Rayleigh Fading Multiple Access Channel without Channel State Information

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Abstract — **In this paper we determine bounds of the capacity region of a two-user multiple-access channel with Rayleigh fading when neither the transmitters nor the receiver has channel state information (CSI). We assume that the fading coefficients as well as the additive noise are zero-mean complex Gaussian. Equivalently their amplitudes are Rayleigh-distributed and their phases are uniform. We are interested in cases in which there is an average power constraint at the channel input for both senders. Results that we get show that the lower (inner) and the upper (outer) bound of the capacity region are quite close for low and high signal-to-noise ratio (SNR). As a measure of the tightness we use the volume of the capacity region. Surprisingly, the boundary of the capacity region is achieved by time sharing among users, which is not the case for fading channels with perfect CSI at the receiver. As an additional result we derive a closed form expression for the mutual information if the input is on-off binary.**

Index Terms– **Multiple-access channel, capacity region, Rayleigh fading, channel state information, volume of the capacity region.**

I. Introduction

Wireless communication systems are currently becoming more and more important. We are witnessing a continuous improvement of existing systems and devices, offering new services almost every day. Customers are permanently increasing their demands with respect to new services, quality, reliability, stand-by times of the equipment, at the same time being critically aware of increasing electromagnetic pollution due to wireless communications. A challenging task for operators of mobile communication systems and researchers is the need to constantly improve spectral efficiency, maintain a desirable quality of service, minimize the consumption of transmit power in order to lower electromagnetic radiation and prolong the battery life. In the same time the number of base stations has to be minimized, whilst accommodating as many users as possible.

In fulfilling these requirements, the greatest obstacle is the nature of the mobile communication channel, which is time-varying, due to rapid changes in the environment and mobility of users. Signal strength may drop by several orders of magnitude due to an increase in distance between transmitter and receiver and superposition phenomena in scattering environments. This phenomenon is commonly known as fading and such channels as fading channels. Many modern wireless systems send a training sequence inserted in the data stream in order to provide the receiver with information about the channel. On the other hand, some systems provide a feedback channel from the receiver to the transmitter and this information can help the transmitter to choose an appropriate signal to access the channel. Knowledge of the channel is known as *channel state information* (CSI). Many papers have been written on channels with perfect channel state information at the receiver, at the transmitter, at both and at neither of them. Some work has also been done on imperfect CSI at the receiver and/or the transmitter.

In practical wireless communication systems, whenever there is a large number of independent scatterers and no line-of-sight path between the transmitter and the receiver, the radio link may be modelled as a Rayleigh fading channel. In a multi-user environment the uplink channel is typically modelled as multiple access channel (MAC). The performance of the channel strongly depends on the fact whether the state of the channel is available at the receiver and(or) the transmitter(s).

In this paper we are interested in the *capacity region* of the two-user Rayleigh fading channel without channel state information at the receiver or the transmitter. The capacity region of a multiple access channel is the closure of achievable rates for all users [5]. This channel is of interest since in some cases the channel can vary very quickly and it will be not possible to send any information about the channel. First we review some information theoretic results for single and multi user channels.

Consider the single user fading channel $Y = AX + Z$, with power constraint on the input, that is, $E[|X|^2] \le P$ and Z being zero-mean complex Gaussian additive noise with variance σ_Z^2 that we denote as $Z \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_Z^2)$. Goldsmith and Varaiya [7], determined the channel capacity of the fading channel, with channel state perfectly known at the receiver[1]

$$C_R(P) = E_A \left[\ln(1 + \frac{|A|^2 P}{\sigma_Z^2}) \right] \tag{1}$$

and with channel state known at both transmitter and receiver

$$C_{TR}(P) = \int_{\gamma_0}^{\infty} \ln(\frac{\gamma}{\gamma_0}) p(\gamma) d\gamma. \tag{2}$$

Here $p(\gamma)$ is the probability density function of the received SNR, $\gamma = |A|^2 P / \sigma_Z^2$, and γ_0 has to satisfy $\int_{\gamma_0}^{\infty} (1/\gamma_0 - 1/\gamma)\, p(\gamma) d\gamma = 1$. Note that if A is complex Gaussian $\mathcal{N}_{\mathbb{C}}(0, \sigma_A^2)$, then $|A|$ is Rayleigh and γ is exponentially distributed. Then, denoting $\sigma_A^2 P / \sigma_Z^2$ by ρ, in

[1] All rates are in *nats*. 1 nat equals $1/\ln(2) \approx 1.4427$ bits.

the first case, we can write a closed form solution for the capacity [6]

$$C_R(\rho) = e^{1/\rho} \cdot E_1(1/\rho), \tag{3}$$

where $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$. In the latest case the closed form solution for the capacity is given by

$$C_{TR}(\rho) = E_1(\gamma_0/\rho) \tag{4}$$

where γ_0 is the solution of $e^{-\gamma_0/\rho}/\gamma_0 - E_1(\gamma_0/\rho)/\rho = 1$.

