*This is the version with appendix of the Infocom 2005 paper with same title.*

The implementation in ns2 of the models in this paper (random trip models) can be obtained freely from

http://ica1www.epfl.ch/RandomTrip/

# Perfect Simulation and Stationarity of a Class of Mobility Models

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***Abstract*— We define "random trip", a generic mobility model for independent mobiles that contains as special cases: the random waypoint on convex or non convex domains, random walk with reflection or wrapping, city section, space graph and other models. We use Palm calculus to study the model and give a necessary and sufficient condition for a stationary regime to exist. When this condition is satisfied, we compute the stationary regime and give an algorithm to start a simulation in steady state (perfect simulation). The algorithm does not require the knowledge of geometric constants. For the special case of random waypoint, we provide for the first time a proof and a sufficient and necessary condition of the existence of a stationary regime. Further, we extend its applicability to a broad class of non convex and multi-site examples, and provide a ready-to-use algorithm for perfect simulation. For the special case of random walks with reflection or wrapping, we show that, in the stationary regime, the mobile location is uniformly distributed and is independent of the speed vector, and that there is no speed decay. Our framework provides a rich set of well understood models that can be used to simulate mobile networks with independent node movements.**

***Index Terms*— Simulations, Stochastic processes /Queueing theory**

## I. Introduction

### A. Mobility Models and Stationarity

Our goal is to provide a class of mobility models (1) that is rich enough to accommodate a large variety of examples and (2) whose simulation can easily be mastered. The latter point is motivated by recent findings about the random waypoint, an apparently simple model that fits in our framework. The simulation of the random waypoint poses a surprising number of challenges, such as speed decay, a change in the distribution of location and speed as the simulation progresses [15], [11], [13], [8]. All of these observations are related to the existence of a stationary regime. Camp, Navidi and Bauer [13] point out that if the model has a stationary regime, it is important to simulate it in this regime; otherwise, if the initial configuration is not sampled from the stationary regime, the performance evaluation of a system under study may be biased and non reproducible.

### B. Perfect Simulation

A standard method for avoiding such a bias is to (1) make sure the used model has a stationary regime and (2) remove the beginning of all simulation runs in the hope that long runs converge to stationary regime. However, as we show now, the length of transients may be prohibitively long for even simple mobility models. Our example is the space graph explained in Figure 1. There are a little less than 5000 possible paths; in Figure 1 we show the distribution of the path used by the mobile at time $t$, given that initially a path is selected uniformly among all possible paths (i.e. the mobile is initially placed uniformly among all nodes). This was obtained analytically (see Appendixfor details). Figure 1 illustrates that the transient period may be long compared to typical simulation lengths (for example 900 sec in [5]). A major difficulty with transient removal is to know when the transient ends; if it may be long, as we illustrated, considerable care should be used. An alternative, called "perfect simulation", is to sample the initial simulation state from the stationary regime. For most models this is hard to do, but, as we show, this is quite easy (from an implementation viewpoint) for the random trip model. Perfect simulation for the random waypoint was advocated and solved by Navidi and Camp who gave the stationary distribution in [12].

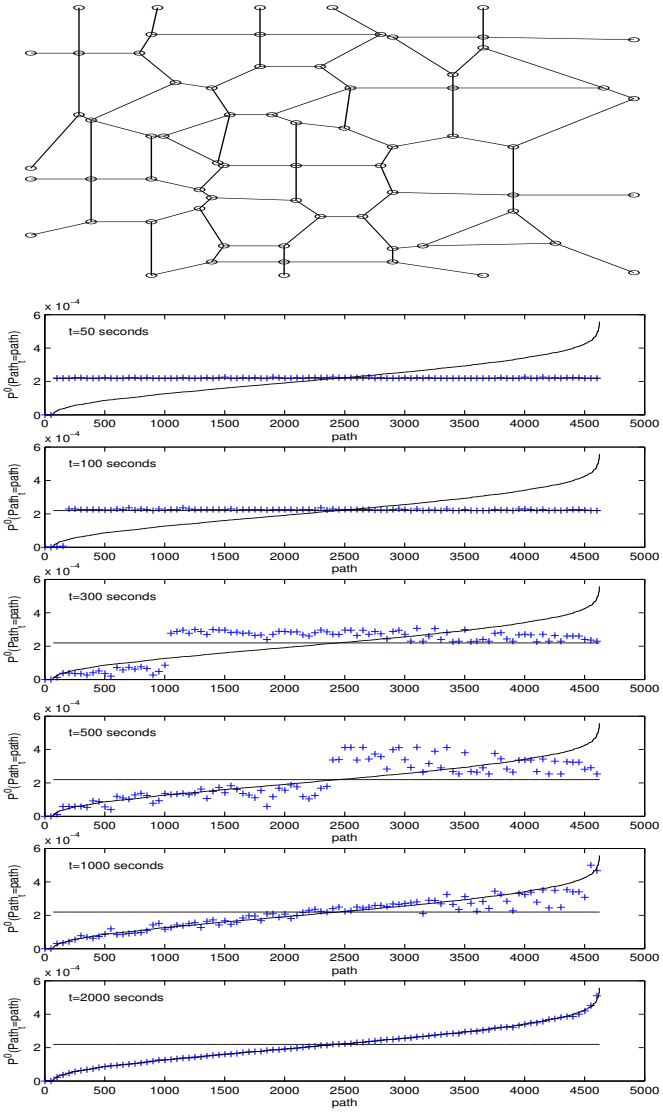


Fig. 1. Top: "Space Graph", a model proposed by Jardosh et al [9]. A mobile starts from a randomly chosen circle and goes along a shortest path towards another randomly chosen circle. Numerical speed is constant = 1.25 m/s. Bounding area 1 km $\times$1 km. Bottom: Probability distribution of the path used by a mobile at time $t$. Initially, the path is chosen uniformly among all possible paths. $x$-axis: path index, sorted by path length; $y$-axis: probability that this path is used at time $t$ for $t = 50, 100, 300, 500, 1000, 2000$ seconds of simulated time. Horizontal solid line: initial distribution; other solid line: time-stationary distribution. The transient lasts for a long time.

### C. The Palm Calculus Framework

The derivations in [12] involve long and sophisticated computations. We use a different approach, based on Palm calculus, a set of formulas that relate time averages to event averages. Palm calculus is now well established, but not widely used or even known in applied areas. For a quick overview of Palm calculus, see [10]; for a full fledged theory, see [1]. This framework allows us to generalize the results in [12] to a broad class of models, as discussed next. Incidentally, even for the original random waypoint model, we provide new elements: a proof that a stationary regime exists when $v_{\min} > 0$, a proof that location and speed are independent in the stationary regime (Navidi and Camp implicitly assume both), and a sampling algorithm that, for complicated, non convex areas, does not require a priori computation of geometric integrals. More fundamentally, the Palm calculus framework allows us derive simple sampling algorithms for the generic random trip model – a task that would be formidable without this tool.

### D. Contributions of This Paper

As a first step towards our goal, we give a model for independent mobiles (leaving group mobility models for further study). The model is called "random trip". In the absence of established properties of real mobility patterns, it is not yet clear today what the requirements on a mobility model should be [6]. We focus here on a model that is able to *synthesize* an a priori assumed mobile behaviour. This leads to examples such as city driving models ("Space Graph" [9], "City section" or "Hierarchical random waypoint", called "restricted random waypoint" in [3]), simple airplane circulation models ("Random Waypoint on Sphere"), or the special purpose "Fish in a Bowl" and "Swiss Flag". In some cases, it is desirable to assume that node location is uniformly distributed in steady-state; this is provided by the two "Random Walk" examples and by "Random Waypoint on a Sphere". We give a definition of the model and a non exhaustive list of examples in Section II.

Our main contributions are:

- a generic model and a framework to analyze it;
- a proven necessary and sufficient condition for a stationary regime to exist; a proof that when the stationary regime exists it is unique. This appears to be new even for the classical random waypoint;
- a generalization of random waypoint perfect simulation to non convex areas;
- a sampling algorithm that does not require the computation of geometric integrals;
- the proof that for three examples (random walk on a rectangle with wrapping or reflection, random waypoint on sphere) the node location is uniform. For the random walk examples, the steady state is essentially the same as the naive initialization (with uniform node placement) and there is no speed decay. In contrast, there *is* speed decay for random waypoint on a sphere.

We focus on perfect simulation and leave for a further paper the study of convergence (and its rate) to the stationary regime when it exists. Due to space limitation, some of the details of the derivations are put in the appendix of the extended version of this paper, available

at [4]. A notation list is given in the next section. Related work is described at the end of the paper.

## II. A General Mobility Model

We consider a generic family of models, defined by the following framework.

1) The **domain** $\mathcal{A}$ is a closed, bounded, connected (not necessarily convex) subset of $\mathbb{R}^2$ or $\mathbb{R}^3$.

2) $\mathcal{P}$ is a set of **paths** on $\mathcal{A}$. A path is a continuous mapping from $[0,1]$ to $\mathcal{A}$ that has a continuous derivative except maybe at a finite number of points (this is necessary to define the speed).

For $p \in \mathcal{P}$, $p(0)$ is the origin of $p$, $p(1)$ is its destination, and $p(u)$ is the point on $p$ attained when a fraction $u \in [0,1]$ of the path is traversed.

3) **Trip Selection Rule**: A *trip* is the combination of a duration and a path. The position $X(t)$ of the mobile at time $t$ is defined iteratively as follows. There is a set $T_n \in \mathbb{R}$, $n \in \mathbb{Z}$ of *transition instants*, such that $T_0 \leq 0 < T_1 < T_2 < ....$ At time $T_n$, a path $P_n \in \mathcal{P}$ and a trip duration $S_n \in \mathbb{R}^+$ are drawn according to some specified *trip selection rule*, specific to the model. The next transition instant is $T_{n+1} = T_n + S_n$ and the position of the mobile is $X(t) = P_n(\frac{t-T_n}{S_n})$ for $T_n \leq t \leq T_{n+1}$.

The trip selection rule is constrained to choose a path $P_n$ such that $P_n(0) = P_{n-1}(1)$. Further, we assume that, with probability 1, the duration of the trip $S_n$ is positive (instantaneous transitions are not allowed).

4) **Default Initialization Rule**: at time $t = 0$, the initial position, path, position on path, and remaining time until the next transition are drawn according to some specified *default initialization rule*. A common default rule considers that time 0 is the first transition instant ($T_0 = 0$), and selects a path and trip duration according to the trip selection rule. However, as shown in Section I, this causes some problems, that are fixed by using the perfect simulation initialization rule, described in Section VI-B.

In addition, we do the following assumptions. They are essential for our model to be tractable, while supporting a very broad class of mobility models.

**(H1)** The trip selection rule depends on the past only through the current mobile location $M_n$ and the state of a Markov chain $I_n$. Further, the evolution of $I_n$ depends on the past only through the last state $I_{n-1}$. More precisely, $I_n$ (the *phase* of the process) is defined on some enumerable set $I$; it changes its value at transition instants $T_n$. Given that the phase selected at $T_n$ is $I_n = i$, and given the mobile location $M_n = m$ at time $T_n$, the path $P_n$ and the trip duration $S_n$ are drawn independently of the past until time $T_n$, with a distribution that may depend on $m$ and $i$ but not on $n$; the new value of the chain $I_{n+1}$ is drawn in a way that depends only on $i$.

