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INVERSE APPROXIMATION THEOREMS FOR DIRICHLET SERIES IN $AC(\overline{D})$

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We consider functions $f \in AC(\overline{D})$ on a convex polygon $D \subset \mathbb{C}$ and their Dirichlet expansion

$$f(z) \sim \sum_{\lambda\in\Lambda} \kappa_f(\lambda)\frac{e^{\lambda z}}{L'(\lambda)}.$$

The order of convergence is related to the regularity of f with respect to Tamrazov's moduli of smoothness. We give an extension of the inverse approximation theorem by Mel'nik in [5] with respect to moduli of arbitrary order $k \in \mathbb{N}$.

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1. Introduction

Let $\overline{D} \subset \mathbb{C}$ be a closed convex polygon with vertices at the points $a_1, \ldots, a_N \in \mathbb{C}$, $N \geq 3$. Let $\partial D = \overline{D} \setminus D$ denote the boundary of $\overline{D}$ and let $D = \overline{D} \setminus \partial D$ be its open part, such that $0 \in D$.

By $AC(\overline{D})$ we denote the Banach space of all functions $f(z)$ regular on D and continuous on $\overline{D}$ with finite norm of uniform convergence

$$\|f\|_{AC(\overline{D})} := \sup_{z\in\overline{D}} |f(z)|.$$

The class $AC^q(\overline{D})$ contains all functions f holomorphic in D with $f^{(q)} \in AC(\overline{D})$.

Consider the quasipolynomial

$$L(z) = \sum_{k=1}^{N} d_k e^{a_k z},$$

where $d_k \in \mathbb{C} \setminus \{0\}$, $k = 1, \dots, N$. We denote by Λ the set of zeros of the quasipolynomial L. We shall need estimates for those zeros as well as for the corresponding complex exponentials. The following results due to Leont'ev are well-known [4, Ch. 1, §2], [6]:

a) The zeros $\lambda_n^{(j)}$ of L with $|\lambda_n^{(j)}| > C$ for sufficient large C have the form $\lambda_n^{(j)} = \widetilde{\lambda}_n^{(j)} + \delta_n^{(j)}$, where

$$\widetilde{\lambda}_n^{(j)} = \frac{2\pi n i}{a_{j+1} - a_j} + q_j e^{i\beta_j} \tag{1}$$

and $|\delta_n^{(j)}| \leq e^{-an}$. Here $0 < a = \text{const}$, $j = 1, \dots, N$, $n > n_0$, and $a_{N+1} := a_1$. The parameters β_j and q_j are given by

$$e^{q_j(a_{j+1}-a_j)e^{i\beta_j}} = -\frac{d_j}{d_{j+1}}, \quad \text{where} \quad d_{N+1} := d_1.$$

Hence, these zeros are simple. The set of zeros Λ can be represented in the form

$$\Lambda = \{\lambda_n\}_{n=1,\dots,n_0} \cup \left(\bigcup_{j=1}^{N} \{\lambda_n^{(j)}\}_{n=n(j),n(j)+1,\dots} \right).$$

b) There are positive constants A_1 and c_1 such that for all $n \geq n(j)$ and all $\xi \in [\,a_j, a_k\,]$ we have

$$\left| e^{-\lambda_n^{(j)}(\xi - a_k)} - e^{-\widetilde{\lambda}_n^{(j)}(\xi - a_k)} \right| \leq A_1 \cdot e^{-c_1 n}.$$

Here $[\,a_j, a_k\,]$ denotes the line between the vertices a_j and a_k in the complex plane.

c) There is a constant $c_2 > 0$ such that there exists a positive constant A with

$$\left| \frac{e^{\lambda_n^{(j)} z}}{L'(\lambda_n^{(j)})} - (-1)^n B_j e^{\widetilde{\lambda}_n^{(j)}\left(z - \frac{a_{j+1}+a_j}{2}\right)} \right| \leq A\, e^{-c_2 n} \quad \text{for all } n > n_0.$$

Here all $B_j \neq 0$ are constant, $j = 1, \dots, N$. This inequality is true for all $z \in \overline{D}$.

The proof can be deduced from the results in [4, Ch. 1, §2]. For simplicity reasons we assume that all these zeros are simple.

The family $\mathcal{E}(\Lambda) := \{e^{\lambda z}\}_{\lambda \in \Lambda}$ of complex exponentials generated from the zeros Λ of L is complete in the subspace of all $f \in AC(\overline{D})$ with $\sum_{k=1}^{N} d_k f(a_k) = 0$. To every such f we can assign the Dirichlet expansion

$$f(z) \sim \sum_{\lambda \in \Lambda} \kappa_f(\lambda) \frac{e^{\lambda z}}{L'(\lambda)}, \tag{2}$$

where

$$\kappa_f(\lambda) = \sum_{k=1}^{N} d_k e^{a_k\lambda} \int_{a_j}^{a_k} f(\eta) e^{-\lambda\eta}\, d\eta \tag{3}$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N} d_k (a_k - a_j) \int_0^{2\pi} f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right) e^{-\lambda\left(\frac{a_j-a_k}{2\pi}\,\theta\right)}\, d\theta \tag{4}$$

are the Leont'ev coefficients. In (3) the index $j = 1, \ldots, N$ is arbitrary, but fixed. Dzjadyk showed in [1] that for those f the series (2) converges absolutely for all $z \in \overline{D}$ and uniformly to f, if $\int_0^\delta \frac{\omega(t)}{t}\, dt < \infty$ is satisfied for the first modulus of continuity of f and all $\delta > 0$. Dzjadyk proved this result for $d_k = 1$ for all k, but this is inessential. Many further deep results on Dirichlet series are due to Leont'ev and can be found in his monograph [4].

We know [3] that the partial series, weighted with the Jackson kernel, approximate $f \in AC(\overline{D})$ in the order of the modulus of continuity. The question in this paper is, if the converse is also true: Let a Dirichlet series approximate some function $f \in AC(\overline{D})$ in a certain order. What can be deduced on the order of the modulus of continuity of f?

