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Superposition Principle for Short-Term Solutions of Richards' Equation: Application to the Interaction of Wetting Fronts with an Impervious Surface

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Abstract. The interaction of a wetting front with an impervious layer is described by adding a reflected solution to the incoming solution for a semi-infinite medium. It is shown and checked by comparison with a numerical solution that the result is accurate during the early times of the interaction between the front and the impervious surface. This superposition principle is quite general and should prove especially useful to initiate numerical schemes by this analytical approximation as in the early times singularities are difficult to describe numerically.

Key words: Infiltration, layered soils, superposition of solutions, impervious surface.

1. Introduction

The aim of this paper is to discuss a superposition principle which can be used to describe the short time behaviour of solutions to Richards' equation. This principle should be especially useful to describe events of short duration. For this reason the principle will be presented and illustrated by considering the interaction of a wetting front with an impervious interface. It is of course trivial to apply this superposition of solution for linear governing equations. However, Richards' equation is strongly nonlinear and *exact* linearisation techniques, e.g. using Bäcklund transformations, apply only for Fujita diffusivities (Rogers *et al.*, 1983). Most systematic studies based on group classification and transformation properties of the equation (Lisle and Parlange, 1993) still lead to generalized Fujita-type diffusivities. Here on the contrary the method applies to arbitrary soil-water diffusivities, but the result is only approximate.

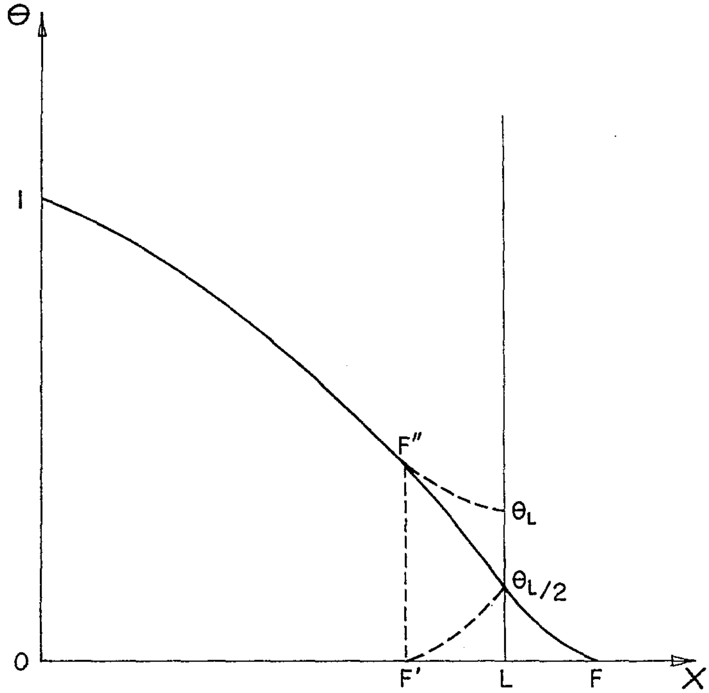

Fig. 1. Illustration of the Superposition principle: The incoming front for a semi-infinite medium is shown by a solid line. The reflected part ending in F', shown by a dashed line is added to the incoming front, resulting in the solution ending at $\theta = \theta_L$ and $x = L$.

Natural soils are usually stratified and the interaction of the wetting front with the interface is difficult to handle mathematically because of the usually large gradients in water concentration at the front. These, in turn, cause water content and potential to increase very rapidly at the interface (Braddock *et al.*, 1982; Parlange *et al.*, 1982; Hornung *et al.*, 1987). The situation is illustrated in Figure 1. The solid line shows the incoming water profile that would exist in the absence of an interface at $x = L$. Due to the interaction with the interface the water content increases from the point F'' to the position $x = L$, as shown by the dashed curve. Another difficulty occurs if there is no well-defined wetting front, but instead a long 'tail' (Braddock and Parlange, 1980). With a well defined wetting front, there is a well-defined transition time such that, in early times, water infiltrates as in a semi-infinite medium (Hornung *et al.*, 1987). If not, some small interaction between the front and the interface takes place at once.

In this paper we consider the early stages of interaction before any significant amount of water has crossed the interface. This is obviously the case when the interface is impervious to water movement. More importantly, this is also the case when the second layer is coarser than the first. Then water will not penetrate the interface until the potential at the interface has reached some critical value: 'the water entry pressure'. This concept is fundamental, for instance, to understand the formation of fingers in the coarse layer (Hillel and Baker, 1988; Baker and Hillel, 1990).