**(H2)** Either of the following is true:

**(H2a)** The distribution of the mobile location $M_{n+1}$ at time $T_{n+1}$, conditional on the past phases up to $T_n$, depends only on the phase $I_n$ and not on $n$. Moreover, there exists a set of selected transitions $I^\circ \subseteq I^2$ of the Markov process of phases such that the distribution of location $M_{n+1}$, given all the past up to time $T_n$ and $(I_n, I_{n-1}) \in I^\circ$, depends only on the phase $I_n$. (The distribution of $M_{n+1}$, given $(I_n, I_{n-1}) = (i,j) \in I^\circ$, and $M_n = x$, may depend on $i$ but not on $x$, $n$ and $j$.); or

**(H2b)** The distribution of the mobile location $M_n$ at time $T_n$ does not depend on $n$, $M_n$ is independent of $I_n$, and $(S_n, I_{n+1})$ depends on the all past only through $I_n$.

**(H3)** The Markov chain $I_n$ is positive recurrent. For example, this is true if $I$ is finite and the graph of the chain $I_n$ is fully connected.

As we show next, these assumptions are verified by a very large class of mobility models.

**Notation List**

- $\mathcal{A}$ ($\subset \mathbb{R}^2$ or $\mathbb{R}^3$): model domain, connected and bounded
- $d(m,n)$ length of shortest path in $\mathcal{A}$ from $m \in \mathcal{A}$ to $n \in \mathcal{A}$; if $\mathcal{A}$ is convex $d(m,n) = \|m-n\|$
- $T_n$: $n$th transition time, at which a new trip is defined
- $I_n \in I, M_n \in \mathcal{A}, P_n \in \mathcal{P}, S_n \in (0,\infty)$: phase, starting point, path, trip duration for the $n$th trip
- $I(t) \in I, M(t) \in \mathcal{A}, P(t) \in \mathcal{P}, S(t) \in (0,\infty), X(t) \in \mathcal{A}$: phase, starting point, path, trip duration for the trip used by mobile at time $t$, location at time $t$. $X(T_n) = M_n$ and if $T_n \leq t < T_{n+1}$ then $I(t) = I_n$, $M(t) = M_n$ and $S(t) = S_n$.
- $U(t) \in [0,1]$: fraction of the current trip that was already traversed. Thus $U(t)S(t)$ is the time elapsed on the current trip and the location of the mobile at time $t$ is $X(t) = p(U(t))$, with $p = P(t)$. We assume that the trip is done at a speed proportional to the default speed of the path, i.e. if $T_n \leq t < T_{n+1}$ then $U(t) = \frac{t-T_n}{T_{n+1}-T_n} = \frac{t-T_n}{S_n}$
- It follows that the speed vector of the mobile at a time $t$ that is not an end of trip is $\vec{V}(t) = \frac{1}{S(t)} \frac{\partial}{\partial u} p(U(t))$, with $p = P(t)$ and the numerical speed is $V(t) = \left\| \vec{V}(t) \right\|$.
- For some random variable $Z$, $\mathbb{E}^0(Z)$ is the "Palm expectation", which can be interpreted as the expectation, conditional to the event that a transition occurs at time 0, when the system has a stationary regime. $\mathbb{E}^0$ denotes the event average viewpoint [1], [10]. For example $\mathbb{E}^0(S_0) = \mathbb{E}^0(S(0))$ is the average trip duration; in contrast, when the system has reached steady-state, $\mathbb{E}(S(0)) = \mathbb{E}(S(t))$ is the average duration of a trip, seen from an observer who samples the system at an arbitrary point in time. Both are usually different because the observer is more likely to sample a large trip duration.

## III. Examples

We give a non exhaustive catalog of examples and show that they all fit in our framework.

### A. Classical Random Waypoint With Pauses.

This is the classical random waypoint model. $\mathcal{A}$ is assumed to be convex ($\mathcal{A}$ is a rectangle or a disk in [8],

[6]). Paths are straight line segments: $p(u) = (1-u)m_0 + um_1$ for the segment with endpoints $m_0, m_1$. Pauses are special cases of paths, when endpoints are equal: $p(u) = m_0$. There are two phases $I = \{pause, move\}$. At a transition instant, the trip selection rule alternates the phase from *pause* to *move* or vice versa. If the new phase is *pause*, the trip duration $S_n$ is picked according to the density $f^0_{pause}(s)$; the path $P_n$ is a pause at the current point. If the new phase is *move*, the trip selection rules picks a point $M_{n+1}$ at random uniformly in $\mathcal{A}$, and a numerical speed $V_n$ according to the density $f^0_V(v)$. A classical choice (uniform speed) is $f^0_V(v) = \frac{1}{v_{\max} - v_{\min}} 1_{\{v_{\min} < v < v_{\max}\}}$. The trip duration is then $S_n = \frac{\|M_{n+1} - M_n\|}{V_n}$ and the path $P_n$ is the segment $[M_n, M_{n+1}]$. The default initialization rule starts the model at the beginning of a pause, at a location uniformly chosen in $\mathcal{A}$.

The trip selection rule makes its choices only based on the current phase, thus (H1) and (H2) are verified. Further, the Markov chain $I_n$ alternates between the two states $\{pause, move\}$, thus hypothesis (H3) is satisfied.

This model is well known; its stationary properties are studied in [13], [8]. However, even for this simple model our framework provides two new results: the proof of existence of a stationary regime, and a sampling algorithm for the stationary distribution over general areas that does not require the computation of geometric integrals.

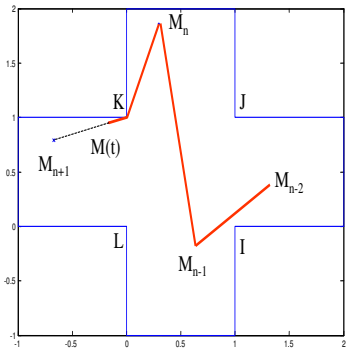


Fig. 2. Random Waypoint on a non convex domain (Swiss Flag). A trip is the shortest path inside the domain from a waypoint $M_n$ to the next. Waypoints $M_n$ are drawn uniformly in the domain. On the figure, the shortest path $M_n, M_{n+1}$ has two segments, with a breakpoint at $K$; the shortest paths $M_{n-1}, M_n$ and $M_{n-2}, M_{n-1}$ have one segment each. $M(t)$ is the current position.

### *B. Random Waypoint on Connected, Non Convex Domain.*

This is a variant of the classical random waypoint (Example III-A), where we relax the assumption that $\mathcal{A}$ is convex, but assume that $\mathcal{A}$ is a connected domain over which a uniform distribution is well defined. For two points $m, n$ in $\mathcal{A}$, we call $d(m,n)$ the distance from $m$ to $n$ in $\mathcal{A}$, i.e. the minimum length of a path entirely inside $\mathcal{A}$ that connects $m$ and $n$. $\mathcal{P}$ is the set of shortest paths between endpoints. The trip selection rule picks a new endpoint uniformly in $\mathcal{A}$, and the next path is the shortest path to this endpoint. If there are several shortest paths, one of them is randomly chosen according to some probability distribution on the set of shortest paths (see "Swiss Flag" on Figure 2 for an example). The set of phases is $I = \{pause, move\}$. This model fits in our framework for the same reasons as the former example.

### *C. City Section.*

This is a special case of random waypoint on a non convex domain (Example III-B). The domain $\mathcal{A}$ is the space graph as in Figure 1). The domain is the union of the segments defined by the edges of the space graph. Arbitrary numeric speeds can be assigned to edges of the graph; the "distance" from one location to another is the travel time.

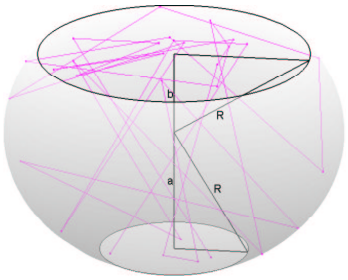


Fig. 3. Fish in a Bowl (Subset random waypoint). $\mathcal{A}$ is the volume of the sphere comprised between two horizontal planes. Waypoints are in the subset $\mathcal{A}_1$ equal to the boundary of the spheric part of $\mathcal{A}$. For perfect sampling we do not need to know average Euclidean distance between two random points on the surface of the bowl. It suffices to know that the distance is at most $2R$.

### *D. Subset Random Waypoint (Fish in a Bowl).*

This is a variant of random waypoint on a non convex domain (Example III-B). The difference is that the waypoints are restricted to a subset $\mathcal{A}_1$ of the domain $\mathcal{A}$ (Figure 3). The set of phases is $I = \{pause, move\}$.

### *E. Space Graph.*

We defined this model in Section I. It is a special case of the subset random waypoint (Example III-D), with $\mathcal{A}$ = the space graph and $\mathcal{A}_1$ = the set of vertices. Note that it differs from the City Section graph in that the waypoints are restricted to be vertices. The set of phases is $I = \{pause, move\}$.

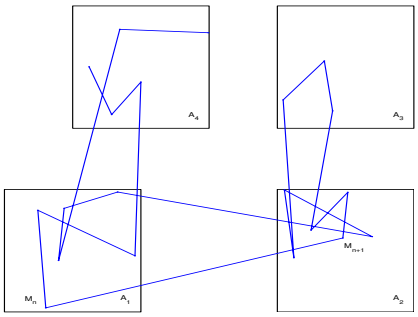

Fig. 4. Restricted random waypoint in the plane with four squares as subdomains. This model was introduced in [3] to simulate a wide-area routing protocol. The model may be interpreted as an idealized view of four towns represented by squares. A mobile moves according to random waypoint within a square for a random number of visits and then picks a point uniformly at random in another randomly chosen square as a destination. The figure shows a sample path of the mobile movement. The speed on the trip is chosen according to a distribution that depends on the origin and destination squares

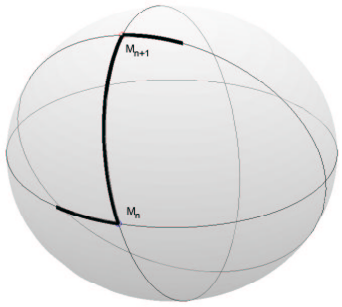


Fig. 5. Random waypoint on a sphere.

### F. Restricted Random Waypoint.

This model was originally introduced by Blažević et al [3]; see Figure 4 for a description. We define it slightly more generally as follows. As before, the domain $\mathcal{A}$ is connected, but not necessarily convex.

There are $L$ subdomains $\mathcal{A}_\ell \subset \mathcal{A}$, $\ell = 1,2,\ldots,L$. (In the original model [3], $\mathcal{A}_\ell$ is a square, $\ell = 1,2,3,4$, the subdomains are disjoint and $\mathcal{A}$ is the convex closure of $\bigcup_\ell \mathcal{A}_\ell$). The mobile executes a number of trips with endpoints in the same subdomain, then picks a new endpoint in some other subdomain $\ell'$ and goes there along a shortest path. $\ell'$ is chosen according to the transition matrix $Q(\ell,\ell')$, assumed to be irreducible and such that $Q(\ell,\ell) = 0$. There is a pause between trips.

More precisely, a phase is a quadruple $I_n = (\ell,\ell',r,\phi)$ with $\ell,\ell' \in \{1,...,L\}$ (origin and destination subdomains), $r \in \mathbb{N}$ (residual number of trips in the same subdomain, including this one) and $\phi \in \{pause, move\}$. If $\ell \neq \ell'$ then $r = 0$ else $r \geq 1$. The trip selection rule is executed at the end of a trip as follows. If $\phi = move$ then $\phi$ is set to *pause*, a pause is executed at the current location, for a duration drawn from a distribution that depends on the current subdomain, and $\ell,\ell',r$ are unchanged. Else $\phi$ is set to *move*, and $\ell,\ell',r$ are updated as follows. If $r \geq 1$, $r$ is decremented by 1. If $r \geq 2$, $\ell$ and $\ell'$ are unchanged (they must be equal). If $r = 1$ (the previous trip was the last with endpoints in the current subdomain), $\ell'$ is set to a new destination subdomain chosen according to the transition matrix $Q(\ell,\ell')$. If $r = 0$ (the previous trip was between subdomains) $\ell$ is set to the value of $\ell'$ and a new value of $r$ is drawn from a probability distribution that depends on $\ell'$. Then a new endpoint is selected uniformly in $\mathcal{A}_{\ell'}$ and the next trip is a shortest path from the current endpoint to this endpoint. For every trip, the numerical speed is selected according to a density that may depend on the origin and destination subdomains of the trip endpoints.

