OPTIMAL SOLUTIONS OF THE LINEARISED DIFFUSION ROUTING MODEL

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1. INTRODUCTION

The solution of complete dynamic wave models is computationally extensive [1]. To simplify the problem, inertial forces may be neglected since they are negligible for most flood waves. On the other hand, pressure forces are quite important [2]. The diffusion models so obtained are then solved by computationally efficient schemes. Further simplification is obtained when both inertial and pressure forces are neglected, giving the kinematic model. The finite-difference solution of the latter model induces numerical dispersion. Solutions of the full diffusion wave model can be simulated by controlling the numerical dispersion. This is the basis of the Muskingum-Cunge (M-C) method [3], a well known flood routing model. Since the method is conceptually simple, computationally efficient and relatively accurate, it has been widely used in practice.

In the M-C model, the physical parameters are dependent on grid characteristics, particularly the spatial and temporal step sizes. The question of how to choose the optimal spatial and temporal steps in the numerical scheme then arises. Ponce and Theurer [4] state that the effect of the spatial step size on the accuracy of the M-C model is not completely understood. Indeed, there is still continuing controversy regarding this approach [5]. The main purpose of this study is to show that solutions of the kinematic wave equation following the M-C approach always give accurate results once a straightforward condition is imposed. For this purpose, the kinematic wave problem is analysed starting from a general scheme. Additionally, the problem is reanalysed to confirm whether the general scheme can attain third-order accuracy. We show that the Courant number is the only determining criterion in fixing the accuracy of the results. There is one Courant number for which the solution is optimal. Given this Courant number and a value for the physical diffusion coefficient, the optimal spatial and temporal step sizes are fixed. This puts an end to the controversy regarding the selection of size of the spatial and temporal steps in the solution of the diffusive wave equation.

2. THEORY

2.1 Models

We briefly describe the complete dynamic routing model and the various approximations applied to it. The continuity equation which describes the conservation of fluid mass in a gradually varied, unsteady open channel flow (with no lateral inflow) is [1]

$$B\frac{\partial y}{\partial t}+\frac{\partial (AV)}{\partial x}=0, \qquad (1)$$

where A, B, V and y are, respectively, the cross-sectional area of flow, surface width of flow, velocity and depth of flow, x is the distance along the channel and t the time. The momentum equation is given by [1]:

$$\frac{1}{g}\frac{\partial V}{\partial t}+\frac{V}{g}\frac{\partial V}{\partial x}+\frac{\partial y}{\partial x}+S_f-S_o=0, \tag{2}$$

where g, S_f and S_o are the magnitude of the acceleration due to gravity, friction slope and bed slope, respectively. The continuity equation (1) and the momentum equation (2) constitute the Saint-Venant equations for gradually varied, unsteady open channel flow. By neglecting the inertial term it is possible to combine equations (1) and (2) to give an advection-dispersion equation [1]:

$$\frac{\partial Q}{\partial t}=D\frac{\partial^2 Q}{\partial x^2}-c\frac{\partial Q}{\partial x}, \tag{3}$$

where c = (dQ/dy)/B and D = $Q/2BS_0$ for regular channels. Often, both c and D are linearised about some selected value of Q, in which case equation (3) becomes linear.

When both inertial and pressure forces are neglected then the equations (1) and (2) reduce to the kinematic wave equation [6]:

$$\frac{\partial Q}{\partial t}=-c\frac{\partial Q}{\partial x}. \tag{4}$$

When the celerity, c, is constant, equation (4) becomes linear else it is nonlinear. It is intended here to study the solution of equation (3), the approximate dynamic model, using the general scheme described in next section. Diffusive waves apply to a wider range of practical problems than kinematic waves because most flood waves have a small amount of physical diffusion [3]. The initial and boundary conditions used in this study are:

$$\text{Case 1: } Q(x,0)=0,\ Q(0,t)=Q_o;\ \text{Case 2: } Q(x,0)=Q_o,\ Q(0,t)=Q(t). \tag{5}$$

We address two issues here: (i) the order of accuracy of the general scheme used to solve equation (4) and (ii) the optimal temporal and spatial steps used in the numerical solution of equation (3) derived making use of (i). Like other comparable analyses [7], the present study is limited to the linear case.