For the Gaussian multiple access channel $Y = \sum_{i=1}^M X_i + Z$, where $Z \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_Z^2)$, and each user has input power constraint $E[|X_i|^2] \leq P_i$, the capacity region is defined by the set of positive rates $(R_1, R_2, ..., R_M)$ that satisfy

$$\sum_{i \in \mathcal{S}} R_i \leq \ln\left(1 + \frac{\sum_{i \in \mathcal{S}} P_i}{\sigma_Z^2}\right), \tag{5}$$

for all $\mathcal{S} \subseteq \{1, 2, \ldots, M\}$. For this channel time sharing among users can achieve a point on the boundary of the capacity region. In this point each user is given an amount of time proportional to its input power constraint $P_i / \sum_{i=1}^M P_i$. The maximum sum-rate is $\ln\left(1 + \sum_{i=1}^M P_i/\sigma_Z^2\right)$.

For the multi-access fading channel $Y = \sum_{i=1}^M A_i X_i + Z$, with additive Gaussian noise $Z \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_Z^2)$, if the channel state is perfectly known at the receiver, the capacity region is already convex and it is the set of rates $(R_1, R_2, ..., R_M)$ that satisfy

$$\sum_{i \in \mathcal{S}} R_i \leq E\left[\ln\left(1 + \frac{\sum_{i \in \mathcal{S}} |A_i|^2 P_i}{\sigma_Z^2}\right)\right], \tag{6}$$

for all $\mathcal{S} \subseteq \{1, 2, \ldots, M\}$. The expected value is taken with respect to $\{A_i\}_{i \in \mathcal{S}}$. Here, the maximum sum-rate achieved by time sharing is considerably smaller than the maximum sum-rate in the capacity region, which is not the case with the channel without CSI.

The multi-access fading channel $Y = \sum_{i=1}^M \sqrt{H_i} X_i + Z$, with additive white Gaussian noise $Z \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_Z^2)$ and an average power constraint to every user $E[|X_i|^2] \leq P_i$, with perfect channel state information at all transmitters and the receiver, is studied in [12]. The solution may be seen as generalization of the water-filling construction for single-user to multi-user channels.

The case without channel state information for the single user channel has been studied in [11, 9, 3, 8] and for the multi-user channel in [10, 2]. In [3], authors show that without channel state information, the optimal input is discrete with a mass point at zero. In [11] it is shown that without channel state information, the capacity at high SNR depends double-logarithmically on the SNR. A more general result on the double logarithmic behavior at high SNR is given in [8].

In Section II we analyze the single user case with no channel state information, where we review some results obtained in [3]. We establish a closed form solution for the mutual information when the input is binary on-off, in Section III. In Section IV we find lower and upper bounds of the capacity region of a two-user Rayleigh fading channel. We compare the bounds in Section V. Finally we generalize the results for $M-$users in Section VI and we give conclusions in Section VII.

II. Rayleigh Fading Channels without CSI

In this section we explain the method of computing the capacity $C_{su}(\rho)$ of the single user channel $Y' = AX' + Z$, where $A, Z \sim \mathcal{N}_{\mathbb{C}}(0,1)$ and $E[|X|^2] \leq \rho$. Having $\rho = \sigma_A^2 P/\sigma_Z^2$, the capacity of this channel is the same as the capacity of the channel with $A \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_A^2)$, $Z \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_Z^2)$ and $E[|X|^2] \leq P$. According to [3], the capacity achieving input distribution is discrete $X' \in \{0, x_1, \ldots, x_N\}$ with $\Pr\{X' = x_i\} = p_i, \forall i = 0, 1, \ldots, N$, and the constraint is $\sum_{i=1}^N p_i |x_i|^2 \leq \rho$, with $\sum_{i=0}^N p_i = 1$ and $p_0 > 0$. A proof of this can be found in [3]. The number N of input levels with non-zero probability, depends on the SNR and it increases with the SNR. For low SNR, the mutual information for binary inputs is not far from the capacity, [3]. For extremely high SNR, higher than the fading number, defined in [8], the capacity behaves as $\log(\log(\text{SNR}))$. Conditioned on the input, the sufficient statistic of the output is $|Y'|^2$ and it has a central chi-square distribution with two degrees of freedom. It is clear that the influence of X' on the channel is only through its magnitude. Therefore, we introduce $X = |X'| > 0$. Letting $Y = |Y'|^2$, we obtain an equivalent channel with real non-negative input X, non-negative output Y, transition probability

$$p_{Y|X}(y|x) = (1 + x^2)^{-1} \exp\{-y/(1 + x^2)\} \tag{7}$$

and an average power constraint $E[X^2] \leq \rho$. Since x appears only via its square it is convenient to make an invertible change of variables $S = 1/(1 + X^2)$, so that