Note that all models III-A to III-E and III-G are special cases of the restricted random waypoint, with $L = 1$, $r = 0$, and $\mathcal{A}_1 = \mathcal{A}$ for examples III-A to III-C, $\mathcal{A}$ a strict subset of $\mathcal{A}$ for examples III-D and III-E. Note that the subdomains $\mathcal{A}_\ell$ may be convex as in Figure 4 or not as in Figure 3.

### G. Random Waypoint on Sphere.

Here $\mathcal{A}$ is the unit sphere of $\mathbb{R}^3$. $\mathcal{P}$ is the set of shortest paths plus pauses. The shortest path between two points is the shortest of the arcs on the great circle that contains the two points. If the two points are on the same great circle diameter, the two arcs have same length (this occurs with probability 0). The trip transition rule picks a path endpoint uniformly on the sphere, and the path is the shortest path to it (if there are two, one is chosen with probability 0.5). The set of phases is $I = \{pause, move\}$. The numerical speed is chosen independently. Initially, a point is chosen uniformly.

This model is in fact a special case of the random waypoint on a connected, non convex domain. However, we mention it separately as it enjoys special properties (the stationary location is uniform, unlike for the random waypoint models described earlier).

### H. Random Walk with Wrapping.

This model is viewed as a random waypoint on a torus in [11]. It has similarity with the *Random Direction* in [6]. It is used primarily because of its simplicity: unlike for the random waypoint, the distribution of location and speed at a random instant are the same as at a transition instant, as we show later.

The domain $\mathcal{A}$ is the rectangle $[0,a_1] \times [0,a_2]$. Paths are *wrapped* segments, defined as follows. The trip selection rule chooses a speed vector $\vec{V}_n$ and a trip duration

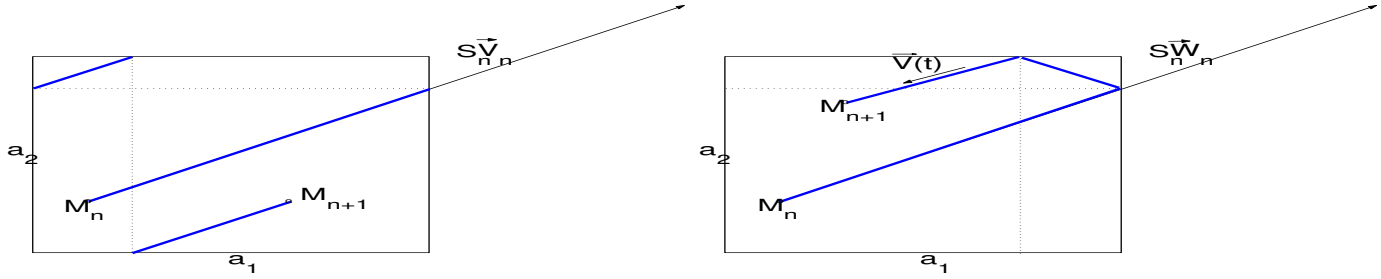


Fig. 6. Definition of Random Walk (Random Direction) with wrapping (left) or billiard-like reflection (right) at the edge of the domain.

$S_n$ independently, according to some fixed distributions. Choosing a speed vector $\vec{V}_n$ is the same as choosing a direction of movement and a numerical speed. The mobile moves from the endpoint $M_n$ in the direction and at the rate given by the speed vector. When it hits the boundary of $\mathcal{A}$, say for example at a location $(x_0, a_2)$, it is wrapped to the other side, to location $(x_0, 0)$, from where it continues the trip (Figure 6). Let $w : \mathbb{R}^2 \to \mathcal{A}$ be the wrapping function : $w(x,y) = (x \mod a_1, y \mod a_2)$. The path $P_n$ (if not a pause) is defined by $(M_n, \vec{V}_n, S_n)$, such that $P_n(u) = w\left(M_n + uS_n\vec{V}_n\right)$. Note that wrapping does not modify the speed vector (Figure 6). After a trip, a pause time is drawn independent of the past from some fixed distribution. Initially, the first endpoint is chosen uniformly in $\mathcal{A}$. As we show next, this implies that all endpoints are in turn uniformly distributed (when sampled at transition instants). This model obviously satisfies assumptions H1 and H3 with set of phases $I = \{pause, move\}$. We now show that it satisfies H2.

*Lemma 1: Let $X$ be a random point, uniformly distributed in $\mathcal{A} = [0,a_1] \times [0,a_2] \times ...[0,a_d]$. For any non random vector $\vec{v} \in \mathbb{R}^d$, the distribution of $w(X+\vec{v})$ is also uniform in $\mathcal{A}$.*

(Proof in Appendix)

*Theorem 1: The distribution of points $M_1, M_2, ...,$ is uniform in $\mathcal{A}$.*

**Proof.** $M_0$ is uniform by assumption. By Lemma 1, the distribution of $M_1$ is also uniform, and recursively, so is the distribution of $M_n$.□

This shows H2b. Note that this is true regardless of the distribution with which $\vec{V}_n$ and $S_n$ are chosen.

### *I. Random Walk with Reflection.*

This is similar to example III-H, but with billiard-like reflections instead of wrapping (Figure 6). It enjoys some of the same final simplicity, but the intermediate steps are more elaborate. The definition is identical to example III-H, with the three following differences:

1) The wrapping function is replaced by the *billiard reflection* function $b : \mathbb{R}^2 \to \mathcal{A}$, defined by $b(x,y) = (a_1 b_1(x/a_1), a_2 b_1(y/a_2))$ where $b_1 : \mathbb{R} \to [0,1]$ is the 2-periodic function defined by $b_1(x) = |x|$ for $-1 \le x \le 1$.

2) Unlike the wrapping function, the billiard reflection may alter the speed vector (Figure 6). Therefore we differentiate the *unreflected speed vector* $\vec{W}_n$ from the instantaneous speed vector $\vec{V}(t)$ at time $t$. The path $P_n$ (if not a pause) is defined by $(M_n, \vec{W}_n, S_n)$, such that $P_n(u) = b\left(M_n + uS_n\vec{W}_n\right)$.

3) We assume that the distribution of the unreflected speed vector $\vec{W}_n$ chosen by the trip selection rule has a density and is *completely symmetric*. We say that a random vector $(X,Y)$ has a completely symmetric distribution iff $(-X,Y)$ and $(X,-Y)$ have the same distribution as $(X,Y)$. This is true for example if the direction of $\vec{W}$ is uniformly chosen on the unit circle, or if the two coordinates of $\vec{W}$ are independent and have even distributions.

This model obviously satisfies assumptions H1 and H3 with set of phases $I = \{pause, move\}$. It also satisfies H2:

*Lemma 2: Let $X$ be a random point, uniformly distributed in $\mathcal{A} = [0,a_1] \times [0,a_2] \times ...[0,a_d]$. Let $\vec{V}$ be a random vector in $\mathbb{R}^d$ with a completely symmetric density. The distribution of the reflection $b(X+\vec{V})$ is also uniform in $\mathcal{A}$.*

**Proof.** Follows from Lemma 5 in Section VII. □

*Theorem 2: The distribution of points $M_1, M_2, ...,$ is uniform in $\mathcal{A}$.*

**Proof.** Similar to Theorem 1, using Lemma 2. □

## IV. Existence and Uniqueness of Stationary Distribution

*Theorem 3: With the model defined in Section II, there is a time-stationary regime if and only if the expected time between transitions $\mathbb{E}^0(S_0)$ is finite. If it exists, the stationary regime is unique.*

**Proof.** The model is $(I(t), P(t), S(t), U(t))$.

**Step 1.** We first show that if the condition is true, a stationary regime exists.

**Step 1a.** Distribution of $(I_n, M_n)$: Let $\pi^0$ be the stationary distribution of the Markov chain $I_n$. By assumption (H3), $\pi^0$ exists and it is unique. If (H2b) holds, then $M_n$ has a distribution independent of $n$ and is independent of $I_n$. Since $(I_n, M_n)$ is Markov (H1) it follows that the distribution of $(I_n, M_n)$ is independent of $n$ if we pick $I_n$ in steady state.

Else (H2a) holds. By a Markov renewal argument, for any bounded function $\phi$ and (H2):

$$\mathbb{E}^0(\phi(I_n,M_n))|I_0=i) = \sum_{j\in I} P^{n-1}(i,j)\mathbb{E}^0(\phi(I_n,M_n)|I_{n-1}=j,I_0=i)$$
$$= \sum_{j\in I} P^{n-1}(i,j)\mathbb{E}^0(\phi(I_n,M_n)|I_{n-1}=j) = \sum_{j\in I} P^{n-1}(i,j)G(j)$$

where $P^n(i,j) = \mathbb{P}^0(I_n = j|I_0 = i)$ and $G(j) = \mathbb{E}^0(\phi(I_1,M_1)|I_0 = j)$. Assume that $I_0$ is in steady-state; hence, $\mathbb{E}^0(\phi(I_n,M_n)) = \pi^0 P^{n-1}G^T = \pi^0 G^T = \mathbb{E}^0(\phi(I_0,M_0))$. Thus $(I_n,M_n)$ has a distribution independent of $n$.

**Step 1b.** $(I_n,P_n,S_n)$ can be made (event-) stationary. By (H1), $(I_n,P_n,S_n)$ is Markov renewal with respect to $I_n, M_n = P_n(0)$ thus, by Step 1a, it is a stationary sequence.

**Step 1c.** We use Slivnyak's inverse construction [1], which says that a stationary regime exists for $(I(t),P(t),S(t))$ if the following conditions hold:

(S1) $0 < \mathbb{E}^0(T_1) < \infty$
(S2) $\mathbb{P}^0(T_1 > 0) = 1$
(S3) $\mathbb{E}^0(N(0,t]) < \infty$, for all $t < t_0$, for some $t_0 > 0$,

where $N(s,t]$ is the number of transitions in $(s,t]$. S2 is assumed by the model, which also implies $0 < \mathbb{E}^0(T_1)$; the rest of S1 is hypothesis of the theorem. It remains to verify S3.

