Network Calculus viewed as a Min-plus System Theory applied to Communication Networks

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Abstract

We analyze some queueing problems arising in guaranteed service and controlled load networks using min-plus algebra. We find an explicit representation for the sub-additive closure of the minimum of two operators, and we introduce a new, useful family of idempotent, time-varying, and min-plus linear operators. We model queueing systems arising in networks networks as non-linear min-plus systems that can be bounded by linear systems, and apply our concepts to: the optimal shaper studied by Anantharam and Konstantopoulos, the window flow control problem previously studied by: Cruz and Okino; Chang; Agrawal and Rajan. In all these cases we explain the existing bounds and in the latter case derive another bound. We then show how the same method enables us to give a representation for the losses in a shaper with finite buffer constraints or with delay constraints. We apply the result to bound the losses in a variable bit rate (VBR) trunk system by the losses in simpler, constant bit rate trunk systems (CBR) systems. Finally, as a by-product of the concepts proposed in the paper, we show how it provides an explicit solution to the deterministic Skorokhod reflection mapping problem with two boundaries.

Keywords Guaranteed Quality of Service; ATM; Queueing Systems; Network Calculus; Min-Plus Algebra; Reflection Mapping; Window Flow Control; Optimal Leaky Bucket;

1 Introduction

A number of recent papers [6, 7, 8, 9] has brought together a set of calculus rules, called network calculus, for networks with guaranteed quality of service. They extend the original theory of service curves introduced by Parek and Gallager [1, 2] and Cruz [3, 4, 5] by placing it in the general context of min-plus algebra [10]. The results give bounds for such quantities as delays, backlogs in networks which offer guaranteed service, with or without flow control.

The starting point for this paper is a central result of min-plus algebra which describes the solution of a system of inequations using the concept of closure of an operator [10] (Theorem 1 in Section 4). We use this result to propose a systematic method for modeling a number of situations arising in communication networks, not only for guaranteed service.

In Section 4 we present our framework for system modelling; it relies on standard concepts and results from min-plus algebra. Then we introduce a family of min-plus linear, time-varying and idempotent operators which are useful for modelling a number of systems. They can be used in particular to represent systems with losses. Then we focus on the closure of the minimum of two operators, for which we propose an explicit representation. If one of the operators is idempotent, we obtain a closed form representation.

In Section 5 we apply the results from Section 4 to Examples 1 to 3 (lossless systems). Our analysis not only explains in a systematic way some results which were previously obtained, but it also provides new results. Then in Section 6 we apply our framework to Examples 4 and 5 (lossy systems). We obtain a representation of the losses in these systems, and apply it to provide bounds for variable bit rate (VBR) shapers. VBR shapers are found in many instances of communication systems, for example with packet video servers or in hierarchical reservation systems [21]. In Section 6.1.1 we first compare a VBR shaper to a virtual system made of two simpler, constant bit rate (CBR) shapers in parallel; we show that, for every sample path, the amount of losses in the VBR shaper is bounded by the amount of losses in the virtual system. We show a similar result for the total amount of time spent in congestion. Then we use a virtual system made of two CBR shapers in sequence and obtain a similar result for the amount of losses. Such a result may exist for the time spent in congestion but we did not investigate it. In Section 6.1.2 we analyze a virtual system consisting in splitting the original input flow into component flows, and allocating to every component flow a virtual, segregated system. Such an analysis by segregation is used for example in [13]. We show that, for every sample path, the amount of losses in the original system is always bounded by the amount of losses in the virtual systems. For those systems, it is known that a similar result for the time spent in congestion is not true. In Section 6.2 we analyze Example 5 and obtain a similar result. We expect the results in Section 6 to form the starting point for obtaining bounds on probabilities of loss or congestion for shapers with losses; the method would consist in applying known bounds to virtual systems and take the minimum over a set of virtual systems. This application of our framework is outside the scope of this paper.

Finally, as a by-product of our framework, we give in Section 7 the explicit solution to the deterministic Skorokhod reflection mapping problem with two boundaries. Section 8 concludes the paper.

2 Notation and Background

We consider a discrete or continuous time system, described by $\vec{x}(t)$ where t is the time. We assume that for all t the values $\vec{x}(t)$ are in $(\mathbb{R}^+ \cup \{\infty\})^J$, where J is a fixed, finite integer.

For $\vec{z}, \vec{z'} \in (\mathbb{R}^+ \cup \{\infty\})^J$, we define $\vec{z} \wedge \vec{z}'$ as the coordinate-wise minimum of $\vec{z}$ and $\vec{z}'$, and similarly for the + operator. We write $\vec{z} \leq \vec{z'}$ with the meaning that $\vec{z}_j \leq \vec{z'}_j$ for $j = 1 \ldots J$. Note that the comparison so defined is not a total order, namely, we cannot guarantee that either $\vec{z} \leq \vec{z'}$ or $\vec{z'} \leq \vec{z}$ holds. For a constant K, we note $\vec{z} + K$ the vector defined by adding K to *all* elements of $\vec{z}$.

For sequences or functions, we note similarly $(\vec{x} \wedge \vec{y})(t) = \vec{x}(t) \wedge \vec{y}(t)$ and $(\vec{x} + K)(t) = \vec{x}(t) + K$ for all $t \geq 0$, and write $\vec{x} \leq \vec{y}$ with the meaning that $\vec{x}(t) \leq \vec{y}(t)$ for all t.

We further call $\mathcal{F}_J$ the set of sequences or functions $\vec{x}$ that are wide-sense increasing and non-negative, namely, for which $0 \leq x_j(0) \leq x_j(t_1) \leq x_j(t_2)$ for any $0 < t_1 < t_2$ and all $j = 1 \ldots J$.

We denote vectors with the arrow symbol as in $\vec{x}$ or with a non-arrowed greek letter.

The min-plus convolution operation, which we note $\otimes$ is defined as follows [6, 8, 9, 10].

$$(\vec{x} \otimes \vec{y})(t) = \inf_{u \text{ such that } 0 \leq u \leq t} \{\vec{x}(u) + \vec{y}(t-u))\}$$

Note that $(\vec{x} \otimes \vec{y}) + K = (\vec{x} + K) \otimes \vec{y} = \vec{x} \otimes (\vec{y} + K)$.

In network calculus, when the dimension $J = 1$, $\vec{x}(t) = x(t)$ is for example the number of bits, or ATM cells, counted from time 0 to t at a given observation point. We say that a network element guarantees to a flow $x(t)$ a *service curve* β if there exists some $u \in [0,t]$ such that the output $y(t)$ satisfies $y(t) - x(s) \geq \beta(t-s)$, or equivalently

$$y \geq \beta \otimes x.$$

For example, an ideal constant bit rate (CBR) server with rate c offers a service curve defined by $\beta(t) = ct$. More realistically, the IETF assumes that guaranteed service nodes offer a service curve $\beta(t) = R[t-T]^+$ [6]. Similarly, we say that a flow, described by $x(t)$, is constrained by α if for any $t \geq 0$, any $0 \leq s \leq t$, $x(t) - x(s) \leq \alpha(t-s)$, or equivalently

$$x \leq \alpha \otimes x.$$

We say that α is an *arrival curve* for the flow [6, 7, 8, 9]. The concept of arrival curve generalizes that of leaky buckets. Service and arrival curves are the key concept for modelling guaranteed service schedulers and computing useful bounds.

3 Examples considered in this Paper

We consider the following examples.

3.1 Example 1: Optimal traffic shaper

This example is found in [12]. Let σ_0 and σ be two increasing, concave functions, such that $\sigma_0 \leq \sigma$. We want to find an operator $\varphi(\cdot)$ which

- is *causal*, meaning that $\varphi(a)(t)$ depends only on $a(s)$ for $0 \leq s \leq t$,
- is *realizable*, meaning that $\varphi(a)(t) \leq a(t)$ for any $t \geq 0$, and $a(t) \geq 0$,
- satisfies the *burstiness constraints* specified by (σ_0, σ), namely for any $t \geq s \geq 0$, $a(t) \geq 0$, $\varphi(a)(t) \leq \sigma_0(t)$ and $\varphi(a)(t) - \varphi(a)(s) \leq \sigma(t-s)$,

and which is *optimal* in the sense that if ψ is another operator satisfying the above requirements then $\psi(a)(t) \leq \varphi(a)(t)$ for any $t \geq 0$, $a(\cdot) \geq 0$.

The solution to this problem is found in [12] using reflection mappings, and is summarized as follows.

Let $g^\star$ denote the convex transform of a function $g \in \mathcal{F}_1$, defined as

$$g^\star(\rho) = \sup_{t \geq 0}\{g(t) - \rho t\}. \tag{1}$$

with $\rho \geq 0$. The inverse transform is

$$g(t) = \inf_{\rho \geq 0}\{g^\star(\rho) + \rho t\}. \tag{2}$$

Theorem 5 of [12] defines the optimal map φ by the following equations, with $\rho \geq 0$.

$$\begin{aligned} z^\rho(t) &= \sigma^\star(\rho) - \sigma_0^\star(\rho) + a(t) - \rho t & (3)\\ N^\rho(t) &= -\inf_{0 \leq s \leq t}\{z^\rho(s)\} \wedge 0 & (4)\\ \varphi(a)(t) &= a(t) \wedge \inf_{\rho \geq 0}\{\sigma_0^\star(\rho) + \rho t - N^\rho(t)\}. & (5) \end{aligned}$$

We will show that these equations can be obtained and recast in a simpler form using min-plus methods.

In general, the burstiness constraints of a shaper are expressed only over sliding temporal windows, i.e. by σ, which means that $\sigma_0 = \sigma$. We find then back the results of [6]. For the following Examples 4 and 5, we will assume that for a shaper $\sigma_0 = \sigma$, and moreover that $\sigma(0) = 0$ and that σ is sub-additive, which is not a restriction [6].

3.2 Example 2: Window flow controller

This example is found independently in [8] and [9]. A data flow $a(t)$ is fed via a window flow controller to a network offering a service curve of β. The window flow controller limits the amount of data admitted into the network in such a way that the total backlog is less than or equal to K, where K (the window size) is a fixed number (Figure 1).

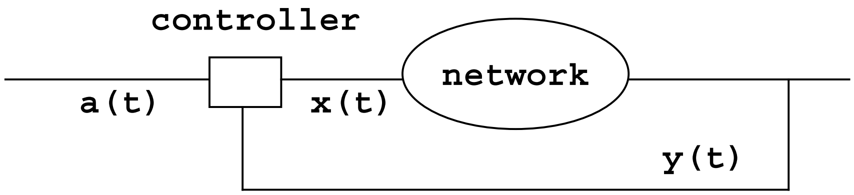

Figure 1: Example 2, from [8] or [9]

Call $x(t)$ the flow admitted to the network, and $y(t)$ the output. The definition of the controller means that $x(t)$ is the maximum solution to

$$\begin{cases} x(t) \leq a(t) \\ x(t) \leq y(t) + K \end{cases} \tag{6}$$

which implies that $x(t) = a(t) \wedge (y(t) + K)$. Note that we do not know the mapping $x(t) \to y(t)$, but we do know that

$$y(t) \geq (\beta \otimes x)(t). \tag{7}$$

In [8], (6) and (7) are used to derive that

$$x \geq \overline{(\beta + K)} \otimes a \tag{8}$$

In the formula, $\overline{(\beta + K)}$ is the sub-additive closure of $\beta + K$ [6, 8, 10]. The sub-additive closure $\overline{\alpha}$ of a vector or function α is defined by

$$\overline{\alpha}(t) = \inf\{\delta, \alpha, \alpha^{(2)}, \alpha^{(3)}, \dots\}$$

with $\alpha^{(i)} = \alpha \otimes \dots \otimes \alpha$ (i times) and δ is the fixed function defined by $\delta(t) = \infty$ for $t > 0$ and $\delta(0) = 0$. Equation (8) means that the complete system offers a service curve equal to $\overline{(\beta + K)}$. We show in this paper that this result is indeed obtained by min-plus methods.

3.3 Example 3: Detailed window flow control

This example is a more detailed representation of window-flow control. Compared to Example 1, the additional modelling assumptions in [7] can be re-formulated as follows (see Figure 2 for the notation).

- the output of the window flow controller (marked Network element 1 on the figure) is constrained by an arrival curve α.
- the output of network elements 1 and 2 is constrained by a maximum service rate . More precisely, the number of bits output at station i ($i = 1, 2$) during time interval $(s, t]$ is bounded by $M_i(t) - M_i(s)$ for some fixed functions M_1 and M_2. This models the fact that the server is busy serving other flows.

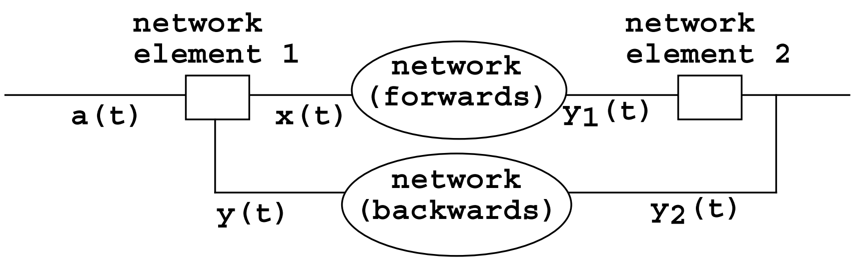

Figure 2: Example 3, from [7]

As with Example 2, the controller guarantees that a maximum of K bits are backlogged in the loop. Network elements f (forwards) and b (backwards) are assumed to offer service curves S_f and S_b.

With these assumptions, the admitted flow x is the maximum solution to the system

$$\begin{cases} x(t) \leq a(t) \\ x(t) \leq (\alpha \otimes x)(t) \\ x(t) \leq \inf_{u \text{ such that } 0 \leq u \leq t} \{x(u) + M_1(t) - M_1(u)\} \\ x(t) \leq y(t) + K \end{cases} \tag{9}$$

As with Example 1, we do not know the exact mapping $x(t) \to y(t)$, but we do know that

$$\begin{cases} y_1(t) \leq x(t) \\ y_1(t) \geq (S_f \otimes x)(t) \\ y_2(t) \leq y_1(t) \\ y_2(t) \leq \inf_{u \text{ such that } 0 \leq u \leq t} \{y_2(u) + M_2(t) - M_2(u)\} \\ y(t) \leq y_2(t) \\ y(t) \geq (S_b \otimes y_2)(t) \end{cases} \tag{10}$$

and that (x, y_2) is the maximum couple of functions such that (9) and (10) hold.

Additional assumptions made in [7] are that for all $t \geq 0$, $0 \leq s \leq t$

$$\begin{cases} \alpha(t) \leq C_1 \cdot t \\ \hat{S}_i(t-s) \leq M_i(t) - M_i(s) \leq C_i \cdot (t-s), \qquad i = 1, 2 \end{cases} \tag{11}$$

In (11), C_1 and C_2 represent the maximum line rates at network elements 1 and 2, while functions $\hat{S}_1$ and $\hat{S}_2$ give minimum guarantees on the service rates. With these assumptions, Cruz and Okino derive in [7] a service curve S_1 for network element 1, namely, they find an S_1 such that $x \geq S_1 \otimes a$. We give more detail about S_1 in Section 5 and show that it is obtained by min-plus algebra methods. Further, we are able to improve S_1.

3.4 Example 4: Losses in a shaper with finite buffer

In this example, one considers a shaper with a buffer X and a shaping curve σ, which is a system that forces an input flow $a(t)$ flow to have an output $y(t)$ which has σ as arrival curve, at the expense of possibly delaying bits in the buffer. The shaping curve is assumed to be sub-additive and such that $\sigma(0) = 0$ (If it is not the case, σ should be replaced by its sub-additive closure). If the traffic a enters the shaper, its output is $y = \sigma \otimes a$ [8, 9].

