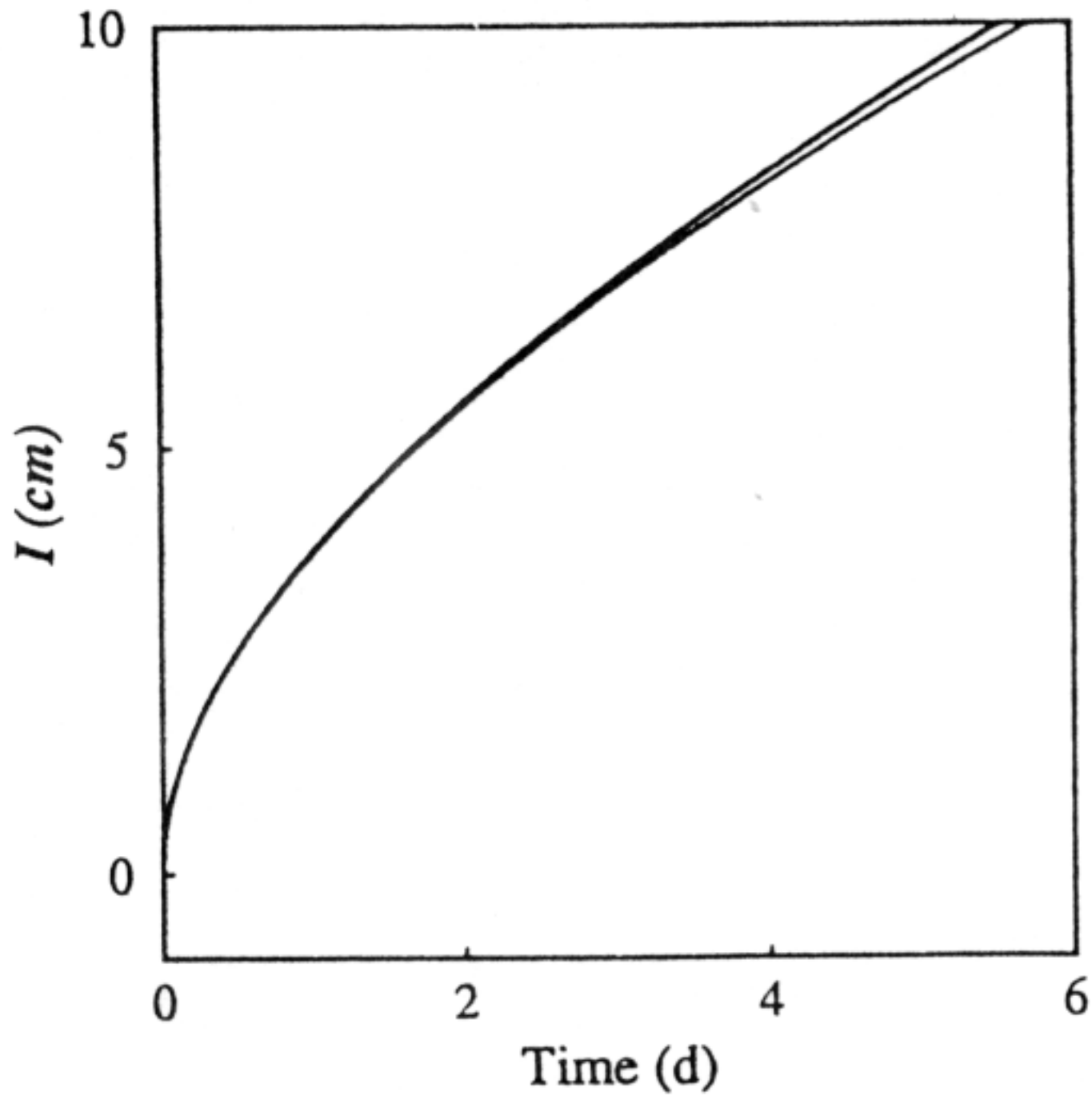


Figure 3. Comparison of exact (thick, upper) and approximate (thin, lower) cumulative infiltration curves for falling head infiltration.

is not a particularly judicious choice for δ , as it corresponds to $t = 0.6$ d.

