[2] I. Daubechies, *Ten Lectures on Wavelets*. Philadelphia, PA: Soc. Ind. Appl. Math., 1992.
[3] M. J. T. Smith and T. P. Barnwell, "Exact reconstruction techniques for tree-structured subband coders," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 434–441, June 1986.
[4] P. P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and applications: A tutorial," *Proc. IEEE*, vol. 78, pp. 56–93, Jan. 1990.
[5] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*. Englwood Cliffs, NJ: Prentice-Hall, 1975.
[6] E. I. Jury, *Theory and Application of the z-Transform Method*. New York: Wiley, 1964.
[7] W. Lawton, "Application of complex valued wavelet transforms to subband decomposition," *IEEE Trans. Signal Processing*, vol. 41, pp. 3566–3568, Dec. 1993.
[8] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Wellesley, MA: Wellesley-Cambridge, 1996.
[9] C. Herley and M. Vetterli, "Wavelets and recursive filter banks," *IEEE Trans. Signal Processing*, vol. 41, pp. 2536–2556, Aug. 1993.
[10] A. Cohen, I. Daubechies, and J. C. Feauveau, "Biorthogonal bases of compactly supported wavelets," *Commun. Pure Appl. Math.*, vol. 45, pp. 485–560, 1992.
[11] M. Doroslovački and H. Fan, "Generalized Heisenberg's uncertainty relation for signals," in *Proc. Conf. Inform. Sci. Syst.*, Baltimore, MD, Mar. 1993, pp. 464–469.
[12] M. Doroslovački, H. Fan, and P. M. Djurić, "Time-frequency localization for sequences," in *Proc. IEEE-SP Int. Sym. Time-Frequency, Time-Scale Anal.*, Victoria, BC, Canada, Oct. 1992, pp. 159–162.
[13] M. Doroslovački, "Discrete-time signals: Uncertainty relations, wavelets, and linear system modeling," Ph.D. dissertation, Dept. Elect. Comput. Eng., Univ. Cincinnati, Cincinnati, OH, Nov. 1993.
[14] M. Doroslovački, H. Fan, and P. M. Djurić, "Discrete-time wavelets: Time-frequency localization, shift-invariance, modeling of linear time-invariant systems," in *Proc. Conf. Inform. Sci. Syst.*, Princeton, NJ, Mar. 1992, vol. 1, pp. 21–26.

Construction of Biorthogonal Wavelets Starting from Any Two Multiresolutions

Akram Aldroubi, Patrice Abry, and Michael Unser

***Abstract*— Starting from any two given multiresolution analyses of L_2, $\{V_j^1\}_{j\in\mathbf{Z}}$, and $\{V_j^2\}_{j\in\mathbf{Z}}$, we construct biorthogonal wavelet bases that are associated with this chosen pair of multiresolutions. Thus, our construction method takes a point of view opposite to the one of Cohen–Daubechies–Feauveau (CDF), which starts from a well-chosen pair of biorthogonal discrete filters. In our construction, the necessary and sufficient condition is the nonperpendicularity of the multiresolutions.**

I. Motivation

Our goal is to construct biorthogonal wavelets starting from any two given multiresolutions $\{V_j^1\}_{j\in\mathbf{Z}}$ and $\{V_j^2\}_{j\in\mathbf{Z}}$ instead of the usual approach that starts from the specification of a pair of biorthogonal filters [7] or the more recent lifting scheme approach of Sweldens [9]. For example, we may want to choose the analyzing multiresolution (MR)$\{V_j^1\}_{j\in\mathbf{Z}}$ to be the Haar MR and $\{V_j^2\}_{j\in\mathbf{Z}}$ to be a smoother spline MR. Using ideas similar to those in [1], [2], and [6], it is possible to construct wavelets with a variety of desired properties and/or time shape and to construct their biorthogonal duals. In our method, even if both scaling functions ϕ_1 and ϕ_2 have compact support, the analysis filters that implement the wavelet transform (WT) need not be FIR. However, they can still be implemented exactly using recursive filtering techniques, as in [10], or truncated, as has been done in the original paper of Mallat [8].

