Ultrawide Bandwidth Signals as Shot-Noise: a Unifying Approach

Andrea Ridolfi, and Moe Z. Win.

École Polytechnique Fédérale de Lausanne - EPFL
School of Computer and Communication Sciences

Technical Report IC/2005/037
June, 2005

This research was supported, in part, by the Swiss National Science Foundation under Grant 21-65187.01, the Office of Naval Research Young Investigator Award N00014-03-1-0489, the National Science Foundation under Grant ANI-0335256, the Charles Stark Draper Endowment, and the Institute of Advanced Study Natural Science & Technology Fellowship.

Andrea Ridolfi is with the Laboratory of AudioVisual Communications (LCAV), School of Computer and Communication Sciences, École Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland (e-mail: andrea.ridolfi@epfl.ch).

Moe Z. Win is with the Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology, Room 32-D658, 77 Massachusetts Avenue, Cambridge, MA 02139 USA (e-mail: moewin@mit.edu).

Abstract

We present a shot-noise based model for a large family of ultrawide bandwidth (UWB) signals. These include time-hopping and direct-sequence signaling with pulse position, interval and amplitude modulations. Each specific signal is constructed by adding features to a basic model in a modular, simple, and tractable way. Our work unifies the contributions scattered in the literature and provides a general approach that allows various extensions of previous works. The exact power spectrum is then evaluated using shot-noise spectral theory, which provides a simpler, systematic, and rigorous approach to the spectra evaluation of complex UWB signals. The strength of our methodology is that different features of the model contribute clearly and separately in the resulting spectral expressions.

I. Introduction

The family of pulse modulation techniques has gathered increasing attention since the introduction of ultrawide bandwidth (UWB) impulse radio. UWB radio communicates with pulses of very short duration, thereby spreading the energy of the radio signal over several GHz. UWB signals are transmitted with spread spectral content while maintaining the average power level required for reliable communications. As a consequence, the spectral characteristics (spectral occupancy and composition) of an UWB transmission has a key role in the design of UWB systems.

Exact spectral evaluation has already received attention in the communications community. Among several contributions, we mention the computation of the power spectrum of a general time-hopping, pulse position modulated signal [1]–[3], and that of the spectral density of the family of pulse interval modulated signals [4]. In these contributions, the signal models are based on Dirac pseudo-functions and lack generality, especially with respect to the type of temporal modulation. The spectrum computation is performed using the classical wide-sense stationary (WSS) approach, *i.e.*, as the Fourier transform of the correlation function. Although this is a common approach, the resulting computations are complex and the introduction of additional random features of the model, such as clock jitter, pulse losses or pulse distortion, requires a new computation from scratch.

Here we show that the large family of UWB signals are aptly modelled as a shot-noise with random excitation, *i.e.*, a filtered stream of spikes or filtered point process with a random filtering function. Shot-noise processes have received much attention in the applied literature (see for instance [5], [6] and the references therein). Concerning communications systems, they have been widely used in queuing and teletraffic theory [7], and a Poisson based model for pulse coded optical transmissions has been proposed in [8], [9].

As we shall see, there are two considerable advantages in modeling UWB signals as shot-noise processes. Firstly, the model is *modular*, *simple*, and *tractable*. Thus, one can construct different UWB signals in a unifying way by simply adding features to a basic model systematically, and can easily take into account random quantities, such as jitter, losses, or pulse distortions, which affect the signals. Secondly, spectra are obtained, in a systematic and rigorous manner from a single general formula. This formula simplifies previous proofs of the existing results and provides spectrum expressions of highly complicated signals where various features of the model appear separately and explicitly, preserving the modularity of the model. As we will see, these advantages have a tremendous impact on the design and the analysis of UWB signal models.

We consider one-dimensional models of signals taking real values. Extension to complex signals is straightforward while the extension to N dimensions can be easily achieved by following the approach presented in [10]. UWB transmission systems employ time-hopping or direct-sequence signals to achieve multiple access with pulse position or pulse amplitude modulation for data transmission (see for instance [1], [11]). Here, we focus on the following schemes: pulse amplitude modulation (PAM), pulse position modulation (PPM), pulse interval modulation (PIM), time-hopping (TH) signals and direct-sequence (DS) signals.

The outline of the paper is as follows. In Section II we recall the definition of a shot-noise process and give the general expression of its power spectrum. In Section III and Section IV we introduce, respectively, UWB pulse modulated signals and UWB multiple access pulse modulated signals, and we describe how they can be expressed using a single shot-noise model. For each signal, we provide the corresponding power spectrum as a particular case of the formula of the spectrum of a shot-noise process.

II. Shot-Noise Modeling and Spectrum Computation

A shot-noise with random excitation is a stochastic process of the form

$$X(t) = \sum_{n\in\mathbb{Z}} h\left(t - T_n, Z_n\right) \,, \quad t \in \mathbb{R} \,. \tag{1}$$

where $\{T_n\}_{n\in\mathbb{Z}}$ is a general sequence of random times, and $\{Z_n\}_{n\in\mathbb{Z}}$ is a sequence of independent and identically distributed (i.i.d.) random variables (each associated to a random time). We remark that the random times are general and are not assumed to be a Poisson process. We shall also refer to the random times $\{T_n\}_{n\in\mathbb{Z}}$ as random spikes or, in mathematical parlance, as a point process, *i.e.*, a collection of random times $N = \{T_n, n \in \mathbb{Z}\}$ (in the following, we shall use the subscript N to indicate functions related to the random times $\{T_n\}_{n\in\mathbb{Z}}$). The function $h\left(t, Z\right)$ can be interpreted as an impulse response function that depends on a random parameter Z. In the deterministic case $h\left(t, Z\right) = h\left(t\right)$, and $X(t)$ can be symbolically obtained as the as the convolution of the impulse response h with a sequence of Dirac pseudo-functions centered at random times $\{T_n\}_{n\in\mathbb{Z}}$, *i.e.*,

$$X\left(t\right) = h * \sum_{n\in\mathbb{Z}} \delta\left(\cdot - T_n\right) \ \left(t\right) .$$

In the case of UWB signals, the sequence of random times $\{T_n\}_{n\in\mathbb{Z}}$ determines the temporal structure of the signal and the random function $h\left(t, Z\right)$ characterizes the shape of the pulses and its random modifications (amplitude, displacement, distortion).

When the amplitudes of the pulses are random and i.i.d., they can be modeled via the random parameter Z (this will be explained later in the paper). In the more general case where the amplitudes of the pulses are modulated by a (correlated) WSS time series $\{A_n\}_{n\in\mathbb{Z}}$, the modulated shot-noise with random excitation can be written as

$$X(t) = \sum_{n\in\mathbb{Z}} A_n h\left(t - T_n, Z_n\right) \,, \quad t \in \mathbb{R} \,, \tag{2}$$

which has (1) as a special case. We refer to $\mathbb{E}\{A\} = \mathbb{E}\{A_n\}$ as the mean, $\mathbb{E}\{A^2\} = \mathbb{E}\{A_n^2\}$ as the second order moment, and $R_A\left(k\right) = \mathbb{E}\{A_{n+k}A_n^*\}$ as the correlation of the WSS time series.

Concerning the power spectral density of a shot-noise with random excitation, we have the following result [10], [12].