Equation (28) was solved numerically using the Runge-Kutta method, with the parameter values discussed above. The prediction from (28), and the exact falling head solution (calculated following the discussion in section 3.2), is plotted for comparison in Fig. 3. The predicted infiltration is slightly less than the exact value, with a maximum relative error of approximately 2% occurring at the conclusion of the infiltration event. Given the artificial nature of the 'soil' used in this comparison, this result suggests that (28) is a robust approximation.

3.3.2. Application to a time-dependent surface condition. As a second example, we take the following condition:

$$\psi_s = \psi_a + \gamma + \beta \exp(-\epsilon t) \quad (36)$$

with the same conductivity function as above. In addition, γ and β were taken as 6 and 5 cm, respectively, with $\epsilon = 1 \text{ d}^{-1}$. Again, there is nothing particularly significant about this surface condition. As above, it was found from the exact solution that μ and δ varied little. Thus, X was approximated by ψ_s . The calculations were performed as already described. The cumulative infiltration is shown in Fig. 4. The error, like that in Fig. 3, grows with time at a small rate. The maximum relative error in the figure is less than 2%.

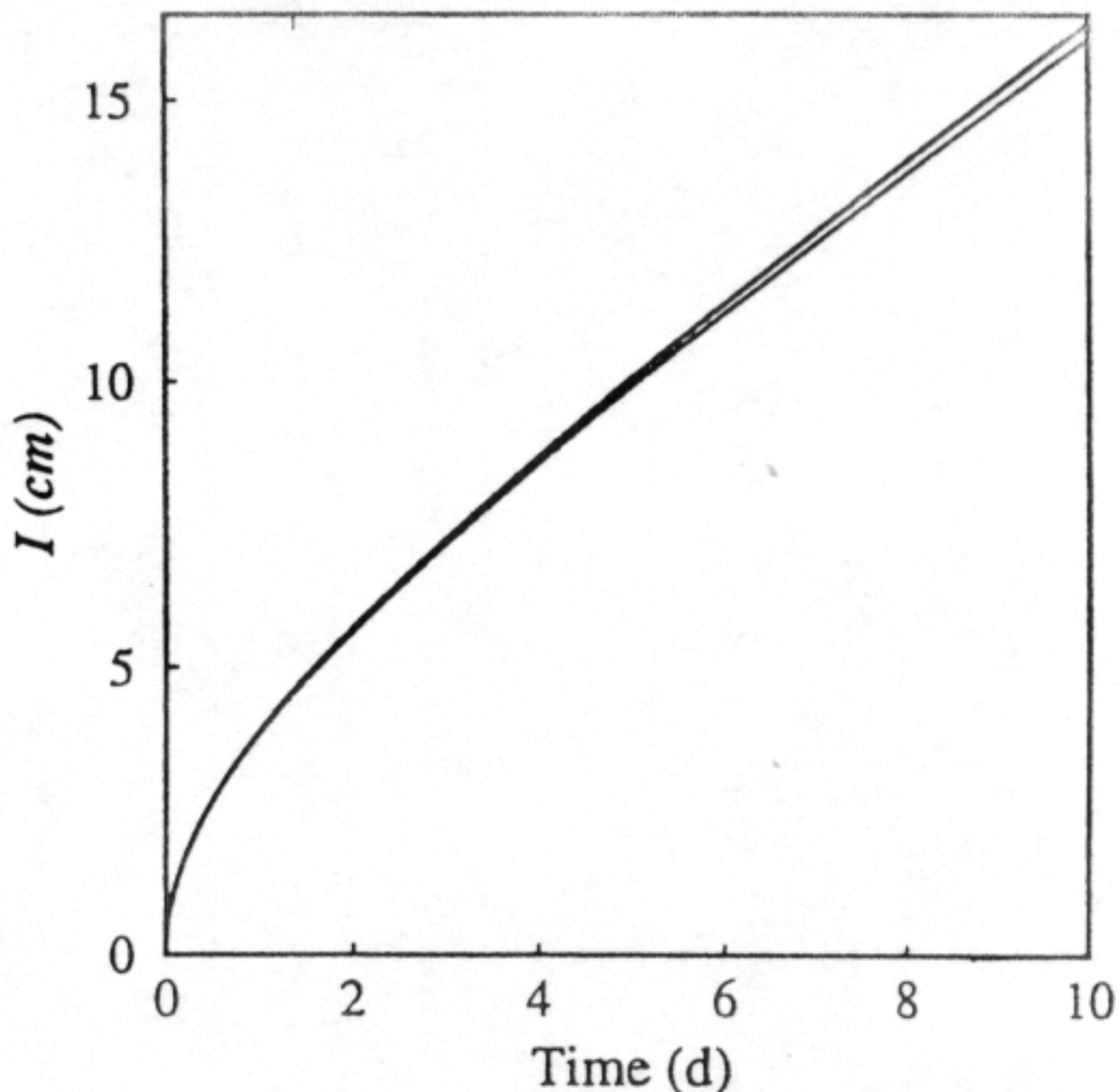


Figure 4. Comparison of exact (thick, upper) and approximate (thin, lower) cumulative infiltration curves for the time-dependent surface condition (36).

3.4 Drainage from a Soil Profile

We consider the saturated-unsaturated soil profile shown in Fig. 5. Initially, the external reservoir, which controls the head at the profile base, is located such that the air-water interface is z_f above the base. The z -axis has its origin at the profile and, as above, is oriented so that positive z is downwards. Within the soil profile, the soil is saturated from $z = 0$ up to $z = \psi_a - z_f$. Dagan and Kroszynski (1973) and Dagan (1989) consider the drainage that will occur from the soil profile if the external reservoir is

suddenly dropped so that the phreatic surface in the reservoir is aligned with the bottom of the profile, i.e., $z = 0$.

The drainage of the column is governed by Richards' equation (1). The initial time, $t = 0$, is taken as the instant the external reservoir is lowered. Then, the initial pressure head gradient is

$$\frac{\partial \psi}{\partial z} = 1 - \frac{z_I}{z_I - \psi_a}, \quad t = 0. \quad (37)$$

The solution to (1) is given by (3) with $\psi_s = 0$. Equations (3) and (37) show that