II. Basic Wavelets

We choose two arbitrary and *a priori* independent MR's

$$V_j^m = \left\{ \sum_{k\in\mathbf{Z}} c_j(k)\phi_{m(j,\,k)}(t),\ c_j \in l_2 \right\}$$

where $\phi_{m(j,\,k)}(t) = 2^{-j/2}\phi_m(2^{-j}t - k)$, and $m = 1,\,2$. We use the notation ϕ_m for $\phi_{m(0,\,0)}$. The set $\left\{\phi_{m(j,\,k)}\right\}_{k\in\mathbf{Z}}$ is a Riesz basis of V_j^m, and we have

$$\phi_m(t/2) = 2\sum_{k\in\mathbf{Z}} h_m(k)\phi_m(t-k). \tag{1}$$

A pair of biorthogonal wavelets ψ_m, $m = 1,\,2$ associated with the MR's $\{V_j^m\}_{j\in\mathbf{Z}}$ must have the property that their translations and dilations $\psi_{m(j,\,k)}$ form Riesz bases of the spaces

$$W_j^m = \left\{ \sum_{k\in\mathbf{Z}} d_j(k)\psi_{m(j,\,k)}(t);\ d_j \in l_2 \right\}$$

that complement the spaces V_j^m, i.e., $W_j^m + V_j^m = V_{j-1}^m$ ($m = 1,\,2$). We have

$$\psi_m\left(\frac{t}{2}\right) = 2\sum_{k\in\mathbf{Z}} g_m(k)\phi_m(t-k). \tag{2}$$

The wavelet bases $\psi_{1(j,\,k)}$ and $\psi_{2(j,\,k)}$ must satisfy the biorthogonality condition $\langle\psi_{1(j,\,k)},\,\psi_{2(m,\,n)}\rangle = \delta_0(j-m)\delta_0(k-n)$, where $\delta_p(k)$ is the pulse sequence located at $k = p$. Our first goal is to construct a pair of wavelet spaces $\{W_j^1\}_{j\in\mathbf{Z}}$ and $\{W_j^2\}_{j\in\mathbf{Z}}$ such that $W_j^1 \perp V_j^2$ and $W_j^2 \perp V_j^1$ $\forall\, j \in \mathbf{Z}$. The requirement that $W_j^1 \perp V_j^2$ combined with the facts that $W_{l+1}^2 \subset V_l^2 \subset V_j^2$ $\forall l > j$ implies that $W_l^1 \perp W_j^2$ for $l > j$. Since switching the roles of the wavelet spaces does not change the previous argument, we get the following orthogonality between wavelet spaces: $W_l^1 \perp W_j^2$, $\quad l \neq j$. Since $W_1^1 \subset V_0^1$ and $W_1^2 \subset V_0^2$, $\psi_1^b(t/2)$ and $\psi_2^b(t/2)$ must satisfy (the superscript "b" stands for "basic")

$$\begin{aligned} \psi_1^b\left(\frac{t}{2}\right) &= 2\sum_{k\in\mathbf{Z}} g_1(k)\phi_1(t-k) \\ \psi_2^b\left(\frac{t}{2}\right) &= 2\sum_{k\in\mathbf{Z}} g_2(k)\phi_2(t-k) \end{aligned} \tag{3}$$

where g_1 and g_2 are to be determined so that $W_1^1 \perp V_1^2$ and $W_1^2 \perp V_1^1$. These cross-orthogonality requirements are satisfied if and only if the bases of the wavelet spaces W_1^1 and W_1^2 are orthogonal to the bases of the spaces V_1^2 and V_1^1, respectively: $\langle\psi_{1(1,\,0)}^b(\cdot),\,\phi_{2(1,\,k)}(\cdot)\rangle_{L_2} = 0$, $\langle\psi_{2(1,\,0)}^b(\cdot),\,\phi_{1(1,\,k)}(\cdot)\rangle_{L_2} = 0$, $\forall k \in \mathbf{Z}$. Using (3), a simple calculation shows that the two equations above can be written as

$$\downarrow_2 [g_1 * h_2^\vee * a_{21}^\vee] = 0, \quad \downarrow_2 [g_2 * h_1^\vee * a_{21}] = 0 \tag{4}$$

