High Order Balanced MultiWavelets: Theory, Factorization and Design

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Abstract

This paper deals with multiwavelets and the different properties of approximation and smoothness that are associated with them. In particular, we focus on the important issue of the preservation of discrete time polynomial signals by multiwavelet based filter banks. We give here a precise definition of *balancing* for higher degree discrete time polynomial signals and link it to a very natural factorization of the lowpass refinement mask that is the counterpart of the well-known *zeros at π* condition on the scaling function in the usual wavelet framework. This property of balancing proves then to be central to the issues of the preservation of smooth signals by the filter bank, the approximation power of the multiresolution analysis and the smoothness of the scaling functions and wavelets. Using these new results, we are able to construct a family of orthogonal multiwavelets with symmetries and compact support that is indexed by the order of balancing. We also give the minimum length orthogonal multiwavelets for any balancing order.

Keywords

Multiwavelet, time-varying filter bank, high order balancing, factorization, design.

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I. INTRODUCTION

Wavelet constructions from iterated filter banks, as pioneered by Daubechies [4], have become a standard way to derive orthogonal and biorthogonal wavelet bases. The underlying filter banks are well studied, and thus, the design procedure is well understood. By the structure of the problem, certain issues are ruled out: the impossibility of constructing orthogonal FIR linear phase filter banks implies that there is no orthogonal wavelet with compact support and symmetry. Nevertheless, by relaxing the requirement of time invariance, it is easy to see that new solutions are possible. As mentioned in [31], such filter banks are closely related to some matrix 2-scale equations leading to multiwavelets.

In the usual framework of wavelets, the two important issues of the reproduction of continuous-time polynomials in the associated multiresolution analysis (approximation theory issue) and the preservation/cancellation of discrete time polynomial signals by the associated filter bank (subband coding and compression issue) are tightly connected since they have been proved to be equivalent to the same condition, namely the number of zeros at π in the factorization of the lowpass filter $m_0(z)$ of the filter bank. Namely, in the orthogonal case, the lowpass filter $m_0(z)$ is said to be of *regularity* N iff one of the following equivalent conditions [4] holds:

- The lowpass filter $m_0(z)$ has a zero of order N at π .
- The corresponding highpass filter $m_1(z)$ has a zero of order N at 0.
- Discrete time polynomial sequences of degree $n < N$ are perfectly represented by shifts of the scaling sequence $m_0[k]$ (i.e. the discrete time polynomial signals are preserved by the lowpass branch of the filter bank).
- The associated wavelet function $\psi(t)$ has N vanishing moments.
- Continuous time polynomials of degree $n < N$ are perfectly represented by shifts of the scaling function $\phi(t)$ (i.e. the associated multiresolution analysis has approximation power N).

Also, the smoothness of the scaling function $\phi(t)$ (and of the wavelet $\psi(t)$ if the filters are FIR) is governed in a certain sense [7] by the regularity of the lowpass filter. Similar relations also hold for the biorthogonal case. However, we will not be concerned here with that case since the interest of biorthogonality in the multiwavelet framework is not obvious (one shall remember that here there is no theoretical result [6] preventing us from constructing orthogonal, FIR, linear phase filters).

The regularity issue is however different for multiwavelets. In [16], [17], interested in the subband coding issue in general and the problem of processing one dimensional signals with multiwavelets in particular, we showed that the approximation power property was not assuring preservation of discrete time polynomial signals by the lowpass branch of the filter bank. Conse-

quently, we introduced the concept of balanced multiwavelets which was also further investigated in several other papers [14], [21], [24]. One of the goals of this concept was to avoid the artificial step of prefiltering [34], [16] in multiwavelet based systems. Here, based on results from [3], [19], [20], we will clarify the links between balancing order and approximation power and so prove that the notion of balancing order is in fact central to the whole issue of *regularity* for multiwavelets.

Furthermore, the introduction of the balancing order property will enable us to construct a family of orthogonal compactly supported multiwavelets with symmetries that is naturally indexed by the balancing order.

The outline of the paper is as follows. In Section II, the fundamentals of multiwavelet theory are reviewed with a special highlight on the connection to time-varying filter banks. Section III reviews balancing and describes equivalence results between balancing and factorization of the refinement mask. These results are the key in the construction of balanced multiwavelets families. Section IV relates balancing order, approximation power and smoothness. This leads in Section V to the construction of a balanced and smooth family of orthonormal multiwavelets with symmetries and also a surprising result on the shortest length orthonormal multiwavelets of balancing order p which happen to be the Daubechies wavelets.

II. MULTIWAVELETS

Generalizing the wavelet case, one can allow a multiresolution analysis $\{V_n\}_{n \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ to be generated by a finite number of scaling functions $\phi_0(t), \phi_1(t), \dots, \phi_{r-1}(t)$ and their integer translates (the multiresolution analysis is then said to be of *multiplicity* r). Then, the multiscaling function $\boldsymbol{\phi}(t) := [\phi_0(t), \dots, \phi_{r-1}(t)]^\top$ verifies a two-scale equation

$$\boldsymbol{\phi}(t) = \sum_k \mathbf{M}[k] \boldsymbol{\phi}(2t - k) \quad (1)$$

where now $\{\mathbf{M}[k]\}_k$ is a sequence of $r \times r$ matrices of real coefficients. The multiresolution analysis structure gives $V_1 = V_0 \oplus W_0$ where W_0 is the orthogonal complement of V_0 in V_1 . We can construct an orthonormal basis of W_0 generated by $\psi_0(t), \psi_1(t), \dots, \psi_{r-1}(t)$ and their integer translates with $\boldsymbol{\psi}(t) := [\psi_0(t), \dots, \psi_{r-1}(t)]^\top$ derived by

$$\boldsymbol{\psi}(t) := \sum_k \mathbf{N}[k] \boldsymbol{\phi}(2t - k) \quad (2)$$

where $\{\mathbf{N}[k]\}_k$ is a sequence of $r \times r$ matrices of real coefficients obtained by *completion* of $\{\mathbf{M}[k]\}_k$ (a detailed exposition of the completion scheme is given in [15]). Introducing in the z -domain the refinement masks $\mathbf{M}(z) := \frac{1}{2} \sum_n \mathbf{M}[n] z^{-n}$ and $\mathbf{N}(z) := \frac{1}{2} \sum_n \mathbf{N}[n] z^{-n}$, the equations (1) and (2) translate in Fourier domain into

$$\Phi(2\omega) = \mathbf{M}(e^{j\omega}) \Phi(\omega) \quad \text{and} \quad \Psi(2\omega) = \mathbf{N}(e^{j\omega}) \Phi(\omega) \quad (3)$$

We can then derive the behavior of the multiscaling function by iterating the first product above. If this iterated matrix product converges, we get in the limit

$$\Phi(\omega) = \mathbf{M}_\infty(\omega)\Phi(0) = \prod_{i=1}^{\infty} \mathbf{M}(e^{j\omega/2^i}) \Phi(0) \quad (4)$$

In the sequel, we will assume that the sequences $\{\mathbf{M}[k]\}_k$ and $\{\mathbf{N}[k]\}_k$ are finite and thus that $\phi(t)$ and $\psi(t)$ have compact support [22]. Accordingly to the results given in [3], [27], [23], [13], we will also assume that $\mathbf{M}(z)$ verifies the two *basic conditions*:

Condition E (Existence): $\mathbf{M}(1)$ has spectral radius $\rho(\mathbf{M}(1)) \leq 1$, 1 is the only eigenvalue of $\mathbf{M}(1)$ on the unit circle and moreover 1 is a simple eigenvalue.

Condition A1 (Approximation of order 1): There exists $\mathbf{r}_0$ left eigenvector of $\mathbf{M}(1)$ for the eigenvalue 1 such that $\mathbf{r}_0\mathbf{M}(-1) = \mathbf{0}$.