2.2 Finite difference solution

We seek the solution for equation (3) using the kinematic wave equation (4). Equation (4) is hyperbolic and when solved numerically by finite differencing, numerical dispersion is induced. This numerical dispersion is equated to the physical diffusion D in equation (3). Space-weighted forward differencing for the temporal derivative and time-weighted backward differencing for the spatial derivative reduces equation (4) to:

$$Q(i,j+1)=\frac{1-\theta-C_r(1-\omega)}{1-\theta+\omega C_r}Q(i,j)+\frac{\omega C_r-\theta}{1-\theta+\omega C_r}Q(i-1,j+1)+\frac{\theta+C_r(1-\omega)}{1-\theta+\omega C_r}Q(i-1,j), \tag{6}$$

where i and j are the spatial and temporal coordinates ($x = i\Delta x$, $t = j\Delta t$), θ is the spatial weighting factor, ω is the temporal weighting factor, C_r (= $c\Delta t/\Delta x$) is the Courant number, and Δt and Δx are the temporal and spatial steps in the finite difference grid. The scheme is stable subject to the condition $1 - 2\theta + C_r(2\omega - 1) \geq 0$ [8]. The M-C method is a special case of equation (6) for $\omega = 1/2$. We next present relevant relations to arrive at optimal values of the spatial and temporal steps.

2.3 Consistency analysis

Taylor series expansions about the (i,j) grid point where the mixed derivatives are expanded in terms of spatial derivatives [9], give the following third-order accurate equation:

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} = \frac{c\,\Delta x}{2}\left[1 - 2\theta + (2\omega - 1)\,C_r\right]\frac{\partial^2 Q}{\partial x^2} +$$
$$\frac{c\,\Delta x^2}{2}\left\{\left[(1-2\omega)\,C_r + 2\theta\right]\left[\frac{1}{2} - \theta + \left(\omega - \frac{1}{2}\right)C_r\right] + \left(\frac{1}{2} - \omega\right)C_r - \frac{C_r^2}{6} + \theta - \frac{1}{3}\right\}\frac{\partial^3 Q}{\partial x^3} + O\left(\Delta x^3, \Delta t^3\right). \tag{7}$$

Note that cross-derivative terms arising in the Taylor series expansion in equation (6) were eliminated using the approximated equation itself rather than the original exact equation (4) [9]. Two other third-order error analyses of equation (6) have been reported [3,7], neither of which agree with equation (7). Ponce [3] reported that for $C_r = 1$, and $\theta = \omega = 1/2$, the scheme given by equation (6) is third-order accurate. We note that the truncation error reported in [7] does not comply with that of [3]. In [7], for any value of θ and ω, the scheme is third-order accurate if $C_r = 1$. However, from above, the scheme is third-order accurate if $C_r = 1$ and $\theta = \omega$. Under these conditions there is no numerical dispersion and the solution coincides with equation (4). We are interested in the solution of equation (3) and hence the scheme for equation (4) must include numerical dispersion. Clearly, there is great freedom in choosing the parameters C_r, θ and ω so as to ensure the $O(\Delta x^2)$ term in equation (7) is removed.

It is evident from equation (7) that numerical dispersion is induced by the scheme in equation (6). To approximate the solution to equation (3), the numerical dispersion coefficient (of the second-order spatial derivative) is equated to the physical diffusion coefficient in equation (3). Third-order accuracy can be obtained by equating the coefficient of the third-order spatial derivative to zero, i.e., if

$$\omega = \frac{C_r + \sqrt{(1 - C_r^2)/3} + 2\theta - 1}{2C_r}. \tag{8}$$

Note that, by imposing equation (8), one of the three parameters C_r, θ and ω in equation (7) can be removed.