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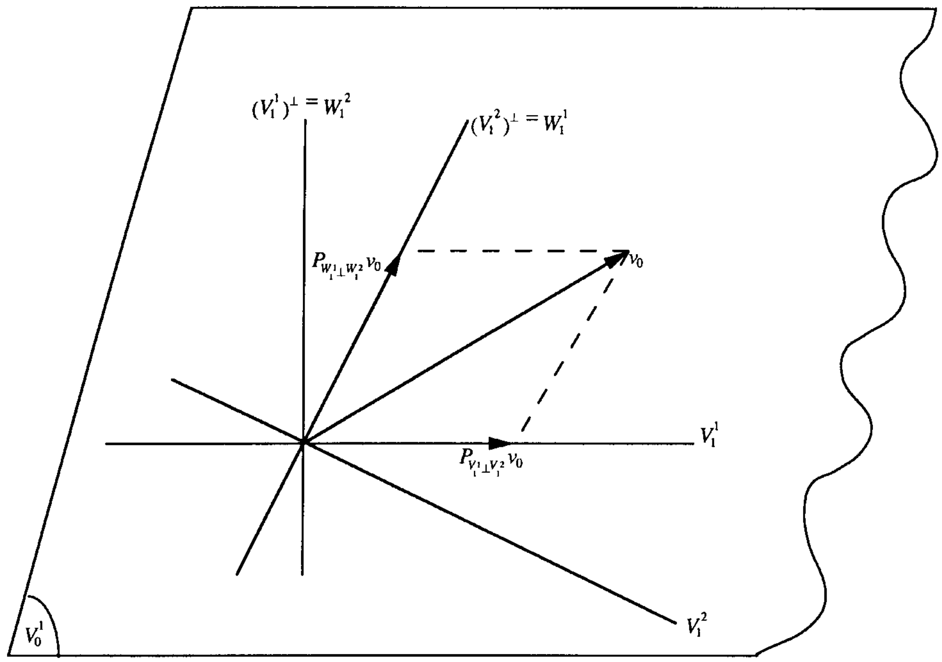

Fig. 1. Oblique projection $P_{V_1^1 \perp V_1^2}$ of $v_0 \in V_0^1$ onto V_1^1 in a direction orthogonal to V_1^2, and the projection $P_{W_1^1 \perp W_1^2}$ onto W_1^1 in a direction orthogonal to W_1^2.

where

- $\downarrow_2$ downsampling operator;
- $\vee$ reflection operator $[h_1^\vee(k) = h_1(-k)]$;
- a_{21} sampled cross correlation function between $\phi_1(t)$ and $\phi_2(t)$: $a_{21}(k) = \int \phi_1(x-k)\phi_2(x)\,dx = (\phi_2 * \phi_1^\vee)(t)|_{t=k}$.

Again, $\phi_1^\vee(t) = \phi_1(-t)$. Using the fact that $\langle \phi_2(t), \phi_1(t-l)\rangle = 2^{-1}\langle \phi_2(t/2), \phi_1((t-2l)/2)\rangle$, we get that

$$a_{21}(k) = 2 \downarrow_2 [h_2 * a_{21} * h_1^\vee](k). \tag{5}$$

To solve for g_1 and g_2, we use the well-known fact that $\downarrow_2 [\delta_1 * b^\pm * b](k) = 0 \quad \forall k \in \mathbf{Z}$, where $b(k)$ is any sequence, and where $b^\pm(k) = (-1)^k b(k)$. We immediately obtain solutions

$$g_1 = \delta_1 * (a_{21}^\vee)^\pm * (h_2^\vee)^\pm, \quad g_2 = \delta_1 * a_{21}^\pm * (h_1^\vee)^\pm. \tag{6}$$

The functions ψ_1^b and ψ_2^b in (3) are indeed wavelets generating the wavelet spaces $\{W_j^1\}_{j\in\mathbf{Z}}$ and $\{W_j^2\}_{j\in\mathbf{Z}}$, as shown below. Here, it is important to note that although the wavelets ψ_1^b and ψ_2^b generate the desired wavelet spaces, they do not necessarily form a biorthogonal pair. Moreover, ψ_1^b and ψ_2^b depend on the simultaneous choice of $\{V_j^1\}_{j\in\mathbf{Z}}$ and $\{V_j^2\}_{j\in\mathbf{Z}}$.

Theorem 1.1 below relies on the notion of angle $\theta(V_0^1, V_0^2)$ between the two MR spaces V_0^1 and V_0^2, which is defined using the orthogonal projection operator $P_{V_0^1}$ on the space V_0^1 (see [4], [11])

$$\begin{aligned}\cos[\theta(V_0^1, V_0^2)] &= \inf\left\{\left\|P_{V_0^1} v\right\|_{L_2};\ v \in V_0^2,\ \|v\|_{L_2} = 1\right\}\\ &= \text{ess-inf}_{f\in[0,1]} \frac{|\hat{a}_{21}(f)|}{[\hat{a}_{11}(f)\hat{a}_{22}(f)]^{1/2}}\end{aligned} \tag{7}$$

where, for all practical purposes, the ess-inf of a function is its minimum, and where $\hat{a}_{ij}(f),\ i = 1, 2,\ j = 1, 2$ are the Fourier transforms of the sampled correlation functions $a_{ij}(k) = (\phi_i * \phi_j^\vee)(k)$ [the Fourier transform of a sequence $b(k)$ is by definition $\hat{b}(f) = \sum_k b(k) z^{-k}|_{z=e^{i2\pi f}}$]. We have the following theorem.