These two conditions with $\Phi(0)$ being a right eigenvector of $\mathbf{M}(1)$ for the eigenvalue 1 assure convergence in the weak sense of the infinite matrix product (4) to a compactly supported distributional solution of (1). Now, if $\mathbf{M}(z)$ also verifies the matrix Smith-Barnwell orthogonality condition (also called **Condition O**):

$$\mathbf{M}(z)\mathbf{M}^\top(z^{-1}) + \mathbf{M}(-z)\mathbf{M}^\top(-z^{-1}) = \mathbf{I} \quad (5)$$

for all z on the unit circle, then the convergence is also in the L^2 sense to a bona-fide L^2 solution. For more details about these results, the reader is referred to [27], [23], [13].

Now, assuming all these conditions, the scaling functions and their integer translates form an orthonormal basis of V_0 . If we also impose orthogonality conditions on $\mathbf{N}(z)$, i.e.

$$\begin{aligned} \mathbf{N}(z)\mathbf{N}^\top(z^{-1}) + \mathbf{N}(-z)\mathbf{N}^\top(-z^{-1}) &= \mathbf{I} \\ \mathbf{M}(z)\mathbf{N}^\top(z^{-1}) + \mathbf{M}(-z)\mathbf{N}^\top(-z^{-1}) &= \mathbf{0} \end{aligned} \quad (6)$$

then we get a fully orthonormal multiresolution analysis. Thus, for $\mathbf{s}(t) \in V_0$, we have

$$\mathbf{s}(t) = \sum_n \mathbf{s}_0^\top[n] \phi(t - n) \quad (7)$$

then from $V_0 = V_{-1} \oplus W_{-1}$, we get

$$\mathbf{s}(t) = \sum_n \mathbf{s}_{-1}^\top[n] \phi\left(\frac{t}{2} - n\right) + \mathbf{d}_{-1}^\top[n] \psi\left(\frac{t}{2} - n\right) \quad (8)$$

and we have the well known relations between the coefficients at the analysis step

$$\mathbf{s}_{-1}[n] = \sum_k \mathbf{M}[k - 2n] \mathbf{s}_0[k] \quad (9)$$

$$\mathbf{d}_{-1}[n] = \sum_k \mathbf{N}[k - 2n] \mathbf{s}_0[k] \quad (10)$$

and for the synthesis, we get

$$\mathbf{s}_0[n] = \sum_k \mathbf{M}^\top[n-2k] \mathbf{s}_{-1}[k] + \mathbf{N}^\top[n-2k] \mathbf{d}_{-1}[k] \quad (11)$$

These relations enable us to construct a multi-input multi-output filter bank (abbr. multifilter bank) as seen in Fig. 2 for the case $r = 2$. In case of a one-dimensional signal, it then requires vectorization of this input signal to produce an input signal which is r -dimensional. A simple way to do that is to split a one-dimensional signal into its polyphase components. Introducing

$$\begin{bmatrix} m_0(z) \\ m_1(z) \\ \vdots \\ m_{r-1}(z) \end{bmatrix} := \mathbf{M}(z^r) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(r-1)} \end{bmatrix} \quad (12)$$

and in the same way $n_0(z), n_1(z), \dots, n_{r-1}(z)$, the system can then be seen as a $2r$ channel time-varying filter bank (Fig. 1 for the case $r = 2$).

III. HIGH ORDER BALANCING

In [16], [17], we showed that if the components $m_0(z), m_1(z), \dots, m_{r-1}(z)$ of the lowpass branch have different spectral behavior, e.g. lowpass behavior for one, highpass for another, it then leads to unbalanced channels that mix the coarse resolution and details coefficients and create strong oscillations (Fig. 3) in the signal reconstructed from truncated subbands. One expect then some class of smooth signals to be preserved by the lowpass branch and cancelled by the highpass.

A. Balancing

We define the band-Toeplitz operator $\mathbf{L} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ corresponding to the lowpass analysis

$$\mathbf{L} := \begin{bmatrix} \cdots & \mathbf{M}[0] & \mathbf{M}[1] & \mathbf{M}[2] & \mathbf{M}[3] & \cdots \\ & & \mathbf{M}[0] & \mathbf{M}[1] & \mathbf{M}[2] & \mathbf{M}[3] & \cdots \\ & & & \mathbf{M}[0] & \mathbf{M}[1] & \mathbf{M}[2] & \mathbf{M}[3] & \cdots \\ & & & & \mathbf{M}[0] & \mathbf{M}[1] & \mathbf{M}[2] & \mathbf{M}[3] & \cdots \end{bmatrix} \quad (13)$$

and in the same way, we define $\mathbf{H}$ the band-Toeplitz operator corresponding to the highpass analysis. So, we want $\mathbf{u}_0 := [\dots, 1, 1, 1, 1, \dots]^\top$ to be preserved by the lowpass branch, hence we introduce

Definition III.1 *An orthonormal multiwavelet system is said to be balanced (of order 1) iff the lowpass synthesis operator $\mathbf{L}^\top$ preserves $[\dots, 1, 1, 1, 1, \dots]^\top$ i.e. $\mathbf{L}^\top \mathbf{u}_0 = \mathbf{u}_0$.*

By the orthonormality relations

$$\begin{bmatrix} \mathbf{L}^\top & \mathbf{H}^\top \end{bmatrix} \begin{bmatrix} \mathbf{L} \\ \mathbf{H} \end{bmatrix} = \mathbf{I} \quad \text{and} \quad \begin{bmatrix} \mathbf{L} \\ \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{L}^\top & \mathbf{H}^\top \end{bmatrix} = \mathbf{I}$$

we get $\mathbf{L}^\top \mathbf{L} + \mathbf{H}^\top \mathbf{H} = \mathbf{I}$, $\mathbf{L}\mathbf{L}^\top = \mathbf{I}$, $\mathbf{L}\mathbf{H}^\top = \mathbf{0}$ and $\mathbf{H}\mathbf{H}^\top = \mathbf{I}$. Then $\mathbf{L}^\top \mathbf{u}_0 = \mathbf{u}_0$ implies $\mathbf{L}\mathbf{u}_0 = \mathbf{u}_0$ and $\mathbf{H}\mathbf{u}_0 = \mathbf{0}$ i.e. $\mathbf{u}_0$ is cancelled by the highpass branch.

Now, we can state the following result giving equivalent conditions for balancing, and especially linking balancing to a condition on the factorization of the refinement mask $\mathbf{M}(z)$

Theorem III.1 *The following conditions are equivalent*

B0. $\mathbf{L}^\top \mathbf{u}_0 = \mathbf{u}_0$.

B1. $[1, 1, \dots, 1]$ is a left eigenvector of $\mathbf{M}(1)$ for $\lambda_0(1) = 1$.

B2. $\Phi(0) = [1, 1, \dots, 1]^\top$.

B3. $\sum_{i=0}^{r-1} m_i(z)$ has zeros¹ on the unit circle at $e^{jk\pi/r}$ for $k = 1, \dots, 2r-1$.

B4. One can factorize $\mathbf{M}(z) = \frac{1}{2} \mathbf{T}(z^2) \mathbf{M}_0(z) \mathbf{T}^{-1}(z)$ with

$$\mathbf{T}(z) := \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & -1 \\ -z^{-1} & 0 & \dots & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_0(1) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Proof.

$B0 \Rightarrow B1$: assuming B0, we have $\mathbf{u}_0^\top \mathbf{L} = \mathbf{u}_0^\top$ so we trivially get condition B1.

$B1 \Rightarrow B2$: from (3) at $\omega = 0$, the basic conditions and orthogonality, we get that $\Phi(0)$ is a left and right eigenvector associated to the eigenvalue 1 of $\mathbf{M}(1)$, hence the result.

$B2 \Rightarrow B3$: $\Phi(0)$ is a left and right eigenvector associated to the eigenvalue 1 of $\mathbf{M}(1)$ and from condition **A1** in the basic conditions, we get $\Phi^\top(0) \mathbf{M}(-1) = \mathbf{0}$. Since $\Phi(0) = [1, 1, \dots, 1]^\top$, we get that $\sum_{i=0}^{r-1} m_i(z) = [1, 1, \dots, 1] \mathbf{M}(z^r) [1, z^{-1}, \dots, z^{-(r-1)}]^\top$ has roots at $z = e^{jk\pi/r}$ for $k = 1, \dots, 2r-1$.