Theorem 2.1: Let $\{X\left(t\right)\}_{t\in\mathbb{R}}$ be the shot noise with random excitation of equation (1). Denote with $\widehat{h}(\nu, Z)$ the Fourier transform of $h\left(t, Z\right)$ (with respect to t) and with $\mathcal{S}_N(\nu)$ the power spectral (pseudo) density of the sequence of random times $\{T_n\}_{n\in\mathbb{Z}}$. Call λ the average number of random points, or times, per unit of time. Then

$$\mathcal{S}_X(\nu) = \left|\mathbb{E}\left\{\widehat{h}(\nu, Z)\right\}\right|^2 \mathcal{S}_N(\nu) + \lambda \mathrm{Var}\left\{\widehat{h}(\nu, Z)\right\} \,, \tag{3}$$

Proof: The proof is sketched in the appendix. A complete and more mathematically rigorous proof can be found in [10], [12]. □

Let now $\{X\left(t\right)\}_{t\in\mathbb{R}}$ be the shot noise with random excitation where the amplitudes of the shots are modulated by the time series $\{A_n\}_{n\in\mathbb{Z}}$, as in (2). Assume that the sequence of random times $\{T_n\}_{n\in\mathbb{Z}}$ forms a renewal process and denote with $\phi_S^k\left(u\right)$ the characteristic function of the i.i.d. sequence of inter-arrival times $S_n = T_{n+1} - T_n$, $n \neq 0$ (i.i.d. characteristic of the inter-arrival times is a property of renewal processes [13]). The power spectrum of the modulated shot noise with random excitation (2) is given by

$$\mathcal{S}_X(\nu) = \left|\mathbb{E}\left\{\widehat{h}(\nu, Z)\right\}\right|^2 \mathcal{S}_{N_A}(\nu) + \lambda\mathbb{E}\left\{|A|^2\right\} \mathrm{Var}\left\{\widehat{h}(\nu, Z)\right\} \tag{4}$$

where

$$\mathcal{S}_{N_A}(\nu) = \lambda \left(2\mathrm{Re}\left\{\sum_{k\geq 0} \phi_S^k(2\pi\nu) R_A(k)\right\} - R_A(0)\right) - \lambda^2\mathbb{E}\{A\}^2 \delta(\nu) \ , \tag{5}$$

We recall that the power spectral (pseudo) density of a renewal process is given by [14]

$$\begin{aligned}\mathcal{S}_N(\nu) &= \lambda \left(2\mathrm{Re}\left\{\sum_{k\geq 0} \phi_S^k(2\pi\nu)\right\} - 1\right) - \lambda^2\delta(\nu) \\ &= \begin{cases} \lambda\left(2\mathrm{Re}\left\{\frac{1}{1-\phi_S(2\pi\nu)}\right\} - 1\right), & \forall\ \nu \neq 0\,, \\ \lambda\left(1 + \frac{\mathbb{E}\{S^2\}}{\mathbb{E}\{S\}}\right), & \text{otherwise}\,. \end{cases}\end{aligned} \tag{6}$$

III. UWB Pulse Modulations

We now present the shot-noise models for the family of UWB pulse modulations and their exact power spectrum expressions.

A. Pulse Position Modulation

In pulse position modulation (PPM) the information is carried by the relative positions of the pulses with respect to a regular grid (see for instance [1], [11], [15], [16]). Therefore, a PPM signal can be seen as regularly spaced pulses (regularly spaced point process convoluted with the pulse shape) to which we add positive random i.i.d. displacement to encode the symbols to be transmitted. The sequence of random times is given by

$$T_n = U + nT, \quad n \in \mathbb{N},$$

where U is a uniform-$[0, T]$ random variable, which ensures the stationarity of the sequence, and T is the time-period. Symbolically, we can express the random points as a Dirac comb $\Delta_N(t) = \sum_{n\in\mathbb{Z}} \delta(t - nT - U)$. The random displacements are modelled as the i.i.d. parametrs $\{Z_n\}_{n\in\mathbb{Z}}$. A PPM signal is then modelled as

$$X(t) = \sum_{n\in\mathbb{Z}} w(t - U - nT - Z_n) \ , \tag{7}$$

where $w(t)$, $t \in \mathbb{R}$, denotes the pulse shape. Hence, we have the shot-noise with random excitation in the form of equation (2), where $A_n := 1$, $T_n := U + nT$, and $h(t, Z) := w(t - Z)$. Fig. 1 shows an example of a signal with pulse position modulation.

In the presence of clock jitter, a signal with PPM can be modeled as

$$X(t) = \sum_{n\in\mathbb{Z}} w\left(t - U - nT - Z^{\mathrm{P}} - Z^{\mathrm{J}}\right) \ ,$$

where $\{Z_n^{\mathrm{P}}\}_{n\in\mathbb{Z}}$ is the sequence of random variables modeling the position modulation and $\{Z_n^{\mathrm{J}}\}_{n\in\mathbb{Z}}$ is the sequence of random variables modeling the additional random displacement due to the clock jitter. Therefore, we have the shot-noise process in the form of equation (1) with filtering function

$$h(t, Z) := w\left(t - Z^{\mathrm{P}} - Z^{\mathrm{J}}\right) \ ,$$

where $Z = (Z^{\mathrm{P}}, Z^{\mathrm{J}})$.

Power Spectrum

We have just shown that a signal with PPM is a shot-noise with random excitation where the corresponding sequence of random times forms a regularly T-spaced grid. The power spectral (pseudo) density of a regularly T-spaced grid is given by

$$\mathcal{S}_N\left(\nu\right) = \frac{1}{T}\sum_{n\neq 0}\delta\left(\nu - \frac{n}{T}\right) .$$

Therefore, from equation (3) we have

$$\mathcal{S}_X(\nu) = \left|\widehat{w}(\nu)\right|^2\left|\phi_Z\left(2\pi\nu\right)\right|^2\frac{1}{T}\sum_{n\neq 0}\delta\left(\nu - \frac{n}{T}\right) + \lambda\left|\widehat{w}(\nu)\right|^2\left(1 - \left|\phi_Z\left(2\pi\nu\right)\right|^2\right) ,$$

where $\phi_Z\left(\cdot\right)$ is the characteristic function of the transmitted symbols $\{Z_n\}_{n\in\mathbb{Z}}$.

The effect of clock jitter and pulse losses can be straightforwardly taken into account. For instance, if the transmitted signal is affected by i.i.d. losses, we consider Z in (3) to be random vector $Z = (Z^{\text{P}}, Z^{\text{L}})$ where now Z^{P} and Z^{L} model, respectively, the displacement due to the encoding of the information and the pulse loss. Then,

$$\mathcal{S}_X(\nu) = \left|\widehat{w}(\nu)\right|^2\mathbb{E}\left\{Z^{\text{L}2}\right\}\left|\phi_Z\left(2\pi\nu\right)\right|^2\frac{1}{T}\sum_{n\neq 0}\delta\left(\nu - \frac{n}{T}\right) + \lambda\left|\widehat{w}(\nu)\right|^2\left(\mathbb{E}\left\{Z^{\text{L}2}\right\} - \mathbb{E}\left\{Z^{\text{L}}\right\}^2\left|\phi_Z\left(2\pi\nu\right)\right|^2\right) .$$

If we consider the transmission of M independent symbols, each with equal probability $1/M$, the spikes take, with equal probability, the relative positions $\{0, T/M, \ldots, T(M-1)/M\}$. Therefore, the characteristic function of the "jitters" is given by

$$\phi_Z(2\pi\nu) = \frac{1}{M}e^{i\pi\nu T\frac{M-1}{M}}\frac{\sin\left(\pi\nu T\right)}{\sin\left(\pi\nu T/M\right)} .$$

The spectrum of a PPM signal is known. However, several extensions of known results can be obtained by modification of the PPM signal using our modular approach.