Theorem 1.1: Let V_0^1 and V_0^2 be two MR spaces such that the angle between them satisfies $\cos[\theta(V_0^1, V_0^2)] \neq 0$, and construct W_j^1,W_j^2, ψ_1^b, and ψ_2^b as described above. Then, we have the following.

1) $W_j^1 \cap V_j^1 = \{0\}$, and $W_j^2 \cap V_j^2 = \{0\}$.
2) $V_{j+1}^1 + W_{j+1}^1 = V_j^1$, and $V_{j+1}^2 + W_{j+1}^2 = V_j^2$.
3) $\cos[\theta(W_0^1, W_0^2)] \neq 0$.
4) The sets $\{\psi_{1(j,k)}^b\}_{j\in\mathbf{Z}}$ and $\{\psi_{2(j,k)}^b\}_{j\in\mathbf{Z}}$ are Riesz bases of W_j^1 and W_j^2, respectively.
5) For any $v_0 \in V_0^1$, we have $v_0 = P_{V_1^1 \perp V_1^2} v_0 + P_{W_1^1 \perp W_1^2} v_0$

where $P_{V_1^1 \perp V_1^2}$ is the projection on V_1^1 in a direction orthogonal to V_1^2, and where $P_{W_1^1 \perp W_1^2}$ is the projection on W_1^1 in a direction orthogonal to W_1^2 (see Fig. 1).

The proof of this theorem is given in the Appendix. As a corollary to Theorem 1.1, we immediately obtain the following corollary.

Corollary 1.2: If the angle $\theta(V_0^1, V_0^2)$ is such that $\cos[\theta(V_0^1, V_0^2)] \neq 0$, then we have the following.

1) For any $u \in L_2$, we have

$$u = P_{V_J^1 \perp V_J^2} u + \sum_{j=-\infty}^{J} P_{W_j^1 \perp W_j^2} u = \sum_{j=-\infty}^{\infty} P_{W_j^1 \perp W_j^2} u.$$

2) The sets $\{\psi_{1(j,k)}^b\}_{(j,k)\in\mathbf{Z}^2}$ and $\{\psi_{2(j,k)}^b\}_{(j,k)\in\mathbf{Z}^2}$ are Riesz bases of $L_2(\mathbf{R})$.

III. Dual Wavelets, Scaling Functions, and Generating Sequences

The condition $\cos[\theta(V_0^1, V_0^2)] \neq 0$ in Theorem 1.1 implies that V_0^1 does not contain vectors that are orthogonal to V_0^2 and vice versa and that the projection $P_{V_0^1 \perp V_0^2} u = \sum_{k\in\mathbf{Z}} c_0(k)\phi_1(t-k)$ of a function $u \in L_2$ onto the space V_0^1 in a direction orthogonal to V_0^2 is a well-defined operation [4, Th. 3.2] (see Fig. 1). Thus, the difference $e = u - P_{V_0^1 \perp V_0^2} u$ must be orthogonal to all the basis functions $\{\phi_2(t-k)\}_{k\in\mathbf{Z}}$ of V_0^2: $\langle (u - P_{V_0^1 \perp V_0^2} u)(\cdot), \phi_2(\cdot - k)\rangle = 0 \quad \forall k \in \mathbf{Z}$. A simple calculation using this property shows that the projection is given by $P_{V_0^1 \perp V_0^2} u = \sum_{k\in\mathbf{Z}} \langle u(\cdot), \tilde{\phi}_2(\cdot - k)\rangle \phi_1(t-k)$, where $\tilde{\phi}_2 \in V_0^2$ is given in terms of the convolution inverse $(a_{21})^{-1}$ of a_{21}

$$\tilde{\phi}_2(t) = \sum_{k\in\mathbf{Z}} (a_{21})^{-1}(k)\phi_2(t-k) \tag{8}$$