This question was first posed by Mel'nik in [5] and solved there for the first moduli of continuity. In this paper we answer this question for moduli of arbitrary order $k \in \mathbb{N}$.

In the following section we give the rate of approximation of the series (2) weighted with the Jackson kernel with respect to the moduli of smoothness defined by P. M. Tamrazov. The inverse theorem of Mel'nik for first moduli is given in the next section. Then we shall extend his result to moduli of arbitrary order. The last section contains the proofs.

2. Approximation with Jackson weights

2.1. Moduli of smoothness and classes of regularity

To estimate the regularity of functions in $AC(\overline{D})$ we consider appropriate moduli of smoothness, which were introduced in [10] by Tamrazov. We state the definition for more general compacta than convex polygons, since we shall use this later in our proofs.

Let $\overline{K} \subset \mathbb{C}$ be a connected compactum with rectifiable boundary such that the open interior of $\overline{K}$ is a Jordan domain or the empty set. Let $\xi \in \overline{K}$, $r \in \mathbb{N}$, $\delta > 0$ and $A > 0$. Let $U(\xi, \delta) := \{z \in \mathbb{C} : |z - \xi| \leq \delta\}$ be the closed ball with center ξ and radius δ. We denote by $T(\overline{K}, \xi, r, \delta, A)$ the set of all vectors $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with

(i) $z_i \in \overline{K} \cap U(\xi, \delta)$ for all $i = 1, \ldots, r$, and

(ii) $|z_i - z_j| \geq A\delta$ for all $i \neq j$, $i, j = 1, \ldots, r$.

If there is no vector satisfying these conditions we define $T(\overline{K}, \xi, r, \delta, A) := \emptyset$. Nevertheless, for $A = 2^{-r}$ there is a $\delta > 0$ with $T(\overline{K}, \xi, r, \delta, A) \neq \emptyset$ [7]. Therefore, let $T_1 = T(\overline{K}, \xi, r+1, \delta, 2^{-r})$. Let $L(z, f, z_1, \ldots, z_r)$ be the polynomial in z of degree at most $r-1$ which interpolates f at the points $z_1, \ldots, z_r$. The r–th modulus of f is defined by

$$\omega_r(f,t) = \omega_{r,\overline{K}}(f,t)_\infty := \sup_{0<\delta\leq t} \sup_{\xi \in \overline{K}} \sup_{\substack{\mathbf{z}\in T_1 \\ \mathbf{z}=(z_0,\ldots,z_r)}} |f(z_0) - L(z_0, f, z_1, \ldots, z_r)|.$$

Here the supremum over the empty set is defined as zero. This modulus is normal [8] [9, Thm.1], i.e.,

$$\omega_{r,\overline{K}}(f, t\delta)_\infty \leq C \cdot t^r \cdot \omega_{r,\overline{K}}(f,\delta)_\infty,$$

where $C > 0$ depends on r and the compactum K only.

Using this modulus we define classes of regularity: Consider normal majorants φ, i.e., bounded non-decreasing functions $\varphi :]0,\infty[\to]0,\infty[$ such that for fixed $\sigma \geq 1$ and an exponent $\gamma \geq 0$ the normality condition $\varphi(t\delta) \leq \sigma t^\gamma \varphi(\delta)$ holds for all $\delta > 0$, $t > 1$ [11, §1]. By $AH_r^\varphi(\overline{K})$ we denote the class of all functions $f \in AC(\overline{K})$ with $\omega_{r,\overline{K}}(f,t) \leq \text{const} \cdot \varphi(t)$, and by $AW^q H_r^\varphi(\overline{K})$ the class of all $f \in AC^q(\overline{K})$ with $f^{(q)} \in AH_r^\varphi(\overline{K})$. For compacta $\overline{K}$ with empty open interior we just write $H_r^\varphi(\overline{K})$ and $W^q H_r^\varphi(\overline{K})$ respectively.

2.2. Direct approximation theorem

Mel'nik proved in [5] a direct approximation theorem for Dirichlet series weighted with the Jackson kernel similar to the approximation theorems for Fourier series. He used there first moduli of continuity. In [3] we gave the following extension to moduli of arbitrary order for his result.

Let $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$ be a multi-index. Consider the partial Dirichlet series of order $n \in \mathbb{N}$ of f, weighted with the generalized Jackson kernel's coefficients:

$$\begin{aligned}\mathcal{P}_{q,n,r}(f)(z) \quad &:= \quad \sum_{m=1}^{n_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} \\ &\quad + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}^{q+1}) \kappa_f(\lambda_m^{(j)}) \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}.\end{aligned}$$

Here

$$x_{n_j,r,m} = 1 - \sum_{p=0}^{n_j} (-1)^p \binom{r}{p} J_{n_j,r,mp}$$

are given by the Fourier expansion of the generalized Jackson kernel

$$K_{l,r}(t) := \lambda_{l,r} \left(\frac{\sin Mt/2}{t/2} \right)^{2r} = \frac{J_{l,r,0}}{2} + \sum_{k=1}^{l} J_{l,r,k} \cos kt,$$

$l \in \mathbb{N}$, $r \geq 2$, $M = \lfloor \frac{l}{r} \rfloor + 1$ and $\lambda_{l,r}$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} K_{l,r}(t)\, dt = 1.$$

Let $1 \leq j \leq N$ be fixed and $r \in \mathbb{N}$. Let $f \in AC(\overline{D})$ have $r-1$ existing derivatives at the vertices a_k, $k = 1, \ldots, N$, of the polygon. Consider for $k \neq j+1$ the polynomial $P_{j,k}$ of degree at most r that interpolates f at the vertices a_j and a_k, and $f', \ldots, f^{(r-1)}$ at the vertex a_k. For $k = j+1$ let $P_{j,j+1}$ denote the polynomial of degree at most $2r-1$ that interpolates $f, f', \ldots, f^{(r-1)}$ at both points a_j and a_{j+1}. We define

$$\begin{aligned}
\delta_r(f,h) \;&:=\; \sum_{\substack{k=1\\k\neq j}}^{N} \Bigg\{ \int_0^h \frac{\left| f\left(a_k + \frac{a_j - a_k}{2\pi} u\right) - P_{j,k}\left(a_k + \frac{a_j - a_k}{2\pi} u\right) \right|}{u}\, du \\
&\quad + h^r \cdot \int_h^{2\pi} \frac{\left| f\left(a_k + \frac{a_j - a_k}{2\pi} u\right) - P_{j,k}\left(a_k + \frac{a_j - a_k}{2\pi} u\right) \right|}{u^{r+1}}\, du \Bigg\}.
\end{aligned}$$

It can be shown that this function is a modified version of another class of Tamrazov's moduli given in [10] which – in contrary to the modulus ω_r defined above – allow multiple nodes. In general ω_r and δ_r are not equivalent, i.e., there are no constants $c, C > 0$ such that $c\,\omega_r(f,h) \leq \delta_r(f,h) \leq C\omega_r(f,h)$ for all $h > 0$ [3].