$B3 \Rightarrow B0$: finally, taking $u[n] = [\dots, 1, 1, 1, 1, \dots]$ and $v[n] = [\dots, 0, 0, 0, 0, \dots]$ in Fig. 3, we get that the $2r$ possible outputs $\hat{x}[2rn + p] = \sum_k \sum_{i=0}^{r-1} m_i[2rk + p]$ for $p = 0, \dots, 2r-1$ are equal. So $[\dots, 1, 1, 1, 1, \dots]$ is an eigenvalue of the operator $\mathbf{L}^\top$.

$B1 \Rightarrow B4$: is a direct consequence of Theorem 4.1 in [19].

$B4 \Rightarrow B3$: Assuming B4, since

$$\sum_{i=0}^{r-1} m_i(z) = [1, \dots, 1] \mathbf{M}(z^r) \begin{bmatrix} 1 \\ \vdots \\ z^{-(r-1)} \end{bmatrix} \quad (14)$$

¹Conditions B3 was first given by Selesnick in [24].

we have

$$\sum_{i=0}^{r-1} m_i(z) = \frac{1}{2} \begin{bmatrix} 1, & \dots, & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & -1 \\ -z^{-2r} & 0 & \dots & 0 & 1 \end{bmatrix} \mathbf{M}_0(z^r) \dots$$

$$\dots \frac{1}{1 - z^{-r}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ z^{-r} & 1 & 1 & \ddots & \vdots \\ \vdots & z^{-r} & \ddots & \ddots & 1 \\ \vdots & \vdots & \ddots & 1 & 1 \\ z^{-r} & z^{-r} & \dots & z^{-r} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \\ \vdots \\ z^{-(r-1)} \end{bmatrix} \quad (15)$$

so

$$\sum_{i=0}^{r-1} m_i(z) = \frac{1}{2} \left(\frac{1 - z^{-2r}}{1 - z^{-1}} \right) \begin{bmatrix} 1, & 0, & \dots, & 0 \end{bmatrix} \mathbf{M}_0(z^r) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(r-1)} \end{bmatrix} \quad (16)$$

and this is condition B3. ■

B. High Order Balancing

A natural generalization of the concept of balancing is then to impose higher degree discrete time polynomial signals $\mathbf{u}_n := [\dots, (-2)^n, (-1)^n, 0^n, 1^n, 2^n, \dots]^\top$ to be also preserved by the lowpass branch, hence we introduce

Definition III.2 *An orthonormal multiwavelet system is said to be balanced of order p iff the signals $\mathbf{u}_n$ for $n = 0, \dots, p-1$ are preserved by the operator $\mathbf{L}^\top$ i.e.*

$$\mathbf{L}^\top \mathbf{u}_n = 2^{-n} \mathbf{u}_n \quad \text{for } n = 0, \dots, p-1$$

Similarly to the previous case, $\mathbf{L}^\top \mathbf{u}_n = 2^{-n} \mathbf{u}_n$ implies $\mathbf{L} \mathbf{u}_n = 2^n \mathbf{u}_n$ and $\mathbf{H} \mathbf{u}_n = \mathbf{0}$ for $n = 0, \dots, p-1$. The polynomial structure of the signal is captured up to degree $p-1$ by the lowpass branch coefficients.

In this framework, an important issue is the interpolation of all the polyphase components of a discrete time polynomial signal from a single one. On this subject, the following lemma will then prove useful

Lemma III.1 Let $u_{0,r}^{(n)}(z), u_{1,r}^{(n)}(z), \dots, u_{r-1,r}^{(n)}(z)$ be the double-sided formal series

$$u_{i,r}^{(n)}(z) := \sum_{k \in \mathbb{Z}} (kr + i)^n z^{-k}$$

then for $i = 1, \dots, r-1$, there exists a unique polynomial $\alpha_{i,r}^{(n)}(z)$ of degree $n-1$ such that we have the formal equality $u_{i,r}^{(n)}(z) = \alpha_{i,r}^{(n)}(z)u_{0,r}^{(n)}(z)$ and the normalization $\alpha_{i,r}^{(n)}(1) = 1$.

Proof. Using Padé approximations [1], we get a Hörner scheme of interpolation $\alpha_{i,r}^{(n)}(z)$ of the sequence

$$[\dots, (-2r+i)^n, (-r+i)^n, (i)^n, (r+i)^n, (2r+i)^n, \dots]$$

from the sequence

$$[\dots, (-2r)^n, (-r)^n, 0^n, r^n, (2r)^n, \dots]$$

given by

$$\begin{aligned} \alpha_{i,r}^{(n)}(z) = & 1 + \frac{i}{r}(1-z^{-1})\left(1 + \frac{i+r}{2r}(1-z^{-1})\left(1 + \frac{i+2r}{3r}(1-z^{-1})\left(1 + \dots \right.\right.\right. \\ & \left.\left.\left.\dots\right)(1 + \frac{i+(n-2)r}{(n-1)r}(1-z^{-1})\left(1 + \frac{i+(n-1)r}{nr}(1-z^{-1})\right)\dots\right) \right) \end{aligned} \quad (17)$$

Thus, we have

$$\alpha_{i,r}^{(n)}(z) = 1 + \sum_{k=1}^{n-1} \frac{\Gamma(k + \frac{i}{r})}{\Gamma(k+1)\Gamma(\frac{i}{r})} (1-z^{-1})^k \quad (18)$$

Furthermore, it is easily seen that one can write

$$u_{i,r}^{(n)}(z) = \frac{1}{[(1-z)(1-z^{-1})]^{n+1}} \sum_{k=-n}^n v_{i,r}^{(n)}[k] z^{-k} \quad (19)$$

consequently $u_{i,r}^{(n)}(z)$ cannot be formally cancelled by multiplication with a polynomial, and so we have the uniqueness. ■

Remark III.1 For the usual case $r = 2$, we give $\alpha_{1,2}^{(n)}(z)$ for $n = 1, \dots, 7$.

$$\begin{aligned} \alpha_{1,2}^{(1)}(z) &= 1 \\ \alpha_{1,2}^{(2)}(z) &= \frac{1}{2}(3 - z^{-1}) \\ \alpha_{1,2}^{(3)}(z) &= \frac{1}{8}(15 - 10z^{-1} + 3z^{-2}) \\ \alpha_{1,2}^{(4)}(z) &= \frac{1}{16}(35 - 35z^{-1} + 21z^{-2} - 5z^{-3}) \\ \alpha_{1,2}^{(5)}(z) &= \frac{1}{128}(315 - 420z^{-1} + 378z^{-2} - 180z^{-3} + 35z^{-4}) \\ \alpha_{1,2}^{(6)}(z) &= \frac{1}{256}(693 - 1155z^{-1} + 1386z^{-2} - 990z^{-3} + 385z^{-4} - 63z^{-5}) \\ \alpha_{1,2}^{(7)}(z) &= \frac{1}{1024}(3003 - 6006z^{-1} + 9009z^{-2} - 8580z^{-3} + 5005z^{-4} - 1638z^{-5} + 231z^{-6}) \end{aligned}$$

By natural extension, we will also write $\alpha_{0,r}^{(n)}(z) := 1$.

Generalizing the wavelet case, we say

Definition III.3 *We say that the refinement mask $\mathbf{M}(z)$ associated with an orthonormal multi-wavelet system has balanced vanishing moments of order p iff for $n = 0, \dots, p-1$ we have*

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \mathbf{y}_k^\top (2j)^{k-n} \frac{d^{n-k}}{d\omega^{n-k}} [M(e^{j\omega})] \Big|_{\omega=0} &= 2^{-n} \mathbf{y}_n^\top \\ \sum_{k=0}^n \binom{n}{k} \mathbf{y}_k^\top (2j)^{k-n} \frac{d^{n-k}}{d\omega^{n-k}} [M(e^{j\omega})] \Big|_{\omega=\pi} &= \mathbf{0} \end{aligned} \quad (20)$$

with for $n = 0, \dots, p-1$

$$\mathbf{y}_n^\top := \left[\left(\frac{0}{r}\right)^n, \quad \left(\frac{1}{r}\right)^n, \quad \dots, \quad \left(\frac{r-1}{r}\right)^n \right]$$

We then get

Theorem III.2 *The following conditions are equivalent.*

B0_p. $\mathbf{L}^\top \mathbf{u}_n = 2^{-n} \mathbf{u}_n$ for $n = 0, \dots, p-1$.