$$p_{Y|s}(y|s) = se^{-sy}, \quad s \in (0, 1], \quad y \geq 0, \tag{8}$$

with the constraint $E[1/S - 1] \leq \rho$. Furthermore, since Y is sufficient statistics for $|Y'|$, and S is sufficient statistics for X', then $I(X'; Y') = I(S; Y)$. Since the optimal input probability is discrete, the result obtained for S is the same as that for X' and the optimal levels are $s_i^* = 1/(1 + (x_i^*)^2)$. Note that since the phase doesn't matter, we may choose $X' = X$. The mutual information $I(S;Y) = E_S[D(p_{Y|S} || p_Y)]$[2] is

$$\begin{aligned} I(S;Y) &= \sum_i p(s_i) \int_0^\infty s_i e^{-s_i y} \ln \frac{s_i e^{-s_i y}}{\sum_j p(s_j) s_j e^{-s_j y}} dy \\ &\overset{(a)}{=} \sum_i p(s_i) \int_1^0 \ln\left(\frac{1}{s_i} \sum_j p(s_j) s_j t^{s_j/s_i - 1}\right) dt \\ &= \sum_i p(s_i) \ln E_S[S \cdot t^{S/s_i - 1}] - \sum_i p(s_i) \ln s_i \end{aligned}$$

where $s_i = (1 + x_i)^{-1}$, $i = 1, 2, \ldots, N$. We get (a) by the change $t = e^{-s_i y}$. The capacity is $C_{su}(\rho) = \max_{\mathbf{p}} \sum_i p(s_i) \left(\ln E_S[S \cdot t^{S/s_i - 1}] - \ln s_i\right)$ with $E[1/S - 1] = \rho$. This is an optimization problem with respect to all p_i that have to satisfy $\sum_{i \geq 0} p_i = 1$ and $\sum_{i > 0} p_i (1/s_i - 1) = \rho$, since $s_0 = 1$. An additional problem is to find how many points are with non-zero probabilities. This number depends on the SNR and denoting it by $N(\rho)$ and the maximizing input distribution by $\mathbf{p}^* = [p_0^*, p_1^*, ..., p_{N(\rho)}^*]$, knowing that there is

[2] $D(p||q) = \int p(x) \ln(p(x)/q(x)) dx$ is the Kullback-Leibler distance

a mass point at zero ($s_0 = 1$), the optimal levels are denoted by $\mathbf{s}^* = [1, s_1^*, ..., s_{N(\rho)}^*]$. Thus, the capacity is $C_{su}(\rho) = \sum_{i=0}^{N(\rho)} p_i^* \int_1^0 \ln\left(\frac{1}{s_i^*}\sum_j p_j^* s_j^* t^{s_j^*/s_i^* - 1}\right) dt$ and can be calculated numerically. Note that $N(\rho) \geq 1$, it increases in ρ and it is 1, for small ρ, [3].

III. Channels with Binary On-Off Input

In this section we give a closed form expression for the mutual information between the input and the output, if the input is on-off binary, namely $(0, b)$. The binary on-off input is interesting since for low SNR, it is optimal. At the end of the section we give a numerical result for the capacity.

Proposition 1 (Closed form expression for the mutual information for a particular on-off input probability p and SNR ρ): For the channel $Y = AX + Z$, when the input is binary on-off with $\Pr\{X = 0\} = 1 - p$ and power constraint $E[|X|^2] \leq \rho$, the mutual information between the input X and the output Y, is

$$\begin{aligned} I_{p,\rho}(X;Y) &= \mathrm{h}(p) + p\mathcal{J}\left(\frac{p+\rho}{p^2(1-p)^{-1}}, \frac{\rho}{p}\right) \\ &+ (1-p)\mathcal{J}\left(\frac{p^2(1-p)^{-1}}{p+\rho}, \frac{\rho}{p+\rho}\right) \end{aligned}$$

where $\mathrm{h}(\cdot)$ is the binary entropy function, $\mathcal{J}(c,d) = -\ln(1+c) + \frac{cd}{1+d} \cdot {}_2F_1(1, 1+d^{-1}; 2+d^{-1}; -c)$, and ${}_2F_1(u, v; w; z) = \frac{\Gamma(w)}{\Gamma(u)\Gamma(v)} \sum_{k=0}^{\infty} \frac{\Gamma(u+k)\Gamma(v+k)}{\Gamma(w+k)} \cdot \frac{z^k}{k!}$ is the Gaussian hypergeometric function defined in [4]. $\Gamma(q) = \int_0^\infty x^{q-1}e^{-x}dx$ is the Euler gamma function.

The proof is given in the Appendix.

To compute the capacity of the binary input Rayleigh fading channel without channel state information, denoted by C_b, one has to find the maximum of $I_{p,\rho}(X;Y)$ over p for different ρ. Unfortunately $dI_{p,\rho}(X;Y)/dp = 0$ is a transcendental equation and cannot be solved explicitly. The capacity and the optimizing p^* as functions of ρ are shown in Fig. 1. Note that as the power of the input signal increases the information rate of this channel goes to its limit of $\ln 2$ nats and p^* goes to 0.5. This happens since if ρ goes to ∞, we get the channel $Y = AX$.