We suppose that the buffer is not large enough to avoid losses for all possible input traffic, and we would like to compute the amount of data lost at time t, with the convention that the system is empty at time $t = 0$. We model losses as shown in Figure 3, where $x(t)$ is the data that has actually entered the system in the time interval $[0, t]$. The amount of data lost during the same period is therefore $L(t) = a(t) - x(t)$. The amount of data $(x(t) - x(s))$ that actually entered the system in any time interval $(s, t]$ is always bounded above by the total amount of data $(a(t) - a(s))$ that has arrived in the system during the same period. Therefore, for any $0 \leq s \leq t$, $x(t) \leq x(s) + a(t) - a(s)$ or equivalently

$$x(t) \leq \inf_{u \text{ such that } 0 \leq u \leq t} \{x(u) + a(t) - a(u)\}. \tag{12}$$

On the other hand, x is the part of a that does actually enter the shaper, so the output of the shaper is $y = \sigma \otimes x$. There is no loss for $x(t)$ if $x(t) - y(t) \leq X$ for any t. Thus

$$x(t) \leq y(t) + X = (\sigma \otimes x)(t) + X \tag{13}$$

The data x that actually enters the system is therefore the maximum solution to (12) and (13).

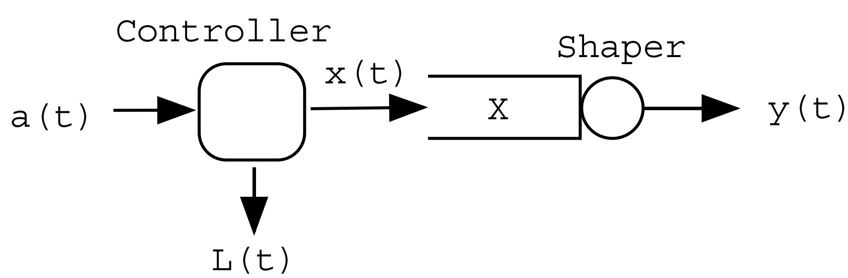

Figure 3: Example 4, shaper with losses

In this paper we will obtain an exact representation of L(t). Then we apply this representation to the case where the shaper represents a variable bit rate (VBR) trunk system. A VBR trunks system is a node which multiplexes a number of flows onto one variable bit rate connection. It can be modelled as a shaper where σ is the minimum of two affine functions. In that case, we are able to compare the amount of lost data $L(t)$, and in some case, the time spent in congestion ("congestion periods") to those of simpler, constant bit rate (CBR) systems. We expect these results to be used in deriving bounds for stochastic VBR trunk systems from known bounds applicable to CBR trunk systems. Stochastic bounds for CBR trunk systems can be found for example in [13].

3.5 Example 5: Losses in a shaper with delay constraints

This last example is similar to the previous one, except that now there is no finite buffer limit. The latter is replaced by a delay constraint: any entering data must have exited the system after at most d unit of times, otherwise it is discarded. Such discarded data will be called losses due to a delay constraint of d time units. As above, let x be the part of a that does actually enter the shaper, so the output of the shaper is $y = \sigma \otimes x$. All the data $x(t)$ that has entered the system during $[0, t]$ must therefore have left at time $t + d$ at the latest, so that $x(t) - y(t + d) \leq 0$ for any t. Thus

$$x(t) \leq y(t + d) = (\sigma \otimes x)(t + d). \tag{14}$$

On the other hand, as in the previous example, the amount of data $(x(t) - x(s))$ that actually entered the system in any time interval $(s, t]$ is always bounded above by the total amount of data $(a(t) - a(s))$ that has arrived in the system during the same period. Therefore the data x that actually enters the system is therefore the maximum solution to (12) and (14).

As with Example 4, we are able to obtain a representation of the flow of lost data which can be used to derive from CBR systems some bounds which are applicable to VBR trunk systems.

4 System Modelling

4.1 General Model

In this Section we show how to use concepts from min-plus algebra in order to model problems such as in the three examples.

For operators $\Pi : \mathcal{F}_J \to \mathcal{F}_J$ we define the following properties, which are direct applications of [10]:

Definition 1 ([10])

- Π *is* isotone *if* $\vec{x}(t) \leq \vec{y}(t)$ *for all* t *always implies* $\Pi(\vec{x})(t) \leq \Pi(\vec{y})(t)$ *for all* t.
- Π *is* causal *if for all* t, $\Pi(\vec{x})(t)$ *depends only on* $\vec{x}(s)$ *for* $0 \leq s \leq t$.
- Π *is* upper-semi-continuous *if for any decreasing sequence of trajectories* $(\vec{x}^i(t))_i$ *we have* $\inf_i \Pi(\vec{x}^i) = \Pi(\inf_i \vec{x}^i)$.
- Π *is* time-invariant *if* $\vec{y}(t) = \Pi(\vec{x})(t)$ *for all* t *and* $\vec{x}'(t) = \vec{x}(t + s)$ *for some* s *always implies that for all* t $\Pi(\vec{x}')(t) = \vec{y}(t + s)$.

  by

We propose to model network elements as *isotone, causal, upper-semicontinuous* operators. The first two properties are intuitive. The third one is a technical assumption required for the Theorem 1 to hold. It is however not a practical restriction.

For Π and Π' we note $\Pi \circ \Pi'$ the compound operator, defined by $(\Pi \circ \Pi')(\vec{x}) = \Pi[\Pi'(\vec{x})]$. We also note $\Pi^{(i)} = \Pi \circ \ldots \circ \Pi$ (i times, $i \geq 1$).

We will use the following definition.

Definition 2 ([10]) *The* closure $\overline{\Pi}$ *of the operator* Π *is defined by*

$$\overline{\Pi}(\vec{x}) = \inf \left\{ \vec{x}, \Pi(\vec{x}), \Pi^{(2)}(\vec{x}), \ldots, \Pi^{(i)}(\vec{x}), \ldots \right\}$$

Lastly, we write $\Pi \leq \Pi'$ to express that $\Pi(\vec{x}) \leq \Pi'(\vec{x})$ for all $\vec{x} \in \mathcal{F}_J$. The following propostion will be useful.

Proposition 1 *If* Π *and* Π' *are isotone and* $\Pi \leq \Pi'$ *then* $\overline{\Pi} \leq \overline{\Pi'}$

The proof is simple and left to the reader.

The main result from min-plus algebra which we use in this paper is the following.

Theorem 1 ([10], Theorem 4.70, item 6) *Let* Π *be an operator* $\mathcal{F}_J \to \mathcal{F}_J$, *and assume it is isotone and upper-semi-continuous. For any fixed function* $\vec{a} \in \mathcal{F}_J$, *the problem*

$$\vec{x} \leq \Pi(\vec{x}) \wedge \vec{a} \tag{15}$$

has one maximum solution, given by $\vec{x} = \overline{\Pi}(\vec{a})$

The theorem is proven in [10]. We give here a direct proof which does not require the pre-requisites in [10].

Proof: (i) Let us first show that $\overline{\Pi}(\vec{a})$ is a solution of (15). Consider the sequence $\{\vec{x}^n\}$ of decreasing sequences defined by

$$\begin{aligned} \vec{x}^0 &= \vec{a} \\ \vec{x}^{n+1} &= \vec{x}^n \wedge \Pi(\vec{x}^n), \qquad n \geq 0. \end{aligned}$$

Then one checks that

$$\vec{x}^\star = \inf_{n \geq 0} \{\vec{x}^n\}$$

is a solution to (15) because $\vec{x}^\star \leq \vec{x}^0 = \vec{a}$ and because Π is upper-semi-continuous so that

$$\Pi(\vec{x}^\star) = \Pi(\inf_{n\geq 0}\{\vec{x}^n\}) = \inf_{n\geq 0}\{\Pi(\vec{x}^n)\} \geq \inf_{n\geq 0}\{\vec{x}^{n+1}\} \geq \inf_{n\geq 0}\{\vec{x}^n\} = \vec{x}^\star.$$

Now, one easily checks that $\vec{x}^n = \inf_{0\leq m\leq n}\{\Pi^{(m)}(\vec{a})\}$, so

$$\vec{x}^\star = \inf_{n\geq 0}\{\vec{x}^n\} = \inf_{n\geq 0}\inf_{0\leq m\leq n}\{\Pi^{(m)}(\vec{a})\} = \inf_{n\geq 0}\{\Pi^{(n)}(\vec{a})\} = \overline{\Pi}(\vec{a}).$$

(ii) Let $\vec{x}$ be a solution of (15). Then $\vec{x} \leq \vec{a}$ and since Π is isotone, $\Pi(\vec{x}) \leq \Pi(\vec{a})$. From (15), $\vec{x} \leq \Pi(\vec{x})$, so that $\vec{x} \leq \Pi(\vec{a})$. Suppose that for some $n \geq 1$, we have shown that $\vec{x} \leq \Pi^{(n-1)}(\vec{a})$. Then as $\vec{x} \leq \Pi(\vec{x})$ and as Π is isotone, it yields that $\vec{x} \leq \Pi^{(n)}(\vec{a})$. Therefore $\vec{x} \leq \inf_{n\geq 0}\{\Pi^{(n)}(\vec{a})\} = \overline{\Pi}(\vec{a})$, which shows that $\vec{x}^\star = \overline{\Pi}(\vec{a})$ is the maximal solution. ■

4.2 Min-Plus linear operators

We also define *min-plus linear* operators:

Definition 3 ([10]) *Operator* Π *is* min-plus linear *if it is upper-semi-continuous and* $\Pi(\vec{x}+K) = \Pi(\vec{x}) + K$ *for all constant* K.

Min-plus operators are the equivalent in min-plus algebra of traditional linear system theory. In particular, it is shown in [10] that an operator is min-plus linear if and only if it can be represented under the form $\Pi(\vec{x})(t) = \inf_u \{H(t,u) + \vec{x}(u)\}$. H is called the matrix representation of the linear operator Π.

In general, a network element cannot be assumed to be a min-plus linear operator on its input. A notable exception is the case of shapers, which are linear and time invariant. We will see later that linear operators can be used to obtain bounds, even when the system is not linear.

The composition of operators translates into min-plus matrix multiplication, namely, if Π and Π' are linear, with matrices H and H', then the compound operator $\Pi \circ \Pi'$ is also linear, with matrix $H \circ H'$, defined by $(H \circ H')(t,s) = \inf_u\{H(t,u) + H'(u,s)\}$.

For a linear operator with matrix H, being causal is equivalent to $H_j(t,s) = \infty$ for $s > t$ and for all coordinates j. Being time-invariant is equivalent to $H_j(t,s) = H_j(t-s)$ for all coordinates j.

As with standard system theory, if Π is time invariant, causal and min-plus linear, then there exists some $\beta \in \mathcal{F}_J$ such that $\Pi(\vec{x}) = \beta \otimes \vec{x}$ [10, 8, 9]. We say that Π is the convolution by β operator and note $\Pi = \mathcal{C}_\beta$. So

$$\mathcal{C}_\beta(\vec{x})(t) = (\beta \otimes \vec{x})(t) = \inf_{u \text{ such that } 0 \le u \le t} \{\beta(t-u) + \vec{x}(u)\}.$$

One easily shows that $\mathcal{C}_\beta$ is isotone and upper-semi continuous. Note that in that case the closure $\overline{\Pi}$ is also time invariant, causal and min-plus linear, with $\overline{\Pi}(\vec{x}) = \overline{\beta} \otimes \vec{x}$. In the formula, $\overline{\beta}$ is the sub-additive closure of β [6, 8].

Two time-invariant, causal and min-plus linear operators Π and Π' commute: $\Pi \circ \Pi' = \Pi' \circ \Pi$.

We introduce the following linear, causal but non-time invariant family of operators, which we will use to model Examples 3, 4 and 5.

Definition 4 *For a given $\alpha \in \mathcal{F}_J$, define the min-plus linear operator h_α by*

$$h_\alpha(\vec{x})(t) = \inf_{u \text{ such that } 0 \le u \le t} \{\alpha(t) - \alpha(u) + \vec{x}(u)\}.$$

It can easily be shown that the h_α operators are isotone, causal and upper-semi-continuous. Moreover, they are idempotent, namely:

$$h_\alpha \circ h_\alpha = h_\alpha \tag{16}$$

and as we have $h_\alpha(\vec{x}) \le \vec{x}$

$$\overline{h_\alpha} = h_\alpha \tag{17}$$

4.3 The closure of the minimum of two operators

In the general case, the closure of the minimum of two operators can be represented from Definition 2 and the following lemma.

Lemma 1 (Representation of $(\Pi_1 \wedge \Pi_2)^{(n)}$) *If $\Pi = \Pi_1 \wedge \Pi_2$, then for any $n \ge 2$*

$$\Pi^{(n)} = \Pi_{11n} \wedge \Pi_{12n} \wedge \Pi_{21n} \wedge \Pi_{22n} \tag{18}$$

where, for $i \in \{1,2\}$,

$$\Pi_{ii,n} = \inf_{1 \le p \le n,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_p=n,\ k_1,\ldots,k_p \ge 1} \left\{ \Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)} \right\} \right\} \tag{19}$$

$$\Pi_{i(3-i),n} = \inf_{1 \le p \le n,\ p \text{ even}} \left\{ \inf_{k_1+\ldots+k_p=n,\ k_1,\ldots,k_p \ge 1} \left\{ \Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_i^{(k_2)} \circ \Pi_{3-i}^{(k_1)} \right\} \right\} \tag{20}$$

Proof: We will first show that the following recurrence relation holds: for any $i, j \in \{1, 2\}$ and any $n > 2$,

$$\Pi_{ij,n} = \Pi_i \circ \left[\Pi_{ij,n-1} \wedge \Pi_{(3-i)j,n-1}\right]. \tag{21}$$

We will only consider the case where $i = j$, as the case where $i = 3 - j$ being handled similarly.