If H2b holds then $(I_n,T_n)$ is a Markov renewal process (Chapter 10, [7]). If H2a holds then we construct an embedded Markov renewal process as follows. Denote the times of the selected transitions of $I_n$, defined by the set $I^\circ$, as $T_n^\circ$ with the usual conventions: $T_0^\circ \le 0 < T_1^\circ < \ldots$, $S_n^\circ := T_{n+1}^\circ - T_n^\circ$. Let $I_n^\circ = I(T_n^\circ)$. From H3, $I_n$ is a positive recurrent Markov chain, so is $I_n^\circ$. Define $\tilde{S}_n^\circ = S(T_n^\circ)$; in other words, this is the time until next transition as seen at time $T_n^\circ$. Now, construct an auxiliary point process $\tilde{T}_{n+1}^\circ := \tilde{T}_n^\circ + \tilde{S}_n^\circ$, with $\tilde{T}_1^\circ = T_1^\circ$. If $(I_{-1},I_0) = (i_{-1},i_0) \in I^\circ$, $(Z_n,\tilde{T}_n^\circ)$ is a Markov renewal process with $Z_n := (I_n^\circ, I_{n-1}^\circ)$. Indeed, with $\mathcal{F}_n := (I_n^\circ, I_{n-1}^\circ, \ldots, I_0^\circ = i_0, I_{-1}^\circ = i_{-1}; \tilde{T}_\ell^\circ, \ell \le n)$,

$$\mathbb{P}^0(I_{n+1}^\circ = j, \tilde{S}_n^\circ \le t|\mathcal{F}_n)$$
$$= \int_{\mathcal{A}} \mathbb{P}^0(I_{n+1}^\circ = j, \tilde{S}_n^\circ \le t|M(T_n^\circ) = x, \mathcal{F}_n) d\mathbb{P}^0(M(T_n^\circ) = x|\mathcal{F}_n)$$
$$= \int_{\mathcal{A}} \mathbb{P}^0(I_{n+1}^\circ = j, \tilde{S}_n^\circ \le t|M(T_n^\circ) = x, I_n^\circ, I_{n-1}^\circ) d\mathbb{P}^0(M(T_n^\circ) = x|I_n^\circ, I_{n-1}^\circ)$$
$$= \mathbb{P}^0(I_{n+1}^\circ = j, \tilde{S}_n^\circ \le t|I_n^\circ, I_{n-1}^\circ)$$

where the second equality is by H1 and H2a. Else, if $(I_{-1},I_0) = (i_{-1},i_0) \in I - I^\circ$, then $(Z_n,\tilde{T}_n^\circ)$ is a delayed Markov renewal process (the event-distribution of $T_1^\circ$ is different from the event-distribution of $S_n^\circ$, $n > 0$). Now, if for the auxiliary process $(Z_n,\tilde{T}_n^\circ)$, S3 is true, in view of the positive recurrence H3, the expected number of transitions between any two selected transitions is finite, so S3 is true for the original transitions $T_n$. It suffices to show that S3 is true under H2b. Note that

$$\mathbb{E}^0(N(0,t]) = \sum_{n\ge 1} n\mathbb{P}^0(N(0,t] = n) = \sum_{n\ge 1} \mathbb{P}^0(T_n \le t) \qquad (1)$$

and we now find a bound for $\mathbb{P}^0(T_n \le t)$. Pick some arbitrary, fixed $s > 0$; by Markov's inequality:

$$\mathbb{P}^0(T_n \le t) \le e^{st}\mathbb{E}^0\left(e^{-sT_n}\right). \qquad (2)$$

We continue the proof in the case where H2b holds, the case H2a is similar with $Z_n$ replacing $I_n$. For any $s > 0$ let

$$H_{i,j}^{(n)}(s) = \mathbb{E}^0\left(e^{-sT_n}1_{\{I_n=j\}}|I_0 = i\right).$$

Condition with respect to $S_0 = s_0$ and obtain, by the Markov renewal property:

$$H_{i,j}^{(n+1)}(s) = \sum_{i,j} \tilde{Q}_{i,k}(s)H_{k,j}^{(n)}(s)$$

where $\tilde{Q}_{i,k}(s) = \mathbb{E}^0\left(e^{-sS_0}1_{\{I_1=j\}}|I_0 = i\right)$. In matrix notation, $H^{(n+1)}(s) = \tilde{Q}(s)H^{(n)}(s)$ and thus $H^{(n)}(s) = (\tilde{Q}(s))^n$. For any $s > 0$ and any $i$: $\sum_j \tilde{Q}_{i,j}(s) = \mathbb{E}^0(e^{-sS_0}|I_0 = i) < 1$ and $\tilde{Q}(s)$ is non-negative, thus, by the Perron-Frobenius theorem, the spectral radius of $\tilde{Q}(s)$ is $< 1$. Thus the series $\sum_{n\ge 0}(\tilde{Q}(s))^n$ is absolutely convergent. Now $\mathbb{E}^0\left(\sum_{n\ge 0} e^{-sT_n}\right) = \pi^0 \sum_{n\ge 0}(\tilde{Q}(s))^n \mathbf{1}^T$ where $\mathbf{1} = (1,1,...,1)$, and thus it is finite. In view of Equation (2), this shows that $\sum_{n\ge 1}\mathbb{P}^0(T_n \le t)$ is finite which, with Equation (1), shows that S3 is verified.

**Step 1d.** We showed that $(I(t),P(t),S(t))$ has a stationary regime; in other words, there exists a stationary marked point process $((I_n,P_n),T_n)$. It remains to see that $(I(t),P(t),S(t),U(t))$ also has. Define $T^-(t) = \sup\{T_n : T_n \le t\}$ and $T^+(t) = \inf\{T_n : T_n > t\}$. The events $\{t - T^-(t) > s\}$ and $\{T^+(t) - t > s\}$ are equal to $\{N[t-s,t] = 0\}$ and $\{N(t,t+s] = 0\}$; these events are determined by $(I(u),P(u),S(u),\ u \in \mathbb{R})$ in a way that is invariant by a shift of time origin; thus if $(I(t),P(t),S(t))$ is stationary, so is $(I(t),P(t),S(t),t-T^-(t),T^+(t)-t)$. Now $U(t) = \frac{t-T-(t)}{(T^+(t)-t)-(t-T^-(t))}$ thus $(I(t),P(t),S(t),U(t))$ is stationary, too.

**Step 2.** The condition is necessary by the intensity formula in [1].

**Step 3.** If a stationary regime exists, then by the inversion formula and the proof of Theorem 4, it is entirely defined by the Palm distribution, which is unique by step 2. □

*Corollary 1: For examples III-A to III-G, there is a stationary regime if and only if the pause time and inverse speed (sampled at a transition) have a finite expectation. For examples III-H and III-I the condition is that the pause time and trip duration (sampled at a transition) have a finite expectation.*

**Comment.** These conditions are known to be necessary for the classical random waypoint to be "harmless". However, it appears to be the first time that the link to the existence of a stationary regime is made rigorously.

## V. Time Stationary Distributions

For a perfect simulation, all we need is to sample from the time stationary distribution of the process state. The state of the process is the phase $(I(t)$, the path $P(t)$, the trip duration $S(t)$ and where on trip $U(t)$. In this section we derive the fundamental relation between the parameters of the random trip model and its stationary distribution. In the next section we apply it to the various examples introduced earlier.

*Theorem 4: Assume the condition for existence and unicity of a stationary distribution in Section IV is satisfied. The time stationary distribution of the process state at an arbitrary time t is defined by the following :*

1) $\mathbb{P}(I(t) = i) = \frac{\pi^0(i)\bar{\tau}_i}{\sum_j \pi^0(j)\bar{\tau}_j}$, *where* $\bar{\tau}_i = \mathbb{E}^0(S_0|I_0 = i)$ *is the mean trip duration for phase i*
2) $d\mathbb{P}(P(t) = p, S(t) = s|I(t) = i) = \frac{s}{\bar{\tau}_i} d\mathbb{P}^0(P_0 = p, S_0 = s|I_0 = i)$
3) $U(t)$ *is independent of* $(I(t),P(t),S(t))$ *and is uniform on* $[0,1]$

**Proof.** We use the inversion formula of Palm calculus [1]. Let $\lambda$ be the intensity of the point process $T_n$, i.e. the average number of transitions per time unit. For any bounded function $\phi$ of the process state: $\mathbb{E}(\phi(I(t),P(t),S(t),U(t))) = \lambda\mathbb{E}^0\left(\int_0^{S_0}\phi(I_0,P_0,S_0,\frac{\tau}{T_1})d\tau\right) =$

$\lambda\mathbb{E}^0\left(S_0\int_0^1\phi(I_0,P_0,S_0,u)du\right)$ where the latter is by the change of variable $\tau=S_0u$ in the integral. Take $\phi(j,p,s,u)=1_{\{j=i\}}$ and obtain

$$\mathbb{P}(I(t)=i)=\lambda\mathbb{E}^0\left(S_0 1_{\{I_0=i\}}\right)=\lambda\pi^0(i)\bar{\tau}_i$$

The condition $\sum_i\mathbb{P}(I(t)=i)=1$ gives $\lambda=1/\sum_i\pi^0(i)\bar{\tau}_i$, which shows item 1. Now take $\phi(j,p,s,u)=\psi(p,s)\xi(u)1_{\{j=i\}}$ and obtain

$$\begin{aligned}&\mathbb{E}(\psi(P(t),S(t))\xi(U(t))1_{\{I(t)=i\}})=\lambda\mathbb{E}^0\left(S_0\psi(P_0,S_0)\int_0^1\xi(u)du1_{\{I_0=i\}}\right)\\&=\lambda\pi^0(i)\mathbb{E}^0\left(S_0\psi(P_0,S_0)\int_0^1\xi(u)du|I_0=i\right)\\&=\lambda\pi^0(i)\int_0^1\xi(u)du\mathbb{E}^0\left(S_0\psi(P_0,S_0)|I_0=i\right)\\&=\mathbb{P}(I(t)=i)\int_0^1\xi(u)du\mathbb{E}^0\left(\frac{S_0}{\bar{\tau}_i}\psi(P_0,S_0)|I_0=i\right)\end{aligned}$$

thus

$$\mathbb{E}(\psi(P(t),S(t))\xi(U(t))|I(t)=i)=\int_0^1\xi(u)du\mathbb{E}^0\left(\frac{S_0}{\bar{\tau}_i}\psi(P_0,S_0)|I_0=i\right)$$

which shows items 2 and 3. □

Note that the factor $1/\sum_i\pi^0(i)\bar{\tau}_i$ in item 1 is precisely the intensity of the point process of transitions [1].

**Special Case.** In many examples (III-A to III-E and III-G) the set of phases is reduced to $\{pause,move\}$ and the model alternates between these two. Then $\pi^0(i)=0.5$ for $i=pause$ or $move$ and item 1 simplifies to $P(I(t)=pause)=\frac{\bar{\tau}_{pause}}{\bar{\tau}_{pause}+\bar{\tau}_{move}}$ and $P(I(t)=move)=\frac{\bar{\tau}_{move}}{\bar{\tau}_{pause}+\bar{\tau}_{move}}$.

## VI. Application to Examples III-A to III-F

In all of this section, we assume that the condition for stationarity in Section IV is satisfied. We focus on restricted random waypoint on general connected area, since examples III-A to III-F are special cases of it.

### A. Time Stationary Distributions

A direct application of the Theorem 4 gives the time stationary distribution of the process. Due to its description complexity, we give it in three pieces, in the following theorems. Special notation local to this section is given below.

**Notation Used in Section VI**

- $Q(\ell,\ell')$: probability that next subdomain is $\mathcal{A}_{\ell'}$ given current subdomain is $\mathcal{A}_\ell$, with $Q(\ell,\ell)=0$. $q^*(\ell)$ is the unique stationary probability of $Q$ ($q^*Q=q^*$).
- For $r\in\mathbb{N}$, $F_\ell(r)$ is the probability that the number of consecutive sojourns in subdomain $\mathcal{A}_\ell$ is $\geq r$. $\bar{R}_\ell=\sum_r F_\ell(r)$ is the average number of consecutive sojourns in subdomain $\mathcal{A}_\ell$.
- $\bar{\Delta}_{\ell,\ell'}$ is the *average distance* in $\mathcal{A}$ for two points chosen uniformly in $\mathcal{A}_\ell$ and $\mathcal{A}_{\ell'}$. $\Delta_{\ell,\ell'}$ is *some upper bound* on the distance in $\mathcal{A}$ between two points in $\mathcal{A}_\ell$ and $\mathcal{A}_{\ell'}$.
- $f^0_{V|i}(v)$ is the Palm (= at a transition instant) distribution of speed, given that phase is $i=(\ell,\ell',r,move)$; $\omega_{\ell,\ell'}=\mathbb{E}^0\left(\frac{1}{V_0}|I_n=(\ell,\ell',r,move)\right)$ is the event average of the inverse of the speed chosen for a trip from subdomain $\mathcal{A}_\ell$ to $\mathcal{A}_{\ell'}$. We have $\omega_{\ell,\ell'}=\int_0^\infty\frac{1}{v}f_{V|\ell,\ell',r,move}(v)dv$, assumed to be independent of $r$.
- $f^0_{S|i}(s)$ is the Palm (= at a transition instant) distribution of pause time, given that phase is $i=(\ell,\ell',r,pause)$; $\tau_{\ell,\ell'}=\mathbb{E}^0(S_0|I_0=(\ell,\ell',r,pause))$ is the average pause time that follows a trip from subdomain $\mathcal{A}_\ell$ to $\mathcal{A}_{\ell'}$. We have $\tau_{\ell,\ell'}=\int_0^\infty sf_{S|\ell,\ell',r,pause}(s)ds$, assumed to be independent of $r$.