With these preliminaries we can formulate the following direct approximation theorem.

Theorem 2.1. *Let $f \in AW^q H_r^{\omega_r}(\overline{D})$, $q \in \mathbb{N}$, $r \geq 2$, where ω_r is a normal majorant with exponent r satisfying the Stechkin condition*

$$\int_0^h \frac{\omega_r(f,t)}{t}\, dt + h^r \cdot \int_h^{2\pi} \frac{\omega_r(f,t)}{t^{r+1}}\, dt \quad \leq \quad c \cdot \omega_r(f,h) \tag{5}$$

for all $0 < h < \frac{2\pi}{r}$ and a positive constant c. Let

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0, \quad 0 \leq s \leq r - 1 + q.$$

Let $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$ *be a multi-index.*

Then the approximation by the quasipolynomial $\mathcal{P}_{q,n,r}(f)$*, weighted with the generalized Jackson kernel, satisfies*

$$\|f - \mathcal{P}_{q,n,r}(f)\|_{AC(\overline{D})} \leq \text{const} \sum_{k=1}^{N} \frac{1}{(n_k)^q} \Omega_r\left(\frac{1}{n_k}\right),$$

where Ω_r *is a normal majorant with exponent* r *and*

$$\Omega_r(h) \leq \text{const} \cdot \{\omega_r(h) + \delta_r(f,h)\}.$$

The proof is given in [3].

In the following sections, we give Mel'nik's approach to the question on a inverse theorem and an extension of his result to moduli of arbitrary order.

3. Mel'nik's inverse approximation theorem

Now we consider the inverse question, namely, if the rate of approximation of a Dirichlet series gives information on the regularity of the limit function f. For the first modulus of continuity Mel'nik proved in [5] the following inverse approximation theorem.

Theorem 3.1. [5, Thm. 2]. *Let* $f \in AC(\overline{D})$ *and let* ω *denote a normal majorant with exponent 1 which satisfies the Zygmund condition*

$$\int_0^h \int_t^{2\pi} \frac{\omega(f,u)}{u^2}\, du\, dt = \int_0^h \frac{\omega(f,t)}{t}\, dt + h \cdot \int_h^{2\pi} \frac{\omega(f,t)}{t^2}\, dt \leq c \cdot \omega(f,h)$$

for all $0 < h < 2\pi$ *and a positive constant* c.

Let $\{\mathcal{P}_n\}_{n \in I \subset \mathbb{N}^{N+1}}$, $n = (n_0, \ldots, n_N)$*, be a sequence of quasipolynomials of the form*

$$\mathcal{P}_n(z) = \sum_{m=1}^{n_0} y_{m,n} \frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=m(j)}^{n_j} y_{j,n_j,m} \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})},$$

such that

$$\|f - \mathcal{P}_n\|_{AC(\overline{D})} \leq \text{const} \sum_{k=1}^{N} \frac{1}{n_k^q} \cdot \omega\left(\frac{1}{n_k}\right)$$

for an appropriate $q \in \mathbb{N}_0$.

Then $f \in AW^q H_1^\omega(\overline{D})$ *and*

$$\sum_{k=1}^{N} d_k f^{(\sigma)}(a_k) = 0 \quad \textit{for all } 0 \leq \sigma \leq q.$$

For the proof see [5].

4. Extension to moduli of higher order

Mel'nik's result can be extended to moduli of arbitrary order $r \in \mathbb{N}$ as follows.

Theorem 4.1. *Let $f \in AC(\overline{D})$ and $\omega_r(f, \cdot)$ be a normal majorant with exponent r which satisfies the r–th Stechkin condition* (5)*. Let f be $r-1$–times continuously differentiable at the vertices a_k, $k = 1, \ldots, N$.*

Furthermore, let there exist a sequence of quasipolynomials $\{\mathcal{P}_n\}_{n \in I}$, $I \subset \mathbb{N}^{N+1}$, $n = (n_0, \ldots, n_N)$, of the form

$$\mathcal{P}_n(z) = \sum_{m=1}^{n_0} y_{m,n} \frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=m(j)}^{n_j} y_{j,n_j,m} \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} \tag{6}$$

such that

$$\|f - \mathcal{P}_n\|_{AC(\overline{D})} \le \text{const} \cdot \sum_{k=1}^{N} \omega_r\left(\frac{1}{n_k}\right). \tag{7}$$

Then $f \in AH_r^{\omega_r}(\overline{D})$ and

$$\sum_{k=1}^{N} d_k f^{(\sigma)}(a_k) = 0 \quad \textit{for all } 0 \le \sigma < r. \tag{8}$$

Furthermore, we can deduce from the order of approximation, how often the function f is continuously differentiable in $\overline{D}$.

Corollary 4.2. *Let $f \in AC(\overline{D})$ and ω_r be a normal majorant with exponent r which satisfies the r–th Stechkin condition* (5)*.*

Let f be $r + q - 1$–times continuously differentiable at the vertices a_k, $k = 1, \ldots, N$.