2.4 Additional constraint

An additional relation can be obtained from the concept of column holdup as often used in solute transport [10]. This concept is equally applicable to this problem. The holdup or storage is calculated for a constant step input of discharge Q_o at the inlet with $Q(x,0) = 0$, as the initial condition. This is the most critical boundary condition because it imposes a discontinuity at the entrance. For a semi-infinite spatial domain, the holdup, H, at a distance Δx is given by [11]

$$H = \frac{c}{\Delta x}\int_0^\infty 1 - Q(\Delta x, t)/Q_o \, dt = 1. \tag{9}$$

We shall force all solutions based on equation (6) to honour equation (9). It is in this sense that our scheme is optimal. To this end, equation (6) is rewritten in the form of a summation. Then, making use of Simpson's rule, the numerically computed holdup is

$$H_{num} = \frac{\omega(2 - 3\omega)C_r^2 + (2\theta - 3)(3\omega - 1)C_r + 3(2 - \theta)(\theta - 1)}{3\left[2(\theta - 1) + (1 - 2\omega)\,C_r\right]}. \tag{10}$$

Note that the above relation was obtained using $Q(0,0) = Q_o/2$, so as to account for the discontinuity in Q at the x,t origin [12]. This numerical holdup is equated to the exact holdup to obtain the following relationship between θ and ω:

$$\theta = C_r\, \omega. \tag{11}$$

For the M-C method, $\omega = 1/2$ so from equation (11) we predict that optimal results will be obtained for $\theta = C_r/2$. Equation (11) reduces equation (6) to the simple explicit scheme,

$$Q(i,j+1) = (1 - C_r)Q(i,j) + C_r\, Q(i-1,j). \tag{12}$$

Equation (12) has quite often been used and is popularly known as a mixing cell model [13]. Also, when equation (11) is used in equation (8), the latter equation becomes independent of both θ and ω. For the special cases of $C_r = 1/2$ or 1, the scheme given by equation (6) is third-order accurate. Observe from equation (7) that, with equation (11), the numerical dispersion coefficient reduces to $D_{num} = c\, \Delta x\, (1 - C_r)/2$. Since $C_r = 1$ does not permit numerical dispersion in equation (7) it is not of interest here. Thus, to maintain third-order accuracy we set $C_r = 1/2$ in which case the spatial step size is $\Delta x = 4D/c$. This relation, if used, and the definition of C_r set both the spatial and temporal steps. There is no flexibility, for example, to adjust Δx to equal the reach length for reasons of computational simplicity. Other step sizes can be selected but only second-order accuracy will be maintained.

3. NUMERICAL RESULTS

We discuss two examples of M-C method here. The first is the case of a step input of constant discharge at the reach entrance. Second, we consider the physically realistic case of a time-dependent entrance condition. The first example was simulated using the parameter values $c = 1$ m d^{-1} and $D = 40$ m^2 d^{-1}. The breakthrough curve was generated taking Δx as the reach length while maintaining third-order accuracy i.e., $C_r = 1/2$. For the M-C model, $\omega = 1/2$ and hence $\theta = 1/4$. The spatial and temporal steps are 160 m and 80 d, respectively. Recall that for $\omega = 2\theta$, equation (6) reduces to equation (12). To see how the results are affected if condition (11) is not maintained, the problem was solved again using $C_r = 1/10$. For $\omega = 1/2$ and $\theta = 1/4$, the spatial step remains 160 m while the time step is reduced to 16 d, i.e., Δt is reduced by a factor of 5. Both sets of results were compared with the exact solution [14]. The breakthrough curves and the relative error (%) versus time are shown in Figure 1. The relative error is defined as the ratio of absolute value of the difference between the exact and the numerically computed value to the exact value. Clearly, the errors are quite significant initially when condition (11) is not satisfied (shown by dashes in Figure 1b). Discharge can attain negative values if condition (11) is not met. We observe that the errors are minimal when condition (11) is satisfied. Also, the column holdup is 0.902 for the case where equation (11) is not satisfied, whereas it is unity, by definition, for the optimal scheme.