B. Pulse Amplitude Modulation

A PAM signal transmits the information through the amplitude of the pulses (see for instance [15], [16]). Hence we have a regularly spaced sequence of pulses with random amplitudes. When the symbols are encoded into a sequence of i.i.d. amplitudes, a PAM signal can be modelled as

$$X\left(t\right) = \sum_{n\in\mathbb{Z}} Z_n w\left(t - U - nT\right) . \tag{8}$$

Clearly, (8) is a particular case of a shot-noise process in the form of equation (2), with $A_n := 1$, $T_n := U + nT$, and $h\left(t, Z\right) := Zw\left(t\right)$. See Fig. 2 for an example of a PAM signal.

If the amplitudes are correlated, we have

$$X\left(t\right) = \sum_{n\in\mathbb{Z}} A_n w\left(t - U - nT\right) , \tag{9}$$

which is a special case of the shot-noise process in (2) with $T_n := U + nT$ and $h\left(t, Z\right) := w\left(t\right)$.

As an example, let us consider a PAM signal with i.i.d. amplitudes affected by random losses of the pulses (the extension to the case of correlated amplitudes is straightforward). In such a case, the PAM signal reads

$$X\left(t\right) = \sum_{n\in\mathbb{Z}} Z^{\text{A}}_n Z^{\text{L}}_n w\left(t - U - nT\right)$$

where Z_n^{A} are the transmitted symbols while Z_n^{L}, with values in $\{0,1\}$, are the random losses. Hence, we have a shot-noise with random excitation with filtering function

$$h(t,Z) := Z^{\text{A}} Z^{\text{L}} w(t) \ ,$$

where the sequence of i.i.d. parameters is now the random vector $Z_n = (Z_n^{\text{A}}, Z_n^{\text{L}})$.

Power Spectrum

For i.i.d. amplitudes, the PAM model is given by equation (8). Hence the sequence of random times forms a regular grid with power spectral pseudo-density

$$\mathcal{S}_N(\nu) = \frac{1}{T}\sum_{n\neq 0} \delta\left(\nu - \frac{n}{T}\right) \ ,$$

and the filtering function is $h(t,Z) := Zw(t)$, with

$$\mathbb{E}\left\{\widehat{h}(\nu,Z)\right\} = \widehat{w}(\nu)\,\mathbb{E}\{Z\}\,, \quad \mathbb{E}\left\{\left|\widehat{h}(\nu,Z)\right|^2\right\} = |\widehat{w}(\nu)|^2\,\mathbb{E}\left\{|Z|^2\right\} \ .$$

The power spectral pseudo-density then reads

$$\mathcal{S}_X(\nu) = |\widehat{w}(\nu)|^2\,|\mathbb{E}\{Z\}|^2\,\frac{1}{T}\sum_{n\neq 0}\delta\left(\nu - \frac{n}{T}\right) + \lambda\,|\widehat{w}(\nu)|^2\,\mathrm{Var}\{Z\} \ .$$

When the amplitudes are correlated the PAM model is given by equation (9). In this case, we have a shot-noise process where the sequence of random times forms a regularly spaced spikes modulated by the time series. Note that a regular grid is a particular case of a renewal process. Therefore, (4) implies that

$$\mathcal{S}_X(\nu) = |\widehat{w}(\nu)|^2\left(\frac{1}{T}\mathcal{S}_A(T\nu) + \frac{1}{T^2}\,|\mathbb{E}\{A\}|^2\sum_{n\in\mathbb{Z}}\delta\left(\nu - \frac{n}{T}\right)\right) \ ,$$

where here $\mathcal{S}_A$ is the spectral density of the time series. The above formula is also known as the "Bennett-Rice" formula [17].

Here again, various extensions of known results can be obtained by modification of the PAM signal using our modular approach. For instance, we can obtain the spectrum of a PAM signal with correlated amplitude and i.i.d. clock jitter

$$\mathcal{S}_X(\nu) = |\phi_Z(\nu)|^2\,|\widehat{w}(\nu)|^2\left(\frac{1}{T}\mathcal{S}_A(T\nu) + \frac{1}{T^2}\,|\mathbb{E}\{A\}|^2\sum_{n\in\mathbb{Z}}\delta\left(\nu - \frac{n}{T}\right)\right) + \lambda\,|\widehat{w}(\nu)|^2\left(1 - |\phi_Z(\nu)|^2\right)d\nu$$

(such a result straightforwardly follows from the computations performed for a PPM signal with i.i.d. jitter).

C. Pulse Interval Modulation

In pulse interval modulated signals (PIM) the information is coded with the relative distance between successive pulses [18]. The sequence of random times $\{T_n\}_{n\in\mathbb{Z}}$ is naturally modeled as a renewal process. A PIM signal reads

$$X(t) = \sum_{n\in\mathbb{Z}} w(t - T_n) \ ,$$

which is a special case of the shot-noise process (2) with $A_n := 1$, $\{T_n\}_{n\in\mathbb{Z}}$ being a renewal process, and $h(t,Z) := w(t)$.

Power Spectrum

The power spectrum of a PIM signal is given by the power spectrum of a renewal process (a well know result; see for instance [14]), times the Fourier transform of the pulse shape, *i.e.*,

$$\mathcal{S}_X(\nu) = |\widehat{w}(\nu)|^2 \lambda \left(2\mathrm{Re}\left\{ \sum_{k\geq 0} \phi_S^k\left(2\pi\nu\right) \right\} - 1 - \lambda\delta\left(\nu\right) \right) ,$$

where $\phi_S\left(\cdot\right)$ is the characteristic function of the inter-arrival times $S_n = T_{n+1} - T_n$, $n \neq 0$.

As an example, we can consider the transmission of i.i.d. symbols from an alphabet size M with equal probabilities. Such symbols can be encoded in the relative distance between two pulses. Hence, it is a shot-noise process where the sequence of random times forms a renewal process with inter-arrivals time taking values with equal probability over $\{T, 2T, \ldots, MT\}$ (T is the basic increment of the relative distance). Then, the spectrum is given by the above formula with

$$\phi_S(2\pi\nu) = \frac{1}{M} e^{i\pi\nu TM} \frac{\sin\left(\pi\nu T(M+1)\right)}{\sin\left(\pi\nu T\right)} .$$

D. Combination of Pulse Modulations

Any combination of the basic modulations we have presented can be easily taken into account. In particular, the spectrum of combined modulations can be computed easily from the spectrum of the basic modulations. This remarkable feature is a direct consequence of the modular approach and it is one of the most important aspects of our contribution.

We shall present some common situations.

Case 1: PPM and PAM. In the case of i.i.d. amplitudes

$$X\left(t\right) = \sum_{n\in\mathbb{Z}} Z_n^{\mathrm{A}} w\left(t - U - nT - Z_n^{\mathrm{P}}\right) ,$$

is a special case of (2) with $A_n := 1$, $T_n = U + nT$, $Z := (Z^{\mathrm{A}}, Z^{\mathrm{P}})$, and $h\left(t, Z\right) := Z^{\mathrm{A}} w\left(t - Z^{\mathrm{P}}\right)$.