Next we look at the mutual information $I_{0.5,\rho}(X;Y)$ achieved using a binary input with equiprobable levels ($p = 0.5$). This uniform binary on-off input is of practical interest since the corresponding rate can be achieved using linear binary codes, for example LDPC. $I_{0.5,\rho}(X;Y)$ is compared to the capacity C_b, in Fig. 2. The capacities of the binary input additive white Gaussian noise channel (BIAWGNC), Rayleigh fading channel with CSI at the receiver and additive white Gaussian channel (AWGN) with general input, are shown in the same figure. Naturally, higher rates are achieved with the same SNR for the non-fading case. For example, for $\rho = 20$ dB, $I_{0.5,\rho}(X;Y)$ approaches 92% of the capacity of the binary input Gaussian channel.

In the same figure we see that as SNR increases the mutual information for $p = 0.5$ approaches the capacity. This is not surprising since $p = 0.5$ is optimal for $SNR \to \infty$. It can be also seen that for $p < 0.5$, $I_{p,\rho}(X;Y) >$

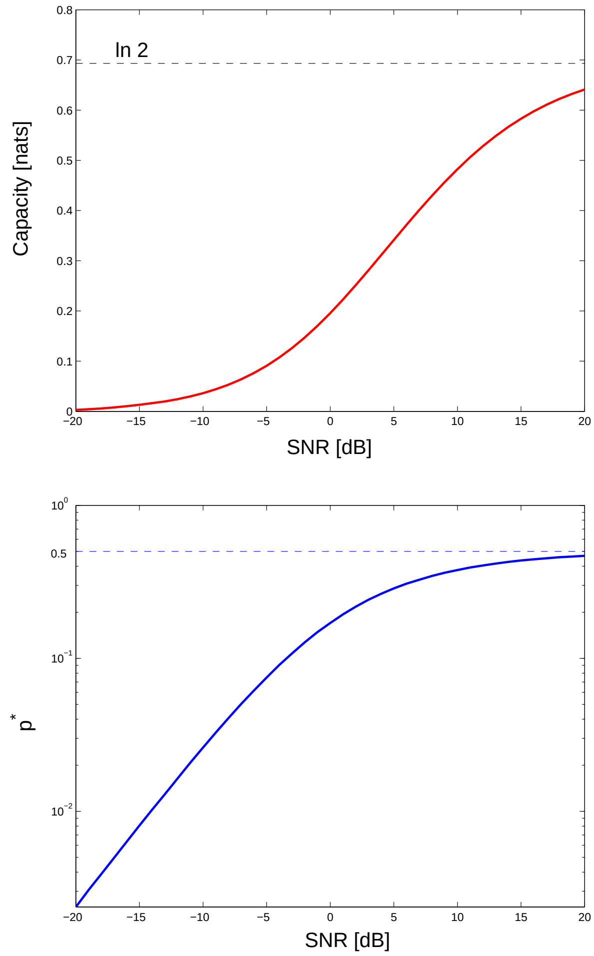

Fig. 1: Capacity C_b and the optimal probability p^* as function of ρ

$I_{0.5,\rho}(X;Y)$ for all ρ smaller than some value. Beyond that value of ρ, $I_{0.5,\rho}(X;Y)$ is dominant. On the other hand, for all $p > 0.5$ the mutual information $I_{0.5,\rho}(X;Y)$ is smaller than $I_{0.5,\rho}(X;Y)$, for any ρ. It is normal, since the maximizing input probability p^* is always smaller than 0.5.

Example 1 For $\rho = 7$ dB, $I_{0.5,\rho}(X;Y) \approx 0.9C_b$, for $\rho = 10$ dB, $I_{0.5,\rho}(X;Y) \approx 0.95C_b$, for $\rho = 16$ dB, $I_{0.5,\rho}(X;Y) \approx 0.99C_b$ and for $\rho = 20$ dB, $I_{0.5,\rho}(X;Y) \approx 0.997C_b$.

IV. Two-User Rayleigh Fading Channel

In this section, we give a lower and an upper bound of the capacity region of a two-user Rayleigh fading channel in the case where the channel is not known either at the transmitters or at the receiver, but all of them know the statistics of the channel exactly. The ratio of the volumes of the lower and the upper bound of the capacity region will serve us as a measure for the proximity of these bounds. More about the volume can be found [1]. Numerical results show that our proposed lower bound is always within 92.6% of the upper bound. The channel is

$$\tilde{Y} = \tilde{A}_1\tilde{X}_1 + \tilde{A}_2\tilde{X}_2 + \tilde{Z}, \quad (9)$$

