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Addendum to 'Interaction of Wetting Fronts with an Impervious Surface'

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In two previous papers (Parlange *et al.*, 1994a,b, referred to as Part I and Part II), the various processes describing the interaction of a wetting front with an impervious layer were described in detail, using numerical and analytical tools. Four stages were recognised:

- at short times, water infiltration as in a semi-infinite medium;
- then, at the early stages of interaction, a superposition principle holds when the incoming water appears to be reflected by the impervious surface without affecting the water intake;
- in the third, intermediary stage, the water intake remains as for a semi-infinite medium, but the water content at the interface grows more slowly than suggested by the superposition principle;
- finally, the water intake itself slows down below that of a semi-infinite medium.

In this addendum, the superposition principle is improved so that it can be used in a trivial manner to describe accurately all three regimes after interation begins. The previous papers, Parts I and II, remain valid in the sense that the four regimes were described properly. Here we give a simpler analytical description which applies for all times and thus should be very helpful for a numerical description of the phenomenon.

In Part I, the superposition principle led to the determination of $\theta_L = \theta(x = L)$ being twice the value of θ at $x = L$ in the absence of the impervious surface. Here θ is the water content, and x is the distance from the surface where water enters the soil layer, the impervious layer being at $x = L$. It was also pointed out in Part I that steady-state linearization of the transport equation is possible when $\int D\,\mathrm{d}\theta$ is used as the independent variable as first demonstated by Gardner (1958), and that linearization holds for all time if D is constant. Thus the superposition principle was expected to hold accurately for θ, initially and near $x = L$ when θ is small and D near constant. In addition, and more importantly, D is near constant at the wall since θ obeys $\partial\theta/\partial x = 0$ at $x = L$ (the case of an impervious wall).

On the other hand, it is also possible to apply the superposition principle on $\int D\,\mathrm{d}\theta$, instead of θ, since $\partial\theta/\partial t$ is usually a small term in the transport equation and the steady-state condition of Gardner is almost satisfied. In this addendum, we will explore the implication of this alternate approach. However, instead of $\int D\,\mathrm{d}\theta$, we shall use $\int D\theta^{-1}\,\mathrm{d}\theta$ because it occurs naturally in the descriptions of the soil-water profile and $\int D\theta^{-1}\,\mathrm{d}\theta$ behaves very much like $\int D\,\mathrm{d}\theta$, except possible near $\theta \sim 0$. The possibility of a singularity at $\theta \sim 0$ (leading to a 'tail' in the profile; see Part I) is removed simply by finally taking G as independent variable within:

$$G(\theta) \equiv \int_0^\theta [D(\bar{\theta}) - D(0)]\bar{\theta}^{-1}\,\mathrm{d}\bar{\theta}. \tag{1}$$

Here, reduced variables are used so that $\theta = 0$ is the initial water content and $\theta = 1$ at $x = 0$. $D(0)$ could of course be zero, but is always finite, whereas $D(1)$ could be infinite. G is essentially Gardner's variable.

The superposition principle based on Gardner's variable is written as:

$$G(\theta_L) = 2G(\theta_{L\infty}), \tag{2}$$

where, again, θ_L is the unknown value of θ at $x = L$,and $\theta_{L\infty}$ is the value of θ at $x = L$ for a semi-infinite medium, i.e. without the impervious wall. For instance, in Parts I and II, θ for a semi-infinite medium was taken simply and accurately as:

$$2\int_\theta^1 D\theta^{-1}\,\mathrm{d}\theta = S\phi + A/2\phi^2, \tag{3}$$

where $\phi = x/\sqrt{t}$ and A is given by:

$$2\int_0^1 D\,\mathrm{d}\theta = \int_0^1 \phi^2\,\mathrm{d}\theta.$$

When the interaction begins, and as long as $\mathrm{d}D/\mathrm{d}\theta$ is well behaved at small θ's, Equations (1) and (2) yield:

$$\theta_L = 2\theta_{L\infty}, \tag{4}$$

which is precisely the superposition principle used in Part I, as expected.

To apply Equation (2), let us use the example of Parts I and II with:

$$D = D_0 \exp 4\theta, \tag{5}$$

and then Equation(2) yields:

$$E_i(4\theta_L) - \ln(4\theta_L) = 2[E_i(4\theta_{L\infty}) - \ln(4\theta_{L\infty})] - \gamma, \tag{6}$$