Concerning the computation of the power spectrum, we have

$$\widehat{h}\left(\nu, Z\right) = \widehat{w}\left(\nu\right) Z^{\mathrm{A}} e^{-i2\pi\nu Z^{\mathrm{P}}} ,$$

and

$$\mathbb{E}\left\{\widehat{h}\left(\nu, Z\right)\right\} = \widehat{w}\left(\nu\right) \mathbb{E}\left\{Z^{\mathrm{A}}\right\} \phi_{Z^{\mathrm{P}}}\left(-2\pi\nu\right) , \quad \mathbb{E}\left\{\left|\widehat{h}\left(\nu, Z\right)\right|^2\right\} = \left|\widehat{w}\left(\nu\right)\right|^2 \mathbb{E}\left\{\left|Z^{\mathrm{A}}\right|^2\right\} ,$$

where $\phi_{Z^{\mathrm{P}}}\left(\cdot\right)$ is the characteristic function of the i.i.d. displacements $\{Z_n^{\mathrm{P}}\}_{n\in\mathbb{Z}}$ encoding the symbols to be transmitted. The sequence of random times, or point process, is a regular grid with spectrum

$$\mathcal{S}_N\left(\nu\right) = \frac{1}{T} \sum_{n\neq 0} \delta\left(\nu - \frac{n}{T}\right) .$$

Hence, the power spectrum of a PPM/PAM signal is given by

$$\mathcal{S}_X(\nu) = \left|\widehat{w}\left(\nu\right)\right|^2 \left|\mathbb{E}\left\{Z^{\mathrm{A}}\right\}\right|^2 \left|\phi_{Z^{\mathrm{P}}}\left(2\pi\nu\right)\right|^2 \frac{1}{T} \sum_{n\neq 0} \delta\left(\nu - \frac{n}{T}\right) + \lambda \left|\widehat{w}\left(\nu\right)\right|^2 \left(\mathbb{E}\left\{\left|Z^{\mathrm{A}}\right|^2\right\} - \left|\mathbb{E}\left\{Z^{\mathrm{A}}\right\}\right|^2 \left|\phi_{Z^{\mathrm{P}}}\left(2\pi\nu\right)\right|^2\right) .$$

Case 2: PIM and PAM. With i.i.d. amplitudes

$$X\left(t\right) = \sum_{n\in\mathbb{Z}} Z_n w\left(t - T_n\right)$$

is a special case of (2) with $A_n := 1$, the point process $\{T_n\}_{n\in\mathbb{Z}}$ being a renewal process, and $h(t,Z) := Zw(t)$.

When the amplitudes are correlated we have

$$X(t) = \sum_{n\in\mathbb{Z}} A_n w(t - T_n) \ ,$$

which is of a form of (2), with the point process $\{T_n\}_{n\in\mathbb{Z}}$ being a renewal process, and $h(t,Z) := w(t)$.

The filtering function is now (i.i.d. amplitudes)

$$h(t,Z) = Zw(t)$$

and the sequence of random times form a renewal process. The spectrum of a PIM/PAM signal is

$$\mathcal{S}_X(\nu) = |\widehat{w}(\nu)|^2 |\mathbb{E}\{Z\}|^2 \mathcal{S}_N(\nu) + \lambda |\widehat{w}(\nu)|^2 \operatorname{Var}\{Z\}$$

where $\mathcal{S}_N$ is the power spectral (pseudo) density of a renewal process (6).

We can consider the case of a PIM/PAM signal for the transmission of i.i.d. symbols from an alphabet with M values, with equal probability. In such a case, Z takes values in $\{1, 2, \ldots, M-1, M\}$ with equal probability, and the sequence of random times $\{T_n\}_{n\in\mathbb{Z}}$ is a discrete renewal process with inter-arrivals taking values in $\{1, 2, \ldots, M-1, M\}$ with equal probability. In particular,

$$\mathbb{E}\left\{\widehat{h}(\nu,Z)\right\} = \widehat{w}(\nu)\frac{M+1}{2}, \quad \mathbb{E}\left\{\left|\widehat{h}(\nu,Z)\right|^2\right\} = |\widehat{w}(\nu)|^2 \frac{M(M+1)(2M+1)}{6}$$

Hence, the power spectrum of a PIM/PAM signal for the transmission of i.i.d. symbols from an alphabet with M values, with equal probability, is given by

$$\mathcal{S}_X(\nu) = |\widehat{w}(\nu)|^2 \left(\frac{M+1}{2}\right)^2 \mathcal{S}_N(\nu) + \lambda |\widehat{w}(\nu)|^2 \left(\frac{M(M+1)(2M+1)}{6} - \left(\frac{M+1}{2}\right)^2\right) ,$$

where the characteristic function of the inter-arrival times, appearing in the expression of $\mathcal{S}_N$ (6), is

$$\phi_S(2\pi\nu) = \frac{1}{M} e^{i\pi\nu TM} \frac{\sin(\pi\nu T(M+1))}{\sin(\pi\nu T)} \ . \tag{10}$$

The correlated amplitude case can be straightforwardly taken into account. Suppose that, as well as by the i.i.d. random variables $\{Z_n\}_{n\in\mathbb{Z}}$, the pulses are modulated by the time series $\{A_n\}_{n\in\mathbb{Z}}$, carrying additional information or modeling correlated losses. The model for the PIM/PAM signal is then

$$X(t) = \sum_{n\in\mathbb{Z}} A_n Z_n w(t - T_n) \ ,$$

and its power spectrum is given by

$$\mathcal{S}_X(\nu) = |\widehat{w}(\nu)|^2 \left(\frac{M+1}{2}\right)^2 \mathcal{S}_{N_A}(\nu) + \lambda \mathbb{E}\left\{|A|^2\right\} |\widehat{w}(\nu)|^2 \left(\frac{M(M+1)(2M+1)}{6} - \left(\frac{M+1}{2}\right)^2\right) ,$$

where $\mathcal{S}_{N_A}$ is given by equation (5) with $\phi_S(\cdot)$ as in (10).

IV. UWB Multiple Access Techniques

Time-Hopping and Direct-Sequence are the common multiple access techniques employed in UWB systems. We present the shot-noise models and the exact power spectrum of pulse modulations combined with such multiple access techniques.

A. Time-Hopping Signals

General time-hopping (TH) signals are characterized by a deterministic periodic pattern $\{c_n\}$ to allow multiple access. Such a pattern, commonly called signature, consists of a sequence of pulses positioned with respect to a regular grid. The temporal structure of such a signal is described with a sequence of random times

$$T_n + \sum_{l=0}^{L_c-1} (lT + c_l T_c) \ , \tag{11}$$

where

- $\{c_n\}$, $n = 0, \ldots, L_c$, is the deterministic sequence characterizing the L_c-periodic pattern;
- T is the period of the regular grid ($T_c < T$);
- T_n is a sequence of random times that depends on the type of temporal modulation (usually PPM).

Fig. 3 depicts a TH signal with a signature $\{c_n\}$ of period $L_c = 3$. We remark that the sequence of random times (11) corresponds to a cluster point process [19] with seeds T_n and deterministic clusters $\sum_{l=0}^{L_c-1} (lT + c_l T_c)$.

A general model for TH signals (without modulation) is given by [2]

$$X(t) = \sum_{n=-\infty}^{\infty} \sum_{l=0}^{L_c-1} w(t - T_n - lT - c_l T_c) \ . \tag{12}$$

Equation (12) is a special case of the shot-noise process (2) with random times T_n, and

$$h(t, Z) := \sum_{l=0}^{L_c-1} w(t - lT - c_l T_c) \ . \tag{13}$$

We consider two cases.