Note first that, from (19) and (20), we have $\Pi_{ii,2} = \Pi_i^{(2)}$ and $\Pi_{i(3-i),2} = \Pi_i \circ \Pi_{3-i}$. For $n = 3$, one easily verifies that

$$\Pi_i \circ \left[\Pi_{ii,2} \wedge \Pi_{i(3-i),2}\right] = \Pi_i \circ \left[\Pi_i^{(2)} \wedge (\Pi_{3-i} \circ \Pi_i)\right] = \Pi_i^{(3)} \wedge [\Pi_i \circ \Pi_{3-i} \circ \Pi_i] = \Pi_{ii,3}.$$

The recurrence relation is thus verified for iterations 2 and 3. Suppose that we have verified that (21) holds until iteration $(n-1)$. Then

$$\begin{aligned}
&\Pi_i \circ \left[\Pi_{ii,n-1} \wedge \Pi_{(3-i)i,n-1}\right] \\
&= \Pi_i \circ \Bigg[\inf_{1\le p\le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_p=n-1,\ k_1,\ldots,k_p\ge 1} \left\{\Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} \\
&\qquad \wedge \inf_{1\le p\le n-1,\ p \text{ even}} \left\{ \inf_{k_1+\ldots+k_p=n-1,\ k_1,\ldots,k_p\ge 1} \left\{\Pi_{3-i}^{(k_p)} \circ \Pi_i^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\}\Bigg] \\
&= \inf_{1\le p\le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_p=n-1,\ k_1,\ldots,k_p\ge 1} \left\{\Pi_i^{(k_p+1)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} \\
&\qquad \wedge \inf_{1\le p\le n-1,\ p \text{ even}} \left\{ \inf_{k_1+\ldots+k_p=n-1,\ k_1,\ldots,k_p\ge 1} \left\{\Pi_i \circ \Pi_{3-i}^{(k_p)} \circ \Pi_i^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} \\
&= \inf_{1\le p\le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k'_p=n,\ k_1,\ldots,k_{p-1}\ge 1, k'_p\ge 2} \left\{\Pi_i^{(k'_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} \\
&\qquad \wedge \inf_{1\le p\le n-1,\ p \text{ even}} \left\{ \inf_{k_1+\ldots+k_p+k_{p+1}=n,\ k_1,\ldots,k_p\ge 1, k_{p+1}=1} \left\{\Pi_i^{(k_{p+1})} \circ \Pi_{3-i}^{(k_p)} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} \\
&= \inf_{1\le p\le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k'_p=n,\ k_1,\ldots,k_{p-1}\ge 1, k'_p\ge 2} \left\{\Pi_i^{(k'_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} \\
&\qquad \wedge \inf_{2\le p''\le n,\ p'' \text{ odd}} \left\{ \inf_{k_1+\ldots+k_{p''-1}+k_{p''}=n,\ k_1,\ldots,k_{p''-1}\ge 1, k_{p''}=1} \left\{\Pi_i^{(k_{p''})} \circ \Pi_{3-i}^{(k_{p''-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\}.
\end{aligned}$$

If n is odd, the latter expression becomes

$$\begin{aligned}
&\Pi_i \circ \left[\Pi_{ii,n-1} \wedge \Pi_{(3-i)i,n-1}\right] \\
&= \Pi_i^{(n)} \wedge \inf_{2\le p\le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_p=n,\ k_1,\ldots,k_{p-1}\ge 1, k_p\ge 2} \left\{\Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} \\
&\qquad \wedge \inf_{2\le p\le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_{p-1}+k_p=n,\ k_1,\ldots,k_{p-1}\ge 1, k_p=1} \left\{\Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} \\
&\qquad \wedge \{\Pi_i \circ \Pi_{3-i} \circ \ldots \Pi_{3-i} \circ \Pi_i\} \\
&= \inf_{1\le p\le n,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_p=n,\ k_1,\ldots,k_p\ge 1} \left\{\Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)}\right\}\right\} = \Pi_{ii,n}
\end{aligned}$$

whereas if n is even, it becomes

$$\begin{aligned}
&\Pi_i \circ \left[\Pi_{ii,n-1} \wedge \Pi_{(3-i)i,n-1}\right] \\
&\quad = \Pi_i^{(n)} \wedge \inf_{2 \le p \le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_p=n,\ k_1,\ldots,k_{p-1}\ge 1, k_p \ge 2} \left\{ \Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)} \right\} \right\} \\
&\qquad \wedge \inf_{2 \le p \le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_{p-1}+k_p=n,\ k_1,\ldots,k_{p-1}\ge 1, k_p = 1} \left\{ \Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \Pi_i^{(k_{p-2})} \circ \ldots \circ \Pi_i^{(k_1)} \right\} \right\} \\
&\quad = \inf_{1 \le p \le n-1,\ p \text{ odd}} \left\{ \inf_{k_1+\ldots+k_p=n,\ k_1,\ldots,k_p\ge 1} \left\{ \Pi_i^{(k_p)} \circ \Pi_{3-i}^{(k_{p-1})} \circ \ldots \circ \Pi_{3-i}^{(k_2)} \circ \Pi_i^{(k_1)} \right\} \right\} = \Pi_{ii,n}
\end{aligned}$$

which in all cases establishes the desired recurrence.

Now, suppose that (18) holds until iteration step $n-1$. Then

$$\begin{aligned}
\Pi^{(n)} &= (\Pi_1 \wedge \Pi_2) \circ \Pi^{(n-1)} = \left(\Pi_1 \circ \Pi^{(n-1)}\right) \wedge \left(\Pi_2 \circ \Pi^{(n-1)}\right) \\
&= (\Pi_1 \circ [\Pi_{11,n-1} \wedge \Pi_{12,n-1} \wedge \Pi_{21,n-1} \wedge \Pi_{22,n-1}]) \\
&\qquad \wedge (\Pi_2 \circ [\Pi_{11,n-1} \wedge \Pi_{12,n-1} \wedge \Pi_{21,n-1} \wedge \Pi_{22,n-1}]) \\
&= (\Pi_1 \circ [\Pi_{11,n-1} \wedge \Pi_{21,n-1}] \wedge \Pi_1 \circ [\Pi_{12,n-1} \wedge \Pi_{22,n-1}]) \\
&\qquad \wedge (\Pi_2 \circ [\Pi_{11,n-1} \wedge \Pi_{21,n-1}] \wedge \Pi_2 \circ [\Pi_{12,n-1} \wedge \Pi_{22,n-1}]) \\
&= (\Pi_{11,n} \wedge \Pi_{12,n}) \wedge (\Pi_{21,n} \wedge \Pi_{22,n}) = \Pi_{11,n} \wedge \Pi_{12,n} \wedge \Pi_{21,n} \wedge \Pi_{22,n}
\end{aligned}$$

which establishes the lemma. ∎

Note that following recurrence also holds for any $n > 2$ and $i, j \in \{1, 2\}$:

$$\Pi_{ij,n} = \left[\Pi_{ij,n-1} \wedge \Pi_{i(3-j),n-1}\right] \circ \Pi_j. \tag{22}$$

Now in the special case where $\Pi = \Pi_1 \wedge h_\alpha$, Lemma 1 leads to the following Theorem.

Theorem 2 *If* $\Pi = \Pi_1 \wedge h_\alpha$ *then*

$$\overline{\Pi} = \inf_{n \ge 1} \left\{ \inf_{1 \le q \le (n-1)/2} \left\{ \inf_{l_1+\ldots+l_q=n-q-1,\ l_1,\ldots,l_q \ge 1} \left\{ h_\alpha \circ \Pi_1^{(l_q)} \circ \ldots \circ h_\alpha \circ \Pi_1^{(l_1)} \circ h_\alpha \right\} \right\} \right\}. \tag{23}$$

Proof: Defining $\Pi_{ij,0}(x) = \Pi^{(0)}(x) = x$ for any $i, j \in \{1, 2\}$, and applying n times (22), we get

$$\begin{aligned}
\Pi_{12,n+1} &= [\Pi_{11,n} \wedge \Pi_{12,n}] \circ h_\alpha = \Pi_{11,n} \circ h_\alpha \wedge \Pi_{12,n} \circ h_\alpha \\
&= \Pi_{11,n} \circ h_\alpha \wedge [\Pi_{11,n-1} \wedge \Pi_{12,n-1}] = [\Pi_{11,n} \wedge \Pi_{11,n-1}] \circ h_\alpha \wedge \Pi_{12,n-1} \\
&= \ldots = \inf_{0 \le k \le n} \{\Pi_{11,k}\} \circ h_\alpha.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\Pi_{21,n+1} &= h_\alpha \circ \inf_{0 \le k \le n} \{\Pi_{11,k}\} \\
\Pi_{22,n+2} &= h_\alpha \circ \inf_{0 \le k \le n} \{\Pi_{11,k}\} \circ h_\alpha.
\end{aligned}$$

The three previous relations together with the fact that $h_\alpha(\vec{x}) \leq \vec{x}$ allows us to write that

$$\begin{aligned}
\overline{\Pi} &= \inf_{n\geq 0}\{\Pi^{(n)}\} = \inf_{n\geq 0}\{\Pi_{11,n} \wedge \Pi_{12,n} \wedge \Pi_{21,n} \wedge \Pi_{22,n}\} \\
&= \inf_{n\geq 0}\{\Pi_{11,n} \wedge \Pi_{12,n+1} \wedge \Pi_{21,n+1} \wedge \Pi_{22,n+2}\} \wedge \Pi_{12,0} \wedge \Pi_{21,0} \wedge \Pi_{22,0} \wedge \Pi_{22,1} \\
&= \inf_{n\geq 0}\{\Pi_{11,n} \wedge \inf_{0\leq k\leq n}\{\Pi_{11,k}\} \circ h_\alpha \wedge h_\alpha \circ \inf_{0\leq k\leq n}\{\Pi_{11,k}\} \wedge h_\alpha \circ \inf_{0\leq k\leq n}\{\Pi_{11,k}\} \circ h_\alpha\} \wedge \Pi_{22,0} \wedge \Pi_{22,1} \\
&= \inf_{n\geq 0}\{h_\alpha \circ \inf_{0\leq k\leq n}\{\Pi_{11,k}\} \circ h_\alpha\} \wedge \Pi_{22,0} \wedge \Pi_{22,1} \\
&= \inf_{n\geq 0}\{\Pi_{22,n+2}\} \wedge \Pi_{22,0} \wedge \Pi_{22,1} = \inf_{n\geq 0}\{\Pi_{22,n}\} \\
&= \inf_{n\geq 0}\left\{\inf_{1\leq p\leq n,\ p \text{ odd}}\left\{\inf_{k_2+\ldots+k_{p-1}=n-\frac{p+1}{2},\ k_2,\ldots,k_{p-1}\geq 1}\left\{h_\alpha \circ \Pi_1^{(k_{p-1})} \circ \ldots \circ h_\alpha \circ \Pi_1^{(k_2)} \circ h_\alpha\right\}\right\}\right\}.
\end{aligned}$$

■

4.4 Examples 1 to 5 as min-plus systems

We now show how Examples 1 to 5 can be represented by the problem in Theorem 1.

In **Example 1**, $J = 1$ and φ is the operator mapping $a(t)$ to $x(t)$ such that $x(t) = \varphi(a)(t)$ is the maximum solution of

$$x \leq a \wedge \sigma_0 \wedge (\sigma \otimes x). \tag{24}$$

Consider now **Example 2**. Here $J = 1$. Define Π as the operator that maps $x(t)$ to $y(t)$. From Equation (6), we derive that $x(t)$ is the maximum solution to

$$x \leq a \wedge (\Pi(x) + K) \tag{25}$$

The operator Π can be assumed to be isotone, causal and upper-semi-continuous, but not necessarily linear. However, we know that $\Pi \geq \mathcal{C}_\beta$. We will exploit this formulation in Section 5.

Consider next **Example 3**. Define Π_f as the one-dimensional operator that maps $x(t)$ to $y_1(t)$ and Π_b the one-dimensional operator that maps $y_2(t)$ to $y(t)$. From equation (9), we derive that $(x(t), y_2(t))$ is the maximum sequence such that

$$\left\{\begin{array}{lll} x & \leq & a \wedge (\beta \otimes x) \wedge h_{M_1}(x) \wedge (\Pi_b(y_2) + K) \\ y_2 & \leq & \Pi_f(x) \wedge h_{M_2}(y_2). \end{array}\right. \tag{26}$$

In Theorem 1 we have shown the existence of such a maximum. Here we have thus $J = 2$. Let $\vec{z} = (x, y_2)$ and define the non-linear operator $\vec{\Pi}(\vec{z}) = (\Pi_1(\vec{z}), \Pi_2(\vec{z}))$ by

$$\left\{\begin{array}{lll} \Pi_1(\vec{z}) & = & (\beta \otimes x) \wedge h_{M_1}(x) \wedge (\Pi_b(y_2) + K) \\ \Pi_2(\vec{z}) & = & \Pi_f(x) \wedge h_{M_2}(y_2). \end{array}\right.$$

The problem in Example 3 is thus equivalent to finding the maximum solution to the problem $\vec{z} \leq \vec{\Pi}(\vec{z}) \wedge (a, \delta)$.

In **Example 4**, $J = 1$ again and now all operators are linear. We know that $x \leq a$. Combining this relation with (12) and (13), we derive that x is the maximum solution to

$$x \leq a \wedge h_a(x) \wedge (\sigma \otimes x + X).$$

Denoting

$$\mathcal{C}_{\sigma_X}(x) = (\sigma \otimes x) + X, \tag{27}$$

this inequality becomes

$$x \leq a \wedge (h_a \wedge \mathcal{C}_{\sigma_X})(x). \tag{28}$$

Example 5 is similar to Example 4: $J = 1$ and combining $x \leq a$ with (12) and (14), we derive that x is the maximum solution to

$$x(t) \leq a(t) \wedge h_a(x)(t) \wedge (\sigma \otimes x)(t + d). \tag{29}$$

We can recast this equation using the operator

$$\mathcal{C}_{\sigma^d}(x)(t) = \inf_{0 \leq s \leq t} \{\sigma(t + d - s) + x(s)\}. \tag{30}$$

and the following lemma.

Lemma 2 *For all* $t \geq 0$

$$x(t) \leq (\sigma \otimes x)(t + d)$$

if and only if for all $t \geq 0$

$$x(t) \leq \mathcal{C}_{\sigma^d}(x)(t).$$

Proof:

($\Rightarrow$) Note that if $x(t) \leq (\sigma \otimes x)(t + d)$ for all $t \geq 0$ then

$$x(t) \leq \inf_{0 \leq s \leq t+d} \{\sigma(t + d - s) + x(s)\} \leq \inf_{0 \leq s \leq t} \{\sigma(t + d - s) + x(s)\} = \mathcal{C}_{\sigma^d}(x)(t).$$

($\Leftarrow$) Suppose now that $x(t) \leq \mathcal{C}_{\sigma^d}(x)(t)$. Let

$$s^\star = \arg \inf_{0 \leq s \leq t+d} \{\sigma(t + d - s) + x(s)\}.$$

(i) Suppose that $0 \leq s^\star \leq t$. Then

$$(\sigma \otimes x)(t + d) = \inf_{0 \leq s \leq t+d} \{\sigma(t + d - s) + x(s)\} = \inf_{0 \leq s \leq t} \{\sigma(t + d - s) + x(s)\} = \mathcal{C}_{\sigma^d}(x)(t) \geq x(t).$$

(ii) Suppose next that $t < s^\star \leq t + d$. Then $x(s^\star) \geq x(t)$ because $s^\star > t$. Since $\sigma(t + d - s^\star) \geq 0$, this implies that

$$(\sigma \otimes x)(t + d) = \sigma(t + d - s^\star) + x(s^\star) \geq 0 + x(t) = x(t).$$

Consequently $x(t) \leq (\sigma \otimes x)(t + d)$ in both cases (i) and (ii). ∎

Equation (29) can therefore be recast as

$$x \leq a \wedge (h_a \wedge \mathcal{C}_{\sigma^d})(x). \tag{31}$$

5 Solution to Examples 1 to 3 (Lossless shaping and window flow control)

5.1 Example 1

From Theorem 1, the maximal solution to (24) is

$$x = \overline{\sigma \otimes (a \wedge \sigma_0)}. \tag{32}$$

As σ and σ_0 are concave and increasing, we can compute that for all $t \geq 0$ and $n \geq 1$

$$\begin{aligned} \sigma^{(n)}(t) &= \sigma(t) + (n-1)\sigma(0) \\ (\sigma \otimes \sigma_0)(t) &= (\sigma(t) + \sigma_0(0)) \wedge (\sigma_0(t) + \sigma(0)). \end{aligned}$$

Hence (32) becomes

$$\begin{aligned} x &= \inf_{n \geq 0} \left\{ \sigma^{(n)} \otimes (a \wedge \sigma_0) \right\} \\ &= a \wedge \sigma_0 \wedge \inf_{n \geq 1} \left\{ (\sigma + (n-1)\sigma(0)) \otimes (a \wedge \sigma_0) \right\} \\ &= a \wedge \sigma_0 \wedge \inf_{n \geq 1} \left\{ (\sigma \otimes a) \wedge (\sigma \wedge \sigma_0) + (n-1)\sigma(0) \right\} \\ &= a \wedge \sigma_0 \wedge \sigma \otimes a \wedge \sigma \otimes \sigma_0 \\ &= a \wedge \sigma_0 \wedge \sigma \otimes a \wedge (\sigma + \sigma_0(0)) \wedge (\sigma_0 + \sigma(0)) \wedge \sigma_0 \end{aligned}$$

Now $\sigma(t) + \sigma_0(0) \geq \sigma(t) \geq \sigma_0(t)$ and $\sigma_0(t) + \sigma(0) \geq \sigma_0(t)$ so that (32) eventually becomes

$$\varphi(a) = x = a \wedge \sigma_0 \wedge \sigma \otimes a. \tag{33}$$

We show now that the solution (5), found using reflection mappings is identical to (33). Indeed, in (4), $N^\rho(t)$ can take two values: 0 or $-\inf_{0 \leq s \leq t}\{z^\rho(s)\}$.
If $N^\rho(t) = 0$, then (5) becomes

$$\varphi(a)(t) = a(t) \wedge \inf_{\rho \geq 0} \{\sigma_0^\star(\rho) + \rho t\} = a(t) \wedge \sigma_0(t).$$

Conversely, if $N^\rho(t) = -\inf_{0 \leq s \leq t}\{z^\rho(s)\}$, then

$$\begin{aligned} \varphi(a)(t) &= a(t) \wedge \inf_{\rho \geq 0} \left\{ \sigma_0^\star(\rho) + \rho t + \inf_{0 \leq s \leq t} \{\sigma^\star(\rho) - \sigma_0^\star(\rho) + a(s) - \rho s\} \right\} \\ &= a(t) \wedge \inf_{\rho \geq 0} \left\{ \inf_{0 \leq s \leq t} \{\rho t + \sigma^\star(\rho) + a(s) - \rho s\} \right\} \\ &= a(t) \wedge \inf_{0 \leq s \leq t} \left\{ \inf_{\rho \geq 0} \{\sigma^\star(\rho) + \rho(t-s)\} + a(s) \right\} \\ &= a(t) \wedge \left\{ \inf_{0 \leq s \leq t} \sigma(t-s) + a(s) \right\} = a(t) \wedge (\sigma \otimes a)(t). \end{aligned}$$

Combining both results, we get (33).