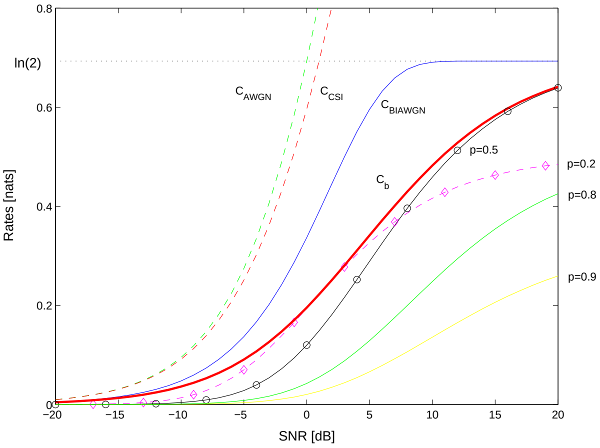

Fig. 2: Comparison of the mutual information for different p's to the capacities C_b, C_{BIAWGN}, C_{CSI} and C_{AWGN}.

where $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{Z}$ are independent and identically distributed (i.i.d.), zero-mean, complex Gaussian random variables of variances $\sigma^2_{A_1}, \sigma^2_{A_2}$ and σ^2_Z respectively. There are input constraints, $E[|\tilde{X}_1|]^2 \leq P_1$ and $E[|\tilde{X}_2|]^2 \leq P_2$. No channel information is provided to the transmitters and the receiver. To find "good" bounds of the capacity region of this channel, we use the results for the single user memoryless Rayleigh fading channel, studied in [3]. Channel (9) has the same capacity as the following channel

$$\frac{\tilde{Y}}{\sigma_Z} = \frac{\tilde{A}_1}{\sigma_{A_1}}\frac{\sigma_{A_1}}{\sigma_Z}\tilde{X}_1 + \frac{\tilde{A}_2}{\sigma_{A_2}}\frac{\sigma_{A_2}}{\sigma_Z}\tilde{X}_2 + \frac{\tilde{Z}}{\sigma_Z}. \tag{10}$$

Letting $Y = \tilde{Y}/\sigma_Z$, $X_1 = \sigma_{A_1}\tilde{X}_1/\sigma_Z$ and $X_2 = \sigma_{A_2}\tilde{X}_2/\sigma_Z$, we get

$$Y = A_1X_1 + A_2X_2 + Z, \tag{11}$$

with $A_1 = \tilde{A}_1/\sigma_{A_1}$, $A_2 = \tilde{A}_2/\sigma_{A_2}$ and $Z = \tilde{Z}/\sigma_Z$ all being $\mathcal{N}_{\mathbb{C}}(0,1)$. Power constraints on the new inputs become $E[|X_1|]^2 \leq \sigma^2_{A_1}P_1/\sigma^2_Z = \rho_1$ and $E[|X_2|]^2 \leq \sigma^2_{A_2}P_2/\sigma^2_Z = \rho_2$. It is clear that doing these transforms, all mutual information in the new channel remain the same. Thus, the capacity region of the channel (9) is the same as the capacity region of the channel (11). For a particular input distribution, the region of achievable rates for the channel (11) is $\mathcal{R}(p_{X_1}, p_{X_2}) = \{(R_1, R_2) \in \mathbb{R}^2_+ : R_1 \leq I(Y; X_1|X_2); R_2 \leq I(Y; X_2|X_1); R_1 + R_2 \leq I(Y; X_1, X_2)\}$. The capacity region is a closure of the convex hull of the union over all possible product input distributions $p_{X_1}(x)p_{X_2}(x)$ of all such regions $\mathcal{R}(p_{X_1}, p_{X_2})$. To compute the maximum mutual information in the capacity region for user 1 and user 2 separately, we need to analyze the single user fading channel, similarly as it is done in [3]. Given $X_2 = x_2$, the equivalent channel is $Y = A_1X_1 + (A_2x_2 + Z)$. This channel is the same as the single user fading channel $Y = A_1X_1 + Z$, with larger variance of the additive noise, that is, $1 + |x_2|^2$. Thus, it behaves as the channel $Y = A_1X_1 + Z$, with different SNR constraint, that is, $\rho' = \rho/(1 + |x|^2)$. It is shown in [3] that the capacity achieving input distribution for this channel has to be discrete with a mass point at the origin. Moreover, it is shown in the same paper that for low SNR, the maximizing input distribution is binary. Thus, the rate of user 1 is bounded by

$$\begin{aligned} R_1 &\leq \sum_{x_2} p_{X_2}(x_2) I(X_1; Y|X_2 = x_2) \\ &\leq \sum_{x \in \mathcal{X}_2} p_{X_2}(x) C_{su}\left(\frac{\rho_1}{1 + |x|^2}\right) \\ &\leq \sum_{x \in \mathcal{X}_2} p_{X_2}(x) C_{su}(\rho_1) = C_{su}(\rho_1), \end{aligned} \tag{12}$$

where the last inequality is achieved with equality if $p_{X_2}(0) = 1$, i.e. if user 2 is silent. By $C_{su}(\rho)$ we denote the capacity of the single user fading channel with no channel state information, for a particular SNR$= \rho$. Thence, the point $C_{su}(\rho_1)$ is achievable and it is the highest rate that can be achieved by user 1, using the channel while user 2 is silent. That is one point on the boundary of the capacity region, namely the extreme point on the R_1-axis. From symmetry, the same is true for user 2, i.e. the extreme point on the R_2-axis is $C_{su}(\rho_2)$.