Furthermore, let there be a sequence of quasipolynomials of the form (6) *such that*

$$\|f - \mathcal{P}_n\|_{AC(\overline{D})} \le \text{const} \sum_{k=1}^{N} \frac{1}{(n_k)^q} \cdot \omega_r\left(\frac{1}{n_k}\right). \tag{9}$$

Then $f \in AW^q H_r^{\omega_r}(\overline{D})$ and

$$\sum_{k=1}^{N} d_k f^{(\sigma)}(a_k) = 0 \quad \textit{for all} \quad 0 \le \sigma < r + q. \tag{10}$$

Thus, we can deduce the order of differentiability and the behavior of $\omega_r(f,h)$ for $h \to 0$ from the order of approximation of the Dirichlet series of f.

5. Proofs

In this section we give the proofs of Theorem 4.1 and Corollary 4.2. First we formulate a result by P. M. Tamrazov, which we shall use in the proofs.

5.1. Preliminaries

We shall make use of a relation between modulus on the boundary ∂K of a compactum $\overline{K}$ and the modulus defined on the whole of $\overline{K}$. The following theorem is due to P. M. Tamrazov.

Theorem 5.1. [8] *Let $\overline{K} \subset \mathbb{C}$ be a simply connected convex compactum, K its open interior and ∂K the rectifiable Jordan boundary in $\mathbb{C}$. Let $f \in AC(\overline{K})$, φ be a a normal majorant and $k \in \mathbb{N}$.*

(i) If a modulus of continuity satisfies $\omega_{r,\partial K}(f,\delta) \le \varphi(\delta)$ for all $\delta > 0$, then also $\omega_{r,\overline{K}}(f,\delta) \le \mathrm{const} \cdot \varphi(\delta)$ for all $\delta > 0$. Here the constant is independent of f, δ and $\overline{K}$. It depends only on φ.

(ii) If $\omega_{r,\partial K}(f,\delta) = o(\varphi(\delta))$ for $\delta \to 0$, then $\omega_{r,\overline{K}}(f,\delta) = o(\varphi(\delta))$ for $\delta \to 0$.

This result enables us to estimate the modulus for the boundary ∂K and deduce the estimate for the whole of $\overline{K}$.

5.2. Proof of Theorem 4.1

The proof is divided into four steps. First we decompose $\mathcal{P}_n$ into appropriate partial quasipolynomials and deduce some fundamental properties. In the second step we shall show that these quasipolynomials form Cauchy series. Then we prove that $f \in AH^{\omega_r}(\overline{D})$. Finally, in the fourth step we show that the equalities (8) are true.

5.2.1. Decomposition of $\mathcal{P}_n$

Let $n = (n_1, \ldots, n_N)$. Using property c) of the set of zeros Λ we decompose the quasipolynomial $\mathcal{P}_n$ (see (6)) in the following terms

$$
\begin{aligned}
&\mathcal{P}_n(z)\\
&\quad= \left\{ \sum_{m=1}^{n_0} y_{m,n} \frac{e^{\lambda_m z}}{L'(\lambda_m)} \right.\\
&\qquad \left. + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j'} y_{j,n_j,m} \left(\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_{j+1}+a_j}{2}\right)} \right) \right\}\\
&\qquad + \sum_{j=1}^{N} B_j \sum_{m=n(j)}^{n_j'} y_{j,n_j,m} (-1)^m e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_{j+1}-a_j}{2}\right)}\\
&\quad= p_n(z) + \sum_{j=1}^{N} p_{j,n_j}(z),
\end{aligned}
\tag{11}
$$

where
$$
p_{j,n_j}(z) := B_j \sum_{m=n(j)}^{n_j} (-1)^m y_{j,n_j,m} e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_j+a_{j+1}}{2}\right)}
$$

and

$$
\begin{aligned}
&p_n(z)\\
&\quad:= \sum_{m=1}^{n_0} y_{m,n} \frac{e^{\lambda_m z}}{L'(\lambda_m)}\\
&\qquad + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} y_{j,n_j,m} \left\{ \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_j+a_{j+1}}{2}\right)} \right\}.
\end{aligned}
\tag{12}
$$

Consider
$$
\Pi_{j,n_j}(w) := \sum_{m=n(j)}^{n_j} y_{j,n_j,m} w^m.
$$

Then

$$
\begin{aligned}
p_{j,n_j}(z) &= B_j \sum_{m=n(j)}^{\infty} y_{j,n_j,m} (-1)^m e^{\left(\frac{2\pi m i}{a_{j+1}-a_j} + q_j e^{i\beta_j}\right)\left(z - \frac{a_{j+1}+a_j}{2}\right)}\\
&= B_j e^{q_j e^{i\beta_j}\left(z - \frac{a_{j+1}+a_j}{2}\right)} \sum_{m=n(j)}^{\infty} y_{j,n_j,m} e^{\pi m i} e^{\frac{2\pi m i}{a_{j+1}-a_j}\left(z - \frac{a_{j+1}+a_j}{2}\right)}\\
&= B_j e^{q_j e^{i\beta_j}(z-a_j)} e^{-q_j e^{i\beta_j} \frac{a_{j+1}-a_j}{2}} \sum_{m=n(j)}^{\infty} y_{j,n_j,m} e^{\frac{2\pi m i}{a_{j+1}-a_j}(z-a_j)}\\
&= B_j e^{q_j e^{i\beta_j}(z-a_j)} e^{-q_j e^{i\beta_j} \frac{a_{j+1}-a_j}{2}} \Pi_{j,n_j}\left(e^{\frac{2\pi i}{a_{j+1}-a_j}(z-a_j)}\right).
\end{aligned}
\tag{13}
$$

Now we consider the coefficients $y_{j,n_j,m}$ and $y_{m,n}$. They can be calculated by (3) via the equations

$$y_{j,n_j,m} = \sum_{k=1}^{N} d_k e^{a_k \lambda_m^{(j)}} \int_{a_j}^{a_k} \mathcal{P}_n(\eta) e^{-\lambda_m^{(j)}\eta}\, d\eta \tag{14}$$

and

$$y_{m,n} = \sum_{k=1}^{N} d_k e^{a_k \lambda_m} \int_{a_j}^{a_k} \mathcal{P}_n(\eta) e^{-\lambda_m \eta}\, d\eta. \tag{15}$$

We show that the sequence $\{y_{j,n_j,m}\}_{m\geq n(j)}$ is bounded. For $\{y_{m,n}\}_{m=1,\ldots,n_0}$ this is obvious. $\mathcal{P}_n$ is bounded since by (7) and the triangle inequality we have

$$\|\mathcal{P}_n\| - \|f\| \leq \text{const} \sum_{k=1}^{N} \omega_r\left(\frac{1}{n_k}\right),$$

and therefore

$$\|\mathcal{P}_n\| \leq \text{const} \sum_{k=1}^{N} \omega_r\left(\frac{1}{n_k}\right) + \|f\| \leq C\|f\|$$

for a positive constant independent of n and f.