5.2 Example 2

We know now that (25) has one maximum solution and that it is given by

$$x(t) = \overline{(\Pi + K)}(a)(t)$$

Now from (7) we have $\Pi(x) + K \geq \beta \otimes x + K$. From Lemma 1, we have:

$$x \geq \overline{(\beta + K)} \otimes a$$

which is Equation (8).

5.3 Example 3

It follows easily from the problem formulation in (26) and the isotony of the operators that if $(x(t), y_2(t))$ is the maximum solution to (26), then $y_2(t)$ is also the maximum the solution to

$$y_2 \leq \Pi_f(x) \wedge h_{M_2}(y_2)$$

and thus, by application of Theorem 1,

$$y_2 = \overline{h_{M_2}} \circ \Pi_f(x)$$

and thus from (17)

$$y_2 = h_{M_2} \circ \Pi_f(x)$$

and thus $x(t)$ is a solution to the one-dimensional problem

$$x \leq a \wedge (\alpha \otimes x) \wedge h_{M_1}(x) \wedge (\Pi_b \circ h_{M_2} \circ \Pi_f(x) + K) \tag{34}$$

Thus we can conclude again from Theorem 1 that

$$x = \overline{Q}(a)$$

with the operator Q defined by

$$Q(x) = (\alpha \otimes x) \wedge h_{M_1}(x) \wedge (\Pi_b \circ h_{M_2} \circ \Pi_f(x) + K)$$

¿From (10) and (11) we can now bound Q from below by

$$Q(x) \geq (G \otimes x) \wedge h_{M_1}(x) \tag{35}$$

with G defined as follows [7]. First let $G_0 = (S_b \otimes \hat{S}_2 \otimes S_f + K) \wedge \alpha$. Then define

$$\Delta = \inf\{t \geq 0 : G_0(t) \leq C_1 t\}$$

and let $G(t) = G_0(t)$ if $t \geq \Delta$ and $G(t) = G_0(\Delta)$ otherwise.
Equation (35) can be re-written $Q \geq \mathcal{C}_G \wedge h_{M_1}$. Thus, from Lemma 1,

$$\overline{Q} \geq \overline{\mathcal{C}_G \wedge h_{M_1}} \tag{36}$$

and from Lemma 2,

$$\overline{\mathcal{C}_G \wedge h_{M_1}} = \inf_{n\geq 1}\left\{\inf_{1\leq q\leq (n-1)/2}\left\{\inf_{l_1+\ldots+l_q=n-q-1,\, l_1,\ldots,l_q\geq 1}\left\{h_{M_1}\circ \mathcal{C}_G^{(l_q)}\circ\ldots\circ\mathcal{C}_G^{(l_1)}\circ h_{M_1}\right\}\right\}\right\}. \tag{37}$$

By using the bound

$$h_{M_1}(x)\geq \hat{S}_1\otimes x$$

and the fact that operators $\mathcal{C}_G$ and $\mathcal{C}_{\hat{S}_1}$ are isotone and commute (because they are linear and time-invariant), the combination of (36) and (37) yields that

$$\begin{aligned}
\overline{Q} &\geq \overline{\mathcal{C}_G \wedge h_{M_1}}\\
&\geq \inf_{n\geq 1}\left\{\inf_{1\leq q\leq (n-1)/2}\left\{\inf_{l_1+\ldots+l_q=n-q-1,\, l_1,\ldots,l_q\geq 1}\left\{\mathcal{C}_{\hat{S}_1}\circ \mathcal{C}_G^{(l_q)}\circ\ldots\circ\mathcal{C}_G^{(l_1)}\circ \mathcal{C}_{\hat{S}_1}\right\}\right\}\right\}\\
&= \inf_{n\geq 1}\left\{\inf_{1\leq q\leq (n-1)/2}\left\{\inf_{l_1+\ldots+l_q=n-q-1,\, l_1,\ldots,l_q\geq 1}\left\{\mathcal{C}_G^{(l_q+\ldots+l_1)}\circ \mathcal{C}_{\hat{S}_1}^{(q+1)}\right\}\right\}\right\}\\
&= \inf_{n\geq 1}\left\{\inf_{1\leq q\leq (n-1)/2}\left\{\mathcal{C}_G^{(n-q-1)}\circ \mathcal{C}_{\hat{S}_1}^{(q+1)}\right\}\right\}\\
&= \inf_{k\geq 0}\left\{\inf_{0\leq q\leq k}\left\{\mathcal{C}_G^{(2k-q)}\circ \mathcal{C}_{\hat{S}_1}^{(q+1)}\right\}\wedge \inf_{0\leq q\leq k-1}\left\{\mathcal{C}_G^{(2k-q-1)}\circ \mathcal{C}_{\hat{S}_1}^{(q+1)}\right\}\right\}\\
&= \inf_{k\geq 0}\left\{\inf_{0\leq q\leq k}\left\{\mathcal{C}_G^{(2k-q)}\circ \mathcal{C}_{\hat{S}_1}^{(q+1)}\right\}\right\}\\
&= \inf_{k\geq 0}\left\{\left\{\mathcal{C}_G^{(k)}\circ \mathcal{C}_{\hat{S}_1}^{(k+1)}\right\}\wedge\left\{\mathcal{C}_G^{(k+1)}\circ \mathcal{C}_{\hat{S}_1}^{(k)}\right\}\wedge\ldots\wedge\left\{\mathcal{C}_G^{(2k)}\circ \mathcal{C}_{\hat{S}_1}^{(1)}\right\}\right\}\\
&\geq \inf_{k\geq 0}\left\{\left\{\mathcal{C}_G^{(k)}\circ \mathcal{C}_{\hat{S}_1}^{(k+1)}\right\}\wedge\left\{\mathcal{C}_G^{(k+1)}\circ \mathcal{C}_{\hat{S}_1}^{(k+2)}\right\}\wedge\ldots\wedge\left\{\mathcal{C}_G^{(2k)}\circ \mathcal{C}_{\hat{S}_1}^{(2k+1)}\right\}\right\}\\
&= \inf_{k\geq 0}\left\{\mathcal{C}_G^{(k)}\circ \mathcal{C}_{\hat{S}_1}^{(k+1)}\right\}.
\end{aligned}$$

This shows that a service curve for network element 1 on Figure 2 is given by

$$S_1(t) = \inf_{k\geq 0}\left\{G^{(k)}\otimes \hat{S}_1^{(k+1)}\right\}(t) \tag{38}$$

which is precisely the service curve found in [7].

We are now also able to derive another bound. Since $\overline{\mathcal{C}_G \wedge h_{M_1}}$ is linear,

$$\overline{\mathcal{C}_G \wedge h_{M_1}}(x)(t) = \inf_s\{\overline{H}(t,s)+x(s)\} \tag{39}$$

with, because of equation (37), $\overline{H}(t,s)$ being such that

$$\begin{aligned}
\overline{H}(t,s) \geq\ & M_1(t)-M_1(u_{2q})+G^{(k_q)}(u_{2q}-u_{2q-1})+M_1(u_{2q-1})-M_1(u_{2q-2})\\
&+G^{(k_{q-1})}(u_{2q-2}-u_{2q-3})+\ldots+M_1(u_3)-M_1(u_2)+G^{(k_1)}(u_2-u_1)+M_1(u_1)-M_1(s)
\end{aligned}$$

for some integer $q>1$ and some sequences of times $s\leq u_1\leq u_2\leq\ldots\leq u_{2q}\leq t$ and of integers $k_1,\ldots,k_q$. Now from (11) we have

$$M_1(t)-M_1(u_{2q})+M_1(u_{2q-1})-M_1(u_{2q-2})+\ldots+M_1(u_1)-M_1(s))\geq M_1(t)-M_1(s)-C_1u$$

where $u = u_{2q} - u_{2q-1} + \ldots + u_2 - u_1$. This shows that

$$\overline{H}(t,s) \geq \hat{S}_1(t-s) + \inf_{n \geq 0, 0 \leq u \leq t-s} \{G^{(n)}(u) - C_1 u\}.$$

Define

$$S_e(v) = \hat{S}_1(v) + \inf_{n \geq 0, 0 \leq u \leq v} \{G^{(n)}(u) - C_1 u\}. \tag{40}$$

We have shown that a service curve for network element 1 is S_e. In general, S_e is better if the delay introduced by the feedback loop in Figure 2 is large compared to the delay parameter of $\hat{S}_1$. Figure 4 shows the values of S_e and S_1 for one example.

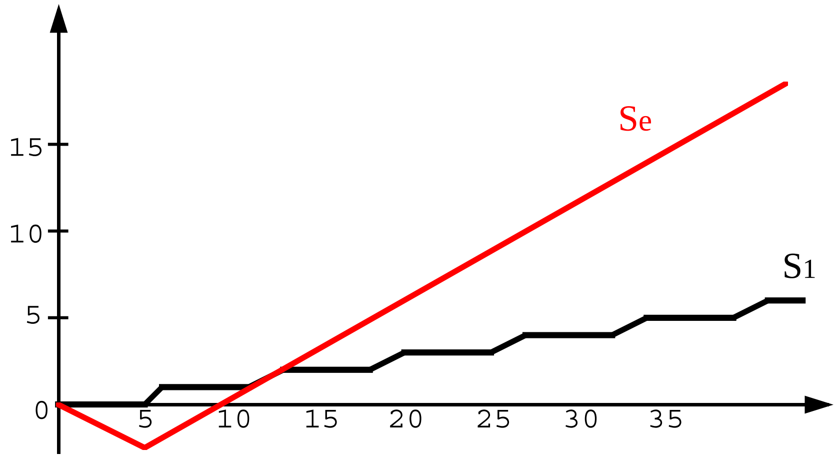

Figure 4: The service curves S_e and S_1 on one example. Here $K = \infty$, $\alpha(t) = \{C_1 t\} wedge \{b + rt\}$, $\hat{S}_1(t) = C_1[t - T_1]^+$, with $T_1 = 5$, $C_1 = 1$, $r = 0.5$ and $b = 0.5$.

6 Solutions to Examples 4 and 5 (Lossy systems)

6.1 Example 4

From Theorem 1, the maximal solution of (28) is

$$x = \overline{h_a \wedge \mathcal{C}_{\sigma_X}}(a) \tag{41}$$

Lemma 2 yields that

$$\begin{aligned}
\overline{h_a \wedge \mathcal{C}_{\sigma_X}} &= \inf_{n \geq 1} \left\{ \inf_{1 \leq q \leq (n-1)/2} \left\{ \inf_{l_1 + \ldots + l_q = n-q-1,\ l_1, \ldots, l_q \geq 1} \left\{ h_a \circ \mathcal{C}_{\sigma_X}^{(l_q)} \circ \ldots \circ h_a \circ \mathcal{C}_{\sigma_X}^{(l_1)} \circ h_a \right\} \right\} \right\} \\
&= \inf_{n \geq 1, n odd} \left\{ \left\{ \inf_{l_1 + \ldots + l_{(n-1)/2} = (n-1)/2,\ l_1, \ldots, l_{(n-1)/2} \geq 1} \left\{ h_a \circ \mathcal{C}_{\sigma_X}^{(l_{(n-1)/2})} \circ \ldots \circ h_a \circ \mathcal{C}_{\sigma_X}^{(l_1)} \circ h_a \right\} \right\} \right. \\
&\quad \left. \wedge \inf_{1 \leq q \leq (n-3)/2} \left\{ \inf_{l_1 + \ldots + l_q = n-q-1,\ l_1, \ldots, l_q \geq 1} \left\{ h_a \circ \mathcal{C}_{\sigma_X}^{(l_q)} \circ \ldots \circ h_a \circ \mathcal{C}_{\sigma_X}^{(l_1)} \circ h_a \right\} \right\} \right\} \\
&\quad \wedge \inf_{n \geq 1, n \text{ even}} \left\{ \inf_{1 \leq q \leq (n-2)/2} \left\{ \inf_{l_1 + \ldots + l_q = n-q-1,\ l_1, \ldots, l_q \geq 1} \left\{ h_a \circ \mathcal{C}_{\sigma_X}^{(l_q)} \circ \ldots \circ h_a \circ \mathcal{C}_{\sigma_X}^{(l_1)} \circ h_a \right\} \right\} \right\}.
\end{aligned}$$

In this latter expression, all the indices $l_1, \ldots, l_{(n-1)/2}$ of the first term must all be equal to 1, because their sum is $(n-1)/2$. Conversely, at least one index among $l_1, \ldots, l_q$ in both the second and third terms must be strictly larger than 1, because their sum always exceeds q. Now, for any integer $k \geq 1$, the sub-additivity of σ yields that

$$\mathcal{C}_{\sigma_X}^{(k)}(r) = \sigma \otimes \sigma \otimes \ldots \otimes \sigma \otimes r + kX = \sigma \otimes r + kX = \mathcal{C}_{\sigma_{kX}}(r) \geq \mathcal{C}_{\sigma_X}(r)$$

so that

$$h_a \circ \mathcal{C}_{\sigma_X}^{(l_q)} \circ \ldots \circ h_a \circ \mathcal{C}_{\sigma_X}^{(l_1)} \circ h_a \geq h_a \circ \mathcal{C}_{\sigma_X} \circ \ldots \circ h_a \circ \mathcal{C}_{\sigma_X} \circ h_a = (h_a \circ \mathcal{C}_{\sigma_X})^{(q)} \circ h_a.$$

Therefore we have that

$$x = \overline{h_a \wedge \mathcal{C}_{\sigma_X}}(a) = \inf_{n \geq 1, n \text{ odd}} \left\{ (h_a \circ \mathcal{C}_{\sigma_X})^{((n-1)/2)} \circ h_a \right\}(a) = \inf_{k \geq 0} \left\{ (h_a \circ \mathcal{C}_{\sigma_X})^{(k)} \right\}(a) = \overline{h_a \circ \mathcal{C}_{\sigma_X}}(a),$$

where we used the fact that $h_a(a) = a$.