$$A(0) = 1 - \frac{z_I}{\psi_a}. \quad (38)$$

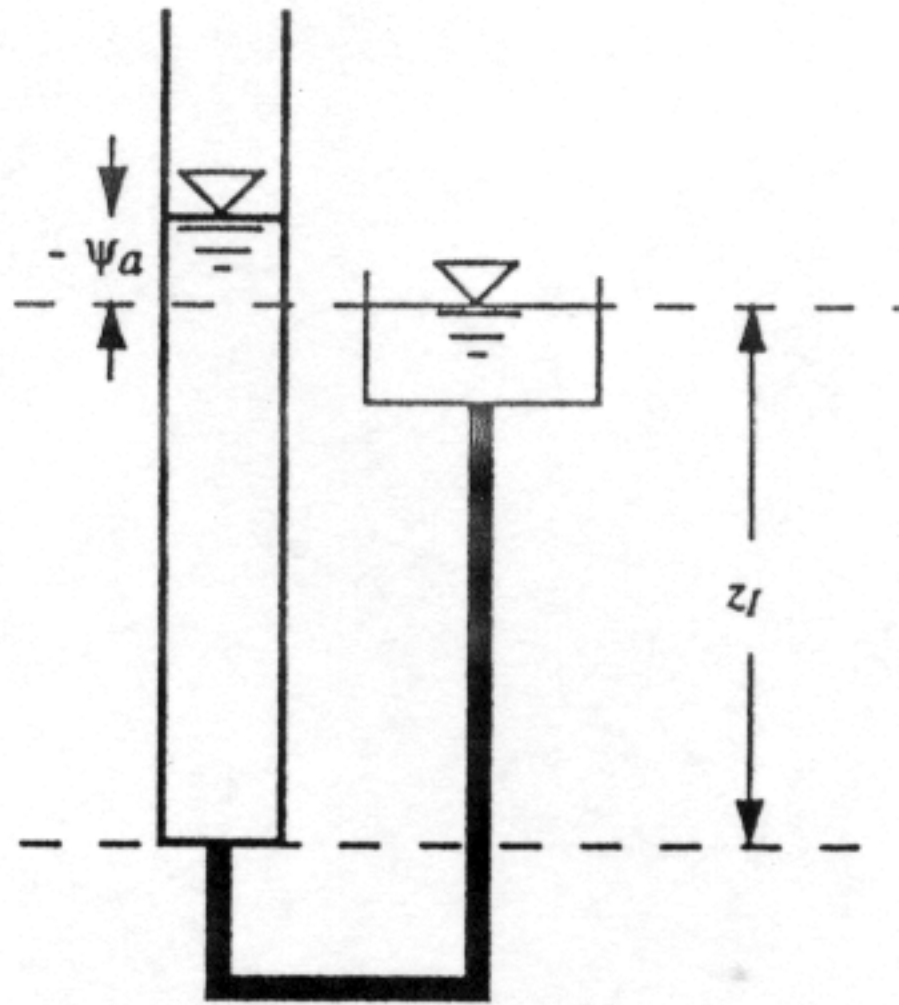


Figure 5. Initial configuration of the drainage problem: Due to the external fluid reservoir, the soil profile is saturated as shown to depth $z_I - \psi_a$. At $t = 0$, the external reservoir is lowered so that the phreatic surface of the reservoir fluid is aligned with the base of the soil profile.

Equation (6) is the solution for $A(t)$. Because ψ_s is constant, $C(t)$, which is calculated from (7), is a constant independent of t . Denote this constant value by C_c . Now, the relationship between the cumulative drainage, D_r and A is just (11), with I replaced by D_r . Integration of (11) yields

$$D_r = \Delta\theta\psi_a[A(t) - A(0)]. \quad (39)$$

With (39) and (6), D_r can be expressed as

$$\frac{D_r}{\Delta\theta\psi_a} + \ln\left(1 - \frac{D_r}{\Delta\theta z_I}\right) + C_c t = 0. \quad (40)$$

An explicit solution for D_r is available using the W function:

$$\frac{D_r}{\Delta\theta z_I} = 1 + \frac{\psi_a}{z_I} W_p\left[-\frac{z_I}{\psi_a} \exp\left(-\frac{z_I}{\psi_a} - C_c t\right)\right]. \quad (41)$$

Dagan (1989) derived an expression which, under suitable notation changes, is identical to (40). The predictions of the formula were compared with the extensive experimental data set of Vachaud (1968) with reasonably good results. However, to derive his result, Dagan (1989) assumed that the capillary fringe remains fixed during the entire drainage process. The derivation given here relies not on this assumption, but rather on the relationship (8). Clearly, the results could be extended to account for a time-dependent bottom boundary condition, along the lines of the analysis of the previous sections.

3.5 Application to Solute Transport

Barry et al. (1991) showed that if K is defined as (Gardner, 1958)

$$K = \frac{v}{B} \exp\left(\frac{v\psi}{D}\right), \quad (42)$$

then with the mapping

$$K = \frac{vc}{B} \quad (43)$$

and

$$\theta = \frac{c + s(c)}{B}, \quad (44)$$

(1) becomes

$$\frac{\partial [c + s(c)]}{\partial t} = D \frac{\partial^2 c}{\partial z^2} - v \frac{\partial c}{\partial z}, \quad (45)$$

which is the governing equation for solute transport with the nonlinear sorption isotherm, s (e.g., Bajracharya and Barry, 1993). Barry et al. (1993b) have presented a new solution for (45) based on (3) for ψ_s constant. Here, this solution is extended to the case of a time-dependent ψ_s .