Case 1: Time-hopping with PPM/PAM. Let $\{Z_n^{\mathrm{P}}\}_{n\in\mathbb{Z}}$ denote a sequence of i.i.d. random variables representing the positions in PPM, and $\{Z_n^{\mathrm{A}}\}_{n\in\mathbb{Z}}$ denote a sequence of i.i.d. random variables representing the amplitudes in PAM. For simplicity we consider the case of i.i.d. amplitudes; the correlated case can be easily extended.

The random times T_n are now given by

$$T_n = U + nL_c T \ , \tag{14}$$

where U is a random variable uniformly distributed over $[0, L_c T]$. A time-hopping pulse and amplitude modulated signal can be then written as

$$X(t) = \sum_{k=-\infty}^{\infty} Z_n^{\mathrm{A}} \sum_{l=0}^{L_c-1} w(t - U - nL_c T - Z_n^{\mathrm{P}} - lT - c_l T_c) \ . \tag{15}$$

The above expression corresponds to the model presented by [2]. Again, we have the shot-noise with random excitation (2), where the basic point process has random times given by (14), with $A_n := 1$, $Z = (Z^{\mathrm{A}}, Z^{\mathrm{P}})$, and

$$h(t, Z) := Z^{\mathrm{A}} \sum_{l=0}^{L_c-1} w(t - Z^{\mathrm{P}} - lT - c_l T_c) \ . \tag{16}$$

Case 2: Time-hopping PPM/PAM signal with jitter and thinning. Consider the case of a time-hopping with PPM and PAM that is affected by i.i.d. clock jitter and i.i.d. random losses. As previously seen for PPM and PAM, jitter and thinning are introduced using the random parameter of the impulse response (we remark that the case of correlated losses can be modeled through the correlated sequence $\{A_n\}_{n\in\mathbb{Z}}$, with values in $\{0,1\}$).

More precisely, we introduce the i.i.d. sequences of i.i.d. L_c-plets

$$\left\{\left(Z^{\mathrm{J}}_{0;n},\ldots,Z^{\mathrm{J}}_{L_\mathrm{c}-1;n}\right)\right\}_{n\in\mathbb{Z}}, \quad \text{and} \quad \left\{\left(Z^{\mathrm{L}}_{0;n},\ldots,Z^{\mathrm{L}}_{L_\mathrm{c}-1;n}\right)\right\}_{n\in\mathbb{Z}},$$

to model, respectively, the random displacements and the random losses. Thus Z^{L} is a binary sequence. We remark that, due to the clustered nature of the random times, the introduction of L_c-plets is necessary in order to consider jitter and thinning of each pulse of the signature.

Then, in the presence of i.i.d. jitter and i.i.d. random losses, the time-hopping pulse position and amplitude modulated signal of equation (15) reads

$$X(t)=\sum_{n\in\mathbb{Z}} Z^{\mathrm{A}}_n \sum_{l=0}^{L_\mathrm{c}-1} Z^{\mathrm{L}}_{l;n} w\left(t-U-nL_\mathrm{c}T-Z^{\mathrm{P}}_n-lT-c_lT_c-Z^{\mathrm{J}}_{l;n}\right). \tag{17}$$

Again, it is a shot-noise with random excitation, where the sequence of random parametres is

$$Z_n=\left(Z^{\mathrm{A}}_n,Z^{\mathrm{P}}_n,\left(Z^{\mathrm{J}}_{0;n},\ldots,Z^{\mathrm{J}}_{L_\mathrm{c}-1;n}\right),\left(Z^{\mathrm{L}}_{0;n},\ldots,Z^{\mathrm{L}}_{L_\mathrm{c}-1;n}\right)\right),$$

the filtering function is

$$h(t,Z)=Z^{\mathrm{A}}\sum_{l=0}^{L_\mathrm{c}-1} Z^{\mathrm{L}}_l w\left(t-Z^{\mathrm{P}}-Z^{\mathrm{J}}_l-lT-c_lT_c\right), \tag{18}$$

and the random times are given by (14). Note that expression (17) is an extension of the model presented in [2] that takes into account random losses of the pulses as well as a general jitter. Indeed, it is the strength of our approach that enables us to account for additional random quantities that affect the signal, such as jitter, losses, and pulse distortion, obtaining spectrum expression that are still explicit.

Power Spectrum

We consider the time-hopping model of equation (12). The spectrum can be straightforwardly obtained by applying equation (3), as we show in the following cases.

CASE 1: Time-hopping PPM/PAM signal. Recall that $\{Z^{\mathrm{P}}_n\}_{n\in\mathbb{Z}}$ denotes the i.i.d. sequence modeling the positions in PPM and $\{Z^{\mathrm{A}}_n\}_{n\in\mathbb{Z}}$ denotes the i.i.d. sequence modeling the i.i.d. amplitudes in PAM (the correlated case can be straightforwardly taken into account).

Now $T_n=U+nL_\mathrm{c}T$, where U is $[0,L_\mathrm{c}T]$-uniformly distributed. Therefore, its spectral pseudo-density is

$$\mathcal{S}_N=\frac{1}{L_\mathrm{c}T}\sum_{n\neq 0}\delta\left(\nu-\frac{n}{L_\mathrm{c}T}\right).$$

The filtering function is given by (16), therefore

$$\mathbb{E}\left\{\widehat{h}(\nu,Z)\right\}=\widehat{w}(\nu)\,\mathbb{E}\{Z^{\mathrm{A}}\}\,\phi_{Z^{\mathrm{P}}}(-2\pi\nu)\sum_{l=0}^{L_\mathrm{c}-1} e^{-i2\pi\nu(lT+c_lT_c)},$$

$$\mathbb{E}\left\{\left|\widehat{h}(\nu,Z)\right|^2\right\}=\left|\widehat{h}(\nu)\right|^2\mathbb{E}\left\{|Z^{\mathrm{A}}|^2\right\}\left|\sum_{l=0}^{L_\mathrm{c}-1} e^{-i2\pi\nu(lT+c_lT_c)}\right|^2.$$

Therefore, the power spectral pseudo-density of a TH PPM/PAM signal is

$$\mathcal{S}_X(\nu) = \frac{1}{L_c T} |\widehat{w}(\nu)|^2 \left| \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT+c_l T_c)} \right|^2 \left(|\mathbb{E}\{Z^{\mathrm{A}}\}|^2 |\phi_{Z^{\mathrm{P}}}(2\pi\nu)|^2 \sum_{n\neq 0} \delta\left(\nu - \frac{n}{L_c T}\right) + \left[\mathbb{E}\{|Z^{\mathrm{A}}|^2\} - |\mathbb{E}\{Z^{\mathrm{A}}\}|^2 |\phi_{Z^{\mathrm{P}}}(2\pi\nu)|^2\right] \right).$$