The amount of lost data in the interval $[0, t]$ is therefore given by

$$\begin{aligned}
L(t) &= a(t) - x(t) = a(t) - \inf_{k \geq 0} \left\{ (h_a \circ \mathcal{C}_{\sigma_X})^{(k)} \right\}(a)(t) \\
&= \sup_{k \geq 0} \left\{ a(t) - (h_a \circ \mathcal{C}_{\sigma_X})^{(k)}(a)(t) \right\} \\
&= \sup_{k \geq 0} \left\{ a(t) - \inf_{0 \leq s_{2k} \leq \ldots \leq s_2 \leq s_1 \leq t} \{ a(t) - a(s_1) + \sigma(s_1 - s_2) + X + a(s_2) - \ldots + a(s_{2k}) \} \right\} \\
&= \sup_{k \geq 0} \left\{ \sup_{0 \leq s_{2k} \leq \ldots \leq s_2 \leq s_1 \leq t} \{ a(s_1) - \sigma(s_1 - s_2) - a(s_2) + \ldots - a(s_{2k}) - kX \} \right\} \\
&= \sup_{k \geq 0} \left\{ \sup_{0 \leq s_{2k} \leq \ldots \leq s_2 \leq s_1 \leq t} \left\{ \sum_{i=1}^{k} [a(s_{2i-1}) - a(s_{2i}) - \sigma(s_{2i-1} - s_{2i})] \right\} - kX \right\}.
\end{aligned} \quad (42)$$

If we know that the arriving traffic a is constrained by an arrival curve $\alpha(\cdot)$, we can also bound the amount of lost data by

$$L(t) \leq \sup_{k \geq 0} \left\{ \sup_{u_1, \ldots, u_k \geq 0;\ \sum_{i=1}^{k} u_i \leq t} \left\{ \sum_{i=1}^{k} [\alpha(u_i) - \sigma(u_i)] \right\} - kX \right\}. \quad (43)$$

Let us now apply (42) to bound the losses in a buffered shaper by the losses in simpler systems. The first application deals with a VBR shaper, which is compared with two CBR shapers. The second application is the bound of the losses in a shaper by a system that segregates the resources (buffer, bandwidth) between a storage system and a policer. For both applications, the losses in the original shaper are bounded along every sample path by the losses in the simpler systems. For congestion times however, the same conclusion holds only for the first application.

6.1.1 Application 1: Bound on the losses in a VBR shaper

As a particular application we will show how it is possible to bound the losses in a shaper, with a somewhat complex shaping curve σ, by losses in simpler systems. Take the example of a "buffered leaky bucket" shaper [6] with buffer X, whose output must conform to a VBR shaping curve with peak rate P, sustainable rate M and burst tolerance B so that $\sigma_{\text{VBR}}(t) = \{Pt\} \wedge \{Mt + B\}$. We will consider

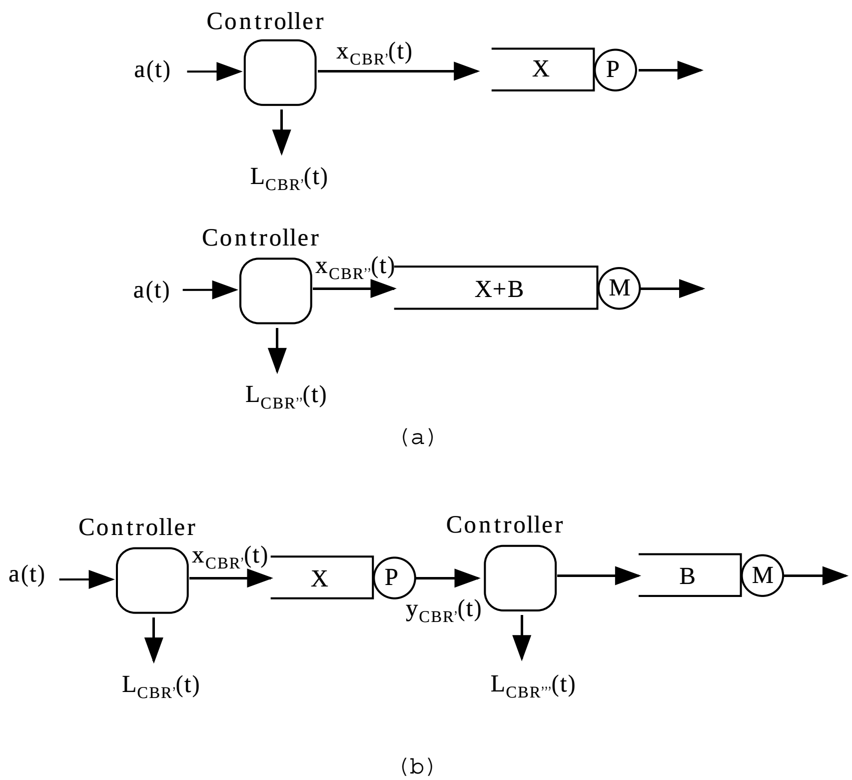

Figure 5: Two CBR shapers in parallel (a) and in tandem (b).

two systems to bound these losses: first two CBR shapers in parallel (Figure 5(a)) and second two CBR shapers in tandem (Figure 5(b)).

i) Bound by two CBR shapers in parallel

We will first show that the *amount of losses* during $[0, t]$ in this system is bounded by the sum of losses in two CBR shapers in parallel, as shown in Figure 5(a): the first one has buffer of size X and rate P, whereas the second one has buffer of size $X + B$ and rate M. Both receive the same arriving traffic a as the original VBR shaper. All variables without prime refer to the original VBR shaper, variable with one prime to the first CBR shaper and variables with a double prime to the second CBR shaper.

Theorem 3 *Let $L_{VBR}(t)$ be the amount of lost data in the time interval $[0, t]$ in a VBR shaper with buffer X and shaping curve $\sigma_{VBR}(t) = \{Pt\} \wedge \{Mt + B\}$, when the data that has arrived in $[0, t]$ is $a(t)$.*

Let $L_{CBR'}(t)$ (resp. $L_{CBR''}(t)$) be the amount of lost data during $[0, t]$ in a CBR shaper with buffer X (resp. $(X + B)$) and shaping curve $\sigma_{CBR'}(t) = Pt$ (resp. $\sigma_{CBR''}(t) = Mt$) with the same incoming traffic $a(t)$.

Then $L_{VBR}(t) \leq L_{CBR'}(t) + L_{CBR''}(t)$.

Proof: Define

$$l^{(k)}(t) = \sup_{0 \leq s_{2k} \leq \ldots \leq s_2 \leq s_1 \leq t} \left\{ \sum_{i=1}^{k} [a(s_{2i-1}) - a(s_{2i}) - \sigma(s_{2i-1} - s_{2i})] \right\} - kX \qquad (44)$$

so that (42) can be rewritten as $L(t) = \sup_{k\geq 0}\{l^{(k)}(t)\}$. Note that relation (44) can be recast recursively as

$$l^{(k)}(t) = \sup_{0\leq s_2\leq s_1\leq t}\left\{a(s_1) - a(s_2) - \sigma(s_1 - s_2) - X + l^{(k-1)}(s_2)\right\},$$

which motivates a proof by induction on k, as follows.

We will show by induction that

$$l^{(k)}_{\mathrm{VBR}}(t) \leq L_{\mathrm{CBR}'}(t) + L_{\mathrm{CBR}''}(t). \tag{45}$$

Clearly, this relation holds (with equality sign) for $k = 0$, as in this case the left hand side of (45) is zero. One easily shows that this relation holds (again with equality sign) for $k = 1$. Indeed,

$$\begin{aligned}
l^{(1)}_{\mathrm{VBR}}(t) &= \sup_{0\leq s_2\leq s_1\leq t}\{a(s_1) - a(s_2) - \sigma_{\mathrm{VBR}}(s_1 - s_2) - X\} \\
&= \sup_{0\leq s_2\leq s_1\leq t}\{a(s_1) - a(s_2) - P(s_1 - s_2) - X\} \\
&\qquad \vee \sup_{0\leq s_2\leq s_1\leq t}\{a(s_1) - a(s_2) - M(s_1 - s_2) - B - X\} \\
&\leq L_{\mathrm{CBR}'}(t) + L_{\mathrm{CBR}''}(t).
\end{aligned}$$

Suppose that (45) holds until iteration k. Then for the VBR system we can be write that

$$\begin{aligned}
l^{(k+1)}_{\mathrm{VBR}}(t) &= \sup_{0\leq s_2\leq s_1\leq t}\left\{a(s_1) - a(s_2) - \sigma_{\mathrm{VBR}}(s_1 - s_2) - X + l^{(k)}_{\mathrm{VBR}}(s_2)\right\} \\
&= \sup_{0\leq s_2\leq s_1\leq t}\left\{a(s_1) - a(s_2) - P(s_1 - s_2) - X + l^{(k)}_{\mathrm{VBR}}(s_2)\right\} \\
&\qquad \vee \sup_{0\leq s_2\leq s_1\leq t}\left\{a(s_1) - a(s_2) - M(s_1 - s_2) - B - X + l^{(k)}_{\mathrm{VBR}}(s_2)\right\} \\
&\leq \sup_{0\leq s_2\leq s_1\leq t}\{a(s_1) - a(s_2) - P(s_1 - s_2) - X + L_{\mathrm{CBR}'}(s_2) + L_{\mathrm{CBR}''}(s_2)\} \\
&\qquad \vee \sup_{0\leq s_2\leq s_1\leq t}\{a(s_1) - a(s_2) - M(s_1 - s_2) - B - X + L_{\mathrm{CBR}'}(s_2) + L_{\mathrm{CBR}''}(s_2)\} \\
&\leq \left\{\sup_{0\leq s_2\leq s_1\leq t}\left\{a(s_1) - a(s_2) - P(s_1 - s_2) - X + \sup_{k\geq 0}\{l^{(k)}_{\mathrm{CBR}'}(s_2)\}\right\} + \sup_{0\leq s_2\leq s_1\leq t}\{L_{\mathrm{CBR}''}(s_2)\}\right\} \\
&\qquad \vee \left\{\sup_{0\leq s_2\leq s_1\leq t}\left\{a(s_1) - a(s_2) - M(s_1 - s_2) - B - X + \sup_{k\geq 0}\{l^{(k)}_{\mathrm{CBR}''}(s_2)\}\right\} + \sup_{0\leq s_2\leq s_1\leq t}\{L_{\mathrm{CBR}'}(s_2)\}\right\} \\
&= \left\{\sup_{k\geq 0}\left\{\sup_{0\leq s_2\leq s_1\leq t}\{a(s_1) - a(s_2) - P(s_1 - s_2) - X + l^{(k)}_{\mathrm{CBR}'}(s_2)\}\right\} + L_{\mathrm{CBR}''}(t)\right\} \\
&\qquad \vee \left\{\sup_{k\geq 0}\left\{\sup_{0\leq s_2\leq s_1\leq t}\{a(s_1) - a(s_2) - M(s_1 - s_2) - B - X + l^{(k)}_{\mathrm{CBR}'}(s_2)\}\right\} + L_{\mathrm{CBR}'}(t)\right\} \\
&= \left\{\sup_{k\geq 0}\{l^{(k+1)}_{\mathrm{CBR}'}(t)\} + L_{\mathrm{CBR}''}(t)\right\} \vee \left\{\sup_{k\geq 0}\{l^{(k+1)}_{\mathrm{CBR}''}(t)\} + L_{\mathrm{CBR}'}(t)\right\} \\
&\leq L_{\mathrm{CBR}'}(t) + L_{\mathrm{CBR}''}(t).
\end{aligned}$$

Therefore (45) holds for $k + 1$, so that

$$L_{\mathrm{VBR}}(t) = \sup_{k\geq 0}\{l^{(k)}_{\mathrm{VBR}}(t)\} \leq L_{\mathrm{CBR}'}(t) + L_{\mathrm{CBR}''}(t),$$

which proves the theorem. ■

We can not only show that the *amount of losses* in the VBR system are bounded by the sum of the amounts of losses in the two CBR systems, but we will also show that the *congestion periods* in the VBR system, that is, the time intervals during which the VBR system suffers losses, is bounded by the sum of times during which at least one of the two CBR shapers is also congested.

To prove this result, we will make use of the following lemma:

Lemma 3 *Let $W_{VBR}(t)$, $W_{CBR'}(t)$ and $W_{CBR''}(t)$ denote respectively the buffer contents of the VBR shaper and the two CBR shapers at time t. Then*

(i) $W_{CBR'}(t) \leq W_{VBR}(t)$,

(ii) $W_{CBR''}(t) \geq \sup_{0\leq s\leq t}\{x_{VBR}(t) - x_{VBR}(s) - M(t-s)\}$.

Proof: (i) Let t be a given time, and let $0 \leq v \leq t$ be the smallest time such that the VBR system has no loss in the system in $[v, t]$. The traffic that actually entered the VBR system during $[v, t]$ is therefore identical to the traffic that arrived during this time interval: $x_{\mathrm{VBR}}(t) - x_{\mathrm{VBR}}(v) = a(t) - a(v)$.

If $v = 0$, then the backlogged data in the first CBR system at time t is given by

$$\begin{aligned} W_{\mathrm{CBR}'}(t) &= \sup_{0\leq s\leq t}\{x_{\mathrm{CBR}'}(t) - x_{\mathrm{CBR}'}(s) - P(t-s)\} \leq \sup_{0\leq s\leq t}\{a(t) - a(s) - P(t-s)\} \\ &\leq \sup_{0\leq s\leq t}\{a(t) - a(s) - \sigma_{\mathrm{VBR}}(t-s)\} = \sup_{0\leq s\leq t}\{x_{\mathrm{VBR}}(t) - x_{\mathrm{VBR}}(s) - \sigma_{\mathrm{VBR}}(t-s)\} \\ &= W_{\mathrm{VBR}}(t). \end{aligned}$$

If $v > 0$, it means that the VBR system was congested at time v, and so that $W_{\text{VBR}}(v) = X$. Then

$$\begin{aligned}
W_{\text{CBR}'}(t) &= \sup_{0\le s\le t}\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(s) - P(t-s)\} \\
&= \sup_{0\le s\le v}\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(s) - P(t-s)\} \vee \sup_{v\le s\le t}\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(s) - P(t-s)\} \\
&= \sup_{0\le s\le v}\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(v) - P(t-v) + x_{\text{CBR}'}(v) - x_{\text{CBR}'}(s) - P(v-s)\} \\
&\qquad\qquad \vee \sup_{v\le s\le t}\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(s) - P(t-s)\} \\
&= \left\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(v) - P(t-v) + \sup_{0\le s\le v}\{x_{\text{CBR}'}(v) - x_{\text{CBR}'}(s) - P(v-s)\}\right\} \\
&\qquad\qquad \vee \sup_{v\le s\le t}\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(s) - P(t-s)\} \\
&= \{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(v) - P(t-v) + W_{\text{CBR}'}(v)\} \vee \sup_{v\le s\le t}\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(s) - P(t-s)\} \\
&\le \{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(v) - P(t-v) + X\} \vee \sup_{v\le s\le t}\{x_{\text{CBR}'}(t) - x_{\text{CBR}'}(s) - P(t-s)\} \\
&\le \{a(t) - a(v) - P(t-v) + X\} \vee \sup_{v\le s\le t}\{a(t) - a(s) - P(t-s)\} \\
&= \{x_{\text{VBR}}(t) - x_{\text{VBR}}(v) - P(t-v) + X\} \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - P(t-s)\} \\
&\le \{x_{\text{VBR}}(t) - x_{\text{VBR}}(v) - \sigma_{\text{VBR}}(t-v) + X\} \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(t-s)\} \\
&= \{x_{\text{VBR}}(t) - x_{\text{VBR}}(v) - \sigma_{\text{VBR}}(t-v) + W_{\text{VBR}}(v)\} \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(t-s)\} \\
&= \left\{x_{\text{VBR}}(t) - x_{\text{VBR}}(v) - \sigma_{\text{VBR}}(t-v) + \sup_{0\le s\le v}\{x_{\text{VBR}}(v) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(v-s)\}\right\} \\
&\qquad\qquad \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(t-s)\} \\
&= \sup_{0\le s\le v}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(t-v) - \sigma_{\text{VBR}}(v-s)\} \\
&\qquad\qquad \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(t-s)\} \\
&\le \sup_{0\le s\le v}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(t-s)\} \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(t-s)\} \\
&= \sup_{0\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(t-s)\} \\
&= W_{\text{VBR}}(t),
\end{aligned}$$

which proves part (i) of the lemma.