B1_p. $\mathbf{M}(z)$ has balanced vanishing moments of order p .

B3_p. $\sum_{k=0}^{r-1} \alpha_{k,r}^{(p)}(z^{2r}) m_k(z)$ has zeros² of order p at $e^{jk\pi/r}$ for $k = 1, \dots, 2r-1$.

B4_p. $\mathbf{M}(z)$ can be factored as

$$\mathbf{M}(z) = \frac{1}{2^p} \mathbf{T}^p(z^2) \mathbf{M}_{p-1}(z) \mathbf{T}^{-p}(z) \quad (21)$$

$$\text{with } \mathbf{M}_{p-1}(1) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \text{ and } \mathbf{T}(z) \text{ defined as before.}$$

Proof.

- [B0_p⇒B1_p]: $\mathbf{M}(z)$ satisfies the conditions of Theorem 2.1 in [20] with

$$\mathbf{y}_n^\top := \left[\left(\frac{0}{r}\right)^n, \quad \left(\frac{1}{r}\right)^n, \quad \dots, \quad \left(\frac{r-1}{r}\right)^n \right]$$

for $n = 0, \dots, p-1$. The balanced vanishing moments of order p are just a rewriting of these conditions.

- [B1_p⇒B4_p]: Applying Corollary 4.3. from [19], we get the factorization

$$\mathbf{M}(z) = \frac{1}{2^p} \mathbf{C}_0(z^2) \dots \mathbf{C}_{p-1}(z^2) \mathbf{M}_{p-1}(z) \mathbf{C}_{p-1}^{-1}(z) \dots \mathbf{C}_0^{-1}(z) \quad (22)$$

²Conditions B3_p was first given by Selesnick in [24].

with

$$\mathbf{C}_n(z) := \begin{bmatrix} c_{0,n}^{-1} & -c_{0,n}^{-1} & 0 & \dots & 0 \\ 0 & c_{1,n}^{-1} & -c_{1,n}^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & c_{r-2,n}^{-1} & -c_{r-2,n}^{-1} \\ -z^{-1}c_{r-1,n}^{-1} & 0 & \dots & 0 & c_{r-1,n}^{-1} \end{bmatrix} \quad (23)$$

and the FIR refinement mask $\mathbf{M}_{p-1}(z)$ verifying $\mathbf{M}_{p-1}(1)\mathbf{c}_{p-1} = \mathbf{c}_{p-1}$ where

$$\mathbf{c}_n^\top := [c_{0,n}, \dots, c_{r-1,n}] = 2^{-n} [1, \dots, 1]$$

obtained recursively from $\mathbf{y}_n$ for $n = 0, \dots, p-1$. Thus $\mathbf{C}_n(z) = 2^n \mathbf{T}(z)$ and $\mathbf{M}_{p-1}(1)[1, \dots, 1]^\top = [1, \dots, 1]^\top$.

- [B4_p⇒B3_p]: First, we give an idea of the proof in the case $r = 2$, for $p = 2, 3$ (case $p = 1$ is a consequence of Theorem III.1).

- For $p = 2$, we have

$$\begin{aligned} 2(m_0(z) + \alpha_{1,2}^{(2)}(z^4)m_1(z)) &= 2m_0(z) + (3 - z^{-4})m_1(z) \\ &= [2, \quad 3 - z^{-4}] \mathbf{M}(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \\ &= \frac{1}{4} \left(\frac{1 - z^{-4}}{1 - z^{-1}} \right)^2 [2, \quad -1] \mathbf{M}_1(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \end{aligned} \quad (24)$$

- For $p = 3$,

$$\begin{aligned} 8(m_0(z) + \alpha_{1,2}^{(3)}(z^4)m_1(z)) &= 8m_0(z) + (15 - 10z^{-4} + 3z^{-8})m_1(z) \\ &= \frac{1}{8} \left(\frac{1 - z^{-4}}{1 - z^{-1}} \right)^3 [8 + 3z^{-4}, \quad -9] \mathbf{M}_2(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \end{aligned} \quad (25)$$

In the general case

$$\begin{aligned} \sum_{i=0}^{r-1} \alpha_{i,r}^{(p)}(z^{2r})m_i(z) &= \frac{1}{2} [\alpha_{0,r}^{(p)}(z^{2r}), \dots, \alpha_{r-1,r}^{(p)}(z^{2r})] \dots \\ &\dots \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & -1 \\ -z^{-2r} & 0 & \dots & 0 & 1 \end{bmatrix} \mathbf{M}_0(z^r) \frac{1}{1 - z^{-r}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ z^{-r} & 1 & 1 & \ddots & \vdots \\ \vdots & z^{-r} & \ddots & \ddots & 1 \\ \vdots & \vdots & \ddots & 1 & 1 \\ z^{-r} & z^{-r} & \dots & z^{-r} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ \vdots \\ z^{-(r-1)} \end{bmatrix} \end{aligned} \quad (26)$$

$$\sum_{i=0}^{r-1} \alpha_{i,r}^{(p)}(z^{2r}) m_i(z) = \frac{1}{2} \left(\frac{1}{1-z^{-1}} \right) \left[\alpha_{0,r}^{(p)}(z^{2r}) - z^{-2r} \alpha_{r-1,r}^{(p)}(z^{2r}), \alpha_{1,r}^{(p)}(z^{2r}) - \alpha_{0,r}^{(p)}(z^{2r}), \dots \right. \\ \left. \dots, \alpha_{r-1,r}^{(p)}(z^{2r}) - \alpha_{r-2,r}^{(p)}(z^{2r}) \right] \mathbf{M}_0(z^r) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(r-1)} \end{bmatrix} \quad (27)$$

Let $\gamma_{i,r}^{(l)}[k] := \frac{\Gamma(l+1)\Gamma(k+\frac{i}{r})}{\Gamma(k+1)\Gamma(l+\frac{i}{r})}$, then

$$\sum_{i=0}^{r-1} \alpha_{i,r}^{(p)}(z^{2r}) m_i(z) = \frac{1}{2} \left(\frac{1-z^{-2r}}{1-z^{-1}} \right) \left[1 - z^{-2r} \left(\frac{r-1}{r} \sum_{k=1}^{p-2} \gamma_{r-1,r}^{(1)}[k+1](1-z^{-2r})^k \right), \right. \\ \left. \frac{1}{r} \sum_{k=1}^{p-2} \gamma_{1,r}^{(1)}[k+1](1-z^{-2r})^k, \frac{i}{r} \sum_{k=1}^{p-2} \gamma_{i,r}^{(1)}[k+1](1-z^{-2r})^k - \frac{i-1}{r} \sum_{k=1}^{p-2} \gamma_{i-1,r}^{(1)}[k+1](1-z^{-2r})^k, \dots \right. \\ \left. \frac{r-1}{r} \sum_{k=1}^{p-2} \gamma_{r-1,r}^{(1)}[k+1](1-z^{-2r})^k - \frac{r-2}{r} \sum_{k=1}^{p-2} \gamma_{r-2,r}^{(1)}[k+1](1-z^{-2r})^k \right] \mathbf{M}_0(z^r) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(r-1)} \end{bmatrix} \quad (28)$$

so we have the first factor. Moreover, since

$$\mathbf{M}_0(z) = \frac{1}{2} \mathbf{T}(z^2) \mathbf{M}_1(z) \mathbf{T}^{-1}(z) \quad (29)$$

then, by a similar argument involving the coefficients $\gamma_{i,r}^{(2)}$ (as clearly seen from (17)), we can extract another factor $\left(\frac{1-z^{-2r}}{1-z^{-1}} \right)$, and so iterating until there is no factor anymore, we get the result that one can factor

$$\sum_{i=0}^{r-1} \alpha_{i,r}^{(p)}(z^{2r}) m_i(z) = \left(\frac{1-z^{-2r}}{1-z^{-1}} \right)^p q_{p-1}(z) \quad (30)$$

with $q_p(z)$ a Laurent polynomial, so we have $\mathbf{B3}_p$.