Case 2: Time-hopping PPM/PAM signal with jitter and thinning. Recall that the filtering function is given by (18). Hence,

$$\mathbb{E}\left\{\widehat{h}(\nu, Z)\right\} = L_c \widehat{w}(\nu) \mathbb{E}\{Z^{\mathrm{A}}\} \mathbb{E}\{Z^{\mathrm{L}}\} \phi_{Z^{\mathrm{P}}}(-2\pi\nu) \phi_{Z^{\mathrm{J}}}(-2\pi\nu) \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT+c_l T_c)}$$

$$\mathbb{E}\left\{\left|\widehat{h}(\nu, Z)\right|^2\right\} = |\widehat{w}(\nu)|^2 \mathbb{E}\{|Z^{\mathrm{A}}|^2\} \left[L_c \mathbb{E}\{|Z^{\mathrm{L}}|^2\} + |\mathbb{E}\{Z^{\mathrm{L}}\}|^2 |\phi_{Z^{\mathrm{P}}}(2\pi\nu)|^2 \left(\left| \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT+c_l T_c)} \right|^2 - L_c \right) \right].$$

The power spectral pseudo-density is then

$$\mathcal{S}_X(\nu) = |\widehat{w}(\nu)|^2 \frac{1}{L_c T} \Bigg(|\mathbb{E}\{Z^{\mathrm{A}}\}|^2 |\mathbb{E}\{Z^{\mathrm{L}}\}|^2 |\phi_{Z^{\mathrm{P}}}(2\pi\nu)|^2 |\phi_{Z^{\mathrm{J}}}(2\pi\nu)|^2 \left| \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT+c_l T_c)} \right|^2 \sum_{n\neq 0} \delta\left(\nu - \frac{n}{L_c T}\right)$$
$$+ L_c \mathbb{E}\{|Z^{\mathrm{A}}|^2\} \left(\mathbb{E}\{|Z^{\mathrm{L}}|^2\} - |\mathbb{E}\{Z^{\mathrm{L}}\}|^2 |\phi_{Z^{\mathrm{J}}}(2\pi\nu)|^2\right)$$
$$+ |\mathbb{E}\{Z^{\mathrm{L}}\}|^2 |\phi_{Z^{\mathrm{J}}}(2\pi\nu)|^2 \left| \sum_{l=0}^{L_c-1} e^{-i2\pi\nu(lT+c_l T_c)} \right|^2 \left(\mathbb{E}\{|Z^{\mathrm{A}}|^2\} - |\mathbb{E}\{Z^{\mathrm{A}}\}|^2 |\phi_{Z^{\mathrm{P}}}(2\pi\nu)|^2\right) \Bigg).$$

B. Direct-Sequence Signals

Like the time-hopping signal, a direct-sequence signal can be used for multiple access. It consists of multiplying a stream of pulses by a periodic deterministic sequence of +1 and −1 (the signature). It can be generally modeled as

$$X(t) = \sum_{n\in\mathbb{N}} \sum_{k=0}^{L_d-1} a_k^{\mathrm{DS}} w\left(t - T_{nL_d+k}\right) \tag{19}$$

where

- L_d is the period of the signature sequence;
- a_k^{DS}, $k = 0, \ldots, L_d - 1$, is the signature sequence, where $a^{\mathrm{DS}} \in \{+1, -1\}$.

Fig. 4 depicts a direct-sequence signal.

Next, we consider the DS/TH with PPM/PAM in the presence of jitter and random losses. To model a DS/TH with PPM and PAM in the presence of jitter and random losses, we combine the results from the previous subsection.

Let $\mathrm{L} = \max(L_c, L_d)$. Using a similar approach of that of the last section, we introduce the i.i.d. sequences of i.i.d. L-plets

$$\left\{\left(Z_{0;n}^{\mathrm{J}}, \ldots, Z_{L-1;n}^{\mathrm{J}}\right)\right\}_{n\in\mathbb{Z}} \text{ and } \left\{\left(Z_{0;n}^{\mathrm{L}}, \ldots, Z_{L-1;n}^{\mathrm{L}}\right)\right\}_{n\in\mathbb{Z}},$$

to model, respectively, the random displacements and the random losses. Let $\{Z_n^{\mathrm{P}}\}_{n\in\mathbb{Z}}$ denote the i.i.d. sequence modeling the positions in PPM and let $\{Z_n^{\mathrm{A}}\}_{n\in\mathbb{Z}}$ denote the i.i.d. sequence modeling the amplitudes in PAM.

We assume that the period L_{d} of the DS signature and the period L_{c} of the TH signature are multiples of each other. We consider three cases separately.

1) $L_{\mathrm{c}} = L_{\mathrm{d}}$: Then,

$$X(t) = \sum_{n\in\mathbb{N}} Z_n^{\mathrm{A}} \sum_{k=0}^{L_{\mathrm{d}}-1} Z_{k;n}^{\mathrm{L}} a_k^{\mathrm{DS}} w\left(t - U - nL_{\mathrm{c}}T - Z_n^{\mathrm{P}} - kT - c_k T_c - Z_{k;n}^{\mathrm{J}}\right). \tag{20}$$

2) $L_{\mathrm{c}} > L_{\mathrm{d}}$: Then,

$$X(t) = \sum_{n\in\mathbb{N}} Z_n^{\mathrm{A}} \sum_{l=0}^{\frac{L_{\mathrm{c}}}{L_{\mathrm{d}}}-1} \sum_{k=0}^{L_{\mathrm{d}}-1} Z_{lL_{\mathrm{d}}+k;n}^{\mathrm{L}} a_k^{\mathrm{DS}} w\left(t - U - nL_{\mathrm{c}}T - Z_n^{\mathrm{P}} - (lL_{\mathrm{d}}+k)T - c_{lL_{\mathrm{d}}+k} T_c - Z_{lL_{\mathrm{d}}+k;n}^{\mathrm{J}}\right). \tag{21}$$

3) $L_{\mathrm{c}} < L_{\mathrm{d}}$: Then,

$$X(t) = \sum_{n\in\mathbb{N}} Z_n^{\mathrm{A}} \sum_{k=0}^{\frac{L_d}{L_c}-1} \sum_{l=0}^{L_c-1} Z_{kL_c+l;n}^{\mathrm{L}} a_{kL_c+l}^{\mathrm{DS}} w\left(t - U - nL_d T - Z_n^{\mathrm{P}} - (kL_c+l)T - c_k T_c - Z_{kL_c+l;n}^{\mathrm{J}}\right) \tag{22}$$

All three cases correspond to the shot-noise with random excitation (2) where we have $A_n := 1$, $Z := \left(Z^{\mathrm{A}}, Z^{\mathrm{P}}, \left(Z_0^{\mathrm{J}}, \ldots, Z_{L_{\mathrm{c}}-1}^{\mathrm{J}}\right), \left(Z_0^{\mathrm{L}}, \ldots, Z_{L_{\mathrm{c}}-1}^{\mathrm{L}}\right)\right)$, and

$$h(t,Z) := \begin{cases} Z^{\mathrm{A}} \sum_{k=0}^{L_d-1} Z_k^{\mathrm{L}} a_k^{\mathrm{DS}} w\left(t - Z^{\mathrm{P}} - kT - c_k T_c - Z_k^{\mathrm{J}}\right) & \text{if } L_{\mathrm{c}} = L_{\mathrm{d}}\,; \\ Z^{\mathrm{A}} \sum_{l=0}^{\frac{L_c}{L_d}-1} \sum_{k=0}^{L_d-1} Z_{lL_d+k}^{\mathrm{L}} a_k^{\mathrm{DS}} w\left(t - Z^{\mathrm{P}} - (lL_d+k)T - c_{lL_d+k} T_c - Z_{lL_d+k}^{\mathrm{J}}\right) & \text{if } L_{\mathrm{c}} > L_{\mathrm{d}}\,; \\ Z^{\mathrm{A}} \sum_{k=0}^{\frac{L_d}{L_c}-1} \sum_{l=0}^{L_c-1} Z_{kL_c+l}^{\mathrm{L}} a_{kL_c+l}^{\mathrm{DS}} w\left(t - Z^{\mathrm{P}} - (kL_c+l)T - c_k T_c - Z_{kL_c+l}^{\mathrm{J}}\right) & \text{if } L_{\mathrm{c}} < L_{\mathrm{d}}\,. \end{cases}$$

Power Spectrum

We consider the spectrum of DS/TH with PPM/PAM in the presence of jitter and random losses. Without loss of generality we assume that the DS and the TH signatures have the same period, *i.e.*, $L_{\mathrm{c}} = L_{\mathrm{d}}$ (the other cases can be easily derived). We call L such a common period.