(ii) Let t be a given time, and let $0 \le v \le t$ be the smallest time such that the second CBR system (with service curve $\sigma_{\text{CBR}''}(t) = Mt$) has no loss in the system in $[v, t]$. The traffic that actually entered this system during $[v, t]$ is therefore identical to the traffic that arrived during this time interval: $x_{\text{CBR}''}(t) - x_{\text{CBR}''}(v) = a(t) - a(v)$.

If $v = 0$, then the backlogged data in the second CBR system at time t is given by

$$\begin{aligned}
W_{\text{CBR}''}(t) &= \sup_{0\le s\le t}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(s) - M(t-s)\} = \sup_{0\le s\le t}\{a(t) - a(s) - M(t-s)\} \\
&\ge \sup_{0\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-s)\}.
\end{aligned}$$

If $v > 0$, it means that the Second CBR system was congested at time v, and so that $W_{\text{CBR}''}(v) = X + B$. Then

$$
\begin{aligned}
W_{\text{CBR}''}(t) &= \sup_{0\le s\le t}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(s) - M(t-s)\} \\
&= \sup_{0\le s\le v}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(s) - M(t-s)\} \vee \sup_{v\le s\le t}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(s) - M(t-s)\} \\
&= \sup_{0\le s\le v}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(v) - M(t-v) + x_{\text{CBR}''}(v) - x_{\text{CBR}''}(s) - M(v-s)\} \\
&\qquad \vee \sup_{v\le s\le t}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(s) - M(t-s)\} \\
&= \left\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(v) - M(t-v) + \sup_{0\le s\le v}\{x_{\text{CBR}''}(v) - x_{\text{CBR}''}(s) - M(v-s)\}\right\} \\
&\qquad \vee \sup_{v\le s\le t}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(s) - M(t-s)\} \\
&= \{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(v) - M(t-v) + W_{\text{CBR}''}(v)\} \vee \sup_{v\le s\le t}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(s) - M(t-s)\} \\
&= \{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(v) - M(t-v) + X + B\} \vee \sup_{v\le s\le t}\{x_{\text{CBR}''}(t) - x_{\text{CBR}''}(s) - M(t-s)\} \\
&= \{a(t) - a(v) - M(t-v) + X + B\} \vee \sup_{v\le s\le t}\{a(t) - a(s) - M(t-s)\} \\
&\ge \{a(t) - a(v) - M(t-v) + W_{\text{VBR}}(v) + B\} \vee \sup_{v\le s\le t}\{a(t) - a(s) - M(t-s)\} \\
&\ge \{x_{\text{VBR}}(t) - x_{\text{VBR}}(v) - M(t-v) + W_{\text{VBR}}(v) + B\} \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-s)\} \\
&= \left\{x_{\text{VBR}}(t) - x_{\text{VBR}}(v) - M(t-v) + \sup_{0\le s\le v}\{x_{\text{VBR}}(v) - x_{\text{VBR}}(s) - \sigma_{\text{VBR}}(v-s)\} + B\right\} \\
&\qquad \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-s)\} \\
&= \sup_{0\le s\le v}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-v) - \sigma_{\text{VBR}}(v-s) + B\} \\
&\qquad \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-s)\} \\
&\ge \sup_{0\le s\le v}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-v) - (M(v-s) + B) + B\} \\
&\qquad \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-s)\} \\
&= \sup_{0\le s\le v}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-s)\} \vee \sup_{v\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-s)\} \\
&= \sup_{0\le s\le t}\{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t-s)\},
\end{aligned}
$$

which proves part (ii) of the lemma. ■

We can now establish the following result.

Theorem 4 *Suppose that at time t a VBR shaper with buffer X and shaping curve $\sigma_{VBR}(t) = \{Pt\} \wedge \{Mt + B\}$ is congested (i.e. is suffering losses) when the data that has arrived in $[0, t]$ is $a(t)$.*

Then at least one of the following CBR systems is also congested at time t when the data that has arrived in $[0, t]$ is $a(t)$: either a shaper with buffer X and shaping curve $\sigma_{CBR'}(t) = Pt$, or a shaper with buffer $(X + B)$ and shaping curve $\sigma_{CBR''}(t) = Mt$.

Proof: If the VBR system is congested at time t, there exists some time $s^\star$, $0 \le s^\star < t$, such that

$$x_{\text{VBR}}(t) = x_{\text{VBR}}(s^\star) + \sigma_{\text{VBR}}(t - s^\star) + X. \tag{46}$$

Note first that the buffer of the VBR system is empty at time $s^\star$. Indeed, suppose that at this time $s^\star$, the backlogged data in the buffer is nonzero, i.e. that

$$0 < W_{\text{VBR}}(s^\star) = x_{\text{VBR}}(s^\star) - \inf_{0 \le u \le s^\star} \{x_{\text{VBR}}(u) - \sigma_{\text{VBR}}(s^\star - u)\}.$$

Then, since $\sigma_{\text{VBR}}(\cdot)$ is sub-additive,

$$\begin{aligned}
W_{\text{VBR}}(t) &= x_{\text{VBR}}(t) - x_{\text{VBR}}(s^\star) - \sigma_{\text{VBR}}(t - s^\star) \\
&< x_{\text{VBR}}(t) - \inf_{0 \le u \le s^\star} \{x_{\text{VBR}}(u) - \sigma_{\text{VBR}}(s^\star - u)\} - \sigma_{\text{VBR}}(t - s^\star) \\
&= \sup_{0 \le u \le s^\star} \{x_{\text{VBR}}(t) - x_{\text{VBR}}(u) - \sigma_{\text{VBR}}(s^\star - u) - \sigma_{\text{VBR}}(t - s^\star)\} \\
&\le \sup_{0 \le u \le s^\star} \{x_{\text{VBR}}(t) - x_{\text{VBR}}(u) - \sigma_{\text{VBR}}(t - u)\} \\
&\le \sup_{0 \le u \le t} \{x_{\text{VBR}}(t) - x_{\text{VBR}}(u) - \sigma_{\text{VBR}}(t - u)\} = W_{\text{VBR}}(t),
\end{aligned}$$

a contradiction. Therefore $W_{\text{VBR}}(s^\star) = 0$.

We consider now two cases: (i) $s^\star \ge t - B/(P - M)$ and (ii) $s^\star < t - B/(P - M)$.

(i) If $s^\star \ge t - B/(P - M)$, one easily checks that $\sigma_{\text{VBR}}(t - s^\star) = P(t - s^\star)$. Hence, for all times in $[s^\star, t]$, the service curve of the VBR system is identical to the service curve of the of the first CBR system:

$$\sigma_{\text{VBR}}(t - s^\star) = P(t - s^\star) = \sigma_{\text{CBR}'}(t - s^\star).$$

The buffer of the VBR system at time $s^\star$ is empty, as we have seen above. Because of Part (i) of Lemma 3, the buffer of the first CBR system must also be empty:

$$W_{\text{VBR}}(s^\star) = W_{\text{CBR}'}(s^\star) = 0.$$

Finally, both systems receive during $[s^\star, t]$ an identical amount of traffic given by $(a(t) - a(s^\star))$. Consequently, all variables in the systems are identical in the time interval $[s^\star, t]$, because the input and characteristics of both systems are identical during this time interval. In particular,

$$W_{\text{CBR}'}(t) = W_{\text{VBR}}(t) = X$$

which shows that the first CBR system is congested.

(ii) If $s^\star < t - B/(P - M)$, one easily checks that $\sigma_{\text{VBR}}(t - s^\star) = M(t - s^\star) + B$.

In this case, by making use of Part (ii) of Lemma 3 and of (46), we can write that,

$$\begin{aligned}
W_{\text{CBR}''}(t) &\ge \sup_{0 \le s \le t} \{a(t) - a(s) - (L_{\text{VBR}}(t) - L_{\text{VBR}}(s)) - M(t - s)\} \\
&= \sup_{0 \le s \le t} \{x_{\text{VBR}}(t) - x_{\text{VBR}}(s) - M(t - s)\} \\
&\ge x_{\text{VBR}}(t) - x_{\text{VBR}}(s^\star) - M(t - s^\star) \\
&= \sigma_{\text{VBR}}(t - s^\star) + X - M(t - s^\star) \\
&= M(t - s^\star) + B + X - M(t - s^\star) = B + X.
\end{aligned}$$

As $W_{\text{CBR}''}(t) \leq B + X$, the latter indicates that $W_{\text{CBR}''}(t) = B + X$ and shows that the second CBR system is congested. ■

ii) Bound by two CBR shapers in tandem

We will now derive another bound on *amount of losses* during $[0, t]$ in the VBR system by computing the sum of losses in two CBR shapers in cascade as shown in Figure 5(b): the first one has buffer of size X and rate P, and receives the same arriving traffic a as the original VBR shaper, whereas its output is fed into the second one with buffer of size B and rate M. All variables without prime refer to the original VBR shaper, variables with one prime to the first CBR shaper and variables with a triple prime to the second CBR shaper.

Theorem 5 *Let $L_{VBR}(t)$ be the amount of lost data in the time interval $[0, t]$ in a VBR shaper with buffer X and shaping curve $\sigma_{VBR}(t) = \{Pt\} \wedge \{Mt + B\}$, when the data that has arrived in $[0, t]$ is $a(t)$.*

Let $L_{CBR'}(t)$ (resp. $L_{CBR'''}(t)$) be the amount of lost data during $[0, t]$ in a CBR shaper with buffer X (resp. B) and shaping curve $\sigma_{CBR'}(t) = Pt$ (resp. $\sigma_{CBR'''}(t) = Mt$) fed by the same incoming traffic $a(t)$ (resp. the output traffic of the first CBR shaper)

Then $L_{VBR}(t) \leq L_{CBR'}(t) + L_{CBR'''}(t)$.

Proof: The proof is very similar to the proof of Theorem 3. With $l^{(k)}(t)$ defined by (44), we will show by induction that

$$l^{(k)}_{\text{VBR}}(t) \leq L_{\text{CBR}'}(t) + L_{\text{CBR}'''}(t). \tag{47}$$

Clearly, this relation holds (with equality sign) for $k = 0$, as in this case the left hand side of (45) is zero.

Call $y_{\text{CBR}'}(t)$ the output of the first shaper system. Note that for any $s \geq 0$

$$a(s) - L_{\text{CBR}'}(s) - X = x_{\text{CBR}'}(s) - X \leq y_{\text{CBR}'}(s) \leq x_{\text{CBR}'}(s) = a(s) - L_{\text{CBR}'}(s)$$

Suppose that (45) holds until iteration k.

Then for the VBR system we can be write that

$$
\begin{aligned}
l_{\mathrm{VBR}}^{(k+1)}(t) &= \sup_{0\le s_2\le s_1\le t}\left\{a(s_1)-a(s_2)-\sigma_{\mathrm{VBR}}(s_1-s_2)-X+l_{\mathrm{VBR}}^{(k)}(s_2)\right\} \\
&= \sup_{0\le s_2\le s_1\le t}\left\{a(s_1)-a(s_2)-P(s_1-s_2)-X+l_{\mathrm{VBR}}^{(k)}(s_2)\right\} \\
&\qquad \vee \sup_{0\le s_2\le s_1\le t}\left\{a(s_1)-a(s_2)-M(s_1-s_2)-B-X+l_{\mathrm{VBR}}^{(k)}(s_2)\right\} \\
&\le \sup_{0\le s_2\le s_1\le t}\{a(s_1)-a(s_2)-P(s_1-s_2)-X+L_{\mathrm{CBR}'}(s_2)+L_{\mathrm{CBR}'''}(s_2)\} \\
&\quad \vee \sup_{0\le s_2\le s_1\le t}\{a(s_1)-a(s_2)-M(s_1-s_2)-B-X+L_{\mathrm{CBR}'}(s_2)+L_{\mathrm{CBR}'''}(s_2)\} \\
&\le \sup_{0\le s_2\le s_1\le t}\{a(s_1)-a(s_2)-P(s_1-s_2)-X+L_{\mathrm{CBR}'}(s_2)+L_{\mathrm{CBR}'''}(s_2)\} \\
&\quad \vee \sup_{0\le s_2\le s_1\le t}\{(y(s_1)+X+L_{\mathrm{CBR}'}(s_1))-(y(s_2)+L_{\mathrm{CBR}'}(s_2)) \\
&\qquad\qquad -M(s_1-s_2)-B-X+L_{\mathrm{CBR}'}(s_2)+L_{\mathrm{CBR}'''}(s_2)\} \\
&\le \left\{\sup_{0\le s_2\le s_1\le t}\left\{a(s_1)-a(s_2)-P(s_1-s_2)-X+\sup_{k\ge 0}\left\{l_{\mathrm{CBR}'}^{(k)}(s_2)\right\}\right\}\right. \\
&\qquad\qquad \left. + \sup_{0\le s_2\le s_1\le t}\{L_{\mathrm{CBR}'''}(s_2)\}\right\} \\
&\quad \vee \left\{\sup_{0\le s_2\le s_1\le t}\left\{y(s_1)-y(s_2)-M(s_1-s_2)-B-X+\sup_{k\ge 0}\left\{l_{\mathrm{CBR}'''}^{(k)}(s_2)\right\}\right\}\right. \\
&\qquad\qquad \left. + \sup_{0\le s_2\le s_1\le t}\{L_{\mathrm{CBR}'}(s_1)\}\right\} \\
&= \left\{\sup_{k\ge 0}\left\{\sup_{0\le s_2\le s_1\le t}\left\{a(s_1)-a(s_2)-P(s_1-s_2)-X+l_{\mathrm{CBR}'}^{(k)}(s_2)\right\}\right\}+L_{\mathrm{CBR}'''}(t)\right\} \\
&\quad \vee \left\{\sup_{k\ge 0}\left\{\sup_{0\le s_2\le s_1\le t}\left\{y(s_1)-y(s_2)-M(s_1-s_2)-B+l_{\mathrm{CBR}'''}^{(k)}(s_2)\right\}\right\}+L_{\mathrm{CBR}'}(t)\right\} \\
&= \left\{\sup_{k\ge 0}\left\{l_{\mathrm{CBR}'}^{(k+1)}(t)\right\}+L_{\mathrm{CBR}'''}(t)\right\} \vee \left\{\sup_{k\ge 0}\left\{l_{\mathrm{CBR}'''}^{(k+1)}(t)\right\}+L_{\mathrm{CBR}'}(t)\right\} \\
&\le L_{\mathrm{CBR}'}(t)+L_{\mathrm{CBR}'''}(t).
\end{aligned}
$$

Therefore (45) holds for $k+1$. By taking the supremum over all k, the theorem is proven. ∎

None of the two systems in Figure 5 gives a better bound for any traffic pattern. For example, suppose that the VBR system parameters are $P = 4$, $M = 1$, $B = 12$ and $X = 4$, and that the traffic is a single burst of data sent at rate R during 4 time units, so that

$$
a(t) = \begin{cases} R\cdot t & \text{if} \quad 0 \le t \le 4 \\ 4R & \text{if} \quad t \ge 4 \end{cases} \tag{48}
$$

If $R = 5$, both the VBR system and the parallel set of the two CBR$'$ and CBR$''$ systems are lossless, whereas the amount of lost data after 5 units of time in the tandem of the two CBR$'$ and CBR$'''$ systems is equal to 3.