where the convolution inverse of a sequence a is the sequence $(a)^{-1}$ satisfying $[(a)^{-1} * a](k) = \delta_0(k)$. We note that $(a_{21})^{-1}$ exists. To see why, we simply observe that if $\cos[\theta(V_0^1, V_0^2)] \neq 0$, then (7) implies that $\hat{a}_{21}(f)$ is nonzero for f, a.e. Thus, $[\hat{a}_{21}(f)]^{-1}$ is well defined, and its inverse Fourier transform is precisely the sequence $(a_{21})^{-1}$. It is not difficult to check that $\langle \tilde{\phi}_2(\cdot), \phi_1(\cdot - k)\rangle = \delta_0(k)$. Because of this relation, $\tilde{\phi}_2(t) \in V_0^2$ is the *biorthogonal dual* with respect to V_0^2 of $\phi_1(t) \in V_0^1$. Since the spaces $\{V_j^1\}_{j\in\mathbf{Z}}$ are copies of each other at different scales, it follows that for any fixed j, we have $\langle \tilde{\phi}_{2(j,0)}(\cdot), \phi_{1(j,k)}(\cdot)\rangle = \delta_0(k) \quad \forall k \in \mathbf{Z}$. Calculations similar to those for computing $P_{V_0^1 \perp V_0^2} u$ yield $P_{W_0^1 \perp W_0^2} u = \sum_{k\in\mathbf{Z}} \langle u(\cdot), \tilde{\psi}_2^b(\cdot - k)\rangle \psi_1^b(t-k)$, where $\tilde{\psi}_2^b \in W_0^2$ is given by

$$\tilde{\psi}_2^b(t) = 2^{-1} \sum_{k\in\mathbf{Z}} (\downarrow_2 [g_2 * a_{21} * g_1^\vee])^{-1}(k)\psi_2^b(t-k). \tag{9}$$

The function $\tilde{\psi}_2^b(t) \in V_0^2$ satisfies $\langle \tilde{\psi}_2^b(\cdot), \psi_1^b(\cdot - k)\rangle_{L_2} = \delta_0(k)$ and is the biorthogonal dual of ψ_1^b. From this property and the fact that $W_j^2 \perp W_l^1$ for $j \neq l$, we also deduce that $\langle \tilde{\psi}_{2(j,k)}^b(\cdot), \psi_{1(m,n)}^b(\cdot)\rangle = \delta_0(j-m)\delta_0(k-n)$. This property means that the two sets $\{\psi_{1(j,k)}^b\}_{(j,k)\in\mathbf{Z}^2}$ and $\{\tilde{\psi}_{2(j,k)}^b\}_{(j,k)\in\mathbf{Z}^2}$ form two biorthogonal (or dual) Riesz bases of L_2. Here, we would like to emphasize that it is the pair $\{\tilde{\psi}_2^b, \psi_1^b\}$ that constitutes the biorthogonal wavelets and not the pair $\{\psi_2^b, \psi_1^b\}$ that we constructed in Section II. Since (8) states that $\tilde{\phi}_2$ is a linear combination of ϕ_2, we conclude that $\tilde{\phi}_2 \in V_0^2$. In a similar fashion, from (9), we can deduce that $\tilde{\psi}_2^b \in W_0^2$. Therefore, $\exists\, \tilde{h}_2(k)$ such that $\tilde{\phi}_2(t/2) = 2\sum_{k\in\mathbf{Z}} \tilde{h}_2(k)\tilde{\phi}_2(t-k)$, and similarly, $\exists\, \tilde{g}_2(k)$ such that $\tilde{\psi}_2^b(t/2) = 2\sum_{k\in\mathbf{Z}} \tilde{g}_2(k)\tilde{\phi}_2(t-k)$. Using (5), (8), and (9), we get

$$\begin{aligned}\tilde{h}_2 &= \uparrow_2 [(a_{21})^{-1}] * a_{21} * h_2\\ \tilde{g}_2 &= \delta_1 * \uparrow_2 [a_{21}^{-1}] * h_1^{\vee\pm} = 2^{-1} \uparrow_2 [(\downarrow_2 [g_1^\vee * a_{21} * g_2])^{-1}]\\ &\quad * a_{21} * g_2.\end{aligned} \tag{10}$$

From our construction, we note that $\tilde{\phi}_2(t)$ is another scaling function for the spaces $\{V_j^2\}_{j\in\mathbf{Z}}$ and that $\tilde{\psi}_2^b(t)$ is another wavelet generating the spaces $\{W_j^2\}_{j\in\mathbf{Z}}$.

IV. Biorthogonal Wavelet Decomposition and Reconstruction

By combining Corollary 1.2 and the biorthogonality relations $\langle \tilde{\phi}_{2(j,0)}(\cdot), \phi_{1(j,k)}(\cdot)\rangle = \delta_0(k)$ and $\langle \tilde{\psi}_{2(j,k)}^b(\cdot), \psi_{1(m,n)}^b(\cdot)\rangle = \delta_0(j-m)\delta_0(k-n)$, we are able to decompose any vector $u \in L_2$ as

$$\begin{aligned} u(t) &= \sum_{k\in\mathbf{Z}} \langle u(t), \tilde{\phi}_{2(J,k)}(t)\rangle \phi_{1(J,k)}(t)\\ &\qquad + \sum_{j=-\infty}^{J} \sum_{k\in\mathbf{Z}} \langle u(t), \tilde{\psi}_{2(j,k)}^b(t)\rangle \psi_{1(j,k)}^b(t)\\ &= \sum_{j,k} \langle u(t), \tilde{\psi}_{2(j,k)}^b(t)\rangle_{L_2} \psi_{1(j,k)}^b(t)\\ &= \sum_{(j,k)\in\mathbf{Z}^2} d_j(k)\psi_{1(j,k)}^b(t). \end{aligned} \tag{11}$$