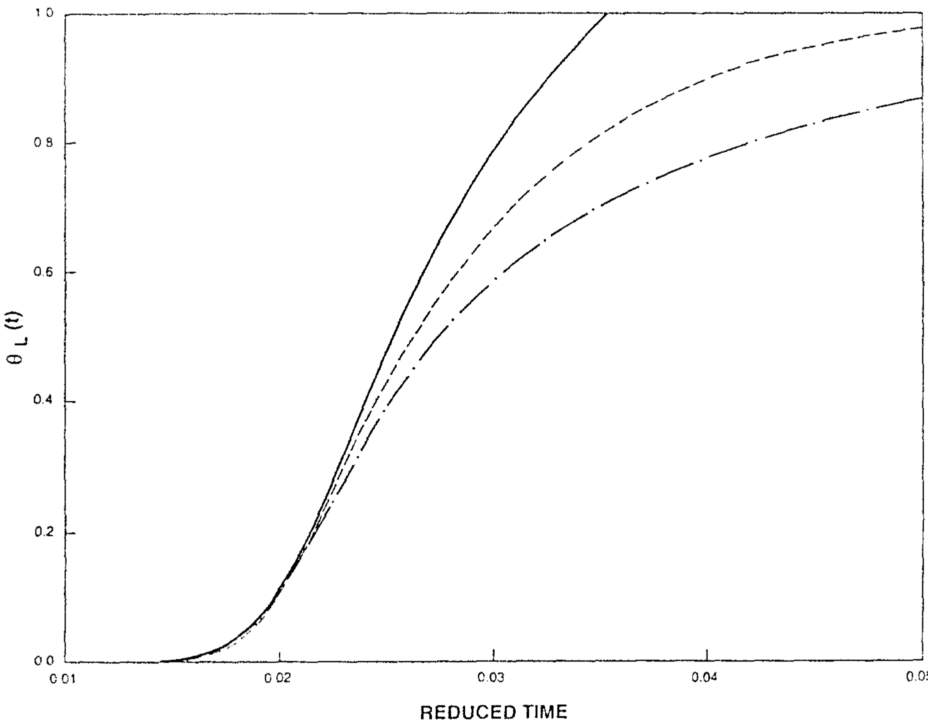

Fig. 1. A comparison of $\theta_L(t)$ obtained numerically (– – –), with $\theta_L(t)$ obtained using Equation (4) (—) and $\theta_L(t)$ obtained using Equation (2) or Equation (6) (– · –).

where γ is Euler's constant and E_i the exponential integral (Abramowitz and Stegun, 1968) and $\theta_{L\infty}$ is given by Equation(3) for $x = L$.

Figure 1 gives the exact $\theta_L(t)$ obtained numerically in Part I, θ_L given by Equation (4) following the theory of Part I, and θ_L given by Equation (2) or Equation (6). We see once that all of them merge at the beginning of the interaction and diverge as time increases, θ_L given by Equation (4) increasing too fast, θ_L given by Equation (2) increasing too slowly. This was to be expected since, as explained earlier Equation (4) was based on the fact that θ varies little when x is closed to L, whereas Equation (2) works for θ, varying rapidly in the wetting front, which is the case unless x is very close to L. Thus, since the wetting front has both features, near constant for x near L and varying rapidly slightly away from the wall, it is natural that θ_L should vary between the two extremes given by Equations (2) and (4).

By interpolating between the two extremes, one could obtain a prediction close to the numerical solution. This can be achieved by using an interpolated diffusivity, D_i, varying with θ half-way between a constant and the true D, e.g. the square root of D, so that G in Equation (1) is replaced by:

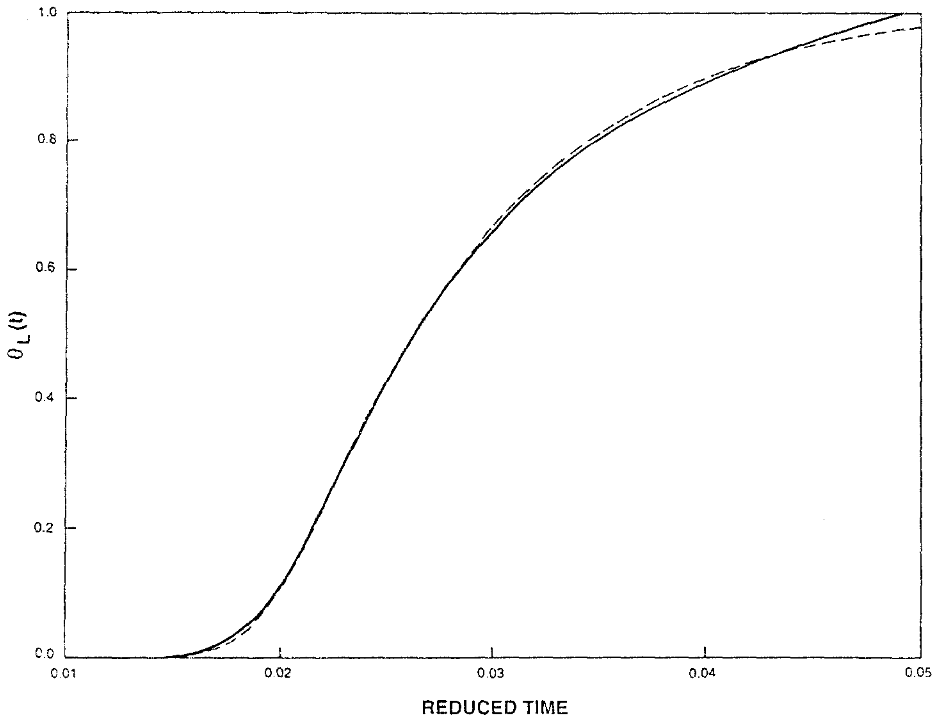

Fig. 2. A comparison of $\theta_L(t)$ obtained numerically (– – –), with $\theta_L(t)$ obtained using Equation (9) (—).

$$G_{\text{int}}(\theta) = \int_0^{\theta} [\sqrt{D} - \sqrt{D_0}]\bar{\theta}^{-1}\,\mathrm{d}\bar{\theta}, \tag{7}$$

and then Equation (2) gives:

$$G_{\text{int}}(\theta_L) = 2G_{\text{int}}(\theta_{L\infty}). \tag{8}$$

Now using Equation (5) in Equation (8) yields:

$$E_i(2\theta_L) - \ln(2\theta_L) = 2[E_i(2\theta_{L\infty}) - \ln(2\theta_{L\infty})] - \gamma. \tag{9}$$

Figure 2 shows the solution obtained from Equation (9) and the numerical solution. It can be seen that the agreement between the two solutions is excellent.

In conclusion, we have improved the previous superposition principle to take into better account the mixed behaviour of the profile near an impervious wall. The result is quite accurate essentially for all times and should prove of great help to describe the interaction of steep fronts with each other, or surfaces when the numerical solution becomes difficult to handle.

References

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