The filtering function is now given by

$$h(t,Z) = Z^{\mathrm{A}} \sum_{k=0}^{L-1} Z_k^{\mathrm{L}} a_k^{\mathrm{DS}} w\left(t - Z^{\mathrm{P}} - kT - c_k T_c - Z_k^{\mathrm{J}}\right),$$

where a_k^{DS} and c_k represents, respectively, the direct-sequence and time-hopping signatures, Z^{A} models random amplitudes in PAM, Z^{L} models the random losses, Z^{P} models random positions in PPM, and

Z^{J} models the jitter. Therefore

$$\mathbb{E}\left\{\widehat{h}\left(\nu,Z\right)\right\} = \widehat{w}\left(\nu\right)\mathbb{E}\left\{Z^{\mathrm{A}}\right\}\mathbb{E}\left\{Z^{\mathrm{L}}\right\}\phi_{Z^{\mathrm{P}}}\left(-2\pi\nu\right)\phi_{Z^{\mathrm{J}}}\left(-2\pi\nu\right)\sum_{k=0}^{L-1}a_k^{\mathrm{DS}}e^{-i2\pi\nu(lT+c_lT_c)}$$

$$\mathbb{E}\left\{\left|\widehat{h}\left(\nu,Z\right)\right|^2\right\} = \left|\widehat{w}\left(\nu\right)\right|^2\mathbb{E}\left\{\left|Z^{\mathrm{A}}\right|^2\right\}\left(\mathrm{L}\mathbb{E}\left\{\left|Z^{\mathrm{L}}\right|^2\right\}+\left|\mathbb{E}\left\{Z^{\mathrm{L}}\right\}\right|^2\left|\phi_{Z^{\mathrm{P}}}\left(2\pi\nu\right)\right|^2\left(\left|\sum_{l=0}^{L_c-1}a_l^{\mathrm{DS}}e^{-i2\pi\nu(lT+c_lT_c)}\right|^2-L_c\right)\right).$$

The sequence of random times, or spikes, has spectrum

$$\mathcal{S}_N\left(\nu\right)=\frac{1}{LT}\sum_{n\neq 0}\delta\left(\nu-\frac{n}{LT}\right).$$

Finally, the power spectral pseudo-density of a DS/TH with PPM/PAM in the presence of jitter and random losses (20) is given by

$$\begin{aligned}\mathcal{S}_X\left(\nu\right)=&\left|\widehat{w}\left(\nu\right)\right|^2\frac{1}{L_{\mathrm{c}}T}\\&\Bigg(\left|\mathbb{E}\left\{Z^{\mathrm{A}}\right\}\right|^2\left|\mathbb{E}\left\{Z^{\mathrm{L}}\right\}\right|^2\left|\phi_{Z^{\mathrm{P}}}\left(2\pi\nu\right)\right|^2\left|\phi_{Z^{\mathrm{J}}}\left(2\pi\nu\right)\right|^2\left|\sum_{l=0}^{L_{\mathrm{c}}-1}a_l^{\mathrm{DS}}e^{-i2\pi\nu(lT+c_lT_c)}\right|^2\sum_{n\neq 0}\delta\left(\nu-\frac{n}{L_{\mathrm{c}}T}\right)\\&+L_c\mathbb{E}\left\{\left|Z^{\mathrm{A}}\right|^2\right\}\left(\mathbb{E}\left\{\left|Z^{\mathrm{L}}\right|^2\right\}-\left|\mathbb{E}\left\{Z^{\mathrm{L}}\right\}\right|^2\left|\phi_{Z^{\mathrm{J}}}\left(2\pi\nu\right)\right|^2\right)\\&+\left|\mathbb{E}\left\{Z^{\mathrm{L}}\right\}\right|^2\left|\phi_{Z^{\mathrm{J}}}\left(2\pi\nu\right)\right|^2\left|\sum_{l=0}^{L_c-1}a_l^{\mathrm{DS}}e^{-i2\pi\nu(lT+c_lT_c)}\right|^2\left(\mathbb{E}\left\{\left|Z^{\mathrm{A}}\right|^2\right\}-\left|\mathbb{E}\left\{Z^{\mathrm{A}}\right\}\right|^2\left|\phi_{Z^{\mathrm{P}}}\left(2\pi\nu\right)\right|^2\right)\Bigg).\end{aligned}$$

We remark that the parameters characterizing the different features of the model (signature of DS and TH, type of modulations, as well as the characteristic of the jitter and the random losses) appear clearly and separately.

Spectrum of DS/TH with PPM/PAM was not available before in the literature. As already mentioned, a significant contribution is that our method allows us to obtain such formula. We emphasize the generality of the result that presents a generic DS and TH, a generic jitter and generic losses. Moreover, different pulse modulations can be taken into account by simply replacing the spectrum $\mathcal{S}_N$ and the random pulse function.

V. Conclusions

Shot-noise processes aptly model the large family of UWB signals and associated spectrum expressions provide a general formula for evaluating the spectra of UWB transmissions. The spectral formulas we have presented unify well-known results and provide spectral expressions for highly complicated UWB signals. Moreover, the corresponding approach is modular allowing us to progressively add specific features to the model in a systematic manner. In addition the exact power spectrum expression we have derived are easy to understand since the contribution of the various features of the model appears explicitly and separately. Such a feature of our approach is useful in the design and the analysis of UWB system.

Appendix

Proof: (Theorem 2.1)

Given a WSS stochastic process $\{X(t)\}_{t\in\mathbb{R}}$, we recall that its power spectral density (or pseudo density) can be defined as the function (or pseudo function) S_x such that

$$\operatorname{cov}\left(X\left(t+\tau\right),X\left(t\right)\right)=\int_{\mathbb{R}} e^{i2\pi\nu\tau} S_X\left(\nu\right)d\nu \tag{23}$$

(see for instance [20], [21]).