- $[\mathbf{B3}_p \Rightarrow \mathbf{B0}_p]$: As mentioned in [24], condition $\mathbf{B3}_p$ says that the multirate system

$$-(\uparrow 2r) - \boxed{\sum_{k=0}^{r-1} \alpha_{k,r}^{(p)}(z^{2r}) m_k(z)} -$$

has zeros of order p at the roots of the unity $e^{jk\pi/r}$ with $k = 1, \dots, 2r-1$. So, from the rank M wavelets theory (Theorem 2.1. in [11]), we get that this system preserves discrete polynomial sequences of degree $n = 0, \dots, p-1$, and since this multirate system is equivalent to the lowpass synthesis branch for polynomial sequences of degree up to $p-1$, this translates in time domain into condition $\mathbf{B0}_p$. ■

IV. BALANCING ORDER, APPROXIMATION POWER AND SMOOTHNESS

Here, we will clarify how the results obtained above relate to the classical notions of *regularity*, i.e. approximation power and smoothness.

A. Approximation Power and Balancing Order

First, let us recall

Definition IV.1 *One says that a multiscaling function $\phi(t)$ has approximation power m if one can exactly decompose polynomials $1, t, t^2, \dots, t^{m-1}$ using only $\phi_0(t), \phi_1(t), \dots, \phi_{r-1}(t)$ and their integer translates, i.e. for $n = 0, \dots, p-1$, there exists a sequence $\mathbf{x}[n]$ such that*

$$t^n = \sum_k \mathbf{x}_n^\top[k] \phi(t-k) \quad (31)$$

Then, assuming that $\phi(t)$ is balanced of order p , we get that $\mathbf{M}(z)$ factorizes as in (21), so applying p times Theorem 2.6. from [20], we get that $\phi(t)$ has at least an approximation power of p . So,

Proposition IV.1 *If an orthonormal multiwavelet system is balanced of order p , then the associated multiscaling function $\phi(t)$ has an approximation power of at least p .*

We can notice that the converse is false: the DGHM [6] multiscaling function has an approximation power of 2 but is not even balanced [17].

B. Smoothness and Balancing Order

Introducing the Sobolev spaces

$$H^s := \{f \in L^2(\mathbb{R}) \mid \int \|F(\omega)\|^2 (1 + |\omega|^2)^s d\omega < \infty\} \quad (32)$$

we are interested in studying the relation between balancing order and smoothness in that sense.

To characterize the smoothness, we are interested in the decay as $N \rightarrow \infty$ of $\Phi(2^{kN}\omega_0)$ for $\omega_0 \in [0, 2\pi]$. From the convergence (4), we form the truncated products $\mathbf{M}_N(\omega) := \prod_{i=1}^N \mathbf{M}(e^{-j\omega/2^i})$, then evaluating these on the invariant cycle $\{\omega_0, \dots, \omega_{k-1}\}$ of $\omega \mapsto 2\omega \pmod{2\pi}$ [12], we get

$$\begin{aligned} \mathbf{M}_{kN}(2^{kN}\omega_0) &= \prod_{i=1}^{kN} \mathbf{M}(e^{-j2^{-i}2^{kN}\omega_0}) \\ &= (\mathbf{M}(e^{-j\omega_{k-1}}) \dots \mathbf{M}(e^{-j\omega_0}))^N \end{aligned} \quad (33)$$

then we study the asymptotic behavior of this product by looking at the eigenvalues of

$$\mathbf{M}(e^{-j\omega_{k-1}}) \dots \mathbf{M}(e^{-j\omega_0}) = \mathbf{U}_k \Lambda_k \mathbf{U}_k^\top \quad (34)$$

where $\Lambda_k = \text{diag}(\lambda_0^{(k)}, \lambda_1^{(k)}, \dots, \lambda_{r-1}^{(k)})$. Then if $\rho(\Lambda_k) = \max\{|\lambda_0^{(k)}|, |\lambda_1^{(k)}|, \dots, |\lambda_{r-1}^{(k)}|\} \geq 2^{-ki}$ then the scaling functions cannot have Sobolev exponent of more than i and so cannot be more than $\lfloor i - 1/2 \rfloor$ times continuously differentiable [7]. Applying this to the factorization (21), we get the upper-bounds on smoothness:

Proposition IV.2 *If an orthonormal multiwavelet system has balancing order p and the spectral radius of $\mathbf{M}_{p-1}(z)$ in the factorization (21) verifies $\rho(\mathbf{M}_{p-1}(1)) < 2$, then defining*

$$\gamma_k := \frac{1}{k} \log_2 \rho(\mathbf{M}_{p-1}(e^{-j\omega_{k-1}}) \dots \mathbf{M}_{p-1}(e^{-j\omega_0})) \quad (35)$$

with $\{\omega_0, \dots, \omega_{k-1}\}$ invariant cycles of $\omega \mapsto 2\omega \pmod{2\pi}$, and $\gamma := \inf_k \gamma_k$, we get that $\phi(t)$ is at most $\lfloor p - \gamma - \frac{1}{2} \rfloor$ times continuously differentiable (and has at most Sobolev exponent $s = p - \gamma$).

Proof. Because $\{\mathbf{M}[k]\}_k$ is of finite length, without lost of generality one can write $\mathbf{M}_{p-1}(z) = \sum_{k=0}^L \mathbf{M}[k]z^{-k}$, then let $\mathcal{E}$ be the space of trigonometric polynomials of degree at most L , i.e.

$$\mathcal{E} := \{u \mid u(\omega) = \sum_{k=-L}^L u_k e^{-jk\omega}\} \quad (36)$$

Introducing the transfer operator

$$\mathbf{T}_{p-1} \mathbf{X}(\omega) := \sum_{k=0}^{r-1} \mathbf{M}_{p-1}(e^{-j\frac{\omega+k2\pi}{r}}) \mathbf{X}(\frac{\omega+k2\pi}{r}) \mathbf{M}_{p-1}^\top(e^{j\frac{\omega+k2\pi}{r}}) \quad (37)$$

operating on the space of 2π -periodic $r \times r$ matrices $\mathbf{X}(\omega)$. Then, if we restrict $\mathbf{T}_{p-1}$ to the invariant space $\mathcal{E}$ and let $\sigma := \rho(\mathbf{T}_{p-1}|_{\mathcal{E}})$, we get by theorem 5.1 from [3] that $\phi \in H^s$ for all $s < p - \frac{1}{2} \log_2(\sigma)$. Also from ergodic theory [7], we have that $\gamma \leq \sqrt{\sigma}$, hence the result. ■

Using results from the Perron-Frobenius theory [12], one can also find lower-bounds and prove that $s = p - \gamma$ is in fact a good estimate of the Sobolev exponents of $\phi(t)$ and $\psi(t)$.

For example in the case of the Haar multiwavelet, with $\omega_0 = 2\pi/3$, $\lambda_0 = 0, \lambda_1 = \frac{1}{4}$, it then proves that the scaling functions cannot be continuous. In the case of the DGHM multiwavelet, $\lambda_0 = \frac{1}{100}, \lambda_1 = \frac{1}{4^2}$, it proves that the scaling functions can be at most C^1 . DGHM scaling functions and wavelets are in fact Lipschitz.

V. CONSTRUCTION OF HIGH ORDER BALANCED MULTIWAVELETS

A. Bat family

Using the results above, we are now able to investigate the construction of orthonormal multiwavelets of arbitrary balancing order in a similar way to what Daubechies did for her well-know wavelet family. The scheme of construction is the following:

1. Impose the order of balancing to be p by writing (21)

$$\mathbf{M}(z) = \frac{1}{2^p} \mathbf{T}^p(z^2) \mathbf{M}_{p-1}(z) \mathbf{T}^{-p}(z)$$

with $\mathbf{M}_{p-1}(1)[1, \dots, 1]^\top = [1, \dots, 1]^\top$. This way we reduce the number of degrees of freedom in the design.