To estimate the exponentials in (14) and (15) we use that for $k \neq j+1$

$$\operatorname{Re}\left(i\,\frac{a_j - a_k}{a_{j+1} - a_j}\right) > 0.$$

Therefore

$$\operatorname{Re}\left(i\,\frac{z - a_k}{a_{j+1} - a_j}\right) \geq 0 \quad \text{for all } z \in [\,a_j, a_k\,].$$

Hence, by (1),

$$\begin{aligned}\left|e^{-\lambda_m^{(j)}(z-a_k)}\right| &= \left|e^{-\frac{2\pi m i}{a_{j+1}-a_j}(z-a_k)}\right| \cdot \left|e^{-q_j e^{i\beta_j}(z-a_k)}\right| \cdot \left|e^{-\delta_m^{(j)}(z-a_k)}\right| \\ &\leq \text{const}\end{aligned}$$

for all $z \in [\,a_j, a_k\,]$ and all $m \geq n(j)$. Thus $\{y_{j,n_j,m}\}_{m\geq n(j)}$ is a bounded sequence.

Since there are only finitely many $y_{m,n}$, $m = 1, \ldots, n_0$, there exists a positive constant $A > 0$ such that

$$\begin{cases} |y_{j,n_j,m}| \leq A & \text{for all } m \geq n(j) \quad \text{and} \\ |y_{m,n}| \leq A & \text{for all } m = 1, \ldots, n_0. \end{cases} \tag{16}$$

Let $\mu = (\mu_1, \ldots, \mu_N)$ be another multi-index and let, as usual, $n + \mu := (n_1 + \mu_1, \ldots, n_N + \mu_N)$ define the addition for multi-indices. Then (7) and (14) yield

$$
\begin{aligned}
&|y_{j,n_j+\mu_j,m} - y_{j,n_j,m}| \\
&\quad \leq \sum_{k=1}^{N} d_k \int_{a_j}^{a_k} |\mathcal{P}_{n+\mu}(\eta) - \mathcal{P}_n(\eta)| \cdot |e^{-\lambda_m^{(j)}(\eta - a_k)}|\,|d\eta| \\
&\quad \leq \text{const} \sum_{k=1}^{N} d_k \int_{a_j}^{a_k} \left(|\mathcal{P}_{n+\mu}(\eta) - f(\eta)| + |\mathcal{P}_n(\eta) - f(\eta)|\right) |d\eta| \\
&\quad \leq \text{const} \sum_{k=1}^{N} \omega_r\left(\frac{1}{n_k}\right),
\end{aligned}
\tag{17}
$$

and by similar arguments, with (7) and (15),

$$
|y_{m,n+\mu} - y_{m,n}| \leq \text{const} \sum_{k=1}^{N} \omega_r\left(\frac{1}{n_k}\right) \tag{18}
$$

for positive constants independent of n and m.

For later use we finally estimate the series

$$
\sum_{l=0}^{\infty} \omega_r(2^{-l}) \quad \text{and} \quad \sum_{l=K+1}^{\infty} \omega_r(2^{-l}), \qquad K \in \mathbb{N}.
$$

We have

$$
\begin{aligned}
\int_{2^{-l}}^{2^{-l+1}} \frac{\omega_r(u)}{u}\,du &\geq \omega_r(2^{-l}) \int_{2^{-l}}^{2^{-l+1}} \frac{1}{u}\,du = \omega_r(2^{-l}) \ln\left(\frac{2^{-l+1}}{2^{-l}}\right) \\
&= \omega_r(2^{-l}) \ln(2).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\ln(2) \sum_{l=0}^{\infty} \omega_r(2^{-l}) &\leq \sum_{l=0}^{\infty} \int_{2^{-l}}^{2^{-l+1}} \frac{\omega_r(u)}{u}\,du \\
&= \int_0^2 \frac{\omega_r(u)}{u}\,du \leq \text{const} \cdot \omega_r(2)
\end{aligned}
\tag{19}
$$

using the Stechkin condition (5). By the same arguments,

$$
\begin{aligned}
\ln(2) \sum_{l=K+1}^{\infty} \omega_r(2^{-l}) &\leq \sum_{l=K+1}^{\infty} \int_{2^{-l}}^{2^{-l+1}} \frac{\omega_r(u)}{u}\,du \\
&= \int_0^{2^{-K}} \frac{\omega_r(u)}{u}\,du \leq \text{const} \cdot \omega_r(2^{-K}).
\end{aligned}
$$

5.2.2. Cauchy sequences of quasipolynomials

Let $\widetilde{\mu}_j = (0,\ldots,\mu_j,\ldots,0)$ be the multi-index with entry μ_j at the j-th position and zero everywhere else. Let us set $\widetilde{n}_j := n + \widetilde{\mu}_j$. Then, by (11) and (12) we have