For the second example the following variable boundary condition was used [15]

$$Q(0,t) = 125 - 75\, \mathrm{Cos}\left(\frac{\pi t}{48}\right),\ 0 \le t \le 96; \quad Q(0,t) = 50,\ t > 96. \tag{13}$$

In this case, the values of c and D were assumed to be 1 m d^{-1} and 2 m^2 d^{-1} respectively. We consider only second-order accuracy and hence any value of C_r can be taken. For $\omega = 1/2$ and $C_r = 4/5$, we have $\theta = 2/5$ to maintain condition (11). For this case $\Delta x = 20$ m and $\Delta t = 16$ d. Similarly, the solution was calculated by reducing Δt, i.e., equation (11) was not satisfied. The time step, Δt, was reduced to 1 d, so that $C_r = 1/20$. Both solutions were compared with an "exact" numerical solution computed using a standard Crank-Nicolson scheme. Figure 2b shows the relative error versus time for this case. It is seen that most of the time the errors are minimal when equation (11) is satisfied. We note that the difference in the breakthrough curve is significant in the first example (Figure 1a) because of the more critical boundary condition. In the second example, equation (13) is continuous and the difference is less severe (Figure 2a). However, the reduced relative error in each case underlines the robustness of equation (11). Note that the discharge values are below the base flow of 50 m^3 d^{-1} initially when equation (11) was not satisfied.

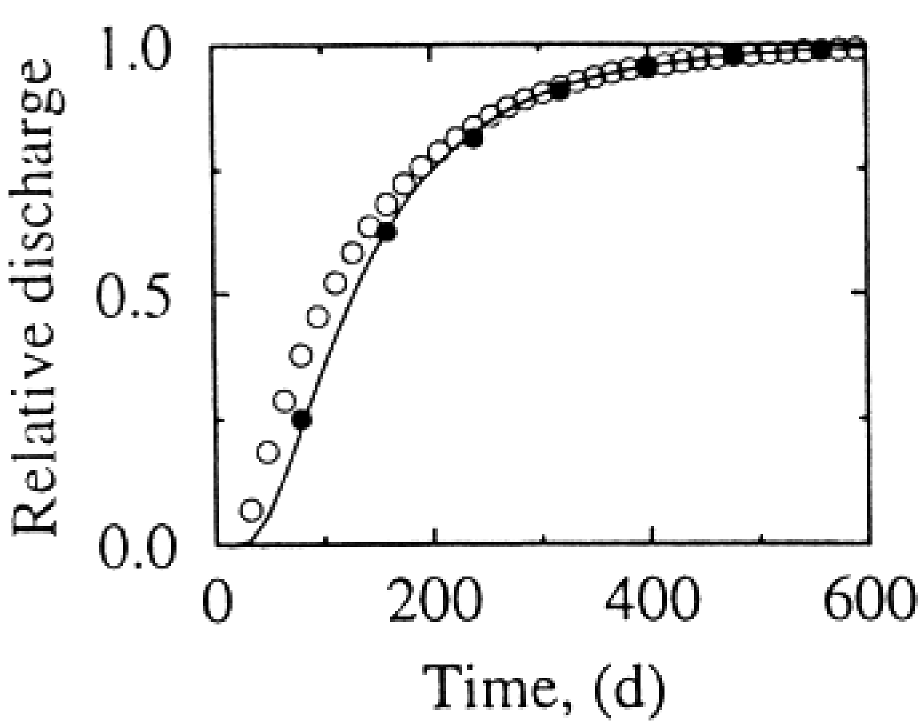

Figure 1a.
Discharge versus time computed by maintaining (C_r = 1/2, solid circles), and without maintaining (C_r = 1/10, open circles) equation (11). Line shows the exact solution.