2. Impose the condition **O** (orthonormality) (5) on $\mathbf{M}(z)$, this gives quadratic equations on the free variables of $\mathbf{M}_{p-1}(z)$.

3. Impose flipping property on $m_0(z), m_1(z)$ (i.e. $m_1(z) = z^{-2L+1}m_0(z^{-1})$) and linear phase on $n_0(z)$ and $n_1(z)$. (The flipping property enables an easy lossless symmetrization [33] of finite length input signals both for the lowpass filters and the highpass.)

4. Solve the system of equations using a Gröbner bases approach (here, we used the program Singular [8]).

5. The highpass filters are easily derived from the lowpass by imposing $n_0(z)$ to be symmetric and $n_1(z)$ to be antisymmetric. The orthonormality conditions (6) give unique solutions up to a change of sign.

Using this approach, we have been able to construct all the multiwavelets of compact support $\subset [0, 10]$ with flipped scaling functions and symmetric/antisymmetric wavelets for order 1, 2, 3 and 4 of balancing. Figures 4, 5 and 6 show the smoothest high order balanced multiwavelets with these properties. Tables I, II and III give the closed form expressions of the coefficients obtained. For order 4 of balancing, only numerical solutions have been obtained and the optimization process does not assure us that the solution obtained here (Fig. 7) is the smoothest one.

B. Minimal Length BMW

In this section, assuming $r = 2$, we will prove the following surprising result:

Theorem V.1 *The shortest length orthonormal multiwavelets of balancing order p are the Daubechies wavelets of length $2p$.*

Remark V.1 Here we define the length of a matrix Laurent polynomial $\mathbf{M}(z) = \sum_{k=-N_1}^{N_2} \mathbf{M}[k]z^{-k}$ with $\mathbf{M}[-N_1] \neq \mathbf{0}$ and $\mathbf{M}[N_2] \neq \mathbf{0}$, to be $\text{length}(\mathbf{M}(z)) = N_2 - N_1 + 1 = \deg(z^{N_2}\mathbf{M}(z)) + 1$. One verifies easily that $\text{length}(\mathbf{M}(z)\mathbf{N}(z)) \leq \text{length}(\mathbf{M}(z)) + \text{length}(\mathbf{N}(z)) - 1$.

In order to prove the theorem, we need the following lemma:

Lemma V.1 *Let $\mathbf{M}(z)$ be the refinement mask associated to an orthonormal multiwavelet system of balancing order p . If $\mathbf{M}(z)$ is of minimal length, then $m_1(z) = z^{-2}m_0(z)$.*

Proof. Assuming $\mathbf{M}(z)$ is the refinement mask associated with an orthonormal multiscaling function of balancing order p , we then have by orthonormality condition:

$$\mathbf{M}(z)\mathbf{M}^\top(z^{-1}) + \mathbf{M}(-z)\mathbf{M}^\top(-z^{-1}) = \mathbf{I} \quad (38)$$

Besides, balancing of order p gives us

$$\mathbf{M}(z) = \frac{1}{2^p} \mathbf{T}^p(z^2) \mathbf{M}_{p-1}(z) \mathbf{T}^{-p}(z) \quad (39)$$

with $\mathbf{M}_{p-1}(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{T}(z) = \begin{bmatrix} 1 & -1 \\ -z^{-1} & 1 \end{bmatrix}$.

Introducing $\mathbf{V}_{p-1}(z) := 2^{-p}(1 - z^{-2})^p \mathbf{M}_{p-1}(z) \mathbf{T}^{-p}(z)$, one gets

$$\mathbf{V}_{p-1}(z)\mathbf{V}_{p-1}^\top(z^{-1}) + \mathbf{V}_{p-1}(-z)\mathbf{V}_{p-1}^\top(-z^{-1}) = \begin{bmatrix} 2 & 1+z^2 \\ 1+z^{-2} & 2 \end{bmatrix}^p \quad (40)$$

Furthermore, one can write

$$\mathbf{U}_{p-1}(z) := \begin{bmatrix} 2 & 1+z^2 \\ 1+z^{-2} & 2 \end{bmatrix}^p = \sum_{k=-\lceil p/2 \rceil}^{k=\lceil p/2 \rceil} \mathbf{U}_{p-1}[k] z^{-2k} \quad (41)$$

with $\mathbf{U}_{p-1}[-k] = \mathbf{U}_{p-1}^\top[k]$.

Thus for $\mathbf{W}(z) = \sum_{k=-N_1}^{k=N_2} \mathbf{W}[k] z^{-k}$ to verify

$$\mathbf{W}(z) + \mathbf{W}(-z) = \mathbf{U}_{p-1}(z) \quad (42)$$

one needs $N_1, N_2 \geq 2\lceil p/2 \rceil$. Introducing $\mathbf{u}(z) := \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$, then

$$\mathbf{W}_{p-1}(z) := (1 + z^{-1})^p (1 + z)^p \mathbf{u}(z) \mathbf{u}^\top(z) = (1 + z^{-1})^p (1 + z)^p \begin{bmatrix} 1 & z \\ z^{-1} & 1 \end{bmatrix} \quad (43)$$

is an obvious minimal length solution of (42). So one has to prove now that there is no other minimal length solution. Since all even degrees of $\mathbf{W}(z)$ are uniquely determined by (42), all the other minimal length solutions will be of the form $\mathbf{W}(z) = \mathbf{W}_{p-1}(z) + z^{-1} \mathbf{R}(z^2)$ and should factorize as $\mathbf{W}(z) = \mathbf{V}(z) \mathbf{V}^\top(z^{-1})$. We have then

$$z^{-1} \mathbf{R}(z^2) = \mathbf{V}(z) \mathbf{V}^\top(z^{-1}) - (1 + z^{-1})^p (1 + z)^p \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \begin{bmatrix} 1 & z \end{bmatrix} \quad (44)$$

Since $\mathbf{u}^\top(z) \mathbf{u}(-z^{-1}) = \mathbf{u}^\top(-z^{-1}) \mathbf{u}(z) = 0$, multiplying (44) by $\mathbf{u}^\top(-z^{-1})$ on the left and by $\mathbf{u}(-z)$ on the right, we have

$$z^{-1} \mathbf{u}^\top(-z^{-1}) \mathbf{R}(z^2) \mathbf{u}(-z) = \mathbf{u}^\top(-z^{-1}) \mathbf{V}(z) \mathbf{V}^\top(z^{-1}) \mathbf{u}(-z) \quad (45)$$

For $|z| = 1$, we obtain

$$z^{-1}\mathbf{u}^\top(-z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z) = |\mathbf{V}^\top(z^{-1})\mathbf{u}(-z)|^2 \quad (46)$$

Changing $z \rightarrow -z$, we also get

$$-z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(z) = |\mathbf{V}^\top(-z^{-1})\mathbf{u}(z)|^2 \quad (47)$$

Now again, multiplying (44) by $\mathbf{u}^\top(z^{-1})$ on the left and by $\mathbf{u}(z)$ on the right, we get

$$\begin{aligned} z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(z) &= \mathbf{u}^\top(z^{-1})\mathbf{V}(z)\mathbf{V}^\top(z^{-1})\mathbf{u}(z) \\ &\quad - (1+z^{-1})^p(1+z)^p \begin{bmatrix} 1 & z \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \begin{bmatrix} 1 & z \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \end{aligned} \quad (48)$$

i.e. for $|z| = 1$,

$$z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(z) = |\mathbf{V}^\top(z^{-1})\mathbf{u}(z)|^2 - 4(1+z^{-1})^p(1+z)^p \quad (49)$$