$$\begin{aligned}
&\|p_{j,n_j+\mu_j} - p_{j,n_j}\| \\
(20)\qquad &\leq \|\mathcal{P}_{\widetilde{n}_j} - \mathcal{P}_n\| + \|p_{\widetilde{n}_j} - p_n\| \\
&\leq \|\mathcal{P}_{\widetilde{n}_j} - f\| + \|f - \mathcal{P}_n\| + \left\| \sum_{m=1}^{n_0} (y_{m,\widetilde{n}_j} - y_{m,n}) \frac{e^{\lambda_m \cdot}}{L'(\lambda_m)} \right\| \\
&\quad + \Bigg\| \sum_{m=n(j)}^{n_j} (y_{j,n_j,m} - y_{j,n_j+\mu_j,m}) \\
&\qquad \times \left\{ \frac{e^{\lambda_m^{(j)}\cdot}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(\cdot - \frac{a_j+a_{j+1}}{2}\right)} \right\} \Bigg\| \\
&\quad + \left\| \sum_{m=n_j+1}^{n_j+\mu_j} y_{j,n_j,m} \left\{ \frac{e^{\lambda_m^{(j)}\cdot}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(\cdot - \frac{a_j+a_{j+1}}{2}\right)} \right\} \right\| \\
(21)\qquad &\leq \text{const} \sum_{k=1}^{N} \omega_r\left(\frac{1}{n_k}\right),
\end{aligned}$$

where in addition we used (7), (16), (18), (18) and property c) of the set of zeros Λ. The constant in (21) is positive and independent of n. For $n_k \to \infty$, $k \neq j$, we obtain

$$(22)\qquad \|p_{j,n_j+\mu_j} - p_{j,n_j}\|_{AC(\overline{D})} \leq \text{const} \cdot \omega_r\left(\frac{1}{n_j}\right).$$

Hence $\{p_{j,n_j}\}_{n_j}$ is a Cauchy sequence for $n_j \to \infty$. Since $AC(\overline{D})$ is complete as being a Banach space, there is a function $\Phi_j \in AC(\overline{D})$ with

$$\lim_{n_j\to\infty} \|\Phi_j - p_{j,n_j}\|_{AC(\overline{D})} = 0.$$

From (22) we obtain for $\mu_j \to \infty$

$$(23)\qquad \|\Phi_j - p_{j,n_j}\|_{AC(\overline{D})} \leq \text{const} \cdot \omega_r\left(\frac{1}{n_j}\right).$$

By (20) and (21) the same argument gives the existence of a function $\Phi \in AC(\overline{D})$ such that

$$(24)\qquad \|\Phi - p_n\|_{AC(\overline{D})} \leq \text{const} \sum_{j=1}^{N} \omega_r\left(\frac{1}{n_j}\right).$$

We have

$$f(z) = \sum_{j=1}^{N} \Phi_j(z) + \Phi(z) \in AC(\overline{D}), \tag{25}$$

because, in view of (7), (11), (23) and (24),

$$\begin{aligned}
&\left\| f - \sum_{j=1}^{N} \Phi_j - \Phi \right\| \\
&\quad \leq \|f - \mathcal{P}_n\| \quad + \quad \left\| \mathcal{P}_n - \sum_{j=1}^{N} \Phi_j - \Phi \right\| \\
&\quad \leq \|f - \mathcal{P}_n\| \quad + \quad \sum_{j=1}^{N} \|p_{j,n_j} - \Phi_j\| \quad + \quad \|p_n - \Phi\| \quad \to \quad 0
\end{aligned}$$

as $n \to \infty$.

5.2.3. Conclusion on the properties of f

In this part we show that $f \in AH_r^{\omega_r}(\overline{D})$. We first consider the transformation

$$z = a_j + \frac{a_{j+1} - a_j}{2\pi}\theta, \quad 0 \leq \theta \leq 2\pi,$$

as in (4). If z runs through the straight-line interval $[\, a_j, a_{j+1}\,]$, then

$$w = e^{2\pi i \frac{z-a_j}{a_{j+1}-a_j}} = e^{i\theta}$$

describes a torus. From equations (14) and (22) we deduce that for all $\mu_j \in \mathbb{N}$ and $B := \{w \mid |w| < 1\}$ the estimate

$$\|\Pi_{j,n_j+\mu_j} - \Pi_{j,n_j}\|_{AC(\overline{B})} \leq \text{const} \cdot \omega_r\left(\frac{1}{n_j}\right).$$

holds. Since $AC(\overline{B})$ is a Banach space, there are functions $F_j(w)$, $j = 1, \ldots, N$, such that

$$\|F_j - \Pi_{j,n_j}\|_{AC(\overline{B})} \leq \text{const} \cdot \omega_r\left(\frac{1}{n_j}\right). \tag{26}$$

Equation (14), (23) and the estimates (26) yield, for all $z \in \overline{D}$,

$$\Phi_j(z) = B_j e^{q_j e^{i\beta_j}(z-a_j)} e^{-q_j e^{i\beta_j}\frac{a_{j+1}-a_j}{2}} F_j\left(e^{\frac{2\pi i}{a_{j+1}-a_j}(z-a_j)}\right).$$

With these results we now can prove that $\Phi, \Phi_1, \ldots, \Phi_N \in AH_r^{\omega_r}(\overline{D})$. Then (25) gives $f \in AH_r^{\omega_r}(\overline{D})$. We begin with the proof of $\Phi \in AH_r^{\omega_r}(\overline{D})$. By (12) and (24) we have