We now consider $\{X(t)\}_{t\in\mathbb{R}}$ to be the shot noise of equation (1). We have

$$\begin{aligned}\operatorname{cov}\left(X\left(t+\tau\right),X\left(t\right)\right)=&\mathbb{E}\left\{\sum_{n\in\mathbb{Z}} h\left(t+\tau-T_n,Z_n\right)\sum_{k\in\mathbb{Z}} h^*\left(t-T_k,Z_k\right)\right\}\\&-\mathbb{E}\left\{\sum_{n\in\mathbb{Z}} h\left(t+\tau-T_n,Z_n\right)\right\}\mathbb{E}\left\{\sum_{k\in\mathbb{Z}} h^*\left(t-T_k,Z_k\right)\right\}.\end{aligned}$$

Let Z be a random variable with distribution equal to the the common distribution of the i.i.d. random variables $\{Z_n\}_{n\in\mathbb{Z}}$, and let $\mathbb{E}_Z$ be the expectation with respect to such a distribution. Then,

$$\begin{aligned}\operatorname{cov}\left(X\left(t+\tau\right),X\left(t\right)\right)=&\mathbb{E}\left\{\sum_{n\in\mathbb{Z}}\mathbb{E}_Z\left\{h\left(t+\tau-T_n,Z\right)h^*\left(t-T_n,Z\right)\right\}\right\}\\&+\mathbb{E}\left\{\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\mathbb{E}_Z\left\{h\left(t+\tau-T_n,Z\right)\right\}\mathbb{E}_Z\left\{h^*\left(t-T_k,Z\right)\right\}\right\}\\&-\mathbb{E}\left\{\sum_{n\in\mathbb{Z}}\mathbb{E}_Z\left\{h\left(t+\tau-T_n,Z\right)\right\}\mathbb{E}_Z\left\{h^*\left(t-T_n,Z\right)\right\}\right\}\\&-\mathbb{E}\left\{\sum_{n\in\mathbb{Z}}\mathbb{E}_Z\left\{h\left(t+\tau-T_n,Z_n\right)\right\}\right\}\mathbb{E}\left\{\sum_{k\in\mathbb{Z}}\mathbb{E}_Z\left\{h^*\left(t-T_k,Z_k\right)\right\}\right\}\\=&A+B-C-D\,.\end{aligned}$$

By Campbell's theorem [14]

$$\begin{aligned}A=&\lambda\int_{\mathbb{R}}\mathbb{E}\left\{h\left(t+\tau-s,Z\right)h^*\left(t-s,Z\right)\right\}ds\\C=&\lambda\int_{\mathbb{R}}\mathbb{E}\left\{h\left(t+\tau-s,Z\right)\right\}\mathbb{E}\left\{h^*\left(t-s,Z\right)\right\}ds\,,\end{aligned}$$

and by Parseval's equality

$$\begin{aligned}A=&\lambda\int_{\mathbb{R}} e^{i2\pi\nu\tau}\mathbb{E}\left\{\widehat{h}\left(\nu,Z\right)\widehat{h}^*\left(\nu,Z\right)\right\}d\nu\\C=&\lambda\int_{\mathbb{R}} e^{i2\pi\nu\tau}\mathbb{E}\left\{\widehat{h}\left(\nu,Z\right)\right\}\mathbb{E}\left\{\widehat{h}^*\left(\nu,Z\right)\right\}d\nu\,.\end{aligned}$$

Therefore

$$A+C=\lambda\int_{\mathbb{R}} e^{i2\pi\nu\tau}\operatorname{Var}\left\{\widehat{h}\left(\nu,Z\right)\right\}d\nu\,.$$

By the definition of the Bartlett spectrum [14] (or Bartlett pseudo spectral density) S_N

$$B-D=\int_{\mathbb{R}} e^{i2\pi\nu\tau}\left|\mathbb{E}\left\{\widehat{h}\left(\nu,Z\right)\right\}\right|^2 S_N\left(\nu\right)d\nu\,.$$

Finally

$$\operatorname{cov}\left(X\left(t+\tau\right),X\left(t\right)\right)=\int_{\mathbb{R}}e^{i2\pi\nu\tau}\left|\mathbb{E}\left\{\widehat{h}\left(\nu,Z\right)\right\}\right|^{2}S_{N}\left(\nu\right)d\nu+\lambda\int_{\mathbb{R}}e^{i2\pi\nu\tau}\mathrm{Var}\left\{\widehat{h}\left(\nu,Z\right)\right\}d\nu\,,$$

and by identification with (23) we obtain

$$S_{X}\left(\nu\right)=\left|\mathbb{E}\left\{\widehat{h}\left(\nu,Z\right)\right\}\right|^{2}S_{N}\left(\nu\right)+\lambda\mathrm{Var}\left\{\widehat{h}\left(\nu,Z\right)\right\}\,,$$

i.e., equation (3).

Similarly, we obtain the proof of (4) (see [10], [12] for more details). □

Acknowledgments

The authors wish to thank M. Vetterli, P. Brémaud, W. Suwansantisuk and F. M. Kashif for their helpful suggestions and careful reading of the manuscript.

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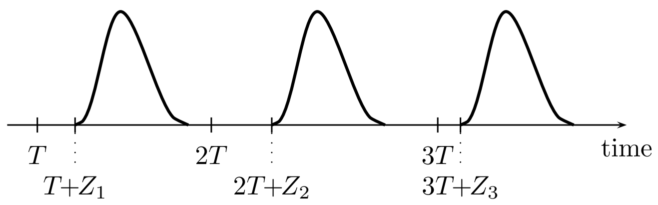

Fig. 1. Pulse position modulation (PPM), where the random parameters model the coded symbols.

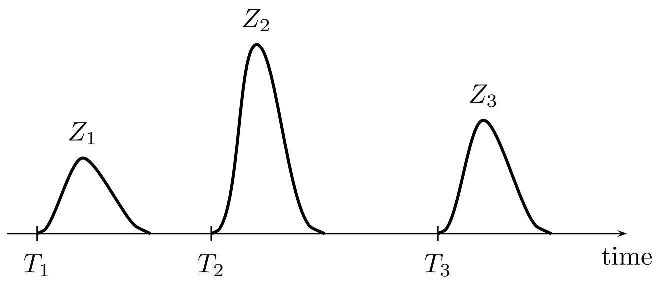

Fig. 2. Pulse amplitude modulation with i.i.d. amplitudes.

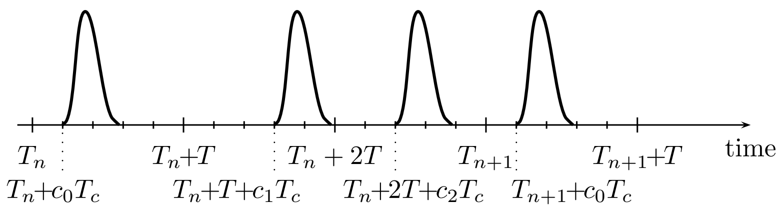

Fig. 3. Time-hopping signal with signature $\{c_0, c_1, c_2\}$, where $T_c = T/4$, $c_0 = 1$, $c_1 = 3$, and $c_2 = 3$.

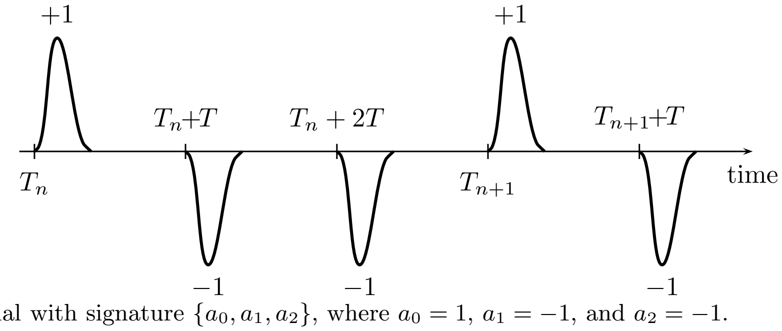

Fig. 4. Direct-sequence signal with signature $\{a_0, a_1, a_2\}$, where $a_0 = 1$, $a_1 = -1$, and $a_2 = -1$.