2. Superposition Principle

The initial water content is taken as Θ_i and the surface water content as Θ_s, which could be below saturation. In the following, we shall use a reduced water content

$$\theta = (\Theta - \Theta_i)/(\Theta_s - \Theta_i). \tag{1}$$

Water transport obeys Richards' equation which is a nonlinear diffusion equation

$$\partial\theta/\partial t = \partial[D\partial\theta/\partial x]/\partial x - \partial K/\partial x, \tag{2}$$

where t is the time and the distance x is measured from the surface of the first layer. Equation (2) is to be solved with the initial and boundary conditions

$$\theta = 0, \qquad t \leq 0, \qquad 0 \leq x \leq L, \tag{3}$$

$$\partial\theta/\partial x = 0, \qquad x = L, \tag{4}$$

$$\theta = 1, \qquad x = 0, \qquad t > 0, \tag{5}$$

where L is the thickness of the layer. Equation (4) obviously holds only until water crosses the interface. The two parameters D and K are the soil-water diffusivity and conductivity, respectively. The effect of K, i.e. gravity, affects the shape of the incoming front before interaction with the wall takes place. However, during the interaction, the short time behaviour is unaffected by gravity because the rapid changes of large gradients are dominated by the time derivative and the second derivative in x. Thus gravity is not essential to understand the superposition principle and for that reason gravity is ignored in the following, i.e. the front is moving horizontally.

For linear equations, two independent solutions can be added to obtain a new solution. As is well known, this is a powerful technique to solve Laplace's equation, especially with the use of images when surfaces are present, as here. Of course, it is the nature of Equation (1) to be nonlinear because of D and K.

In a fundamental paper, Gardner (1958) pointed out that the steady-state equation, dropping $\partial\theta/\partial t$ in Equation (1), can be linearized using $\int D \mathrm{d}\theta$ as independent variable if $\ln K$ is a linear function of the soil-water potential. In a series of papers beginning in 1971 (see Parlange (1971)) it was pointed out that even for unsteady problems, the term $\partial\theta/\partial t$ introduces only secondary corrections to the shape of the water profile. Thus we expect that the superposition principle should hold as an approximation, and as explained above the exact relationship between K and the soil-water potential will need be crucial if short time processes are discussed.

We also expect that the approximation will be best if the interaction of a wetting front with a surface introduces small corrections, so that the error on those corrections is negligible, which is another reason to apply the principle for relatively short times only.

Note that the superposition principle should hold whatever the dimensionality of the problem (it is not limited to one dimension as in Equation (2)) and whatever the initial and boundary conditions. The only requirement is that the superposition should be used for short times only. However, we must emphasize that this is a limited and approximate principle. The correction added by superimposition must remain small so that errors in the correction are truly negligible.

3. Application

We shall now check the existence and validity of a superposition principle for the problem summarized in Equations (2) to (5), with the further simplification of dropping the gravity term. This does not change significantly the qualitative features of the incoming front as long as the layer is not too thick (Parlange, 1993). Specifically, the two main features of the solutions, i.e. the presence of an abrupt front and a long tail, result from the nonlinear behavior of D and not from gravity. Hence, the accuracy of the superposition principle should not be affected by gravity, especially in the short time. However, the main reason for dropping gravity at this stage is that to check the accuracy of the theory, we need an (essentially) exact numerical solution with which to compare the theory, and the only one available is without gravity (Braddock *et al.*, 1982; Parlange *et al.*, 1982). To be specific, we consider an exponential soil-water diffusivity

$$D = D_0 \exp n\theta, \tag{6}$$

which gives an excellent correlation for many soils with $n \simeq 8$ (Reichardt *et al.*, 1972). Note that the term $\partial\theta/\partial t$ in Equation (2) is less important as n increases (in general, as D approaches a delta function). It is in that limit that the method introduced by Parlange (1971) is more accurate, and hence when the superposition principle will hold better. The value $n \simeq 8$ is large and thus it is critical of the method to consider lower values of n since, if it is accurate at one value, it is always more accurate for larger n's. Another reason for using lower n's is that the accurate numerical solution is more difficult for larger n as the increasingly large gradients in the incoming front become more difficult to describe. Braddock *et al.* (1982) estimate that there are increasing numerical errors if n is higher than four. Thus, as an illustration, we shall use $n = 4$ as an upper limit where the numerical results are very accurate. Good agreement for $n = 4$ will ensure that the analytical method can be relied on for all realistic soils which have $n > 4$. It is somewhat paradoxical that Equation (2) can be truly linear with D constant, and linearized when D approaches a delta function. Only for those two extremes is superposition of solutions accurate.