By interchanging $\psi_{1(j,k)}^b(t)$ and $\tilde{\psi}_{2(j,k)}^b(t)$ in (11), we also get $u(t) = \sum_{j,k} \langle u(t), \psi_{1(j,k)}^b(t)\rangle_{L_2} \tilde{\psi}_{2(j,k)}^b(t)$.

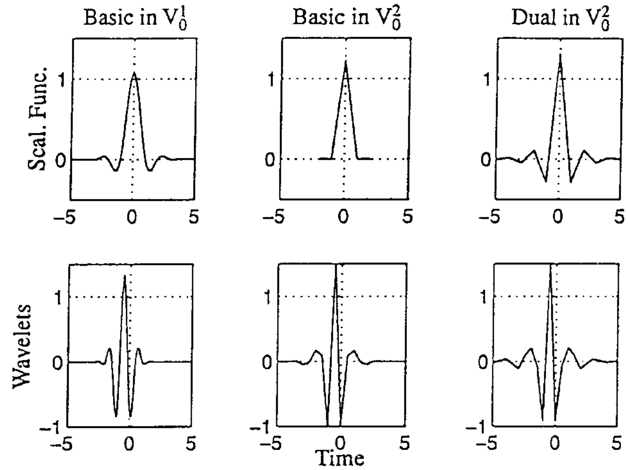

Fig. 2. V_0^1 and V_0^2 are the spline of order 3 and of order 1 MR's. First row of plots shows scaling functions and second row shows wavelets obtained as summarized in Section VI. From left to right we have basic in V_0^1, basic in V_0^2, and dual in V_0^2 of basic in V_0^1.

V. Relation with Cohen–Daubechies–Feauveau Biorthogonal Wavelets

The sets $\{\phi_1(t-k)\}_{k\in\mathbf{Z}}$ and $\{\tilde{\phi}_2(t-k)\}_{k\in\mathbf{Z}}$ are Riesz bases of V_0^1 and V_0^2, respectively. A simple calculation shows that $2 \downarrow_2 [h_1 * \tilde{h}_2^\vee] = 2\sum_k h_1(k)\tilde{h}_2(k-2n) = \delta_0(n)$, which is the starting point for the construction of the biorthogonal wavelets of Cohen–Daubechies–Feauveau (CDF) [7]. Moreover, with the appropriate choice of multiresolutions, the function $\tilde{\psi}_2^b(t)$ and the function $\psi_1^b(t)$ will be the biorthogonal compactly supported wavelets of CDF [7]. In the present context, $\tilde{\psi}_2^b(t)$ and the function $\psi_1^b(t)$ are not necessarily compactly supported since we have chosen the spaces $\{V_j^1\}_{j\in\mathbf{Z}}$ and $\{V_j^2\}_{j\in\mathbf{Z}}$ arbitrarily.

VI. Implementation and Example

Let h_1 and h_2 be defined as in (1). The procedure to obtain the wavelets and associated filter banks is as follows.

1) Compute a_{21} from (5).
2) Check the nonperpendicularity condition from a_{21} (7) and Theorem 1.1.
3) Using (6), compute g_1 and g_2, thus defining the basic wavelets (3).
4) Using (10), compute $\tilde{h}_2$ and $\tilde{g}_2$, thus defining the dual scaling function and wavelet [(8) and (9)].

To compute the WT with the designed wavelet, we can use the filter bank algorithm with filters $\tilde{h}_2$ and $\tilde{g}_2$ for the analysis and h_1 and g_1 for the synthesis. Fig. 2 shows an example where the two starting MR's are the spline of order 3 and spline of order 1 MR's.

VII. Conclusion

We developed a construction of biorthogonal wavelets that starts from any two multiresolution analyzes. Our approach is geometric and can be used to construct biorthogonal wavelets with desired properties and/or time shape that can be implemented using fast filter bank algorithms.

Appendix
Proof of Theorem 1.1

Part 1: By construction, $W_1^1 \perp V_1^2$. Therefore, if $g_1 \in W_1^1 \cap V_1^1$, then $g_1 \in V_1^1$, and $g_1 \perp V_1^2$, but this contradicts the fact that $\cos[\theta(V_0^1, V_0^2)] \neq 0$ [see Definition (7)] unless $g_1 = 0$.