The first theorem generalizes known statements for the classical random waypoint (Example III-A) [14], [13]. It relates the time average speed to the distribution of the speed selected at a waypoint, and contains an exact representation of the time stationary distribution of location. The theorem goes a little further and establishes an independence property, that does not appear to have been demonstrated so far, even in the special case of classical random waypoint.

*Theorem 5: Under the time stationary distribution, conditional to phase $I(t)=i=(\ell,\ell',r,move)$:*

*1) The numerical speed is independent of the path and the instantaneous location of the mobile at time $t$. Its density is $f_i(v)=\frac{C_i}{v}f^0_{V|i}(v)$, where $f^0_{V|i}(v)$ is the density of the numerical speed sampled at a transition instant and $C_i$ is a normalizing constant.*

*2) The path endpoints $(P(t)(0),P(t)(1))$ have a joint density over $\mathcal{A}_\ell\times\mathcal{A}_{\ell'}$ given by $d\mathbb{P}(P(t)(0)=m_0,P(t)(1)=m_1|I(t)=i)=K_{\ell,\ell'}d(m_0,m_1)$, where $K_{\ell,\ell'}$ are normalizing constants and $d()$ is the distance in $\mathcal{A}$.*

*3) The distribution of $X(t)$ given $P(t)(0)=p,P(t)(1)=n$ is uniform on the segment $[p,n]$.*

**Proof.** Apply Theorem 4 to obtain the joint distribution of the path, location and speed $V(t)$, by noting that $V(t)=d(P(t)(0),P(t)(1))/S(t)$. □

**Comment 1.** As we show later, there is no need to know the value of the constants $K_{\ell,\ell'}$ to use the theorem in a simulation. [1]

**Comment 2.** The distribution of path endpoints $P(t)(0)$ and $P(t)(1)$ is *not* uniform, and the two endpoints are correlated (they tend to be far apart), contrary to what happens when sampled at transition instants. This was found already for Example III-A in [12].

**Comment 3.** The relation between time stationary and event stationary distribution of speed is sometimes interpreted as "speed decay" since it is more likely to produce low speed values than the density $f^0_i(v)$. If one desires a uniform speed distribution in time average, then the density of speed at transition instants should be $f^0_i(v)=K'_iv1_{\{v_{\min}<v<v_{\max}\}}$. Note that such a speed distribution satisfies the stability condition in Section IV

[1] However, in the special case of convex domains where $d(m,n)$ is the usual euclidian distance, it is worth noting that there are known formulae: $K^{-1}_{\ell,\ell'}=\text{vol}(\mathcal{A}_\ell)\text{vol}(\mathcal{A}'_\ell)\bar{\Delta}_{\ell,\ell'}$ where $\text{vol}(\mathcal{A}_\ell)$ is the area or volume of $\mathcal{A}_\ell$ (in square or cubic meters) and $\bar{\Delta}_{\ell,\ell'}$ is the average distance in $\mathcal{A}$ between two points drawn uniformly in $\mathcal{A}_\ell$ and $\mathcal{A}_{\ell'}$. For $\ell=\ell'$ and $\mathcal{A}_\ell=$ a square of a size $a$, $K^{-1}_{\ell,\ell}\approx 0.5214a^5$; for a disk of radius $a$, $K^{-1}_{\ell,\ell}\approx 0.9054\pi^2a^5$ [8]. For an arbitrary case, it is generally not possible to obtain either $\text{vol}(\mathcal{A}_\ell)$ or $\bar{\Delta}_{\ell,\ell'}$ in closed form, but $K^{-1}_{\ell,\ell'}$ can be obtained directly by Monte Carlo simulation.

even if $v_{\min} = 0$.

*Theorem 6: Under the time stationary distribution, conditional to phase $I(t) = i = (\ell, \ell', r, pause)$, the location $X(t)$ and the time $R(t)$ until end of pause are independents; $X(t)$ is uniform in $\mathcal{A}_{\ell'}$; $R(t)$ has density $f_i(r) = 1/\bar{\tau}_i \int_r^\infty f^0_{S|i}(s)\,ds$ where $f^0_{S|i}(s)$ is the density of the pause time selected at a transition.*

**Proof.** Similar to (but simpler than) Theorem 5. □

*Theorem 7: The time stationary distribution $\pi(i)$ to be in phase $I(t) = i = (\ell, \ell', r, pause)$ is*

$$\begin{cases} \pi(\ell,\ell',0,pause) = \lambda q^*(\ell) Q(\ell,\ell') \tau_{\ell,\ell'} \\ \pi(\ell,\ell,r,pause) = \lambda q^*(\ell) F_\ell(r) \tau_{\ell,\ell} \text{ for } r \geq 1 \\ \pi(\ell,\ell,r,move) = \lambda q^*(\ell) Q(\ell,\ell') \bar{\Delta}_{\ell,\ell'} \omega_{\ell,\ell'} \\ \pi(\ell,\ell,r,move) = \lambda q^*(\ell) F_\ell(r) \bar{\Delta}_{\ell,\ell} \omega_{\ell,\ell} \text{ for } r \geq 1 \end{cases}$$

*where $\lambda$ is a normalizing constant, defined by the above equation and $\sum_i \pi(i) = 1$.*

**Proof.** By substitution in the balance equation, we can verify that the event-stationary distribution of the phase $I_n$ is given by

$$\begin{cases} \pi^0(\ell,\ell',r,pause) = \pi^0(\ell,\ell',r,move) \\ \pi^0(\ell,\ell',0,move) = \alpha q^*(\ell) Q(\ell,\ell') \\ \pi^0(\ell,\ell,r,move) = \alpha q^*(\ell) F_\ell(r) \text{ for } r \geq 1 \end{cases} \quad (3)$$

with $\alpha$ a normalizing constant. The rest follows from Theorem 4. □

As with Theorem 5, we show later that we do not need to know $\lambda$ and $\bar{\Delta}_{\ell,\ell'}$ to use this theorem for sampling.

**Special Case.** For examples III-A to III-E and III-G, Theorem 7 reduces to $\pi(pause) = \frac{\tau_{pause}}{\tau_{pause} + \bar{\Delta}\omega}$ and $\pi(move) = 1 - \pi(pause)$, where $\tau_{pause}$ is the average pause time, $\bar{\Delta}$ the average distance in $\mathcal{A}$ between two points in $\mathcal{A}_1$, and $\omega = \mathbb{E}^0(1/V_0 | I_0 = move)$ the event average of the inverse of the speed.

### *B. Perfect Simulation Without Computing Geometric Integrals*

A straightforward application of the previous section poses the problem of how to sample $m_0, m_1$ from the density in Theorem 5. Further, in order to sample the phase in Theorem 7 one needs to compute the geometric integrals $\bar{\Delta}_{\ell,\ell'}$; for simple cases ($L = 1$ and $\mathcal{A}_1$ is a rectangle or disk) there exist closed forms, as mentioned in Comment 1 after Theorem 5. Otherwise, one needs to compute them offline by Monte Carlo simulation. For cases like Figure 4, this is time consuming (see an analysis at the end of this section). There is a generally more efficient procedure, which avoids computing the geometric integrals when they are not known, as we show now. The solution of these two problems is based on the following sampling lemma.

Let $(J, Y)$ be a random vector, where $J$ is in a discrete set $\mathcal{J}$ and $Y \in \mathbb{R}^d$. Assume that $\mathbb{P}(J(t) = j) = \lambda \mu(j) \varpi_j$ and the distribution of $Y$ conditional to $J = j$ has a density $\frac{f_j(y)}{\varpi_j}$ The problem is to sample from $(J, Y)$ without having to compute the normalizing constants of the densities $\varpi_j$ for all $j$.

Assume we know factorizations of the form $f_j(y) = k_j(y) g_j(y)$ where $g_j(y)$ is a probability density ($\int g_j(y) dy = 1$, i.e. there is no normalizing constant to compute for $g_j(y)$) and assume that we know upper bounds $\kappa_j$ such that $0 \leq k_j(y) \leq \kappa_j$.

*Lemma 3: Let $\nu$ be the probability on $\mathcal{J}$ defined by: if $\varpi_j$ is known $\nu(j) = \alpha \mu(j) \varpi_j$ else $\nu(j) = \alpha \mu(j) \kappa_j$, where $\alpha$ is a normalizing constant, defined by the condition $\sum_j \nu(j) = 1$. The following algorithm draws a sample from $(J, Y)$:*

**do forever**
  *draw $j$ with probability $\nu(j)$*
  ***if** $\varpi_j$ is known*
    *draw $y$ from the density $f_j(y)/\varpi_j$;**leave***
  ***else***
    *draw $y$ from the density $g_j(y)$*
    *draw $U \sim Unif(0, \kappa_j)$*
    ***if** $U \leq \frac{k_j(y)}{\kappa_j}$ **leave***
***end do***

(Proof in Appendix) The following theorem gives the sampling method. The details for the general case have some description complexity, and are given in Appendix. We show all details here for the case $L = 1$.

*Theorem 8: (Perfect Simulation of Restricted Random Waypoint) The following algorithm draws a sample of the time stationary state of the restricted random waypoint: 1) Sample a phase $I(t) = i = (\ell, \ell', r, \phi)$ from the algorithm in Figure 7 (simple case) or in Appendix(general case))*

*2) If $\phi = pause$ sample a time $t$ from the distribution with density $f_i(t) = 1/\bar{\tau}_i \int_t^\infty f^0_{S|i}(s)\,ds$. Sample a point $M$ uniformly in $\mathcal{A}_{\ell'}$. Start the simulation in pause phase at location $M$ and schedule the end of pause at $t$.*

*3) If $\phi = move$ sample a speed $v$ from the distribution with density proportional to $\frac{1}{v} f^0_{V|i}(v)$. Set $M_0, M_1$ to the value returned by the algorithm in Figure 7 (simple case) or in Appendix(general case). Sample $u$ uniformly in $(0,1)$. Start the simulation in move phase, with initial position $(1-u)M_0 + uM_1$, next trip endpoint $= M_1$, and speed$= v$.*

**Proof.** First note (Theorem 5) that we need only to consider path and location. Then apply of Theorems 5, 6 and 7. When $\bar{\Delta}_{\ell,\ell'}$ is known, we solve the first problem of sampling $m_0, m_1$ from the density in Theorem 5 by applying Lemma 3 with $\mathcal{J} = \{1\}$, $y = (m_0, m_1)$, $\varpi_1 = \bar{\Delta}_{\ell,\ell'}$, $f_1(m_0, m_1) = d(m_0, m_1) \text{unif}_{\mathcal{A}_\ell}(m_0) \text{unif}_{\mathcal{A}_{\ell'}}(m_1)$, $\kappa_1 = \Delta_{\ell,\ell'}$. The second problem ($\bar{\Delta}_{\ell,\ell'}$ not known) is solved by setting $\mathcal{J} = I$ and $\varpi_i = \bar{\tau}_i$. □

Note that the algorithm solves both problems men-

If $\bar{\Delta}$ is known

$q_0 = \tau_{pause}/(\tau_{pause} + \omega\bar{\Delta})$
Draw $U_1 \sim U(0,1)$
**if** $U_1 \leq q_0$ $I(t) = pause$
**else**
  $I(t) = move$
  **do**
    Draw $M_0 \sim$ Unif$(\mathcal{A}_1), M_1 \sim$ Unif$(\mathcal{A}_1)$
    Draw $U_2 \sim$ Unif$(0,\Delta)$
  **until** $U_2 < d(M_0, M_1)$

else (i.e. $\bar{\Delta}$ is not known)