To describe the incoming front we start with the approximation (Parlange *et al.*, 1992):

$$2\int_{\theta}^{1} D\bar{\theta}^{-1}\,\mathrm{d}\bar{\theta} = S\phi + A/2\phi^2, \tag{7}$$

which is simple and yet extremely accurate ($\bar{\theta}$ is the variable of integration). Here ϕ is defined as

$$\phi = x/\sqrt{t}. \tag{8}$$

Mathematical details justifying Equation (7) can be found in Parlange *et al.* (1992).

Alternatively the slightly less convenient but equally accurate Equation (13) in Parlange *et al.* (1993) could be used instead of Equation (7) above.

Equation (7) is the solution when the layer is infinitely thick (Parlange *et al.*, 1993). In the original paper (Parlange, 1971), the ϕ^2 term in Equation (7) was not included. The coefficient S is the usual sorptivity, which, for an exponential diffusivity, is known exactly (Braddock *et al.*, 1981) and in general can be calculated with any degree of accuracy very simply (Parlange, 1975; Elrick and Robin, 1981). The coefficient A is calculated so that Equation (7) satisfies the exact integral condition:

$$2\int_0^1 D\,\mathrm{d}\theta = \int_0^1 \phi^2\,\mathrm{d}\theta, \tag{9}$$

or

$$S^2 = 2\int_0^1 D\,\mathrm{d}\theta(1 - A/2). \tag{10}$$

Equation (7) provides us with a solution which obviously cannot satisfy Equation (4) at any time, since $\phi \sim \ln\theta^{-1}$ for θ small, i.e. the profile extends to infinity due to the tail (Parlange, 1972). The superposition principle is quite obvious: We take the mirror image of the solution given by Equation (7) for $x > L$ and add it to the solution for $x < L$ (see Figure 1). By construction, Equation (4) is now satisfied and a composite profile is created. We are now going to estimate its accuracy. First, it is clear that, as time progresses, the point F in Figure 1, which represents the wetting front (ignoring the tail), moves further right and F' (its mirror image) moves further left. When F' is at the origin, the boundary condition $\theta = 1$ at $x = 0$ will not be satisfied accurately and the solution will fail. In a similar vein, the constructed solution gives for water intake I