$$
\begin{aligned}
&\Phi(z)\\
(27)= \quad & p_{1,\ldots,1}(z) \quad + \quad \sum_{l=1}^{\infty}\left(p_{2^l,\ldots,2^l}(z)-p_{2^{l-1},\ldots,2^{l-1}}(z)\right)\\
= \quad & \sum_{m=1}^{n_0} y_{m,(1,\ldots,1)}(z)\frac{e^{\lambda_m z}}{L'(\lambda_m)}\\
&+\sum_{j=1}^{N}\sum_{m=n(j)}^{1} y_{j,1,m}\left\{\frac{e^{\lambda_m^{(j)}z}}{L'(\lambda_m^{(j)})}-(-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(z-\frac{a_j+a_{j+1}}{2}\right)}\right\}\\
&+\sum_{l=1}^{\infty}\left\{\sum_{m=1}^{n_0} y_{m,(2^l,\ldots,2^l)}\frac{e^{\lambda_m z}}{L'(\lambda_m)} \quad - \quad \sum_{m=1}^{n_0} y_{m,(2^{l-1},\ldots,2^{l-1})}\frac{e^{\lambda_m z}}{L'(\lambda_m)}\right.\\
&+\sum_{j=1}^{N}\sum_{m=n(j)}^{2^l} y_{j,2^l,m}\left\{\frac{e^{\lambda_m^{(j)}z}}{L'(\lambda_m^{(j)})}-(-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(z-\frac{a_j+a_{j+1}}{2}\right)}\right\}\\
&\left.-\sum_{j=1}^{N}\sum_{m=n(j)}^{2^{l-1}} y_{j,2^{l-1},m}\left\{\frac{e^{\lambda_m^{(j)}z}}{L'(\lambda_m^{(j)})}-(-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(z-\frac{a_j+a_{j+1}}{2}\right)}\right\}\right\}\\
= \quad & \sum_{m=1}^{n_0} y_{m,(1,\ldots,1)}\frac{e^{\lambda_m z}}{L'(\lambda_m)}\\
&+\sum_{l=1}^{\infty}\sum_{m=1}^{n_0}\left(y_{m,(2^l,\ldots,2^l)}-y_{m,(2^{l-1},\ldots,2^{l-1})}\right)\frac{e^{\lambda_m z}}{L'(\lambda_m)}\\
&+\sum_{j=1}^{N}\sum_{m=n(j)}^{1} y_{j,1,m}\left\{\frac{e^{\lambda_m^{(j)}z}}{L'(\lambda_m^{(j)})}-(-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(z-\frac{a_j+a_{j+1}}{2}\right)}\right\}\\
&+\sum_{l=1}^{\infty}\sum_{j=1}^{N}\sum_{m=n(j)}^{2^{l-1}}\left(y_{j,2^l,m}-y_{j,2^{l-1},m}\right)\left\{\frac{e^{\lambda_m^{(j)}z}}{L'(\lambda_m^{(j)})}\right.\\
&\qquad\left.-(-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(z-\frac{a_j+a_{j+1}}{2}\right)}\right\}\\
(28) \quad &+\sum_{l=1}^{\infty}\sum_{j=1}^{N}\sum_{m=2^{l-1}+1}^{2^l} y_{j,2^l,m}\left\{\frac{e^{\lambda_m^{(j)}z}}{L'(\lambda_m^{(j)})}-(-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(z-\frac{a_j+a_{j+1}}{2}\right)}\right\}.
\end{aligned}
$$

The termwise z-derivatives of the first, third and fifth series can be estimated for all $z \in \overline{D}$ by an appropriate positive constant using (16) and property c) of the set of zeros Λ.

For the derivatives of order $\sigma \in \mathbb{N}_0$ of the second term the estimates (18) and (19) yield

$$\left|\sum_{l=1}^{\infty}\sum_{m=1}^{n_0}(y_{m,(2^l,\dots,2^l)}-y_{m,(2^{l-1},\dots,2^{l-1})})\cdot\frac{e^{\lambda_m z}}{L'(\lambda_m)}\cdot(\lambda_m)^\sigma\right|$$
$$\leq \left|\sum_{l=1}^{\infty}\sum_{m=1}^{n_0}\sum_{k=1}^{N}\omega_r\left(\frac{1}{2^{l-1}}\right)\frac{e^{\lambda_m z}}{L'(\lambda_m)}\cdot(\lambda_m)^\sigma\right|$$
$$\leq \text{const}\cdot\sum_{m=1}^{n_0}\sum_{k=1}^{N}\sum_{l=1}^{\infty}\omega_r\left(\frac{1}{2^{l-1}}\right) \quad\leq\quad \text{const}.$$

For the derivative of the fourth term, by (18) and property c) of the set of zeros Λ,

$$\sum_{l=1}^{\infty}\sum_{j=1}^{N}\sum_{m=n(j)}^{2^{l-1}}(y_{j,2^l,m}-y_{j,2^{l-1},m})\left\{\frac{e^{\lambda_m^{(j)}z}}{L'(\lambda_m)}(\lambda_m^{(j)})^\sigma\right.$$
$$\left.-(-1)^m B_j(\widetilde{\lambda}_m^{(j)})^\sigma e^{\widetilde{\lambda}_m^{(j)}(z-\frac{a_j+a_{j+1}}{2})}\right\}$$
$$\leq \sum_{l=1}^{\infty}\sum_{j=1}^{N}\sum_{m=n(j)}^{2^{l-1}}\sum_{k=1}^{N}\omega_r\left(\frac{1}{2^{l-1}}\right)A(\sigma)e^{-cm}\quad\leq\quad\text{const}.$$

Hence, the series (28) allows term by term differentiation of arbitrary order $\sigma\in\mathbb{N}_0$ for all $z\in\overline{D}$. Thus $\Phi\in AH_r^{\omega_r}(\overline{D})$ and

$$\Phi^{(\sigma)}(z)=p^{(\sigma)}_{(1,\dots,1)}(z)+\sum_{l=1}^{\infty}\left(p^{(\sigma)}_{(2^l,\dots,2^l)}(z)-p^{(\sigma)}_{(2^{l-1},\dots,2^{l-1})}(z)\right),\quad z\in\overline{D},\ \sigma\in N_0.$$

We now show that $\Phi_j\in AH_r^{\omega_r}(\overline{D})$ for all $1\leq j\leq N$. By (26) we have

$$|F_j(e^{i\theta})-\Pi_{j,n_j}(e^{i\theta})|\leq\text{const}\cdot\omega_r(\frac{1}{n_j})\quad\text{for all}\quad\theta\in[0,2\pi].$$

Thus, the application of Bernstein's inverse approximation theorem for Fourier series yields $F_j\in H_r^{\omega_r}(\mathbb{T})$. Besides, by Tamrazov's Theorem 5.1, $F_j\in AH_r^{\omega_r}(\overline{B})$ and thus $\Phi_j\in AH_r^{\omega_r}(\overline{D})$. Hence, by (25), $f\in AH_r^{\omega_r}(\overline{D})$.