After finding both extreme points, let us find the maximum sum rate. It is shown in [10] that if the propagation coefficients take on new independent values for every symbol (i.i.d.), then the total throughput capacity for any number of users larger than 1, is equal to the capacity if there is only one user. Hence, time division multiple access (TDMA) is an optimal scheme for multiple users. In that case the sum rate is given by $\Theta = aC_{su}(\rho_1/a) + (1-a)C_{su}(\rho_2/(1-a)) \leq C_{su}(\rho_1 + \rho_2)$, with $a \in [0, 1]$. Note that the maximum throughput cannot be larger than $C_{su}(\rho_1 + \rho_2)$, the capacity which is achieved if both users fully cooperate, and is equivalent to the single user capacity for SNR$= \rho_1 + \rho_2$. The latest is achieved with equality for $a = \rho_1/(\rho_1 + \rho_2)$. This is an upper bound of the capacity region, namely the pentagon $\{(R_1, R_2) \in \mathbb{R}^2_+ : R_1 \leq C_{su}(\rho_1), R_2 \leq C_{su}(\rho_2), R_1 + R_2 \leq C_{su}(\rho_1 + \rho_2)$ (Fig. 3). A straightforward lower bound

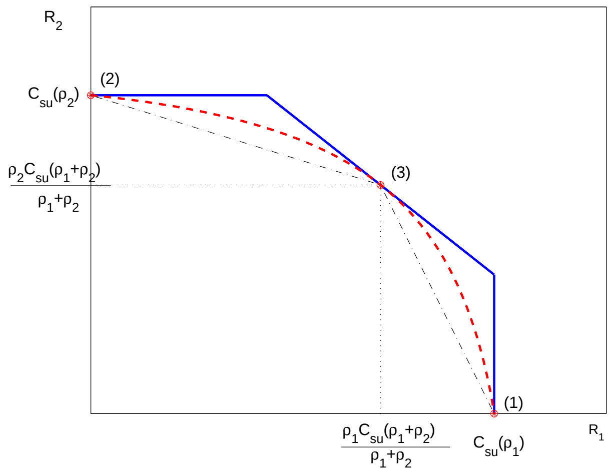

Fig. 3: Lower and upper bounds of the capacity region, for given ρ_1 and ρ_2.

is the region obtained by connecting the points that are achievable (dash-dot line in Fig. 3). Better lower bound is the time-sharing region (dashed line in Fig. 3). It is obtained by allowing user 1 to use the channel aT seconds and user 2, $(1-a)T$ seconds. Because of the average power constraint the power used during the active period is normalized. Hence, the proposed lower bound of the capacity region parameterized by $a \in [0,1]$ is given by

$$\begin{aligned} R_1(a) &= a \cdot C_{su}\left(\rho_1/a\right) \\ R_2(a) &= (1-a) \cdot C_{su}\left(\rho_2/(1-a)\right). \end{aligned} \tag{13}$$

The capacity region touches the upper bound in the following three points $(C_{su}(\rho_1), 0)$, $(0, C_{su}(\rho_2))$ and $(\frac{\rho_1 C_{su}(\rho_1+\rho_2)}{\rho_1+\rho_2}, \frac{\rho_2 C_{su}(\rho_1+\rho_2)}{\rho_1+\rho_2})$. Note that a trivial upper bound that is much looser is the capacity region of the same channel, with perfect channel state information at the receiver.

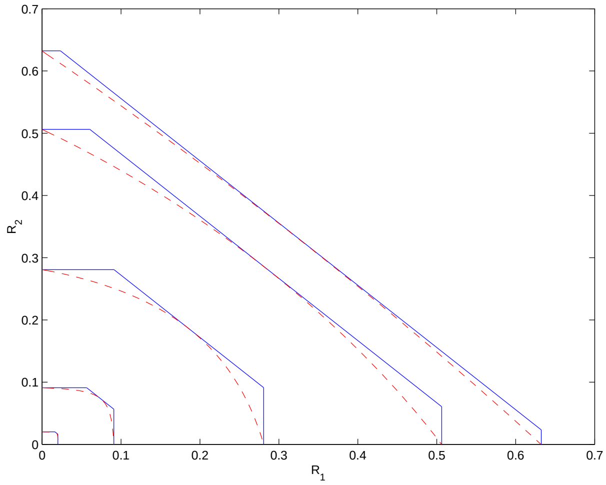

Fig. 4: Lower and upper bounds of the capacity region for low and high SNRs.