$$I = S\sqrt{t}, \tag{11}$$

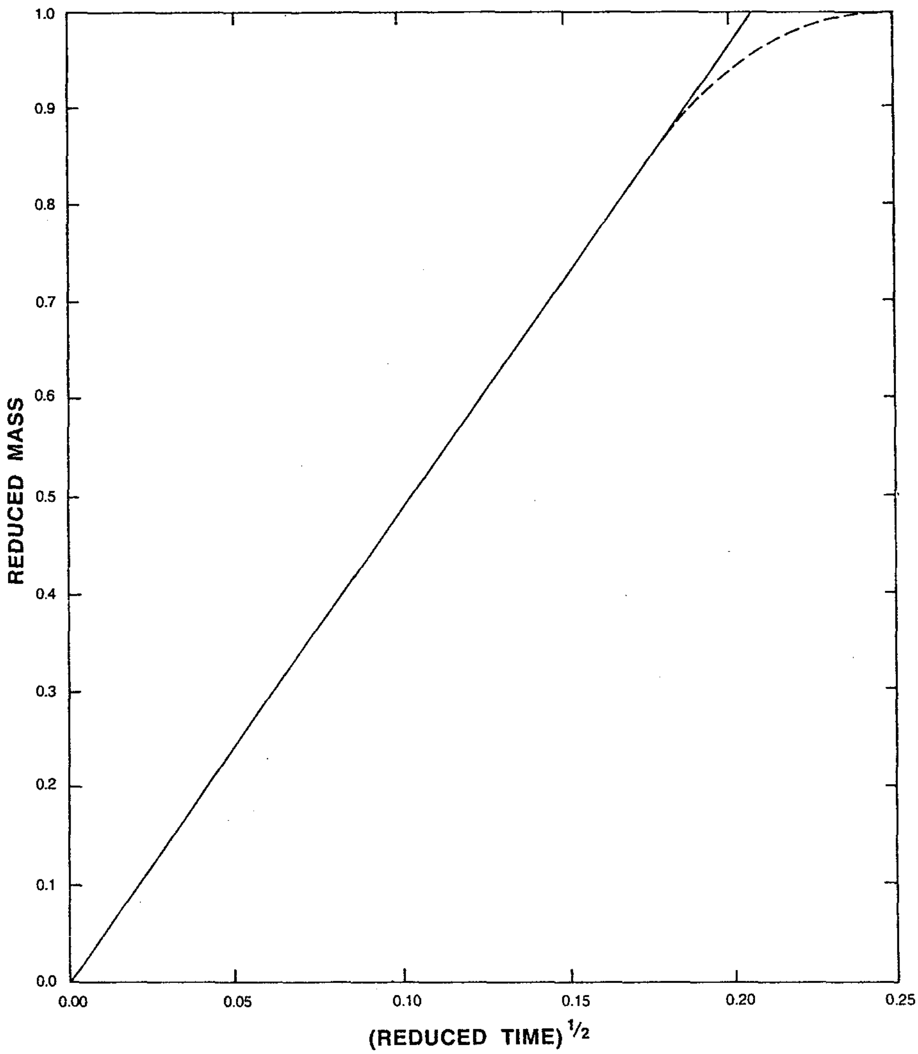

Fig. 2. Cumulative infiltration given as the reduced mass I/L, where $I = \int_0^L \theta \, dx$, as a function of the square root of the reduced time $\sqrt{tD_0/L^2}$. $S = 4.8331\sqrt{D_0}$ and $A = 0.1284$.

as for a semi-infinite medium. Clearly, at some time, I must fall below $S\sqrt{t}$. Similarly, $\theta_L = \theta(x = L)$ is twice the value of θ at $x = L$ given by Equation (7). This must lead to a θ_L which increases too fast with time, and may reach values above one and the superposition fails. Figure 1 shows very simply why the superposition principle is more accurate when D approaches a delta-function. In that limit, the profile is a step function, so that there is no interaction at all with the wall at $x = L$ until it reaches it and then the process stops suddenly (Equation (11) holds exactly until that time and then I remains constant).

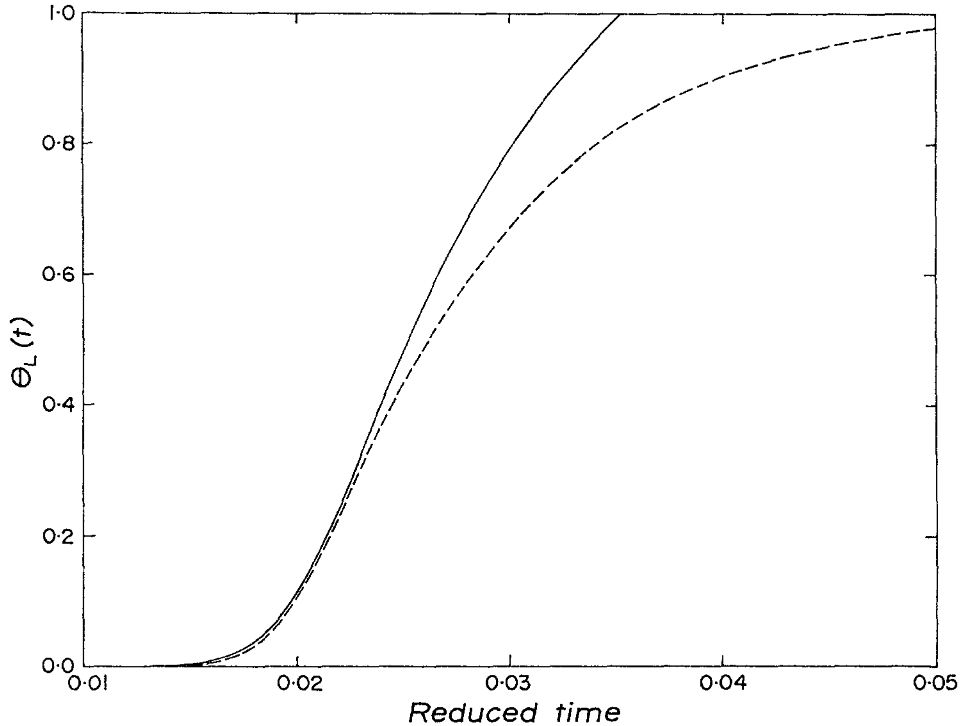

Fig. 3. Values of θ_L as a function of the reduced time tD_0/L^2. The dashed line is the numerical solution and the solid line results from the superposition principle as illustrated in Figure 1. Solid line is $I/L = 4.8331\sqrt{tD_0/L^2}$, and dashed line is obtained numerically.