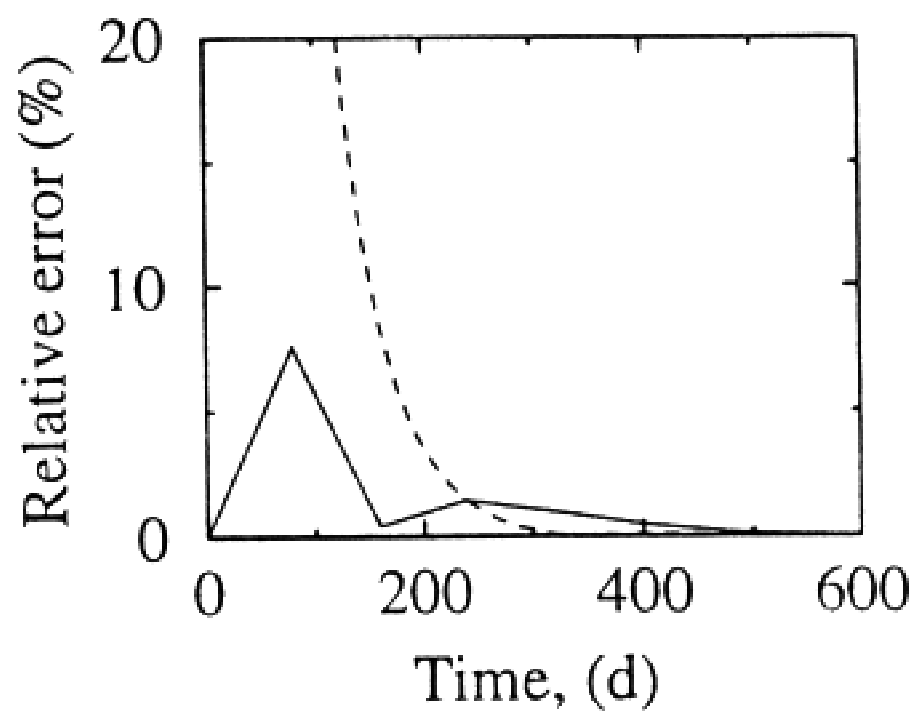

Figure 1b.
Relative error (%) versus time for a step input of discharge maintaining (C_r = 1/2, line) and without maintaining (C_r = 1/10, dashes) equation (11).

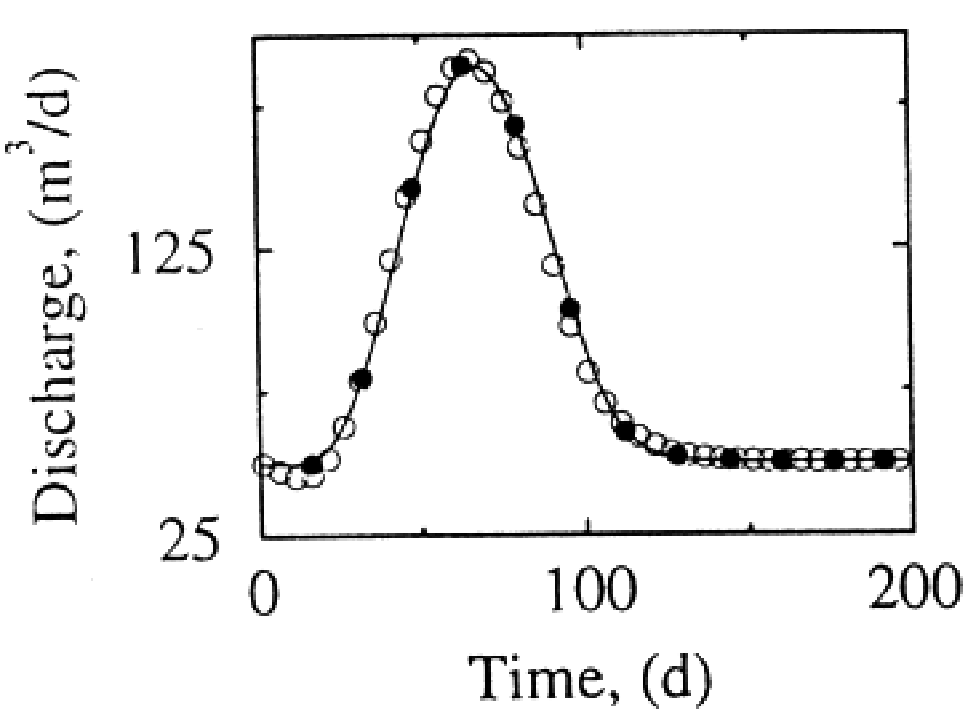

Figure 2a
Discharge versus time computed by maintaining (C_r = 4/5, solid circles), and without maintaining (C_r = 1/20, open circles) equation (11). Line shows the Crank-Nicolson solution.