The soil profile is considered to be initially solute free, i.e.

$$c(z,0) = 0, \quad z > 0, \quad (46)$$

while at the profile surface the solute concentration is a known function of time:

$$c(0,t) = c_0(t), \quad t > 0. \quad (47)$$

The solution to (45)-(47) is, from (3) and (43)

$$c = \begin{cases} c_B \exp\left(\frac{vz}{DA}\right), & z > z_f, \\ c_0, & z < z_f \end{cases} \quad (48)$$

where $c_B(t)$ is an isotherm fitting parameter defined by

$$c_B = \exp\left(\frac{v\psi_s}{D}\right) \quad (49)$$

and z_f is the position in the profile locating the solute front position where $c = c_0$,

$$z_f = \frac{AD}{v} \ln\left(\frac{c_0}{c_B}\right).$$

The surface concentration, c_0 , is related to the air-entry pressure head via

$$c_0 = \exp\left(\frac{v\psi_a}{D}\right). \quad (51)$$

Because c_0 is a time dependent, so to is ψ_a in (51). We note that this presents no difficulties in obtaining the solution from (43) as this time dependence enters in the calculation of $C(t)$, as given by (7). Equations (48)–(51) relate various quantities needed in the solution for (45)–(47) to the solution already obtained for Richards' equation (1). The isotherm, s , for which (48) is an exact solution is then

$$\frac{s(c) + c}{s(c_0) + c_0} = \frac{Ei\left[\ln\left(\frac{c}{c_B}\right) - \frac{A \frac{dc_B}{dt}}{c_B \frac{dA}{dt}}\right]}{Ei\left[\ln\left(\frac{c_0}{c_B}\right) - \frac{A \frac{dc_B}{dt}}{c_B \frac{dA}{dt}}\right]}, \quad (52)$$

where Ei is the exponential integral (Spanier and Oldham, 1987). Note that, to derive (52), we used the relationships:

$$\theta(c) = -Ei\left[\ln\left(\frac{c}{c_B}\right) - \frac{A \frac{dc_B}{dt}}{c_B \frac{dA}{dt}}\right], \quad (53)$$

and

$$B = \frac{-[s(c_0) + c_0]}{Ei\left[\ln\left(\frac{c_0}{c_B}\right) - \frac{A \frac{dc_B}{dt}}{c_B \frac{dA}{dt}}\right]}. \quad (54)$$

Since c_0 is defined by the boundary condition (47), the isotherm given by (52) has c_B as a fitting parameter, as already noted. Barry et al. (1993b) have shown that this isotherm, for c_B constant, is similar to the lower portion of the S-curve isotherm (Sposito, 1989; Barry, 1992). Such isotherms occur, e.g., in cation exchange processes (Schweich et al., 1983). In addition, Barrow (1989) has pointed out that time-dependence of solute sorption in soils is much more common than equilibrium sorption. Thus, it appears that the time dependence of the adsorption isotherm, (52), allows for more realistic modelling of solute sorption behaviour. Finally, the total amount of solute, I_c , that has entered the soil at any time is found by integrating the solution (48) over the spatial domain. The result is

$$I_c = c_0 \left(z_f - \frac{AD}{v}\right). \quad (55)$$

4. Conclusions

There are very few exact solutions for Richards' equation, the governing equation for unsaturated water flow. Recently, some recent exact solutions have been derived using specific forms of the diffusivity and conductivity functions (Rogers et al., 1983; Broadbridge and White, 1988; Sander et al., 1988; Barry and Sander, 1991; Warrick et al., 1990, 1991). The solution presented here, and that of Barry et al. (1993b), do not depend on specific soil hydraulic functions, but on a relationship between the soil moisture characteristic curve, the hydraulic conductivity, and the surface pressure head. Of course, this relationship is constraining, and the soil to which the solution applies is, consequently, of an artificial nature. On the other hand, the solution obtained is relatively simple. Hence, it should be of use in checking numerical schemes, where no other checks are possible due to the lack of analytical solutions for nonlinear forms of Richards' equation. For the draining soil profile, a particularly simple drainage formula can be derived. Finally, the solution for infiltration has been used to derive an exact solution for nonlinear sorption of a solute moving in a soil profile. The sorption isotherm depends on time through the surface condition, as well as through a fitting parameter. Again, this solution is one of a very few exact solutions for nonlinear solute adsorption and transport.

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