5.2.4. Vertex conditions

Similarly to (27), the inequality (7) yields the convergent telescope expansion

$$f(z)=\mathcal{P}_{(1,\dots,1)}(z)+\sum_{l=1}^{\infty}(\mathcal{P}_{(2^l,\dots,2^l)}(z)-\mathcal{P}_{(2^{l-1},\dots,2^{l-1})}(z))$$

for all $z \in \overline{D}$. By (6)

$$\sum_{k=1}^{N} d_k \mathcal{P}_n(a_k) = \sum_{m=1}^{n_0} y_{m,n} \frac{L(\lambda_m)}{L'(\lambda_m)} + \sum_{m=n(j)}^{n_j} y_{j,n_j,m} \frac{L(\lambda_m^{(j)})}{L'(\lambda_m^{(j)})} = 0 \tag{29}$$

for all $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$, since $\lambda_m, \lambda_m^{(j)} \in \Lambda$ are the zeros of the quasipolynomial L. Hence

$$\begin{aligned}
&\sum_{k=1}^{N} d_k f(a_k) \\
&\quad = \sum_{k=1}^{N} d_k \mathcal{P}_{(1,\ldots,1)}(a_k) + \sum_{k=1}^{N} d_k \sum_{l=1}^{\infty} \left(\mathcal{P}_{(2^l,\ldots,2^l)}(a_k) - \mathcal{P}_{(2^{l-1},\ldots,2^{l-1})}(a_k)\right) \\
&\quad = 0.
\end{aligned}$$

By the assumptions of the theorem, f is $(r-1)$-times continuously differentiable at the vertices a_k, $k = 1, \ldots, N$. Hence, for $k = 1, \ldots, N$ and $\sigma = 1, \ldots, r-1$,

$$f^{(\sigma)}(a_k) = \mathcal{P}^{(\sigma)}_{(1,\ldots,1)}(a_k) + \sum_{l=1}^{\infty} \left(\mathcal{P}^{(\sigma)}_{(2^l,\ldots,2^l)}(a_k) - \mathcal{P}^{(\sigma)}_{(2^{l-1},\ldots,2^{l-1})}(a_k)\right),$$

and thus,

$$\sum_{k=1}^{N} d_k \mathcal{P}_n^{(\sigma)}(a_k) = \sum_{m=1}^{n_0} (\lambda_m)^{\sigma} y_{m,n} \frac{L(\lambda_m)}{L'(\lambda_m)} + \sum_{m=n(j)}^{n_j} (\lambda_m^{(j)})^{\sigma} y_{j,n_j,m} \frac{L(\lambda_m^{(j)})}{L'(\lambda_m^{(j)})} = 0$$

by arguments similar to those yielding (29). This gives (8).

5.3. Proof of Corollary 4.2

We proceed as in the proof of Theorem 4.1. We start with the estimate of the quasipolynomial using (9) as in the first step 5.2.1. The estimates (17) and (18) yield

$$|y_{j,n_j+\mu_j,m} - y_{j,n_j,m}| \le \text{const} \sum_{k=1}^{N} \frac{1}{(n_k)^q} \cdot \omega_r\left(\frac{1}{n_k}\right)$$

and

$$|y_{m,n+\mu} - y_{m,n}| \le \text{const} \sum_{k=1}^{N} \frac{1}{(n_k)^q} \cdot \omega_r\left(\frac{1}{n_k}\right).$$

Thus, we can show in the second step that

$$\|p_{j,n_j+\mu_j} - p_{j,n_j}\|_{AC(\overline{D})} \leq \text{const}\frac{1}{(n_j)^q} \cdot \omega_r\left(\frac{1}{n_j}\right)$$

for $n_k \to \infty$ and $k \neq j$.

Hence $\{p_{j,n_j}\}_{n_j \geq n(j)}$ forms a Cauchy sequence as $n_j \to \infty$ (see (20), (21) and (22)). Thus, there is a function $\Phi_j \in AC(\overline{D})$ such that

$$\lim_{n_j \to \infty} \|\Phi_j - p_{j,n_j}\|_{AC(\overline{D})} = 0.$$

The limit $\mu_j \to \infty$ yields

$$\|\Phi_j - p_{j,n_j}\|_{AC(\overline{D})} \leq \text{const}\frac{1}{(n_j)^q} \cdot \omega_r\left(\frac{1}{n_j}\right).$$

In the same way we deduce the existence of a function $\Phi \in AC(\overline{D})$ with

$$\|\Phi - p_n\|_{AC(\overline{D})} \leq \text{const}\sum_{j=1}^{N} \frac{1}{(n_j)^q} \cdot \omega_r\left(\frac{1}{n_j}\right)$$

(see (24)). We have $f(z) = \sum_{j=1}^{N} \Phi_j(z) + \Phi(z) \in AC(\overline{D})$.

In the third step we show that $f \in AW^q H_r^{\omega_r}(\overline{D})$. First, consider Φ: We decompose Φ as in (28) and deduce as in the proof of Theorem 4.1 that $\Phi \in AW^q H_r^{\omega_r}(\overline{D})$, as well as

$$\Phi^{(\sigma)}(z) = p^{(\sigma)}_{(1,\dots,1)}(z) + \sum_{l=1}^{\infty}\left(p^{(\sigma)}_{(2^l,\dots,2^l)}(z) - p^{(\sigma)}_{(2^{l-1},\dots,2^{l-1})}(z)\right), \quad z \in \overline{D},\ \sigma \in \mathbb{N}_0.$$

Now we consider Φ_j. As in (26) we find

$$|F_j(e^{i\theta}) - \Pi_{j,n_j}(e^{i\theta})| \leq \|F_j - \Pi_{j,n_j}\|_{AC(\overline{B})} \leq \text{const} \cdot \frac{1}{(n_j)^q} \cdot \omega_r\left(\frac{1}{n_j}\right)$$

for all $\theta \in [0, 2\pi]$. Bernstein's inverse approximation theorem for Fourier series yields $F_j \in W^q H_r^{\omega_r}(\mathbb{T})$. Tamrazov's Theorem 5.1 gives $F_j \in AW^q H_r^{\omega_r}(\overline{B})$. Altogether we deduce $f \in AW^q H_r^{\omega_r}(\overline{D})$.

The properties (10) at the vertices of D can be shown as in the step 5.2.4. of the proof of Theorem 4.1.

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