V. Comparison

In Fig. 4, we see the proposed lower and upper bounds of the capacity region for different SNRs. It can be seen that for low and high SNR they are closer then for the some medium SNR. How to measure the "tightness" of the bounds? We propose comparing the *volumes* of the corresponding regions. For the two user case the volume is $\mathcal{V}_2 = \int_{R_1} R_2 dR_1$. It is easy to compute the volume of the upper bound in terms of the single-user capacities. For $\rho_1 = \rho_2 = \rho$,

$$\mathcal{V}_{UB}(\rho) = \frac{1}{2} \cdot \left[C_{su}(2\rho)\right]^2 - \left[C_{su}(2\rho) - C_{su}(\rho)\right]^2. \tag{14}$$

The volume of the lower bound is $\mathcal{V}_{LB}(\rho) = \int_0^{C_{su}(\rho)} R_2 dR_1 = \int_0^1 R_2(a)\dot{R}_1(a)da$, where $R_1(a)$ and $R_2(a)$ are given by (13), and $\dot{R}_1(a)$ is the first derivative of R_1 with respect to a. It is determined numerically. The ratio $\mathcal{V}_{LB}/\mathcal{V}_{UB}$ as a function of ρ is shown in Fig. 5. It can be seen that for low SNR the lower and the upper bound are very close and as the SNR increases they diverge up to some SNR (~ 3 dB), where the lower and the upper bound are at maximum "distance". In this case $\mathcal{V}_{LB} \simeq 0.925\mathcal{V}_{UB}$. As SNR increases above 3 dB, the bounds approach again, i.e. the ratio of $\mathcal{V}_{LB}$ and $\mathcal{V}_{UB}$ increases and tends to 1. The results can be easily extended for an $M-$user case.

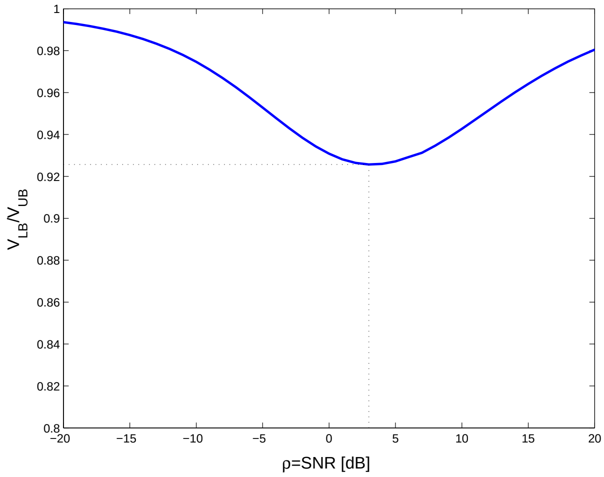

Fig. 5: Ratio $\mathcal{V}_{LB}/\mathcal{V}_{UB}$, as function of ρ.

VI. $M-$User Case

We can easily generalize the bounds for the $M-$user case. The upper bound of the capacity region is given by the following $2^M - 1$ hyperplanes

$$\sum_{i\in\mathcal{S}} R_= C_{su}\left(\sum_{i\in\mathcal{S}} \rho_i\right)$$

for all non-empty subsets $\mathcal{S}$ of $\{1,2,\ldots,M\} \equiv [M]$. Note that the maximal sum-rate is given by

$$\sum_{i=1}^{M} R_i = C_{su}\left(\sum_{i=1}^{M} \rho_i\right).$$

The inner bound of the capacity region (time sharing region) is given by the following M equations

$$R_i = a_i \cdot C_{su}\left(\rho_i/a_i\right), i = 1,2,\ldots,M,$$

where $a_i \in [0,1]$ and $\sum_{i=1}^{M} a_i = 1$. It is not difficult to see that there are $2^M - 1$ points in which these two bounds touch. For any non-empty set $\mathcal{S} \subseteq [M]$, the coordinates of all these points are given by

$$R_i^{\{\mathcal{S}\}} = \frac{\rho_i}{\sum_{j\in\mathcal{S}} \rho_j} \cdot C_{su}\left(\sum_{j\in\mathcal{S}} \rho_j\right),$$

for all $i \in \mathcal{S}$ and $\mathcal{S} \subseteq \{1,2,...,M\}$. Note that the maximal sum-rate is also achievable by time division. In that case $a_i = \rho_i/(\sum_{j=1}^{M}\rho_j)$, for $i = 1,...,M$, and the rate tuple on the boundary of the capacity region has coordinates

$$R_i^{\{[M]\}} = \frac{\rho_i}{\sum_{j=1}^{M} \rho_j} \cdot C_{su}\left(\sum_{j=1}^{M} \rho_j\right).$$

VII. Conclusions

The single user Rayleigh fading channel with no side information has attracted some attention since it is useful for modelling different wireless channels. In this paper we get some insight for the multiple access Rayleigh fading channel with no CSI. We give bounds of the capacity region of the two-user Rayleigh multiple access channel. We see that the sum rate is maximized by time-sharing among users and in that case we achieve the boundary of the capacity region by giving to each user an amount of time that is proportional to its input average power constraint multiplied by the variance of the fading. This is not the case with multiple access channels with perfect CSI at the receiver. We also see that the inner bound is always within 92.6 % (in terms of the volume of the capacity region) of the outer bound. As an open problem for future research we leave the improvement of the inner and the outer bound.