On the other hand, if $R = 6$, the amount of lost data after 5 units of time in the VBR system, the parallel system (CBR$'$ and CBR$''$) and the tandem system (CBR$'$ and CBR$'''$) are respectively equal to 4, 8 and 7.

6.1.2 Application 2: Bound of losses by segregation between buffer and policer

As second application, we will compare the losses in two systems, having the same input flow $a(t)$.

The first system is the shaper of Figure 3 with shaping curve σ and buffer X, whose losses $L(t)$ are therefore given by (42).

The second system is made of two parts, as shown in Figure 6(a). The first part is a system with storage capacity X, that realizes some mapping $\Pi(\cdot)$ of the input which is not explicitly given. We know however that a first controller discards data as soon as the total backlogged data in this system exceeds X. This operation is called *buffer discard*, the amount of buffer discarded data in $[0,t]$ is denoted by $L_{\text{Buf}}(t)$. The second part is a policer without buffer, and with shaping curve σ. Similarly, a second controller discards data as soon as the total output flow of the storage system exceeds the maximum output allowed by the policer. This operation is called *policing discard*, the amount of discarded data by policing in $[0,t]$ is denoted by $L_{\text{Pol}}(t)$.

Theorem 6 *Let $L(t)$ be the amount of lost data in a shaper with shaping curve $\sigma(t)$ and buffer X.*

Let $L_{\text{Buf}}(t)$ (resp. $L_{\text{Pol}}(t)$ be the amount of data lost in the time interval $[0,t]$ by buffer (resp. policing) discard, as defined above.

Then $L(t) \leq L_{\text{Buf}}(t) + L_{\text{Pol}}(t)$.

Proof: Let x and y denote respectively the admitted and output flows of the buffered part of the second system. They are linked by the following constraints: for any $s \geq 0$,

$$a(s) - L_{\text{Buf}}(s) - X = x(s) - X \leq y(s) \leq x(s) = a(s) - L_{\text{Buf}}(s)$$

which, together with the fact that $L_{\text{Buf}}(t)$ is non decreasing, implies that

$$\begin{aligned}
L_{\text{Pol}}(t) &= \sup_{k\geq 0}\left\{\sup_{0\leq s_{2k}\leq\ldots\leq s_2\leq s_1\leq t}\left\{\sum_{i=1}^{k}\left[y(s_{2i-1}) - y(s_{2i}) - \sigma(s_{2i-1}-s_{2i})\right]\right\}\right\}\\
&\geq \sup_{k\geq 0}\left\{\sup_{0\leq s_{2k}\leq\ldots\leq s_1\leq t}\left\{\sum_{i=1}^{k}\left[(a(s_{2i-1}) - L_{\text{Buf}}(s_{2i-1}) - X) - (a(s_{2i}) - L_{\text{Buf}}(s_{2i})) - \sigma(s_{2i-1}-s_{2i})\right]\right\}\right\}\\
&= \sup_{k\geq 0}\left\{\sup_{0\leq s_{2k}\leq\ldots\leq s_1\leq t}\left\{\sum_{i=1}^{k}\left[a(s_{2i-1}) - a(s_{2i}) - \sigma(s_{2i-1}-s_{2i})\right] - \sum_{i=1}^{k}\left[L_{\text{Buf}}(s_{2i-1}) - L_{\text{Buf}}(s_{2i})\right]\right\} - kX\right\}\\
&\geq \sup_{k\geq 0}\left\{\sup_{0\leq s_{2k}\leq\ldots\leq s_1\leq t}\left\{\sum_{i=1}^{k}\left[a(s_{2i-1}) - a(s_{2i}) - \sigma(s_{2i-1}-s_{2i})\right] - L_{\text{Buf}}(t)\right\} - kX\right\}\\
&= \sup_{k\geq 0}\left\{\sup_{0\leq s_{2k}\leq\ldots\leq s_1\leq t}\left\{\sum_{i=1}^{k}\left[a(s_{2i-1}) - a(s_{2i}) - \sigma(s_{2i-1}-s_{2i})\right]\right\} - kX\right\} - L_{\text{Buf}}(t)\\
&= L(t) - L_{\text{Buf}}(t),
\end{aligned}$$

which establishes the desired result. ■

Such a separation of resources between "buffered system" and "policing system" is used in the estimation of loss probability for devising statistical CAC (Call Acceptance Control) algorithms as proposed by Lo

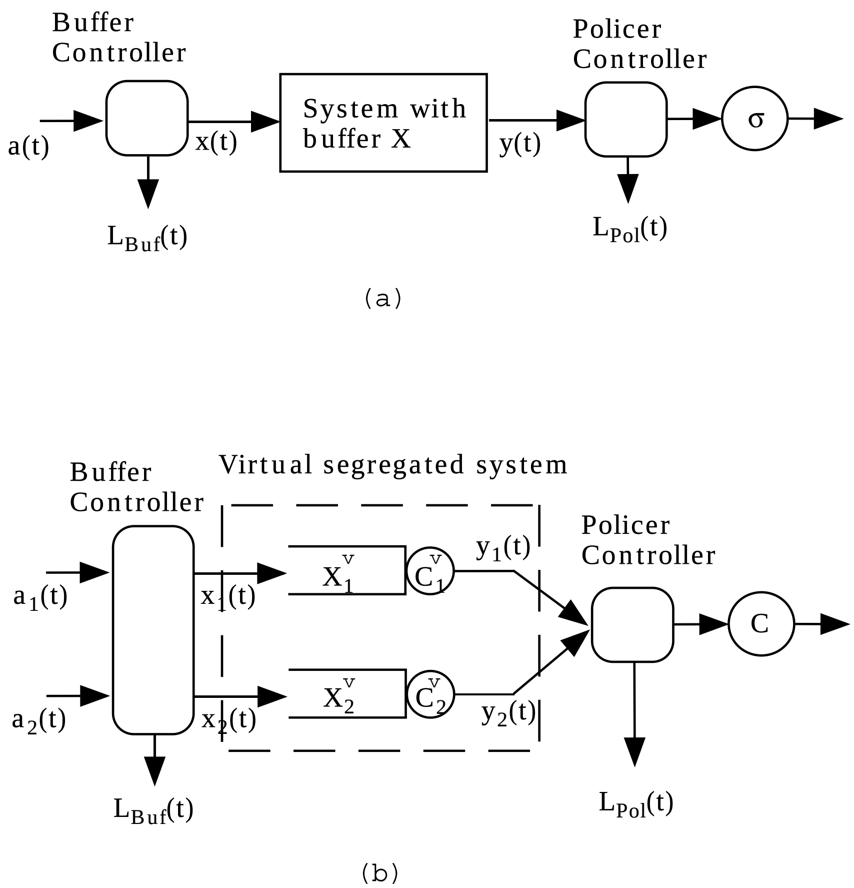

Figure 6: A storage/policer system with separation between losses due to buffer discard and to policing discard (a) A virtual segregated system for 2 classes of traffic, with buffer discard and policing discard, as used by Lo Presti et al [13] (b)

Presti et al. [13] (see also Elwalid et al [14]). The incoming traffic is separated in two classes. All variables relating to the first (resp. second) class are marked with a index 1 (resp. 2), so that $a(t) = a_1(t) + a_2(t)$. The shaper is of CBR type ($\sigma(t) = Ct$) and the storage system is a virtually segregated system as in Figure 6(b), made of 2 shapers with rates C_1^v and C_2^v and buffers X_1^v and X_2^v. The virtual shapers are large enough to ensure that no loss occurs for all possible arrival functions $a_1(t)$ and $a_2(t)$. The total buffer space (resp. bandwidth) is larger than the original buffer space (resp. bandwidth): $X_1^v + X_2^v \geq X$ ($C_1^v + C_2^v \geq C$). However, the buffer controller discards data as soon as the total backlogged data in the virtual system exceeds X and the policer controller discards data as soon as the total output rate of the virtual system exceeds C.

We have shown that the losses in this system are indeed an upper bound on the losses in the original CBR shaper with rate C and buffer X. However, contrary to the first application, this is no longer true for congestion times, at least along all sample paths of the process. Ross [15] has indeed provided an example where the sum of congestion times in the virtual system due to buffer and policing discards are not an upper bound to the sum of congestion times in the original system.

6.2 Example 5

Equation (31) for the last example is very similar to (31), with the operator $\mathcal{C}_{\sigma_X}$ replaced by $\mathcal{C}_{\sigma^d}$. ¿From Theorem 1, its solution is

$$x = \overline{h_a \wedge \mathcal{C}_{\sigma^d}}(a) \tag{49}$$

Lemma 2 yields that

$$\begin{aligned}
\overline{h_a \wedge \mathcal{C}_{\sigma^d}} &= \inf_{n\geq 1}\left\{\inf_{1\leq q\leq (n-1)/2}\left\{\inf_{l_1+\ldots+l_q=n-q-1,\ l_1,\ldots,l_q\geq 1}\left\{h_a\circ\mathcal{C}_{\sigma^d}^{(l_q)}\circ\ldots\circ h_a\circ\mathcal{C}_{\sigma^d}^{(l_1)}\circ h_a\right\}\right\}\right\}\\
&= \inf_{n\geq 1, n \text{ odd}}\left\{\left\{\inf_{l_1+\ldots+l_{(n-1)/2}=(n-1)/2,\ l_1,\ldots,l_{(n-1)/2}\geq 1}\left\{h_a\circ\mathcal{C}_{\sigma^d}^{(l_{(n-1)/2})}\circ\ldots\circ h_a\circ\mathcal{C}_{\sigma^d}^{(l_1)}\circ h_a\right\}\right\}\right.\\
&\qquad \left.\wedge \inf_{1\leq q\leq (n-3)/2}\left\{\inf_{l_1+\ldots+l_q=n-q-1,\ l_1,\ldots,l_q\geq 1}\left\{h_a\circ\mathcal{C}_{\sigma^d}^{(l_q)}\circ\ldots\circ h_a\circ\mathcal{C}_{\sigma^d}^{(l_1)}\circ h_a\right\}\right\}\right\}\\
&\qquad \wedge \inf_{n\geq 1, n \text{ even}}\left\{\inf_{1\leq q\leq (n-2)/2}\left\{\inf_{l_1+\ldots+l_q=n-q-1,\ l_1,\ldots,l_q\geq 1}\left\{h_a\circ\mathcal{C}_{\sigma^d}^{(l_q)}\circ\ldots\circ h_a\circ\mathcal{C}_{\sigma^d}^{(l_1)}\circ h_a\right\}\right\}\right\}.
\end{aligned}$$

In this latter expression, all the indices $l_1,\ldots,l_{(n-1)/2}$ of the first term must all be equal to 1, because their sum is $(n-1)/2$. Conversely, at least one index among $l_1,\ldots,l_q$ in both the second and third terms must be strictly larger than 1, because their sum always exceeds q.

Now, for any integer $k\geq 1$, note that

$$\mathcal{C}_{\sigma^d}^{(k)}(r) \geq \mathcal{C}_{\sigma^{kd}}(r) \geq \mathcal{C}_{\sigma^d}(r) \tag{50}$$

Indeed, since σ is sub-additive, we have that

$$\begin{aligned}
\{\mathcal{C}_{\sigma^d}\circ\mathcal{C}_{\sigma^d}\}(r)(t) &= \inf_{0\leq s_2\leq s_1\leq t}\{\sigma(t+d-s_1)+\sigma(s_1+d-s_2)+r(s_2)\}\\
&\geq \inf_{0\leq s_2\leq s_1\leq t}\{\sigma(t+2d-s_2)+r(s_2)\} = \mathcal{C}_{\sigma^{2d}}(r)(t) \geq \mathcal{C}_{\sigma^d}(r).
\end{aligned}$$

Relation (50) easily follows by induction for any $k>2$, and yields that

$$h_a\circ\mathcal{C}_{\sigma^d}^{(l_q)}\circ\ldots\circ h_a\circ\mathcal{C}_{\sigma^d}^{(l_1)}\circ h_a \geq h_a\circ\mathcal{C}_{\sigma^d}\circ\ldots\circ h_a\circ\mathcal{C}_{\sigma^d}\circ h_a = (h_a\circ\mathcal{C}_{\sigma^d})^{(q)}\circ h_a,$$

and so that

$$x = \overline{h_a\wedge\mathcal{C}_{\sigma^d}}(a) = \inf_{n\geq 1, n \text{ odd}}\left\{(h_a\circ\mathcal{C}_{\sigma^d})^{((n-1)/2)}\circ h_a\right\}(a) = \inf_{k\geq 0}\left\{(h_a\circ\mathcal{C}_{\sigma^d})^{(k)}\right\}(a) = \overline{h_a\circ\mathcal{C}_{\sigma^d}}(a).$$

Making similar computations as in Example 4, the amount of lost data in the interval $[0,t]$ is found to be

$$L(t) = \sup_{k\geq 0}\left\{\sup_{0\leq s_{2k}\leq\ldots\leq s_2\leq s_1\leq t}\left\{\sum_{i=1}^{k}\left[a(s_{2i-1})-a(s_{2i})-\sigma(s_{2i-1}+d-s_{2i})\right]\right\}\right\}. \tag{51}$$

If we know that the arriving traffic a is constrained by an arrival curve $\alpha(\cdot)$, we can also bound the amount of lost data by

$$L(t) \leq \sup_{k\geq 0}\left\{\sup_{u_1,\ldots,u_k\geq 0;\ \sum_{i=1}^{k}u_i\leq t}\left\{\sum_{i=1}^{k}\left[\alpha(u_i)-\sigma(u_i+d)\right]\right\}\right\}. \tag{52}$$

Note that for a CBR shaping curve $\sigma(t)=Pt$, (42) and (51) are identical if $X=Pd$. In other words, the losses in a CBR shaper, with output rate P and buffer X, are identical to the losses a CBR shaper (with the same output rate) discarding any piece of data that would suffer a delay larger than $d=X/P$ units of time. This is no longer true for a VBR shaper.

Application: Bound on the losses in a VBR shaper

Let us return to the application of the bound on losses in a VBR shaper. In the previous example, we had shown how it is possible to bound the losses in a "buffered leaky bucket" shaper by losses in two CBR systems.

We now show that a dual result holds when the losses are caused not by a finite buffer overflow, but by the enforcement of a maximum delay tolerance d. We take the same leaky bucket shaper, whose output must conform to a VBR shaping curve with peak rate P, sustainable rate M and burst tolerance B, but where the buffer is now infinite. Instead, any data that would not exit the VBR system after at most d unit of time is discarded.

We will show that the *amount of losses* during $[0,t]$ in this system is bounded by the sum of losses in two CBR shapers: the first one, having a rate P enforces the same maximum delay tolerance d as the original VBR system, whereas the second one, having rate M, enforces a looser maximum delay tolerance equal to $d+B/M$. Both receive the same arriving traffic a as the original VBR shaper. All variables without prime refer to the original VBR shaper, variable with one prime to the first CBR shaper and variables with a double prime to the second CBR shaper.