$q_0 = \tau_{pause}/(\tau_{pause} + \omega\Delta)$
**do forever**
  Draw $U_1 \sim U(0,1)$
  **if** $U_1 \leq q_0$ $I(t) = pause$; **leave**
  **else**
    Draw $M_0 \sim$ Unif$(\mathcal{A}_1), M_1 \sim$ Unif$(\mathcal{A}_1)$
    Draw $U_2 \sim$ Unif$(0,\Delta)$
    **if** $U_2 < d(M_0, M_1)$
    $I(t) = move$; **leave**
**end do**

Fig. 7. Sampling algorithm for restricted random waypoint with $L = 1$, supporting both cases where the average distance between points in $\mathcal{A}_1$ is known or not. $\tau_{pause}$ is the average pause time, $\bar{\Delta}$ the average distance in $\mathcal{A}$ between two points in $\mathcal{A}_1$, $\Delta$ an upper bound on the distance in $\mathcal{A}$ between two points in $\mathcal{A}_1$ and $\omega = \mathbb{E}^0(1/V_0 | I_0 = move)$

tioned at the introduction of this section. If $\bar{\Delta}$ is known with little computational cost (i.e. when $\mathcal{A}$ is a rectangle or a disk) it is always preferable to use the former case ("$\bar{\Delta}$ is known"). Else there are two options: (1) compute $\bar{\Delta}$ offline by Monte-Carlo simulation and use the case $\bar{\Delta}$ is known", or (2) use the case ("$\bar{\Delta}$ is not known"). Apart from unusually long simulation campaigns with the same model, the optimal choice, in terms of number of operations is to use the latter case (see Appendix). Furthermore, using the latter case simplifies the overall simulation code development. Figure 8 illustrate the sampling method on Examples III-B to III-F.

## VII. Application to Examples III-G to III-I

In all of this section, we assume that the condition for stationarity in Section IV is satisfied.

### A. Random Waypoint on Sphere (Example III-G)

This model is a special case of restricted random waypoint over a non convex area, with $L = 1$ and $\mathcal{A}_1 = \mathcal{A}$. Thus all findings of Section VI apply, in particular, the time stationary speed is independent of location and is given by Theorem 5.

*Theorem 9: For the random waypoint on the sphere, the time stationary distribution of the mobile location is uniform.*

**Proof.** Apply Theorem 5. The distribution of $X(t)$ is invariant under any rotation of the sphere around an axis that contains the center of the sphere, and any distribution that has such an invariance property must be uniform. □

Note that, with the same argument, we can show that, given we are in a move phase, the time stationary distribution of each path endpoint (previous and next) separately is also uniform, but the two endpoints are correlated (it is more likely that they are far apart). This is because, from Theorem 5, a typical path seen in time average is drawn with a probability proportional to its length. This implies that, though the time stationary distribution of points is uniform, *it is not sufficient for perfect simulation to draw an initial position uniformly on the sphere* and start as if it would be a path endpoint (we need in addition to sample a path and where on path according to Theorem 5).

### B. Random Walk with Wrapping (Example III-H)

Let $f^0_{pause}(t)$ [resp. $f^0_{move}(t)$] be the density of the pause [resp. move] duration, sampled at a transition time. Both densities are model parameters. Also let $\bar{\tau}_{pause}, \bar{\tau}_{move}$ be the corresponding averages (thus for example $\bar{\tau}_{pause} = \mathbb{E}^0(S_0 | I_0 = pause) = \int_0^\infty t f^0_{pause}(t) dt$). Finally, let $f^0_{\vec{V}}(\vec{v})$ be the density of the distribution of the speed vector (sampled at trip endpoints).

*Theorem 10: For random walk with wrapping, under the time stationary distribution:*

1) *the process state at time t is fully described by the phase $I(t)$, the location $X(t)$, the speed vector $\vec{V}(t)$ ($= \vec{0}$ if phase=pause) and the residual time until end of trip $R(t)$*
2) *the location $X(t)$ is uniform*
3) $P(I(t) = pause) = \frac{\bar{\tau}_{pause}}{\bar{\tau}_{pause} + \bar{\tau}_{move}}$ *and* $P(I(t) = move) = \frac{\bar{\tau}_{move}}{\bar{\tau}_{pause} + \bar{\tau}_{move}}$
4) *conditional to phase=pause:*
   - *the residual pause duration $R(t)$ has density $f_{pause}(r) = 1/\bar{\tau}_{pause} \int_r^\infty f^0_{pause}(s)\, ds$*
   - *$X(t)$ and $R(t)$ are independent*
5) *conditional to phase=move:*
   - *$\vec{V}(t)$ has density $f^0_{\vec{V}}(\vec{v})$*
   - *the residual trip duration $R(t)$ has density $f_{move}(r) = 1/\bar{\tau}_{move} \int_r^\infty f^0_{move}(s)\, ds$*
   - *$X(t), \vec{V}(t)$ and $R(t)$ are independent*

**Proof.** Item 1 follows from the fact that the speed vector is not altered by wrapping. Item 3 directly follows from Theorem 4 and the discussion after it. We now show item 5. Recall $P(t)(0)$ is the start position of the current path. By Theorem 4, the time stationary joint density of $P(t)(0) = m, \vec{V}(t) = \vec{v}, S(t) = s$, conditional to a move phase is $\frac{s}{\bar{\tau}_{move}} f^0_{\vec{V}}(\vec{v}) f^0_{move}(s)$unif$(m)$, where unif$(m)$ is the uniform density on $\mathcal{A}$. Now $X(t) = w(M_n + U(t) S_n \vec{V}_n)$, $T_n \leq t < T_{n+1}$, and $R(t) = (1 - U(t)) S(t)$. Take any three bounded functions $\phi, \psi, \xi$. By Theorem 4 we have:

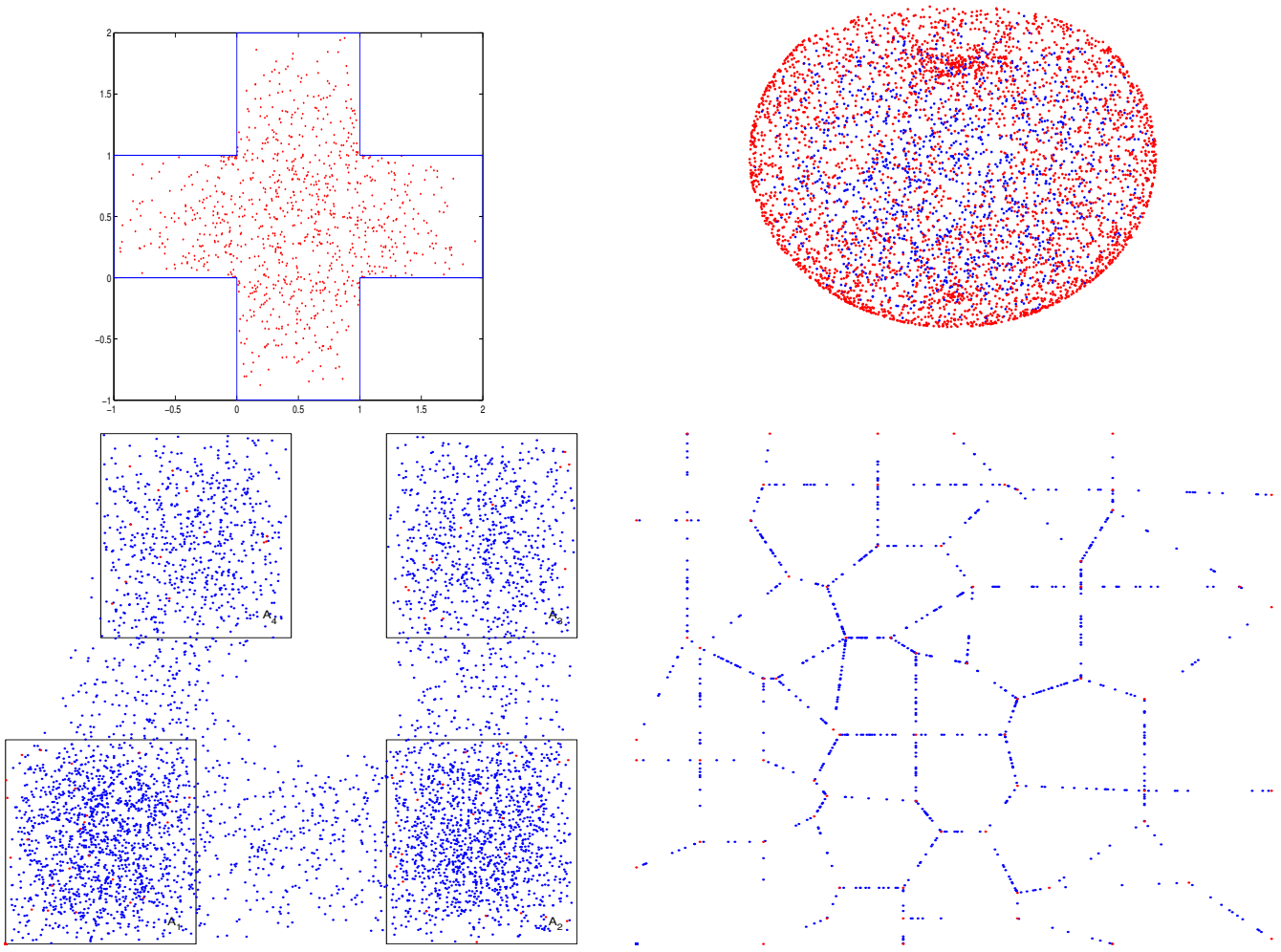


Fig. 8. Perfect sampling of node position from time-stationary distribution for swiss flag (1000 samples), fish in a bowl (5000 samples), four-town-restricted random waypoint (5000 samples) and space graph (10000 samples). Densities are not uniform, with bias towards central areas and interior corner points.

$$\mathbb{E}\left(\phi(X(t))\psi(\vec{V}(t))\xi(R(t))|I(t)=move)\right) =$$
$$\int_0^1 \int_{\vec{v},s} \int_m \phi(w(m+us\vec{v}))\text{unif}(m)dm\psi(\vec{v})\xi(us)\tfrac{s}{\bar{\tau}_{move}} f^0_{\vec{V}}(\vec{v}) f^0_{move}(s)d\vec{v}dsdu$$
$$= \int_{\mathcal{A}} \phi(m)dm \int_0^1 \int_{\vec{v},s} \psi(\vec{v})\xi(us)\tfrac{s}{\bar{\tau}_{move}} f^0_{\vec{V}}(\vec{v}) f^0_{move}(s)d\vec{v}dsdu$$

The last equality is because by Lemma 1, the interior integral with respect to $m$ is $\int_{\mathcal{A}} \phi(m)dm$. Thus

$$\mathbb{E}\left(\phi(X(t)),\psi(\vec{V}(t),\xi(R(t))|I(t)=move\right) =$$
$$\int_{\mathcal{A}} \phi(m)dm \int_{\mathbb{R}^2} \psi(\vec{v}) f^0_{\vec{V}}(\vec{v})d\vec{v} \int_0^1 \int_0^\infty \tfrac{s}{\bar{\tau}_{move}} \xi(us) f^0_{move}(s)dsdu$$
$$= \int_{\mathbb{R}^2} \psi(\vec{v}) f^0_{\vec{V}}(\vec{v})d\vec{v} \int_0^\infty \xi(r) f_{move}(r)dr$$

where the last equality is by the change of variable $(s,u)$ to $(r=us,s)$. This shows item 5. Item 4 is analog. Now conditional to the phase being either move or pause, the location $X(t)$ is uniform. Item 2 follows. □

Thus, contrary to random waypoint on sphere, perfect simulation of this model is very simple. Pick a phase in proportion to the average time spent in the phase. Pick a point and, for move phase, a speed vector as if at a transition point, and pick a remaining trip duration according to the general formula for the density of the time until next transition, in any stationary system. Also, there is no speed decay [15] as with random waypoint on a sphere.