Part 3: Using (3) and (6), the sampled cross correlation function $X_{21}(k)$ between ψ_1^b and ψ_2^b is

$$X_{21}(k) = 2 \downarrow_2 [(h_1^\vee)^\pm * (a_{21})^\pm * (h_2)^\pm * (a_{21})^\pm * a_{21}]. \quad (12)$$

We use (5) and the fact that $[(a_{21})^\pm * a_{21}](2k) = 0$ to rewrite (12) as $X_{21} = \downarrow_2 [(a_{21})^\pm * a_{21}] * (a_{21})^\pm$. By taking the Fourier transform of this last equation, we obtain $\hat{X}_{21}(f) = [\hat{a}_{21}[(f-1)/2]\hat{a}_{21}(f/2)]\hat{a}_{21}[f-(1/2)]$. Since $\cos[\theta(V_0^1, V_0^2)] \neq 0$, $\exists A > 0$ s.t. $|\hat{a}_{21}(f)| \geq A \; \forall f$ [see (7)]. Thus, $\exists$ Const > 0 s.t. $|\hat{X}_{21}(f)| \geq$ Const a.e. f. Therefore, from (7), we get $\cos[\theta(W_0^1, W_0^2)] \neq 0$.

Parts 2 and 5: For $v_0 = \sum_{k\in\mathbf{Z}} c_0(k)\phi_1(t-k) \in V_0^1$, we use (3), (10), and (11) to write $v^\approx = P_{V_1^1 \perp V_1^2} v_0 + P_{W_1^1 \perp W_1^2} v_0$ in the basis of V_0^1 as $v^\approx = \sum_{k\in\mathbf{Z}} c_0^\approx(k)\phi_1(t-k)$, where $c_0^\approx = 2 \uparrow_2 [\downarrow_2 (c_0 * \tilde{g}_2^\vee)] * g_1 + 2 \uparrow_2 [\downarrow_2 (c_0 * \tilde{h}_2^\vee)] * h_1$. Taking the Z transform of the previous equation, we obtain

$$\begin{aligned} C_0^\approx(z) &= C_0(z)[\tilde{G}_2(z^{-1})G_1(z) + \tilde{H}_2(z^{-1})H_1(z)] \\ &\quad + C_0(-z)[\tilde{G}_2(-z^{-1})G_1(-z) + \tilde{H}_2(-z^{-1})H_1(-z)]. \end{aligned}$$

Using the Z transform of (5), $A_{21}(z) = H_2(z^{1/2})H_1(z^{-1/2})A_{21}(z^{1/2}) + H_2(-z^{1/2})H_1(-z^{-1/2})A_{21}(-z^{1/2})$, in the last equation together with the Z transforms of (6) and (10), we get that $C_0^\approx(z) = C_0(z)$. Thus, we have proven (2) and (5).

Part 4: It is necessary and sufficient to show that the Fourier transform $\hat{a}(f)$ of the sampled autocorrelation $a(k)$ of ψ_1^b is bounded for f a.e. by two positive constants $C_2 \geq C_1 > 0$ [5, Th. 2]. Using the fact that $\langle \psi_1^b(t), \psi_1^b(t-l)\rangle = 2^{-1}\langle \psi_1^b(t/2), \psi_1^b((t-2l)/2)\rangle_{L_2}$, and (3), we obtain $a = 2 \downarrow_2 [h_2 * h_2^\vee * a_{11} * a_{21} * a_{21}^\vee]$. We do not change the last identity if we convolve it with $a_{22} * a_{22}^{-1}$ to obtain $a = 2 \downarrow_2 [h_2 * h_2^\vee * a_{11} * a_{21} * a_{21}^\vee * a_{22} * a_{22}^{-1}]$. Because $\{\phi_{m(0,k)}\}_{k\in\mathbf{Z}}$ are Riesz bases for V_0^m and because $\cos[\theta(V_0^1, V_0^2)) \neq 0$, the Fourier transforms $\hat{a}_{ij}(f)$ of all the sequences a_{ij} that appear in the expression of a are bounded above and below by positive constants $0 < \alpha_1 \leq \hat{a}_{ij}(f) \leq \alpha_2 < \infty$ for f a.e. Using this fact and the fact that $a_{22} = 2 \downarrow_2 [h_2 * h_2^\vee * a_{22}]$, we deduce that $\hat{a}(f)$ is bounded above and below almost everywhere, which completes the proof. A different proof can also be obtained using [3, Th. 3.3]. $\square$

REFERENCES

[1] P. Abry and A. Aldroubi, "Designing multiresolution analysis-type wavelets and their fast algorithms," *J. Fourier Anal. Appl.,* vol. 2, no. 2, pp. 136–159, 1995.