Figure 2 gives $I(t)$ obtained numerically for $n = 4$ (Braddock *et al.*, 1982). Remarkably, Equation (11) holds for a long time, in agreement with the superposition principle. A more critical feature is $\theta_L(t)$. Figure 3 shows that θ_L is essentially zero until the dimensionless time D_0t/L^2 is about 0.015. Then the increase is rather slow (interaction with the tail), until it increases very rapidly (interaction with the main wetting front). Up to times of 0.025, the superposition principle yields a very accurate θ_L. Only for greater times is θ_L increasing less rapidly. This accuracy is surprising, considering how simple the analytical solution is. Solving Equation (7) at $x = 1$ gives, at once, $\theta = \theta_L/2$ as a function of time. To obtain $\theta(x,\ t)$ for $x < 1$ simply requires solving a quadratic equation, Equation (7). Figure 4 gives a few profiles for reduced times up to 0.0245. Again, the accuracy is excellent. We notice that the area under the curves is exact, i.e. $I = S\sqrt{t}$. Thus the higher value for the analytical θ_L results in a profile crossing very slightly under the numerical results away from the wall. The prediction of θ_L is crucial: It is at $x = L$ that the prediction of θ is the worst.

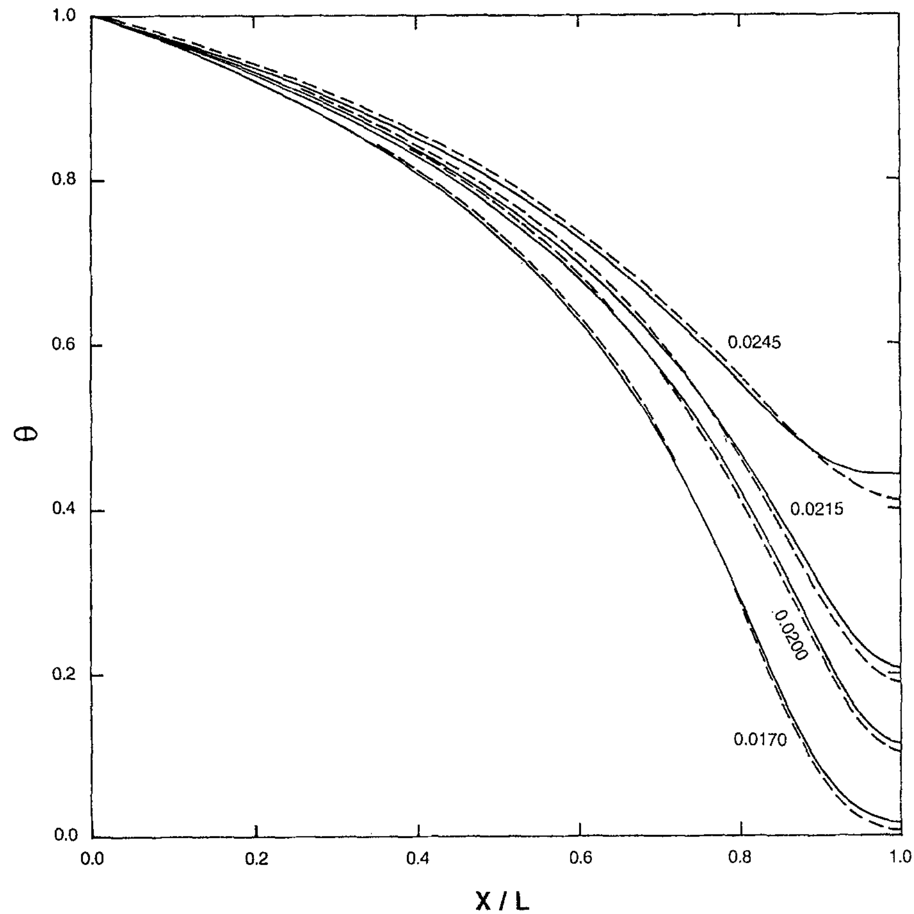

Fig. 4. Profiles calculated from Equation (7) with the mirror image added, following the construction of Figure 1 are shown by the solid lines. Numerical profiles are shown by the dashed lines. Reduced times are indicated alongside each profile.

4. Conclusion

From the superposition principle, we have been able to obtain very simply some valuable information about a wetting front interaction with a surface. The analytical solution was shown to be extremely accurate in the early stages of the interaction when the principle is accurate. It is also at that stage that numerical analysis is difficult due to rapid changes in space and time. In general, the superposition principle might be most practical to initiate numerical solutions accurately by the use of the analytical solution for short times.

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