### C. Random Walk with Reflection (Example III-I)

There is a similar result for random walk with billiard reflection, but its proof if more elaborate. We use continue with the same notation, with the difference that the instantaneous speed $\vec{V}(t)$ may differ from the unreflected speed $\vec{W}_n$ chosen at the beginning of the trip. Let $f^0_{\vec{W}}(\vec{w})$ be the density of the distribution of the non reflected speed vector (sampled at trip endpoints).

The following lemma expresses that, in order to continue a path from an intermediate point $m$ it is not needed to know the unreflected speed vector, the instantaneous speed is enough (Proof in Appendix):

*Lemma 4: For any non random point $m \in \mathcal{A}$ and vector $\vec{v} \in \mathbb{R}^2$: $b(m+\vec{v}) = b\left(b(m) + \nabla b_{m+\vec{v}}\vec{v}\right)$.*

The following lemma says that, at the end of trip that starts from a uniform point $M$ and a completely sym-

metric initial speed vector $\vec{W}$, the reflected destination point $M'$ and speed vector $\vec{W}'$ are independent and have same distribution as initially (Proof in Appendix).

*Lemma 5: Let $M$ be a random point, uniform in $\mathcal{A}$. Let $\vec{W}$ be a random vector in $\mathbb{R}^2$ independent of $M$ and with completely symmetric distribution under reflections. Let $M' = b\left(M + \vec{W}\right)$ and $\vec{W}' = \nabla b_{M+\vec{W}}\vec{W}$. $M'$ and $\vec{W}'$ are independent and have the same distribution as $M$ and $\vec{W}$.*

*Theorem 11: For the random walk with reflection, the same holds as in Theorem 10 after replacing $f^0_{\vec{V}}(\vec{v})$ by $f^0_{\vec{W}}(\vec{v})$ in the first bullet of item 5.*

**Proof.** Item 1 follows from Lemma 4. The rest follows from Theorem 4 and Lemma 5, in a similar way as for Theorem 10. □

**Remark.** The location $X(t)$ and the path $M(t), \vec{W}(t), S(t)$ are *not* independent. For example, given that the unreflected speed vector is $\vec{W}(t) = (0.5a_1, 0)$ and the trip duration is $S(t) = 1$, it is more likely that $X(t)$ is in the second right half of the rectangle. However, independence is true if, instead of the path descriptor, we take as simulation state the current position, speed and time to next endpoint, as justified by the theorem. Perfect simulation of this random walk is similar to the random walk with wrapping.

## VIII. Related Work

For a survey of existing mobility models, see the work by Camp, Boleng, and Davies [6] and the references therein. Bettstetter, Harnstein, and Pérez-Costa [8] studied the time-stationary distribution of a node position for classical random-waypoint model. They observed that the time-stationary node position is non-uniform and it has more mass in the center of a rectangle. A similar problem has been further studied by Bettstetter, Resta, and Santi [2]. A closed-form expression for the time-stationary density of a node position is obtained only for random-waypoint on a one-dimensional interval; for two dimensions only approximations are obtained. Note that in Theorem 5 we do have an exact representation of the distribution of mobile location as a marginal of a distribution with a known density. Neither [8] nor [2] consider how to run perfect simulations. It is the original finding of Yoon, Liu, and Noble [14] that the default setting of the classical random-waypoint is in fact ill-defined. The default random-waypoint assumes the event-stationary distribution of the speed to be uniform on an interval $(0, vmax]$. The authors found that if a node is initialized such that origin is a waypoint, the expected speed decreases with time to 0. This in fact corresponds to an infinite event-average time between two waypoints, which as we show in Section IV, corresponds to the absence of stationary regime. In a subsequent work [15], the same authors advocate to run sound mobility models by initializing a simulation by drawing a sample of the speed according to its time-stationary distribution. We remark that speed is only a partial state of a node; in this paper, we look at the complete state of the node mobility. For the last reason, the authors in [15] do not completely solve the problem of running perfect simulations. Another related work is that of Lin, Noubir, and Rajaraman [11] that studies a class of mobility models where travel distance and travel speed between transition points can be modeled as a renewal process. The renewal assumption was also made in [14], [15]. We note that this assumption is *not* verified with mobility models such as classical random-waypoint on any non-isotropic domain, such as rectangle, for example. The renewal assumption has been made largely to make use of a "cycle" formula from renewal theory. An elementary knowledge of Palm calculus tells us that "cycle" formula is in fact Palm inversion formula, which we used extensively throughout the paper, and that applies more generally; this renders the renewal assumption unnecessary. Perhaps the work closest to ours is that of Navidi, Camp, and Bauer in [13], [12]. As discussed in Section I-C, we provide a systematic framework that allows to formally prove some of the implicit statements in [12] and generalize to a broader class. Further, our perfect sampling algorithm differs in that it works even when geometric constants are not a priori known.

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## References

[1] François Baccelli and Pierre Brémaud. *Palm Probabilities and Stationary Queues*. Springer LNS, 1987.

[2] Christian Bettstetter, Giovanni Resta, and Paolo Santi. The node distribution of the random waypoint mobility model for wireless ad hoc networks. *IEEE Transactions On Mobile Computing*, 2(3), July-September 2003.

[3] Ljubica Blazevic, Jean-Yves Le Boudec, and Silvia Giordano. A location based routing method for mobile ad hoc networks. *IEEE Transactions on Mobile Computing*, 3(4), December 2004.

[4] Jean-Yves Le Boudec and Milan Vojnović. Perfect simulation and stationarity of a class of mobility models. Technical report, EPFL, July 2004. http://lcawww.epfl.ch/Publications/LeBoudec/lebvoj04.pdf.

[5] Josh Broch, David A. Maltz, David B. Johnson, Yih-Chun Hu, and Jorjeta Jetcheva. A performance comparison of multi-hop wireless ad hoc network routing protocols. In *Mobile Computing and Networking*, pages 85–97, 1998.

[6] T Camp, J. Boleng, and V. Davies. A survey of mobility models for ad hoc network research. *WCMC: Special issue on Mobile Ad Hoc Networking: Research, Trends and Applications*, 2(5):483–502, 2002.

[7] Erhan Çinlar. *Introduction to Stochastic Processes*. Prentice Hall, 1975.

[8] Hannes Hartenstein Christian Bettstetter and Xavier Pérez-Costa. Stochastic properties of the random waypoint mobility model. *ACM/Kluwer Wireless Networks*, Special Issue on Modeling and Analysis of Mobile Networks 2003.

[9] Amit Jardosh, Elizabeth M. Belding-Royer, Kevin C. Almeroth, and Subhash Suri. Towards realistic mobility models for mobile ad-hoc networking. In *Proc. of ACM Mobicom 2003*, pages 217–229, San Diego, CA, 2003.

[10] Jean-Yves Le Boudec. Understand the simulation of mobility models with palm calculus. Technical Report EPFL/IC/2004/53, EPFL, 06 2004.

[11] Guolong Lin, Guevara Noubir, and Rajmohan Rajamaran. Mobility models for ad-hoc network simulation. In *Proceedings of Infocom 2004*, April 2004.

[12] William Navidi and Tracy Camp. Stationary distributions for the random waypoint model. *IEEE Transactions on Mobile Computing*, 3(1), 2004.

[13] T. Camp W. Navidi and N. Bauer. Improving the accuracy of random waypoint simulations through steady-state initialization. In *Proceedings of the 15th International Conference on Modeling and Simulation (MS '04)*, pages 319–326, March 2004.

[14] Jungkeun Yoon, Mingyan Liu, and Brian Noble. Random waypoint considered harmful. In *Proc of Infocom*, 2003.

[15] Jungkeun Yoon, Mingyan Liu, and Brian Noble. Sound mobility models. In *Proceedings of Mobicom*, 2003.

## APPENDIX

### *Evaluating a Transient Distribution*

We discuss details on obtaining the transient distribution shown in the example of Section I (Figure 1). Define the set of phases as $I = \mathcal{A}_1 \times \mathcal{A}_1 \times \{pause, move\}$, where $\mathcal{A}_1$ are points in $\mathcal{A}$ defined by the location of the vertices of the graph. At time $t$, $I(t) = (v_0, v_1, \phi)$ denotes the origin vertex is $v_0$, the destination vertex is $v_1$, $\phi$ is either *move* or *pause*. Assume $I_n$ verifies H3, $I$ is finite, the transition matrix is $Q$ and the stationary distribution $\pi^0$. Our goal is to compute $\mathbb{P}^0(I(t) = i)$, $i \in I$, the distribution of the phase at time $t$, given that the initial phase $I_0$ is drawn from the event-stationary distribution. The distribution $\mathbb{P}^0(I(t) = i)$, $i \in I$, is a *transient* distribution, which we know how to compute in the view that $(I_n, T_n)$ is a Markov renewal process. To that end, let $A(t)$ be an event determined by $(I(t), S(t))$. By Markov renewal property, we have

$$\mathbb{P}^0(A(t)|I_0 = i) = \mathbb{P}^0(A(t), T_1 > t|I_0 = i) + \quad (4)$$
$$+ \sum_{j \in I} \int_0^t Q(i,j) G_i^0(j, ds) \mathbb{P}^0(A(t-s)|I_0 = j), \ i \in I$$

where $G_i^0(j,s) := \mathbb{P}^0(I_1 = j, T_1 \leq s | I_0 = i)$. From [7] (Proposition 4.9, Chapter 10, Section 4)

$$\begin{aligned} & \lim_{t \uparrow \infty} \mathbb{P}^0(A(t)|I_0 = i) \\ = \ & \lambda \sum_{j \in I} \pi^0(j) \int_0^\infty \mathbb{P}^0(A(s), T_1 > s | I_0 = j) ds \\ = \ & \lambda \sum_{j \in I} \pi^0(j) \mathbb{E}^0 \left( \int_0^\infty 1_{A(s)} 1_{\{T_1 > s\}} ds | I_0 = j \right) \\ = \ & \lambda \sum_{j \in I} \pi^0(j) \mathbb{E}^0 \left( \int_0^{T_1} 1_{A(s)} ds | I_0 = j \right) \end{aligned}$$

where $\lambda = 1 / \sum_{j \in I} \pi^0(i) \mathbb{E}^0(T_1 | I_0 = i)$. By Palm inversion formula, we recognise that the right-hand side in the last equality of the above display is $\mathbb{P}(A(t))$; in other words, the transient distribution converges to the time-stationary distribution as $t$ goes to infinity. A question is how long it takes for the transient distribution to converge close the time-stationary distribution. This is evaluated numerically in Section I.

The system of equations (4) is known as Markov renewal equation; in principle, it is numerically solved routinely. In our example this is in particular simple for the reasons explained now. In the example, we assumed the mobile always moves and travel time on a path $j$ is fixed to an integer number of seconds $\tau_j > 0$. This allows us to instantiate (4) as: $A(t) := \{I(t) = k\}$ and $G_i^0(j, ds) = \delta_{\tau_j}(s)$. With the notation $\pi^0(i,k,t) := \mathbb{P}^0(I(t) = k | I(0) = i)$, (4) boils down to

$$\pi^0(i,k,t) = a(i,k,t) + \sum_{j \in I} Q(i,j) b(j,t) \pi^0(j,k,t - \tau_j) \quad (5)$$

where $a(i,k,t) = 1_{\{\tau_i > t, i = k\}}$ and $b(j,t) = 1_{\{t \geq \tau_j\}}$, $i, j, k \in I$.