Appendix
Proof of Proposition 1

In the case when the input is binary, since there is a power constraint and from the fact that one input level has to be zero, the other level is $b = \sqrt{\rho/p}$, assuming that the probability of zero is $1-p$. According to the analysis in Section II, $s_0 = 1$ with probability $1-p$ and $s_1 = (1+\rho/p)^{-1}$ with probability p. Therefore, since $p_{Y|S}(y|s) = se^{-sy}$ and $p_Y(y) = (1-p)e^{-y} + ps_1e^{-s_1 y}$, the mutual information $I(S;Y)$ is

$$\begin{aligned}
(1-p) &\quad D(p_{Y|1}\|p_Y) + pD(p_{Y|s_1}\|p_Y) \\
= &\quad (1-p)\int_0^\infty e^{-y}\ln\frac{e^{-y}}{p_Y(y)}dy \\
+ &\quad p\int_0^\infty s_1e^{-s_1 y}\ln\frac{s_1e^{-s_1y}}{p_Y(y)}dy \\
= &\quad (1-p)\cdot\int_1^0 \ln[(1-p)+ps_1t^{1-s_1}]dt \\
+ &\quad p\cdot\int_1^0 \ln[(1-p)t^{1/s_1-1}/s_1+p]dt \\
= &\quad \mathrm{h}(p) + (1-p)\mathcal{J}(\frac{ps_1}{1-p},\frac{\rho}{p+\rho}) \\
+ &\quad p\cdot\mathcal{J}(\frac{1-p}{ps_1},\frac{\rho}{p}) \qquad (15)
\end{aligned}$$

To show $\mathcal{J}(c,d)$ we start with

$$\begin{aligned}
\Phi(t,c,d) &= \int \ln(1+ct^d)dt \\
&= -\int\sum_{k=1}^\infty \frac{(-ct^d)^k}{k}dt \\
&\overset{(a)}{=} -\sum_{k=1}^\infty \frac{(-c)^k}{k}\int t^{dk}dt \\
&= -\sum_{k=1}^\infty \frac{(-c)^k t^{dk+1}}{k(dk+1)} \\
&= -\sum_{k=1}^\infty (-c)^k t^{dk+1}\left(\frac{1}{k}-\frac{1}{k+1/d}\right) \\
&= -t\sum_{k=1}^\infty \frac{(-ct^d)^k}{k} + t\sum_{k=1}^\infty \frac{(-ct^d)^k}{k+1/d} \\
&\overset{(b)}{=} t\ln(1+ct^d) \\
&+ \frac{t(-ct^d)}{1+1/d}\cdot {}_2F_1(1,1+\frac{1}{d};2+\frac{1}{d};-ct^d) \\
&= t\ln(1+ct^d) \\
&- \frac{cdt^{1+d}}{1+d}\cdot {}_2F_1(1,1+\frac{1}{d};2+\frac{1}{d};-ct^d),
\end{aligned}$$

where (a) follows from the convergence of the sum and (b) from the definition of the hypergeometric function, defined in [4], that is ${}_2F_1(1,1+x;2+x;z)$ is equal to

$$\begin{aligned}
&\frac{\Gamma(x+2)}{\Gamma(1)\Gamma(x+1)}\cdot\sum_{k=0}^\infty \frac{\Gamma(k+1)\Gamma(1+x+k)}{\Gamma(2+x+k)}\frac{z^k}{k!} \\
= &\frac{(x+1)}{z}\cdot\sum_{k=0}^\infty \frac{z^{k+1}}{x+k+1} \\
= &\frac{(x+1)}{z}\cdot\sum_{k=0}^\infty \frac{z^{k+1}}{x+k+1} \\
= &\frac{(x+1)}{z}\cdot\sum_{k=1}^\infty \frac{z^k}{x+k}.
\end{aligned}$$

We also use the fact that $\Gamma(k+1) = k!, \forall k \in \mathbb{N}$ and $\Gamma(x+2) = (x+1)\Gamma(x+1), \forall x \in \mathbb{R}$. Finally it is easy to see that

$$\begin{aligned}
\mathcal{J}(c,d) &= \int_1^0 \ln(1+ct^d)dt \\
&= \Phi(0,c,d) - \Phi(1,c,d) = -\Phi(1,c,d) \\
&= -\ln(1+c) \\
&+ \frac{cd}{1+d}\cdot {}_2F_1(1,1+d^{-1};2+d^{-1};-c).
\end{aligned}$$

□

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