So adding equations (47) and (49), one gets

$$|\mathbf{V}^\top(z^{-1})\mathbf{u}(z)|^2 + |\mathbf{V}^\top(-z^{-1})\mathbf{u}(z)|^2 = 4(1+z^{-1})^p(1+z)^p \quad (50)$$

then

$$|\mathbf{V}^\top(z^{-1})\mathbf{u}(z)| = O((1+z^{-1})^p) \quad (51)$$

$$|\mathbf{V}^\top(-z^{-1})\mathbf{u}(z)| = O((1+z^{-1})^p) \quad (52)$$

then $|\mathbf{V}^\top(z^{-1})\mathbf{u}(-z)| = O((1-z^{-1})^p)$

Besides multiplying (44) by $\mathbf{u}^\top(z^{-1})$ on the left and by $\mathbf{u}(-z)$ on the right, we get

$$z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z) = \mathbf{u}^\top(z^{-1})\mathbf{V}(z)\mathbf{V}^\top(z^{-1})\mathbf{u}(-z) \quad (53)$$

then

$$z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z) = O((1-z^2)^p) \quad (54)$$

i.e. there exists a unique Laurent polynomial $q(z)$ such that

$$k(z) := z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z) = (1-z^2)^p q(z) \quad (55)$$

Furthermore, from (44), $z\mathbf{R}^\top(z^{-2}) = z^{-1}\mathbf{R}(z^2)$, then $k(-z^{-1}) = -k(z)$, and so $q(-z^{-1}) = -q(z)$. Hence, $q(z)$ is at least of length 3. Thus $z^{-1}\mathbf{R}(z^2)$ is at least of length $2p$ and because it is of odd length by structure, so it is at least of length $2p+1$. Hence, we have that $\mathbf{W}_{p-1}(z)$ is the unique minimal length solution.

Furthermore, since $\mathbf{V}_{p-1}(z) := 2^{-p}(1 - z^{-2})^p \mathbf{M}_{p-1}(z) \mathbf{T}^{-p}(z)$, and $\mathbf{V}_{p-1}(z) \mathbf{V}_{p-1}^\top(z^{-1}) = \mathbf{W}_{p-1}(z)$, then

$$\mathbf{M}(z) \mathbf{M}^\top(z^{-1}) = 2^{-2p} (1 - z^{-2})^{-p} (1 - z^2)^{-p} \mathbf{T}^p(z^2) \mathbf{W}_{p-1}(z) \mathbf{T}^{p\top}(z^{-2}) \quad (56)$$

Now, since $\mathbf{T}(z^2) \mathbf{u}(z) = (1 - z^{-1}) \mathbf{u}(z)$, we have

$$\mathbf{M}(z) \mathbf{M}^\top(z^{-1}) = \frac{1}{2} \begin{bmatrix} 1 & z \\ z^{-1} & 1 \end{bmatrix} \quad (57)$$

On the other side, since $\det(\mathbf{W}_{p-1}(z)) = 0$, one can write

$$\mathbf{M}(z) = \begin{bmatrix} 1 \\ \lambda(z) \end{bmatrix} \begin{bmatrix} m_{00}(z) & m_{01}(z) \end{bmatrix} \quad (58)$$

so

$$\mathbf{M}(z) \mathbf{M}^\top(z^{-1}) = (m_{00}(z)m_{00}(z^{-1}) + m_{01}(z)m_{01}(z^{-1})) \begin{bmatrix} 1 & \lambda(z^{-1}) \\ \lambda(z) & \lambda(z)\lambda(z^{-1}) \end{bmatrix} \quad (59)$$

Hence $\lambda(z) = z^{-1}$ and so $m_1(z) = z^{-2}m_0(z)$. ■

We are now able to prove the Theorem V.1

Proof. Using lemma V.1 and the balancing order condition **B3**, we get that

$$m_0(z) + \alpha_{1,2}^{(p)}(z^4)m_1(z) = (1 + z^{-2}\alpha_{1,2}^{(p)}(z^4))m_0(z)$$

must have zeros of order p at $j, -1, -j$.

Moreover for $z = -1$, we have

$$1 + z^{-2}\alpha_{1,2}^{(p)}(z^4) = 1 + (-1)^{-2}\alpha_{1,2}^{(p)}((-1)^4) = 1 + \alpha_{1,2}^{(p)}(1) = 2$$

which implies that $m_0(z)$ must p zeros at $z = -1$. Since

$$\mathbf{M}(z) = \begin{bmatrix} m_{00}(z) & m_{01}(z) \\ z^{-1}m_{00}(z) & z^{-1}m_{01}(z) \end{bmatrix}$$

with $m_{00}(z)$ and $m_{01}(z)$ the polyphase components of $m_0(z)$, then the orthonormality condition (5) gives that $m_0(z)$ is a real conjugate mirror filter. Then, from the well-know theorem of Daubechies [4], this implies that $m_0(z)$ has at least $2p$ non-zero coefficients, and that the minimal length filters are the Daubechies filters (i.e. classical D_{2p} and Symlets of order p). ■

This also imply that:

Corollary V.1 *An orthonormal multiwavelet system of balancing order p has a refinement mask $\mathbf{M}(z)$ with at least $p + 1$ non-zero (2×2) taps.*

VI. CONCLUSION

By introducing the concept of high order balancing, we have clarified an important issue in the design of multiwavelets. We have proved that this concept was equivalent to a natural counterpart of the *zeros at π* condition in the standard wavelet theory. With these results, we made it possible to design general families of high order balanced multiwavelets with the required properties for practical signal processing (preservation/cancellation of discrete-time polynomial signals in the lowpass/highpass subbands, FIR, linear phase and orthogonality). However the scheme of construction exposed is making use of computationally heavy methods (Gröbner basis decomposition) and it is not clear at this point that closed form designs will be feasible for multiwavelets of balancing order $p \geq 4$. For $p = 4$, only numerical solutions have been obtained (no algebraic form for the coefficients). Matrix spectral factorization could be a way to overcome this limitation. Another subject of interest would also be to use the invariant cycles scheme developed for estimating the smoothness so as to link the smoothness of the scaling functions to a particular factorization of the refinement mask (the counterpart of the *zeros at pre-periodic points* condition [12] in the standard wavelet theory). This problem remains open.

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LIST OF FIGURES

1. Multifilter bank seen as a time-varying filter bank.
2. Orthogonal multifilter bank for $r = 2$.
3. Left: Reproduction of the input signal $[1, 1, \dots, 1]$ by a DGHM (Donovan, Geronimo, Hardin, Massopust) multiwavelet based filter bank, Right: Reconstruction from the lowpass coefficients only of a polynomial test signal; it shows the poor robustness of a system based on the DGHM multiwavelet.
4. First order balanced orthogonal multiwavelet: the scaling functions are flipped around 1, the wavelets are symmetric/antisymmetric, the length is 3 taps (2x2) and an estimate of the smoothness using Proposition IV.2 gives the Sobolev exponent $s = 0.64$.
5. Order 2 balanced orthogonal multiwavelet: the scaling functions are flipped around 2, the wavelets are symmetric/antisymmetric, the length is 5 taps (2x2) and an estimate of the smoothness using Proposition IV.2 gives the Sobolev exponent $s = 1.15$.
6. Order 3 balanced orthogonal multiwavelet: the scaling functions are flipped around 3, the wavelets are symmetric/antisymmetric, the length is 7 taps (2x2) and an estimate of the smoothness using Proposition IV.2 gives the Sobolev exponent $s = 1.71$.
7. Order 4 balanced orthogonal multiwavelet: the scaling functions are flipped around 5, the wavelets are symmetric/antisymmetric, the length is 11 taps (2x2) and an estimate of the smoothness using Proposition IV.2 gives the Sobolev exponent $s = 1.63$.

LIST OF TABLES

- I.* Coefficients of BAT O1: first order balanced orthogonal multiwavelet.
- II.* Coefficients of BAT O2: order 2 balanced orthogonal multiwavelet.
- III.* Coefficients of BAT O3: order 3 balanced orthogonal multiwavelet.

Fig. 1. Multifilter bank seen as a time-varying filter bank.

Fig. 2. Orthogonal multifilter bank for $r = 2$.