The equations (5) are difference equations, which we run recursively for $t = 1, 2, \ldots$ to obtain numerical values in Figure 1.

### *Proof of Lemma 1*

First we show the lemma for $d = 1$. It is also sufficient to show this for $a_1 = 1$. We have

$$X' = X + v \mod 1 \quad (6)$$

Since $X'$ is limited to the interval $[0,1]$, its distribution is entirely defined by its Fourier coefficients for $n \in \mathbb{Z}$: $c'_n = \mathbb{E}\left(e^{2i\pi n X'}\right)$. By Equation (6) $c'_n = e^{2i\pi n v} c_n$, where $c_n$ is the $n$th Fourier coefficient of the distribution of $X$. Now $X$ is uniform over $[0,a]$ thus $c_n = 0$ for $n \neq 0$ and $c_0 = 1$. It follows that $c'_n = c_n$ for all $n$.

Now back to the general case, we have shown that all coordinates are uniformly distributed. Further, they are independent because $X$ is uniform and $\vec{v}$ is constant.

*Proof of Lemma 3*

Let $I_k$ be the phase drawn at the $k$ iteration of the loop and $T$ be the number of iterations when we exit the loop (if ever). Assume first that $\bar{\tau}_i$ is unknown for all $i$. We have $\mathbb{P}(T=k) = q_1(1-q_1)^{k-1}$ with

$$q_1 = \sum_i \int_{\mathbb{R}^d} \frac{k_i(y)}{\kappa_i} g_i(y)dy = \sum_i \nu(i)\frac{\bar{\tau}_i}{\kappa_i} = \alpha \sum_i \pi^0(i)\bar{\tau}_i$$

Note that $0 < q_1 \leq 1$ thus the loop terminates with probability 1. $I_T$ is the value of $i$ when we exit the loop and

$$\mathbb{P}(I_T = i) = \sum_{k\geq 1} \mathbb{P}(I_T = i \text{ and } T = k | T \geq k)(1-q_1)^{k-1}$$
$$= \sum_{k\geq 1} \nu(i)\frac{\bar{\tau}_i}{\kappa_i}(1-q_1)^{k-1} = \frac{\frac{\bar{\tau}_i}{\kappa_i}}{q_1} = \frac{\pi^0(i)\bar{\tau}_i}{\sum_j \pi^0(j)\bar{\tau}_j}$$

which shows the result in this case. Second, consider some $i$ for which $\bar{\tau}_i$ is known. Let $g_i = f_i/\bar{\tau}_i$, $k_i(y) = \bar{\tau}_i$ and $\kappa_i = \bar{\tau}_i$. When $I = i$ is drawn, it is kept with probability 1. Thus the case $\bar{\tau}_i$ is a special case of the previous one.

*Proof of Lemma 4*

It is enough to show the lemma in dimension 1 and for $a_1 = 1$. In this case, the result to prove is

$$b_1(x+v) = b_1\left(b_1(x) + b_1'(x)v\right)$$

for any $x \in \mathbb{R}\backslash\mathbb{Z}, v \in \mathbb{R}$. Both sides of the equation are 2-periodic in $x$, thus we can restrict to $-1 < x < 0$ and $0 < x < 1$. In the former case, the equation is trivial. In the latter, it becomes $b_1(x+v) = b_1(-x-v)$, which is true because $b_1()$ is even.

*Details of Perfect Sampling for Restricted Random Waypoint*

**Complexity.** We compare the complexity of the two branches of the algorithm in numbers of calls to the random number generator. Let $a$ be the number of such calls required to simulate one sample $(M_0, M_1)$ uniformly in the $\mathcal{A}_1$ plus one ($a = 5$ for a rectangle or a disk, usually more for non convex domains). By an analysis similar to the proof of Lemma 3, we find, for the former case $C_1 = \frac{\alpha+\Delta a}{\alpha+\bar{\Delta}}$ and for the latter $C_2 = \frac{\Delta-\bar{\Delta}}{\alpha+\bar{\Delta}}(1+a) + \frac{\alpha+(1+a)\Delta}{\alpha+\Delta}$, with $\alpha = \tau_{pause}/\omega$.

We always have $C_2 > C_1$; thus if $\bar{\Delta}$ is known with little computational cost, it is always preferable to use the former case ("$\bar{\Delta}$ is known"). In contrast, if $\bar{\Delta}$ is not known, there are two options: (1) compute $\bar{\Delta}$ offline by Monte-Carlo simulation and use the former case ("$\bar{\Delta}$ is known"), or (2) use the latter case ("$\bar{\Delta}$ is not known"). The optimal choice depends on the number $N$ of mobiles that need to be initialized by the sampling procedure ($N$ includes the number of replications of the simulation).

- $\mathcal{L} =$ is the set of $(\ell, \ell')$ for which $\bar{\Delta}_{\ell,\ell'}$ is known in advance. For $(\ell, \ell') \in \mathcal{L}$, $K_{\ell,\ell'} = \bar{\Delta}_{\ell,\ell'}$ and otherwise $K_{\ell,\ell'} = \Delta_{\ell,\ell'}$.
- The following four distributions are used:

$$\begin{cases} E_1(\ell,\ell') = q^*(\ell)Q(\ell,\ell')\tau_{\ell,\ell'}/e_1 \\ E_2(\ell) = q^*(\ell)\bar{R}_\ell\tau_{\ell,\ell}/e_2 \\ E_3(\ell,\ell') = q^*(\ell)Q(\ell,\ell')K_{\ell,\ell'}\omega_{\ell,\ell'}/e_3 \\ E_4(\ell) = q^*(\ell)\bar{R}_\ell K_{\ell,\ell'}\omega_{\ell,\ell}/e_4 \end{cases}$$

  where $e_1, e_2, e_3, e_4$ are normalizing c

**do forever**
  Draw $U_1 \sim U(0,1)$
  **if** $U_1 \leq \frac{e_1+e_2}{e_1+e_2+e_3+e_4}$ // decide $\phi(t) = pause$
    **if** $U_1 \leq \frac{e_1}{e_1+e_2+e_3+e_4}$
      Draw $(\ell, \ell')$ from the distribution $E_1(\ell, \ell')$
      $I(t) = (\ell, \ell', 0, pause)$; **leave**
    **else**
      Draw $\ell$ from the distribution $e_2(\ell)$
      Draw $r \in \mathbb{Z}$ with probability $\frac{F_\ell(r)}{\bar{R}_\ell}$
      $I(t) = (\ell, \ell, r, pause)$; **leave**
  **else** // try $\phi(t) = move$
    // first sample $\ell, \ell'$
    **if** $U_1 \leq \frac{e_1+e_2+e_3}{e_1+e_2+e_3+e_4}$
      Draw $(\ell, \ell')$ from the distribution $E_3(\ell, \ell')$; $r = 0$
    **else**
      Draw $\ell$ from the distribution $E_4(\ell)$, $\ell' = \ell$
      Draw $r \in \mathbb{Z}$ with probability $\frac{F_\ell(r)}{\bar{R}_\ell}$
    **if** $(\ell, \ell') \in \mathcal{L}$
      $I(t) = (\ell, \ell', r, move)$
      **do**
        Draw $M_0 \sim$ Unif$(\mathcal{A}_\ell), M_1 \sim$ Unif$(\mathcal{A}_{\ell'})$
        Draw $U_2 \sim$ Unif$(0, \Delta_{\ell,\ell'})$
      **until** $U_2 < d(M_0, M_1)$
      **leave**
    **else** // $(\ell, \ell') \notin \mathcal{L}$
      Draw $M_0 \sim$ Unif$(\mathcal{A}_\ell), M_1 \sim$ Unif$(\mathcal{A}_{\ell'})$
      Draw $U_2 \sim$ Unif$(0, \Delta_{\ell,\ell'})$
      **if** $U_2 < d(M_0, M_1)$
        $I(t) = (\ell, \ell', r, move)$; **leave**
**end do**

Fig. 9. Sampling algorithm for restricted random waypoint, supporting both cases where the average distance between $\mathcal{A}_\ell$ and $\mathcal{A}_{\ell'}$ is known or not.

Clearly, since $C_2 > C_1$, as $N$ goes to $\infty$, and since the cost of the Monte Carlo simulation is incurred only once for all simulation runs, there is a breakpoint $N_0$ such that for $N \leq N_0$ it is optimal to use the first option, and vice versa. The complexity of Monte Carlo to compute $\bar{\Delta}$ with 99.99% confidence interval and a relative accuracy of $1-\varepsilon$ is of the order of $a(6\frac{\sigma}{\Delta\varepsilon})^2$, where $\sigma^2$ is the variance of the distance between two points in $\mathcal{A}_1$. $\sigma$ depends on the regularity of the domain $\mathcal{A}$. For restricted random waypoint or city graph, it is large compared to the mean value. For more regular areas, a crude approximation of $\sigma$ is $\Delta - \bar{\Delta}$. Comparing $C_2/C_1$ to this complexity, we find that $N_0$ is of the order of 10 to 1000 times $\frac{1}{\varepsilon^2}$. In practice, $\varepsilon = 10^{-4}$ and thus $N_0$ is of the order of $10^9$ to $10^{11}$ for $L = 1$, which is probably larger than the number of simulation runs performed in a campaign by several orders of magnitude. Thus, it should generally be much more efficient to consider the second option.

*Proof of Lemma 5*

It is sufficient to consider the case $a_1 = a_2 = 1$. The mapping that transforms $(M = (x,y), \vec{W} = (u,v))$ into $(M' = (x',y'), \vec{W}' = (u',v'))$ is such that $x = \varepsilon_1(x'-u') + 2n_1$, $y = \varepsilon_2(y'-v') + 2n_2$, $u = \varepsilon_1 u'$ and $v = \varepsilon_2 v'$, where $\varepsilon_1, \varepsilon_2 \in \{-1,1\}$ and $n_1, n_2 \in \mathbb{Z}$. It is differentiable almost everywhere and its Jacobian is 1. Thus, the joint density of $(M', \vec{W}')$ is

$$f_{M',\vec{W}'}(x',y',u',v') = \sum_{\varepsilon_1,\varepsilon_2 \in \{-1,1\}, n_1,n_2 \in \mathbb{Z}} f_{\vec{W}}(\varepsilon_1 u', \varepsilon_2 v') 1_{\{\varepsilon_1(x'-u')+2n_1 \in (0,1)\}} 1_{\{\varepsilon_2(y'-v')+2n_2 \in (0,1)\}}$$

Since $f_{\vec{W}}$ is completely symmetric:

$$= f_{\vec{W}}(u',v') \sum_{\varepsilon_1,\varepsilon_2 \in \{-1,1\}, n_1,n_2 \in \mathbb{Z}} 1_{\{\varepsilon_1(x'-u')+2n_1 \in (0,1)\}} 1_{\{\varepsilon_2(y'-v')+2n_2 \in (0,1)\}}$$
$$= \left(\sum_{\varepsilon_1 \in \{-1,1\}, n_1 \in \mathbb{Z}} 1_{\{\varepsilon_1(x'-u')+2n_1 \in (0,1)\}}\right) \left(\sum_{\varepsilon_2 \in \{-1,1\}, n_2 \in \mathbb{Z}} 1_{\{\varepsilon_2(y'-v')+2n_2 \in (0,1)\}}\right)$$

Now for any $x \in \mathbb{R} \backslash \mathbb{Z}$:

$$\sum_{\varepsilon_1 \in \{-1,1\}, n_1 \in \mathbb{Z}} 1_{\{\varepsilon_1 x + 2n_1 \in (0,1)\}} = 1$$

It follows that for all $u', v'$ and $x', y' \in (0,1)$ except on a set of zero mass $f_{M',\vec{W}'}(x',y',u',v') = f_{\vec{W}}(u',v')$.