Theorem 7 *Let $L_{VBR}(t)$ be the amount of losses in the time interval $[0,t]$ due a delay constraint of d in a VBR shaper with shaping curve $\sigma_{VBR}(t)=\{Pt\}\wedge\{Mt+B\}$, when the data that has arrived in $[0,t]$ is $a(t)$.*

Let $L_{CBR'}(t)$ (resp. $L_{CBR''}(t)$) be the amount of lost data during $[0,t]$ due a delay constraint of d (resp. $d+B/M$) in a CBR shaper with shaping curve $\sigma_{CBR'}(t)=Pt$ (resp. $\sigma_{CBR''}(t)=Mt$) with the same incoming traffic $a(t)$.

Then $L_{VBR}(t)\le L_{CBR'}(t)+L_{CBR''}(t)$.

Proof: Define

$$l^{(k)}(t)=\sup_{0\le s_{2k}\le\ldots\le s_2\le s_1\le t}\left\{\sum_{i=1}^{k}\left[a(s_{2i-1})-a(s_{2i})-\sigma(s_{2i-1}+d-s_{2i})\right]\right\}\tag{53}$$

so that (42) can be rewritten as $L(t)=\sup_{k\ge 0}\{l^{(k)}(t)\}$. Note that relation (44) can be recast recursively as

$$l^{(k)}(t)=\sup_{0\le s_2\le s_1\le t}\left\{a(s_1)-a(s_2)-\sigma(s_1+d-s_2)+l^{(k-1)}(s_2)\right\}.$$

We will show by induction that

$$l^{(k)}_{\text{VBR}}(t)\le L_{\text{CBR}'}(t)+L_{\text{CBR}''}(t).\tag{54}$$

Clearly, this relation holds (with equality sign) for $k=0$, as in this case all terms are zero ($l^{(0)}(t)=0$). One easily shows that this relation holds (again with equality sign) for $k=1$. Indeed,

$$\begin{aligned}
l^{(1)}_{\text{VBR}}(t) &= \sup_{0\le s_2\le s_1\le t}\{a(s_1)-a(s_2)-\sigma_{\text{VBR}}(s_1+d-s_2)\}\\
&= \sup_{0\le s_2\le s_1\le t}\{a(s_1)-a(s_2)-P(s_1+d-s_2)\}\\
&\qquad \vee \sup_{0\le s_2\le s_1\le t}\{a(s_1)-a(s_2)-M(s_1+d-s_2)-X-B\}\\
&\le L_{\text{CBR}'}(t)+L_{\text{CBR}''}(t).
\end{aligned}$$

Suppose that (45) holds until iteration k. Then for the VBR system we can be write that

$$
\begin{aligned}
l_{\text{VBR}}^{(k+1)}(t) &= \sup_{0 \le s_2 \le s_1 \le t} \left\{ a(s_1) - a(s_2) - \sigma_{\text{VBR}}(s_1 + d - s_2) + l_{\text{VBR}}^{(k)}(s_2) \right\} \\
&= \sup_{0 \le s_2 \le s_1 \le t} \left\{ a(s_1) - a(s_2) - P(s_1 + d - s_2) + l_{\text{VBR}}^{(k)}(s_2) \right\} \\
&\qquad \vee \sup_{0 \le s_2 \le s_1 \le t} \left\{ a(s_1) - a(s_2) - M(s_1 + d - s_2) + l_{\text{VBR}}^{(k)}(s_2) \right\} \\
&\le \sup_{0 \le s_2 \le s_1 \le t} \{ a(s_1) - a(s_2) - P(s_1 + d - s_2) + L_{\text{CBR}'}(s_2) + L_{\text{CBR}''}(s_2) \} \\
&\quad \vee \sup_{0 \le s_2 \le s_1 \le t} \{ a(s_1) - a(s_2) - M(s_1 + (d + B/M) - s_2) + L_{\text{CBR}'}(s_2) + L_{\text{CBR}''}(s_2) \} \\
&\le \left\{ \sup_{0 \le s_2 \le s_1 \le t} \left\{ a(s_1) - a(s_2) - P(s_1 + d - s_2) + \sup_{k \ge 0} \{ l_{\text{CBR}'}^{(k)}(s_2) \} \right\} + \sup_{0 \le s_2 \le s_1 \le t} \{ L_{\text{CBR}''}(s_2) \} \right\} \\
&\quad \vee \left\{ \sup_{0 \le s_2 \le s_1 \le t} \left\{ a(s_1) - a(s_2) - M(s_1 + (d + B/M) - s_2) + \sup_{k \ge 0} \{ l_{\text{CBR}''}^{(k)}(s_2) \} \right\} \right. \\
&\qquad\qquad \left. + \sup_{0 \le s_2 \le s_1 \le t} \{ L_{\text{CBR}'}(s_2) \} \right\} \\
&= \left\{ \sup_{k \ge 0} \left\{ \sup_{0 \le s_2 \le s_1 \le t} \{ a(s_1) - a(s_2) - P(s_1 + d - s_2) - X + l_{\text{CBR}'}^{(k)}(s_2) \} \right\} + L_{\text{CBR}''}(t) \right\} \\
&\quad \vee \left\{ \sup_{k \ge 0} \left\{ \sup_{0 \le s_2 \le s_1 \le t} \{ a(s_1) - a(s_2) - M(s_1 + (d + B/M) - s_2) + l_{\text{CBR}'}^{(k)}(s_2) \} \right\} + L_{\text{CBR}'}(t) \right\} \\
&= \left\{ \sup_{k \ge 0} \{ l_{\text{CBR}'}^{(k+1)}(t) \} + L_{\text{CBR}''}(t) \right\} \vee \left\{ \sup_{k \ge 0} \{ l_{\text{CBR}''}^{(k+1)}(t) \} + L_{\text{CBR}'}(t) \right\} \\
&\le L_{\text{CBR}'}(t) + L_{\text{CBR}''}(t).
\end{aligned}
$$

Therefore (45) holds for $k+1$, and similarly to the proof of Theorem 3, we have $L_{\text{VBR}}(t) \le L_{\text{CBR}'}(t) + L_{\text{CBR}''}(t)$. ∎

7 A solution to Skorokhod's reflection problem with two boundaries

Not only Example 1, but Examples 4 and 5 can also be studied using reflection mappings [11]. However, the latter study, restricted to a specific set of shaping functions (such as CBR shaping curve, but excluding a VBR shaping curve like $\sigma(t) = \{Pt\} \wedge \{Mt + B\}$), does not result in explicit formulas such as (42) or (51). On the contrary, by formulating the problem in the network calculus framework, and by applying methods from min-plus algebra, an explicit and more general solution has been obtained. We will see that, in fact, an explicit solution to the general problem of reflection mapping known as Skorokhod's reflection problem can be obtained as a by-product of network calculus, for a large class of processes and 2 fixed boundaries [20, 11, 19, 17]

Let us first review this reflection mapping problem following the exposition of [11]. We are given a lower boundary that will be taken here as the origin, an upper boundary $X > 0$ and a *free process* $z(t)$ such that $0 \le z(0-) \le X$. Skorokhod's reflection problem looks for functions $N(t)$ (*lower boundary process*) and $L(t)$ (*upper boundary process*) such that

1. The *reflected process*

$$W(t) = z(t) + N(t) - L(t) \tag{55}$$

is in $[0, X]$ for all $t \geq 0$.

2. Both $N(t)$ and $L(t)$ are non decreasing with $N(0-) = L(0-) = 0$, and $N(t)$ (resp. $L(t)$) increases only when $W(t) = 0$ (resp. $W(t) = X$), i.e., with 1_A denoting the indicator function of A

$$\int_0^\infty 1_{\{W(t)>0\}} dN(t) = 0 \tag{56}$$

$$\int_0^\infty 1_{\{W(t)<X\}} dL(t) = 0 \tag{57}$$

The solution to this problem exists and is unique. When only one boundary is present, explicit formulas are available. For instance, if $X \to \infty$, then there is only one lower boundary, and the solution is easily found to be

$$\begin{aligned} N(t) &= -\inf_{0 \leq s \leq t} \{z(s)\} \\ L(t) &= 0. \end{aligned}$$

If $X < \infty$, then the solution can be constructed by successive approximations but no explicit solution exists to our knowledge. The following theorem gives such explicit solutions for a continuous VF functions $z(t)$. A VF function (VF standing for Variation Finie [17, 18]) $z(t)$ on $\mathbb{R}^+$ is a function such that for all $t > 0$

$$\sup_{n \geq 1} \sup_{0 = s_n < s_{n-1} < \ldots < s_1 < s_0 = t} \left\{ \sum_{i=0}^{n-1} |z(s_i) - z(s_{i+1})| \right\} < \infty.$$

VF functions have the following property [18]: $z(t)$ is a VF function on $\mathbb{R}^+$ if and only if it can be written as the difference of two increasing functions on $\mathbb{R}^+$.

Theorem 8 *Let the free process $z(t)$ be a continuous VF function on $\mathbb{R}^+$. Then the solution to Skorokhod's reflection problem on $[0, X]$ is*

$$N(t) = \sup_{k \geq 0} \left\{ \sup_{0 \leq s_{2k+1} \leq \ldots \leq s_2 \leq s_1 \leq t} \left\{ \sum_{i=1}^{2k+1} (-1)^i z(s_i) \right\} - kX \right\} \tag{58}$$

$$L(t) = \sup_{k \geq 0} \left\{ \sup_{0 \leq s_{2k} \leq \ldots \leq s_2 \leq s_1 \leq t} \left\{ \sum_{i=1}^{2k} (-1)^{i+1} z(s_i) \right\} - kX \right\}. \tag{59}$$

Proof: As $z(t)$ is a VF function on $[0, \infty)$, there exist two increasing functions $a(t)$ and $b(t)$ such that $z(t) = a(t) - b(t)$ for all $t \geq 0$. As $z(0) \geq 0$, we can take $b(0) = 0$ and $a(0) = z(0)$. Note that $a(t), b(t) \in \mathcal{F}_1$.

Let $(x(t), y(t))$ be the maximal solution of the following system of inequalities:

$$x(t) \leq \inf_{0 \leq s \leq t} \{a(t) - a(s) + x(s)\} \tag{60}$$

$$x(t) \leq y(t) + X \tag{61}$$

$$y(t) \leq x(t) \tag{62}$$

$$y(t) \leq \inf_{0 \leq s \leq t} \{b(t) - b(s) + y(s)\}. \tag{63}$$

Define

$$N(t) = b(t) - y(t)$$
$$L(t) = a(t) - x(t)$$

and let us show that these two functions are indeed the solutions to Skorokhod's reflection problem.

First note that

$$W(t) = z(t) + N(t) - L(t) = (a(t) - b(t)) + (b(t) - y(t)) - (a(t) - x(t)) = x(t) - y(t)$$

is in $[0, X]$ for all $t \geq 0$ because of (61) and (62).

Second, because of (63), note that $N(0) = b(0) - y(0) = 0$ and that for any $t > 0$ and $0 \leq s < t$, $N(t) - N(s) = b(t) - b(s) + y(s) - y(t) \geq 0$, which shows that $N(t)$ is non decreasing. The same properties can be deduced for $L(t)$ from (60).

Finally, if $W(t) = x(t) - y(t) > 0$, there exists some $s^\star \in [0, t]$ such that $y(t) = y(s^\star) + b(t) - b(s^\star)$ because y is the maximal solution satisfying (62) and (63). Therefore for all $s \in [s^\star, t]$,

$$0 \leq N(t) - N(s) \leq N(t) - N(s^\star) = b(t) - b(s^\star) + y(s^\star) - y(t) = 0$$

which shows that $N(t) - N(s) = 0$ and so that $N(t)$ is non increasing if $W(t) > 0$. A similar reasoning shows that $L(t)$ is non increasing if $W(t) < X$.

Consequently, $N(t)$ and $L(t)$ are the lower and upper reflected processes that we are looking for. We will now use the methods developed in this paper to compute these two functions explicitly. It is worth noting at this point that all variables defined above can be represented as in Figure 7 by a storage system of limited capacity X, with an arrival process $a(t)$, a departure process $b(t)$, the adjunction of a flow $N(t)$ if the buffer of the storage system is empty and the removal (loss) of a flow $L(t)$ if the storage system is full. The reflected process $W(t)$ is then the backlog in the storage system.

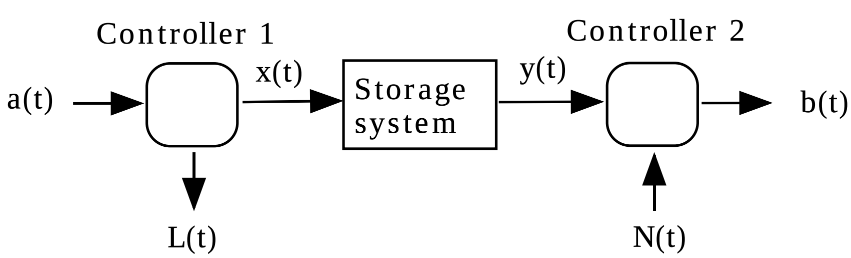

Figure 7: A storage system representing the variables used to solve Skorokhod's reflection problem with two boundaries

Let us now recast (60) to (63) in a framework well suited for applying Theorem 1. From (62) and (63), we get $y \leq x \wedge h_b(y)$ whose solution reads $y = \overline{h_b}(x) = h_b(x)$ because of Theorem 1 and of (16). Hence (61) is equivalent to $x \leq h_b(x) + X$. Noting that $x \leq a$ and defining $h_{b_X} = h_b + X$, we then get

$$x \leq a \wedge (h_a \wedge h_{b_X})(x).$$

One easily checks that $h_{b_X}^{(k)} = h_{b_{kX}} \geq h_{b_X}$ so that a similar development as the one made for Example 4 yields that

$$x = \overline{(h_a \circ h_{b_X})}(a)$$
$$y = h_b \circ \overline{(h_a \circ h_{b_X})}(a)$$

and after some manipulations that

$$
\begin{aligned}
N(t) &= \sup_{k\geq 0}\left\{\sup_{0\leq s_{2k+1}\leq\ldots\leq s_2\leq s_1\leq t}\left\{\sum_{i=1}^{2k+1}(-1)^i(a(s_i)-b(s_i))\right\}-kX\right\}\\
L(t) &= \sup_{k\geq 0}\left\{\sup_{0\leq s_{2k}\leq\ldots\leq s_2\leq s_1\leq t}\left\{\sum_{i=1}^{2k}(-1)^{i+1}(a(s_i)-b(s_i))\right\}-kX\right\}
\end{aligned}
$$

which establishes (58) and (59) since $a(s_i)-b(s_i)=z(s_i)$. ∎

8 Conclusion

We have proposed a methodology for solving a number of problems arising in communication networking using min-plus system theory. We have given an explicit expression of the sub-additive closure of the minimum of two operators, and we have introduced a new family of idempotent, linear and time-varying operators. These results enable us to compute service curves for lossless nonlinear systems with feedback control and to compute bounds for losses in linear systems.

In the lossless case, we have obtained in a systematic way the same service curves for two window flow control models [8, 7], while having new results for the second one. We have also obtained an explicit formulation of the optimal traffic shaper as defined in [12].

We have also modelled a lossy shaper by introducing a controlling device, and we have provided an explicit representation for the losses in a shaping device with either finite buffer or delay constraints. We have applied it for bounding the losses in a VBR shaper by those in two CBR systems, in series or in parallel. We have shown that the two CBR systems in parallel not only bound the amounts of lost data but also the congestion times of the VBR system. We have then shown that a separation between buffer and policer, as used in connection admission control (CAC) algorithms proposed in [13], also gives a bound on the losses in a buffered shaper. However, this is no longer true for congestion periods.

Finally, as a by-product of our method, we have obtained a closed-form solution of Skorokhod's deterministic reflection problem with two boundaries.

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