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$m_0[k]$	0	$\frac{1}{2} + \frac{1}{4}\sqrt{7}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2} - \frac{1}{4}\sqrt{7}$	0
$m_1[k]$	0	$\frac{1}{2} - \frac{1}{4}\sqrt{7}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2} + \frac{1}{4}\sqrt{7}$	0
$n_0[k]$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
$n_1[k]$	0	$\frac{1}{4}$	$-\frac{1}{4}\sqrt{7}$	$\frac{1}{4}\sqrt{7}$	$-\frac{1}{4}$	0

TABLE I

COEFFICIENTS OF BAT O1: FIRST ORDER BALANCED ORTHOGONAL MULTIWAVELET

$m_0[k]$	0	$-\frac{31}{640} + \frac{1}{640}\sqrt{31}$	$\frac{93}{640} - \frac{13}{640}\sqrt{31}$	$\frac{217}{640} + \frac{23}{640}\sqrt{31}$	$\frac{341}{640} - \frac{11}{640}\sqrt{31}$
$m_1[k]$	0	$-\frac{13}{640} + \frac{3}{640}\sqrt{31}$	$-\frac{1}{640} + \frac{1}{640}\sqrt{31}$	$\frac{11}{640} - \frac{11}{640}\sqrt{31}$	$\frac{23}{640} + \frac{7}{640}\sqrt{31}$
$n_0[k]$	0	$\frac{23}{160} - \frac{3}{160}\sqrt{31}$	$\frac{11}{160} - \frac{1}{160}\sqrt{31}$	$-\frac{91}{160} + \frac{1}{160}\sqrt{31}$	$\frac{57}{160} + \frac{3}{160}\sqrt{31}$
$n_1[k]$	0	$\frac{47}{320} - \frac{7}{320}\sqrt{31}$	$\frac{9}{320} + \frac{1}{320}\sqrt{31}$	$-\frac{159}{320} + \frac{9}{320}\sqrt{31}$	$\frac{103}{320} + \frac{17}{320}\sqrt{31}$

$\frac{23}{640} + \frac{7}{640}\sqrt{31}$	$\frac{11}{640} - \frac{11}{640}\sqrt{31}$	$-\frac{1}{640} + \frac{1}{640}\sqrt{31}$	$-\frac{13}{640} + \frac{3}{640}\sqrt{31}$	0
$\frac{341}{640} - \frac{11}{640}\sqrt{31}$	$\frac{217}{640} + \frac{23}{640}\sqrt{31}$	$\frac{93}{640} - \frac{13}{640}\sqrt{31}$	$-\frac{31}{640} + \frac{1}{640}\sqrt{31}$	0
$\frac{57}{160} + \frac{3}{160}\sqrt{31}$	$-\frac{91}{160} + \frac{1}{160}\sqrt{31}$	$\frac{11}{160} - \frac{1}{160}\sqrt{31}$	$\frac{23}{160} - \frac{3}{160}\sqrt{31}$	0
$-\frac{103}{320} - \frac{17}{320}\sqrt{31}$	$\frac{159}{320} - \frac{9}{320}\sqrt{31}$	$-\frac{9}{320} - \frac{1}{320}\sqrt{31}$	$-\frac{47}{320} + \frac{7}{320}\sqrt{31}$	0

TABLE II

COEFFICIENTS OF BAT O2: ORDER 2 BALANCED ORTHOGONAL MULTIWAVELET

$m_0[k]$	0	$-\frac{2989}{2232320} + \frac{97}{2232320}\sqrt{15199}$	$\frac{537}{2232320} + \frac{69}{2232320}\sqrt{15199}$	$-\frac{75969}{2232320} - \frac{105}{446464}\sqrt{15199}$
$m_1[k]$	0	$\frac{2481}{446464} - \frac{21}{446464}\sqrt{15199}$	$-\frac{4701}{446464} + \frac{39}{446464}\sqrt{15199}$	$\frac{13769}{2232320} - \frac{43}{2232320}\sqrt{15199}$
$n_0[k]$	0	$-\frac{2871}{279040} + \frac{33}{279040}\sqrt{15199}$	$-\frac{1177}{279040} + \frac{1}{279040}\sqrt{15199}$	$-\frac{12001}{279040} - \frac{11}{55808}\sqrt{15199}$
$n_1[k]$	0	$\frac{12021}{1116160} - \frac{63}{1116160}\sqrt{15199}$	$\frac{7697}{1116160} - \frac{101}{1116160}\sqrt{15199}$	$-\frac{25921}{1116160} - \frac{331}{1116160}\sqrt{15199}$

$\frac{14785}{446464} + \frac{551}{2232320}\sqrt{15199}$	$\frac{601461}{1116160} - \frac{441}{1116160}\sqrt{15199}$	$\frac{450639}{1116160} + \frac{819}{1116160}\sqrt{15199}$
$\frac{22083}{2232320} - \frac{111}{2232320}\sqrt{15199}$	$-\frac{96093}{1116160} + \frac{45}{223232}\sqrt{15199}$	$\frac{30005}{223232} - \frac{667}{1116160}\sqrt{15199}$
$-\frac{1555}{55808} - \frac{71}{279040}\sqrt{15199}$	$\frac{75083}{139520} + \frac{19}{139520}\sqrt{15199}$	$-\frac{63171}{139520} + \frac{27}{139520}\sqrt{15199}$
$-\frac{44869}{1116160} - \frac{241}{1116160}\sqrt{15199}$	$\frac{150307}{558080} + \frac{743}{558080}\sqrt{15199}$	$-\frac{348777}{558080} + \frac{333}{558080}\sqrt{15199}$

$\frac{30005}{223232} - \frac{667}{1116160}\sqrt{15199}$	$-\frac{96093}{1116160} + \frac{45}{223232}\sqrt{15199}$	$\frac{22083}{2232320} - \frac{111}{2232320}\sqrt{15199}$
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$-\frac{63171}{139520} + \frac{27}{139520}\sqrt{15199}$	$\frac{75083}{139520} + \frac{19}{139520}\sqrt{15199}$	$-\frac{1555}{55808} - \frac{71}{279040}\sqrt{15199}$
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$\frac{13769}{2232320} - \frac{43}{2232320}\sqrt{15199}$	$-\frac{4701}{446464} + \frac{39}{446464}\sqrt{15199}$	$\frac{2481}{446464} - \frac{21}{446464}\sqrt{15199}$	0
$-\frac{75969}{2232320} - \frac{105}{446464}\sqrt{15199}$	$\frac{537}{2232320} + \frac{69}{2232320}\sqrt{15199}$	$-\frac{2989}{2232320} + \frac{97}{2232320}\sqrt{15199}$	0
$-\frac{12001}{279040} - \frac{11}{55808}\sqrt{15199}$	$-\frac{1177}{279040} + \frac{1}{279040}\sqrt{15199}$	$-\frac{2871}{279040} + \frac{33}{279040}\sqrt{15199}$	0
$\frac{25921}{1116160} + \frac{331}{1116160}\sqrt{15199}$	$-\frac{7697}{1116160} + \frac{101}{1116160}\sqrt{15199}$	$-\frac{12021}{1116160} + \frac{63}{1116160}\sqrt{15199}$	0

TABLE III

COEFFICIENTS OF BAT O3: ORDER 3 BALANCED ORTHOGONAL MULTIWAVELET

Fig. 4. First order balanced orthogonal multiwavelet: the scaling functions are flipped around 1, the wavelets are symmetric/antisymmetric, the length is 3 taps (2×2) and an estimate of the smoothness using Proposition IV.2 gives the Sobolev exponent $s = 0.64$.

Fig. 5. Order 2 balanced orthogonal multiwavelet: the scaling functions are flipped around 2, the wavelets are symmetric/antisymmetric, the length is 5 taps (2×2) and an estimate of the smoothness using Proposition IV.2 gives the Sobolev exponent $s = 1.15$.

Fig. 6. Order 3 balanced orthogonal multiwavelet: the scaling functions are flipped around 3, the wavelets are symmetric/antisymmetric, the length is 7 taps (2×2) and an estimate of the smoothness using Proposition IV.2 gives the Sobolev exponent $s = 1.71$.

Fig. 7. Order 4 balanced orthogonal multiwavelet: the scaling functions are flipped around 5, the wavelets are symmetric/antisymmetric, the length is 11 taps (2×2) and an estimate of the smoothness using Proposition IV.2 gives the Sobolev exponent $s = 1.63$.