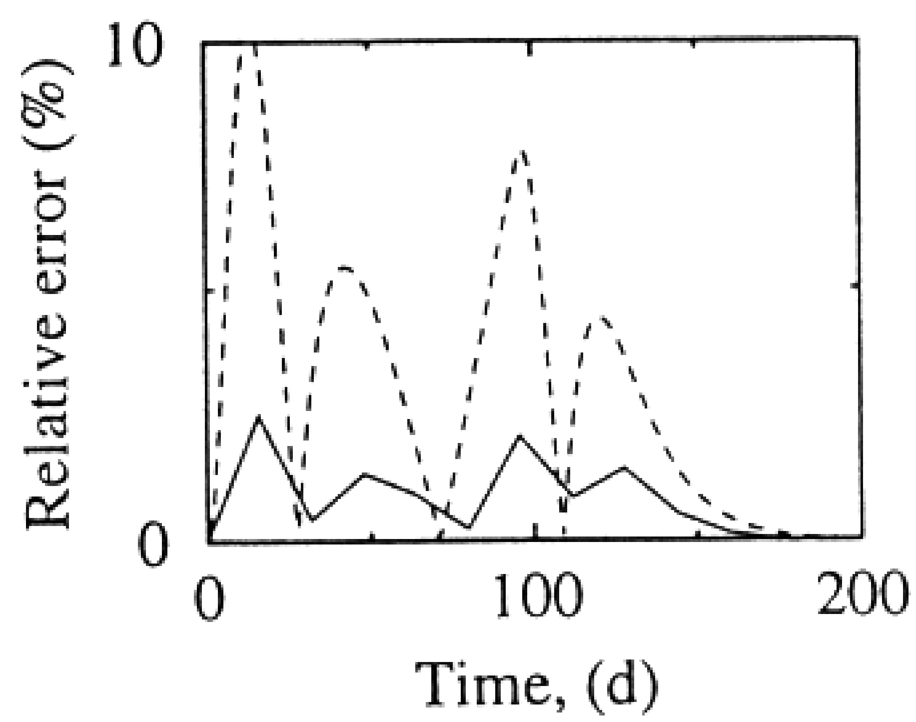

Figure 2b.
Relative error (%) versus time for continuous time dependent input equation (13) maintaining (C_r = 4/5, line) and without maintaining (C_r = 1/20, dashes) equation (11).

4. CONCLUSIONS

The truncation error of a widely used hydraulic routing numerical scheme has been calculated to third order. The scheme, which depends on three parameters, θ, ω and C_r, will be third-order accurate if condition (8) is imposed. Furthermore, forcing the numerical solution to produce the correct holdup leads to condition (11), relating these parameters. With condition (11), it has been shown that equation (6) is third-order accurate only for $C_r = 1/2$ and 1. When $C_r = 1$, there is no numerical dispersion and hence equation (3) cannot be satisfied. In that case $C_r = 1/2$ is the optimal Courant number, giving the numerical dispersion of $c\Delta x/4$. These conditions thus fix both the spatial and temporal steps. The consequence of maintaining equation (11) is that the only determining factor controlling the accuracy of solutions for equation (3) (solved with equation 6) is the Courant number, C_r. In other words, the simple explicit equation (12) is the best scheme to solve equation (3). In flood routing calculations, engineers routinely fix the reach length, Δx. In that case the results are second-order accurate unless C_r happens to be 1/2.

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