[2] ——, "Semi- and bi-orthogonal MRA-type wavelet design and their fast algorithms," in *Mathematical Imaging: Wavelet Applications in Signal and Image Processing,* A. F. Laine and M. Unser, Eds. Bellingham, WA: SPIE, 1995, vol. 2569, pp. 452–463.

[3] A. Aldroubi, "Oblique and hierarchical multiwavelet bases," *Appl. Comput. Harmon. Anal.,* vol. 4, no. 3, pp. 231–263, 1997.

[4] ——, "Oblique projections in atomic spaces," *Proc. Amer. Math. Soc.,* vol. 124, pp. 2051–2060, 1996.

[5] A. Aldroubi and M. Unser, "Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory," *Numer. Funct. Anal. Optim.,* vol. 15, nos. 1/2, pp. 1–21, 1994.

[6] ——, "Families of multiresolution and wavelet spaces with optimal properties," *Numer. Funct. Anal. Optim.,* vol. 14, no. 5, pp. 417–446, 1993.

[7] A. Cohen, I. Daubechies, and J. C. Fauveau, "Biorthogonal bases of compactly supported wavelets," *Commun. Pure Appl. Math.,* vol. 45, pp. 485–560, 1992.

[8] S. Mallat, "Multiresolution approximations and wavelet orthonormal bases of L^2," *Trans. Amer. Math. Soc.,* vol. 315, pp. 69–97, 1989.

[9] W. Sweldens, The lifting scheme: A custom-design construction of biorthogonal wavelets, *Appl. Comput. Harmon. Anal.,* vol. 3, no. 2, pp. 186–200, 1996.

[10] M. Unser, A. Aldroubi, and M. Eden, "Fast B-spline transforms for continuous image representation and interpolation," *IEEE Trans. Pattern Anal. Machine Intell.,* vol. 13, pp. 277–285, 1991.

[11] M. Unser and A. Aldroubi, "A general sampling theory for non-ideal acquisition devices," *IEEE Trans. Signal Processing,* vol. 42, pp. 2915–2925, Nov. 1994.

Sampling Approximation of Smooth Functions via Generalized Coiflets

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***Abstract*—We present the sampling approximation power of a newly constructed class of compactly supported orthonormal wavelets called *generalized coiflets*. We study the accuracy of generalized coiflets-based sampling approximation of smooth functions by developing convergence rates for the pointwise approximation error as well as its L^p-norm. We show i) that the L^2-error due to the approximation of expansion coefficients by function samples is asymptotically negligible compared with that due to projection and ii) that generalized coiflets can achieve asymptotically better approximation than the original coiflets.**

***Index Terms*— Approximation methods, signal reconstruction, signal sampling wavelet transforms.**

I. INTRODUCTION

During the past decade, the theory of wavelets and multiresolution analysis has established itself firmly as one of the most successful methods for a broad range of signal processing applications. We first review some fundamentals from wavelet theory on which this paper is based. For a more detailed discussion, see related literature (e.g., [1]–[3]).

Let $h: \mathbb{Z} \to \mathbb{R}$ be the impulse response of the lowpass filter associated with an orthonormal wavelet $\psi: \mathbb{R} \to \mathbb{R}$. The scaling function $\phi: \mathbb{R} \to \mathbb{R}$ is recursively defined by the *dilation equation* (or *refinement equation*)

$$\hat{\phi}(\omega) = H(e^{j\omega/2})\hat{\phi}(\omega/2) \quad (1)$$

where $\hat{\phi}(\omega) = \int_{\mathbb{R}} \phi(t)e^{-j\omega t}\,dt$ and $H(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n}$. The scaled and translated versions of the wavelet $\{\psi(2^i t - k)\}_{i,k}$ constitute an orthonormal basis of $L^2(\mathbb{R})$. Most families of wavelet bases are indexed by the number of *vanishing moments* for wavelets (e.g., [4]–[6]).

An important problem in wavelet-based multiresolution approximation theory is to measure the decay of the approximation error as resolution increases, given some *a priori* knowledge on the smoothness of the function being approximated [7]–[13]. Let f be a smooth L^2 function in the sense that $f^{(L)}$ is square integrable, and let ϕ be an Lth-order orthonormal scaling function. Define $\mathcal{P}_i f$ to be the approximation of f at resolution 2^{-i}, i.e., the orthogonal

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